

Masterarbeit

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Abstract

Contents

1	Preliminaries	3
2	partial consistency improving	4
2.1	partial-conditions and -satisfiability	4
2.2	partial consistency improving	4
3	application condition	6

1 Preliminaries

Definition 1.1 (subgraph). Let G_1 and G_2 be graphs. The graph G_2 is called a subgraph of G_1 if an injective morphism $f : G_2 \rightarrow G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective. The set of all subgraphs of G_1 is denoted by $\text{sub}(G_1)$.

Definition 1.2 (overlap). Let G and G' be graphs. A graph H is called an overlap of G and G' if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by $\text{ol}(G, G')$.

Definition 1.3 (overlap at morphism). Let C, G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an overlap of G and C' at p if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by $\text{ol}_p(G, C')$.

Definition 1.4 (partial morphism). Let $f : G_1 \rightarrow G_2$ and $g : G_3 \rightarrow G_4$ be morphisms. The morphism g is called a partial morphism of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.

Definition 1.5 (nested graph condition). A graph condition over a graph C_0 is inductively defined as follows:

- *true* is a graph condition over every graph.
- $\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is an injective graph morphism and d is a graph condition over C_1 .
- $\neg d$ is a graph condition over C_0 if d is a graph condition over C_0 .
- $d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0 .

Conditions over the empty graph \emptyset are called constraints. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies *true*. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c . An injective morphism satisfies $\neg d$ if it does not satisfy d , it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \vee d_2$ if it satisfies d_1 or d_2 . A graph G satisfies a constraint c , $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c . We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg \exists(a : C_0 \hookrightarrow C_1, \neg d)$.

The nesting level nl of a condition is defined as $\text{nl}(\text{true}) = 0$ and $\text{nl}(\exists(a : P \rightarrow Q, d)) := \text{nl}(d) + 1$.

Definition 1.6 (alternating quantifier normal form (ANF)[1]). A graph condition c is in alternating normal form (ANF) if it is of the form

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \dots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 partial consistency improving

2.1 partial-conditions and -satisfiability

Definition 2.1 (partial condition). Let c be a condition over C_0 . A partial condition of c over $C'_0 \subseteq C_0$ is defined as:

1. *true* is the partial condition of *true* for every morphism.
2. if $c = Q(a : C_0 \hookrightarrow C_1, d)$, with $Q \in \{\exists, \forall\}$, a partial condition of c over C'_0 is given by $\exists(a' : C'_0 \hookrightarrow C'_1, d')$ with a'_1 being a partial morphism of a , $C'_1 \subseteq C_1 \setminus a(C_0 \setminus C'_0)$ and d' is a partial condition of d over C'_1 .
3. if $c = d_1 \wedge d_2$ or $c = d_1 \vee d_2$ the partial condition of c is given by $d'_1 \wedge d'_2$ and $d'_1 \vee d'_2$, respectively, with d'_1 and d'_2 being partial conditions of d_1 and d_2 over C'_0 .
4. if $c = \neg d$ the partial condition of c is given by $\neg d'$ with d' being a partial condition of d over C'_0 .

A partial condition $Q(a : C'_0 \hookrightarrow C'_1, d)$ of c over C'_0 is called the closest partial condition of c over C'_0 if $C'_1 = C_1 \setminus a(C_0 \setminus C'_0)$ and d is the closest partial condition of d over C'_1 .

We use the notation $c' \leq c$ if c' is partial condition of c over $C'_0 \subseteq C_0$ and $c' < c$ if $C'_i \subset C_i$ for any i .

Definition 2.2 (partial satisfiability). Let a condition c over C_0 and a graph G be given. A morphism $p_0 : C'_0 \rightarrow G$, with $C'_0 \subseteq C_0$, partial satisfies c , $p_0 \models_p c$, if a partial morphism $p'_0 : C''_0 \rightarrow G$ of p_0 satisfies a partial condition of c over C''_0 .

Note, that $p_0 \models c$ implies $p_0 \models_p c$

2.2 partial consistency improving

Definition 2.3 (improving atomic transformation). Given a graph G a condition c over C_0 and a morphism $p : C_0 \rightarrow G$ with $p \not\models c$. An atomic transformation is a rule application that inserts or deletes one single element. A set of atomic transformations is called improving set if the morphism $p' : C_0 \rightarrow H$ with $p = p'$ and H being the graph derived by applications of all transformations in A satisfies c . A is called a minimal improving set if no improving set $B \subset A$ exists. Let $I_{c,p}$ be the set of all minimal improving sets for G with respect to c and p . A set $A \in I_{G,c}$ with $|A| \leq |A'|$ for all $A' \in I_{c,p}$ is called a least changing minimal improving set. The total number of improving atomic transformations for G and p with respect to c is given by

$$\text{iat}_p(G, c) := \sum_{A \in I_{G,c}} |A|$$

If c is a constraint the following notation is used:

$$\text{iat}(G, c) := \sum_{A \in I_{G,c}} |A|$$

Definition 2.4 (partial consistency improving transformation). Given a constraint c and a rule r , a transformation $G \xrightarrow{r,m} H$ is partial consistency improving with respect to c if $\text{iat}(H, c) < \text{iat}(G, c)$.

Construction 2.5. Let a graph G and a condition c in ANF be given.

1. $c = \text{true}$ and $p : C_0 \rightarrow G$:

$$\text{repair}(c, p) := 0$$

2. $c = \text{false}$ and $p : C_0 \rightarrow G$:

$$\text{repair}(c, p) := 0$$

3. $c = \forall(a : C_0 \hookrightarrow C_1, d)$ and $p : C_0 \rightarrow G$:

$$\text{repair}(c, p) := \begin{cases} 0 & , p \models c \\ \sum_{\substack{q: C_1 \rightarrow G \\ q \not\models d \\ p = q \circ a}} |C_1 \setminus C_0| + \text{repair}(q, d) & , p \not\models c \end{cases}$$

4. $c = \exists(a : C_0 \hookrightarrow C_1, d)$ and $p : C_0 \rightarrow G$:

$$\text{repair}(c, p) := \begin{cases} 0 & , p \models c \\ \sum_{\substack{q: C'_1 \rightarrow G \\ p = q \circ a \\ C'_1 \subseteq C_1}} \sum_{\substack{H \in \text{ol}_q(G, C_1) \\ q': C'_1 \rightarrow H \\ q \leq q'}} |C_1| - |C'_1| + \text{repair}(q', d) & , p \not\models c \end{cases}$$

Lemma 2.6. Let a graph G and a condition c over C_1 in ANF and a morphism $p : C_1 \rightarrow G$ be given. The value $\text{repair}(c, p)$ of construction 2.5 is an upper bound for $\text{iat}_p(G, c)$.

Proof. 1. $c = \text{true}$: The morphism p satisfies true and therefore $\text{iat}(p, c) = 0 = \text{repair}(c, p)$.

2. $c = \text{false}$: There is no set of improving transformations, such that $p \models c$ and therefore $\text{iat}(p, c) = 0 = \text{repair}(c, p)$.

3. $c = \forall(a : C_0 \hookrightarrow C_1, d)$: For every morphism $q : C_1 \rightarrow G$ with $q \not\models d$ and $p = q \circ a$ repair sets exists that either (a) delete nodes or edges of $C_1 \setminus C_0$ or (b) repair, such that $q \models d$.

- (a) For every q minimal improving sets exists that delete exactly one edge or isolated node of G . The sum of these minimal improving sets is therefore bound by

$$\sum_{\substack{q: C_1 \rightarrow G \\ p = q \circ a \\ q \not\models d}} |C_1 \setminus C_0|$$

- (b) For every q minimal improving sets exists that repair d . By the induction hypothesis the sum of these sets has the upper bound:

$$\text{repair}(d, q)$$

With (a) and (b) $\text{iat}(p, c) \leq \text{repair}(c, p)$ follows.

4. $c = \exists(a : C_0 \hookrightarrow C_1, d)$: For every morphism $q : C'_1 \rightarrow G$ with $p = q \circ a$ and $C'_1 \subseteq C_1$ minimal improving sets exist that (a) insert the necessary elements of $C_1 \setminus C'_1$ and afterwards (b) repair, such that $q \models d$.

- (a) For every q every possible insertion of the elements of $C_1 \setminus C'_1$ is performed in one minimal improving set. Therefore the sum of these operations is exactly

$$(|C_1| - |C'_1|) |\text{ol}_q(G, C_1)|$$

- (b) After the insertions in (a) a minimal improving set exists that repairs d for the morphism $q' : C_1 \rightarrow H$ with $q \leq q'$ for every $H \in \text{ol}_q(G, C_1)$. Therefore, the sum of these sets has, by induction hypothesis, the upper bound

$$\sum_{\substack{H \in \text{ol}_p(G, C_1) \\ q' : C_1 \rightarrow G \\ q \leq q'}} \text{repair}(q', d)$$

With (a) and (b) follows:

$$\begin{aligned} \text{iat}(p, c) &\leq \sum_{\substack{q : C'_1 \rightarrow G \\ p = q \circ a \\ C'_1 \subseteq C_1}} (|C_1| - |C'_1|) |\text{ol}_q(G, C_1)| + \sum_{\substack{H \in \text{ol}_p(G, C_1) \\ q' : C_1 \rightarrow G \\ q \leq q'}} \text{repair}(q', d) \\ &= \sum_{\substack{q : C'_1 \rightarrow G \\ p = q \circ a \\ C'_1 \subseteq C_1}} \sum_{\substack{H \in \text{ol}_q(G, C_1) \\ q' : C_1 \rightarrow H \\ q \leq q'}} |C_1| - |C'_1| + \text{repair}(q', d) \\ &= \text{repair}(c, p) \end{aligned}$$

□

Lemma 2.7. *Given a constraint c , a rule r and a graph G . A transformation $G \xrightarrow{r, m} H$ is partial consistency improving if $\text{repair}_G(c) < \text{repair}_H(c)$.*

Proof. Noch ein ganz super duper noicer proofböart

□

3 application condition

Construction 3.1. *Given a rule $r = L \xleftarrow{l} K \xrightarrow{r} R$ and a constraint c in ANF with $\text{nl}(c) = 1$. Let G and H be graphs. We denote the set of all overlaps of G and H as $\text{ol}(G, H)$. We construct the following application conditions:*

1. if $c = \forall(a_0 : C_0 \rightarrow C_1, false)$:

$$\bigvee_{Q \in \text{ol}(L, C_1)} \left(\exists(a : L \hookrightarrow Q, true) \wedge \text{Left}(\neg \exists(b : R \rightarrow Q', true) \right)$$

with $Q' \setminus (R \setminus (K \cup C_1)) = Q \setminus (L \setminus (K \cup C_1))$.

2. if $c = \exists(a_0 : C_0 \rightarrow C_1, true)$:

$$\bigvee_{Q \in \text{ol}(L, C_1)} \bigvee_{Q' \subset Q} \left(\exists(a : L \rightarrow Q', true) \wedge \left(\bigvee_{\substack{\bar{Q} \subset Q \\ Q' \subset \bar{Q}}} \text{Left}(\exists(a : R \rightarrow \bar{Q}, true)) \right) \right)$$

References

- [1] C. Sandmann and A. Habel. [Rule-based graph repair](#). *arXiv preprint arXiv:1912.09610*, 2019.