

Masterarbeit

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Abstract

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1 Preliminaries

Definition 1.1 (subgraph). Let G_1 and G_2 be graphs. The graph G_2 is called a subgraph of G_1 if an injective morphism $f : G_2 \rightarrow G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective. The set of all subgraphs of G_1 is denoted by $\text{sub}(G_1)$.

Definition 1.2 (overlap). Let G and G' be graphs. A graph H is called an overlap of G and G' if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by $\text{ol}(G, G')$.

Definition 1.3 (overlap at morphism). Let C, G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an overlap of G and C' at p if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by $\text{ol}_p(G, C')$.

Definition 1.4 (partial morphism). Let $f : G_1 \rightarrow G_2$ and $g : G_3 \rightarrow G_4$ be morphisms. The morphism g is called a partial morphism of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.

Definition 1.5 (nested graph condition). A graph condition over a graph C_0 is inductively defined as follows:

- *true* is a graph condition over every graph.
- $\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is an injective graph morphism and d is a graph condition over C_1 .
- $\neg d$ is a graph condition over C_0 if d is a graph condition over C_0 .
- $d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0 .

Conditions over the empty graph \emptyset are called constraints. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies *true*. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c . An injective morphism satisfies $\neg d$ if it does not satisfy d , it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \vee d_2$ if it satisfies d_1 or d_2 . A graph G satisfies a constraint c , $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c . We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg \exists(a : C_0 \hookrightarrow C_1, \neg d)$.

The nesting level nl of a condition is defined as $\text{nl}(\text{true}) = 0$ and $\text{nl}(\exists(a : P \rightarrow Q, d)) := \text{nl}(d) + 1$.

Definition 1.6 (alternating quantifier normal form (ANF)[1]). A graph condition c is in alternating normal form (ANF) if it is of the form

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \dots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 partial consistency improving

2.1 conditions up to layer

Definition 2.1 (Layer of a subcondition). Let c be a condition and d a subcondition of c . The layer of d is defined as $\text{lay}(d) := \text{nl}(c) - \text{nl}(d) - 1$.

Definition 2.2 (substitution at layer). Let c be a condition, such that the subcondition of c with layer $0 \leq k \leq \text{nl}(c)$ is an condition over C_k . Let e be a condition over C_k . The substitution of $c = Q(a : C_0 \hookrightarrow C_1, d)$ at layer k with e , $\text{sub}(k, c, e)$, is recursively defined as:

1. If $k = 0$:

$$\text{sub}(0, c, e) := e$$

2. If $k > 0$:

$$\text{sub}(k, c, e) := Q(a : C_0 \hookrightarrow C_1, \text{sub}(k-1, d, e))$$

Definition 2.3 (Condition up to layer). Let c be a condition and d be the subcondition of c at layer $0 \leq k \leq \text{nl}(c)$. The condition up to layer k of c , $\text{cond}(k, c)$ is defined as

$$\text{cond}(k, c) := \begin{cases} \text{sub}(k, c, \text{true}) & , \text{ if } k = 0 \vee d \text{ is existentially bound} \\ \text{sub}(k, c, \text{false}) & , \text{ if } d \text{ is universally bound.} \end{cases}$$

Definition 2.4 (Satisfaction up to layer). Let G be a graph and c be a condition over C_0 . A morphism $p : C_0 \hookrightarrow G$ satisfies c up to layer k , $p \models_k c$, if

$$p \models \text{cond}(k, c).$$

A graph G satisfies a constraint c up to layer k , $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies $\text{cond}(k, c)$.

Lemma 2.5. Let G be a graph $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, then for any condition f over C_k holds:

$$p \models \text{sub}(k, c, f)$$

Proof. Let k be the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound, let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be this subcondition. Let $q : G_{k-1} \rightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$. This must exist, since $p \models_k c$ and k is the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound.

Therefore, there does not exist a morphism $q' : C_k \rightarrow G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \rightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$. \square

Lemma 2.6. *Let G be a graph, $p : C_0 \rightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\text{lay}(d) = k$ is universally bound,*

$$p \models_k c \implies p \models c.$$

Proof. Follows immediately by using lemma 2.5 and setting f to the subcondition of c with layer $k + 1$. \square

Lemma 2.7. *Let c be a condition in ANF over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$. Let $d = Q(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k + 2$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that*

$$p \models \text{sub}(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f))$$

with f being a \bar{Q} bound condition over C' .

Proof. If $p \models c$, we can choose $C' = C_{k+2}$ and $f = e$.

If $p \not\models c$, there does not exist a j with $p \models_j c$ and the subcondition of c with layer j is universally bound and $Q = \exists$ follows immediately. We choose $C' = C_{k+1}$ and $f = \text{true}$. Let $q : C_k \rightarrow G$ with $p = q \circ a_k \circ \dots \circ a_1$ and $q \circ \dots \circ a_\ell$ satisfying the condition up to $\ell - k$ of the subcondition of c at layer ℓ for all $0 \leq \ell \leq k$. This morphism must exist since $p \models_k c$ and $p \not\models c$. Let $q' : C_{k+1} \rightarrow G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a'_{k+2} has to be the identity and therefore $q' = q' \circ a'_{k+2}$. It follows that $q' \models \exists(a'_{k+2} : C_{k+1} \hookrightarrow C', \text{true})$ and therefore $p \models \text{sub}(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f))$. \square

Definition 2.8 (partial condition). *Let c be a condition in ANF over C_0 . Let d be the subcondition of c at layer $k + 1$. The partial condition of c at layer k with C' , $\text{part}(k, c, C')$ is defined as:*

1. *If d is universally bound, let $e = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$ be the subcondition of c at layer $k + 2$ with $C_{k+1} \subseteq C' \subseteq C_{k+2}$:*

$$\text{part}(k, c, C') := \text{sub}(k + 2, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true}))$$

2. *If $d = \exists(a : C_k \hookrightarrow C_{k+1}, f)$ is existentially bound with $C_k \subseteq C' \subseteq C_{k+1}$:*

$$\text{part}(k, c, C') := \text{sub}(k + 1, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true}))$$

Definition 2.9 (biggest partially satisfying condition). *Let G be a graph, c a condition over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$.*

A partial condition $c = \text{part}(k, c, C')$ with $p \models c$ is a biggest partially satisfying condition if there does not exist a graph $C' \subset C''$ with $p \models \text{part}(k, c, C'')$.

The set of graphs C' such that $\text{part}(k, c, C')$ is a biggest partially satisfying condition is denoted by $\mathcal{G}_{k,c}$.

2.2 minimal consistency improving

Definition 2.10 (number of violations). Let G be a graph and c a constraint in ANF, such that $G \models_k c$ and $G \not\models c$. The number of violations $\text{nvc}(k, G)$ at layer k in G is set to 0 if there does exist a $j > k$ with $G \models_j c$ and otherwise:

1. If $k = 0$ and $c = \exists(a : C_0 \hookrightarrow C_1, e)$ is existentially bound:

$$\text{nvc}(k, G) := |\{C' \mid G \not\models \text{part}(0, c, C') \wedge C_0 \subseteq C' \subseteq C_1\}|$$

2. If $k \neq 0$, let $d = \forall(a_k : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c at layer $k + 1$.

$$\text{nvc}(k, G) := \sum_{C \in \mathcal{G}_{k,c}} \sum_{C \subseteq C'} |\{q \mid q : C_{k+2} \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}|$$

Definition 2.11 (minimal consistency improving). Let G be a graph, r a rule and c a constraint in ANF with $G \models_k c$ and $G \not\models c$.

A transformation $G \xrightarrow{r,m} H$ is called minimal consistency improving, if

$$\text{nvc}(k, H) < \text{nvc}(k, G).$$

A rule r is called minimal consistency improving, if all of its applications are.

Lemma 2.12. Let G be $p : C_0 \rightarrow G$ a morphism, c a constraint in ANF over C_0 with $p \models_k c$. Then $p \models_j c$ for all $j < k$ such that the subcondition of c at layer j is existentially bound.

Proof. 1. The subcondition of c at layer k is existentially bound: If an $j < k$ with $p \models_j c$ exists such that the subcondition of c at layer j is universally bound, let j_1 be the smallest of these. With lemma 2.5 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that the subcondition of c at layer j is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of the subcondition of c at layer ℓ . Since $\ell < j_1$, a morphism $q : C_\ell \rightarrow G$ with $q \models d$ must exist and therefore a morphism $q' : C_{\ell+1} \rightarrow G$ with $q = q' \circ a_k$ must exist. It follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \text{true})$ and with that $p \models_\ell c$.

2. The subcondition of c at layer k is universally bound: With lemma 2.5 follows that $p \models_{k+1} c$. Since c is in ANF 1. can be applied to $k + 1$.

□

Lemma 2.13. Let G be a graph, r a rule and c a constraint in ANF with $G \not\models c$. Let k be the biggest number, such that $G \models_k c$. A transformation $G \xrightarrow{r,m} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.

Proof. Since $G \not\models c$, with lemma 2.6 follows that the subcondition of c at layer k has to be existentially bound and since k is the biggest number such that $G \models_k c$ it follows that $\text{nvc}(k+1, G) > 0$. If the subcondition of c at layer j is universally bound, $H \models c$ follows with lemma 2.6 and lemma 2.12 $H \models_{k+2} c$. Therefore $\text{nvc}(k+2, H) = 0$. Otherwise the subcondition of c at layer j is existentially bound and therefore $G \models_{k+2} c$ and $\text{nvc}(k+2, H) = 0$ follows immediately. \square

Definition 2.14 (direct minimal consistency improving). Let G be a graph, r a plain rule and c a constraint in ANF with $G \models_k$ and $G \not\models c$. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ be the condition at layer k of c . A transformation $t : G \xrightarrow{r,m} H$ is called direct minimal consistency improving if the transformation is minimal consistency improving and

$$\begin{aligned} \forall p : C_{k+1} \hookrightarrow G ((p \models e \wedge \text{tr}_t \circ p \text{ is total}) \implies \text{tr}_t \circ p \models p) \wedge \\ \forall p' : C_{k+1} \hookrightarrow H (\neg \exists p : C \hookrightarrow G (p' = \text{tr}_t \circ p) \implies p' \models d). \end{aligned} \quad (2.1)$$

3 application condition

Definition 3.1 (extended overlap). Let G and $C_0 \subset C_1$ be graphs and C' an overlap of G and C_0 with overlap morphisms $p : G \hookrightarrow C'$ and $q : C_0 \hookrightarrow C'$. An overlap C'' of G and C_1 is called the extended overlap of C' with C'' if $C' \subset C''$ and a morphism $q' : C_1 \hookrightarrow C''$ with $q'|_{C_0} = q$ exists.

Definition 3.2 (overlap shift). Let $r = L \xleftarrow{l} K \xrightarrow{r} R$ be a plain rule, C a graph and C' an overlap of C and L with morphisms $p : L \hookrightarrow C'$, $k : K \hookrightarrow C'$, $c : C \hookrightarrow C'$ and the partial morphism $q : R \hookrightarrow C'$. We define

$$\begin{aligned} D := \{e \in C' \mid (\exists e' \in L : p(e') = e \\ \vee \exists e' \in R : q(e') = e) \\ \wedge \exists e' \in C : c(e') = e\} \end{aligned} \quad (3.1)$$

Let $r = L \xleftarrow{l} K' \xrightarrow{r} R$ be the rule with

$$K' := K \cup D$$

The graph H , of the transformation $G \xrightarrow{r,p} H$ is called the overlap shifted graph of C' .

Definition 3.3 (minimal consistency improving application condition). Let $r = L \xleftarrow{l} K \xrightarrow{r} R$ be a plain rule and c a constraint in ANF. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(b : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the condition at layer k of c . The application condition ap_k of the condition at layer k of c and $C_{k+1} \subset C' \subseteq C_{k+2}$ is defined as:

$$\begin{aligned} \text{ap}(k, C') := & \left(\bigvee_{P \in \text{ol}(L, C_{k+1})} \text{nex}(P, C') \wedge (\text{rep}(P, C') \vee \text{del}(P, C')) \right) \wedge \\ & \left(\bigwedge_{P \in \text{ol}(L, C_{k+1})} \text{ex}(P, C') \right) \wedge \left(\bigwedge_{P \in \text{ol}(L, C_{k+1})} \text{rem}(P, C') \right) \end{aligned} \quad (3.2)$$

with

1. Let Q be the extended overlap of P with C' .

$$\text{nex}(P, C') := \exists(a : L \hookrightarrow P, \neg \exists(b : P \hookrightarrow Q, \text{true}))$$

2. Let Q be the extended overlap of P with C' , Q' the overlap shifted graph of Q and P' the overlap shifted graph of P .

$$\text{rep}(P, C') := \text{Left}(\forall(a : L \hookrightarrow P', \neg \exists(b : P' \hookrightarrow Q', \text{true})), r)$$

3. Let P' be the overlap shifted graph of P .

$$\text{del}(P, C') := \text{Left}(\neg \exists(a : R \hookrightarrow P', \text{true}), r)$$

4. Let Q be the extended overlap of P with C' and Q' the overlap shifted graph of Q .

$$\text{ex}(P, C') := \exists(a : L \hookrightarrow Q, \text{true}) \implies \text{Left}(\exists(a : R \hookrightarrow Q', \text{true}), r)$$

5. Q be the extended overlap of P with C' and Q' the overlap shifted graph of Q .

$$\text{rem}(P, C') := \exists(a : L \hookrightarrow Q, \text{true}) \implies \text{Left}(\exists(b : R \hookrightarrow Q', \text{true}), r)$$

Lemma 3.4. Let G be a graph, c a constraint in ANF, such that $G \models_k c$ and r a plain rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(b : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the condition at layer k of c . Then, every application of r , equipped with $\text{ap}(k, C')$ with $C' \notin \mathcal{B}_{k, \text{true}}$, such that a $C \in \mathcal{B}_{k, \text{true}}$ with $C \subset C'$ exists is minimal consistency improving.

Proof.

□

Lemma 3.5. Let G be a graph, c a constraint in ANF, such that $G \models_k c$, and $r = L \xleftarrow{l} K \xrightarrow{r} R$ a plain rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(b : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the condition at layer k of c and $\text{ap}(k, C')$ with $C' \notin \mathcal{B}_{k, \text{true}}$, such that a $C \in \mathcal{B}_{k, \text{true}}$ with $C \subset C'$ exists. The following simplifications apply:

1. Let $P \in \text{ol}(L, C_{k+1})$. If an injective morphism $p : P \hookrightarrow G$ does not exist, $\text{nex}(P, C')$, $\text{rep}(P, C')$, $\text{del}(P, C')$ and $\text{ex}(P, C')$ can be replaced by *false*.
2. If $(L \setminus K) \cap C_{k+1} = \emptyset$, every $\text{del}(P, C')$ can be replaced by *false*.
3. If $(R \setminus K) \cap C' = \emptyset$, every $\text{rep}(P, C')$ can be replaced by *false*.
4. If 1. and 2. apply, $\text{ap}(k, C')$ can be replaced by *false*.
5. If $(R \setminus K) \cap C_{k+1} = \emptyset$, every $\text{rem}(P, C')$ can be replaced by *true*.

Proof.

□

References

- [1] C. Sandmann and A. Habel. [Rule-based graph repair](#). *arXiv preprint arXiv:1912.09610*, 2019.