# Rule-based Graph Repair using Minimally Restricted Consistency-Improving Transformations

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**Abstract**

# Contents

## 1 Preliminaries

**Definition 1.1** (**subgraph**). *Let $G_1$ and $G_2$ be graphs. The graph $G_2$ is called a* sub-graph *of $G_1$ if an injective morphism $f : G_2 \to G_1$ exists. We use the notation $G_2 \subseteq G_1$ if $G_2$ is a subgraph of $G_1$ and $G_2 \subset G_1$ if $f$ is not bijective.*

**Definition 1.2** (**uppergraph**). *Let $G_1$ and $G_2$ be graphs with $G_1 \subseteq G_2$. A graph $C$ is called an* upper-graph *of $G_1$ w.r.t $G_2$, if $G_1 \subset C \subseteq G_2$ The set of uppergraphs of $G_1$ w.r.t. $G_2$ is denoted by $\mathcal{U}(G_1, G_2)$. If $G_1 = G_2$, we set $\mathcal{U}(G_1, G_2) = \{G_1\}$.*

**Definition 1.3** (**overlap**). *Let $G$ and $G'$ be graphs. A graph $H$ is called an* overlap of $G$ and $G'$ *if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that $p$ and $p'$ are jointly surjective. The set of all overlaps of $G$ and $G'$ is denoted by* $\mathrm{ol}(G, G')$.

**Definition 1.4** (**overlap at morphism**). *Let $C$,$G$ and $C'$ with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph $H$ is called an* overlap of $G$ and $C'$ at $p$ *if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of $G$ and $C'$ at $p$ is denoted by* $\mathrm{ol}_p(G, C')$.

**Definition 1.5** (**partial morphism**). *Let $f : G_1 \to G_2$ and $g : G_3 \to G_4$ be morphisms. The morphism $g$ is called a* partial morphism *of $f$ if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.*

**Definition 1.6** (**nested graph condition**). *A* graph condition *over a graph $C_0$ is inductively defined as follows:*

- *true is a graph condition over every graph.*
- *$\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over $C_0$ if $a$ is a injective graph morphism and $d$ is a graph condition over $C_1$.*
- *$\neg d$ is a graph condition over $C_0$ if $d$ is a graph condition over $C_0$.*
- *$d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over $C_0$ if $d_1$ and $d_2$ are graph conditions over $C_0$.*

*Conditions over the empty graph $\emptyset$ are called* constraints*. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies true. An injective morphism $p$ satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and $q$ satisfies $c$. An injective morphism satisfies $\neg d$ if it does not satisfy $d$, it satisfies $d_1 \wedge d_2$ if it satisfies $d_1$ and $d_2$ and it satisfies $d_1 \wedge d_2$ if it satisfies $d_1$ or $d_2$. A graph $G$ satisfies a constraint $c$, $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies $c$. We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg\exists(a : C_0 \hookrightarrow C_1, \neg d)$.*

*The* nesting level nl *of a condition is defined as* $\mathrm{nl}(\mathit{true} = 0$ *and* $\mathrm{nl}(\exists(a : P \to Q, d)) := \mathrm{nl}(d) + 1$.

**Definition 1.7** (**alternating quantifier normal form (ANF)**[1]). *A graph condition $c$ is in* alternating normal form *(ANF) if it is of the form*

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \ldots))))$$

*with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.*

## 2 consistency increasing

In this section, we introduce the notion of *consistency increasing* transformations and rules, which allows to increase the consistency of a constraint layer by layer.

### 2.1 extended alternating quantifier normal form

To prevent the need of case discrimination, a new normal form for conditions, called *extended alternating quantifier normal form* (EANF), will be introduced. The sets of conditions in ANF and EANF do intersect and we show that both sets are expressively equivalent.

**Definition 2.1** (**extended alternating quantifier normal form**). *A conditions $c$ is in* extended alternating quantifier normal form *(EANF) if it is of the form $\forall(a_0 : C_0 \hookrightarrow C_1, d)$ and $d$ is a condition in ANF.*

Note that, given a condition $c$ in EANF, every subcondition of $c$ at layer $1 \leq k \leq \mathrm{nl}(c)$ is universally bound, if $k$ is an odd number and existentially bound, if $k$ is an even number.

**Lemma 2.2.** *Any condition in ANF can be transformed into an equivalent condition in EANF and vice versa.*

*Proof.* "$\Longrightarrow$": Let a graph $G$ and a constraint $c$ in ANF be given. If $c$ is universally, $c$ is already in EANF.

If $c = \exists(a_0 : C_0 \hookrightarrow C_1, d)$, we show that $c$ is equivalent to $c' := \forall(\mathrm{id}_{C_0} : C_0 \hookrightarrow C_0, c)$.

1. Let $p : C_0 \hookrightarrow G$ be a morphism, such that $q \models c$. Therefore a morphism $q : C_0 \to G$ with $q \models e$ and $p = q \circ a_1$ exists. Then, $p \models d$, since $p$ is the only morphism from $C_0$ to $G$ with $p = p \circ \mathrm{id}_{C_0}$ and $p \models c$.

2. Let $p : C_0 \hookrightarrow G$ be a morphism with $p \models c'$, therefore all morphisms $q : C_0 \hookrightarrow G$ with $p = q \circ \mathrm{id}_{C_0}$ satisfy $c$. Since $p = p \circ \mathrm{id}_{C_0}$, $p \models c$ follows immediately.

"$\Longleftarrow$": Let a graph $G$ and a constraint $c$ in EANF be given. If $c = \forall(a_0 : C_0 \hookrightarrow C_1, d)$ with $C_0 \neq C_1$, $c$ is already in ANF.

Otherwise, the equivalence of $c$ with $d$ can be shown analogously to the first case.

□

### 2.2 conditions up to layer

**Definition 2.3** (**Layer of a subcondition**). *Let $c$ be a condition and $d$ a subcondition of $c$. The layer of $d$ is defined as* $\mathrm{lay}(d) := \mathrm{nl}(c) - \mathrm{nl}(d) - 1$.

**Definition 2.4** (**Subcondition at layer**). *Let $c$ be a condition. The* subcondition at layer $k$*, denoted by* $\mathrm{sco}_c(k)$ *, is the subcondition $d$ of $c$ with* $\mathrm{lay}(d) = k$.

We define a notion of partial consistency, called *satisfaction at layer*, which will be used for the definition of consistency increasing. First, two operators are introduced to modify given constraints on a certain layer.

**Definition 2.5** (**substitution at layer**). *Let $c = Q(a : C_0 \hookrightarrow C_1, d)$ be a condition in ANF, such that the subcondition of $c$ with layer $0 \leq k \leq \mathrm{nl}(c)$ is a condition over $C_k$. Let $e$ be a condition over $C_k$. The* substitution in $c$ at layer $k$ with $e$, $\mathrm{sub}(k, c, e)$, *is recursively defined as:*

1. *If $k \leq 1$:*
$$\mathrm{sub}(0, c, e) := e$$

2. *If $k > 1$:*
$$\mathrm{sub}(k, c, e) := Q\big(a : C_0 \hookrightarrow C_1, \mathrm{sub}(k-1, d, e)\big)$$

**Example 2.1.** *Let the conditions $c := \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1 : C_1 \hookrightarrow C_2, \mathsf{true}))$ and $d = \exists(a_1' : C_1 \hookrightarrow C_3, e)$ be given. Then,*

$$\mathrm{sub}(2, c, d) = \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1' : C_1 \hookrightarrow C_3, e)).$$

Through this, we define *condition up to layer*. Intuitively, a condition is cut off at a certain layer, by replacing the subcondition at this layer by $\mathsf{true}$ or $\mathsf{false}$, depending on the quantifier, the replaced subcondition is bound by.

**Definition 2.6** (**Condition up to layer**). *Let $c$ be a condition in EANF and $d$ be the subcondition of $c$ at layer $0 \leq k \leq \mathrm{nl}(c)$. The* condition up to layer $k$ *of $c$,* $\mathrm{cond}(k, c)$, *is defined as*

$$\mathrm{cond}(k, c) := \begin{cases} \mathrm{sub}(k+1, c, \mathsf{true}) & \text{if } d \text{ is existentially bound} \\ \mathrm{sub}(k+1, c, \mathsf{false}) & \text{if } d \text{ is universally bound.} \end{cases}$$

**Example 2.2.** *Let the condition $c = \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1 : C_1 \hookrightarrow C_2, d))$ be given. Then,*

$$\mathrm{cond}(2, c) = \forall(a_0 : C_0 \hookrightarrow C_1, \mathsf{true}).$$

Now, we are ready to define satisfaction at layer, which is a key ingredient to define consistency increasing.

**Definition 2.7** (**Satisfaction at layer**). *Let $G$ be a graph and $c$ be a condition over $C_0$. A morphism $p : C_0 \hookrightarrow G$* satisfies $c$ at layer $k$, $p \models_k c$, *if*

$$p \models \mathrm{cond}(k, c).$$

*A graph $G$ satisfies a constraint $c$ at layer $k$, $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies* $\mathrm{cond}(k, c)$. *The biggest $k$ with $G \models_k c$ such that no $j > k$ with $G \models_j c$ exists is denoted by $c^G_{\max}$.*

**Example 2.3.**

The following lemmas arise as a direct consequence of the definition of satisfaction at layer. If a graph satisfies a constraint up to a certain layer, let $c$ be this condition up to this layer, that ends with $\forall(a : C \hookrightarrow C', \mathsf{false})$, the graph satisfies all constraints starting with $c$.

**Lemma 2.8.** *Let $G$ be a graph $p : C_0 \hookrightarrow G$ a morphism and $c$ a condition over $C_0$ in EANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of $c$ at layer $k$ is universally bound, then for any condition $f$ over $C_k$ it holds that*

$$p \models \mathrm{sub}(k+1, c, f).$$

*Proof.* Let $k$ be the smallest number with $\mathrm{sco}_c(k) = \forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false})$ being universally bound and $p \models_k c$. Let $q : G_{k-1} \hookrightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false})$. This must exist, since $p \models_k c$ and $k$ is the smallest even number such that $p \models_k c$.

Therefore, there does not exist a morphism $q' : C_k \hookrightarrow G$ with $q = q' \circ a_k$. Hence, for every condition $f$ over $C_k$ a morphism $q' : C_k \hookrightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$ and with that $p \models \mathrm{sub}(k+1, c, f)$. For every $j > k$ even, such that $p \models_j c$, and every condition $d$ over $C_j$, with the first part, it holds that $p \models \mathrm{sub}(k+1, c, f)$ with $f = \mathrm{sub}(j-k+1, \mathrm{sco}_c(k+1), d)$ and since $\mathrm{sub}(k+1, c, f) = \mathrm{sub}(j+1, c, d)$ it follows that $p \models \mathrm{sub}(j+1, c, d)$. □

As a direct consequence of the previous lemma, a graph satisfying a condition up to layer ending with $\forall(a : C \hookrightarrow C', \mathsf{false})$ also satisfies the whole constraint.

**Lemma 2.9.** *Let $G$ be a graph, $p : C_0 \hookrightarrow G$ a morphism and $c$ a condition over $C_0$ in EANF with $p \models_k c$. If $\mathrm{sco}_c(k)$ is universally bound,*

$$p \models_k c \implies p \models c.$$

*Proof.* Follows immediately by using lemma 2.8 and setting $f$ to $\mathrm{sco}_c(k+1)$. □

**Lemma 2.10.** *Let a graph $G$, a morphism $p : C_0 \hookrightarrow G$ and a constraint $c$ in EANF be given. Then, $p \models_j c$ for all $j < c^G_{\max}$ such that $\mathrm{sco}_c(j)$ is existentially bound.*

*Proof.*

1. The subcondition of $c$ at layer $c^G_{\max}$ is existentially bound: If a $j < c^G_{\max}$ with $p \models_j c$ exists such that $\mathrm{sco}_c(j)$ is universally bound, let $j_1$ be the smallest of these. With lemma 2.8 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that $\mathrm{sco}_c(\ell)$ is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of $\mathrm{sco}_c(\ell)$. Since $\ell < j_1$, a morphism $q : C_\ell \hookrightarrow G$ with $q \models d$ must exists and therefore a morphism $q' : C_{\ell+1} \hookrightarrow G$ with $q = q' \circ a_k$ must exists. It follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \mathsf{true})$ and with that $p \models_\ell c$.

2. The subcondition of $c$ at layer $c^G_{\max}$ is universally bound: With lemma 2.8 follows that $p \models_{k+1} c$. Since $c$ is in EANF, 1. can be applied to $k+1$.

□

Through satisfaction at layer an increase of consistency can be detected in the following way. Let $G \Longrightarrow H$ be a transformation. If $c^G_{\max} < c^H_{\max}$, the transformation can be considered as consistency increasing, since $H$ satisfies more layers of the constraint than $G$. The notion of consistency increasing should also be able to detect the smallest transformations that lead to a increase of consistency, namely the inserting of a single edge or node of an existentially bound graph. To remedy this issue, we introduce *partial conditions*. Intuitively, given a constraint $c$ in EANF ending with $\mathrm{sco}_c(k) = \exists(a_k : C_k \hookrightarrow C_{k+1}, d)$, the condition $\mathrm{sco}_c(k)$ is replaced by $\exists(a^p_k : C_k \hookrightarrow C', \mathsf{true})$ with $C' \in \mathcal{U}(C_k, C_{k+1})$.

The construction of partial conditions is designed to only replace graphs in existentially bound layers, since the replacement in an universally bound layer would lead to a more restrictive constraint than the original condition up to layer. That means, given the condition $c = \forall(a_0 : C_0 \hookrightarrow C_1, \mathsf{false})$ and let $C' \in \mathcal{U}(C_0, C_1)$. If the condition $c' = \forall(a^p_0 : C_0 \hookrightarrow C_1, \mathsf{false})$ is satisfied this implies that $c$ is also satisfied but the backwards implication does not hold.

**Definition 2.11 (partial condition).** *Let $c$ be a condition in EANF.*

*The* partial condition of $c$ at layer $k < \mathrm{nl}(c)$ with *with*

$$C' \in \begin{cases} \mathcal{U}(C_k, C_{k+1}) & \text{if } k \text{ is even} \\ \mathcal{U}(C_{k+1}, C_{k+2}) & \text{if } k \text{ is odd,} \end{cases}$$

$\mathrm{part}(k, c, C')$*, is defined as:*

1. *If $k$ is odd, let* $\mathrm{sco}_c(k+1) = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$*:*

$$\mathrm{part}(k, c, C') := \mathrm{sub}\big(k+1, c, \exists(a^p : C_{k+1} \hookrightarrow C', \mathit{true})\big)$$

2. *If $k$ is even, let* $\mathrm{sco}_c(k+1) = \exists(a : C_k \hookrightarrow C_{k+1}, f)$*:*

$$\mathrm{part}(k, c, C') := \mathrm{sub}\big(k, c, \exists(a^p : C_{k+1} \hookrightarrow C', \mathit{true})\big)$$

**Example 2.4.**

If a graph $G$ does not satisfy the condition up to layer $k := c^G_{\max} + 2$ of a given constraint $c$, with $k$ being even, there does exists a graph $C' \in \mathcal{U}(C_{k-1}, C_k)$, such that $G$ satisfies $\mathrm{part}(k, c, C')$, note that $G$ always satisfies $\mathrm{part}(k, c, C_{k-1})$. For the biggest of these graphs, we define the notions of *biggest partially satisfying conditions and graphs.*

**Definition 2.12 (biggest partially satisfying condition).** *Let $G$ be a graph, $c$ a condition in EANF over $C_0$ and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$ and $p \not\models c$.*

*A partial condition $c = \mathrm{part}(k+2, c, C')$ with $p \models c$ is a* biggest partially satisfying condition w.r.t $c$ of $p$ *if there does not exist a graph $C''$, such that $C'$ is a subgraph of $C''$, with $p \models \mathrm{part}(c_{max}, c, C'')$. The graph $C'$ is then called a* biggest partially satisfying graph w.r.t $c$ of $p$.

*Given a constraint $c$, the set of biggest partially satisfying conditions w.r.t $c$ of $p$ : $\emptyset \inf G$ is denoted by $\mathcal{P}^G_c$.*

*The set of all biggest partially satisfying graphs w.r.t $c$ of $p$ is denoted by $\mathcal{G}^G_c$.*

## 2.3 consistency increasing transformations and rules

With the results above, we are now ready to define the notions of *consistency increasing* and *direct consistency increasing*, with direct increasing being a more restrictive version of increasing, yielding the advantage that second-order logic formulas can be used in order to determine whether a transformation is (direct) consistency increasing or not.

These notions are designed to only detect transformations that increase the consistency of the first two unsatisfied layer of a constraint $c$. That means, given a graph $G$ and a constraint $c$, let $k = c^G_{\max}$ and $\forall(a_{k+1} : C_k \hookrightarrow C_{k+1}, \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, d)) =: \mathrm{sco}_c(k+1)$. A transformation $t : G \Longrightarrow H$ is considered as (direct) consistency increasing if $c^G_{\max} \leq c^H_{\max}$, i.e. the satisfaction up to layer is not decreased, and more increasing insertions or deletions have been performed than decreasing ones. An increasing deletion is the deletion of an occurrence of $C_{k+1}$ that does not satisfy $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \mathsf{true})$, an increasing insertion is the insertion of elements of $C_{k+2}$, such that for at least one occurrence $p$ of $C_{k+1}$ it holds that $p \not\models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ and $\mathrm{tr}_t \circ p \models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ for an $C' \in \mathcal{U}(C_{k+1}, C_{k+2})$. A decreasing insertion is the creation of an occurrence of $C_{k+1}$ not satisfying $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \mathsf{true})$ and a decreasing deletion is the deletion of elements of $C_{k+2}$ such that for an occurrence $p$ of $C_{k+1}$ with $p \models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ it holds that $\mathrm{tr}_t \circ p \not\models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$.

To evaluate this, we define the *number of violations*. Intuitively, for all occurrences $p$ of $C_{k+1}$ the number of graphs $C' \in \mathcal{U}(C_{k+1}, C_{k+2})$ with $p \not\models \exists C'$ is add up and it can be determined whether more increasing or decreasing actions have been performed by a transformation.

Note, that the number of violations is defined for each layer of the constraint, but only for the first unsatisfied layer the sum is calculated as described above. For all layer $k$ with $k \leq c^G_{\max}$ it is set to 0 and for all layer $k$ with $k > c^G_{\max} + 1$ it is set to $\infty$. Through this, a transformation $tG \Longrightarrow H$ that increases the partial consistency can easily detected since the number of violations in $H$ at layer $c^G_{\max} + 1$ will be set to 0.

**Definition 2.13 (number of violations).** *Let $G$ be a graph and $c$ a constraint in EANF. The* number of violations $\mathrm{nvc}(j, G)$ at layer $j$ in $G$ *is defined as:*

1. *If $j < c^G_{\max} + 1$:*
$$\mathrm{nvc}(j, G) := 0$$

2. *If $j = c_{\max} + 1$, let $d = \forall(a_k : C_j \hookrightarrow C_{j+1}, e)$ be the subcondition of $c$ at layer $j + 1$.*
$$\mathrm{nvc}(j, G) := \begin{cases} \sum_{C' \in \mathcal{U}(C_j, C_{j+1})} |\{q \mid q : C_{j+1} \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\}| & \text{if } e \neq \mathsf{false} \\ |\{q \mid q : C_{j+1} \hookrightarrow G\}| & \text{if } e = \mathsf{false} \end{cases}$$

3. *If $j > c^G_{\max} + 1$:*
$$\mathrm{nvc}(j, G) := \infty$$

Via the number of violations, we now define *consistency increasing* transformations and rules, by checking whether the number of violations has decreased for any layer of the constraint.

**Definition 2.14 (consistency increasing).** *Let a graph $G$, a rule $\rho$ and a constraint $c$ in EANF be given.*

*A transformation $G \Longrightarrow_{\rho,m} H$ is called* consistency increasing w.r.t $c$, *if*

$$\mathrm{nvc}(k, H) < \mathrm{nvc}(k, G)$$

*for any $0 \leq k \leq \mathrm{nl}(c)$. A rule $r$ is called* consistency increasing w.r.t $c$, *if all of its applications are.*

Note that if $G \models c$ there does not exist a consistency increasing transformation w.r.t $c$, since $\mathrm{nvc}(j, G) = 0$ for all $1 \leq j \leq \mathrm{nl}(c)$. Also, no plain rule $\rho$ is consistency increasing w.r.t $c$, since a graph $G$ satisfying $c$, such that a transformation $t : G \Longrightarrow_{\rho,m} H$ exists can always be constructed. Therefore, each consistency increasing rule has to be equipped with at least one application condition.

As mentioned above, a transformation should be detected as consistency increasing if it increases the partial consistency, which is shown by the following theorem.

**Theorem 2.1.** *Let a graph $G$, a rule $\rho$ and a constraint $c$ in EANF be given. A transformation $t : G \Longrightarrow_{\rho,m} H$ is consistency increasing if $c^G_{\max} < c^H_{\max}$.*

*Proof.* No $\ell > c^G_{\max}$ with $G \models_\ell c$ exists. Hence, $\mathrm{nvc}(c^G_{\max} + 1, G) > 0$ and $\mathrm{nvc}(c^G_{\max} + 1, G) \neq \infty$. Since $c^H_{\max} > c^G_{\max}$, $\mathrm{nvc}(k + 1, H) = 0$ and it follows immediately that $t$ is consistency increasing. $\square$

Since no consistency increasing transformation originating in consistent graphs exist, there do not exist infinite long sequences of consistency increasing transformations. Additionally, if a set of rules $\mathcal{R}$ is given, and a sequence of consistency increasing transformations with rules of $\mathcal{R}$ ends with a graph $G$, then either $G$ satisfies the constraint or there do not exist any consistency increasing transformations $G \Longrightarrow_{\rho,m} H$ with $\rho \in \mathcal{R}$.

**Theorem 2.2.** *Let a constraint $c$ in EANF and a set of rules $\mathcal{R}$ be given. Every sequence of minimal consistency improving transformation w.r.t $c$ with rules in $\mathcal{R}$ is finite.*

*Proof.* Let $G_0$ be a graph and

$$G_0 \Longrightarrow_{\rho_1} G_1 \Longrightarrow_{\rho_2} G_2 \Longrightarrow_{\rho_3} \dots$$

be a sequence of minimal consistency improving transformations w.r.t $c$ with $\rho_i \in \mathcal{R}$. We assume that $c^{G_0}_{\max} < \mathrm{nl}(c)$, otherwise $\mathrm{nvc}(j, G_0) = 0$ for all $j \in \{0, \dots, \mathrm{nl}(c)\}$ and no transformation $G_0 \Longrightarrow H$ is consistency increasing.

We show that after at most $j := \mathrm{nvc}(\mathrm{c}_{\max G_0} + 1, G_0)$ transformations $G_j \models_{c^{G_0}_{\max}+2} c$ holds. Note that $j$ has to be finite, since $G_0$ contains only a finite number of occurrences of $C_{j+1}$. After each transformation it holds that $\mathrm{nvc}(c^{G_{i+1}}_{\max} + 1, G_{i+1}) \leq \mathrm{nvc}(c^{G_i}_{\max} + 1, G_i) -$

1. Therefore, after $j$ transformations, $\text{nvc}(c^{G_0}_{\max}+1, G_j) \leq \text{nvc}(c^{G_0}_{\max}+1, G_0) - j = 0$ holds and with that $G_j \models_{c^{G_0}_{\max}+2} c$. By iteratively applying this, it follows that after a finite number of transformations a graph $G_k$ with $G_k \models c$ has to exist. Since no consistency increasing transformation $G_k \Longrightarrow G_{k+1}$ exists, the sequence has to be finite. Also, for some $G_{k'}$ there may not exist a consistency increasing transformation $G_{k'} \Longrightarrow_\rho H$ with $\rho \in \mathcal{R}$ and therefore the sequence is also finite, but does not end with a consistent graph. □

Now, we define *direct consistency increasing* transformations, a stricter version of consistency increasing.

**Definition 2.15 (direct minimal consistency improving).** *Let $G$ be a graph, $\rho$ a plain rule and $c$ a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be the condition at layer $k = c^G_{\max} + 1 \leq \text{nl}(c)$ of $c$. A transformation $t : G \Longrightarrow_{\rho,m} H$ is called* direct minimal consistency improving *if the following equations hold.*

*Every occurrence of $C_k$ in $G$ that satisfies $\text{part}(1, e, C')$ for any $C' \in \mathcal{U}(C_k, C_{k+1})$ still satisfies $\text{part}(1, e, C')$ in $H$.*

$$\forall p : C_k \hookrightarrow G\Big(\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} \big(p \models \text{part}(1, e, C') \wedge \text{tr}_t \circ p \textit{ is total}\big) \Longrightarrow \text{tr}_t \circ p \models \text{part}(1, e, C')\Big) \tag{2.1}$$

*Every new inserted occurrence of $C_k$ by $t$ satisfies $\text{part}(1, e, C_{k+1})$.*

$$\forall p' : C_k \hookrightarrow H\big(\neg \exists p : C_k \hookrightarrow G(p' = \text{tr}_t \circ p) \implies p' \models \text{part}(1, e, C_{k+1})\big) \tag{2.2}$$

*At least one occurrence of $C_k$ in $G$ that does not satisfy $\text{part}(1, e, C')$, for any $C' \in \mathcal{U}(C_k, C_{k+1})$, either has been destroyed by $t$ or satisfies $\text{part}(1, e, C')$ in $H$.*

$$\exists p : C_k \hookrightarrow G\Big(\bigvee_{C' \in \mathcal{U}(C_k, C_{k+1})} \big(p \not\models \text{part}(1, e, C') \wedge (\text{tr}_t \circ p \textit{ is not total} \vee (\text{tr}_t \circ p \textit{ is total } \wedge \text{tr}_t \circ p \models \text{part}(1, e, C'))))\Big) \tag{2.3}$$

*No occurrence of a universally bound graph $C_j$ with $j < k$ gets inserted.*

$$\bigwedge_{\substack{i<k \\ i \textit{ even}}} \forall p : C_i \hookrightarrow H(\exists p' : C_i \hookrightarrow G(p' = \text{tr}_t \circ p)) \tag{2.4}$$

*No occurrence of an existentially bound graph $C_j$ with $j < k$ gets deleted.*

$$\bigwedge_{\substack{i<k \\ i \textit{ odd}}} \forall p : C_i \hookrightarrow G(\text{tr}_t \circ p \textit{ is total}) \tag{2.5}$$

**Lemma 2.16.** *Let a transformation $t : G \Longrightarrow H$ and a constraint $c$ in EANF be given, such that (2.4) and (2.5) of definition 2.15 are satisfied. Then,*

$$H \models_{c^G_{\max}} c.$$

*Proof.* Assume that $G \not\models_{c^G_{\max}} c$. Then, either a new occurrence of an universally bound graph of $c$ has been inserted or an occurrence of an existentially bound graph of $c$ has been destroyed. Therefore, the following holds:

$$\exists p : C_i \hookrightarrow G(\neg\exists p' : C_i \hookrightarrow (p' = \mathrm{tr}_t \circ p) \vee \exists p : C_j \hookrightarrow G(\mathrm{tr}_t \circ p \textit{ is not total})$$

with $i, j < k$, $i$ being odd and $j$ being even. It follows immediately that either (2.4) or (2.5) is not satisfied. This is a contradiction. $\square$

**Lemma 2.17.** *Let a graph $G$, a constraint $c$ in EANF and a direct minimal improving transformation $t : G \Rightarrow_{r,m} H$ w.r.t. $c$ be given. Then, $t$ is also a minimal improving transformation.*

*Proof.* Let $G$ be a graph with $k = c^G_{\max} + 1$ and $G \models \mathrm{part}(k, c, C)$ with $\mathrm{part}(k, c, C) \in \mathcal{P}^G_c$ Let $d$ be the subcondition of $c$ at layer $k$. With lemma 2.16 follows that $c^H_{\max} \geq c^G_{\max}$ and with that $\mathrm{nvc}(k, H) \neq \infty$.

1. We show that equations (2.1) and (2.2) imply that $\mathrm{nvc}(k, H) \leq \mathrm{nvc}(k, G)$. Assume that $\mathrm{nvc}(k, H) > \mathrm{nvc}(k, G)$. Therefore, a morphism $p : C_k \hookrightarrow H$ with $p \not\models \mathrm{part}(1, d, C')$ for any $C' \in \mathcal{U}(C, C_{k+1})$ exists, such that either 1a or 1b is satisfied.

   (a) There does exist a morphism $q' : C_k \hookrightarrow G$ with $q' \models \mathrm{part}(1, d, C')$ and $p = \mathrm{tr}_t \circ q'$.

   (b) There does not exist a morphism $q : C_k \hookrightarrow G$, such that $p = \mathrm{tr}_t \circ q$.

   This is a contradiction, if 1a is satisfied, $q'$ does not satisfy equation (2.1) and if 1b is satisfied $q$ does not satisfy equation (2.2). It also follows that

   $$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\}| \leq |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\}|$$

   for all $C' \in \mathcal{U}(C_k, C_{k+1})$.

2. Since (2.3) is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \mathrm{part}(1, d, C')$, such that either $\mathrm{tr} \circ p$ is total and $\mathrm{tr}_t \circ p \models \mathrm{part}(1, d, C')$ or $\mathrm{tr} \circ p$ is not total exists, for any $C' \in \mathcal{U}(C_k, C_{k+1})$. In both cases the following holds

   $$p \in \{q \mid q : C_k \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\} \wedge$$
   $$\mathrm{tr} \circ p \notin \{q \mid q : C_k \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\}.$$

   With that and 1. it follows that

   $$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\}| < |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\}|.$$

In total $\mathrm{nvc}(k, H) < \mathrm{nvc}(k, G)$ follows and $t$ is a minimal improving transformation. $\square$

## 2.4 Comparison with other consistency concepts

**Lemma 2.18.** *Let a constraint $c$ in ANF, graphs $G$ and $H$ with $G \not\models c$, and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \text{ is c-guaranteeing} \implies t \text{ is minimal consistency improving w.r.t } c$$

*,*

$$t \text{ is c-guaranteeing} \;\not\Longrightarrow\; t \text{ is direct minimal consistency improving w.r.t } c$$

*and*

$$t \text{ is minimal consistency improving w.r.t } c \;\not\Longrightarrow\; t \text{ is c-guaranteeing}$$

*Proof.* Let $c'$ be the equivalent constraint in EANF.

1. Let $t$ be c-guaranteeing transformation, then $H \models c$. Since $G \not\models c$, $G \not\models c'$ follows and $\mathrm{nvc}(c^G_{\max} + 1, G) > 0$ and $\mathrm{nvc}(c^G_{\max} + 1, H) = 0$. Therefore $t$ is minimal consistency improving.

2. Let a constraint $c = \exists(a_0 : \emptyset \hookrightarrow C_0, \forall(a_1 : C_0 \hookrightarrow C_1, \exists(a_2 : C_1 \hookrightarrow C_2, \mathsf{true})))$ and a graph $G = C' \dot\cup C'$ with $C' = C_2 \setminus \{e\}$ for an $e \in E_{C_2 \setminus C_1}$ and the occurrences $p_1 : C_1 \hookrightarrow G$ and $p_2 : C_1 \hookrightarrow G$ be given. Let $t : G \Longrightarrow H$ be a transformation with $H = C_2 \dot\cup C''$ and $C'' = C' \setminus \{e'\}$ for an $e' \in E_{C_2 \setminus C_1}$, such that $\mathrm{tr}_t \circ p_1$ and $\mathrm{tr}_t \circ p_2$ are total. Then, $t$ is $c$-guaranteeing and not direct minimal consistency improving, since $p_i \models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$ for $i = 1, 2$ and either $\mathrm{tr}_t \circ p_1 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$ or $\mathrm{tr}_t \circ p_2 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$.

3. Let $t$ be a minimal consistency improving transformation w.r.t $c'$, such that $\mathrm{nvc}(c^G_{\max} + 1, H) > \mathrm{nvc}(c^G_{\max} + 1, H) > 1$ and $H \not\models_{c^G_{\max}+2} c'$. Then,$H \not\models c'$ and $t$ is not a guaranteeing transformation.

□

**Corollary 2.19.** *Let $c$ be an existentially bound constraint in ANF. Then, a transformation $t$ is c-guaranteeing if and only if it is consistency improving. With lemma 2.18 follows:*

$$t \text{ is consistency guaranteeing w.r.t } c \implies t \text{ is minimal consistency improving w.r.t } c$$

*and*

$$t \text{ is minimal consistency improving w.r.t } c \;\not\Longrightarrow\; t \text{ is consistency improving w.r.t } c$$

**Lemma 2.20.** *let a universally bound constraint $c$ with $\mathrm{nl}(c) = 1$ in ANF, graphs $G$ and $H$ with $G \not\models c$, and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \text{ is consistency improving} \iff \text{is minimal consistency improving}$$

*Proof.* Let $c = \forall(a : \emptyset \hookrightarrow C, \mathsf{false})$. Then, the equivalent constraint in EANF is $c' = \forall(a : \emptyset \hookrightarrow C, \exists(\mathrm{id} : C \hookrightarrow C, \mathsf{false})$. Since $\mathcal{U}(C, C) = \{C\}$, $\mathrm{nvc}(1, G)$ is the number of occurrences of $C$ in $G$. This is exactly the definition of the number of violations for consistency improving transformations and

$$t \text{ is minimal consistency improving} \iff t \text{ is consistency improving}$$

follows immediately. □

**Lemma 2.21.** *Let a universally bound constraint $c$ with $\mathrm{nl}(c) \geq 2$ in ANF, graphs $G$ and $H$ with $G \not\models c$ and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \textit{ is direct consistency improving} \not\Longrightarrow t \textit{ is minimal consistency improving}$$

*and*

$$t \textit{ is direct minimal consistency improving} \not\Longrightarrow t \textit{ is consistency sustaining}$$

*Proof.* 1. Let $c = \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \mathsf{true}))$ be a constraint. Let $V_{C_0} = V_{C_1}$ and $|E_{C_1}| - |E_{C_0}| = 2$. Let $G = C' \dot\cup C'$ with $C' = C_1 \setminus \{e\}$ with $e \in E_{C_1} \setminus E_{C_0}$ and the occurrences $p_1 : C_0 \hookrightarrow G$ and $p_2 : C_0 \hookrightarrow G$ be given. It follows that $\mathrm{nvc}(1, G) = 2$. Let $t : G \Longrightarrow H$ be a transformation, such that $H = C_0$. Then, $t$ is a direct consistency improving transformation, since $H$ contains only one occurrence of $C_0$ not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1\ \mathsf{true})$. But, $t$ is not minimal consistency improving since $\mathrm{nvc}(1, H) = 3$.

2. Let $c := \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \mathsf{true}))))$ be a constraint.

   Let a graph $G = C_0$ with the morphism $q : C_0 \hookrightarrow G$ and a transformation $t : G \Longrightarrow H$ with $H := C_2 \dot\cup C_2$ be given, such that $\mathrm{tr}_t \circ q$ is total. Then $t$ is a direct minimal consistency improving transformation but not a consistency sustaining one, since $H$ contains more occurrences of $C_0$ not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \mathsf{true})))$ than $G$.

□

**Lemma 2.22.** *Let a universally bound constraint $c$ with $\mathrm{nl}(c) \geq 2$ in ANF, graphs $G$ and $H$ with $G \not\models c$ and a transformation $t : G \Longrightarrow H$ be given. If $t$ satisfies (2.1) and (2.2), $H \models_{c^G_{\max}} c$ and*

$$\exists p : C_0 \hookrightarrow G(p \not\models \mathrm{part}(c^G_{\max} + 2, c, \textit{true}) \wedge \mathrm{tr}_t \circ p \models c).$$

*Then,*

$$t \textit{ is consistency improving} \Longrightarrow t \textit{ is minimal consistency improving.}$$

*Proof.* □

## 3 application condition

**Definition 3.1** (**extended overlaps**). *Let $G$ and $C_0 \subseteq C_1$ with the inclusion $i : C_0 \hookrightarrow C_1$ be graphs. Let $C$ be an overlap of $C_0$ and $G$ with the inclusion $q : C_0 \hookrightarrow C$. The set of* extended overlaps of $C$ with $i$, $\mathrm{eol}(C, i)$, *is the set of all overlaps $C'$ of $G$ and $C_1$, such that $C \subseteq C'$ and $q \models \exists(i : C_0 \hookrightarrow C_1, \mathsf{true})$.*

**Definition 3.2** (**overlap shift**). *Let $\rho = L \hookleftarrow K \hookrightarrow R$ be a plain rule, $C$ a graph and $C'$ an overlap of $C$ and $L$ with morphisms $p : L \hookrightarrow C'$, $k : K \hookrightarrow C'$, $c : C \hookrightarrow C'$ and the partial morphism $q : R \hookrightarrow C'$. We define*

$$\begin{aligned} D := \{e \in C' \mid &(\exists e' \in L : p(e') = e \\ &\vee \exists e' \in R : q(e') = e) \\ &\wedge \exists e' \in C : c(e') = e\} \end{aligned} \tag{3.1}$$

*Let $r = L \hookleftarrow K' \hookrightarrow R$ be the rule with*

$$K' := K \cup D$$

*The graph $H$ derived by the transformation $G \Rightarrow_{r,p} H$ is called the* overlap shifted graph of $C'$ with $C$ and $\rho$. *The overlap shifted graph of an graph $C$ is denoted by $\mathrm{ols}_\rho(C, C')$.*

**Definition 3.3** (**application condition**). *Let $\rho = L \hookleftarrow K \hookrightarrow R$ be a plain rule and $c$ a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of $c$ at layer $k$ with $k$ being and even number.*

*The application condition $\mathrm{ap}_k$ of $c$ at layer $k$ with $C' \in \mathcal{U}(C_k, C_{k+1})$ and $i : C_k \hookrightarrow C'$ is defined as:*

$$\mathrm{ap}(k, C') := \Big( \bigvee_{P \in \mathrm{ol}(L, C_k)} \mathrm{nex}(P, C') \wedge \mathrm{rep}(P, C') \Big) \wedge \mathrm{nin} \wedge \mathrm{rem} \wedge \mathrm{nwo} \tag{3.2}$$

*with*

1. $$\mathrm{nex}(P, C') := \begin{cases} \exists(i_L : L \hookrightarrow P, \mathsf{true}) & , \textit{if } C_k = C_{k+1} \\ \bigwedge_{Q \in \mathrm{eol}(P,i)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow Q, \mathsf{true})) & , \textit{otherwise} \end{cases}$$

2. *Let $\mathbf{P}$ be the set of all overlaps of $R$ and $C'$, such that $i_R(R \setminus K) \cap i_{C'}(C') \neq \emptyset$:*

   (a) *If $C_k = C_{k+1}$:*

   $$\mathrm{rep}(P, C') := \begin{cases} \mathsf{true} & , \textit{if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \mathsf{false} & , \textit{otherwise} \end{cases}$$

(b) *Otherwise:*

$$\operatorname{rep}(P, C') := \begin{cases} \textit{true} & \text{, if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \bigvee_{P' \in \mathbf{P}} \operatorname{Left}(\exists(i_R : R \hookrightarrow P', \textit{true}), \rho) & \text{, otherwise} \end{cases}$$

3. *Let $E$ be the set of all graphs $C_j$ of $c$ with $j \leq k$ and $j$ being odd and let $\mathbf{P}_{C_j}$ be the set all overlaps of $L$ and $C_j$ with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.*

$$\operatorname{rem} := \bigwedge_{C \in E} \bigwedge_{C' \in \mathbf{P}_{C_j}} \neg\exists(i_L : L \hookrightarrow C', \textit{true})$$

4. *Let $U$ be the set of all graphs $C_j$ of $c$ with $j \leq k$ and $j$ being even and let $\mathbf{P_{C_j}}$ be the set of all overlaps of $R$ and $C_j$ with $i_R(R \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.*

$$\operatorname{nin} := \bigwedge_{C \in U} \bigwedge_{C' \in \mathbf{P}_{C_j}} \operatorname{Left}(\neg\exists(i_R : R \hookrightarrow C', \textit{true}), \rho)$$

5. *Let $E$ be the set of all overlaps of $L$ and $C_k$, such that each $P \in E$ is also an overlap of $L$ and $C'' \in \mathcal{U}(C_k, C_{k+1})$, $i_{C_k} \models \exists(a'_k : C_k \hookrightarrow C'', \textit{true})$ and $i_L(L \setminus K) \cap i_{C''}(C'' \setminus C_k) \neq \emptyset$.*

$$\operatorname{nwo} := \bigwedge_{P \in E} \neg\exists(i_L : L \hookrightarrow P, \textit{true})$$

**Lemma 3.4.** *Let a graph $G$, a constraint $c$ in EANF, with $G \not\models c$, and a plain rule $\rho$ be given. Then, the rule $\rho'(\rho, \operatorname{ap}(c^G_{\max}, C))$ for a graph $C \in \mathcal{U}(C_{c^G_{\max}}, C_{c^G_{\max}+1})$ is a minimal consistency improving rule.*

*Proof.* Let $t : G \Longrightarrow_{\rho',m} H$ be a transformation and $k = c^G_{\max} + 1$. We show, by contradiction, that this transformation is direct minimal consistency improving by showing that (2.1), (2.2), (2.3), (2.4) and (2.5) are satisfied. With that, it follows that $\rho'$ is a minimal consistency improving rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ be the subcondition of $c$ at layer $k$.

1. Assume that (2.1) does not hold. Then, there does exist an morphism $p : C_k \hookrightarrow G$, such that $p \models \operatorname{part}(1, e, C')$, $\operatorname{tr}_t \circ p$ is total and $\operatorname{tr}_t \circ p \not\models \operatorname{part}(1, e, C')$ for a graph $C' \in \mathcal{U}(C_k, C_{k+1})$. There has to exist an overlap $P$ of $L$ and $C'$ such that $i_{C_k} \models \exists(a^p_k : C_k \hookrightarrow C', \mathsf{true})$ and $m \models \exists(i_L : L \hookrightarrow P, \mathsf{true})$. Then, nwo and with that $\operatorname{ap}(c^G_{\max}, C)$ is not satisfied.
2. Assume that (2.2) does not hold. Then, a morphism $p' : C_k \hookrightarrow H$ with $p' \not\models \operatorname{part}(1, e, C_{k+1})$ exists, such that there does not exist a morphism $p : C_k \hookrightarrow G$ with $\operatorname{tr}_t \circ p = p'$. Then, an overlap $P$ of $R$ and $C_k$ with $i_R(R \setminus K) \cap i_R(C_k) \neq \emptyset$ exists, such that $m \models \operatorname{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$. Then, $m \not\models \operatorname{ap}(c^G_{\max}, C)$.

3. Assume that (2.3) does not hold. Then, there does not exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \mathrm{part}(1, e, C)$, such that $\mathrm{tr}_t \circ p$ is not total or $\mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C)$ and $\mathrm{tr}_t \circ p$ is total. Then, no overlap $P$ of $L$ and $C_k$ with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ exists, such that $m \models \mathrm{nex}(P, C)$. Also, no overlap $P$ of $C$ and $R$ with $i_R(R \setminus K) \cap i_C(C) \neq \emptyset$ and $m \models \mathrm{nex}(P, C)$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$ and therefore $\mathrm{rep}(P, C) = \mathsf{false}$. It follows that for all $P \in \mathrm{ol}(L, C_k)$ it holds that $\mathrm{nex}(P, C) \wedge \mathrm{rep}(P, C) = \mathsf{false}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

4. Assume that (2.4) does not hold. Then, there does exist a morphism $p : C_j \hookrightarrow G$ with $j < k$ and $i$ being even, such that no morphism $p' : C_j \hookrightarrow G$ with $\mathrm{tr}_t \circ p' = p$ exists. Then, an overlap $P$ of $C_j$ and $R$ with $i_R(R \setminus K) \cap i_{C_j} \neq \emptyset$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$. It follows that $m \not\models \mathrm{rem}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

5. Assume that (2.5) does not hold. Then, there does exist and morphism $p : C_j \hookrightarrow G$ with $j < k$ and $j$ being odd, such that $\mathrm{tr}_t \circ p$ is not total. Then, an overlap $P$ of $C_j$ and $L$ with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$ exists, such that $m \models \exists(i_L : L \hookrightarrow P, \mathsf{true})$. It follows that $m \not\models \mathrm{rem}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

In total follows that if $m \models \mathrm{ap}(c^G_{\max}, C)$, then $t$ is a direct minimal improving transformation. □

**Lemma 3.5.** *Let $G$ be a graph, $c$ a constraint in EANF, with $c^G_{\max} < \mathrm{nl}(c)$, and $\rho = L \hookleftarrow K \hookrightarrow R$ a plain rule. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of $c$ at layer $\mathrm{c}_{\max} + 1$ and $\mathrm{ap}(\mathrm{c}_{\max}, C')$ the application condition constructed by definition 3.3 with $C' \in \mathcal{U}(C, C_{k+1})$ for any $C \in \mathcal{G}^G_c$. If*

$$((R \setminus K) \cap C_{k+1}) \cup ((L \setminus K) \cap C_{k+1}) = \emptyset$$

*$\mathrm{ap}(\mathrm{c}_{\max}, C')$ can be replaced by $\mathsf{false}$.*

*Proof.* There does not exist an overlap $P$ of $C_k$ and $L$ with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ and $\mathrm{rep}(P, C')$ will be equal to $\mathsf{false}$, if $C_k = C_{k+1}$, or equal to $\bigvee_{P' \in \mathbf{P}} \mathrm{Left}(\exists(i_R : R \hookrightarrow P', \mathsf{true}), \rho)$. Since the set $\mathbf{P}$ has to be empty, this expression can be replaced by $\mathsf{false}$. If follows that $\mathrm{rep}(P, C') = \mathsf{false}$ for all $P \in \mathrm{ol}(L, C_k)$ and therefore $\mathrm{ap}(\mathrm{c}_{\max}, C')$ will always be evaluated to $\mathsf{false}$. □

## 3.1 potentially minimal improving rules

**Definition 3.6** (**basic improving rule**)**.** *Let a constraint $c$ and a plain rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ be given. The rule $\rho$ is called* basic improving *w.r.t $c$ at layer $k$ with $C_k \subset P \subseteq C_{k+1}$, $p : C_k \hookrightarrow P$ being the inclusion, and $k \in \{1, 3, \dots, \mathrm{nl}(c)\}$, if*

$$(L \setminus K) \cap (C_{k-1} \cup (C_{k+1} \setminus C_k)) = \emptyset \tag{3.3}$$

*and*

$$(R \setminus K) \cap C_k = \emptyset \tag{3.4}$$

*and either 1. or 2. applies.*

1. *The rule $\rho$ deletes elements of $C_k \setminus C_{k-1}$:*

$$L \subseteq C_k \wedge L \setminus K \neq \emptyset \tag{3.5}$$

*Then, $\rho$ is called a* deleting basic improving rule.

2. *The rule $\rho$ creates an instance of $P$:*

$$L = C_k \wedge P \subseteq R \tag{3.6}$$

*and $p$ is a partial morphism of $r$. Then, $\rho$ is called an* inserting basic improving rule.

**Definition 3.7 (application conditions for basic improving rules).** *Let a constraint $c$ in EANF and a basic improving rule $\rho = L \xhookleftarrow{l} K \xhookrightarrow{r} R$ w.r.t $c$ at layer $k$ with $C_k \subseteq P \subseteq C_{k+1}$ be given. We define the application condition for $r$ as:*

1. *If $\rho$ is a deleting potentially minimal improving rule:*

$$\mathrm{ap}_{\mathrm{pi}}(j, P) := \begin{cases} \bigvee_{P \in \mathrm{ol}(L, C_k)} \bigwedge_{P' \in \mathrm{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \mathsf{true})) & , \text{ if } j = k \\ \mathsf{false} & , \text{ if } j \neq k. \end{cases}$$

2. *If $\rho$ is an inserting potentially minimal improving rule:*

$$\mathrm{ap}_{\mathrm{pi}}(j, P) := \begin{cases} \neg\exists(a_k^p : L \hookrightarrow P, \mathsf{true}) & , \text{ if } j = k \\ \mathsf{false} & , \text{ if } j \neq k \end{cases}$$

**Theorem 3.1.** *Let a graph $G$, a constraint $c$ in EANF, with $G \models \mathrm{part}(c^G_{\max}, c, C)$ and $\mathrm{part}(c^G_{\max}, c, C) \in \mathcal{P}^G_c$, and a potentially minimal improving rule $\rho = L \xhookleftarrow{l} K \xhookrightarrow{r} R$ at layer $c^G_{\max}$ with $C_{c_{\max}+1} \subseteq P \subseteq C_{c_{\max}+2}$ be given. Then, $\rho' = (\rho, \mathrm{ap}_{\mathrm{pi}}(c^G_{\max} + 1, P))$ is a direct minimal consistency improving rule.*

*Proof.* Let $t : G \Longrightarrow_{\rho', m} H$ be a transformation, $k = c^G_{\max} + 1$ and $e$ be the subcondition of $c$ at layer $k$. We show that $t$ is a direct minimal consistency improving transformation. First, we show that equation (2.1) is satisfied. Let $p : C_k \hookrightarrow G$ be a morphism. If $\rho$ is a deleting potentially minimal improving rule, either 1. or 2. applies, if $\rho$ is an inserting and not a deleting potentially minimal improving rule, only 2. applies, because $\rho$ cannot destroy any occurrences of $C_k$ in $G$.

1. If $p(C_k) \cap m(L \setminus K) \neq \emptyset$, $\mathrm{tr}_t \circ p$ is not total, since at least one element of $p(C_k)$ has been deleted by $t$ and $p$ does satisfy $\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \models \mathrm{part}(1, e, C') \wedge \mathrm{tr}_t \circ p \text{ is total}) \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C'))$.
2. If $p(C_k) \cap m(L \setminus K) = \emptyset$, $\mathrm{tr}_t \circ p$ is total, since no element of $p(C_k)$ has been deleted by $t$. Because (3.3) holds, $t$ does not delete any elements of $C_{k+1} \setminus C_k$ and $p \models \mathrm{part}(1, e, C') \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C')$ for all $C' \in (U)(C_k, C_{k+1})$.

With 1. and 2. follows that (2.1) is satisfied.

Second, we show that equation (2.2) is satisfied. Let $p' : C_k \hookrightarrow H$ be a morphism. Because (3.4) is satisfied, $t$ does not create any elements of $C_k$ and there must exist a morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p = p'$. If follows that (2.2) is satisfied.

Third, we show that (2.3) is satisfied. We consider the cases that firstly, $\rho$ is a deleting minimal potentially improving rule and secondly, that $\rho$ is an inserting and not a deleting minimal potentially improving rule.

1. If $\rho$ is a deleting potentially minimal improving rule, the condition $\mathrm{ap}_{\mathrm{pi}}(k, P) = \exists\big(b : L \hookrightarrow C_k, \neg\exists(a_{k+2} : C_k \hookrightarrow C_{k+1}, \mathsf{true})\big)$ is satisfied by $m$. Therefore a morphism $p : C_k \hookrightarrow G$ with $p \not\models \neg\exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, \mathsf{true}) = \mathrm{part}(1, e, C_{k+1})$ and $m = p \circ b$ must exist. Since $\rho$ is a deleting rule, at least one element of $p(C_k)$ has been deleted by $t$ and $\mathrm{tr}_t \circ p$ is not total. It follows that (2.3) is satisfied.
2. If $\rho$ is an inserting and not a deleting potentially minimal improving rule, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \mathsf{true})$ is satisfied by $m$. Because $L = C_k$, $m \models \neg\exists(b : C_k \hookrightarrow P, \mathsf{true}) = \mathrm{part}(1, e, P)$. Since (3.6) is satisfied, $\mathrm{tr} \circ p$ is total and $\mathrm{tr}_t \circ p \models \mathrm{part}(1, e, P)$. Therefore, (2.3) is satisfied.

Last, since (3.4) and (3.3) are satisfied, $\rho$ cannot create any occurrence of $C_j$ with $j \leq k$ and $j$ being even and $\rho$ cannot delete any occurrences of $C_j$ with $j \leq k$ and $j$ being odd. Therefore, (2.4) and (2.5) are satisfied.

In total follows that $t$ is a direct minimal consistency improving transformation and $\rho'$ is a direct minimal consistency improving rule. □

**Theorem 3.2.** *Let a constraint $c$ in EANF, a basic increasing rule $\rho$ with $P$ and a consistency increasing transformation $t : G \Longrightarrow_{\rho,m} H$ w.r.t $c$ be given. Then, $m \models \mathrm{ap}_{\mathrm{pi}}(c^G_{\max}, P)$.*

*Proof.* Let $k = c^G_{\max}$.

1. If $c$ is a inserting rule, then $\mathrm{ap}_{\mathrm{pi}}(c^G_{\max}, P) = \neg\exists(a^p_k : L \hookrightarrow P, \mathsf{true})$. Let $n : R \hookrightarrow H$ be the comatch of $t$. Since $t$ is a consistency increasing transformation, there does exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$, $\mathrm{tr}_t \circ p \models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$ such that $p'(P) \cap n(R \setminus K)$ for the morphism $p' : P \hookrightarrow H$ with $p = p \circ a^p_k$. Assume that $m \models \exists(a^p_k : L \hookrightarrow P, \mathsf{true})$. Then, every morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p \models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$ with $p'(P) \cap n(R \setminus K)$ for the morphism $p' : P \hookrightarrow H$ with $p = p' \circ a^p_k$ already satisfies $\exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$. This is a contradiction.
2. If $\rho$ is a deleting rule, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \bigvee_{P \in \mathrm{ol}(L, C_k)} \bigwedge_{P' \in \mathrm{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \mathsf{true}))$. Since $t$ is a consistency increasing transformation, there does exist a morphism $p : C_k \hookrightarrow G$, such that $p \not\models \exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ and $\mathrm{tr}_t \circ p$ is not total. Assume that $m \not\models \mathrm{ap}_{\mathrm{pi}}(k, P)$. Then, for each overlap $Q$ of $L$ and $C_k$, the inclusion $i_{C_k}$ does satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$. This is a contradiction, because $p(C_k) \cap m(L \setminus K) \neq \emptyset$ has to hold.

$\square$

**Lemma 3.8.** *Let a constraint $c$ in EANF and inserting basic increasing rules $\rho_1$, $\rho_2$ at layer $k$ with $P_1$ and $P_2$, respectively, be given. Then, the rules $\rho_1' = (\rho_1, \mathrm{ap}_{\mathrm{pi}}(k, P_1))$ and $\rho_2' = (\rho_2, \mathrm{ap}_{\mathrm{pi}}(k, P_2))$ are sequentially independent.*

*Proof.* Let $G_1 \Longrightarrow_{\rho_1', m_1} G_2 \Longrightarrow_{\rho_2', m_2} G_3$ be a sequence of transformations. First, note that this sequnce can only exist if $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$ Since both rules are do not insert elements of $C_k$, a morphism $d_1 : L_2 \hookrightarrow D_1$ with $m_2 = h_1 \circ d_1$ exists. Because both rules do not delete any elements a morphism $d_2 : R_1 \hookrightarrow D_2$ with $m_1^* = g_2 \circ d_2$ exists. Because $\rho_2$ is not a basic increasing rule with $P_1$, $h_2 \circ d_2 \models R(\mathrm{ap}_{\mathrm{pi}})$ and because $\rho_1$ does not delete any elements of $P_2$ is holds that $g_1 \circ d_1 \models R(\mathrm{ap}_{\mathrm{pi}})$. $\square$

**Definition 3.9** (**repairing rule set**). *Let a constraint $c$ in EANF and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is called a* repairing rule set for $c$ at layer $k$ *if for all graphs $G$ with $k = \mathrm{c}_{\max}$ a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

*exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models_{k+2} c$.*

**Corollary 3.10.** *Let a constraint $c$ in EANF and a set of rules $\mathcal{R}$ be given. If $\mathcal{R}$ is a repairing rule set for $c$ at layer $k$, $\mathcal{R}$ is a repairing rule set w.r.t $c$ at layer $j$ for all $k < j \leq \mathrm{nl}(c)$.*

**Corollary 3.11.** *Let a constraint $c$ in EANF and a repairing rule set $\mathcal{R}$ for $c$ at layer $k$, for all $k \in \{1, 3, \ldots, \mathrm{nl}(c)\}$, be given. Then, for all graphs $G$, a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

*exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models c$.*

**Definition 3.12** (**decomposition of a graph**). *Let graphs $G_0 \subset G_1$ be given.*

*A decomposition of $G_1$ with $G_0$ is a minimal set*

$$\mathbf{P} \subseteq \{P_v \mid v \in V_{C_1 \setminus C_0}\}$$

*of subgraphs of $G_1$, such that every element of $G_1$ is contained in at least one $P \in \mathbf{P}$ and every $P_v$ is constructed in the following way: $G_0 \subset P_v$, $v \in P$ and for all nodes $v' \in P \setminus C_k$ is holds that $P$ contains all edges $e \in E_{G_1 \setminus G_0}$ and all nodes $u \in V_{G_1}$ such that either $\mathrm{tar}(e) = v' \wedge \mathrm{src}(e) = u$ or $\mathrm{src}(e) = u \wedge \mathrm{tar}(e) = v'$ holds.*

**Lemma 3.13.** *Let graphs $G_0 \subset G_1$ and a decomposition $\mathbf{P}$ of $G_1$ with $G_0$ be given. Then, for each pair $P, P' \in \mathbf{P}$ with $P \neq P'$ the following holds:*

$$(P \setminus C_k) \cap (P' \setminus C_k) = \emptyset$$

*Proof.* Assume that $(P \setminus C_k) \cap (P' \setminus C_k) \neq \emptyset$, therefore a node $v \in G_1 \setminus G_0$ with $v \in P \cap P'$ exists. By the construction of $P$ and $P'$ it follows that $P = P_v$ and $P' = P_v$ and therefore $P = P'$. This is a contradiction. $\square$

**Lemma 3.14.** *Let graphs $G_0 \subset G_1$ and a decomposition $\mathbf{P}$ of $G_1$ with $G_0$ be given. Then,*

$$G_1 = \bigcup_{P \in \mathbf{P}} P.$$

*Proof.* Let $H := \bigcup_{P \in \mathbf{P}} P$. Firstly, we show that $H \subseteq G_1$. Since every $P \in \mathbf{P}$ is a subgraph of $G_1$ it follows that $V_H \subseteq V_{G_1}$ and $E_H \subseteq E_{G_1}$

Secondly, we show that $G_1 \subseteq H$. Let $u \in V_{G_1}$ be a node, if $u \in V_{G_0}$, then $u$ is contained in every $P \in \mathbf{P}$ and therefore $u \in V_H$. Otherwise, if $u \notin V_{G_0}$, then $u$ has to be, by the definition of $\mathbf{P}$, contained in at least one $P \in \mathbf{P}$ and $V_{G_1} \subseteq V_H$ follows. Let $e \in E_{G_1}$ be an edge. If $e \in E_{G_0}$, then $e$ is contained in every $P \in \mathbf{P}$ and $e \in E_H$. Otherwise, if $e \notin E_{G_0}$, by the definition of $\mathbf{P}$, $e$ has to be contained in at least one $P \in \mathbf{P}$. It follows that $e \in E_H$ and with that $E_{G_1} \subseteq E_H$. □

**Theorem 3.3.** *Let a constraint $c$ in EANF and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is a repairing set of $c$ at layer $k \leq \mathrm{nl}(c)$ if either 1 or 2 applies.*

1. *For any universally bound graph $C_j$ at layer $j \leq k$ of $c$, $(\rho, \mathrm{app}_{\mathrm{pi}}(j, C_{j+1})) \in \mathcal{R}$ and $\rho$ is a deleting potentially minimal improving rule at layer $j$ with $C_{j+1}$, such that $\rho$ only deletes edges of $C_j$.*
2. *A decomposition $\mathbf{P}$ of $C_k$ with $C_{k-1}$ exists, such that for each $P \in \mathbf{P}$ a rule $(\rho, \mathrm{app}_{\mathrm{pi}}(k, P)) \in \mathcal{R}$ exists, such that $\rho$ is an inserting basic improving rule at layer $k$ with $P$.*

*Proof.* Let a constraint $c$ in EANF, a rule set $\mathcal{R}$ and a graph $G$ with $k = \mathrm{c}_{\max}$ and $\mathrm{c}_{\max} < \mathrm{nl}(c)$ be given. We show that a sequence $G = C'_0 \Rightarrow \ldots \Rightarrow C'_n = H$ with rules of $\mathcal{R}$ exists, such that $H \models_{k+2} c$ if 1. or 2. of theorem 3.3 is satisfied.

1. Assume that 1. of theorem 3.3 holds. Let $(\rho, \mathrm{app}_{\mathrm{pi}}(j, C_{j+1})) \in \mathcal{R}$, such that $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ is a deleting potentially minimal improving rule at layer $j \leq k$ with $C_{j+1}$ and $C_j$ is a universally bound graph of $c$. Then, $\mathrm{app}_{\mathrm{pi}}(j, C_{j+1}) = \exists\big(b : L \hookrightarrow C_j, \neg\exists(a_j : C_j \hookrightarrow C_{j+1}, \mathsf{true})\big)$. Let $q : C_j \hookrightarrow G$ be a morphism such that $q \not\models \exists(a_j : C_j \hookrightarrow C_{j+1}, \mathsf{true})$. Since $L \subseteq C_j$, we can construct a morphism $m_1 : L \hookrightarrow G$ with $m_1 = q \circ b$ and therefore $m_1 \models \mathrm{app}_{\mathrm{pi}}(j, C_{j+1})$. Since $r$ only deletes edges, a transformation $t : G = G_0 \Rightarrow_{r,m_1} G_1$ exists and $\mathrm{tr}_t \circ p$ is not total. Because $r$ does not insert any elements of $C_j$:

   $$|\{q : C_k \hookrightarrow G_0 \mid q \not\models d\}| < |\{q : C_k \hookrightarrow G_1 \mid q \not\models d\}|$$

   with $d = \exists(a_j : C_j \hookrightarrow C_{j+1}, \mathsf{true})$. By iteratively applying this construction, we can generate a finite sequence of transformations

   $$G = G_0 \Rightarrow_{r,m_1} G_1 \Rightarrow_{r,m_2} \ldots \Rightarrow_{r,m_n} G_n = H$$

   such that $|\{q : C_k \hookrightarrow G_n \mid q \not\models d\}| = 0$ and therefore $H \models_j c$. With lemma 2.9, $H \models_{k+2} c$ and $H \models c$ follows.

2. Assume that 2. of theorem 3.3 holds. Let $\rho_0 = (\rho, \text{ap}_{\text{pi}}(k, P)) \in \mathcal{R}$, such that $\rho$ is an inserting basic improving rule of $c$ at layer $k$ with $P \in \mathbf{P}$. Then, $\text{ap}_{\text{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \mathsf{true})$. Let $q_0 : C_k \hookrightarrow G$ be a morphism, such that $q_0 \not\models \exists(a'_k : C_k \hookrightarrow P, \mathsf{true})$ with $a'_k$ being a partial morphism of $a_k$. Since $L = C_k$, we set $m_0 : C_k \hookrightarrow G$ with $m_0 = q_0$. It follows that $m_0 \models \neg\exists(a'_k : C_k \hookrightarrow P, \mathsf{true}) = \text{ap}_{\text{pi}}(k, P)$. Because $r$ does not delete any elements, a transformation $t_0 : G \Longrightarrow_{r_0, m_0} G_1$ exists and $\text{tr}_t \circ q \models \exists(a'_k : C_k \hookrightarrow P, \mathsf{true})$. We set $q_1 = \text{tr}_{t_0} \circ q_0$ and apply the same method to $q_1$.

   By iteratively applying this, we can construct a finite sequence of transformations

   $$G \Longrightarrow_{r_0, m_0} G_0 \Longrightarrow_{r_1, m_1} \ldots \Longrightarrow_{r_n, m_n} G_n$$

   such that $m_i = \text{tr}_{t_{i-1}} \circ \ldots \circ \text{tr}_{t_0} \circ m_0$ and $q \models \exists(b_i : C_k \hookrightarrow P_i, \mathsf{true})$ for all $P_i \in \mathbf{P}$ with $q = \text{tr}_{t_n} \circ q_n$. Let $p_i : P_i \hookrightarrow G_n$ be the morphism, such that $q = p_i \circ b_i$.

   Now, we can construct a morphism $p : C_{k+1} \hookrightarrow G$ with

   $$p(e) := \begin{cases} p_1(e) & ,\text{if } e \in P_1 \\ & \vdots \\ p_j(e) & ,\text{if } e \in P_j. \end{cases}$$

   Let $e \in C_k$, because $q(e) = p_i \circ b_i(e)$ and $q(e) = p_\ell \circ b_\ell(e)$ and $b_i$ and $b_\ell$ are both partial morphisms of $a_k$, it follows that $b_i(e) = b_\ell(e)$ and therfore $p_i(e) = p_\ell(e)$. Because $(P_i \cap P_\ell) \setminus C_k = \emptyset$ for all $i \neq \ell$, $p$ is a morphism and by the definition of $p$ it follows that $q = p \circ a_k$ and therefore $q \models \exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$.

   By iteratively applying this whole construction to all occurrences of $C_k$ that do not satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ the derived graph graph $H$ does not contain any occurrences of $C_k$ not satisfying $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ and therefore $H \models_{k+2} c$.

□

**Corollary 3.15.** *If a set of rule $\mathcal{R}$ is a repairing set of $c$ at layer $k \leq \text{nl}(c)$ and 1. of theorem 3.3 applies, then $\mathcal{R}$ is a repairing set of $c$ at layer $j$ for all $k \leq j \leq \text{nl}(c)$.*

**Lemma 3.16.** *Let a graph $G_0$, a constraint $c$ in EANF and a repairing set $\mathcal{R}$ at layer $c^{G_0}_{\max}+1$ be given, such that each rule in $\mathcal{R}$ is a basic increasing. Then, for every sequence*

$$G_0 \Longrightarrow_{\rho_0, m_0} G_1 \Longrightarrow_{\rho, m_0} \ldots \Longrightarrow_{\rho_n, m_n} G_n$$

*such that $\rho_i = (\rho, \text{ap}_{\text{pi}}(c^{G_0}_{\max} + 1, P)$ with $\rho \in \mathcal{R}$ being a consistency increasing rule at layer $c^{G_0}_{\max} + 1$ with $P$, it holds that*

## References

[1] C. Sandmann and A. Habel. Rule-based graph repair. *arXiv preprint arXiv:1912.09610*, 2019.