

Masterarbeit

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Abstract

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1 Preliminaries

Definition 1.1 (subgraph). Let G_1 and G_2 be graphs. The graph G_2 is called a subgraph of G_1 if an injective morphism $f : G_2 \rightarrow G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective. The set of all subgraphs of G_1 is denoted by $\text{sub}(G_1)$.

Definition 1.2 (overlap). Let G and G' be graphs. A graph H is called an overlap of G and G' if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by $\text{ol}(G, G')$.

Definition 1.3 (overlap at morphism). Let C, G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an overlap of G and C' at p if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by $\text{ol}_p(G, C')$.

Definition 1.4 (partial morphism). Let $f : G_1 \rightarrow G_2$ and $g : G_3 \rightarrow G_4$ be morphisms. The morphism g is called a partial morphism of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.

Definition 1.5 (nested graph condition). A graph condition over a graph C_0 is inductively defined as follows:

- *true* is a graph condition over every graph.
- $\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1 .
- $\neg d$ is a graph condition over C_0 if d is a graph condition over C_0 .
- $d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0 .

Conditions over the empty graph \emptyset are called constraints. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies *true*. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c . An injective morphism satisfies $\neg d$ if it does not satisfy d , it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \vee d_2$ if it satisfies d_1 or d_2 . A graph G satisfies a constraint c , $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c . We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg \exists(a : C_0 \hookrightarrow C_1, \neg d)$.

The nesting level nl of a condition is defined as $\text{nl}(\text{true}) = 0$ and $\text{nl}(\exists(a : P \rightarrow Q, d)) := \text{nl}(d) + 1$.

Definition 1.6 (alternating quantifier normal form (ANF)[1]). A graph condition c is in alternating normal form (ANF) if it is of the form

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \dots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 partial consistency improving

2.1 partial-conditions and -satisfiability

Definition 2.1 (partial condition). Let c be a condition over C_0 . A partial condition of c over $C'_0 \subseteq C_0$ is defined as:

1. *true* is the partial condition of *true* for every morphism.
2. if $c = Q(a : C_0 \hookrightarrow C_1, d)$, with $Q \in \{\exists, \forall\}$, a partial condition of c over C'_0 is given by $\exists(a' : C'_0 \hookrightarrow C'_1, d')$ with a'_1 being a partial morphism of a , $C'_1 \subseteq C_1 \setminus a(C_0 \setminus C'_0)$ and d' is a partial condition of d over C'_1 .
3. if $c = d_1 \wedge d_2$ or $c = d_1 \vee d_2$ the partial condition of c is given by $d'_1 \wedge d'_2$ and $d'_1 \vee d'_2$, respectively, with d'_1 and d'_2 being partial conditions of d_1 and d_2 over C'_0 .
4. if $c = \neg d$ the partial condition of c is given by $\neg d'$ with d' being a partial condition of d over C'_0 .

A partial condition $Q(a : C'_0 \hookrightarrow C'_1, d)$ of c over C'_0 is called the closest partial condition of c over C'_0 if $C'_1 = C_1 \setminus a(C_0 \setminus C'_0)$ and d is the closest partial condition of d over C'_1 .

We use the notation $c' \leq c$ if c' is partial condition of c over $C'_0 \subseteq C_0$ and $c' < c$ if $C'_i \subset C_i$ for any i .

Definition 2.2 (partial satisfiability). Let a condition c over C_0 and a graph G be given. A morphism $p_0 : C'_0 \rightarrow G$, with $C'_0 \subseteq C_0$, partial satisfies c , $p_0 \models_p c$, if a partial morphism $p'_0 : C''_0 \rightarrow G$ of p_0 satisfies a partial condition of c over C''_0 .

Note, that $p_0 \models c$ implies $p_0 \models_p c$

2.2 minimal consistency improving

Definition 2.3 (Condition up to layer). Let c be a condition and d a subcondition of c . The layer of d is defined as $\text{lay}(d) := \text{nl}(c) - \text{nl}(d) - 1$.

Let L be the set of subconditions of c with layer k . The condition up to layer k of c is the condition which is obtained by replacing each $d = Q(a : P \hookrightarrow P', e) \in L$ by $d = Q(a : P \hookrightarrow P', \text{true})$ if $Q = \exists$ and by $Q(a : P \hookrightarrow P', \text{false})$ if $Q = \forall$ in c . If $k = 0$, the condition up to layer k of every condition c is *true*.

Definition 2.4 (layered consistency). Let G be a graph, $p : C_0 \rightarrow G$ a morphism, c a condition over C_0 and d the condition up to layer k of c . The morphism p satisfies c up to layer k if $p \models d$. The notation $p \models_k c$ is used if p satisfies c up to layer k . The notation $G \models_k c$ is used if c is a constraint and $p \models_k c$ for $p : \emptyset \rightarrow G$.

Lemma 2.5. Let G be a graph $p : C_0 \rightarrow G$ a morphism and $c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, \dots))$ a condition over C_0 in ANF with $p \models_k c$. If the subcondition $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, $p \models Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, \dots \forall(a_k : C_{k-1} \hookrightarrow C_k, f)))$ for every condition f over C_k .

Proof. Let k be the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound, let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be this subcondition. Let $q : G_{k-1} \rightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$. This must exist, since $p \models_k c$ and k is the smallest number such that $p \models_k v$ and the subcondition of c with layer k is universally bound.

Therefore, there does not exist a morphism $q' : C_k \rightarrow G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \rightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$. \square

Lemma 2.6. *Let G be a graph, $p : C_0 \rightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\text{lay}(d) = k$ is universally bound, $p \models_k c$ implies $p \models c$.*

Proof. Follows immediately by using lemma 2.5 and setting f to the subcondition of c with layer $k + 1$. \square

Lemma 2.7. *Let $c = Q(a_1 : C_0, \hookrightarrow C_1, \dots)$ be a condition in ANF over C_0 and $p : C_0 \rightarrow G$ a morphism with $p \models_k c$. Let $d = Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q}'(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the subcondition of c with layer $k + 1$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that $p \models Q(a_1 : C_0, \hookrightarrow C_1, \dots, Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q}'(a'_{k+2} : C_{k+1} \hookrightarrow C', f)))$ with f being a Q' bound condition over C' and a'_{k+2} being a partial morphism of a_{k+2} .*

Proof. If $p \models c$, we can choose $C' = C_{k+2}$ and $f = e$. If $p \not\models c$, there does not exist a j with $p \models_j c$ and the subcondition of c with layer j is universally bound and $Q' = \forall$ follows. We choose $C' = C_{k+1}$ and $f = \text{true}$. Let $q : C_k \rightarrow G$ with $p = q \circ a_k \circ \dots \circ a_1$ and $q \circ \dots \circ a_\ell$ satisfying the condition up to $\ell - k$ of the subcondition of c at layer ℓ for all $0 \leq \ell \leq k$. This morphism must exist since $p \models_k c$ and $p \not\models c$. Let $q' : C_{k+1} \rightarrow G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a'_{k+2} has to be the identity and therefore $q' = q' \circ a'_{k+2}$. It follows that $q' \models \exists(a_{k+2} : C_{k+1} \hookrightarrow C', \text{true})$ and therefore $p \models Q(a_1 : C_0, \hookrightarrow C_1, \dots, \forall(a_{k+1} : C_k \hookrightarrow C_{k+1}, \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+1}, \text{true})))$. \square

Definition 2.8 (biggest partially satisfying graph). *Let G be a graph, $c = Q(a_1 : C_0, \hookrightarrow C_1, \dots)$ a condition in ANF and $p : C_0 \rightarrow G$ a morphism with $p \models_k c$. Let $d = Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q}'(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the subcondition of c with $\text{lay}(d) = k + 1$. With lemma 2.7 there exists a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ and an Q' bound condition e over C' in ANF with $p \models Q(a_1 : C_0, \hookrightarrow C_1, \dots, Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q}'(a_{k+2} : C_{k+1} \hookrightarrow C', e))) =: d_{C', e}$. Let $\mathcal{C}_{k, p, e}$ be the set of graphs $C_{k+1} \subseteq C' \subseteq C_{k+2}$ with $p \models d_{C', e}$.*

A graph $C' \in \mathcal{C}_{k, p, e}$, such that no $C'' \in \mathcal{C}$ with $C' \subseteq C''$ exists is called a biggest partially satisfying graph of c at level k with p and e . The set of these graphs is denoted by $\mathcal{B}_{k, p, e}$.

Definition 2.9. *Let G be a graph and $c = Q(a_1 : \emptyset \hookrightarrow C_1, \dots)$ a constraint in ANF, such that $G \models_k c$ and $G \not\models c$. Let $d = Q(a_1 : \emptyset \hookrightarrow C_1, \dots, \forall(a_{k+1} : C_k \hookrightarrow C_{k+1}, \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \text{true})))$ be the condition up to layer $k + 2$ of c . The number of violations of c up to layer $k + 2$ in G is defined as the number of morphisms $q : C_{k+2} \rightarrow G$ that do not satisfy $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \text{true})$. This number is denoted by $\text{nv}_c(k + 2, G)$.*

Definition 2.10 (minimal consistency improving). Let G be a graph, r a rule and c a constraint in ANF with $G \models_k c$ and $G \not\models c$. With lemma 2.6 follows that the subcondition of c with layer k has to be existentially bound.

A transformation $G \xrightarrow{r,m} H$ is called minimal consistency improving if either $H \models c$ or $\text{nv}_c(k+2, G) < \text{nv}_c(k+2, H)$ or every $C' \in \mathcal{B}_{k,G,\text{true}}$ is also an element of $\mathcal{C}_{k+2,H,e}$ and for at least one C' an $C'' \in \mathcal{C}_{k+2,H,\text{true}}$ with $C' \subseteq C''$ exists. The rule r is called minimal consistency improving if all of its applications are.

Lemma 2.11. Let G be $p : C_0 \rightarrow G$ a morphism, c a constraint in ANF over C_0 with $p \models_k c$. Then $p \models_j c$ for all $j < k$ such that the subcondition of c at layer j is existentially bound.

Proof. 1. The subcondition of c at layer k is existentially bound: If an $j < k$ with $p \models_j c$ exists such that the subcondition of c at layer j is universally bound, let j_1 be the smallest of these. With lemma 2.5 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that the subcondition of c at layer j is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of the subcondition of c at layer ℓ . Since $\ell < j_1$, a morphism $q : C_\ell \rightarrow G$ with $q \models d$ must exists and therefore a morphism $q' : C_{\ell+1} \rightarrow G$ with $q = q' \circ a_k$ must exists. It follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \text{true})$ and with that $p \models_\ell c$.

2. The subcondition of c at layer k is universally bound: With lemma 2.5 follows that $p \models_{k+1} c$. Since c is in ANF 1. can be applied to $k+1$. □

Lemma 2.12. Let G be a graph, r a rule and c a constraint in ANF with $G \not\models c$. Let k be the biggest number, such that $G \models_k c$. A transformation $G \xrightarrow{r,m} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.

Proof. Since $G \not\models c$, with lemma 2.6 follows that the subcondition of c at layer k has to be existentially bound.

The subcondition of c at layer j is existentially bound: With lemma 2.11 follows that $H \models_{k+2} c$. Let $d = \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k+2$. The set $\mathcal{C}_{k+2,H,\text{true}}$ contains every graph C' with $C_{k+1} \subseteq C' \subseteq C_{k+2}$. Since $G \not\models_{k+2} c$, $C_{k+2} \notin \mathcal{C}_{k+2,G,\text{true}}$ and for every $C'' \in \mathcal{B}_{k+2,G,\text{true}}$ it holds that $C'' \in \mathcal{C}_{k+2,H,\text{true}}$ and $C'' \subset C_{k+2} \in \mathcal{C}_{k+2,H,\text{true}}$.

2. The subcondition of c at layer j is universally bound: With lemmas 2.5 and 2.6, $H \models_{k+2} c$ follows and 1. can be applied. □

3 application condition

Construction 3.1. Given a rule $r = L \xleftarrow{l} K \xrightarrow{r} R$ and a constraint c in ANF with $\text{nl}(c) = 1$. Let G and H be graphs. We denote the set of all overlaps of G and H as $\text{ol}(G, H)$. We construct the following application conditions:

1. if $c = \forall(a_0 : C_0 \rightarrow C_1, \text{false})$:

$$\bigvee_{Q \in \text{ol}(L, C_1)} \left(\exists(a : L \hookrightarrow Q, \text{true}) \wedge \text{Left}(\neg \exists(b : R \rightarrow Q', \text{true})) \right)$$

with $Q' \setminus (R \setminus (K \cup C_1)) = Q \setminus (L \setminus (K \cup C_1))$.

2. if $c = \exists(a_0 : C_0 \rightarrow C_1, \text{true})$:

$$\bigvee_{Q \in \text{ol}(L, C_1)} \bigvee_{\substack{Q' \subset Q \\ \overline{Q} \subset Q \\ Q' \subset \overline{Q}}} \left(\exists(a : L \rightarrow Q', \text{true}) \wedge \left(\bigvee_{\substack{\overline{Q} \subset Q \\ Q' \subset \overline{Q}}} \text{Left}(\exists(a : R \rightarrow \overline{Q}, \text{true})) \right) \right)$$

References

- [1] C. Sandmann and A. Habel. [Rule-based graph repair](#). *arXiv preprint arXiv:1912.09610*, 2019.