Rule-based Graph Repair using Minimally Restricted Consistency-Improving Transformations

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November 18, 2022

Abstract

Contents

1 Preliminaries

Definition 1.1 (**subgraph**)**.** *Let G_1 and G_2 be graphs. The graph G_2 is called a* subgraph *of G_1 if an injective morphism $f : G_2 \rightarrow G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective.*

Definition 1.2 (**uppergraph**)**.** *Let G_1 and G_2 be graphs with $G_1 \subseteq G_2$. A graph C is called an* upper-graph *of G_1 w.r.t G_2, if $G_1 \subset C \subseteq G_2$ The set of uppergraphs of G_1 w.r.t. G_2 is denoted by $\mathcal{U}(G_1, G_2)$. If $G_1 = G_2$, we set $\mathcal{U}(G_1, G_2) = \{G_1\}$.*

Definition 1.3 (**overlap**)**.** *Let G and G' be graphs. A graph H is called an* overlap of G and G' *if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by* $\mathrm{ol}(G, G')$.

Definition 1.4 (**overlap at morphism**)**.** *Let C,G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an* overlap of G and C' at p *if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by* $\mathrm{ol}_p(G, C')$.

Definition 1.5 (**partial morphism**)**.** *Let $f : G_1 \rightarrow G_2$ and $g : G_3 \rightarrow G_4$ be morphisms. The morphism g is called a* partial morphism *of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.*

Definition 1.6 (**nested graph condition**)**.** *A* graph condition *over a graph C_0 is inductively defined as follows:*

- *true is a graph condition over every graph.*
- *$\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1.*
- *$\neg d$ is a graph condition over C_0 if d is a graph condition over C_0.*
- *$d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0.*

Conditions over the empty graph $\emptyset$ are called constraints*. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies* true*. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c. An injective morphism satisfies $\neg d$ if it does not satisfy d, it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \wedge d_2$ if it satisfies d_1 or d_2. A graph G satisfies a constraint c, $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c. We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg\exists(a : C_0 \hookrightarrow C_1, \neg d)$.*

The nesting level nl *of a condition is defined as* $\mathrm{nl}(\mathit{true} = 0$ *and* $\mathrm{nl}(\exists(a : P \rightarrow Q, d)) := \mathrm{nl}(d) + 1$.

Definition 1.7 (**alternating quantifier normal form (ANF)**[3])**.** *A graph condition c is in* alternating normal form *(ANF) if it is of the form*

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \ldots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 consistency increasing

In this section, we introduce the notion of *consistency increasing* transformations and rules, which allows to increase the consistency of a constraint layer by layer.

2.1 extended alternating quantifier normal form

To prevent the need of case discrimination, a new normal form for conditions, called *extended alternating quantifier normal form* (EANF), will be introduced. The sets of conditions in ANF and EANF do intersect and we show that both sets are expressively equivalent.

Definition 2.1 (**extended alternating quantifier normal form**). *A conditions c is in* extended alternating quantifier normal form *(EANF) if it is of the form $\forall(a_0 : C_0 \hookrightarrow C_1, d)$ and d is a condition in ANF.*

Note that, given a condition c in EANF, every subcondition of c at layer $1 \leq k \leq \mathrm{nl}(c)$ is universally bound, if k is an odd number and existentially bound, if k is an even number.

Lemma 2.2. *Any condition in ANF can be transformed into an equivalent condition in EANF and vice versa.*

Proof. "$\Longrightarrow$": Let a graph G and a constraint c in ANF be given. If c is universally, c is already in EANF.

If $c = \exists(a_0 : C_0 \hookrightarrow C_1, d)$, we show that c is equivalent to $c' := \forall(\mathrm{id}_{C_0} : C_0 \hookrightarrow C_0, c)$.

1. Let $p : C_0 \hookrightarrow G$ be a morphism, such that $q \models c$. Therefore a morphism $q : C_0 \to G$ with $q \models e$ and $p = q \circ a_1$ exists. Then, $p \models d$, since p is the only morphism from C_0 to G with $p = p \circ \mathrm{id}_{C_0}$ and $p \models c$.

2. Let $p : C_0 \hookrightarrow G$ be a morphism with $p \models c'$, therefore all morphisms $q : C_0 \hookrightarrow G$ with $p = q \circ \mathrm{id}_{C_0}$ satisfy c. Since $p = p \circ \mathrm{id}_{C_0}$, $p \models c$ follows immediately.

"$\Longleftarrow$": Let a graph G and a constraint c in EANF be given. If $c = \forall(a_0 : C_0 \hookrightarrow C_1, d)$ with $C_0 \neq C_1$, c is already in ANF.

Otherwise, the equivalence of c with d can be shown analogously to the first case.

□

2.2 conditions up to layer

Definition 2.3 (**Layer of a subcondition**). *Let c be a condition and d a subcondition of c. The layer of d is defined as* $\mathrm{lay}(d) := \mathrm{nl}(c) - \mathrm{nl}(d) - 1$.

Definition 2.4 (**Subcondition at layer**). *Let c be a condition. The* subcondition at layer k*, denoted by* $\mathrm{sco}_c(k)$ *, is the subcondition d of c with* $\mathrm{lay}(d) = k$.

We define a notion of partial consistency, called *satisfaction at layer*, which will be used for the definition of consistency increasing. First, two operators are introduced to modify given constraints on a certain layer.

Definition 2.5 (**substitution at layer**). *Let $c = Q(a : C_0 \hookrightarrow C_1, d)$ be a condition in ANF, such that the subcondition of c with layer $0 \leq k \leq \mathrm{nl}(c)$ is a condition over C_k. Let e be a condition over C_k. The* substitution in c at layer k with e, $\mathrm{sub}(k, c, e)$, *is recursively defined as:*

1. *If $k \leq 1$:*
$$\mathrm{sub}(0, c, e) := e$$

2. *If $k > 1$:*
$$\mathrm{sub}(k, c, e) := Q\big(a : C_0 \hookrightarrow C_1, \mathrm{sub}(k-1, d, e)\big)$$

Example 2.1. *Let the conditions $c := \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1 : C_1 \hookrightarrow C_2, \mathit{true}))$ and $d = \exists(a'_1 : C_1 \hookrightarrow C_3, e)$ be given. Then,*

$$\mathrm{sub}(2, c, d) = \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a'_1 : C_1 \hookrightarrow C_3, e)).$$

Through this, we define *condition up to layer*. Intuitively, a condition is cut off at a certain layer, by replacing the subcondition at this layer by true or false, depending on the quantifier, the replaced subcondition is bound by.

Definition 2.6 (**Condition up to layer**). *Let c be a condition in EANF and d be the subcondition of c at layer $0 \leq k \leq \mathrm{nl}(c)$. The* condition up to layer k *of c*, $\mathrm{cond}(k, c)$, *is defined as*

$$\mathrm{cond}(k, c) := \begin{cases} \mathrm{sub}(k+1, c, \mathit{true}) & \textit{if } d \textit{ is existentially bound} \\ \mathrm{sub}(k+1, c, \mathit{false}) & \textit{if } d \textit{ is universally bound.} \end{cases}$$

Example 2.2. *Let the condition $c = \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1 : C_1 \hookrightarrow C_2, d))$ be given. Then,*

$$\mathrm{cond}(2, c) = \forall(a_0 : C_0 \hookrightarrow C_1, \mathit{true}).$$

Now, we are ready to define satisfaction at layer, which is a key ingredient to define consistency increasing.

Definition 2.7 (**Satisfaction at layer**). *Let G be a graph and c be a condition over C_0. A morphism $p : C_0 \hookrightarrow G$* satisfies c at layer k, *$p \models_k c$, if*

$$p \models \mathrm{cond}(k, c).$$

A graph G satisfies a constraint c at layer k, $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies $\mathrm{cond}(k, c)$. *The biggest k with $G \models_k c$ such that no $j > k$ with $G \models_j c$ exists is denoted by $c^G_{\max}$.*

Example 2.3.

The following lemmas arise as a direct consequence of the definition of satisfaction at layer. If a graph satisfies a constraint up to a certain layer, let c be this condition up to this layer, that ends with $\forall(a : C \hookrightarrow C', \mathsf{false})$, the graph satisfies all constraints starting with c.

Lemma 2.8. *Let G be a graph $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in EANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, then for any condition f over C_k it holds that*

$$p \models \mathrm{sub}(k+1, c, f).$$

Proof. Let k be the smallest number with $\mathrm{sco}_c(k) = \forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false})$ being universally bound and $p \models_k c$. Let $q : G_{k-1} \hookrightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false})$. This must exist, since $p \models_k c$ and k is the smallest even number such that $p \models_k c$.

Therefore, there does not exist a morphism $q' : C_k \hookrightarrow G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \hookrightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$ and with that $p \models \mathrm{sub}(k+1, c, f)$. For every $j > k$ even, such that $p \models_j c$, and every condition d over C_j, with the first part, it holds that $p \models \mathrm{sub}(k+1, c, f)$ with $f = \mathrm{sub}(j-k+1, \mathrm{sco}_c(k+1), d)$ and since $\mathrm{sub}(k+1, c, f) = \mathrm{sub}(j+1, c, d)$ it follows that $p \models \mathrm{sub}(j+1, c, d)$. □

As a direct consequence of the previous lemma, a graph satisfying a condition up to layer ending with $\forall(a : C \hookrightarrow C', \mathsf{false})$ also satisfies the whole constraint.

Lemma 2.9. *Let G be a graph, $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in EANF with $p \models_k c$. If $\mathrm{sco}_c(k)$ is universally bound,*

$$p \models_k c \implies p \models c.$$

Proof. Follows immediately by using lemma 2.8 and setting f to $\mathrm{sco}_c(k+1)$. □

Lemma 2.10. *Let a graph G, a morphism $p : C_0 \hookrightarrow G$ and a constraint c in EANF be given. Then, $p \models_j c$ for all $j < c^G_{\max}$ such that $\mathrm{sco}_c(j)$ is existentially bound.*

Proof.

1. The subcondition of c at layer $c^G_{\max}$ is existentially bound: If a $j < c^G_{\max}$ with $p \models_j c$ exists such that $\mathrm{sco}_c(j)$ is universally bound, let j_1 be the smallest of these. With lemma 2.8 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that $\mathrm{sco}_c(\ell)$ is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of $\mathrm{sco}_c(\ell)$. Since $\ell < j_1$, a morphism $q : C_\ell \hookrightarrow G$ with $q \models d$ must exists and therefore a morphism $q' : C_{\ell+1} \hookrightarrow G$ with $q = q' \circ a_k$ must exists. It follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \mathsf{true})$ and with that $p \models_\ell c$.

2. The subcondition of c at layer $c^G_{\max}$ is universally bound: With lemma 2.8 follows that $p \models_{k+1} c$. Since c is in EANF, 1. can be applied to $k+1$.

□

Through satisfaction at layer an increase of consistency can be detected in the following way. Let $G \Longrightarrow H$ be a transformation. If $c^G_{\max} < c^H_{\max}$, the transformation can be considered as consistency increasing, since H satisfies more layers of the constraint than G. The notion of consistency increasing should also be able to detect the smallest transformations that lead to a increase of consistency, namely the inserting of a single edge or node of an existentially bound graph. To remedy this issue, we introduce *partial conditions*. Intuitively, given a constraint c in EANF ending with $\mathrm{sco}_c(k) = \exists(a_k : C_k \hookrightarrow C_{k+1}, d)$, the condition $\mathrm{sco}_c(k)$ is replaced by $\exists(a^p_k : C_k \hookrightarrow C', \mathsf{true})$ with $C' \in \mathcal{U}(C_k, C_{k+1})$.

The construction of partial conditions is designed to only replace graphs in existentially bound layers, since the replacement in an universally bound layer would lead to a more restrictive constraint than the original condition up to layer. That means, given the condition $c = \forall(a_0 : C_0 \hookrightarrow C_1, \mathsf{false})$ and let $C' \in \mathcal{U}(C_0, C_1)$. If the condition $c' = \forall(a^p_0 : C_0 \hookrightarrow C_1, \mathsf{false})$ is satisfied this implies that c is also satisfied but the backwards implication does not hold.

Definition 2.11 (**partial condition**)**.** *Let c be a condition in EANF.*

The partial condition of c at layer $k < \mathrm{nl}(c)$ with *with*

$$C' \in \begin{cases} \mathcal{U}(C_k, C_{k+1}) & \textit{if } k \textit{ is even} \\ \mathcal{U}(C_{k+1}, C_{k+2}) & \textit{if } k \textit{ is odd,} \end{cases}$$

$\mathrm{part}(k, c, C')$*, is defined as:*

1. *If k is odd, let* $\mathrm{sco}_c(k+1) = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$*:*

$$\mathrm{part}(k, c, C') := \mathrm{sub}\big(k+1, c, \exists(a^p : C_{k+1} \hookrightarrow C', \textsf{\textit{true}})\big)$$

2. *If k is even, let* $\mathrm{sco}_c(k+1) = \exists(a : C_k \hookrightarrow C_{k+1}, f)$*:*

$$\mathrm{part}(k, c, C') := \mathrm{sub}\big(k, c, \exists(a^p : C_{k+1} \hookrightarrow C', \textsf{\textit{true}})\big)$$

Example 2.4.

If a graph G does not satisfy the condition up to layer $k := c^G_{\max} + 2$ of a given constraint c, with k being even, there does exists a graph $C' \in \mathcal{U}(C_{k-1}, C_k)$, such that G satisfies $\mathrm{part}(k, c, C')$, note that G always satisfies $\mathrm{part}(k, c, C_{k-1})$. For the biggest of these graphs, we define the notions of *biggest partially satisfying conditions and graphs.*

Definition 2.12 (**biggest partially satisfying condition**)**.** *Let G be a graph, c a condition in EANF over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$ and $p \not\models c$.*

A partial condition $c = $ $\mathrm{part}(k+2, c, C')$ *with $p \models c$ is a* biggest partially satisfying condition w.r.t c of p *if there does not exist a graph C'', such that C' is a subgraph of C'', with $p \models$* $\mathrm{part}(c_{max}, c, C'')$*. The graph C' is then called a* biggest partially satisfying graph w.r.t c of p.

Given a constraint c, the set of biggest partially satisfying conditions w.r.t c of p : $\emptyset \inf G$ is denoted by $\mathcal{P}^G_c$.

The set of all biggest partially satisfying graphs w.r.t c of p is denoted by $\mathcal{G}^G_c$.

2.3 consistency increasing transformations and rules

With the results above, we are now ready to define the notions of *consistency increasing* and *direct consistency increasing*, with direct increasing being a more restrictive version of increasing, yielding the advantage that second-order logic formulas can be used in order to determine whether a transformation is (direct) consistency increasing or not.

These notions are designed to only detect transformations that increase the consistency of the first two unsatisfied layer of a constraint c. That means, given a graph G and a constraint c, let $k = c^G_{\max}$ and $\forall(a_{k+1} : C_k \hookrightarrow C_{k+1}, \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, d)) =: \operatorname{sco}_c(k+1)$. A transformation $t : G \Longrightarrow H$ is considered as (direct) consistency increasing if $c^G_{\max} \leq c^H_{\max}$, i.e. the satisfaction up to layer is not decreased, and more increasing insertions or deletions have been performed than decreasing ones. An increasing deletion is the deletion of an occurrence of C_{k+1} that does not satisfy $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \mathsf{true})$, an increasing insertion is the insertion of elements of C_{k+2}, such that for at least one occurrence p of C_{k+1} it holds that $p \not\models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ and $\operatorname{tr}_t \circ p \models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ for an $C' \in \mathcal{U}(C_{k+1}, C_{k+2})$. A decreasing insertion is the creation of an occurrence of C_{k+1} not satisfying $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \mathsf{true})$ and a decreasing deletion is the deletion of elements of C_{k+2} such that for an occurrence p of C_{k+1} with $p \models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ it holds that $\operatorname{tr}_t \circ p \not\models \exists(a^p_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$.

To evaluate this, we define the *number of violations*. Intuitively, for all occurrences p of C_{k+1} the number of graphs $C' \in \mathcal{U}(C_{k+1}, C_{k+2})$ with $p \not\models \exists C'$ is add up and it can be determined whether more increasing or decreasing actions have been performed by a transformation.

Note, that the number of violations is defined for each layer of the constraint, but only for the first unsatisfied layer the sum is calculated as described above. For all layer k with $k \leq c^G_{\max}$ it is set to 0 and for all layer k with $k > c^G_{\max} + 1$ it is set to ∞. Through this, a transformation $tG \Longrightarrow H$ that increases the partial consistency can easily detected since the number of violations in H at layer $c^G_{\max} + 1$ will be set to 0.

Definition 2.13 (**number of violations**). *Let G be a graph and c a constraint in EANF. The* number of violations $\operatorname{nvc}(j, G)$ at layer j in G *is defined as:*

1. *If $j < c^G_{\max} + 1$:*
$$\operatorname{nvc}(j, G) := 0$$

2. *If $j = c_{\max} + 1$, let $d = \forall(a_k : C_j \hookrightarrow C_{j+1}, e)$ be the subcondition of c at layer $j + 1$.*
$$\operatorname{nvc}(j, G) := \begin{cases} \sum_{C' \in \mathcal{U}(C_j, C_{j+1})} |\{q \mid q : C_{j+1} \hookrightarrow G \wedge q \not\models \operatorname{part}(1, e, C')\}| & \textit{if } e \neq \mathsf{false} \\ |\{q \mid q : C_{j+1} \hookrightarrow G\}| & \textit{if } e = \mathsf{false} \end{cases}$$

3. *If $j > c^G_{\max} + 1$:*
$$\operatorname{nvc}(j, G) := \infty$$

Via the number of violations, we now define *consistency increasing* transformations and rules, by checking whether the number of violations has decreased for any layer of the constraint.

Definition 2.14 (consistency increasing). *Let a graph G, a rule ρ and a constraint c in EANF be given.*

A transformation $G \Longrightarrow_{\rho,m} H$ is called consistency increasing w.r.t c*, if*

$$\mathrm{nvc}(k, H) < \mathrm{nvc}(k, G)$$

for any $0 \leq k \leq \mathrm{nl}(c)$. A rule r is called consistency increasing w.r.t c*, if all of its applications are.*

Note that if $G \models c$ there does not exist a consistency increasing transformation w.r.t c, since $\mathrm{nvc}(j, G) = 0$ for all $1 \leq j \leq \mathrm{nl}(c)$. Also, no plain rule ρ is consistency increasing w.r.t c, since a graph G satisfying c, such that a transformation $t : G \Longrightarrow_{\rho,m} H$ exists can always be constructed. Therefore, each consistency increasing rule has to be equipped with at least one application condition.

As mentioned above, a transformation should be detected as consistency increasing if it increases the partial consistency, which is shown by the following theorem.

Theorem 2.1. *Let a graph G, a rule ρ and a constraint c in EANF be given. A transformation $t : G \Longrightarrow_{\rho,m} H$ is consistency increasing if $c^G_{\mathrm{max}} < c^H_{\mathrm{max}}$.*

Proof. No $\ell > c^G_{\mathrm{max}}$ with $G \models_\ell c$ exists. Hence, $\mathrm{nvc}(c^G_{\mathrm{max}} + 1, G) > 0$ and $\mathrm{nvc}(c^G_{\mathrm{max}} + 1, G) \neq \infty$. Since $c^H_{\mathrm{max}} > c^G_{\mathrm{max}}$, $\mathrm{nvc}(k + 1, H) = 0$ and it follows immediately that t is consistency increasing. □

Since no consistency increasing transformation originating in consistent graphs exist, there do not exist infinite long sequences of consistency increasing transformations. Additionally, if a set of rules $\mathcal{R}$ is given, and a sequence of consistency increasing transformations with rules of $\mathcal{R}$ ends with a graph G, then either G satisfies the constraint or there do not exist any consistency increasing transformations $G \Longrightarrow_{\rho,m} H$ with $\rho \in \mathcal{R}$.

Theorem 2.2. *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. Every sequence of minimal consistency improving transformation w.r.t c with rules in $\mathcal{R}$ is finite.*

Proof. Let G_0 be a graph and

$$G_0 \Longrightarrow_{\rho_1} G_1 \Longrightarrow_{\rho_2} G_2 \Longrightarrow_{\rho_3} \dots$$

be a sequence of minimal consistency improving transformations w.r.t c with $\rho_i \in \mathcal{R}$. We assume that $c^{G_0}_{\mathrm{max}} < \mathrm{nl}(c)$, otherwise $\mathrm{nvc}(j, G_0) = 0$ for all $j \in \{0, \dots, \mathrm{nl}(c)\}$ and no transformation $G_0 \Longrightarrow H$ is consistency increasing.

We show that after at most $j := \mathrm{nvc}(\mathrm{c}_{\mathrm{max}G_0} + 1, G_0)$ transformations $G_j \models_{c^{G_0}_{\mathrm{max}}+2} c$ holds. Note that j has to be finite, since G_0 contains only a finite number of occurrences of C_{j+1}. After each transformation it holds that $\mathrm{nvc}(c^{G_{i+1}}_{\mathrm{max}} + 1, G_{i+1}) \leq \mathrm{nvc}(c^{G_i}_{\mathrm{max}} + 1, G_i) -$

1. Therefore, after j transformations, $\mathrm{nvc}(c^{G_0}_{\max}+1, G_j) \leq \mathrm{nvc}(c^{G_0}_{\max}+1, G_0) - j = 0$ holds and with that $G_j \models_{c^{G_0}_{\max}+2} c$. By iteratively applying this, it follows that after a finite number of transformations a graph G_k with $G_k \models c$ has to exist. Since no consistency increasing transformation $G_k \Longrightarrow G_{k+1}$ exists, the sequence has to be finite. Also, for some $G_{k'}$ there may not exist a consistency increasing transformation $G_{k'} \Longrightarrow_\rho H$ with $\rho \in \mathcal{R}$ and therefore the sequence is also finite, but does not end with a consistent graph. □

The replacement of a graph not satisfying a constraint c by a c satisfying graph via a transformation is a consistency increasing transformation. Therefore, a consistency increasing transformation can perform insertions or deletions that are unnecessary in order to increase the consistency. That is, the deletion of occurrences of existentially bound graphs, the deletions of occurrences of graphs C_k satisfying $\exists C_{k+1}$ or the insertion of occurrences of universally bound graphs and the insertion of existentially bound graphs C_k, such that the corresponding occurrence of C_{k-1} already satisfied $\exists C_k$.

Direct consistency increasing transformations are more restricted, in the sense, that these unnecessary deletions and insertions are leading a transformation to not being direct increasing. The presence of these unnecessary actions can be checked via second-order logic formulas. Additionally, it is secured that no new violations are introduced since these can always be considered as a unnecessary insertion or deletion. With that, the removal of one violation is sufficient to state that the transformation is (direct) consistency increasing, which can be checked via an additional second order logic formula. Intuitively, it is checked by the first two formulas, that no new violations are introduced. The third formula secures that at least one violation has been removed and the last formulas secure that the partial consistency is not decreased.

Definition 2.15 (direct consistency increasing). *Let G be a graph, ρ a plain rule, c a constraint in EANF and $\mathrm{sco}_c(c^G_{\max}+1) = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$. A transformation $t : G \Longrightarrow_{\rho,m} H$ is called* direct consistency increasing *if the following equations hold. Let*

$$\mathbf{G} = \begin{cases} \mathcal{U}(C_k, C_{k+1}) & \text{if } e \neq \mathsf{false} \\ \{C_k\} & \text{otherwise} \end{cases}$$

Every occurrence of C_k in G that satisfies $\mathrm{part}(1, e, C')$ for any $C' \in \mathbf{G}$ still satisfies $\mathrm{part}(1, e, C')$ in H.

$$\forall p : C_k \hookrightarrow G\Big(\bigwedge_{C' \in \mathbf{G}} \big(p \models \mathrm{part}(1, e, C') \wedge \mathrm{tr}_t \circ p \text{ is total}\big) \Longrightarrow \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C')\big)\Big) \tag{2.1}$$

Let $C' = C_{k+1}$ if $e \neq \mathsf{false}$ and $C' = C_k$ otherwise. Every new inserted occurrence of C_k by t satisfies $\mathrm{part}(1, e, C')$.

$$\forall p' : C_k \hookrightarrow H\big(\neg\exists p : C_k \hookrightarrow G(p' = \mathrm{tr}_t \circ p) \implies p' \models \mathrm{part}(1, e, C')\big) \tag{2.2}$$

At least one occurrence of C_k in G that does not satisfy $\mathrm{part}(1, e, C')$*, for any* $C' \in \mathbf{G}$*, either has been removed or satisfies* $\mathrm{part}(1, e, C')$ *in* H*.*

$$\exists p : C_k \hookrightarrow G\Big(\bigvee_{C' \in \mathbf{G}} (p \not\models \mathrm{part}(1, e, C') \wedge (\mathrm{tr}_t \circ p \textit{ is not total} \\ \vee (\mathrm{tr}_t \circ p \textit{ is total } \wedge \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C'))))\Big) \tag{2.3}$$

No occurrence of a universally bound graph C_j with $j < k$ gets inserted.

$$\bigwedge_{\substack{i<k \\ i \textit{ odd}}} \forall p : C_i \hookrightarrow H(\exists p' : C_i \hookrightarrow G(p' = \mathrm{tr}_t \circ p)) \tag{2.4}$$

No occurrence of an existentially bound graph C_j with $j < k$ gets deleted.

$$\bigwedge_{\substack{i<k \\ i \textit{ even}}} \forall p : C_i \hookrightarrow G(\mathrm{tr}_t \circ p \textit{ is total}) \tag{2.5}$$

Note that (2.4) and (2.5) not only secure that the partial consistency does not decrease, as shown in the following lemma, but also prevent further unnecessary insertions and deletions, since the insertion of a universally and the deletion of a existentially bound graph will never lead to an increase of consistency.

Lemma 2.16. *Let a transformation $t : G \Longrightarrow H$ and a constraint c in EANF be given, such that (2.4) and (2.5) of definition 2.15 are satisfied. Then,*

$$H \models_{c^G_{\max}} c.$$

Proof. Assume that $G \not\models_{c^G_{\max}} c$. Then, either a new occurrence of an universally bound graph C_k, with $k \leq c^G_{\max}$, of c has been inserted or an occurrence of an existentially bound graph C_k, with $k \leq c^G_{\max}$, of c has been destroyed. Therefore, the following holds:

$$\exists p : C_i \hookrightarrow G(\neg \exists p' : C_i \hookrightarrow (p' = \mathrm{tr}_t \circ p)) \vee \exists p : C_j \hookrightarrow G(\mathrm{tr}_t \circ p \textit{ is not total})$$

with $i, j \leq c^G_{\max}$, i being odd and j being even. It follows immediately that either (2.4) or (2.5) is not satisfied. This is a contradiction. □

Now, we will show the already indicated relationship between direct increasing and increasing, namely that a direct increasing transformation is also increasing. Counterexamples showing that the inversion of the implication does not hold can easily be constructed, showing that these notions are not identical, but related.

Lemma 2.17. *Let a graph G, a constraint c in EANF and a direct increasing transformation $t : G \Longrightarrow_{\rho,m} H$ w.r.t. c be given. Then, t is also a increasing transformation.*

Proof. Let G be a graph, $k = c^G_{\max} + 1 \leq \mathrm{nl}(c)$ and $d = \mathrm{sco}_c(k)$. With lemma 2.16 follows that $c^H_{\max} \geq c^G_{\max}$ and with that $\mathrm{nvc}(k, H) \neq \infty$.

1. We show that equations (2.1) and (2.2) imply that $\mathrm{nvc}(k, H) \leq \mathrm{nvc}(k, G)$. Assume that $\mathrm{nvc}(k, H) > \mathrm{nvc}(k, G)$. Therefore, a morphism $p : C_k \hookrightarrow H$ with $p \not\models \mathrm{part}(1, d, C')$ for any $C' \in \mathcal{U}(C, C_{k+1})$ exists, such that either 1a or 1b is satisfied.

   (a) There does exist a morphism $q' : C_k \hookrightarrow G$ with $q' \models \mathrm{part}(1, d, C')$ and $p = \mathrm{tr}_t \circ q'$.

   (b) There does not exist a morphism $q : C_k \hookrightarrow G$, such that $p = \mathrm{tr}_t \circ q$.

   This is a contradiction, if 1a is satisfied, q' does not satisfy equation (2.1) and if 1b is satisfied q does not satisfy equation (2.2). It follows that

   $$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\}| \leq |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\}|$$

   for all $C' \in \mathcal{U}(C_k, C_{k+1})$.

2. Since (2.3) is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \mathrm{part}(1, d, C')$, such that either $\mathrm{tr} \circ p$ is total and $\mathrm{tr}_t \circ p \models \mathrm{part}(1, d, C')$ or $\mathrm{tr} \circ p$ is not total exists, for any $C' \in \mathcal{U}(C_k, C_{k+1})$. In both cases the following holds:

   $$\begin{aligned} p \in \{q \mid q : C_k \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\} \wedge \\ \mathrm{tr} \circ p \notin \{q \mid q : C_k \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\} \end{aligned}$$

   With that and 1. it follows that

   $$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \mathrm{part}(1, e, C')\}| < |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\}|.$$

In total, $\mathrm{nvc}(k, G) < \mathrm{nvc}(k, G)$ and t is an increasing transformation.

For the special case that $k = \mathrm{nl}(c) - 1$ and c ends with $\forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false})$ it can be shown in a similar way that (2.1) and (2.2) imply that

$$|\{q \mid q : C_k \hookrightarrow G\}| \leq |\{q \mid q : C_k \hookrightarrow H\}|$$

and that (2.3) implies

$$|\{q \mid q : C_k \hookrightarrow G\}| < |\{q \mid q : C_k \hookrightarrow H\}|.$$

It follows that t is an increasing transformation. □

2.4 Comparison with other consistency concepts

In this chapter, the notions of (direct) consistency increasing are compared to the already known concepts of consistency guaranteeing, consistency preserving [1], (direct) consistency increasing and sustaining [2] in order to reveal relationships between them and secure that (direct) increasing is indeed a new notion of consistency. These relations are displayed in figure 1.

$$\begin{tikzcd}
 & c\text{-guaranteeing} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{direct increasing w.r.t } c \arrow[dl, Rightarrow] \\
c\text{-preserving} & & \text{increasing w.r.t } c \arrow[ll, Rightarrow] &
\end{tikzcd}$$

Figure 1: General relations of consistency notions.

First we compare (direct) consistency increasing with the notions of consistency-guaranteeing and -preserving transformations. As already mentioned, given a constraint c and a graph G with $G \models c$, there does not exist a (direct) increasing transformation $t : G \Longrightarrow H$ and it follows immediately that (direct) increasing w.r.t c implies c-preserving.

Obviously, guaranteeing implies increasing, this property is embedded in the definition of increasing. The inversion of this implication does not holds, since increasing is a way stricter notion, in the sense, that the number of removed violations has to be greater than the number of introduced violations. For guaranteeing transformations this is not the case, an arbitrary number of violations can be inserted, long as the derived graph satisfies the constraint and therefore guaranteeing does not imply direct increasing, since a direct increasing transformations is not allowed to introduce any new violations.

Lemma 2.18. *Let a constraint c in ANF, graphs G and H with $G \not\models c$, and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$\begin{array}{lcl}
t \textit{ is c-guaranteeing} & \Longrightarrow & t \textit{ is increasing w.r.t c} \quad \wedge \\
t \textit{ is c-guaranteeing} & \not\Longrightarrow & t \textit{ is direct increasing w.r.t c} \wedge \\
t \textit{ is increasing w.r.t c} & \not\Longrightarrow & t \textit{ is c-guaranteeing.}
\end{array}$$

Proof. Let c' be the equivalent constraint in EANF.

1. Let t be a c-guaranteeing transformation, then $H \models c$. Since $G \not\models c$, $G \not\models c'$ follows and $\mathrm{nvc}(c^G_{\max} + 1, G) > 0$ and $\mathrm{nvc}(c^G_{\max} + 1, H) = 0$. Therefore t is consistency increasing.

2. Let a constraint $c = \exists(a_0 : \emptyset \hookrightarrow C_0, \forall(a_1 : C_0 \hookrightarrow C_1, \exists(a_2 : C_1 \hookrightarrow C_2, \mathsf{true})))$ and a graph $G = C' \dot{\cup} C'$ with $C' = C_2 \setminus \{e\}$ for one edge $e \in E_{C_2 \setminus C_1}$ and the occurrences $p_1 : C_1 \hookrightarrow G$ and $p_2 : C_1 \hookrightarrow G$ be given. Let $t : G \Longrightarrow H$ be a transformation with $H = C_2 \dot{\cup} C''$ and $C'' = C' \setminus \{e'\}$ for an $e' \in E_{C_2 \setminus C_1}$, such that $\mathrm{tr}_t \circ p_1$ and $\mathrm{tr}_t \circ p_2$ are total. Then, t is c-guaranteeing and not direct increasing, since $p_i \models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$ for $i = 1, 2$ and either $\mathrm{tr}_t \circ p_1 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$ or $\mathrm{tr}_t \circ p_2 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$.

3. Let t be a consistency increasing transformation w.r.t c', such that $\mathrm{nvc}(c^G_{\max} + 1, G) > \mathrm{nvc}(c^G_{\max} + 1, H) > 1$ and $H \not\models_{c^G_{\max}+2} c'$. Then,$H \not\models c'$ and t is not a c-guaranteeing transformation.

□

The definition of consistency improving only differs from guaranteeing if the corresponding constraint is universally bound and these notions are identical for existentially bound constraint. Therefore, with lemma 2.18, we can state the following.

Corollary 2.19. *Let c be an existentially bound constraint in ANF. Then,*

$$\begin{aligned} &t \textit{ is consistency improving w.r.t } c &&\Longrightarrow t \textit{ is consistency increasing w.r.t } c \wedge \\ &t \textit{ is consistency increasing w.r.t } c &&\not\Longrightarrow t \textit{ is consistency improving w.r.t } c \end{aligned}$$

The notions of increasing and improving are equivalent for universally bound constraints with nesting level 1. Note that, with corollary 2.19, this equivalence does not hold for existentially bound constraints with nesting level 1.

Lemma 2.20. *let a universally bound constraint c with $\mathrm{nl}(c) = 1$ in ANF, a graph G with $G \not\models c$ and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \textit{ is consistency improving w.r.t } c \iff t \textit{ is consistency increasing w.r.t } c$$

Proof. Let $c = \forall(a : \emptyset \hookrightarrow C, \mathsf{false})$. Since $\mathcal{U}(C, C) = \{C\}$, $\mathrm{nvc}(1, G)$ is the number of occurrences of C in G. This is exactly the definition of the number of violations for consistency improving transformations and the statement follows immediately. □

For universally bound constraints c with $\mathrm{nl}(c) \geq 2$, (direct) consistency increasing is not related to (direct) consistency improving and sustaining. As shown in [2], (direct) consistency improving implies (direct) consistency sustaining. Therefore it is sufficient to show that direct improving does not imply increasing and that direct increasing does not imply consistency sustainment.

Lemma 2.21. *Let a universally bound constraint c with $\mathrm{nl}(c) \geq 2$ in ANF, a graph G with $G \not\models c$ and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$\begin{aligned} &t \textit{ is direct consistency improving w.r.t } c &&\not\Longrightarrow t \textit{ is consistency increasing w.r.t } c \wedge \\ &t \textit{ is direct increasing w.r.t } c &&\not\Longrightarrow t \textit{ is consistency sustaining w.r.t } c \end{aligned}$$

Proof.

1. Let $c = \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \mathsf{true}))$ be a constraint. Let $V_{C_0} = V_{C_1}$ and $|E_{C_1}| - |E_{C_0}| = 2$. Let $G = C' \dot\cup C'$ with $C' = C_1 \setminus \{e\}$ with $e \in E_{C_1} \setminus E_{C_0}$ and the occurrences $p_1 : C_0 \hookrightarrow G$ and $p_2 : C_0 \hookrightarrow G$ be given. It follows that $\mathrm{nvc}(1, G) = 2$. Let $t : G \Longrightarrow H$ be a transformation, such that $H = C_0$. Then, t is a direct consistency improving transformation, since H contains only one occurrence of C_0 not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1 \, \mathsf{true})$. But, t is not consistency increasing, since $\mathrm{nvc}(1, H) = 3$.

2. Let $c := \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \mathsf{true}))))$ be a constraint.

Let a graph $G = C_0$ with the morphism $q : C_0 \hookrightarrow G$ and a transformation $t : G \Longrightarrow H$ with $H := C_2 \dot{\cup} C_2$ be given, such that $\mathrm{tr}_t \circ q$ is total. Then t is a direct consistency increasing transformation but not a consistency sustaining one, since H contains more occurrences of C_0 not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \mathsf{true})))$ than G.

□

3 general application conditions

To guarantee that each transformation $t : G \Longrightarrow_{\rho,m} H$, given a rule $\rho = L \hookleftarrow K \hookrightarrow R$, is (direct) consistency increasing w.r.t to a constraint c, we present applications conditions ensuring this property. It has to be ensured that at least one violation will be removed. For this, it is necessary (a) to check that an occurrence p of the the universally bound graph $C_{c^G_{\max}+1}$ exists, such that p and the match m do overlap, i.e $p(C_{c^G_{\max}+1}) \cap m(L) \neq \emptyset$, and (b) that p does not satisfy $c' = \exists C_{c^G_{\max}+2}$. Only in this case, it is possible that the transformation does remove a violation. To ensure that p does not satisfy c', the non-existence of all possible overlaps of L and $C_{c^G_{\max}+2}$ such that $p \models c'$ has to be checked. For this, we introduce *extended overlaps*. Intuitively, given an overlap C of L and $C_{c^G_{\max}+1}$ with $p : C_{c^G_{\max}+1} \hookrightarrow C$, the set of extended overlaps contains all overlaps C' of L and $C_{c^G_{\max}+2}$, such that $p \models c'$ if one of these overlaps exists.

Definition 3.1 (extended overlaps). *Let G, C_0 and C_1 with $i : C_0 \hookrightarrow C_1$ be graphs. Let C be an overlap of C_0 and G with the inclusion $q : C_0 \hookrightarrow C$. The set of* extended overlaps of C with i, $\mathrm{eol}(C, i)$, *is the set of all overlaps C' of G and C_1, such that an injective morphism $i_C : C \hookrightarrow C'$ with $i_C \circ q \models \exists(i : C_0 \hookrightarrow C_1, \mathit{true})$ exists.*

A direct consistency increasing application condition has to, as mentioned above, (a) ensure that at least one violation will be removed, (b) ensure that no new violations get inserted and (c) secure that the partial consistency is not decreased. In the definition below, nex() and rep() ensure that (a) is met, with nex() ensuring that a violation is present and rep() ensuring that this violation will be removed. Note, that the construction of these is divided in two cases. Firstly, either $k < \mathrm{nl}(c) - 1$ or the constraints ends with a condition of the form $\exists(a : C \hookrightarrow C', \mathsf{true})$ and secondly, $k = \mathrm{nl}(c) - 1$ and the constraint ends with a condition of the form $\forall(a : C_0 \hookrightarrow C_1, \mathsf{false})$.

For nex(), the first case will be checked as already described above. In the second case, it is sufficient to check whether an occurrence p of C_k with $m(L) \cap p(C_k) \neq \emptyset$ exists.

For rep(), in the first case, a violation can be removed by either deleting an occurrence p of C_k or inserting elements of C_{k+1}, such that $p \not\models \exists C'$ and $\mathrm{tr}_t \circ p \models \exists C'$ for a graph $C' \in \mathcal{U}(C_k, C_{k+1})$. In the second case, a violation can only be removed by deleting an occurrence p of C_1. This will only occur if $m(L \setminus K) \cap p(C_1) \neq \emptyset$.

The conditions constructed by nin() and nwo() ensure that (2) is me, with nin() checking that no new occurrence of C_k will be inserted and nwo() ensuring that for an occurrence p of C_k if $p \models \exists C'$ it is secured that $\mathrm{tr}_t \circ p \models \exists C'$ for any $C' \in \mathcal{U}(C_k, C_{k+1})$.

Additionally, the conditions constructed by rem() and nin() ensure (3), with rem() ensuring that no occurrence of existentially bound graphs C_j with $j \leq k$ will be deleted and nin() ensuring that no occurrences of universally bound graphs C_j with $j \leq k$ will be inserted. With this, it is guaranteed that the partial consistency will not be decreased.

Definition 3.2 (general application condition). *Let $\rho = L \hookleftarrow K \hookrightarrow R$ be a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be the subcondition of c at layer k with k being an odd number.*

The application condition ap_k of c at layer k with $C' = C_k$, if $e = \mathsf{false}$, and $C' \in \mathcal{U}(C_k, C_{k+1})$ otherwise is defined as:

$$\mathrm{ap}(k, C') := \big(\bigvee_{P \in \mathrm{ol}(L, C_k)} \mathrm{nex}(P, C') \wedge \mathrm{rep}(P, C')\big) \wedge \mathrm{nin} \wedge \mathrm{rem} \wedge \mathrm{nwo} \tag{3.1}$$

with

1. *Let $a^p : C' \hookrightarrow C_{k+1}$ be a partial morphism of a_{k+1}.*

$$\mathrm{nex}(P, C') := \begin{cases} \exists(i_L : L \hookrightarrow P, \mathsf{true}) & \text{if } e = \mathsf{false} \\ \bigwedge_{Q \in \mathrm{eol}(P, a^p)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow Q, \mathsf{true})) & \text{otherwise} \end{cases}$$

2. *Let $\mathbf{P}$ be the set of all overlaps of R and C', such that $i_R(R \setminus K) \cap i_{C'}(C') \neq \emptyset$:*

   (a) If $e = \mathsf{false}$:

$$\mathrm{rep}(P, C') := \begin{cases} \mathsf{true} & , \text{ if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \mathsf{false} & , \text{ otherwise} \end{cases}$$

   (b) Otherwise:

$$\mathrm{rep}(P, C') := \begin{cases} \mathsf{true} & , \text{ if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \bigvee_{P' \in \mathbf{P}} \mathrm{Left}(\exists(i_R : R \hookrightarrow P', \mathsf{true}), \rho) & , \text{ otherwise} \end{cases}$$

3. *Let E be the set of all graphs C_j of c with $j \leq k$ and j being odd and let $\mathbf{P}_{C_j}$ be the set all overlaps of L and C_j with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.*

$$\mathrm{rem} := \bigwedge_{C \in E} \bigwedge_{C' \in \mathbf{P}_{C_j}} \neg\exists(i_L : L \hookrightarrow C', \mathsf{true})$$

4. *Let U be the set of all graphs C_j of c with $j \leq k$ and j being even and let $\mathbf{P}_{\mathbf{C_j}}$ be the set of all overlaps of R and C_j with $i_R(R \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.*

$$\mathrm{nin} := \bigwedge_{C \in U} \bigwedge_{C' \in \mathbf{P}_{C_j}} \mathrm{Left}(\neg\exists(i_R : R \hookrightarrow C', \mathsf{true}), \rho)$$

5. *Let E be the set of all overlaps of L and C_k, such that each $P \in E$ is also an overlap of L and $C'' \in \mathcal{U}(C_k, C_{k+1})$, $i_{C_k} \models \exists(a'_k : C_k \hookrightarrow C'', \mathit{true})$ and $i_L(L \setminus K) \cap i_{C''}(C'' \setminus C_k) \neq \emptyset$.*
$$\mathrm{nwo} := \bigwedge_{P \in E} \neg\exists(i_L : L \hookrightarrow P, \mathit{true})$$

Lemma 3.3. *Let a graph G, a constraint c in EANF, with $G \not\models c$, and a plain rule ρ be given. Then, the rule $\rho'(\rho, \mathrm{ap}(c^G_{\max}, C))$ for a graph $C \in \mathcal{U}(C_{c^G_{\max}}, C_{c^G_{\max}+1})$ is a minimal consistency improving rule.*

Proof. Let $t : G \Longrightarrow_{\rho',m} H$ be a transformation and $k = c^G_{\max} + 1$. We show, by contradiction, that this transformation is direct minimal consistency improving by showing that (2.1), (2.2), (2.3), (2.4) and (2.5) are satisfied. With that, it follows that ρ' is a minimal consistency improving rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ be the subcondition of c at layer k.

1. Assume that (2.1) does not hold. Then, there does exist an morphism $p : C_k \hookrightarrow G$, such that $p \models \mathrm{part}(1, e, C')$, $\mathrm{tr}_t \circ p$ is total and $\mathrm{tr}_t \circ p \not\models \mathrm{part}(1, e, C')$ for a graph $C' \in \mathcal{U}(C_k, C_{k+1})$. There has to exist an overlap P of L and C' such that $i_{C_k} \models \exists(a^p_k : C_k \hookrightarrow C', \mathsf{true})$ and $m \models \exists(i_L : L \hookrightarrow P, \mathsf{true})$. Then, nwo and with that $\mathrm{ap}(c^G_{\max}, C)$ is not satisfied.

2. Assume that (2.2) does not hold. Then, a morphism $p' : C_k \hookrightarrow H$ with $p' \not\models \mathrm{part}(1, e, C_{k+1})$ exists, such that there does not exist a morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p = p'$. Then, an overlap P of R and C_k with $i_R(R \setminus K) \cap i_R(C_k) \neq \emptyset$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$. Then, $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

3. Assume that (2.3) does not hold. Then, there does not exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \mathrm{part}(1, e, C)$, such that $\mathrm{tr}_t \circ p$ is not total or $\mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C)$ and $\mathrm{tr}_t \circ p$ is total. Then, no overlap P of L and C_k with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ exists, such that $m \models \mathrm{nex}(P, C)$. Also, no overlap P of C and R with $i_R(R \setminus K) \cap i_C(C) \neq \emptyset$ and $m \models \mathrm{nex}(P, C)$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$ and therefore $\mathrm{rep}(P, C) = \mathsf{false}$. It follows that for all $P \in \mathrm{ol}(L, C_k)$ it holds that $\mathrm{nex}(P, C) \wedge \mathrm{rep}(P, C) = \mathsf{false}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

4. Assume that (2.4) does not hold. Then, there does exist a morphism $p : C_j \hookrightarrow G$ with $j < k$ and i being even, such that no morphism $p' : C_j \hookrightarrow G$ with $\mathrm{tr}_t \circ p' = p$ exists. Then, an overlap P of C_j and R with $i_R(R \setminus K) \cap i_{C_j} \neq \emptyset$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$. It follows that $m \not\models \mathrm{rem}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

5. Assume that (2.5) does not hold. Then, there does exist and morphism $p : C_j \hookrightarrow G$ with $j < k$ and j being odd, such that $\mathrm{tr}_t \circ p$ is not total. Then, an overlap P of C_j and L with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$ exists, such that $m \models \exists(i_L : L \hookrightarrow P, \mathsf{true})$. It follows that $m \not\models \mathrm{rem}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

In total follows that if $m \models \text{ap}(c^G_{\max}, C)$, then t is a direct minimal improving transformation. □

Lemma 3.4. *Let G be a graph, c a constraint in EANF, with $c^G_{\max} < \text{nl}(c)$, and $\rho = L \hookleftarrow K \hookrightarrow R$ a plain rule. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer $\text{c}_{\max} + 1$ and $\text{ap}(\text{c}_{\max}, C')$ the application condition constructed by definition 3.2 with $C' \in \mathcal{U}(C, C_{k+1})$ for any $C \in \mathcal{G}^G_c$. If*

$$((R \setminus K) \cap C_{k+1}) \cup ((L \setminus K) \cap C_{k+1}) = \emptyset$$

$\text{ap}(\text{c}_{\max}, C')$ can be replaced by false.

Proof. There does not exist an overlap P of C_k and L with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ and $\text{rep}(P, C')$ will be equal to false, if $C_k = C_{k+1}$, or equal to $\bigvee_{P' \in \mathbf{P}} \text{Left}(\exists(i_R : R \hookrightarrow P', \mathsf{true}), \rho)$. Since the set $\mathbf{P}$ has to be empty, this expression can be replaced by false. If follows that $\text{rep}(P, C') = \mathsf{false}$ for all $P \in \text{ol}(L, C_k)$ and therefore $\text{ap}(\text{c}_{\max}, C')$ will always be evaluated to false. □

3.1 potentially minimal improving rules

Definition 3.5 (**basic improving rule**)**.** *Let a constraint c and a plain rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ be given. The rule ρ is called* basic improving *w.r.t c at layer k with $C_k \subset P \subseteq C_{k+1}$, $p : C_k \hookrightarrow P$ being the inclusion, and $k \in \{1, 3, \dots, \text{nl}(c)\}$, if*

$$(L \setminus K) \cap (C_{k-1} \cup (C_{k+1} \setminus C_k)) = \emptyset \tag{3.2}$$

and

$$(R \setminus K) \cap C_k = \emptyset \tag{3.3}$$

and either 1. or 2. applies.

1. *The rule ρ deletes elements of $C_k \setminus C_{k-1}$:*

$$L \subseteq C_k \wedge L \setminus K \neq \emptyset \tag{3.4}$$

Then, ρ is called a deleting basic improving rule.

2. *The rule ρ creates an instance of P:*

$$L = C_k \wedge P \subseteq R \tag{3.5}$$

and p is a partial morphism of r. Then, ρ is called an inserting basic improving rule.

Definition 3.6 (**application conditions for basic improving rules**)**.** *Let a constraint c in EANF and a basic improving rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ w.r.t c at layer k with $C_k \subseteq P \subseteq C_{k+1}$ be given. We define the application condition for r as:*

1. *If ρ is a deleting potentially minimal improving rule:*

$$\mathrm{ap}_{\mathrm{pi}}(j, P) := \begin{cases} \bigvee_{P \in \mathrm{ol}(L, C_k)} \bigwedge_{P' \in \mathrm{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \textsf{\textit{true}})) & , \textit{if } j = k \\ \textsf{\textit{false}} & , \textit{if } j \neq k. \end{cases}$$

2. *If ρ is an inserting potentially minimal improving rule:*

$$\mathrm{ap}_{\mathrm{pi}}(j, P) := \begin{cases} \neg\exists(a_k^p : L \hookrightarrow P, \textsf{\textit{true}}) & , \textit{if } j = k \\ \textsf{\textit{false}} & , \textit{if } j \neq k \end{cases}$$

Theorem 3.1. *Let a graph G, a constraint c in EANF, with $G \models \mathrm{part}(c^G_{\max}, c, C)$ and $\mathrm{part}(c^G_{\max}, c, C) \in \mathcal{P}^G_c$, and a potentially minimal improving rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ at layer $c^G_{\max}$ with $C_{\mathrm{c_{max}}+1} \subseteq P \subseteq C_{\mathrm{c_{max}}+2}$ be given. Then, $\rho' = (\rho, \mathrm{ap}_{\mathrm{pi}}(c^G_{\max} + 1, P))$ is a direct minimal consistency improving rule.*

Proof. Let $t : G \Longrightarrow_{\rho', m} H$ be a transformation, $k = c^G_{\max} + 1$ and e be the subcondition of c at layer k. We show that t is a direct minimal consistency improving transformation. First, we show that equation (2.1) is satisfied. Let $p : C_k \hookrightarrow G$ be a morphism. If ρ is a deleting potentially minimal improving rule, either 1. or 2. applies, if ρ is an inserting and not a deleting potentially minimal improving rule, only 2. applies, because ρ cannot destroy any occurrences of C_k in G.

1. If $p(C_k) \cap m(L \setminus K) \neq \emptyset$, $\mathrm{tr}_t \circ p$ is not total, since at least one element of $p(C_k)$ has been deleted by t and p does satisfy $\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \models \mathrm{part}(1, e, C') \wedge \mathrm{tr}_t \circ p \text{ is total}) \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C'))$.

2. If $p(C_k) \cap m(L \setminus K) = \emptyset$, $\mathrm{tr}_t \circ p$ is total, since no element of $p(C_k)$ has been deleted by t. Because (3.2) holds, t does not delete any elements of $C_{k+1} \setminus C_k$ and $p \models \mathrm{part}(1, e, C') \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C')$ for all $C' \in (U)(C_k, C_{k+1})$.

With 1. and 2. follows that (2.1) is satisfied.

Second, we show that equation (2.2) is satisfied. Let $p' : C_k \hookrightarrow H$ be a morphism. Because (3.3) is satisfied, t does not create any elements of C_k and there must exist a morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p = p'$. If follows that (2.2) is satisfied.

Third, we show that (2.3) is satisfied. We consider the cases that firstly, ρ is a deleting minimal potentially improving rule and secondly, that ρ is an inserting and not a deleting minimal potentially improving rule.

1. If ρ is a deleting potentially minimal improving rule, the condition $\mathrm{ap}_{\mathrm{pi}}(k, P) = \exists\big(b : L \hookrightarrow C_k, \neg\exists(a_{k+2} : C_k \hookrightarrow C_{k+1}, \mathsf{true})\big)$ is satisfied by m. Therefore a morphism $p : C_k \hookrightarrow G$ with $p \not\models \neg\exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, \mathsf{true}) = \mathrm{part}(1, e, C_{k+1})$ and $m = p \circ b$ must exist. Since ρ is a deleting rule, at least one element of $p(C_k)$ has been deleted by t and $\mathrm{tr}_t \circ p$ is not total. It follows that (2.3) is satisfied.

2. If ρ is an inserting and not a deleting potentially minimal improving rule, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \mathsf{true})$ is satisfied by m. Because $L = C_k$, $m \models \neg\exists(b : C_k \hookrightarrow P, \mathsf{true}) = \mathrm{part}(1, e, P)$. Since (3.5) is satisfied, $\mathrm{tr} \circ p$ is total and $\mathrm{tr}_t \circ p \models \mathrm{part}(1, e, P)$. Therefore, (2.3) is satisfied.

Last, since (3.3) and (3.2) are satisfied, ρ cannot create any occurrence of C_j with $j \leq k$ and j being even and ρ cannot delete any occurrences of C_j with $j \leq k$ and j being odd. Therefore, (2.4) and (2.5) are satisfied.

In total follows that t is a direct minimal consistency improving transformation and ρ' is a direct minimal consistency improving rule. □

Theorem 3.2. *Let a constraint c in EANF, a basic increasing rule ρ with P and a consistency increasing transformation $t : G \Longrightarrow_{\rho,m} H$ w.r.t c be given. Then, $m \models \mathrm{ap}_{\mathrm{pi}}(c^G_{\mathrm{max}}, P)$.*

Proof. Let $k = c^G_{\mathrm{max}}$.

1. If c is a inserting rule, then $\mathrm{ap}_{\mathrm{pi}}(c^G_{\mathrm{max}}, P) = \neg\exists(a^p_k : L \hookrightarrow P, \mathsf{true})$. Let $n : R \hookrightarrow H$ be the comatch of t. Since t is a consistency increasing transformation, there does exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$, $\mathrm{tr}_t \circ p \models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$ such that $p'(P) \cap n(R \setminus K)$ for the morphism $p' : P \hookrightarrow H$ with $p = p \circ a^p_k$. Assume that $m \models \exists(a^p_k : L \hookrightarrow P, \mathsf{true})$. Then, every morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p \models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$ with $p'(P) \cap n(R \setminus K)$ for the morphism $p' : P \hookrightarrow H$ with $p = p' \circ a^p_k$ already satisfies $\exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$. This is a contradiction.

2. If ρ is a deleting rule, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \bigvee_{P \in \mathrm{ol}(L, C_k)} \bigwedge_{P' \in \mathrm{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \mathsf{true}))$. Since t is a consistency increasing transformation, there does exist a morphism $p : C_k \hookrightarrow G$, such that $p \not\models \exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ and $\mathrm{tr}_t \circ p$ is not total. Assume that $m \not\models \mathrm{ap}_{\mathrm{pi}}(k, P)$. Then, for each overlap Q of L and C_k, the inclusion i_{C_k} does satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$. This is a contradiction, because $p(C_k) \cap m(L \setminus K) \neq \emptyset$ has to hold.

□

Lemma 3.7. *Let a constraint c in EANF and inserting basic increasing rules ρ_1, ρ_2 at layer k with P_1 and P_2, respectively, be given. Then, the rules $\rho'_1 = (\rho_1, \mathrm{ap}_{\mathrm{pi}}(k, P_1))$ and $\rho'_2 = (\rho_2, \mathrm{ap}_{\mathrm{pi}}(k, P_2))$ are sequentially independent.*

Proof. Let $G_1 \Longrightarrow_{\rho'_1, m_1} G_2 \Longrightarrow_{\rho'_2, m_2} G_3$ be a sequence of transformations. First, note that this sequnce can only exist if $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$ Since both rules are do not insert elements of C_k, a morphism $d_1 : L_2 \hookrightarrow D_1$ with $m_2 = h_1 \circ d_1$ exists. Because both rules do not delete any elements a morphism $d_2 : R_1 \hookrightarrow D_2$ with $m^*_1 = g_2 \circ d_2$ exists. Because ρ_2 is not a basic increasing rule with P_1, $h_2 \circ d_2 \models R(\mathrm{ap}_{\mathrm{pi}})$ and because ρ_1 does not delete any elements of P_2 is holds that $g_1 \circ d_1 \models R(\mathrm{ap}_{\mathrm{pi}})$. □

Definition 3.8 (**repairing rule set**). *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is called a* repairing rule set for c at layer k *if for all graphs G with $k = \mathrm{c}_{\max}$ a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models_{k+2} c$.

Corollary 3.9. *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. If $\mathcal{R}$ is a repairing rule set for c at layer k, $\mathcal{R}$ is a repairing rule set w.r.t c at layer j for all $k < j \leq \mathrm{nl}(c)$.*

Corollary 3.10. *Let a constraint c in EANF and a repairing rule set $\mathcal{R}$ for c at layer k, for all $k \in \{1, 3, \ldots, \mathrm{nl}(c)\}$, be given. Then, for all graphs G, a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models c$.

Definition 3.11 (**decomposition of a graph**). *Let graphs $G_0 \subset G_1$ be given.*

A decomposition of G_1 with G_0 is a minimal set

$$\mathbf{P} \subseteq \{P_v \mid v \in V_{C_1 \setminus C_0}\}$$

of subgraphs of G_1, such that every element of G_1 is contained in at least one $P \in \mathbf{P}$ and every P_v is constructed in the following way: $G_0 \subset P_v$, $v \in P$ and for all nodes $v' \in P \setminus C_k$ is holds that P contains all edges $e \in E_{G_1 \setminus G_0}$ and all nodes $u \in V_{G_1}$ such that either $\mathrm{tar}(e) = v' \wedge \mathrm{src}(e) = u$ or $\mathrm{src}(e) = u \wedge \mathrm{tar}(e) = v'$ holds.

Lemma 3.12. *Let graphs $G_0 \subset G_1$ and a decomposition $\mathbf{P}$ of G_1 with G_0 be given. Then, for each pair $P, P' \in \mathbf{P}$ with $P \neq P'$ the following holds:*

$$(P \setminus C_k) \cap (P' \setminus C_k) = \emptyset$$

Proof. Assume that $(P \setminus C_k) \cap (P' \setminus C_k) \neq \emptyset$, therefore a node $v \in G_1 \setminus G_0$ with $v \in P \cap P'$ exists. By the construction of P and P' it follows that $P = P_v$ and $P' = P_v$ and therefore $P = P'$. This is a contradiction. □

Lemma 3.13. *Let graphs $G_0 \subset G_1$ and a decomposition $\mathbf{P}$ of G_1 with G_0 be given. Then,*

$$G_1 = \bigcup_{P \in \mathbf{P}} P.$$

Proof. Let $H := \bigcup_{P \in \mathbf{P}} P$. Firstly, we show that $H \subseteq G_1$. Since every $P \in \mathbf{P}$ is a subgraph of G_1 it follows that $V_H \subseteq V_{G_1}$ and $E_H \subseteq E_{G_1}$

Secondly, we show that $G_1 \subseteq H$. Let $u \in V_{G_1}$ be a node, if $u \in V_{G_0}$, then u is contained in every $P \in \mathbf{P}$ and therefore $u \in V_H$. Otherwise, if $u \notin V_{G_0}$, then u has to be, by the definition of $\mathbf{P}$, contained in at least one $P \in \mathbf{P}$ and $V_{G_1} \subseteq V_H$ follows. Let $e \in E_{G_1}$ be an edge. If $e \in E_{G_0}$, then e is contained in every $P \in \mathbf{P}$ and $e \in E_H$. Otherwise, if $e \notin E_{G_0}$, by the definition of $\mathbf{P}$, e has to be contained in at least one $P \in \mathbf{P}$. It follows that $e \in E_H$ and with that $E_{G_1} \subseteq E_H$. □

Theorem 3.3. *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is a repairing set of c at layer $k \leq \text{nl}(c)$ if either 1 or 2 applies.*

1. *For any universally bound graph C_j at layer $j \leq k$ of c, $(\rho, \text{ap}_{\text{pi}}(j, C_{j+1})) \in \mathcal{R}$ and ρ is a deleting potentially minimal improving rule at layer j with C_{j+1}, such that ρ only deletes edges of C_j.*
2. *A decomposition $\mathbf{P}$ of C_k with C_{k-1} exists, such that for each $P \in \mathbf{P}$ a rule $(\rho, \text{ap}_{\text{pi}}(k, P)) \in \mathcal{R}$ exists, such that ρ is an inserting basic improving rule at layer k with P.*

Proof. Let a constraint c in EANF, a rule set $\mathcal{R}$ and a graph G with $k = \text{c}_{\text{max}}$ and $\text{c}_{\text{max}} < \text{nl}(c)$ be given. We show that a sequence $G = C'_0 \Rightarrow \ldots \Rightarrow C'_n = H$ with rules of $\mathcal{R}$ exists, such that $H \models_{k+2} c$ if 1. or 2. of theorem 3.3 is satisfied.

1. Assume that 1. of theorem 3.3 holds. Let $(\rho, \text{ap}_{\text{pi}}(j, C_{j+1})) \in \mathcal{R}$, such that $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ is a deleting potentially minimal improving rule at layer $j \leq k$ with C_{j+1} and C_j is a universally bound graph of c. Then, $\text{ap}_{\text{pi}}(j, C_{j+1}) = \exists\big(b : L \hookrightarrow C_j, \neg\exists(a_j : C_j \hookrightarrow C_{j+1}, \textsf{true})\big)$. Let $q : C_j \hookrightarrow G$ be a morphism such that $q \not\models \exists(a_j : C_j \hookrightarrow C_{j+1}, \textsf{true})$. Since $L \subseteq C_j$, we can construct a morphism $m_1 : L \hookrightarrow G$ with $m_1 = q \circ b$ and therefore $m_1 \models \text{ap}_{\text{pi}}(j, C_{j+1})$. Since r only deletes edges, a transformation $t : G = G_0 \Rightarrow_{r,m_1} G_1$ exists and $\text{tr}_t \circ p$ is not total. Because r does not insert any elements of C_j:
$$|\{q : C_k \hookrightarrow G_0 \mid q \not\models d\}| < |\{q : C_k \hookrightarrow G_1 \mid q \not\models d\}|$$
with $d = \exists(a_j : C_j \hookrightarrow C_{j+1}, \textsf{true})$. By iteratively applying this construction, we can generate a finite sequence of transformations
$$G = G_0 \Rightarrow_{r,m_1} G_1 \Rightarrow_{r,m_2} \ldots \Rightarrow_{r,m_n} G_n = H$$
such that $|\{q : C_k \hookrightarrow G_n \mid q \not\models d\}| = 0$ and therefore $H \models_j c$. With lemma 2.9, $H \models_{k+2} c$ and $H \models c$ follows.

2. Assume that 2. of theorem 3.3 holds. Let $\rho_0 = (\rho, \text{ap}_{\text{pi}}(k, P)) \in \mathcal{R}$, such that ρ is an inserting basic improving rule of c at layer k with $P \in \mathbf{P}$. Then, $\text{ap}_{\text{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \textsf{true})$. Let $q_0 : C_k \hookrightarrow G$ be a morphism, such that $q_0 \not\models \exists(a'_k : C_k \hookrightarrow P, \textsf{true})$ with a'_k being a partial morphism of a_k. Since $L = C_k$, we set $m_0 : C_k \hookrightarrow G$ with $m_0 = q_0$. It follows that $m_0 \models \neg\exists(a'_k : C_k \hookrightarrow P, \textsf{true}) = \text{ap}_{\text{pi}}(k, P)$. Because r does not delete any elements, a transformation $t_0 : G \Longrightarrow_{r_0,m_0} G_1$ exists and $\text{tr}_t \circ q \models \exists(a'_k : C_k \hookrightarrow P, \textsf{true})$. We set $q_1 = \text{tr}_{t_0} \circ q_0$ and apply the same method to q_1.

   By iteratively applying this, we can construct a finite sequence of transformations
$$G \Longrightarrow_{r_0,m_0} G_0 \Longrightarrow_{r_1,m_1} \cdots \Longrightarrow_{r_n,m_n} G_n$$

$$\begin{tikzcd}
c\text{-guaranteeing} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \text{direct improving w.r.t } c \arrow[dl, Rightarrow] \\
& \text{direct increasing w.r.t } c \arrow[d, Rightarrow] & \\
& \text{sustaining w.r.t } c &
\end{tikzcd}$$

Figure 2: consistency relations for basic rules.

such that $m_i = \text{tr}_{t_{i-1}} \circ \ldots \circ \text{tr}_{t_0} \circ m_0$ and $q \models \exists(b_i : C_k \hookrightarrow P_i, \mathsf{true})$ for all $P_i \in \mathbf{P}$ with $q = \text{tr}_{t_n} \circ q_n$. Let $p_i : P_i \hookrightarrow G_n$ be the morphism, such that $q = p_i \circ b_i$.

Now, we can construct a morphism $p : C_{k+1} \hookrightarrow G$ with

$$p(e) := \begin{cases} p_1(e) & \text{,if } e \in P_1 \\ & \vdots \\ p_j(e) & \text{,if } e \in P_j. \end{cases}$$

Let $e \in C_k$, because $q(e) = p_i \circ b_i(e)$ and $q(e) = p_\ell \circ b_\ell(e)$ and b_i and b_ℓ are both partial morphisms of a_k, it follows that $b_i(e) = b_\ell(e)$ and therfore $p_i(e) = p_\ell(e)$. Because $(P_i \cap P_\ell) \setminus C_k = \emptyset$ for all $i \neq \ell$, p is a morphism and by the definition of p it follows that $q = p \circ a_k$ and therefore $q \models \exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$.

By iteratively applying this whole construction to all occurrences of C_k that do not satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ the derived graph graph H does not contain any occurrences of C_k not satisfying $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ and therefore $H \models_{k+2} c$.

□

Corollary 3.14. *If a set of rule $\mathcal{R}$ is a repairing set of c at layer $k \leq \text{nl}(c)$ and 1. of theorem 3.3 applies, then $\mathcal{R}$ is a repairing set of c at layer j for all $k \leq j \leq \text{nl}(c)$.*

Lemma 3.15. *Let a graph G_0, a constraint c in EANF and a repairing set $\mathcal{R}$ at layer $c_{\max}^{G_0}+1$ be given, such that each rule in $\mathcal{R}$ is a basic increasing. Then, for every sequence*

$$G_0 \Longrightarrow_{\rho_0,m_0} G_1 \Longrightarrow_{\rho,m_0} \ldots \Longrightarrow_{\rho_n,m_n} G_n$$

such that $\rho_i = (\rho, \text{app}_{\text{pi}}(c_{\max}^{G_0} + 1, P)$ with $\rho \in \mathcal{R}$ being a consistency increasing rule at layer $c_{\max}^{G_0} + 1$ with P, it holds that

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