

# Rule-based Graph Repair using Minimally Restricted Consistency-Improving Transformations

Alexander Lauer

November 22, 2022

## **Abstract**

# Contents

<b>1 Preliminaries</b>	<b>3</b>
<b>2 consistency increasing</b>	<b>4</b>
2.1 extended alternating quantifier normal form . . . . .	4
2.2 conditions up to layer . . . . .	4
2.3 consistency increasing transformations and rules . . . . .	8
2.4 Comparison with other consistency concepts . . . . .	12
<b>3 general application conditions</b>	<b>15</b>
3.1 potentially minimal improving rules . . . . .	19

## 1 Preliminaries

**Definition 1.1 (subgraph).** Let  $G_1$  and  $G_2$  be graphs. The graph  $G_2$  is called a subgraph of  $G_1$  if an injective morphism  $f : G_2 \rightarrow G_1$  exists. We use the notation  $G_2 \subseteq G_1$  if  $G_2$  is a subgraph of  $G_1$  and  $G_2 \subset G_1$  if  $f$  is not bijective.

**Definition 1.2 (uppergraph).** Let  $G_1$  and  $G_2$  be graphs with  $G_1 \subseteq G_2$ . A graph  $C$  is called an upper-graph of  $G_1$  w.r.t  $G_2$ , if  $G_1 \subset C \subseteq G_2$ . The set of uppergraphs of  $G_1$  w.r.t.  $G_2$  is denoted by  $\mathcal{U}(G_1, G_2)$ . If  $G_1 = G_2$ , we set  $\mathcal{U}(G_1, G_2) = \{G_1\}$ .

**Definition 1.3 (overlap).** Let  $G$  and  $G'$  be graphs. A graph  $H$  is called an overlap of  $G$  and  $G'$  if morphisms  $p : G \hookrightarrow H$  and  $p' : G' \hookrightarrow H$  such that  $p$  and  $p'$  are jointly surjective. The set of all overlaps of  $G$  and  $G'$  is denoted by  $\text{ol}(G, G')$ .

**Definition 1.4 (overlap at morphism).** Let  $C, G$  and  $C'$  with  $C \subset C'$  be graphs and  $p : C \hookrightarrow G$  a morphism. A graph  $H$  is called an overlap of  $G$  and  $C'$  at  $p$  if a morphism  $p' : C' \hookrightarrow H$  with  $p'|_C = p$  exists. The set of all overlaps of  $G$  and  $C'$  at  $p$  is denoted by  $\text{ol}_p(G, C')$ .

**Definition 1.5 (partial morphism).** Let  $f : G_1 \rightarrow G_2$  and  $g : G_3 \rightarrow G_4$  be morphisms. The morphism  $g$  is called a partial morphism of  $f$  if  $G_3 \subseteq G_1$ ,  $G_4 \subseteq G_2$  and  $f|_{G_3} = g$ .

**Definition 1.6 (nested graph condition).** A graph condition over a graph  $C_0$  is inductively defined as follows:

- *true* is a graph condition over every graph.
- $\exists(a : C_0 \hookrightarrow C_1, d)$  is a graph condition over  $C_0$  if  $a$  is a injective graph morphism and  $d$  is a graph condition over  $C_1$ .
- $\neg d$  is a graph condition over  $C_0$  if  $d$  is a graph condition over  $C_0$ .
- $d_1 \wedge d_2$  and  $d_1 \vee d_2$  are graph conditions over  $C_0$  if  $d_1$  and  $d_2$  are graph conditions over  $C_0$ .

Conditions over the empty graph  $\emptyset$  are called constraints. Every injective morphism  $p : C_0 \hookrightarrow G$  satisfies *true*. An injective morphism  $p$  satisfies  $\exists(a : C_0 \hookrightarrow C_1, d)$  if there exists an injective morphism  $q : C_1 \hookrightarrow G$  such that  $q \circ a = p$  and  $q$  satisfies  $d$ . An injective morphism satisfies  $\neg d$  if it does not satisfy  $d$ , it satisfies  $d_1 \wedge d_2$  if it satisfies  $d_1$  and  $d_2$  and it satisfies  $d_1 \wedge d_2$  if it satisfies  $d_1$  or  $d_2$ . A graph  $G$  satisfies a constraint  $c$ ,  $G \models c$ , if  $p : \emptyset \hookrightarrow G$  satisfies  $c$ . We use the abbreviation  $\forall(a : C_0 \hookrightarrow C_1, d) := \neg \exists(a : C_0 \hookrightarrow C_1, \neg d)$ .

The nesting level  $\text{nl}$  of a condition is defined as  $\text{nl}(\text{true} = 0$  and  $\text{nl}(\exists(a : P \rightarrow Q, d)) := \text{nl}(d) + 1$ .

**Definition 1.7 (alternating quantifier normal form (ANF))**[3]. A graph condition  $c$  is in alternating normal form (ANF) if it is of the form

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \dots))))$$

with  $Q \in \{\exists, \forall\}$  and  $\overline{Q} = \exists$  if  $Q = \forall$ ,  $\overline{Q} = \forall$  if  $Q = \exists$ .

## 2 consistency increasing

In this section, we introduce the notion of *consistency increasing* transformations and rules, which allows to increase the consistency of a constraint layer by layer.

### 2.1 extended alternating quantifier normal form

To prevent the need of case discrimination, a new normal form for conditions, called *extended alternating quantifier normal form* (EANF), will be introduced. The sets of conditions in ANF and EANF do intersect and we show that both sets are expressively equivalent.

**Definition 2.1 (extended alternating quantifier normal form).** A conditions  $c$  is in extended alternating quantifier normal form (EANF) if it is of the form  $\forall(a_0 : C_0 \hookrightarrow C_1, d)$  and  $d$  is a condition in ANF.

Note that, given a condition  $c$  in EANF, every subcondition of  $c$  at layer  $1 \leq k \leq \text{nl}(c)$  is universally bound, if  $k$  is an odd number and existentially bound, if  $k$  is an even number.

**Lemma 2.2.** Any condition in ANF can be transformed into an equivalent condition in EANF and vice versa.

*Proof.* “ $\Rightarrow$ ”: Let a graph  $G$  and a constraint  $c$  in ANF be given. If  $c$  is universally,  $c$  is already in EANF.

If  $c = \exists(a_0 : C_0 \hookrightarrow C_1, d)$ , we show that  $c$  is equivalent to  $c' := \forall(\text{id}_{C_0} : C_0 \hookrightarrow C_0, c)$ .

1. Let  $p : C_0 \hookrightarrow G$  be a morphism, such that  $q \models c$ . Therefore a morphism  $q : C_0 \rightarrow G$  with  $q \models e$  and  $p = q \circ a_1$  exists. Then,  $p \models d$ , since  $p$  is the only morphism from  $C_0$  to  $G$  with  $p = p \circ \text{id}_{C_0}$  and  $p \models c$ .
2. Let  $p : C_0 \hookrightarrow G$  be a morphism with  $p \models c'$ , therefore all morphisms  $q : C_0 \hookrightarrow G$  with  $p = q \circ \text{id}_{C_0}$  satisfy  $c$ . Since  $p = p \circ \text{id}_{C_0}$ ,  $p \models c$  follows immediately.

“ $\Leftarrow$ ”: Let a graph  $G$  and a constraint  $c$  in EANF be given. If  $c = \forall(a_0 : C_0 \hookrightarrow C_1, d)$  with  $C_0 \neq C_1$ ,  $c$  is already in ANF.

Otherwise, the equivalence of  $c$  with  $d$  can be shown analogously to the first case. □

### 2.2 conditions up to layer

**Definition 2.3 (Layer of a subcondition).** Let  $c$  be a condition and  $d$  a subcondition of  $c$ . The layer of  $d$  is defined as  $\text{lay}(d) := \text{nl}(c) - \text{nl}(d) - 1$ .

**Definition 2.4 (Subcondition at layer).** Let  $c$  be a condition. The subcondition at layer  $k$ , denoted by  $\text{sco}_c(k)$ , is the subcondition  $d$  of  $c$  with  $\text{lay}(d) = k$ .

We define a notion of partial consistency, called *satisfaction at layer*, which will be used for the definition of consistency increasing. First, two operators are introduced to modify given constraints on a certain layer.

**Definition 2.5 (substitution at layer).** Let  $c = Q(a : C_0 \rightarrow C_1, d)$  be a condition in ANF, such that the subcondition of  $c$  with layer  $0 \leq k \leq \text{nl}(c)$  is a condition over  $C_k$ . Let  $e$  be a condition over  $C_k$ . The substitution in  $c$  at layer  $k$  with  $e$ ,  $\text{sub}(k, c, e)$ , is recursively defined as:

1. If  $k \leq 1$ :

$$\text{sub}(0, c, e) := e$$

2. If  $k > 1$ :

$$\text{sub}(k, c, e) := Q(a : C_0 \rightarrow C_1, \text{sub}(k - 1, d, e))$$

**Example 2.1.** Let the conditions  $c := \forall(a_0 : C_0 \rightarrow C_1, \exists(a_1 : C_1 \rightarrow C_2, \text{true}))$  and  $d = \exists(a'_1 : C_1 \rightarrow C_3, e)$  be given. Then,

$$\text{sub}(2, c, d) = \forall(a_0 : C_0 \rightarrow C_1, \exists(a'_1 : C_1 \rightarrow C_3, e)).$$

Through this, we define *condition up to layer*. Intuitively, a condition is cut off at a certain layer, by replacing the subcondition at this layer by `true` or `false`, depending on the quantifier, the replaced subcondition is bound by.

**Definition 2.6 (Condition up to layer).** Let  $c$  be a condition in EANF and  $d$  be the subcondition of  $c$  at layer  $0 \leq k \leq \text{nl}(c)$ . The condition up to layer  $k$  of  $c$ ,  $\text{cond}(k, c)$ , is defined as

$$\text{cond}(k, c) := \begin{cases} \text{sub}(k + 1, c, \text{true}) & \text{if } d \text{ is existentially bound} \\ \text{sub}(k + 1, c, \text{false}) & \text{if } d \text{ is universally bound.} \end{cases}$$

**Example 2.2.** Let the condition  $c = \forall(a_0 : C_0 \rightarrow C_1, \exists(a_1 : C_1 \rightarrow C_2, d))$  be given. Then,

$$\text{cond}(2, c) = \forall(a_0 : C_0 \rightarrow C_1, \text{true}).$$

Now, we are ready to define satisfaction at layer, which is a key ingredient to define consistency increasing.

**Definition 2.7 (Satisfaction at layer).** Let  $G$  be a graph and  $c$  be a condition over  $C_0$ . A morphism  $p : C_0 \rightarrow G$  satisfies  $c$  at layer  $k$ ,  $p \models_k c$ , if

$$p \models \text{cond}(k, c).$$

A graph  $G$  satisfies a constraint  $c$  at layer  $k$ ,  $G \models_k c$ , if  $q : \emptyset \rightarrow G$  satisfies  $\text{cond}(k, c)$ . The biggest  $k$  with  $G \models_k c$  such that no  $j > k$  with  $G \models_j c$  exists is denoted by  $c_{\max}^G$ .

**Example 2.3.**

The following lemmas arise as a direct consequence of the definition of satisfaction at layer. If a graph satisfies a constraint up to a certain layer, let  $c$  be this condition up to this layer, that ends with  $\forall(a : C \hookrightarrow C', \text{false})$ , the graph satisfies all constraints starting with  $c$ .

**Lemma 2.8.** *Let  $G$  be a graph  $p : C_0 \hookrightarrow G$  a morphism and  $c$  a condition over  $C_0$  in EANF with  $p \models_k c$ . If the subcondition  $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$  of  $c$  at layer  $k$  is universally bound, then for any condition  $f$  over  $C_k$  it holds that*

$$p \models \text{sub}(k+1, c, f).$$

*Proof.* Let  $k$  be the smallest number with  $\text{sco}_c(k) = \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$  being universally bound and  $p \models_k c$ . Let  $q : G_{k-1} \hookrightarrow G$  be a morphism such that  $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$ . This must exist, since  $p \models_k c$  and  $k$  is the smallest even number such that  $p \models_k c$ .

Therefore, there does not exist a morphism  $q' : C_k \hookrightarrow G$  with  $q = q' \circ a_k$ . Hence, for every condition  $f$  over  $C_k$  a morphism  $q' : C_k \hookrightarrow G$  with  $q \not\models f$  and  $q = q' \circ a_k$  cannot exist. It follows immediately that  $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$  and with that  $p \models \text{sub}(k+1, c, f)$ . For every  $j > k$  even, such that  $p \models_j c$ , and every condition  $d$  over  $C_j$ , with the first part, it holds that  $p \models \text{sub}(k+1, c, f)$  with  $f = \text{sub}(j-k+1, \text{sco}_c(k+1), d)$  and since  $\text{sub}(k+1, c, f) = \text{sub}(j+1, c, d)$  it follows that  $p \models \text{sub}(j+1, c, d)$ .  $\square$

As a direct consequence of the previous lemma, a graph satisfying a condition up to layer ending with  $\forall(a : C \hookrightarrow C', \text{false})$  also satisfies the whole constraint.

**Lemma 2.9.** *Let  $G$  be a graph,  $p : C_0 \hookrightarrow G$  a morphism and  $c$  a condition over  $C_0$  in EANF with  $p \models_k c$ . If  $\text{sco}_c(k)$  is universally bound,*

$$p \models_k c \implies p \models c.$$

*Proof.* Follows immediately by using lemma 2.8 and setting  $f$  to  $\text{sco}_c(k+1)$ .  $\square$

**Lemma 2.10.** *Let a graph  $G$ , a morphism  $p : C_0 \hookrightarrow G$  and a constraint  $c$  in EANF be given. Then,  $p \models c$  for all  $j < c_{\max}^G$  such that  $\text{sco}_c(j)$  is existentially bound.*

*Proof.* 1. The subcondition of  $c$  at layer  $c_{\max}^G$  is existentially bound: If a  $j < c_{\max}^G$  with  $p \models_j c$  exists such that  $\text{sco}_c(j)$  is universally bound, let  $j_1$  be the smallest of these. With lemma 2.8 follows that  $p \models_{j_2} c$  for all  $j_1 < j_2$ . Let  $\ell < j_1$ , such that  $\text{sco}_c(\ell)$  is existentially bound and let  $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$  be the condition up to layer  $j_1 - \ell$  of  $\text{sco}_c(\ell)$ . Since  $\ell < j_1$ , a morphism  $q : C_\ell \hookrightarrow G$  with  $q \models d$  must exists and therefore a morphism  $q' : C_{\ell+1} \hookrightarrow G$  with  $q = q' \circ a_k$  must exists. It follows that  $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \text{true})$  and with that  $p \models_\ell c$ .

2. The subcondition of  $c$  at layer  $c_{\max}^G$  is universally bound: With lemma 2.8 follows that  $p \models_{k+1} c$ . Since  $c$  is in EANF, 1. can be applied to  $k+1$ .  $\square$

Through satisfaction at layer an increase of consistency can be detected in the following way. Let  $G \implies H$  be a transformation. If  $c_{\max}^G < c_{\max}^H$ , the transformation can be considered as consistency increasing, since  $H$  satisfies more layers of the constraint than  $G$ . The notion of consistency increasing should also be able to detect the smallest transformations that lead to a increase of consistency, namely the inserting of a single edge or node of an existentially bound graph. To remedy this issue, we introduce *partial conditions*. Intuitively, given a constraint  $c$  in EANF ending with  $\text{sco}_c(k) = \exists(a_k : C_k \hookrightarrow C_{k+1}, d)$ , the condition  $\text{sco}_c(k)$  is replaced by  $\exists(a_k^p : C_k \hookrightarrow C', \text{true})$  with  $C' \in \mathcal{U}(C_k, C_{k+1})$ .

The construction of partial conditions is designed to only replace graphs in existentially bound layers, since the replacement in an universally bound layer would lead to a more restrictive constraint than the original condition up to layer. That means, given the condition  $c = \forall(a_0 : C_0 \hookrightarrow C_1, \text{false})$  and let  $C' \in \mathcal{U}(C_0, C_1)$ . If the condition  $c' = \forall(a_0^p : C_0 \hookrightarrow C_1, \text{false})$  is satisfied this implies that  $c$  is also satisfied but the backwards implication does not hold.

**Definition 2.11 (partial condition).** Let  $c$  be a condition in EANF.

The partial condition of  $c$  at layer  $k < \text{nl}(c)$  with with

$$C' \in \begin{cases} \mathcal{U}(C_k, C_{k+1}) & \text{if } k \text{ is even} \\ \mathcal{U}(C_{k+1}, C_{k+2}) & \text{if } k \text{ is odd,} \end{cases}$$

$\text{part}(k, c, C')$ , is defined as:

1. If  $k$  is odd, let  $\text{sco}_c(k+1) = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$ :

$$\text{part}(k, c, C') := \text{sub}(k+1, c, \exists(a^p : C_{k+1} \hookrightarrow C', \text{true}))$$

2. If  $k$  is even, let  $\text{sco}_c(k+1) = \exists(a : C_k \hookrightarrow C_{k+1}, f)$ :

$$\text{part}(k, c, C') := \text{sub}(k, c, \exists(a^p : C_{k+1} \hookrightarrow C', \text{true}))$$

#### Example 2.4.

If a graph  $G$  does not satisfy the condition up to layer  $k := c_{\max}^G + 2$  of a given constraint  $c$ , with  $k$  being even, there does exists a graph  $C' \in \mathcal{U}(C_{k-1}, C_k)$ , such that  $G$  satisfies  $\text{part}(k, c, C')$ , note that  $G$  always satisfies  $\text{part}(k, c, C_{k-1})$ . For the biggest of these graphs, we define the notions of *biggest partially satisfying conditions and graphs*.

**Definition 2.12 (biggest partially satisfying condition).** Let  $G$  be a graph,  $c$  a condition in EANF over  $C_0$  and  $p : C_0 \hookrightarrow G$  a morphism with  $p \models_k c$  and  $p \not\models c$ .

A partial condition  $c = \text{part}(k+2, c, C')$  with  $p \models c$  is a biggest partially satisfying condition w.r.t  $c$  of  $p$  if there does not exist a graph  $C''$ , such that  $C'$  is a subgraph of  $C''$ , with  $p \models \text{part}(c_{\max}, c, C'')$ . The graph  $C'$  is then called a biggest partially satisfying graph w.r.t  $c$  of  $p$ .

Given a constraint  $c$ , the set of biggest partially satisfying conditions w.r.t  $c$  of  $p : \emptyset \inf G$  is denoted by  $\mathcal{P}_c^G$ .

The set of all biggest partially satisfying graphs w.r.t  $c$  of  $p$  is denoted by  $\mathcal{G}_c^G$ .

### 2.3 consistency increasing transformations and rules

With the results above, we are now ready to define the notions of *consistency increasing* and *direct consistency increasing*, with direct increasing being a more restrictive version of increasing, yielding the advantage that second-order logic formulas can be used in order to determine whether a transformation is (direct) consistency increasing or not.

These notions are designed to only detect transformations that increase the consistency of the first two unsatisfied layer of a constraint  $c$ . That means, given a graph  $G$  and a constraint  $c$ , let  $k = c_{\max}^G$  and  $\forall(a_{k+1} : C_k \hookrightarrow C_{k+1}, \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, d)) =: \text{sco}_c(k+1)$ . A transformation  $t : G \implies H$  is considered as (direct) consistency increasing if  $c_{\max}^G \leq c_{\max}^H$ , i.e. the satisfaction up to layer  $k$  is not decreased, and more increasing insertions or deletions have been performed than decreasing ones. An increasing deletion is the deletion of an occurrence of  $C_{k+1}$  that does not satisfy  $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \text{true})$ , an increasing insertion is the insertion of elements of  $C_{k+2}$ , such that for at least one occurrence  $p$  of  $C_{k+1}$  it holds that  $p \not\models \exists(a_{k+2}^p : C_{k+1} \hookrightarrow C', \text{true})$  and  $\text{tr}_t \circ p \models \exists(a_{k+2}^p : C_{k+1} \hookrightarrow C', \text{true})$  for an  $C' \in \mathcal{U}(C_{k+1}, C_{k+2})$ . A decreasing insertion is the creation of an occurrence of  $C_{k+1}$  not satisfying  $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \text{true})$  and a decreasing deletion is the deletion of elements of  $C_{k+2}$  such that for an occurrence  $p$  of  $C_{k+1}$  with  $p \models \exists(a_{k+2}^p : C_{k+1} \hookrightarrow C', \text{true})$  it holds that  $\text{tr}_t \circ p \not\models \exists(a_{k+2}^p : C_{k+1} \hookrightarrow C', \text{true})$ .

To evaluate this, we define the *number of violations*. Intuitively, for all occurrences  $p$  of  $C_{k+1}$  the number of graphs  $C' \in \mathcal{U}(C_{k+1}, C_{k+2})$  with  $p \not\models \exists C'$  is add up and it can be determined whether more increasing or decreasing actions have been performed by a transformation.

Note, that the number of violations is defined for each layer of the constraint, but only for the first unsatisfied layer the sum is calculated as described above. For all layer  $k$  with  $k \leq c_{\max}^G$  it is set to 0 and for all layer  $k$  with  $k > c_{\max}^G + 1$  it is set to  $\infty$ . Through this, a transformation  $tG \implies H$  that increases the partial consistency can easily detected since the number of violations in  $H$  at layer  $c_{\max}^G + 1$  will be set to 0.

**Definition 2.13 (number of violations).** Let  $G$  be a graph and  $c$  a constraint in EANF. The number of violations  $\text{nvc}(j, G)$  at layer  $j$  in  $G$  is defined as:

1. If  $j < c_{\max}^G + 1$ :

$$\text{nvc}(j, G) := 0$$

2. If  $j = c_{\max}^G + 1$ , let  $d = \forall(a_k : C_j \hookrightarrow C_{j+1}, e)$  be the subcondition of  $c$  at layer  $j + 1$ .

$$\text{nvc}(j, G) := \begin{cases} \sum_{C' \in \mathcal{U}(C_j, C_{j+1})} |\{q \mid q : C_{j+1} \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| & \text{if } e \neq \text{false} \\ |\{q \mid q : C_{j+1} \hookrightarrow G\}| & \text{if } e = \text{false} \end{cases}$$

3. If  $j > c_{\max}^G + 1$ :

$$\text{nvc}(j, G) := \infty$$

Via the number of violations, we now define *consistency increasing* transformations and rules, by checking whether the number of violations has decreased for any layer of the constraint.

**Definition 2.14 (consistency increasing).** Let a graph  $G$ , a rule  $\rho$  and a constraint  $c$  in EANF be given.

A transformation  $G \Rightarrow_{\rho,m} H$  is called consistency increasing w.r.t  $c$ , if

$$\text{nvc}(k, H) < \text{nvc}(k, G)$$

for any  $0 \leq k \leq \text{nl}(c)$ . A rule  $r$  is called consistency increasing w.r.t  $c$ , if all of its applications are.

Note that if  $G \models c$  there does not exist a consistency increasing transformation w.r.t  $c$ , since  $\text{nvc}(j, G) = 0$  for all  $1 \leq j \leq \text{nl}(c)$ . Also, no plain rule  $\rho$  is consistency increasing w.r.t  $c$ , since a graph  $G$  satisfying  $c$ , such that a transformation  $t : G \Rightarrow_{\rho,m} H$  exists can always be constructed. Therefore, each consistency increasing rule has to be equipped with at least one application condition.

As mentioned above, a transformation should be detected as consistency increasing if it increases the partial consistency, which is shown by the following theorem.

**Theorem 2.1.** Let a graph  $G$ , a rule  $\rho$  and a constraint  $c$  in EANF be given. A transformation  $t : G \Rightarrow_{\rho,m} H$  is consistency increasing if  $c_{\max}^G < c_{\max}^H$ .

*Proof.* No  $\ell > c_{\max}^G$  with  $G \models_{\ell} c$  exists. Hence,  $\text{nvc}(c_{\max}^G + 1, G) > 0$  and  $\text{nvc}(c_{\max}^G + 1, H) \neq \infty$ . Since  $c_{\max}^H > c_{\max}^G$ ,  $\text{nvc}(k + 1, H) = 0$  and it follows immediately that  $t$  is consistency increasing.  $\square$

Since no consistency increasing transformation originating in consistent graphs exist, there do not exist infinite long sequences of consistency increasing transformations. Additionally, if a set of rules  $\mathcal{R}$  is given, and a sequence of consistency increasing transformations with rules of  $\mathcal{R}$  ends with a graph  $G$ , then either  $G$  satisfies the constraint or there do not exist any consistency increasing transformations  $G \Rightarrow_{\rho,m} H$  with  $\rho \in \mathcal{R}$ .

**Theorem 2.2.** Let a constraint  $c$  in EANF and a set of rules  $\mathcal{R}$  be given. Every sequence of minimal consistency improving transformation w.r.t  $c$  with rules in  $\mathcal{R}$  is finite.

*Proof.* Let  $G_0$  be a graph and

$$G_0 \Rightarrow_{\rho_1} G_1 \Rightarrow_{\rho_2} G_2 \Rightarrow_{\rho_3} \dots$$

be a sequence of minimal consistency improving transformations w.r.t  $c$  with  $\rho_i \in \mathcal{R}$ . We assume that  $c_{\max}^{G_0} < \text{nl}(c)$ , otherwise  $\text{nvc}(j, G_0) = 0$  for all  $j \in \{0, \dots, \text{nl}(c)\}$  and no transformation  $G_0 \Rightarrow H$  is consistency increasing.

We show that after at most  $j := \text{nvc}(c_{\max}^{G_0} + 1, G_0)$  transformations  $G_j \models_{c_{\max}^{G_0} + 2} c$  holds. Note that  $j$  has to be finite, since  $G_0$  contains only a finite number of occurrences of  $C_{j+1}$ . After each transformation it holds that  $\text{nvc}(c_{\max}^{G_{i+1}} + 1, G_{i+1}) \leq \text{nvc}(c_{\max}^{G_i} + 1, G_i) -$

1. Therefore, after  $j$  transformations,  $\text{nvc}(c_{\max}^{G_0} + 1, G_j) \leq \text{nvc}(c_{\max}^{G_0} + 1, G_0) - j = 0$  holds and with that  $G_j \models_{c_{\max}^{G_0} + 2} c$ . By iteratively applying this, it follows that after a finite number of transformations a graph  $G_k$  with  $G_k \models c$  has to exist. Since no consistency increasing transformation  $G_k \implies G_{k+1}$  exists, the sequence has to be finite. Also, for some  $G_{k'}$  there may not exist a consistency increasing transformation  $G_{k'} \implies_{\rho} H$  with  $\rho \in \mathcal{R}$  and therefore the sequence is also finite, but does not end with a consistent graph.  $\square$

The replacement of a graph not satisfying a constraint  $c$  by a  $c$  satisfying graph via a transformation is a consistency increasing transformation. Therefore, a consistency increasing transformation can perform insertions or deletions that are unnecessary in order to increase the consistency. That is, the deletion of occurrences of existentially bound graphs, the deletions of occurrences of graphs  $C_k$  satisfying  $\exists C_{k+1}$  or the insertion of occurrences of universally bound graphs and the insertion of existentially bound graphs  $C_k$ , such that the corresponding occurrence of  $C_{k-1}$  already satisfied  $\exists C_k$ .

*Direct consistency increasing* transformations are more restricted, in the sense, that these unnecessary deletions and insertions are leading a transformation to not being direct increasing. The presence of these unnecessary actions can be checked via second-order logic formulas. Additionally, it is secured that no new violations are introduced since these can always be considered as an unnecessary insertion or deletion. With that, the removal of one violation is sufficient to state that the transformation is (direct) consistency increasing, which can be checked via an additional second order logic formula. Intuitively, it is checked by the first two formulas, that no new violations are introduced. The third formula secures that at least one violation has been removed and the last formulas secure that the partial consistency is not decreased.

**Definition 2.15 (direct consistency increasing).** Let  $G$  be a graph,  $\rho$  a plain rule,  $c$  a constraint in EANF and  $\text{sco}_c(c_{\max}^G + 1) = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ . A transformation  $t : G \implies_{\rho, m} H$  is called direct consistency increasing if the following equations hold. Let

$$\mathbf{G} = \begin{cases} \mathcal{U}(C_k, C_{k+1}) & \text{if } e \neq \text{false} \\ \{C_k\} & \text{otherwise} \end{cases}$$

Every occurrence of  $C_k$  in  $G$  that satisfies  $\text{part}(1, e, C')$  for any  $C' \in \mathbf{G}$  still satisfies  $\text{part}(1, e, C')$  in  $H$ .

$$\begin{aligned} \forall p : C_k \hookrightarrow G \left( \bigwedge_{C' \in \mathbf{G}} (p \models \text{part}(1, e, C') \wedge \text{tr}_t op \text{ is total}) \right. \\ \left. \implies \text{tr}_t op \models \text{part}(1, e, C') \right) \end{aligned} \tag{2.1}$$

Let  $C' = C_{k+1}$  if  $e \neq \text{false}$  and  $C' = C_k$  otherwise. Every new inserted occurrence of  $C_k$  by  $t$  satisfies  $\text{part}(1, e, C')$ .

$$\forall p' : C_k \hookrightarrow H (\neg \exists p : C_k \hookrightarrow G (p' = \text{tr}_t op) \implies p' \models \text{part}(1, e, C')) \tag{2.2}$$

At least one occurrence of  $C_k$  in  $G$  that does not satisfy  $\text{part}(1, e, C')$ , for any  $C' \in \mathbf{G}$ , either has been removed or satisfies  $\text{part}(1, e, C')$  in  $H$ .

$$\exists p : C_k \hookrightarrow G \left( \bigvee_{C' \in \mathbf{G}} (p \not\models \text{part}(1, e, C') \wedge (\text{tr}_t \circ p \text{ is not total} \vee (\text{tr}_t \circ p \text{ is total} \wedge \text{tr}_t \circ p \models \text{part}(1, e, C')))) \right) \quad (2.3)$$

No occurrence of a universally bound graph  $C_j$  with  $j < k$  gets inserted.

$$\bigwedge_{\substack{i < k \\ i \text{ odd}}} \forall p : C_i \hookrightarrow H (\exists p' : C_i \hookrightarrow G (p' = \text{tr}_t \circ p)) \quad (2.4)$$

No occurrence of an existentially bound graph  $C_j$  with  $j < k$  gets deleted.

$$\bigwedge_{\substack{i < k \\ i \text{ even}}} \forall p : C_i \hookrightarrow G (\text{tr}_t \circ p \text{ is total}) \quad (2.5)$$

Note that (2.4) and (2.5) not only secure that the partial consistency does not decrease, as shown in the following lemma, but also prevent further unnecessary insertions and deletions, since the insertion of a universally and the deletion of a existentially bound graph will never lead to an increase of consistency.

**Lemma 2.16.** *Let a transformation  $t : G \implies H$  and a constraint  $c$  in EANF be given, such that (2.4) and (2.5) of definition 2.15 are satisfied. Then,*

$$H \models_{c_{\max}^G} c.$$

*Proof.* Assume that  $G \not\models_{c_{\max}^G} c$ . Then, either a new occurrence of an universally bound graph  $C_k$ , with  $k \leq c_{\max}^G$ , of  $c$  has been inserted or an occurrence of an existentially bound graph  $C_k$ , with  $k \leq c_{\max}^G$ , of  $c$  has been destroyed. Therefore, the following holds:

$$\exists p : C_i \hookrightarrow G (\neg \exists p' : C_i \hookrightarrow (p' = \text{tr}_t \circ p)) \vee \exists p : C_j \hookrightarrow G (\text{tr}_t \circ p \text{ is not total})$$

with  $i, j \leq c_{\max}^G$ ,  $i$  being odd and  $j$  being even. It follows immediately that either (2.4) or (2.5) is not satisfied. This is a contradiction.  $\square$

Now, we will show the already indicated relationship between direct increasing and increasing, namely that a direct increasing transformation is also increasing. Counterexamples showing that the inversion of the implication does not hold can easily be constructed, showing that these notions are not identical, but related.

**Lemma 2.17.** *Let a graph  $G$ , a constraint  $c$  in EANF and a direct increasing transformation  $t : G \implies_{\rho, m} H$  w.r.t.  $c$  be given. Then,  $t$  is also a increasing transformation.*

*Proof.* Let  $G$  be a graph,  $k = c_{\max}^G + 1 \leq \text{nl}(c)$  and  $d = \text{sco}_c(k)$ . With lemma 2.16 follows that  $c_{\max}^H \geq c_{\max}^G$  and with that  $\text{nvc}(k, H) \neq \infty$ .

1. We show that equations (2.1) and (2.2) imply that  $\text{nvc}(k, H) \leq \text{nvc}(k, G)$ . Assume that  $\text{nvc}(k, H) > \text{nvc}(k, G)$ . Therefore, a morphism  $p : C_k \hookrightarrow H$  with  $p \not\models \text{part}(1, d, C')$  for any  $C' \in \mathcal{U}(C, C_{k+1})$  exists, such that either 1a or 1b is satisfied.
  - (a) There does exist a morphism  $q' : C_k \hookrightarrow G$  with  $q' \models \text{part}(1, d, C')$  and  $p = \text{tr}_t \circ q'$ .
  - (b) There does not exist a morphism  $q : C_k \hookrightarrow G$ , such that  $p = \text{tr}_t \circ q$ .

This is a contradiction, if 1a is satisfied,  $q'$  does not satisfy equation (2.1) and if 1b is satisfied  $q$  does not satisfy equation (2.2). It follows that

$$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| \leq |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}|$$

for all  $C' \in \mathcal{U}(C_k, C_{k+1})$ .

2. Since (2.3) is satisfied, a morphism  $p : C_k \hookrightarrow G$  with  $p \not\models \text{part}(1, d, C')$ , such that either  $\text{tr} \circ p$  is total and  $\text{tr}_t \circ p \models \text{part}(1, d, C')$  or  $\text{tr} \circ p$  is not total exists, for any  $C' \in \mathcal{U}(C_k, C_{k+1})$ . In both cases the following holds:

$$\begin{aligned} p \in \{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\} \wedge \\ \text{tr} \circ p \notin \{q \mid q : C_k \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\} \end{aligned}$$

With that and 1. it follows that

$$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| < |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}|.$$

In total,  $\text{nvc}(k, G) < \text{nvc}(k, H)$  and  $t$  is an increasing transformation.

For the special case that  $k = \text{nl}(c) - 1$  and  $c$  ends with  $\forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$  it can be shown in a similar way that (2.1) and (2.2) imply that

$$|\{q \mid q : C_k \hookrightarrow G\}| \leq |\{q \mid q : C_k \hookrightarrow H\}|$$

and that (2.3) implies

$$|\{q \mid q : C_k \hookrightarrow G\}| < |\{q \mid q : C_k \hookrightarrow H\}|.$$

It follows that  $t$  is an increasing transformation.  $\square$

## 2.4 Comparison with other consistency concepts

In this chapter, the notions of (direct) consistency increasing are compared to the already known concepts of consistency guaranteeing, consistency preserving [1], (direct) consistency increasing and sustaining [2] in order to reveal relationships between them and secure that (direct) increasing is indeed a new notion of consistency. These relations are displayed in figure 1.

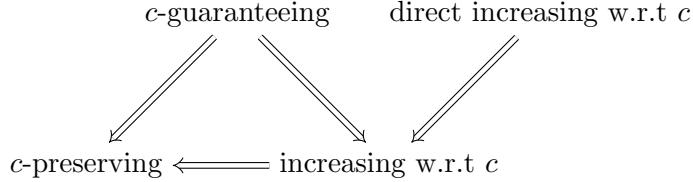


Figure 1: General relations of consistency notions.

First we compare (direct) consistency increasing with the notions of consistency-guaranteeing and -preserving transformations. As already mentioned, given a constraint  $c$  and a graph  $G$  with  $G \models c$ , there does not exist a (direct) increasing transformation  $t : G \Rightarrow H$  and it follows immediately that (direct) increasing w.r.t  $c$  implies  $c$ -preserving.

Obviously, guaranteeing implies increasing, this property is embedded in the definition of increasing. The inversion of this implication does not hold, since increasing is a way stricter notion, in the sense, that the number of removed violations has to be greater than the number of introduced violations. For guaranteeing transformations this is not the case, an arbitrary number of violations can be inserted, long as the derived graph satisfies the constraint and therefore guaranteeing does not imply direct increasing, since a direct increasing transformations is not allowed to introduce any new violations.

**Lemma 2.18.** *Let a constraint  $c$  in ANF, graphs  $G$  and  $H$  with  $G \not\models c$ , and a transformation  $t : G \Rightarrow H$  be given. Then,*

$$\begin{aligned} t \text{ is } c\text{-guaranteeing} &\implies t \text{ is increasing w.r.t } c && \wedge \\ t \text{ is } c\text{-guaranteeing} &\nRightarrow t \text{ is direct increasing w.r.t } c && \wedge \\ t \text{ is increasing w.r.t } c &\nRightarrow t \text{ is } c\text{-guaranteeing}. \end{aligned}$$

*Proof.* Let  $c'$  be the equivalent constraint in EANF.

1. Let  $t$  be a  $c$ -guaranteeing transformation, then  $H \models c$ . Since  $G \not\models c$ ,  $G \not\models c'$  follows and  $\text{nvc}(c_{\max}^G + 1, G) > 0$  and  $\text{nvc}(c_{\max}^G + 1, H) = 0$ . Therefore  $t$  is consistency increasing.
2. Let a constraint  $c = \exists(a_0 : \emptyset \hookrightarrow C_0, \forall(a_1 : C_0 \hookrightarrow C_1, \exists(a_2 : C_1 \hookrightarrow C_2, \text{true})))$  and a graph  $G = C' \dot{\cup} C'$  with  $C' = C_2 \setminus \{e\}$  for one edge  $e \in E_{C_2 \setminus C_1}$  and the occurrences  $p_1 : C_1 \hookrightarrow G$  and  $p_2 : C_1 \hookrightarrow G$  be given. Let  $t : G \Rightarrow H$  be a transformation with  $H = C_2 \dot{\cup} C''$  and  $C'' = C' \setminus \{e'\}$  for an  $e' \in E_{C_2 \setminus C_1}$ , such that  $\text{tr}_t \circ p_1$  and  $\text{tr}_t \circ p_2$  are total. Then,  $t$  is  $c$ -guaranteeing and not direct increasing, since  $p_i \models \exists(a'_2 : C_1 \hookrightarrow C', \text{true})$  for  $i = 1, 2$  and either  $\text{tr}_t \circ p_1 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \text{true})$  or  $\text{tr}_t \circ p_2 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \text{true})$ .
3. Let  $t$  be a consistency increasing transformation w.r.t  $c'$ , such that  $\text{nvc}(c_{\max}^G + 1, G) > \text{nvc}(c_{\max}^G + 1, H) > 1$  and  $H \not\models_{c_{\max}^G + 2} c'$ . Then,  $H \not\models c'$  and  $t$  is not a  $c$ -guaranteeing transformation.

□

The definition of consistency improving only differs from guaranteeing if the corresponding constraint is universally bound and these notions are identical for existentially bound constraint. Therefore, with lemma 2.18, we can state the following.

**Corollary 2.19.** *Let  $c$  be an existentially bound constraint in ANF. Then,*

$$\begin{aligned} t \text{ is consistency improving w.r.t } c &\implies t \text{ is consistency increasing w.r.t } c \wedge \\ t \text{ is consistency increasing w.r.t } c &\not\implies t \text{ is consistency improving w.r.t } c \end{aligned}$$

The notions of increasing and improving are equivalent for universally bound constraints with nesting level 1. Note that, with corollary 2.19, this equivalence does not hold for existentially bound constraints with nesting level 1.

**Lemma 2.20.** *let a universally bound constraint  $c$  with  $\text{nl}(c) = 1$  in ANF, a graph  $G$  with  $G \not\models c$  and a transformation  $t : G \implies H$  be given. Then,*

$$t \text{ is consistency improving w.r.t } c \iff t \text{ is consistency increasing w.r.t } c$$

*Proof.* Let  $c = \forall(a : \emptyset \hookrightarrow C, \text{false})$ . Since  $\mathcal{U}(C, C) = \{C\}$ ,  $\text{nvc}(1, G)$  is the number of occurrences of  $C$  in  $G$ . This is exactly the definition of the number of violations for consistency improving transformations and the statement follows immediately. □

For universally bound constraints  $c$  with  $\text{nl}(c) \geq 2$ , (direct) consistency increasing is not related to (direct) consistency improving and sustaining. As shown in [2], (direct) consistency improving implies (direct) consistency sustaining. Therefore it is sufficient to show that direct improving does not imply increasing and that direct increasing does not imply consistency sustainment.

**Lemma 2.21.** *Let a universally bound constraint  $c$  with  $\text{nl}(c) \geq 2$  in ANF, a graph  $G$  with  $G \not\models c$  and a transformation  $t : G \implies H$  be given. Then,*

$$\begin{aligned} t \text{ is direct consistency improving w.r.t } c &\not\implies t \text{ is consistency increasing w.r.t } c \wedge \\ t \text{ is direct increasing w.r.t } c &\not\implies t \text{ is consistency sustaining w.r.t } c \end{aligned}$$

*Proof.* 1. Let  $c = \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \text{true}))$  be a constraint. Let  $V_{C_0} = V_{C_1}$  and  $|E_{C_1}| - |E_{C_0}| = 2$ . Let  $G = C' \dot{\cup} C''$  with  $C' = C_1 \setminus \{e\}$  with  $e \in E_{C_1} \setminus E_{C_0}$  and the occurrences  $p_1 : C_0 \hookrightarrow G$  and  $p_2 : C_0 \hookrightarrow G$  be given. It follows that  $\text{nvc}(1, G) = 2$ . Let  $t : G \implies H$  be a transformation, such that  $H = C_0$ . Then,  $t$  is a direct consistency improving transformation, since  $H$  contains only one occurrence of  $C_0$  not satisfying  $\exists(a_1 : C_0 \hookrightarrow C_1 \text{ true})$ . But,  $t$  is not consistency increasing, since  $\text{nvc}(1, H) = 3$ .

2. Let  $c := \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \text{true}))))$  be a constraint.

Let a graph  $G = C_0$  with the morphism  $q : C_0 \hookrightarrow G$  and a transformation  $t : G \implies H$  with  $H := C_2 \dot{\cup} C_2$  be given, such that  $\text{tr}_t \circ q$  is total. Then  $t$  is a direct consistency increasing transformation but not a consistency sustaining one, since  $H$  contains more occurrences of  $C_0$  not satisfying  $\exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \text{true})))$  than  $G$ .

□

### 3 general application conditions

To guarantee that each transformation  $t : G \implies_{\rho, m} H$ , given a rule  $\rho = L \hookleftarrow K \hookrightarrow R$ , is (direct) consistency increasing w.r.t to a constraint  $c$ , we present applications conditions ensuring this property. It has to be ensured that at least one violation will be removed. For this, it is necessary (a) to check that an occurrence  $p$  of the the universally bound graph  $C_{c_{\max}^G+1}$  exists, such that  $p$  and the match  $m$  do overlap, i.e  $p(C_{c_{\max}^G+1}) \cap m(L) \neq \emptyset$ , and (b) that  $p$  does not satisfy  $c' = \exists C_{c_{\max}^G+2}$ . Only in this case, it is possible that the transformation does remove a violation. To ensure that  $p$  does not satisfy  $c'$ , the non-existence of all possible overlaps of  $L$  and  $C_{c_{\max}^G+2}$  such that  $p \models c'$  has to be checked. For this, we introduce *extended overlaps*. Intuitively, given an overlap  $C$  of  $L$  and  $C_{c_{\max}^G+1}$  with  $p : C_{c_{\max}^G+1} \hookrightarrow C$ , the set of extended overlaps contains all overlaps  $C'$  of  $L$  and  $C_{c_{\max}^G+2}$ , such that  $p \models c'$  if one of these overlaps exists.

**Definition 3.1 (extended overlaps).** Let  $G$ ,  $C_0$  and  $C_1$  with  $i : C_0 \hookrightarrow C_1$  be graphs. Let  $C$  be an overlap of  $C_0$  and  $G$  with the inclusion  $q : C_0 \hookrightarrow C$ . The set of extended overlaps of  $C$  with  $i$ ,  $\text{eol}(C, i)$ , is the set of all overlaps  $C'$  of  $G$  and  $C_1$ , such that an injective morphism  $i_C : C \hookrightarrow C'$  with  $i_C \circ q \models \exists(i : C_0 \hookrightarrow C_1, \text{true})$  exists.

A direct consistency increasing application condition has to, as mentioned above, (a) ensure that at least one violation will be removed, (b) ensure that no new violations get inserted and (c) secure that the partial consistency is not decreased. In the definition below,  $\text{nex}()$  and  $\text{rep}()$  ensure that (a) is met, with  $\text{nex}()$  ensuring that a violation is present and  $\text{rep}()$  ensuring that this violation will be removed. Note, that the construction of these is divided in two cases. Firstly, either  $k < \text{nl}(c) - 1$  or the constraints ends with a condition of the form  $\exists(a : C \hookrightarrow C', \text{true})$  and secondly,  $k = \text{nl}(c) - 1$  and the constraint ends with a condition of the form  $\forall(a : C \hookrightarrow C_1, \text{false})$ .

For  $\text{nex}()$ , the first case will be checked as already described above. In the second case, it is sufficient to check whether an occurrence  $p$  of  $C_k$  with  $m(L) \cap p(C_k) \neq \emptyset$  exists.

For  $\text{rep}()$ , in the first case, a violation can be removed by either deleting an occurrence  $p$  of  $C_k$  or inserting elements of  $C_{k+1}$ , such that  $p \not\models \exists C'$  and  $\text{tr}_t \circ p \models \exists C'$  for a graph  $C' \in \mathcal{U}(C_k, C_{k+1})$ . In the second case, a violation can only be removed by deleting an occurrence  $p$  of  $C_1$ . This will only occur if  $m(L \setminus K) \cap p(C_1) \neq \emptyset$ .

The conditions constructed by  $\text{nin}()$  and  $\text{nwo}()$  ensure that (2) is met, with  $\text{nin}()$  checking that no new occurrence of  $C_k$  will be inserted and  $\text{nwo}()$  ensuring that for an occurrence  $p$  of  $C_k$  if  $p \models \exists C'$  it is secured that  $\text{tr}_t \circ p \models \exists C'$  for any  $C' \in \mathcal{U}(C_k, C_{k+1})$ .

Additionally, the conditions constructed by  $\text{rem}()$  and  $\text{nin}()$  ensure (3), with  $\text{rem}()$  ensuring that no occurrence of existentially bound graphs  $C_j$  with  $j \leq k$  will be deleted and  $\text{nin}()$  ensuring that no occurrences of universally bound graphs  $C_j$  with  $j \leq k$  will be inserted. With this, it is guaranteed that the partial consistency will not be decreased.

**Definition 3.2 (general application condition).** Let  $\rho = L \hookrightarrow K \hookrightarrow R$  be a plain rule and  $c$  a constraint in EANF. Let  $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$  be the subcondition of  $c$  at layer  $k$  with  $k$  being an odd number.

The application condition  $\text{ap}_k$  of  $c$  at layer  $k$  with  $C' = C_k$ , if  $e = \text{false}$ , and  $C' \in \mathcal{U}(C_k, C_{k+1})$  otherwise is defined as:

$$\text{ap}(k, C') := \left( \bigvee_{P \in \text{eol}(L, C_k)} \text{nex}(P, C') \wedge \text{rep}(P, C') \right) \wedge \text{nin} \wedge \text{rem} \wedge \text{nwo} \quad (3.1)$$

with

- Let  $a^p : C_k \hookrightarrow C'$  be a partial morphism of  $a_{k+1}$  and  $i_L$  and  $i_P$  the inclusions of  $L$  in  $P$  and  $P$  in  $Q$ , respectively.

$$\text{nex}(P, C') := \begin{cases} \exists(i_L^P : L \hookrightarrow P, \text{true}) & \text{if } e = \text{false} \\ \bigwedge_{Q \in \text{eol}(P, a^p)} \exists(i_L^P : L \hookrightarrow P, \neg\exists(i_P^Q : P \hookrightarrow Q, \text{true})) & \text{otherwise} \end{cases}$$

- If  $i_L^P(L \setminus K) \cap i_{C_k}^P(C_k) \neq \emptyset$ , we set

$$\text{rep}(C, C') := \text{true}.$$

Otherwise, let  $P'$  be the graph derived the transformation  $P \Rightarrow_{\rho, m} P'$ . Then,  $P'$  is an overlap of  $R$  and  $C_k$ . If this transformation does not exist, we set  $\text{rep}(C, C') := \text{false}$ . Let  $a^p : C_k \hookrightarrow C'$  be a partial morphism of  $a_{k+1}$ , then

$$\text{rep} := \begin{cases} \text{false} & \text{if } e = \text{false} \\ \bigvee_{Q \in \text{eol}(P', a^p)} \text{Left}(\exists(i_R^Q : R \hookrightarrow Q, \text{true}), \rho) & \text{otherwise} \end{cases}$$

- Let  $E$  be the set of all graphs  $C_j$  of  $c$  with  $j \leq k$ ,  $j$  being odd and let  $\mathbf{P}_{C_j}$  be the set all overlaps  $P'$  of  $L$  and  $C_j$  with  $i_L^{P'}(L \setminus K) \cap i_{C_j}^{P'}(C_j) \neq \emptyset$ .

$$\text{rem} := \bigwedge_{C \in E} \bigwedge_{P' \in \mathbf{P}_{C_j}} \neg\exists(i_L^{P'} : L \hookrightarrow P', \text{true})$$

- Let  $U$  be the set of all graphs  $C_j$  of  $c$  with  $j \leq k$ ,  $j$  being even and let  $\mathbf{P}_{C_j}$  be the set of all overlaps  $P'$  of  $R$  and  $C_j$  with  $i_R^{P'}(R \setminus K) \cap i_{C_j}^{P'}(C_j) \neq \emptyset$ .

$$\text{nin} := \bigwedge_{C \in U} \bigwedge_{P' \in \mathbf{P}_{C_j}} \text{Left}(\neg\exists(i_R^{P'} : R \hookrightarrow P', \text{true}), \rho)$$

5. Let  $E$  be the set of all overlaps of  $L$  and  $C_k$ , such that each  $P' \in E$  is also an overlap of  $L$  and  $C'' \in \mathcal{U}(C_k, C_{k+1})$ ,  $i_{C_k}^{P'} \models \exists(a_k^p : C_k \hookrightarrow C'', \text{true})$  and  $i_L^{P'}(L \setminus K) \cap i_{C''}^{P'}(C'' \setminus C_k) \neq \emptyset$ .

$$\text{nwo} := \bigwedge_{P' \in E} \neg \exists(i_L^{P'} : L \hookrightarrow P', \text{true})$$

**Example 3.1.** Ein tolles Beispiel.

Note that  $\text{ap}(k, C')$ , for any  $C' \in \mathcal{U}(C_k, C_{k+1})$ , will only be evaluated with **true** if an occurrence  $p$  of  $C_k$  with  $p \not\models \exists(a_k^p : C_k \hookrightarrow C', \text{true})$  and  $\text{tr}_t \text{op} \models \exists(a_k^p : C_k \hookrightarrow C', \text{true})$  exists. For any smaller improvements, i.e a similar improvement for a subgraph  $C'' \in \mathcal{U}(C_k, C_{k+1})$  of  $C'$ ,  $\text{ap}(k, C')$  would be evaluated with **false**. For any bigger improvements, i.e the same improvement for a supergraph  $C'' \in \mathcal{U}(C_k, C_{k+1})$  of  $C'$ ,  $\text{ap}(k, C')$  would be evaluated with **false** too, if  $p \models \exists(a_k^p : C_k \hookrightarrow C', \text{true})$ . In both cases, the application condition would prohibit the transformation, even if it would be consistency increasing. To resolve this problem, multiple application conditions can be combined by

$$\bigvee_{C' \in \mathcal{U}(C_k, C_{k+1})} \text{ap}(k, C').$$

This application condition will be evaluated with **true** if the cases described above do appear, with the drawback that this leads to a huge condition, even if duplicate conditions are removed. At least all duplicates of  $\text{rem}()$ ,  $\text{nin}()$  and  $\text{nwo}()$  can be removed, since they are identical for each  $\text{ap}(k, C')$  and only need to be constructed once.

In general, these application conditions are a trade-off between conditions-size and restrictiveness. They are very restrictive, since they do not allow any deletions of occurrences of existentially bound and insertions of universally bound graphs. For example, any of these application conditions with the rule **moveFeature** and constraint  $c_1$  will be equivalent to **false**;  $\text{nwo}()$  will always be evaluated with **false** since **moveFeature** does remove elements of the existentially bound graph  $C_2^1$ . A change of the conditions constructed by  $\text{nwo}()$  such that it is checked whether two nodes of the type **Feature** are connected to a node **Class** will yield application conditions that are satisfiable with **moveFeature**, but for a similar rule moving two nodes of type **Feature**, this newly constructed  $\text{nwo}()$  would still be evaluated with **false**. Therefore, this only leads to a slight decrease of restrictiveness.

The conditions constructed by  $\text{rem}()$  and  $\text{nin}()$  could be changed in a similar fashion. For  $\text{rem}()$  and the universally bound graph  $C_k$ , by checking whether there does exist an additional occurrence  $p$  of  $C_{k+1}$  such that  $p \models \text{sco}_{\text{cond}(c_{\max}^G, c)}(c_{\max}^G - k + 2)$  and for  $\text{nin}()$ , by checking whether an introduced occurrence  $p$  of  $C_k$  does satisfy  $\text{sco}_{\text{cond}(c_{\max}^G, c)}(c_{\max}^G - k + 1)$ . The construction of these is similar to the construction of consistency guaranteeing application conditions as introduced by Habel and Pennemann [1], which is known to construct huge application conditions. Also, they do get more and more restrictive for increasing  $k$ , since the number of conditions constructed by  $\text{rem}()$  and  $\text{nin}()$  also increases.

For constraints of the form  $c := \forall C_1 \exists C_2$  it can be shown that  $\text{ap}(1, C_2)$  is not only a direct increasing, but also a direct improving application condition.

Let us now show that the construction above generates consistency increasing application conditions.

**Theorem 3.1.** *Let a graph  $G$ , a constraint  $c$  in EANF with  $G \not\models c$  and a plain rule  $\rho$  be given. Then,  $\rho' = (\rho, \text{ap}(c_{\max}^G + 1, C))$  with  $C \in \mathcal{U}(C_{c_{\max}^G + 1}, C_{c_{\max}^G + 2})$  is a consistency increasing rule.*

*Proof.* Let  $t : G \implies_{\rho', m} H$  be a transformation and  $k = c_{\max}^G + 1$ . We show, by contradiction, that this transformation is direct consistency increasing by showing that (2.1), (2.2), (2.3), (2.4) and (2.5) are satisfied. With that,  $\rho'$  is a consistency increasing rule. Let  $\text{sco}_c(k) = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$  be the subcondition of  $c$  at layer  $k$ . Let

$$\mathbf{G} := \begin{cases} \mathcal{U}(C_k, C_{k+1}) & \text{if } e \neq \text{false} \\ \{C_k\} & \text{otherwise.} \end{cases}$$

1. Assume that (2.1) does not hold. Then, a morphism  $p : C_k \hookrightarrow G$  exists, such that  $p \models \text{part}(1, e, C')$ ,  $\text{tr}_t \circ p$  is total and  $\text{tr}_t \circ p \not\models \text{part}(1, e, C')$  for a graph  $C' \in \mathbf{G}$ . Therefore, an overlap  $P$  of  $L$  and  $C'$  such that  $i_{C_k}^P \models \exists(a_k^p : C_k \hookrightarrow C', \text{true})$  with  $i_L^P(L \setminus K) \cap i_{C'}^P(C' \setminus C_k) \neq \emptyset$  exists and  $m \models \exists(i_L : L \hookrightarrow P, \text{true})$  holds. Thus  $\text{nwo}()$  and consequently also  $\text{ap}(k, C)$  cannot be satisfied.

2. Assume that (2.2) does not hold and let

$$d := \begin{cases} \text{part}(1, e, C_{k+1}) & \text{if } e \neq \text{false} \\ \text{false} & \text{otherwise.} \end{cases}$$

Then, a morphism  $p' : C_k \hookrightarrow H$  with  $p' \not\models d$  exists, such that no morphism  $p : C_k \hookrightarrow G$  with  $\text{tr}_t \circ p = p'$  exists. Therefore, an overlap  $P$  of  $R$  and  $C_k$  with  $i_R(R \setminus K) \cap i_{C_k}^P(C_k) \neq \emptyset$  exists, such that  $m \models \text{Left}(\exists(i_R^P : R \hookrightarrow P, \text{true}), \rho)$  and  $m$  does not satisfy  $\text{nin}()$ .

3. Assume that (2.3) does not hold. Then, no morphism  $p : C_k \hookrightarrow G$  with  $p \not\models \text{part}(1, e, C)$ , such that  $\text{tr}_t \circ p$  is not total or  $\text{tr}_t \circ p \models \text{part}(1, e, C)$  and  $\text{tr}_t \circ p$  is total exists, for any  $C \in \mathbf{G}$ . Then, no overlap  $P$  of  $L$  and  $C_k$  with  $i_L^P(L \setminus K) \cap i_{C_k}^P(C_k) \neq \emptyset$  exists, such that  $m \models \text{nex}(P, C)$ .

Also, no overlap  $P$  of  $C_k$  and  $L$  with  $i_L^P(L \setminus K) \cap i_{C_k}^P(C_k) = \emptyset$  exists, such that  $m \models \text{nex}(P, C)$  and

Also, no overlap  $P'$  of  $C$  and  $R$  with  $i_R^{P'}(R \setminus K) \cap i_C^{P'}(C) \neq \emptyset$  exists, such that  $m \models \text{Left}(\exists(i_R^{P'} : R \hookrightarrow P', \text{true}), \rho)$  and therefore  $m \not\models \text{rep}(P, C)$  for all overlaps  $P$  of  $L$  and  $C_k$  with  $i_L^P(L \setminus K) \cap i_{C_k}^P(C_k) = \emptyset$ . It follows that  $m \not\models \text{nex}(P, C) \wedge \text{rep}(P, C)$  for all  $P \in \text{ol}(L, C_k)$  and hence  $m \not\models \text{ap}(c_{\max}^G, C)$ .

4. Assume that (2.4) does not hold. Then, there does exist a morphism  $p : C_j \hookrightarrow G$  with  $j < k$  and  $i$  being even, such that no morphism  $p' : C_j \hookrightarrow G$  with  $\text{tr}_t \circ p' = p$  exists. Then, an overlap  $P$  of  $C_j$  and  $R$  with  $i_R(R \setminus K) \cap i_{C_j} \neq \emptyset$  exists, such that  $m \models \text{Left}(\exists(i_R : R \hookrightarrow P, \text{true}), \rho)$ . It follows that  $m \not\models \text{rem}$  and with that  $m \not\models \text{ap}(c_{\max}^G, C)$ .
5. Assume that (2.5) does not hold. Then, there does exist a morphism  $p : C_j \hookrightarrow G$  with  $j < k$  and  $j$  being odd, such that  $\text{tr}_t \circ p$  is not total for any  $C \in \mathbf{G}$ . Then, an overlap  $P$  of  $C_j$  and  $L$  with  $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$  exists, such that  $m \models \exists(i_L : L \hookrightarrow P, \text{true})$ . It follows that  $m \not\models \text{rem}$  and with that  $m \not\models \text{ap}(c_{\max}^G, C)$ .

In total follows that if  $m \models \text{ap}(c_{\max}^G, C)$ , then  $t$  is a direct minimal improving transformation.  $\square$

**Lemma 3.3.** *Let  $G$  be a graph,  $c$  a constraint in EANF, with  $c_{\max}^G < \text{nl}(c)$ , and  $\rho = L \leftarrow K \hookrightarrow R$  a plain rule. Let  $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$  be the subcondition of  $c$  at layer  $c_{\max} + 1$  and  $\text{ap}(c_{\max}, C')$  the application condition constructed by definition 3.2 with  $C' \in \mathcal{U}(C, C_{k+1})$  for any  $C \in \mathcal{G}_c^G$ . If*

$$((R \setminus K) \cap C_{k+1}) \cup ((L \setminus K) \cap C_{k+1}) = \emptyset$$

$\text{ap}(c_{\max}, C')$  can be replaced by *false*.

*Proof.* There does not exist an overlap  $P$  of  $C_k$  and  $L$  with  $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$  and  $\text{rep}(P, C')$  will be equal to *false*, if  $C_k = C_{k+1}$ , or equal to  $\bigvee_{P' \in \mathbf{P}} \text{Left}(\exists(i_R : R \hookrightarrow P', \text{true}), \rho)$ . Since the set  $\mathbf{P}$  has to be empty, this expression can be replaced by *false*. It follows that  $\text{rep}(P, C') = \text{false}$  for all  $P \in \text{ol}(L, C_k)$  and therefore  $\text{ap}(c_{\max}, C')$  will always be evaluated to *false*.  $\square$

### 3.1 potentially minimal improving rules

**Definition 3.4 (basic improving rule).** *Let a constraint  $c$  and a plain rule  $\rho = L \xleftarrow{l} K \xrightarrow{r} R$  be given. The rule  $\rho$  is called basic improving w.r.t  $c$  at layer  $k$  with  $C_k \subset P \subseteq C_{k+1}$ ,  $p : C_k \hookrightarrow P$  being the inclusion, and  $k \in \{1, 3, \dots, \text{nl}(c)\}$ , if*

$$(L \setminus K) \cap (C_{k-1} \cup (C_{k+1} \setminus C_k)) = \emptyset \quad (3.2)$$

and

$$(R \setminus K) \cap C_k = \emptyset \quad (3.3)$$

and either 1. or 2. applies.

1. The rule  $\rho$  deletes elements of  $C_k \setminus C_{k-1}$ :

$$L \subseteq C_k \wedge L \setminus K \neq \emptyset \quad (3.4)$$

Then,  $\rho$  is called a deleting basic improving rule.

2. The rule  $\rho$  creates an instance of  $P$ :

$$L = C_k \wedge P \subseteq R \quad (3.5)$$

and  $p$  is a partial morphism of  $r$ . Then,  $\rho$  is called an inserting basic improving rule.

**Definition 3.5 (application conditions for basic improving rules).** Let a constraint  $c$  in EANF and a basic improving rule  $\rho = L \xleftarrow{l} K \xrightarrow{r} R$  w.r.t  $c$  at layer  $k$  with  $C_k \subseteq P \subseteq C_{k+1}$  be given. We define the application condition for  $r$  as:

1. If  $\rho$  is a deleting potentially minimal improving rule:

$$\text{ap}_{\text{pi}}(j, P) := \begin{cases} \bigvee_{P \in \text{eol}(L, C_k)} \bigwedge_{P' \in \text{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg \exists(i_P : P \hookrightarrow P', \text{true})) & , \text{ if } j = k \\ \text{false} & , \text{ if } j \neq k. \end{cases}$$

2. If  $\rho$  is an inserting potentially minimal improving rule:

$$\text{ap}_{\text{pi}}(j, P) := \begin{cases} \neg \exists(a_k^p : L \hookrightarrow P, \text{true}) & , \text{ if } j = k \\ \text{false} & , \text{ if } j \neq k \end{cases}$$

**Theorem 3.2.** Let a graph  $G$ , a constraint  $c$  in EANF, with  $G \models \text{part}(c_{\max}^G, c, C)$  and  $\text{part}(c_{\max}^G, c, C) \in \mathcal{P}_c^G$ , and a potentially minimal improving rule  $\rho = L \xleftarrow{l} K \xrightarrow{r} R$  at layer  $c_{\max}^G$  with  $C_{c_{\max}^G+1} \subseteq P \subseteq C_{c_{\max}^G+2}$  be given. Then,  $\rho' = (\rho, \text{ap}_{\text{pi}}(c_{\max}^G + 1, P))$  is a direct minimal consistency improving rule.

*Proof.* Let  $t : G \Rightarrow_{\rho', m} H$  be a transformation,  $k = c_{\max}^G + 1$  and  $e$  be the subcondition of  $c$  at layer  $k$ . We show that  $t$  is a direct minimal consistency improving transformation. First, we show that equation (2.1) is satisfied. Let  $p : C_k \hookrightarrow G$  be a morphism. If  $\rho$  is a deleting potentially minimal improving rule, either 1. or 2. applies, if  $\rho$  is an inserting and not a deleting potentially minimal improving rule, only 2. applies, because  $\rho$  cannot destroy any occurrences of  $C_k$  in  $G$ .

1. If  $p(C_k) \cap m(L \setminus K) \neq \emptyset$ ,  $\text{tr}_t \circ p$  is not total, since at least one element of  $p(C_k)$  has been deleted by  $t$  and  $p$  does satisfy  $\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \models \text{part}(1, e, C') \wedge \text{tr}_t \circ p \text{ is total}) \implies \text{tr}_t \circ p \models \text{part}(1, e, C')$ .
2. If  $p(C_k) \cap m(L \setminus K) = \emptyset$ ,  $\text{tr}_t \circ p$  is total, since no element of  $p(C_k)$  has been deleted by  $t$ . Because (3.2) holds,  $t$  does not delete any elements of  $C_{k+1} \setminus C_k$  and  $p \models \text{part}(1, e, C') \implies \text{tr}_t \circ p \models \text{part}(1, e, C')$  for all  $C' \in \mathcal{U}(C_k, C_{k+1})$ .

With 1. and 2. follows that (2.1) is satisfied.

Second, we show that equation (2.2) is satisfied. Let  $p' : C_k \hookrightarrow H$  be a morphism. Because (3.3) is satisfied,  $t$  does not create any elements of  $C_k$  and there must exist a morphism  $p : C_k \hookrightarrow G$  with  $\text{tr}_t \circ p = p'$ . It follows that (2.2) is satisfied.

Third, we show that (2.3) is satisfied. We consider the cases that firstly,  $\rho$  is a deleting minimal potentially improving rule and secondly, that  $\rho$  is an inserting and not a deleting minimal potentially improving rule.

1. If  $\rho$  is a deleting potentially minimal improving rule, the condition  $\text{ap}_{\text{pi}}(k, P) = \exists(b : L \hookrightarrow C_k, \neg\exists(a_{k+2} : C_k \hookrightarrow C_{k+1}, \text{true}))$  is satisfied by  $m$ . Therefore a morphism  $p : C_k \hookrightarrow G$  with  $p \not\models \neg\exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, \text{true}) = \text{part}(1, e, C_{k+1})$  and  $m = p \circ b$  must exist. Since  $\rho$  is a deleting rule, at least one element of  $p(C_k)$  has been deleted by  $t$  and  $\text{tr}_t \circ p$  is not total. It follows that (2.3) is satisfied.
2. If  $\rho$  is an inserting and not a deleting potentially minimal improving rule,  $\text{ap}_{\text{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \text{true})$  is satisfied by  $m$ . Because  $L = C_k$ ,  $m \models \neg\exists(b : C_k \hookrightarrow P, \text{true}) = \text{part}(1, e, P)$ . Since (3.5) is satisfied,  $\text{tr} \circ p$  is total and  $\text{tr}_t \circ p \models \text{part}(1, e, P)$ . Therefore, (2.3) is satisfied.

Last, since (3.3) and (3.2) are satisfied,  $\rho$  cannot create any occurrence of  $C_j$  with  $j \leq k$  and  $j$  being even and  $\rho$  cannot delete any occurrences of  $C_j$  with  $j \leq k$  and  $j$  being odd. Therefore, (2.4) and (2.5) are satisfied.

In total follows that  $t$  is a direct minimal consistency improving transformation and  $\rho'$  is a direct minimal consistency improving rule.  $\square$

**Theorem 3.3.** *Let a constraint  $c$  in EANF, a basic increasing rule  $\rho$  with  $P$  and a consistency increasing transformation  $t : G \implies_{\rho, m} H$  w.r.t  $c$  be given. Then,  $m \models \text{ap}_{\text{pi}}(c_{\max}^G, P)$ .*

*Proof.* Let  $k = c_{\max}^G$ .

1. If  $c$  is a inserting rule, then  $\text{ap}_{\text{pi}}(c_{\max}^G, P) = \neg\exists(a_k^P : L \hookrightarrow P, \text{true})$ . Let  $n : R \hookrightarrow H$  be the comatch of  $t$ . Since  $t$  is a consistency increasing transformation, there does exist a morphism  $p : C_k \hookrightarrow G$  with  $p \not\models \exists(a_k^P : C_k \hookrightarrow P, \text{true})$ ,  $\text{tr}_t \circ p \models \exists(a_k^P : C_k \hookrightarrow P, \text{true})$  such that  $p'(P) \cap n(R \setminus K)$  for the morphism  $p' : P \hookrightarrow H$  with  $p = p \circ a_k^P$ . Assume that  $m \models \exists(a_k^P : L \hookrightarrow P, \text{true})$ . Then, every morphism  $p : C_k \hookrightarrow G$  with  $\text{tr}_t \circ p \models \exists(a_k^P : C_k \hookrightarrow P, \text{true})$  with  $p'(P) \cap n(R \setminus K)$  for the morphism  $p' : P \hookrightarrow H$  with  $p = p' \circ a_k^P$  already satisfies  $\exists(a_k^P : C_k \hookrightarrow P, \text{true})$ . This is a contradiction.
2. If  $\rho$  is a deleting rule,  $\text{ap}_{\text{pi}}(k, P) = \bigvee_{P \in \text{ol}(L, C_k)} \bigwedge_{P' \in \text{ol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \text{true}))$ . Since  $t$  is a consistency increasing transformation, there does exist a morphism  $p : C_k \hookrightarrow G$ , such that  $p \not\models \exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$  and  $\text{tr}_t \circ p$  is not total. Assume that  $m \not\models \text{ap}_{\text{pi}}(k, P)$ . Then, for each overlap  $Q$  of  $L$  and  $C_k$ , the inclusion  $i_{C_k}$  does satisfy  $\exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$ . This is a contradiction, because  $p(C_k) \cap m(L \setminus K) \neq \emptyset$  has to hold.

$\square$

**Lemma 3.6.** *Let a constraint  $c$  in EANF and inserting basic increasing rules  $\rho_1, \rho_2$  at layer  $k$  with  $P_1$  and  $P_2$ , respectively, be given. Then, the rules  $\rho'_1 = (\rho_1, \text{ap}_{\text{pi}}(k, P_1))$  and  $\rho'_2 = (\rho_2, \text{ap}_{\text{pi}}(k, P_2))$  are sequentially independent.*

*Proof.* Let  $G_1 \implies_{\rho'_1, m_1} G_2 \implies_{\rho'_2, m_2} G_3$  be a sequence of transformations. First, note that this sequence can only exist if  $P_1 \not\subseteq P_2$  and  $P_2 \not\subseteq P_1$ . Since both rules are do not

insert elements of  $C_k$ , a morphism  $d_1 : L_2 \hookrightarrow D_1$  with  $m_2 = h_1 \circ d_1$  exists. Because both rules do not delete any elements a morphism  $d_2 : R_1 \hookrightarrow D_2$  with  $m_1^* = g_2 \circ d_2$  exists. Because  $\rho_2$  is not a basic increasing rule with  $P_1$ ,  $h_2 \circ d_2 \models R(\text{app}_\text{pi})$  and because  $\rho_1$  does not delete any elements of  $P_2$  it holds that  $g_1 \circ d_1 \models R(\text{app}_\text{pi})$ .  $\square$

**Definition 3.7 (repairing rule set).** Let a constraint  $c$  in EANF and a set of rules  $\mathcal{R}$  be given. Then,  $\mathcal{R}$  is called a repairing rule set for  $c$  at layer  $k$  if for all graphs  $G$  with  $k = c_{\max}$  a sequence

$$G = G_0 \Rightarrow_{r_0} \dots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that  $r_j \in \mathcal{R}$  for all  $j \in \{0, \dots, n-1\}$  and  $H \models_{k+2} c$ .

**Corollary 3.8.** Let a constraint  $c$  in EANF and a set of rules  $\mathcal{R}$  be given. If  $\mathcal{R}$  is a repairing rule set for  $c$  at layer  $k$ ,  $\mathcal{R}$  is a repairing rule set w.r.t  $c$  at layer  $j$  for all  $k < j \leq \text{nl}(c)$ .

**Corollary 3.9.** Let a constraint  $c$  in EANF and a repairing rule set  $\mathcal{R}$  for  $c$  at layer  $k$ , for all  $k \in \{1, 3, \dots, \text{nl}(c)\}$ , be given. Then, for all graphs  $G$ , a sequence

$$G = G_0 \Rightarrow_{r_0} \dots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that  $r_j \in \mathcal{R}$  for all  $j \in \{0, \dots, n-1\}$  and  $H \models c$ .

**Definition 3.10 (decomposition of a graph).** Let graphs  $G_0 \subset G_1$  be given.

A decomposition of  $G_1$  with  $G_0$  is a minimal set

$$\mathbf{P} \subseteq \{P_v \mid v \in V_{C_1 \setminus C_0}\}$$

of subgraphs of  $G_1$ , such that every element of  $G_1$  is contained in at least one  $P \in \mathbf{P}$  and every  $P_v$  is constructed in the following way:  $G_0 \subset P_v$ ,  $v \in P$  and for all nodes  $v' \in P \setminus C_k$  it holds that  $P$  contains all edges  $e \in E_{G_1 \setminus G_0}$  and all nodes  $u \in V_{G_1}$  such that either  $\text{tar}(e) = v' \wedge \text{src}(e) = u$  or  $\text{src}(e) = u \wedge \text{tar}(e) = v'$  holds.

**Lemma 3.11.** Let graphs  $G_0 \subset G_1$  and a decomposition  $\mathbf{P}$  of  $G_1$  with  $G_0$  be given. Then, for each pair  $P, P' \in \mathbf{P}$  with  $P \neq P'$  the following holds:

$$(P \setminus C_k) \cap (P' \setminus C_k) = \emptyset$$

*Proof.* Assume that  $(P \setminus C_k) \cap (P' \setminus C_k) \neq \emptyset$ , therefore a node  $v \in G_1 \setminus G_0$  with  $v \in P \cap P'$  exists. By the construction of  $P$  and  $P'$  it follows that  $P = P_v$  and  $P' = P_v$  and therefore  $P = P'$ . This is a contradiction.  $\square$

**Lemma 3.12.** Let graphs  $G_0 \subset G_1$  and a decomposition  $\mathbf{P}$  of  $G_1$  with  $G_0$  be given. Then,

$$G_1 = \bigcup_{P \in \mathbf{P}} P.$$

*Proof.* Let  $H := \bigcup_{P \in \mathbf{P}} P$ . Firstly, we show that  $H \subseteq G_1$ . Since every  $P \in \mathbf{P}$  is a subgraph of  $G_1$  it follows that  $V_H \subseteq V_{G_1}$  and  $E_H \subseteq E_{G_1}$ .

Secondly, we show that  $G_1 \subseteq H$ . Let  $u \in V_{G_1}$  be a node, if  $u \in V_{G_0}$ , then  $u$  is contained in every  $P \in \mathbf{P}$  and therefore  $u \in V_H$ . Otherwise, if  $u \notin V_{G_0}$ , then  $u$  has to be, by the definition of  $\mathbf{P}$ , contained in at least one  $P \in \mathbf{P}$  and  $V_{G_1} \subseteq V_H$  follows. Let  $e \in E_{G_1}$  be an edge. If  $e \in E_{G_0}$ , then  $e$  is contained in every  $P \in \mathbf{P}$  and  $e \in E_H$ . Otherwise, if  $e \notin E_{G_0}$ , by the definition of  $\mathbf{P}$ ,  $e$  has to be contained in at least one  $P \in \mathbf{P}$ . It follows that  $e \in E_H$  and with that  $E_{G_1} \subseteq E_H$ .  $\square$

**Theorem 3.4.** *Let a constraint  $c$  in EANF and a set of rules  $\mathcal{R}$  be given. Then,  $\mathcal{R}$  is a repairing set of  $c$  at layer  $k \leq \text{nl}(c)$  if either 1 or 2 applies.*

1. *For any universally bound graph  $C_j$  at layer  $j \leq k$  of  $c$ ,  $(\rho, \text{ap}_{\text{pi}}(j, C_{j+1})) \in \mathcal{R}$  and  $\rho$  is a deleting potentially minimal improving rule at layer  $j$  with  $C_{j+1}$ , such that  $\rho$  only deletes edges of  $C_j$ .*
2. *A decomposition  $\mathbf{P}$  of  $C_k$  with  $C_{k-1}$  exists, such that for each  $P \in \mathbf{P}$  a rule  $(\rho, \text{ap}_{\text{pi}}(k, P)) \in \mathcal{R}$  exists, such that  $\rho$  is an inserting basic improving rule at layer  $k$  with  $P$ .*

*Proof.* Let a constraint  $c$  in EANF, a rule set  $\mathcal{R}$  and a graph  $G$  with  $k = c_{\max}$  and  $c_{\max} < \text{nl}(c)$  be given. We show that a sequence  $G = C'_0 \Rightarrow \dots \Rightarrow C'_n = H$  with rules of  $\mathcal{R}$  exists, such that  $H \models_{k+2} c$  if 1. or 2. of theorem 3.4 is satisfied.

1. Assume that 1. of theorem 3.4 holds. Let  $(\rho, \text{ap}_{\text{pi}}(j, C_{j+1})) \in \mathcal{R}$ , such that  $\rho = L \xleftarrow{l} K \xrightarrow{r} R$  is a deleting potentially minimal improving rule at layer  $j \leq k$  with  $C_{j+1}$  and  $C_j$  is a universally bound graph of  $c$ . Then,  $\text{ap}_{\text{pi}}(j, C_{j+1}) = \exists(b : L \hookrightarrow C_j, \neg \exists(a_j : C_j \hookrightarrow C_{j+1}, \text{true}))$ . Let  $q : C_j \hookrightarrow G$  be a morphism such that  $q \not\models \exists(a_j : C_j \hookrightarrow C_{j+1}, \text{true})$ . Since  $L \subseteq C_j$ , we can construct a morphism  $m_1 : L \hookrightarrow G$  with  $m_1 = q \circ b$  and therefore  $m_1 \models \text{ap}_{\text{pi}}(j, C_{j+1})$ . Since  $r$  only deletes edges, a transformation  $t : G = G_0 \Rightarrow_{r, m_1} G_1$  exists and  $\text{tr}_t \text{op}$  is not total. Because  $r$  does not insert any elements of  $C_j$ :

$$|\{q : C_k \hookrightarrow G_0 \mid q \not\models d\}| < |\{q : C_k \hookrightarrow G_1 \mid q \not\models d\}|$$

with  $d = \exists(a_j : C_j \hookrightarrow C_{j+1}, \text{true})$ . By iteratively applying this construction, we can generate a finite sequence of transformations

$$G = G_0 \Rightarrow_{r, m_1} G_1 \Rightarrow_{r, m_2} \dots \Rightarrow_{r, m_n} G_n = H$$

such that  $|\{q : C_k \hookrightarrow G_n \mid q \not\models d\}| = 0$  and therefore  $H \models_j c$ . With lemma 2.9,  $H \models_{k+2} c$  and  $H \models c$  follows.

2. Assume that 2. of theorem 3.4 holds. Let  $\rho_0 = (\rho, \text{ap}_{\text{pi}}(k, P)) \in \mathcal{R}$ , such that  $\rho$  is an inserting basic improving rule of  $c$  at layer  $k$  with  $P \in \mathbf{P}$ . Then,  $\text{ap}_{\text{pi}}(k, P) = \neg \exists(b : L \hookrightarrow P, \text{true})$ . Let  $q_0 : C_k \hookrightarrow G$  be a morphism, such that  $q_0 \not\models \exists(a'_k : C_k \hookrightarrow P, \text{true})$

with  $a'_k$  being a partial morphism of  $a_k$ . Since  $L = C_k$ , we set  $m_0 : C_k \hookrightarrow G$  with  $m_0 = q_0$ . It follows that  $m_0 \models \neg \exists(a'_k : C_k \hookrightarrow P, \text{true}) = \text{app}_\text{pi}(k, P)$ . Because  $r$  does not delete any elements, a transformation  $t_0 : G \Rightarrow_{r_0, m_0} G_1$  exists and  $\text{tr}_t \circ q \models \exists(a'_k : C_k \hookrightarrow P, \text{true})$ . We set  $q_1 = \text{tr}_{t_0} \circ q_0$  and apply the same method to  $q_1$ .

By iteratively applying this, we can construct a finite sequence of transformations

$$G \Rightarrow_{r_0, m_0} G_0 \Rightarrow_{r_1, m_1} \dots \Rightarrow_{r_n, m_n} G_n$$

such that  $m_i = \text{tr}_{t_{i-1}} \circ \dots \circ \text{tr}_{t_0} \circ m_0$  and  $q \models \exists(b_i : C_k \hookrightarrow P_i, \text{true})$  for all  $P_i \in \mathbf{P}$  with  $q = \text{tr}_{t_n} \circ q_n$ . Let  $p_i : P_i \hookrightarrow G_n$  be the morphism, such that  $q = p_i \circ b_i$ .

Now, we can construct a morphism  $p : C_{k+1} \hookrightarrow G$  with

$$p(e) := \begin{cases} p_1(e) & \text{if } e \in P_1 \\ & \vdots \\ p_j(e) & \text{if } e \in P_j. \end{cases}$$

Let  $e \in C_k$ , because  $q(e) = p_i \circ b_i(e)$  and  $q(e) = p_\ell \circ b_\ell(e)$  and  $b_i$  and  $b_\ell$  are both partial morphisms of  $a_k$ , it follows that  $b_i(e) = b_\ell(e)$  and therefore  $p_i(e) = p_\ell(e)$ . Because  $(P_i \cap P_\ell) \setminus C_k = \emptyset$  for all  $i \neq \ell$ ,  $p$  is a morphism and by the definition of  $p$  it follows that  $q = p \circ a_k$  and therefore  $q \models \exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$ .

By iteratively applying this whole construction to all occurrences of  $C_k$  that do not satisfy  $\exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$  the derived graph  $H$  does not contain any occurrences of  $C_k$  not satisfying  $\exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$  and therefore  $H \models_{k+2} c$ .

□

**Corollary 3.13.** *If a set of rule  $\mathcal{R}$  is a repairing set of  $c$  at layer  $k \leq \text{nl}(c)$  and 1. of theorem 3.4 applies, then  $\mathcal{R}$  is a repairing set of  $c$  at layer  $j$  for all  $k \leq j \leq \text{nl}(c)$ .*

**Lemma 3.14.** *Let a graph  $G_0$ , a constraint  $c$  in EANF and a repairing set  $\mathcal{R}$  at layer  $c_{\max}^{G_0} + 1$  be given, such that each rule in  $\mathcal{R}$  is a basic increasing. Then, for every sequence*

$$G_0 \Rightarrow_{\rho_0, m_0} G_1 \Rightarrow_{\rho_1, m_1} \dots \Rightarrow_{\rho_n, m_n} G_n$$

*such that  $\rho_i = (\rho, \text{app}_\text{pi}(c_{\max}^{G_0} + 1, P))$  with  $\rho \in \mathcal{R}$  being a consistency increasing rule at layer  $c_{\max}^{G_0} + 1$  with  $P$ , it holds that*

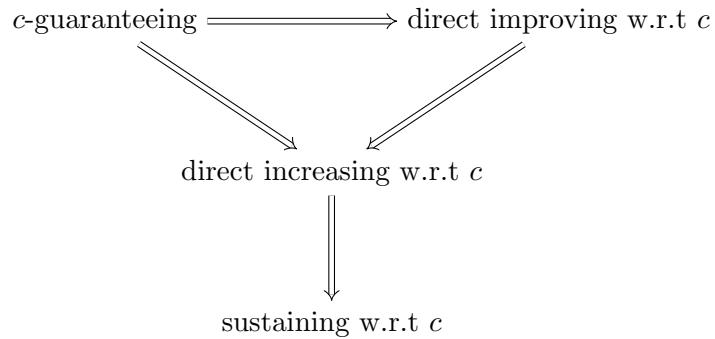


Figure 2: consistency relations for basic rules.

## References

- [1] A. Habel and K.-H. Pennemann. [Correctness of high-level transformation systems relative to nested conditions](#). *Mathematical Structures in Computer Science*, 19(2):245–296, 2009.
- [2] J. Kosiol, D. Strüber, G. Taentzer, and S. Zschaler. [Sustaining and improving graduated graph consistency: A static analysis of graph transformations](#). *Science of Computer Programming*, 214:102729, 2022.
- [3] C. Sandmann and A. Habel. [Rule-based graph repair](#). *arXiv preprint arXiv:1912.09610*, 2019.