Rule-based Graph Repair using Minimally Restricted Consistency-Improving Transformations

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Abstract

Contents

1 Preliminaries

Definition 1.1 (**subgraph**). *Let G_1 and G_2 be graphs. The graph G_2 is called a* subgraph *of G_1 if an injective morphism $f : G_2 \to G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective.*

Definition 1.2 (**uppergraph**). *Let G_1 and G_2 be graphs with $G_1 \subseteq G_2$. A graph C is called an* upper-graph *of G_1 w.r.t G_2, if $G_1 \subset C \subseteq G_2$ The set of uppergraphs of G_1 w.r.t. G_2 is denoted by $\mathcal{U}(G_1, G_2)$. If $G_1 = G_2$, we set $\mathcal{U}(G_1, G_2) = \{G_1\}$.*

Definition 1.3 (**overlap**). *Let G and G' be graphs. A graph H is called an* overlap of G and G' *if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by* $\mathrm{ol}(G, G')$.

Definition 1.4 (**overlap at morphism**). *Let C,G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an* overlap of G and C' at p *if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by* $\mathrm{ol}_p(G, C')$.

Definition 1.5 (**partial morphism**). *Let $f : G_1 \to G_2$ and $g : G_3 \to G_4$ be morphisms. The morphism g is called a* partial morphism *of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.*

Definition 1.6 (**nested graph condition**). *A* graph condition *over a graph C_0 is inductively defined as follows:*

- true *is a graph condition over every graph.*
- $\exists(a : C_0 \hookrightarrow C_1, d)$ *is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1.*
- $\neg d$ *is a graph condition over C_0 if d is a graph condition over C_0.*
- $d_1 \wedge d_2$ *and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0.*

Conditions over the empty graph $\emptyset$ are called constraints. *Every injective morphism $p : C_0 \hookrightarrow G$ satisfies true. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c. An injective morphism satisfies $\neg d$ if it does not satisfy d, it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \wedge d_2$ if it satisfies d_1 or d_2. A graph G satisfies a constraint c, $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c. We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg\exists(a : C_0 \hookrightarrow C_1, \neg d)$.*

The nesting level nl *of a condition is defined as* $\mathrm{nl}(\mathsf{true} = 0$ *and* $\mathrm{nl}(\exists(a : P \to Q, d)) := \mathrm{nl}(d) + 1$.

Definition 1.7 (**alternating quantifier normal form (ANF)**[1]). *A graph condition c is in* alternating normal form *(ANF) if it is of the form*

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \ldots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 consistency increasing

In this section, we introduce the notion of *consistency increasing* transformations and rules, which allows to increase the consistency of a constraint layer by layer.

2.1 extended alternating quantifier normal form

To prevent the need of case discrimination, a new normal form for conditions, called *extended alternating quantifier normal form* (EANF), will be introduced. The sets of conditions in ANF and EANF do intersect and we show that both sets are expressively equivalent.

Definition 2.1 (extended alternating quantifier normal form). *A conditions c is in* extended alternating quantifier normal form *(EANF) if it is of the form $\forall(a_0 : C_0 \hookrightarrow C_1, d)$ and d is a condition in ANF.*

Note that, given a condition c in EANF, every subcondition of c at layer $1 \leq k \leq \mathrm{nl}(c)$ is universally bound, if k is an odd number and existentially bound, if k is an even number.

Lemma 2.2. *Any condition in ANF can be transformed into an equivalent condition in EANF and vice versa.*

Proof. "$\Longrightarrow$": Let a graph G and a constraint c in ANF be given. If c is universally, c is already in EANF.

If $c = \exists(a_0 : C_0 \hookrightarrow C_1, d)$, we show that c is equivalent to $c' := \forall(\mathrm{id}_{C_0} : C_0 \hookrightarrow C_0, c)$.

1. Let $p : C_0 \hookrightarrow G$ be a morphism, such that $q \models c$. Therefore a morphism $q : C_0 \to G$ with $q \models e$ and $p = q \circ a_1$ exists. Then, $p \models d$, since p is the only morphism from C_0 to G with $p = p \circ \mathrm{id}_{C_0}$ and $p \models c$.

2. Let $p : C_0 \hookrightarrow G$ be a morphism with $p \models c'$, therefore all morphisms $q : C_0 \hookrightarrow G$ with $p = q \circ \mathrm{id}_{C_0}$ satisfy c. Since $p = p \circ \mathrm{id}_{C_0}$, $p \models c$ follows immediately.

"$\Longleftarrow$": Let a graph G and a constraint c in EANF be given. If $c = \forall(a_0 : C_0 \hookrightarrow C_1, d)$ with $C_0 \neq C_1$, c is already in ANF.

Otherwise, the equivalence of c with d can be shown analogously to the first case.

□

2.2 conditions up to layer

Definition 2.3 (Layer of a subcondition). *Let c be a condition and d a subcondition of c. The layer of d is defined as* $\mathrm{lay}(d) := \mathrm{nl}(c) - \mathrm{nl}(d) - 1$.

We define a notion of partial consistency, called *satisfaction at layer*, which will be used for the definition of consistency increasing. First, two operators are introduced to modify given constraints on a certain layer.

Definition 2.4 (**substitution at layer**). *Let $c = Q(a : C_0 \hookrightarrow C_1, d)$ be a condition in ANF, such that the subcondition of c with layer $0 \leq k \leq \mathrm{nl}(c)$ is a condition over C_k. Let e be a condition over C_k. The* substitution in c at layer k with e, $\mathrm{sub}(k, c, e)$, *is recursively defined as:*

1. *If $k \leq 1$:*
$$\mathrm{sub}(0, c, e) := e$$

2. *If $k > 1$:*
$$\mathrm{sub}(k, c, e) := Q\big(a : C_0 \hookrightarrow C_1, \mathrm{sub}(k-1, d, e)\big)$$

Example 2.1. *Let the conditions $c := \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1 : C_1 \hookrightarrow C_2, \mathit{true}))$ and $d = \exists(a'_1 : C_1 \hookrightarrow C_3, e)$ be given. Then,*

$$\mathrm{sub}(2, c, d) = \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a'_1 : C_1 \hookrightarrow C_3, e)).$$

Through this, we define *condition up to layer*. Intuitively, a condition is cut off at a certain layer, by replacing the subcondition at this layer by true or false, depending on the quantifier, the replaced subcondition is bound by.

Definition 2.5 (**Condition up to layer**). *Let c be a condition in EANF and d be the subcondition of c at layer $0 \leq k \leq \mathrm{nl}(c)$. The* condition up to layer k *of c,* $\mathrm{cond}(k, c)$, *is defined as*

$$\mathrm{cond}(k, c) := \begin{cases} \mathrm{sub}(k, c, \mathit{true}) & \text{if } d \text{ is existentially bound} \\ \mathrm{sub}(k, c, \mathit{false}) & \text{if } d \text{ is universally bound.} \end{cases}$$

Example 2.2. *Let the condition $c = \forall(a_0 : C_0 \hookrightarrow C_1, \exists(a_1 : C_1 \hookrightarrow C_2, d))$ be given. Then,*

$$\mathrm{cond}(2, c) = \forall(a_0 : C_0 \hookrightarrow C_1, \mathit{true}).$$

Now, we are ready to define satisfaction at layer, which is a key ingredient to define consistency increasing.

Definition 2.6 (**Satisfaction at layer**). *Let G be a graph and c be a condition over C_0. A morphism $p : C_0 \hookrightarrow G$* satisfies c at layer k, *$p \models_k c$, if*

$$p \models \mathrm{cond}(k, c).$$

A graph G satisfies a constraint c at layer k, $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies $\mathrm{cond}(k, c)$. The biggest k with $G \models_k c$ such that no $j > k$ with $G \models_j c$ exists is denoted by $c^G_{\max}$.

Example 2.3.

The following lemmas arise as a direct consequence of the definition of satisfaction at layer. If a graph satisfies a constraint up to a certain layer, let c be the condition up to this layer, such that the subcondition at this layer is universally bound, the graph satisfies all constraints starting with c.

Lemma 2.7. *Let G be a graph $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, then for any condition f over C_k it holds that*

$$p \models \mathrm{sub}(k, c, f).$$

Proof. Let k be the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound, let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be this subcondition. Let $q : G_{k-1} \to G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false})$. This must exist, since $p \models_k c$ and k is the smallest number such that $p \models_k v$ and the subcondition of c with layer k is universally bound.

Therefore, there does not exist a morphism $q' : C_k \to G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \to G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$. □

As a direct consequence of the previous lemma, a graph satisfying a condition up to layer ending with $\forall(a : C \hookrightarrow C', \mathsf{false})$ also satisfies the whole constraint.

Lemma 2.8. *Let G be a graph, $p : C_0 \to G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\mathrm{lay}(d) = k$ is universally bound,*

$$p \models_k c \implies p \models c.$$

Proof. Follows immediately by using lemma 2.7 and setting f to the subcondition of c with layer $k + 1$. □

Lemma 2.9. *Let c be a condition in ANF over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$. Let $d = Q(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k + 2$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that*

$$p \models \mathrm{sub}\big(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f)\big)$$

with f being a $\overline{Q}$ bound condition over C'.

Proof. If $p \models c$, we can choose $C' = C_{k+2}$ and $f = e$.

If $p \not\models c$, there does not exists a j with $p \models_j c$ and the subcondition of c with layer j is universally bound and $Q = \exists$ follows immediately. We choose $C' = C_{k+1}$ and $f = \mathsf{true}$. Let $q : C_k \to G$ with $p = q \circ a_k \circ \ldots \circ a_1$ and $q \circ \ldots \circ a_\ell$ satisfying the condition up to $\ell - k$ of the subcondition of c at layer ℓ for all $0 \leq \ell \leq k$. This morphism must exists since $p \models_k c$ and $p \not\models c$. Let $q' : C_{k+1} \to G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a'_{k+2} has to be the identity and therefore $q' = q' \circ a'_{k+2}$. It follows that $q' \models \exists(a'_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ and therefore $p \models \mathrm{sub}\big(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f)\big)$. □

Definition 2.10 (partial condition). *Let c be a condition in ANF over C_0. Let d be the subcondition of c at layer $k + 1$. The* partial condition of c at layer k with C', $\mathrm{part}(k, c, C')$ *is defined as:*

1. *If d is universally bound, let $e = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$ be the subcondition of c at layer $k+2$ with $C_{k+1} \subseteq C' \subseteq C_{k+2}$:*

$$\operatorname{part}(k, c, C') := \operatorname{sub}\big(k+2, c, \exists(a : C_{k+1} \hookrightarrow C', \textsf{\textit{true}})\big)$$

2. *If $d = \exists(a : C_k \hookrightarrow C_{k+1}, f)$ is existentially bound with $C_k \subseteq C' \subseteq C_{k+1}$:*

$$\operatorname{part}(k, c, C') := \operatorname{sub}\big(k+1, c, \exists(a : C_{k+1} \hookrightarrow C', \textsf{\textit{true}})\big)$$

Definition 2.11 (**biggest partially satisfying condition**). *Let G be a graph, c a condition over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$.*

A partial condition $c = \operatorname{part}(c_{\max}, c, C')$ with $p \models c$ is a biggest partially satisfying condition *if there does not exist a graph $C' \subset C''$ with $p \models \operatorname{part}(c_{max}, c, C'')$. The graph C' is called a* biggest partially satisfying graph.

The set of biggest partially satisfying conditions of c is denoted by $\mathcal{P}_c^G$.

The set of all biggest partially satisfying graphs is denoted by $\mathcal{G}_c^G$.

2.3 minimal consistency improving

Definition 2.12 (**number of violations**). *Let G be a graph and c a constraint in EANF. The* number of violations $\operatorname{nvc}(j, G)$ at layer j in G *is defined as:*

1. *If $j < c_{\max} + 1$:*
$$\operatorname{nvc}(j, G) := 0$$

2. *If $j = c_{\max} + 1$, let $d = \forall(a_k : C_j \hookrightarrow C_{j+1}, e)$ be the subcondition of c at layer $j+1$.*

$$\operatorname{nvc}(j, G) := \sum_{C' \in \mathcal{U}(C_j, C_{j+1})} |\{q \mid q : C_{j+1} \hookrightarrow G \wedge q \not\models \operatorname{part}(1, e, C')\}|$$

3. *If $j > c_{\max}$:*
$$\operatorname{nvc}(j, G) := \infty$$

Definition 2.13 (**minimal consistency improving**). *Let a graph G, a rule r and a constraint c in ANF be given.*

A transformation $G \Longrightarrow_{r,m} H$ is called minimal consistency improving w.rt *c, if*

$$\operatorname{nvc}(k, H) < \operatorname{nvc}(k, G)$$

for any $0 \le k \le \operatorname{nl}(c)$. A rule r is called minimal consistency improving*, if all of its applications are.*

Lemma 2.14. *Let a graph G, a morphism $p : C_0 \to G$ and a constraint c in ANF over C_0 with $p \models_k c$ be given. Then, $p \models_j c$ for all $j < k$ such that the subcondition of c at layer j is existentially bound.*

Proof.

1. The subcondition of c at layer k is existentially bound: If an $j < k$ with $p \models_j c$ exists such that the subcondition of c at layer j is universally bound, let j_1 be the smallest of these. With lemma 2.7 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that the subcondition of c at layer ℓ is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of the subcondition of c at layer ℓ. Since $\ell < j_1$, a morphism $q : C_\ell \to G$ with $q \models d$ must exists and therefore a morphism $q' : C_{\ell+1} \to G$ with $q = q' \circ a_k$ must exists. If follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \mathsf{true})$ and with that $p \models_\ell c$.

2. The subcondition of c at layer k is universally bound: With lemma 2.7 follows that $p \models_{k+1} c$. Since c is in ANF, 1. can be applied to $k+1$.

□

Theorem 2.1. *Let a graph G, a rule r and a constraint c in ANF be given. Let $k < \mathrm{nl}(c)$ be the biggest number, such that $G \models_k c$. A transformation $G \Longrightarrow_{r,m} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.*

Proof. No $\ell > k$ with $G \models_\ell c$ exists and $G \models_k c$. Hence, $\mathrm{nvc}(k, G) > 0$ and $\mathrm{nvc}(k, G) \neq \infty$. Since $j > k$, $\mathrm{nvc}(k, H) = 0$ and it follows immediately that the transformation is minimal consistency improving. □

Theorem 2.2. *Let a constraint c in EANF be given. Every sequence of minimal consistency improving transformation w.r.t c is finite.*

Proof. Let G_0 be a graph and

$$G_0 \Longrightarrow_{\rho_1} G_1 \Longrightarrow_{\rho_2} G_2 \Longrightarrow_{\rho_3} \dots$$

be a sequence of minimal consistency improving transformations w.r.t c. We assume that $c_{\max}^{G_0} < \mathrm{nl}(c)$, otherwise $\mathrm{nvc}(j, G_0) = 0$ for all $j \in \{0, \dots, \mathrm{nl}(c)\}$ and no transformation $G_0 \Longrightarrow H$ is minimal consistency improving.

We show that after at most $j := \mathrm{nvc}(\mathrm{c}_{\max G_0} + 1, G_0)$ transformations $G_j \models_{c_{\max}^{G_0}+2} c$ holds. Note that j has to be finite, since G_0 contains only a finite number of occurrences of C_{j+1}. After each transformation it holds that $\mathrm{nvc}(c_{\max}^{G_{i+1}} + 1, G_{i+1}) \leq \mathrm{nvc}(c_{\max}^{G_i} + 1, G_i) - 1$. Therefore, after j transformations $\mathrm{nvc}(c_{\max}^{G_0} + 1, G_j) \leq \mathrm{nvc}(c_{\max}^{G_0} + 1, G_0) - j = 0$ holds and with that $G_j \models_{c_{\max}^{G_0}+2} c$. By iteratively applying this, it follows that after a finite number of transformations a graph G_k with $G_k \models c$ exists. Since no minimal consistency improving transformation $G_k \Longrightarrow G_{k+1}$ exists, the sequence has to be finite. □

Definition 2.15 (**direct minimal consistency improving**)**.** *Let G be a graph, ρ a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be the condition at layer $k = c_{\max}^G + 1 \leq \mathrm{nl}(c)$ of c. A transformation $t : G \Longrightarrow_{\rho,m} H$ is called* direct minimal consistency improving *if the following equations hold.*

Every occurrence of C_k in G that satisfies part$(1, e, C')$ *for any* $C' \in \mathcal{U}(C_k, C_{k+1})$ *still satisfies* part$(1, e, C')$ *in* H.

$$\forall p : C_k \hookrightarrow G\Big(\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \models \mathrm{part}(1, e, C') \wedge \mathrm{tr}_t \circ p \textit{ is total}) \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C')\Big) \tag{2.1}$$

Every new inserted occurrence of C_k by t satisfies part$(1, e, C_{k+1})$.

$$\forall p' : C_k \hookrightarrow H\big(\neg \exists p : C_k \hookrightarrow G(p' = \mathrm{tr}_t \circ p) \implies p' \models \mathrm{part}(1, e, C_{k+1})\big) \tag{2.2}$$

At least one occurrence of C_k in G that does not satisfy part$(1, e, C')$, *for any* $C' \in \mathcal{U}(C_k, C_{k+1})$, *either has been destroyed by t or satisfies* part$(1, e, C')$ *in* H.

$$\exists p : C_k \hookrightarrow G\Big(\bigvee_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \not\models \mathrm{part}(1, e, C') \wedge (\mathrm{tr}_t \circ p \textit{ is not total} \vee (\mathrm{tr}_t \circ p \textit{ is total } \wedge \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C'))))\Big) \tag{2.3}$$

No occurrence of a universally bound graph C_j with $j < k$ gets inserted.

$$\bigwedge_{\substack{i<k \\ i \textit{ even}}} \forall p : C_i \hookrightarrow H(\exists p' : C_i \hookrightarrow G(p' = \mathrm{tr}_t \circ p)) \tag{2.4}$$

No occurrence of an existentially bound graph C_j with $j < k$ gets deleted.

$$\bigwedge_{\substack{i<k \\ i \textit{ odd}}} \forall p : C_i \hookrightarrow G(\mathrm{tr}_t \circ p \textit{ is total}) \tag{2.5}$$

Lemma 2.16. *Let a transformation $t : G \Longrightarrow H$ and a constraint c in EANF be given, such that (2.4) and (2.5) of definition 2.15 are satisfied. Then,*

$$H \models_{c^G_{\max}} c.$$

Proof. Assume that $G \not\models_{c^G_{\max}} c$. Then, either a new occurrence of an universally bound graph of c has been inserted or an occurrence of an existentially bound graph of c has been destroyed. Therefore, the following holds:

$$\exists p : C_i \hookrightarrow G(\neg \exists p' : C_i \hookrightarrow (p' = \mathrm{tr}_t \circ p) \vee \exists p : C_j \hookrightarrow G(\mathrm{tr}_t \circ p \textit{ is not total})$$

with $i, j < k$, i being odd and j being even. It follows immediately that either (2.4) or (2.5) is not satisfied. This is a contradiction. □

Lemma 2.17. *Let a graph G, a constraint c in EANF and a direct minimal improving transformation $t : G \Rightarrow_{r,m} H$ w.r.t. c be given. Then, t is also a minimal improving transformation.*

Proof. Let G be a graph with $k = c^G_{\max} + 1$ and $G \models \text{part}(k, c, C)$ with $\text{part}(k, c, C) \in \mathcal{P}^G_c$ Let d be the subcondition of c at layer k. With lemma 2.16 follows that $c^H_{\max} \geq c^G_{\max}$ and with that $\text{nvc}(k, H) \neq \infty$.

1. We show that equations (2.1) and (2.2) imply that $\text{nvc}(k, H) \leq \text{nvc}(k, G)$. Assume that $\text{nvc}(k, H) > \text{nvc}(k, G)$. Therefore, a morphism $p : C_k \hookrightarrow H$ with $p \not\models \text{part}(1, d, C')$ for any $C' \in \mathcal{U}(C, C_{k+1})$ exists, such that either 1a or 1b is satisfied.

   (a) There does exist a morphism $q' : C_k \hookrightarrow G$ with $q' \models \text{part}(1, d, C')$ and $p = \text{tr}_t \circ q'$.

   (b) There does not exist a morphism $q : C_k \hookrightarrow G$, such that $p = \text{tr}_t \circ q$.

   This is a contradiction, if 1a is satisfied, q' does not satisfy equation (2.1) and if 1b is satisfied q does not satisfy equation (2.2). It also follows that

   $$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| \leq |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}|$$

   for all $C' \in \mathcal{U}(C_k, C_{k+1})$.

2. Since (2.3) is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \text{part}(1, d, C')$, such that either $\text{tr} \circ p$ is total and $\text{tr}_t \circ p \models \text{part}(1, d, C')$ or $\text{tr} \circ p$ is not total exists, for any $C' \in \mathcal{U}(C_k, C_{k+1})$. In both cases the following holds

   $$\begin{aligned} p &\in \{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\} \wedge \\ \text{tr} \circ p &\notin \{q \mid q : C_k \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}. \end{aligned}$$

   With that and 1. it follows that

   $$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| < |\{q \mid q : C_k \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}|.$$

In total $\text{nvc}(k, H) < \text{nvc}(k, G)$ follows and t is a minimal improving transformation. □

2.4 Comparison with other consistency concepts

Lemma 2.18. *Let a constraint c in ANF, graphs G and H with $G \not\models c$, and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \text{ is } c\text{-guaranteeing} \implies t \text{ is minimal consistency improving w.r.t } c$$

,

$$t \text{ is } c\text{-guaranteeing} \not\Longrightarrow t \text{ is direct minimal consistency improving w.r.t } c$$

and

$$t \text{ is minimal consistency improving w.r.t } c \not\Longrightarrow t \text{ is } c\text{-guaranteeing}$$

Proof. Let c' be the equivalent constraint in EANF.

1. Let t be c-guaranteeing transformation, then $H \models c$. Since $G \not\models c$, $G \not\models c'$ follows and $\mathrm{nvc}(c^G_{\max}+1, G) > 0$ and $\mathrm{nvc}(c^G_{\max}+1, H) = 0$. Therefore t is minimal consistency improving.

2. Let a constraint $c = \exists(a_0 : \emptyset \hookrightarrow C_0, \forall(a_1 : C_0 \hookrightarrow C_1, \exists(a_2 : C_1 \hookrightarrow C_2, \mathsf{true})))$ and a graph $G = C' \dot{\cup} C'$ with $C' = C_2 \setminus \{e\}$ for an $e \in E_{C_2 \setminus C_1}$ and the occurrences $p_1 : C_1 \hookrightarrow G$ and $p_2 : C_1 \hookrightarrow G$ be given. Let $t : G \Longrightarrow H$ be a transformation with $H = C_2 \dot{\cup} C''$ and $C'' = C' \setminus \{e'\}$ for an $e' \in E_{C_2 \setminus C_1}$, such that $\mathrm{tr}_t \circ p_1$ and $\mathrm{tr}_t \circ p_2$ are total. Then, t is c-guaranteeing and not direct minimal consistency improving, since $p_i \models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$ for $i = 1, 2$ and either $\mathrm{tr}_t \circ p_1 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$ or $\mathrm{tr}_t \circ p_2 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \mathsf{true})$.

3. Let t be a minimal consistency improving transformation w.r.t c', such that $\mathrm{nvc}(c^G_{\max}+1, H) > \mathrm{nvc}(c^G_{\max}+1, H) > 1$ and $H \not\models_{c^G_{\max}+2} c'$. Then,$H \not\models c'$ and t is not a guaranteeing transformation.

$\square$

Corollary 2.19. *Let c be an existentially bound constraint in ANF. Then, a transformation t is c-guaranteeing if and only if it is consistency improving. With lemma 2.18 follows:*

$$t \textit{ is consistency guaranteeing w.r.t } c \implies t \textit{ is minimal consistency improving w.r.t } c$$

and

$$t \textit{ is minimal consistency improving w.r.t } c \;\not\Longrightarrow\; t \textit{ is consistency improving w.r.t } c$$

Lemma 2.20. *let a universally bound constraint c with $\mathrm{nl}(c) = 1$ in ANF, graphs G and H with $G \not\models c$, and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \textit{ is consistency improving} \iff \textit{is minimal consistency improving}$$

Proof. Let $c = \forall(a : \emptyset \hookrightarrow C, \mathsf{false})$. Then, the equivalent constraint in EANF is $c' = \forall(a : \emptyset \hookrightarrow C, \exists(\mathrm{id} : C \hookrightarrow C, \mathsf{false})$. Since $\mathcal{U}(C, C) = \{C\}$, $\mathrm{nvc}(1, G)$ is the number of occurrences of C in G. This is exactly the definition of the number of violations for consistency improving transformations and

$$t \text{ is minimal consistency improving} \iff t \text{ is consistency improving}$$

follows immediately. $\square$

Lemma 2.21. *Let a universally bound constraint c with $\mathrm{nl}(c) \geq 2$ in ANF, graphs G and H with $G \not\models c$ and a transformation $t : G \Longrightarrow H$ be given. Then,*

$$t \textit{ is direct consistency improving} \;\not\Longrightarrow\; t \textit{ is minimal consistency improving}$$

and

$$t \textit{ is direct minimal consistency improving} \;\not\Longrightarrow\; t \textit{ is consistency sustaining}$$

Proof. 1. Let $c = \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \mathsf{true}))$ be a constraint. Let $V_{C_0} = V_{C_1}$ and $|E_{C_1}| - |E_{C_0}| = 2$. Let $G = C' \dot\cup C'$ with $C' = C_1 \backslash \{e\}$ with $e \in E_{C_1} \backslash E_{C_0}$ and the occurrences $p_1 : C_0 \hookrightarrow G$ and $p_2 : C_0 \hookrightarrow G$ be given. It follows that $\mathrm{nvc}(1, G) = 2$. Let $t : G \Longrightarrow H$ be a transformation, such that $H = C_0$. Then, t is a direct consistency improving transformation, since H contains only one occurrence of C_0 not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1 \, \mathsf{true})$. But, t is not minimal consistency improving since $\mathrm{nvc}(1, H) = 3$.

2. Let $c := \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \mathsf{true}))))$ be a constraint.

   Let a graph $G = C_0$ with the morphism $q : C_0 \hookrightarrow G$ and a transformation $t : G \Longrightarrow H$ with $H := C_2 \dot\cup C_2$ be given, such that $\mathrm{tr}_t \circ q$ is total. Then t is a direct minimal consistency improving transformation but not a consistency sustaining one, since H contains more occurrences of C_0 not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \mathsf{true})))$ than G.

□

Lemma 2.22. *Let a universally bound constraint c with $\mathrm{nl}(c) \geq 2$ in ANF, graphs G and H with $G \not\models c$ and a transformation $t : G \Longrightarrow H$ be given. If t satisfies (2.1) and (2.2), $H \models_{c^G_{\max}} c$ and*

$$\exists p : C_0 \hookrightarrow G(p \not\models \mathrm{part}(c^G_{\max} + 2, c, \mathit{true}) \wedge \mathrm{tr}_t \circ p \models c).$$

Then,

$$t \text{ is consistency improving} \Longrightarrow t \text{ is minimal consistency improving.}$$

Proof. □

3 application condition

Definition 3.1 (extended overlaps). *Let G and $C_0 \subseteq C_1$ with the inclusion $i : C_0 \hookrightarrow C_1$ be graphs. Let C be an overlap of C_0 and G with the inclusion $q : C_0 \hookrightarrow C$. The set of* extended overlaps of C with i, $\mathrm{eol}(C, i)$, *is the set of all overlaps C' of G and C_1, such that $C \subseteq C'$ and $q \models \exists(i : C_0 \hookrightarrow C_1, \mathit{true})$.*

Definition 3.2 (overlap shift). *Let $\rho = L \hookleftarrow K \hookrightarrow R$ be a plain rule, C a graph and C' an overlap of C and L with morphisms $p : L \hookrightarrow C'$, $k : K \hookrightarrow C'$, $c : C \hookrightarrow C'$ and the partial morphism $q : R \hookrightarrow C'$. We define*

$$\begin{aligned} D := \{e \in C' \mid (&\exists e' \in L : p(e') = e \\ &\vee \exists e' \in R : q(e') = e) \\ &\wedge \exists e' \in C : c(e') = e\} \end{aligned} \tag{3.1}$$

Let $r = L \hookleftarrow K' \hookrightarrow R$ be the rule with

$$K' := K \cup D$$

The graph H derived by the transformation $G \Rightarrow_{r,p} H$ is called the overlap shifted graph of C' with C and ρ. *The overlap shifted graph of an graph C is denoted by* $\mathrm{ols}_\rho(C, C')$.

Definition 3.3 (**application condition**). *Let $\rho = L \hookleftarrow K \hookrightarrow R$ be a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer k with k being and even number.*

The application condition ap_k of c at layer k with $C' \in \mathcal{U}(C_k, C_{k+1})$ and $i : C_k \hookrightarrow C'$ is defined as:

$$\mathrm{ap}(k, C') := \Big(\bigvee_{P \in \mathrm{ol}(L, C_k)} \mathrm{nex}(P, C') \wedge \mathrm{rep}(P, C') \Big) \wedge \mathrm{nin} \wedge \mathrm{rem} \wedge \mathrm{nwo} \tag{3.2}$$

with

1.

$$\mathrm{nex}(P, C') := \begin{cases} \exists(i_L : L \hookrightarrow P, \mathit{true}) & \text{, if } C_k = C_{k+1} \\ \bigwedge_{Q \in \mathrm{eol}(P,i)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow Q, \mathit{true})) & \text{, otherwise} \end{cases}$$

2. *Let $\mathbf{P}$ be the set of all overlaps of R and C', such that $i_R(R \setminus K) \cap i_{C'}(C') \neq \emptyset$:*

   (a) *If $C_k = C_{k+1}$:*

   $$\mathrm{rep}(P, C') := \begin{cases} \mathit{true} & \text{, if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \mathit{false} & \text{, otherwise} \end{cases}$$

   (b) *Otherwise:*

   $$\mathrm{rep}(P, C') := \begin{cases} \mathit{true} & \text{, if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \bigvee_{P' \in \mathbf{P}} \mathrm{Left}(\exists(i_R : R \hookrightarrow P', \mathit{true}), \rho) & \text{, otherwise} \end{cases}$$

3. *Let E be the set of all graphs C_j of c with $j \leq k$ and j being odd and let $\mathbf{P}_{C_j}$ be the set all overlaps of L and C_j with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.*

   $$\mathrm{rem} := \bigwedge_{C \in E} \bigwedge_{C' \in \mathbf{P}_{C_j}} \neg\exists(i_L : L \hookrightarrow C', \mathit{true})$$

4. *Let U be the set of all graphs C_j of c with $j \leq k$ and j being even and let $\mathbf{P}_{\mathbf{C_j}}$ be the set of all overlaps of R and C_j with $i_R(R \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.*

   $$\mathrm{nin} := \bigwedge_{C \in U} \bigwedge_{C' \in \mathbf{P}_{C_j}} \mathrm{Left}(\neg\exists(i_R : R \hookrightarrow C', \mathit{true}), \rho)$$

5. *Let E be the set of all overlaps of L and C_k, such that each $P \in E$ is also an overlap of L and $C'' \in \mathcal{U}(C_k, C_{k+1})$, $i_{C_k} \models \exists(a'_k : C_k \hookrightarrow C'', \mathsf{true})$ and $i_L(L \setminus K) \cap i_{C''}(C'' \setminus C_k) \neq \emptyset$.*
$$\mathrm{nwo} := \bigwedge_{P \in E} \neg\exists(i_L : L \hookrightarrow P, \mathsf{true})$$

Lemma 3.4. *Let a graph G, a constraint c in EANF, with $G \not\models c$, and a plain rule ρ be given. Then, the rule $\rho'(\rho, \mathrm{ap}(c^G_{\max}, C))$ for a graph $C \in \mathcal{U}(C_{c^G_{\max}}, C_{c^G_{\max}+1})$ is a minimal consistency improving rule.*

Proof. Let $t : G \Longrightarrow_{\rho',m} H$ be a transformation and $k = c^G_{\max} + 1$. We show, by contradiction, that this transformation is direct minimal consistency improving by showing that (2.1), (2.2), (2.3), (2.4) and (2.5) are satisfied. With that, it follows that ρ' is a minimal consistency improving rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ be the subcondition of c at layer k.

1. Assume that (2.1) does not hold. Then, there does exist an morphism $p : C_k \hookrightarrow G$, such that $p \models \mathrm{part}(1, e, C')$, $\mathrm{tr}_t \circ p$ is total and $\mathrm{tr}_t \circ p \not\models \mathrm{part}(1, e, C')$ for a graph $C' \in \mathcal{U}(C_k, C_{k+1})$. There has to exist an overlap P of L and C' such that $i_{C_k} \models \exists(a^p_k : C_k \hookrightarrow C', \mathsf{true})$ and $m \models \exists(i_L : L \hookrightarrow P, \mathsf{true})$. Then, nwo and with that $\mathrm{ap}(c^G_{\max}, C)$ is not satisfied.

2. Assume that (2.2) does not hold. Then, a morphism $p' : C_k \hookrightarrow H$ with $p' \not\models \mathrm{part}(1, e, C_{k+1})$ exists, such that there does not exist a morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p = p'$. Then, an overlap P of R and C_k with $i_R(R \setminus K) \cap i_R(C_k) \neq \emptyset$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$. Then, $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

3. Assume that (2.3) does not hold. Then, there does not exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \mathrm{part}(1, e, C)$, such that $\mathrm{tr}_t \circ p$ is not total or $\mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C)$ and $\mathrm{tr}_t \circ p$ is total. Then, no overlap P of L and C_k with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ exists, such that $m \models \mathrm{nex}(P, C)$. Also, no overlap P of C and R with $i_R(R \setminus K) \cap i_C(C) \neq \emptyset$ and $m \models \mathrm{nex}(P, C)$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$ and therefore $\mathrm{rep}(P, C) = \mathsf{false}$. It follows that for all $P \in \mathrm{ol}(L, C_k)$ it holds that $\mathrm{nex}(P, C) \wedge \mathrm{rep}(P, C) = \mathsf{false}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

4. Assume that (2.4) does not hold. Then, there does exist a morphism $p : C_j \hookrightarrow G$ with $j < k$ and i being even, such that no morphism $p' : C_j \hookrightarrow G$ with $\mathrm{tr}_t \circ p' = p$ exists. Then, an overlap P of C_j and R with $i_R(R \setminus K) \cap i_{C_j} \neq \emptyset$ exists, such that $m \models \mathrm{Left}(\exists(i_R : R \hookrightarrow P, \mathsf{true}), \rho)$. It follows that $m \not\models \mathrm{rem}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

5. Assume that (2.5) does not hold. Then, there does exist and morphism $p : C_j \hookrightarrow G$ with $j < k$ and j being odd, such that $\mathrm{tr}_t \circ p$ is not total. Then, an overlap P of C_j and L with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$ exists, such that $m \models \exists(i_L : L \hookrightarrow P, \mathsf{true})$. It follows that $m \not\models \mathrm{rem}$ and with that $m \not\models \mathrm{ap}(c^G_{\max}, C)$.

In total follows that if $m \models \text{ap}(c^G_{\max}, C)$, then t is a direct minimal improving transformation. $\square$

Lemma 3.5. *Let G be a graph, c a constraint in EANF, with $c^G_{\max} < \text{nl}(c)$, and $\rho = L \hookleftarrow K \hookrightarrow R$ a plain rule. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer $\text{c}_{\max} + 1$ and $\text{ap}(\text{c}_{\max}, C')$ the application condition constructed by definition 3.3 with $C' \in \mathcal{U}(C, C_{k+1})$ for any $C \in \mathcal{G}^G_c$. If*

$$((R \setminus K) \cap C_{k+1}) \cup ((L \setminus K) \cap C_{k+1}) = \emptyset$$

$\text{ap}(\text{c}_{\max}, C')$ *can be replaced by* false.

Proof. There does not exist an overlap P of C_k and L with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ and $\text{rep}(P, C')$ will be equal to false, if $C_k = C_{k+1}$, or equal to $\bigvee_{P' \in \mathbf{P}} \text{Left}(\exists(i_R : R \hookrightarrow P', \mathsf{true}), \rho)$. Since the set $\mathbf{P}$ has to be empty, this expression can be replaced by false. If follows that $\text{rep}(P, C') = \mathsf{false}$ for all $P \in \text{ol}(L, C_k)$ and therefore $\text{ap}(\text{c}_{\max}, C')$ will always be evaluated to false. $\square$

3.1 potentially minimal improving rules

Definition 3.6 (**basic improving rule**)**.** *Let a constraint c and a plain rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ be given. The rule ρ is called* basic improving *w.r.t c at layer k with $C_k \subset P \subseteq C_{k+1}$, $p : C_k \hookrightarrow P$ being the inclusion, and $k \in \{1, 3, \dots, \text{nl}(c)\}$, if*

$$(L \setminus K) \cap (C_{k-1} \cup (C_{k+1} \setminus C_k)) = \emptyset \tag{3.3}$$

and

$$(R \setminus K) \cap C_k = \emptyset \tag{3.4}$$

and either 1. or 2. applies.

1. *The rule ρ deletes elements of $C_k \setminus C_{k-1}$:*

$$L \subseteq C_k \wedge L \setminus K \neq \emptyset \tag{3.5}$$

   Then, ρ is called a deleting basic improving rule*.*

2. *The rule ρ creates an instance of P:*

$$L = C_k \wedge P \subseteq R \tag{3.6}$$

   and p is a partial morphism of r. Then, ρ is called an inserting basic improving rule*.*

Definition 3.7 (**application conditions for basic improving rules**)**.** *Let a constraint c in EANF and a basic improving rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ w.r.t c at layer k with $C_k \subseteq P \subseteq C_{k+1}$ be given. We define the application condition for r as:*

1. *If ρ is a deleting potentially minimal improving rule:*

$$\mathrm{ap}_{\mathrm{pi}}(j, P) := \begin{cases} \bigvee_{P \in \mathrm{ol}(L, C_k)} \bigwedge_{P' \in \mathrm{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \textsf{true})) & , \text{ if } j = k \\ \textsf{false} & , \text{ if } j \neq k. \end{cases}$$

2. *If ρ is an inserting potentially minimal improving rule:*

$$\mathrm{ap}_{\mathrm{pi}}(j, P) := \begin{cases} \neg\exists(a_k^p : L \hookrightarrow P, \textsf{true}) & , \text{ if } j = k \\ \textsf{false} & , \text{ if } j \neq k \end{cases}$$

Theorem 3.1. *Let a graph G, a constraint c in EANF, with $G \models \mathrm{part}(c^G_{\max}, c, C)$ and $\mathrm{part}(c^G_{\max}, c, C) \in \mathcal{P}^G_c$, and a potentially minimal improving rule $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ at layer $c^G_{\max}$ with $C_{c_{\max}+1} \subseteq P \subseteq C_{c_{\max}+2}$ be given. Then, $\rho' = (\rho, \mathrm{ap}_{\mathrm{pi}}(c^G_{\max} + 1, P))$ is a direct minimal consistency improving rule.*

Proof. Let $t : G \Longrightarrow_{\rho', m} H$ be a transformation, $k = c^G_{\max} + 1$ and e be the subcondition of c at layer k. We show that t is a direct minimal consistency improving transformation. First, we show that equation (2.1) is satisfied. Let $p : C_k \hookrightarrow G$ be a morphism. If ρ is a deleting potentially minimal improving rule, either 1. or 2. applies, if ρ is an inserting and not a deleting potentially minimal improving rule, only 2. applies, because ρ cannot destroy any occurrences of C_k in G.

1. If $p(C_k) \cap m(L \setminus K) \neq \emptyset$, $\mathrm{tr}_t \circ p$ is not total, since at least one element of $p(C_k)$ has been deleted by t and p does satisfy $\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \models \mathrm{part}(1, e, C') \wedge \mathrm{tr}_t \circ p \text{ is total}) \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C'))$.
2. If $p(C_k) \cap m(L \setminus K) = \emptyset$, $\mathrm{tr}_t \circ p$ is total, since no element of $p(C_k)$ has been deleted by t. Because (3.3) holds, t does not delete any elements of $C_{k+1} \setminus C_k$ and $p \models \mathrm{part}(1, e, C') \implies \mathrm{tr}_t \circ p \models \mathrm{part}(1, e, C')$ for all $C' \in (U)(C_k, C_{k+1})$.

With 1. and 2. follows that (2.1) is satisfied.

Second, we show that equation (2.2) is satisfied. Let $p' : C_k \hookrightarrow H$ be a morphism. Because (3.4) is satisfied, t does not create any elements of C_k and there must exist a morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p = p'$. If follows that (2.2) is satisfied.

Third, we show that (2.3) is satisfied. We consider the cases that firstly, ρ is a deleting minimal potentially improving rule and secondly, that ρ is an inserting and not a deleting minimal potentially improving rule.

1. If ρ is a deleting potentially minimal improving rule, the condition $\mathrm{ap}_{\mathrm{pi}}(k, P) = \exists\big(b : L \hookrightarrow C_k, \neg\exists(a_{k+2} : C_k \hookrightarrow C_{k+1}, \mathsf{true})\big)$ is satisfied by m. Therefore a morphism $p : C_k \hookrightarrow G$ with $p \not\models \neg\exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, \mathsf{true}) = \mathrm{part}(1, e, C_{k+1})$ and $m = p \circ b$ must exist. Since ρ is a deleting rule, at least one element of $p(C_k)$ has been deleted by t and $\mathrm{tr}_t \circ p$ is not total. It follows that (2.3) is satisfied.

2. If ρ is an inserting and not a deleting potentially minimal improving rule, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \mathsf{true})$ is satisfied by m. Because $L = C_k$, $m \models \neg\exists(b : C_k \hookrightarrow P, \mathsf{true}) = \mathrm{part}(1, e, P)$. Since (3.6) is satisfied, $\mathrm{tr} \circ p$ is total and $\mathrm{tr}_t \circ p \models \mathrm{part}(1, e, P)$. Therefore, (2.3) is satisfied.

Last, since (3.4) and (3.3) are satisfied, ρ cannot create any occurrence of C_j with $j \leq k$ and j being even and ρ cannot delete any occurrences of C_j with $j \leq k$ and j being odd. Therefore, (2.4) and (2.5) are satisfied.

In total follows that t is a direct minimal consistency improving transformation and ρ' is a direct minimal consistency improving rule. □

Theorem 3.2. *Let a constraint c in EANF, a basic increasing rule ρ with P and a consistency increasing transformation $t : G \Longrightarrow_{\rho,m} H$ w.r.t c be given. Then, $m \models \mathrm{ap}_{\mathrm{pi}}(c^G_{\max}, P)$.*

Proof. Let $k = c^G_{\max}$.

1. If c is a inserting rule, then $\mathrm{ap}_{\mathrm{pi}}(c^G_{\max}, P) = \neg\exists(a^p_k : L \hookrightarrow P, \mathsf{true})$. Let $n : R \hookrightarrow H$ be the comatch of t. Since t is a consistency increasing transformation, there does exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$, $\mathrm{tr}_t \circ p \models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$ such that $p'(P) \cap n(R \setminus K)$ for the morphism $p' : P \hookrightarrow H$ with $p = p \circ a^p_k$. Assume that $m \models \exists(a^p_k : L \hookrightarrow P, \mathsf{true})$. Then, every morphism $p : C_k \hookrightarrow G$ with $\mathrm{tr}_t \circ p \models \exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$ with $p'(P) \cap n(R \setminus K)$ for the morphism $p' : P \hookrightarrow H$ with $p = p' \circ a^p_k$ already satisfies $\exists(a^p_k : C_k \hookrightarrow P, \mathsf{true})$. This is a contradiction.

2. If ρ is a deleting rule, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \bigvee_{P \in \mathrm{ol}(L, C_k)} \bigwedge_{P' \in \mathrm{eol}(P, a_k)} \exists(i_L : L \hookrightarrow P, \neg\exists(i_P : P \hookrightarrow P', \mathsf{true}))$. Since t is a consistency increasing transformation, there does exist a morphism $p : C_k \hookrightarrow G$, such that $p \not\models \exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ and $\mathrm{tr}_t \circ p$ is not total. Assume that $m \not\models \mathrm{ap}_{\mathrm{pi}}(k, P)$. Then, for each overlap Q of L and C_k, the inclusion i_{C_k} does satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$. This is a contradiction, because $p(C_k) \cap m(L \setminus K) \neq \emptyset$ has to hold.

□

Lemma 3.8. *Let a constraint c in EANF and inserting basic increasing rules ρ_1, ρ_2 at layer k with P_1 and P_2, respectively, be given. Then, the rules $\rho'_1 = (\rho_1, \mathrm{ap}_{\mathrm{pi}}(k, P_1))$ and $\rho'_2 = (\rho_2, \mathrm{ap}_{\mathrm{pi}}(k, P_2))$ are sequentially independent.*

Proof. Let $G_1 \Longrightarrow_{\rho'_1,m_1} G_2 \Longrightarrow_{\rho'_2,m_2} G_3$ be a sequence of transformations. First, note that this sequnce can only exist if $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$ Since both rules are do not insert elements of C_k, a morphism $d_1 : L_2 \hookrightarrow D_1$ with $m_2 = h_1 \circ d_1$ exists. Because both rules do not delete any elements a morphism $d_2 : R_1 \hookrightarrow D_2$ with $m^*_1 = g_2 \circ d_2$ exists. Because ρ_2 is not a basic increasing rule with P_1, $h_2 \circ d_2 \models R(\mathrm{ap}_{\mathrm{pi}})$ and because ρ_1 does not delete any elements of P_2 is holds that $g_1 \circ d_1 \models R(\mathrm{ap}_{\mathrm{pi}})$. □

Definition 3.9 (**repairing rule set**). *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is called a* repairing rule set for c at layer k *if for all graphs G with $k = \mathrm{c}_{\max}$ a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models_{k+2} c$.

Corollary 3.10. *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. If $\mathcal{R}$ is a repairing rule set for c at layer k, $\mathcal{R}$ is a repairing rule set w.r.t c at layer j for all $k < j \leq \mathrm{nl}(c)$.*

Corollary 3.11. *Let a constraint c in EANF and a repairing rule set $\mathcal{R}$ for c at layer k, for all $k \in \{1, 3, \ldots, \mathrm{nl}(c)\}$, be given. Then, for all graphs G, a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models c$.

Definition 3.12 (**decomposition of a graph**). *Let graphs $G_0 \subset G_1$ be given.*

A decomposition of G_1 with G_0 is a minimal set

$$\mathbf{P} \subseteq \{P_v \mid v \in V_{C_1 \setminus C_0}\}$$

of subgraphs of G_1, such that every element of G_1 is contained in at least one $P \in \mathbf{P}$ and every P_v is constructed in the following way: $G_0 \subset P_v$, $v \in P$ and for all nodes $v' \in P \setminus C_k$ is holds that P contains all edges $e \in E_{G_1 \setminus G_0}$ and all nodes $u \in V_{G_1}$ such that either $\mathrm{tar}(e) = v' \wedge \mathrm{src}(e) = u$ or $\mathrm{src}(e) = u \wedge \mathrm{tar}(e) = v'$ holds.

Lemma 3.13. *Let graphs $G_0 \subset G_1$ and a decomposition $\mathbf{P}$ of G_1 with G_0 be given. Then, for each pair $P, P' \in \mathbf{P}$ with $P \neq P'$ the following holds:*

$$(P \setminus C_k) \cap (P' \setminus C_k) = \emptyset$$

Proof. Assume that $(P \setminus C_k) \cap (P' \setminus C_k) \neq \emptyset$, therefore a node $v \in G_1 \setminus G_0$ with $v \in P \cap P'$ exists. By the construction of P and P' it follows that $P = P_v$ and $P' = P_v$ and therefore $P = P'$. This is a contradiction. □

Lemma 3.14. *Let graphs $G_0 \subset G_1$ and a decomposition $\mathbf{P}$ of G_1 with G_0 be given. Then,*

$$G_1 = \bigcup_{P \in \mathbf{P}} P.$$

Proof. Let $H := \bigcup_{P \in \mathbf{P}} P$. Firstly, we show that $H \subseteq G_1$. Since every $P \in \mathbf{P}$ is a subgraph of G_1 it follows that $V_H \subseteq V_{G_1}$ and $E_H \subseteq E_{G_1}$

Secondly, we show that $G_1 \subseteq H$. Let $u \in V_{G_1}$ be a node, if $u \in V_{G_0}$, then u is contained in every $P \in \mathbf{P}$ and therefore $u \in V_H$. Otherwise, if $u \notin V_{G_0}$, then u has to be, by the definition of $\mathbf{P}$, contained in at least one $P \in \mathbf{P}$ and $V_{G_1} \subseteq V_H$ follows. Let $e \in E_{G_1}$ be an edge. If $e \in E_{G_0}$, then e is contained in every $P \in \mathbf{P}$ and $e \in E_H$. Otherwise, if $e \notin E_{G_0}$, by the definition of $\mathbf{P}$, e has to be contained in at least one $P \in \mathbf{P}$. It follows that $e \in E_H$ and with that $E_{G_1} \subseteq E_H$. □

Theorem 3.3. *Let a constraint c in EANF and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is a repairing set of c at layer $k \leq \mathrm{nl}(c)$ if either 1 or 2 applies.*

1. *For any universally bound graph C_j at layer $j \leq k$ of c, $(\rho, \mathrm{ap}_{\mathrm{pi}}(j, C_{j+1})) \in \mathcal{R}$ and ρ is a deleting potentially minimal improving rule at layer j with C_{j+1}, such that ρ only deletes edges of C_j.*

2. *A decomposition $\mathbf{P}$ of C_k with C_{k-1} exists, such that for each $P \in \mathbf{P}$ a rule $(\rho, \mathrm{ap}_{\mathrm{pi}}(k, P)) \in \mathcal{R}$ exists, such that ρ is an inserting basic improving rule at layer k with P.*

Proof. Let a constraint c in EANF, a rule set $\mathcal{R}$ and a graph G with $k = \mathrm{c}_{\mathrm{max}}$ and $\mathrm{c}_{\mathrm{max}} < \mathrm{nl}(c)$ be given. We show that a sequence $G = C'_0 \Rightarrow \ldots \Rightarrow C'_n = H$ with rules of $\mathcal{R}$ exists, such that $H \models_{k+2} c$ if 1. or 2. of theorem 3.3 is satisfied.

1. Assume that 1. of theorem 3.3 holds. Let $(\rho, \mathrm{ap}_{\mathrm{pi}}(j, C_{j+1})) \in \mathcal{R}$, such that $\rho = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ is a deleting potentially minimal improving rule at layer $j \leq k$ with C_{j+1} and C_j is a universally bound graph of c. Then, $\mathrm{ap}_{\mathrm{pi}}(j, C_{j+1}) = \exists\big(b : L \hookrightarrow C_j, \neg\exists(a_j : C_j \hookrightarrow C_{j+1}, \mathsf{true})\big)$. Let $q : C_j \hookrightarrow G$ be a morphism such that $q \not\models \exists(a_j : C_j \hookrightarrow C_{j+1}, \mathsf{true})$. Since $L \subseteq C_j$, we can construct a morphism $m_1 : L \hookrightarrow G$ with $m_1 = q \circ b$ and therefore $m_1 \models \mathrm{ap}_{\mathrm{pi}}(j, C_{j+1})$. Since r only deletes edges, a transformation $t : G = G_0 \Rightarrow_{r,m_1} G_1$ exists and $\mathrm{tr}_t \circ p$ is not total. Because r does not insert any elements of C_j:

$$|\{q : C_k \hookrightarrow G_0 \mid q \not\models d\}| < |\{q : C_k \hookrightarrow G_1 \mid q \not\models d\}|$$

   with $d = \exists(a_j : C_j \hookrightarrow C_{j+1}, \mathsf{true})$. By iteratively applying this construction, we can generate a finite sequence of transformations

$$G = G_0 \Rightarrow_{r,m_1} G_1 \Rightarrow_{r,m_2} \ldots \Rightarrow_{r,m_n} G_n = H$$

   such that $|\{q : C_k \hookrightarrow G_n \mid q \not\models d\}| = 0$ and therefore $H \models_j c$. With lemma 2.8, $H \models_{k+2} c$ and $H \models c$ follows.

2. Assume that 2. of theorem 3.3 holds. Let $\rho_0 = (\rho, \mathrm{ap}_{\mathrm{pi}}(k, P)) \in \mathcal{R}$, such that ρ is an inserting basic improving rule of c at layer k with $P \in \mathbf{P}$. Then, $\mathrm{ap}_{\mathrm{pi}}(k, P) = \neg\exists(b : L \hookrightarrow P, \mathsf{true})$. Let $q_0 : C_k \hookrightarrow G$ be a morphism, such that $q_0 \not\models \exists(a'_k : C_k \hookrightarrow P, \mathsf{true})$ with a'_k being a partial morphism of a_k. Since $L = C_k$, we set $m_0 : C_k \hookrightarrow G$ with $m_0 = q_0$. It follows that $m_0 \models \neg\exists(a'_k : C_k \hookrightarrow P, \mathsf{true}) = \mathrm{ap}_{\mathrm{pi}}(k, P)$. Because r does not delete any elements, a transformation $t_0 : G \Longrightarrow_{r_0,m_0} G_1$ exists and $\mathrm{tr}_t \circ q \models \exists(a'_k : C_k \hookrightarrow P, \mathsf{true})$. We set $q_1 = \mathrm{tr}_{t_0} \circ q_0$ and apply the same method to q_1.

   By iteratively applying this, we can construct a finite sequence of transformations

$$G \Longrightarrow_{r_0,m_0} G_0 \Longrightarrow_{r_1,m_1} \ldots \Longrightarrow_{r_n,m_n} G_n$$

such that $m_i = \mathrm{tr}_{t_{i-1}} \circ \ldots \circ \mathrm{tr}_{t_0} \circ m_0$ and $q \models \exists(b_i : C_k \hookrightarrow P_i, \mathsf{true})$ for all $P_i \in \mathbf{P}$ with $q = \mathrm{tr}_{t_n} \circ q_n$. Let $p_i : P_i \hookrightarrow G_n$ be the morphism, such that $q = p_i \circ b_i$.

Now, we can construct a morphism $p : C_{k+1} \hookrightarrow G$ with

$$p(e) := \begin{cases} p_1(e) & \text{,if } e \in P_1 \\ & \vdots \\ p_j(e) & \text{,if } e \in P_j. \end{cases}$$

Let $e \in C_k$, because $q(e) = p_i \circ b_i(e)$ and $q(e) = p_\ell \circ b_\ell(e)$ and b_i and b_ℓ are both partial morphisms of a_k, it follows that $b_i(e) = b_\ell(e)$ and therfore $p_i(e) = p_\ell(e)$. Because $(P_i \cap P_\ell) \setminus C_k = \emptyset$ for all $i \neq \ell$, p is a morphism and by the definition of p it follows that $q = p \circ a_k$ and therefore $q \models \exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$.

By iteratively applying this whole construction to all occurrences of C_k that do not satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ the derived graph graph H does not contain any occurrences of C_k not satisfying $\exists(a_k : C_k \hookrightarrow C_{k+1}, \mathsf{true})$ and therefore $H \models_{k+2} c$.

$\square$

Corollary 3.15. *If a set of rule $\mathcal{R}$ is a repairing set of c at layer $k \leq \mathrm{nl}(c)$ and 1. of theorem 3.3 applies, then $\mathcal{R}$ is a repairing set of c at layer j for all $k \leq j \leq \mathrm{nl}(c)$.*

Lemma 3.16. *Let a graph G_0, a constraint c in EANF and a repairing set $\mathcal{R}$ at layer $c^{G_0}_{\max}+1$ be given, such that each rule in $\mathcal{R}$ is a basic increasing. Then, for every sequence*

$$G_0 \Longrightarrow_{\rho_0,m_0} G_1 \Longrightarrow_{\rho,m_0} \ldots \Longrightarrow_{\rho_n,m_n} G_n$$

such that $\rho_i = (\rho, \mathrm{app}_{\mathrm{pi}}(c^{G_0}_{\max} + 1, P)$ with $\rho \in \mathcal{R}$ being a consistency increasing rule at layer $c^{G_0}_{\max} + 1$ with P, it holds that

References

[1] C. Sandmann and A. Habel. Rule-based graph repair. *arXiv preprint arXiv:1912.09610*, 2019.