Masterarbeit

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Abstract

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1 Preliminaries

Definition 1.1 (**subgraph**)**.** *Let G_1 and G_2 be graphs. The graph G_2 is called a* subgraph *of G_1 if an injective morphism $f : G_2 \to G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective. The set of all subgraphs of G_1 is denoted by* $\mathrm{sub}(G_1)$*.*

Definition 1.2 (**overlap**)**.** *Let G and G' be graphs. A graph H is called an* overlap of G and G' *if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by* $\mathrm{ol}(G, G')$*.*

Definition 1.3 (**overlap at morphism**)**.** *Let C,G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an* overlap of G and C' at p *if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by* $\mathrm{ol}_p(G, C')$*.*

Definition 1.4 (**partial morphism**)**.** *Let $f : G_1 \to G_2$ and $g : G_3 \to G_4$ be morphisms. The morphism g is called a* partial morphism *of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.*

Definition 1.5 (**nested graph condition**)**.** *A* graph condition *over a graph C_0 is inductively defined as follows:*

- true *is a graph condition over every graph.*
- $\exists(a : C_0 \hookrightarrow C_1, d)$ *is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1.*
- $\neg d$ *is a graph condition over C_0 if d is a graph condition over C_0.*
- $d_1 \wedge d_2$ *and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0.*

Conditions over the empty graph $\emptyset$ are called constraints*. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies true. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c. An injective morphism satisfies $\neg d$ if it does not satisfy d, it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \wedge d_2$ if it satisfies d_1 or d_2. A graph G satisfies a constraint c, $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c. We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg\exists(a : C_0 \hookrightarrow C_1, \neg d)$.*

The nesting level nl *of a condition is defined as* $\mathrm{nl}(\mathsf{true} = 0$ *and* $\mathrm{nl}(\exists(a : P \to Q, d)) := \mathrm{nl}(d) + 1$.

Definition 1.6 (**alternating quantifier normal form (ANF)**[1])**.** *A graph condition c is in* alternating normal form *(ANF) if it is of the form*

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \ldots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 partial consistency improving

2.1 partial-conditions and -satisfiability

Definition 2.1 (**partial condition**). *Let c be a condition over C_0. A partial condition of c over $C_0' \subseteq C_0$ is defined as:*

1. *true is the partial condition of true for every morphism.*
2. *if $c = Q(a : C_0 \hookrightarrow C_1, d)$, with $Q \in \{\exists, \forall\}$, a partial condition of c over C_0' is given by $\exists(a' : C_0' \hookrightarrow C_1', d')$ with a_1' being a partial morphism of a, $C_1' \subseteq C_1 \setminus a(C_0 \setminus C_0')$ and d' is a partial condition of d over C_1'.*
3. *if $c = d_1 \wedge d_2$ or $c = d_1 \vee d_2$ the partial condition of c is given by $d_1' \wedge d_2'$ and $d_1' \vee d_2'$, respectively, with d_1' and d_2' being partial conditions of d_1 and d_2 over C_0'.*
4. *if $c = \neg d$ the partial condition of c is given by $\neg d'$ with d' being a partial condition of d over C_0'.*

A partial condition $Q(a : C_0' \hookrightarrow C_1', d)$ of c over C_0' is called the closest partial condition *of c over C_0' if $C_1' = C_1 \setminus a(C_0 \setminus C_0')$ and d is the closest partial condition of d over C_1'.*

We use the notation $c' \leq c$ if c' is partial condition of c over $C_0' \subseteq C_0$ and $c' < c$ if $C_i' \subset C_i$ for any i.

Definition 2.2 (**partial satisfiability**). *Let a condition c over C_0 and a graph G be given. A morphism $p_0 : C_0' \to G$, with $C_0' \subseteq C_0$, partial satisfies c, $p_0 \models_p c$, if a partial morphism $p_0' : C_0'' \to G$ of p_0 satisfies a partial condition of c over C_0''.*

Note, that $p_0 \models c$ implies $p_0 \models_p c$

2.2 minimal consistency improving

Definition 2.3 (**Condition up to layer**). *Let c be a condition and d a subcondition of c. The layer of d is defined as $\mathrm{lay}(d) := \mathrm{nl}(c) - \mathrm{nl}(d) - 1$.*

Let L be the set of subconditions of c with layer k. The condition up to layer k *of c is the condition which is obtained by replacing each $d = Q(a : P \hookrightarrow P', e) \in L$ by $d = Q(a : P \hookrightarrow P', \mathsf{true})$ if $Q = \exists$ and by $Q(a : P \hookrightarrow P', \mathsf{false})$ if $Q = \forall$ in c. If $k = 0$, the condition up to layer k of every condition c is true.*

Definition 2.4 (**layered consistency**). *Let G be a graph, $p : C_0 \to G$ a morphism, c a condition over C_0 and d the condition up to layer k of c with $k \geq 1$. The morphism p* satisfies c up to layer k *if p satisfies d and there is no condition d' up to layer $j > k$ such that p satisfies d'. The notation $p \models_k c$ is used if p satisfies c up to layer k. The notation $G \models_k c$ is used if c is a constraint and $p \models_k c$ for $p : \emptyset \to G$.*

Lemma 2.5. *Let G be a graph, $p : C_0 \to G$ a morphism and $c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, \ldots))$ a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\mathrm{lay}(d) = k$ is universally bound, $p \models_k c$ implies $p \models c$.*

Proof. Let $c' = Q(a_1 : C_0 \hookrightarrow C_1, \dots \forall(a_k : C_{k-1} \hookrightarrow C_k, \mathsf{false}) \dots)$ be the condition up to layer k of c and d be the subcondition of c with $\mathrm{lay}(d) = k+1$. Since $p \models c'$, there does not exist a morphism $q : C_k \to G$ with $p = q \circ a_k \circ \dots \circ a_1$. Thus, there does not exists a morphism $q : C_k \to G$ with $q \not\models d$ and $p = q \circ a_k \circ \dots \circ a_1$ and $p \models c$ follows immediately. □

Lemma 2.6. *Let $c = Q(a_1 : C_0, \hookrightarrow C_1, \dots)$ be a condition in ANF over C_0 and $p : C_0 \to G$ a morphism with $p \models_k c$. Let $d = Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q'}(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the subcondition of c with layer $k+1$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that $p \models Q(a_1 : C_0, \hookrightarrow C_1, \dots Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q'}(a_{k+2} : C_{k+1} \hookrightarrow C', f))$ with f being a Q' bound condition over C'.*

Proof. If $Q' = \exists$, with lemma 2.5 follows that $p \models c$, therefore we can choose $C' = C_{k+2}$ and $f = e$. If $Q = \forall$, we choose $C' = C_{k+1}$ and $f = \mathsf{true}$. Let $q : C_k \to G$ with $p = q \circ a_k \circ \dots \circ a_1$. This morphism must exists since $p \models_k c$. Let $q' : C_{k+1} \to G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a_{k+2} has to be the identity and therefore $q' = q' \circ a_{k+2}$. It follows that $q' \models \exists(a_{k+2} : C_{k+1} \hookrightarrow C', \mathsf{true})$ and therefore $p \models Q(a_1 : C_0, \hookrightarrow C_1, \dots Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q'}(a_{k+2} : C_{k+1} \hookrightarrow C', f))$. □

Definition 2.7 (biggest partially satisfying graph). *Let G be a graph, $c = Q(a_1 : C_0, \hookrightarrow C_1, \dots)$ a condition in ANF and $p : C_0 \to G$ a morphism with $p \models_k c$. Let $d = Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q'}(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with $\mathrm{lay}(d) = k+1)$. With lemma 2.6 there exists a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ and an Q' bound condition e over C' in ANF with $p \models Q(a_1 : C_0, \hookrightarrow C_1, \dots Q'(a_{k+1} : C_k \hookrightarrow C_{k+1}, \overline{Q}(a_{k+2} : C_{k+1} \hookrightarrow C', e)) =: d_{C',e}$. Let $\mathcal{C}_{k,p,e}$ be the set of graphs $C_{k+1} \subseteq C' \subseteq C_{k+2}$ with $p \models d_{C',e}$.*

A graph $C' \in \mathcal{C}_{k,p,e}$, such that no $C'' \in \mathrm{C}$ with $C' \subseteq C''$ exists is called a biggest partially satisfying graph *of c at level k with p and e. The set of these graphs is denoted by $\mathcal{B}_{k,p,e}$*

Definition 2.8. *Let G be a graph and $c = Q(a_1 : \emptyset \hookrightarrow C_1, \dots)$ a constraint in ANF, such that $G \models_k c$ and $G \not\models c$. Let $d = Q(a_1 : \emptyset \hookrightarrow C_1, \dots \forall(a_{k+1} : C_k \hookrightarrow C_{k+1}, \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \mathsf{true}))$ be the condition up to layer $k+2$ of c. The number of violations of c up to layer $k+2$ in G is defined as the the number of morphisms $q : C_{k+2} \to G$ that do not satisfy $\exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, \mathsf{true})$. This number is denoted by $\mathrm{nv}_c(k+2, G)$.*

Definition 2.9 (minimal consistency improving). *Let G be a graph, r a rule and c a constraint in ANF with $G \models_k c$ and $G \not\models c$. With lemma 2.5 follows that the subcondition of c with layer k has to be existentially bound.*

A transformation $G \xRightarrow{r,m} H$ is called minimal consistency improving *if either $H \models c$ or $\mathrm{nv}_c(k+2, G) < \mathrm{nv}_c(k+2, H)$ or every $C' \in \mathcal{B}_{k,G,\mathsf{true}}$ is also an element of $\mathcal{C}_{k+2,H,e}$ and for at least one C' an $C'' \in \mathcal{C}_{k+2,H,\mathsf{true}}$ with $C' \subseteq C''$ exists. The rule r is called* minimal consistency improving *if all of its applications are.*

Lemma 2.10. *Let G be a graph, r a rule and c a constraint in ANF with $G \models_k c$ and $G \not\models c$. A transformation $G \xRightarrow{r,m} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.*

Proof. Let $G \models_k c$ and $H \models_j c$ with $k < j$. With lemma 2.5 the subcondition of c with layer k is existentially bound. If an $n \in \mathbb{N}$ with $j - k = 2n - 1$ exists, the subcondition of c with layer j has to be universally bound and therefore $H \models c$.

Otherwise, H satisfies the condition up to layer $k+2$ of c, Let $d = \exists(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k+2$, and $\mathcal{C}_{k+2,H,\mathsf{true}}$ contains every graph C' with $C_{k+1} \subseteq C' \subseteq C_{k+2}$. Since $G \not\models_{k+2} c$, $C_{k+2} \notin \mathcal{C}_{k+2,G,\mathsf{true}}$ and for every $C'' \in \mathcal{B}_{k+2,G,\mathsf{true}}$ it holds that $C'' \in \mathcal{C}_{k+2,H,\mathsf{true}}$ and $C'' \subset C_{k+2} \in \mathcal{C}_{k+2,H,\mathsf{true}}$. □

3 application condition

Construction 3.1. *Given a rule $r = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ and a constraint c in ANF with $\mathrm{nl}(c) = 1$. Let G and H be be graphs. We denote the set of all overlaps of G and H as $\mathrm{ol}(G, H)$. We construct the following application conditions:*

1. *if $c = \forall(a_0 : C_0 \to C_1, \textsf{false})$:*

$$\bigvee_{Q \in \mathrm{ol}(L, C_1)} \Big(\exists(a : L \hookrightarrow Q, \textsf{true}) \wedge \mathrm{Left}(\neg\exists(b : R \to Q', \textsf{true}) \Big)$$

   with $Q' \setminus (R \setminus (K \cup C_1)) = Q \setminus (L \setminus (K \cup C_1))$.

2. *if $c = \exists(a_0 : C_0 \to C_1, \textsf{true})$:*

$$\bigvee_{Q \in \mathrm{ol}(L, C_1)} \bigvee_{Q' \subset Q} \Big(\exists(a : L \to Q', \textsf{true}) \wedge (\bigvee_{\substack{\overline{Q} \subset Q \\ Q' \subset \overline{Q}}} \mathrm{Left}(\exists(a : R \to \overline{Q}, \textsf{true})) \Big)$$

References

[1] C. Sandmann and A. Habel. Rule-based graph repair. *arXiv preprint arXiv:1912.09610*, 2019.