Masterarbeit

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Abstract

Contents

1 Preliminaries

Definition 1.1 (**subgraph**). *Let G_1 and G_2 be graphs. The graph G_2 is called a subgraph of G_1 if an injective morphism $f : G_2 \to G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective.*

Definition 1.2 (**minimal uppergraph**). *Let G_1 and G_2 be graphs with $G_1 \subseteq G_2$. A graph C is called an minimal uppergraph of G_1 w.r.t G_2, if $G_1 \subset C \subseteq G_2$ and no graph $C' \subset C$ with $G_1 \subset C \subseteq G_2$ exists. The set of minimal uppergraphs of G_1 w.r.t. G_2 is denoted by $\mathcal{U}_{G_1,G_2}$.*

Definition 1.3 (**overlap**). *Let G and G' be graphs. A graph H is called an overlap of G and G' if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by $\mathrm{ol}(G, G')$.*

Definition 1.4 (**overlap at morphism**). *Let C,G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an overlap of G and C' at p if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by $\mathrm{ol}_p(G, C')$.*

Definition 1.5 (**partial morphism**). *Let $f : G_1 \to G_2$ and $g : G_3 \to G_4$ be morphisms. The morphism g is called a partial morphism of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.*

Definition 1.6 (**nested graph condition**). *A graph condition over a graph C_0 is inductively defined as follows:*

- *true is a graph condition over every graph.*

- *$\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1.*

- *$\neg d$ is a graph condition over C_0 if d is a graph condition over C_0.*

- *$d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0.*

Conditions over the empty graph $\emptyset$ are called constraints. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies true. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c. An injective morphism satisfies $\neg d$ if it does not satisfy d, it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \wedge d_2$ if it satisfies d_1 or d_2. A graph G satisfies a constraint c, $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c. We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg\exists(a : C_0 \hookrightarrow C_1, \neg d)$.

The nesting level nl of a condition is defined as $\mathrm{nl}(\text{true} = 0$ and $\mathrm{nl}(\exists(a : P \to Q, d)) := \mathrm{nl}(d) + 1$.

Definition 1.7 (alternating quantifier normal form (ANF)[1]). *A graph condition* c *is in* alternating normal form *(ANF) if it is of the form*

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \ldots))))$$

with $Q \in \{\exists, \forall\}$ *and* $\overline{Q} = \exists$ *if* $Q = \forall$, $\overline{Q} = \forall$ *if* $Q = \exists$.

2   partial consistency improving

2.1   extended alternating quantifier normal form

Definition 2.1 (extended alternating quantifier normal form). *A constraint* c *is in* extended alternating quantified normal form *(EANF) if* c *is in ANF and universally bound.*

Lemma 2.2. *Any constraint in ANF can be transformed into an equivalent constraint in EANF.*

Proof. Let c be a constraint in ANF. If c is universally bound, c is already in EANF.

If $c = \exists(a_1 : \emptyset \hookrightarrow C_0, e)$ is existentially bound, we show that c is equivalent to $d := \forall(a_0 : \emptyset \hookrightarrow \emptyset, c)$. Let G be a graph.

"$\Longrightarrow$:" Let $p : \emptyset \hookrightarrow G$ be a morphism with $p \models c$, therefore a morphism $q : C_0 \to G$ with $q \models e$ and $p = q \circ a_0$ exists. Then, $p \models d$, since p is the only morphism from $\emptyset$ to G and $p = p \circ a_1$ and $p \models c$.

"$\Longleftarrow$:" Let $p : \emptyset \hookrightarrow G$ be a morphism with $p \models d$, therefore all morphisms $q : \emptyset \hookrightarrow G$ with $p = q \circ a_0$ satisfy c. Since $p = p \circ a_0$, it follows immediately that $p \models c$. $\qquad\square$

2.2   conditions up to layer

Definition 2.3 (Layer of a subcondition). *Let* c *be a condition and* d *a subcondition of* c. *The layer of* d *is defined as* $\mathrm{lay}(d) := \mathrm{nl}(c) - \mathrm{nl}(d) - 1$.

Definition 2.4 (substitution at layer). *Let* c *be a condition, such that the subcondition of* c *with layer* $0 \leq k \leq \mathrm{nl}(c)$ *is an condition over* C_k. *Let* e *be a condition over* C_k. *The* substitution of $c = Q(a : C_0 \hookrightarrow C_1, d)$ *at layer* k *with* e, $\mathrm{sub}(k, c, e)$, *is recursively defined as:*

1. *If* $k = 0$:
$$\mathrm{sub}(0, c, e) := e$$

2. *If* $k > 0$:
$$\mathrm{sub}(k, c, e) := Q\big(a : C_0 \hookrightarrow C_1, \mathrm{sub}(k - 1, d, e)\big)$$

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Definition 2.5 (**Condition up to layer**). *Let c be a condition and d be the subcondition of c at layer $0 \leq k \leq \mathrm{nl}(c)$. The condition up to layer k of c, $\mathrm{cond}(k, c)$ is defined as*

$$\mathrm{cond}(k, c) := \begin{cases} \mathrm{sub}(k, c, \text{\textit{true}}) & \text{, if } k = 0 \vee d \text{ is existentially bound} \\ \mathrm{sub}(k, c, \text{\textit{false}}) & \text{, if } d \text{ is universally bound.} \end{cases}$$

Definition 2.6 (**Satisfaction up to layer**). *Let G be a graph and c be a condition over C_0. A morphism $p : C_0 \hookrightarrow G$ satisfies c up to layer k, $p \models_k c$, if*

$$p \models \mathrm{cond}(k, c).$$

A graph G satisfies a constraint c up to layer k, $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies $\mathrm{cond}(k, c)$.

Lemma 2.7. *Let G be a graph $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, then for any condition f over C_k holds:*

$$p \models \mathrm{sub}(k, c, f)$$

Proof. Let k be the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound, let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be this subcondition. Let $q : G_{k-1} \rightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$. This must exist, since $p \models_k c$ and k is the smallest number such that $p \models_k v$ and the subcondition of c with layer k is universally bound.

Therefore, there does not exist a morphism $q' : C_k \rightarrow G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \rightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$. $\square$

Lemma 2.8. *Let G be a graph, $p : C_0 \rightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\mathrm{lay}(d) = k$ is universally bound,*

$$p \models_k c \implies p \models c.$$

Proof. Follows immediately by using lemma 2.7 and setting f to the subcondition of c with layer $k + 1$. $\square$

Lemma 2.9. *Let c be a condition in ANF over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$. Let $d = Q(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k + 2$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that*

$$p \models \mathrm{sub}\big(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f)\big)$$

with f being a $\overline{Q}$ bound condition over C'.

Proof. If $p \models c$, we can choose $C' = C_{k+2}$ and $f = e$.

If $p \not\models c$, there does not exists a j with $p \models_j c$ and the subcondition of c with layer j is universally bound and $Q = \exists$ follows immediately. We choose $C' = C_{k+1}$ and $f = \text{true}$. Let $q : C_k \to G$ with $p = q \circ a_k \circ \ldots \circ a_1$ and $q \circ \ldots \circ a_\ell$ satisfying the condition up to $\ell - k$ of the subcondition of c at layer ℓ for all $0 \leq \ell \leq k$. This morphism must exists since $p \models_k c$ and $p \not\models c$. Let $q' : C_{k+1} \to G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a'_{k+2} has to be the identity and therefore $q' = q' \circ a'_{k+2}$. It follows that $q' \models \exists(a'_{k+2} : C_{k+1} \hookrightarrow C', \text{true})$ and therefore $p \models \text{sub}\big(k+1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f)\big)$. $\qquad\square$

Definition 2.10 (partial condition). *Let c be a condition in ANF over C_0. Let d be the subcondition of c at layer $k+1$. The* partial condition of c at layer k with C', $\text{part}(k, c, C')$ *is defined as:*

1. *If d is universally bound, let $e = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$ be the subcondition of c at layer $k+2$ with $C_{k+1} \subseteq C' \subseteq C_{k+2}$:*

$$\text{part}(k, c, C') := \text{sub}\big(k+2, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true})\big)$$

2. *If $d = \exists(a : C_k \hookrightarrow C_{k+1}, f)$ is existentially bound with $C_k \subseteq C' \subseteq C_{k+1}$:*

$$\text{part}(k, c, C') := \text{sub}\big(k+1, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true})\big)$$

Definition 2.11 (biggest partially satisfying condition). *Let G be a graph, c a condition over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$.*

A partial condition $c = \text{part}(k, c, C')$ with $p \models c$ is a biggest partially satisfying condition *if there does not exist a graph $C' \subset C''$ with $p \models \text{part}(k, c, C'')$.*

The set of graphs C' such that $\text{part}(k, c, C')$ is a biggest partially satisfying condition is denoted by $\mathcal{G}_{k,c}$.

2.3 minimal consistency improving

Definition 2.12 (number of violations). *Let G be a graph and c a constraint in EANF, such that $G \models_k c$. The* number of violations $\text{nvc}(k, G)$ *at layer k in G is set to 0 if there does exist a $j > k$ with $G \models_j c$ and otherwise:*

1. *If $k \neq 0$, let $d = \forall(a_k : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c at layer $k+1$.*

$$\text{nvc}(k, G) := \sum_{C \in \mathcal{G}_{k,c}} \sum_{C' \in \mathcal{U}_{C, C_{k+2}}} |\{q \mid q : C_{k+2} \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}|$$

2. *For $j > k$:*

$$\text{nvc}(k, G) := \infty$$

Definition 2.13 (**minimal consistency improving**)**.** *Let G be a graph, r a rule and c a constraint in ANF with $G \models_k c$ and an $j > k$ with $G \models_j c$ does not exist.*

A transformation $G \overset{r,m}{\Longrightarrow} H$ is called minimal consistency improving, *if*

$$\text{nvc}(k, H) < \text{nvc}(k, G) \vee (G \models c \implies H \models c)$$

A rule r is called minimal consistency improving, *if all of its applications are.*

Lemma 2.14. *Let G be $p : C_0 \to G$ a morphism, c a constraint in ANF over C_0 with $p \models_k c$. Then $p \models_j c$ for all $j < k$ such that the subcondition of c at layer j is existentially bound.*

Proof.   1. The subcondition of c at layer k is existentially bound: If an $j < k$ with $p \models_j c$ exists such that the subcondition of c at layer j is universally bound, let j_1 be the smallest of these. With lemma 2.7 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that the subcondition of c at layer j is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of the subcondition of c at layer ℓ. Since $\ell < j_1$, a morphism $q : C_\ell \to G$ with $q \models d$ must exists and therefore a morphism $q' : C_{\ell+1} \to G$ with $q = q' \circ a_k$ must exists. If follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \mathsf{true})$ and with that $p \models_\ell c$.

    2. The subcondition of c at layer k is universally bound: With lemma 2.7 follows that $p \models_{k+1} c$. Since c is in ANF 1. can be applied to $k + 1$.
    □

Lemma 2.15. *Let G be a graph, r a rule and c a constraint in ANF with $G \not\models c$. Let k be the biggest number, such that $G \models_k c$. A transformation $G \overset{r,m}{\Longrightarrow} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.*

Proof. Since $G \not\models c$, with lemma 2.8 follows that the subcondition of c at layer k has to be existentially bound and since k is the biggest number such that $G \models_k c$ it follows that $\text{nvc}(k + 1, G) > 0$. If the subcondition of c at layer j is universally bound, $H \models c$ follows with lemma 2.8 and lemma 2.14 $H \models_{k+2} c$. Therefore $\text{nvc}(k + 2, H) = 0$. Otherwise the subcondition of c at layer j is existentially bound and therefore $G \models_{k+2} c$ and $\text{nvc}(k + 2, H) = 0$ follows immediately.
    □

Definition 2.16 (**direct minimal consistency improving**)**.** *Let G be a graph, r a plain rule and c a constraint in ANF with $G \models \text{part}(k, c, C')$ and $G \not\models_j c$ for all $j < k$. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ be the condition at layer k of c. A transformation $t : G \overset{r,m}{\Longrightarrow} H$ is called* direct minimal consistency improving *if $H \models_k c$ and*

$$\begin{aligned}
&\forall p : C_{k+1} \hookrightarrow G((p \models \text{part}(1, d, C') \wedge \text{tr}_t \circ p \text{ is total}) \implies \text{tr}_t \circ p \models \text{part}(1, d, C')) \wedge \\
&\forall p' : C_{k+1} \hookrightarrow H(\neg \exists p : C \hookrightarrow G(p' = \text{tr}_t \circ p) \implies p' \models \text{part}(1, d, C'))
\end{aligned} \quad (2.1)$$

and

$$\exists p' : C_k \hookrightarrow G\Big(p \not\models \mathrm{part}(1, d, C'') \implies \big((\mathrm{tr}_t \circ p' \text{ is total } \wedge \mathrm{tr}_t \circ p \models \mathrm{part}(1, d, C''))$$
$$\vee \; \mathrm{tr}_t \circ p \text{ is not total}\big)\Big). \tag{2.2}$$

Lemma 2.17. *Let a graph G, a constraint c and a direct minimal improving transformation $t : G \Rightarrow_{r,m} H$ w.r.t. c be given. Then, t is also a minimal improving transformation.*

Proof. Let k be such that $G \models_k c$ and no $j > k$ with $G \models_j c$ exists and let $C_{k+1} \subseteq C \subseteq C_{k+2}$ such that $G \models \mathrm{part}(k, c, C)$ and no $C \subset C' \subseteq C_{k+2}$ with $G \models \mathrm{part}(k, c, C')$ exists. Let d be the subcondition of c at layer $k + 1$.

1. We show that equation (2.1) implies that $\mathrm{nvc}(H, k) \leq \mathrm{nvc}(G, k)$. Assume that $\mathrm{nvc}(H, c) > \mathrm{nvc}(G, c)$. Therefore a morphism $p : C_k \to G$ with $p \not\models \mathrm{part}(1, d, C')$ for a $C' \in \mathcal{U}_C$ exists, such that either 1a or 1b is satisfied.

   (a) There does not exist a morphism $q : C_k \to G$ such that $p = \mathrm{tr}_t \circ q$.
   (b) There does exist a morphism $q' : C_k \to G$ with $q' \models \mathrm{part}(1, d, C')$ and $p = \mathrm{tr}_t \circ q'$.

   This is a contradiction, since if 1a is satisfied, q does not satisfy the first part of equation 2.1 and if 1b is satisfied q' does not satisfy the second part of equation 2.1.

2. Since (2.2) is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \mathrm{part}(1, d, C'')$, such that either $\mathrm{tr} \circ p$ is total and $p \models \mathrm{part}(1, d, C'')$ or $\mathrm{tr} \circ p$ is not total. In both cases the following holds

   $$\mathrm{tr} \circ p \notin \{q \mid q : C_{k+2} \hookrightarrow H \wedge q \not\models \mathrm{part}(1, e, C')\}.$$

With 1 and 2 follows that $\mathrm{nvc}(k, G) < \mathrm{nvc}(k, G)$ and therefore t is a minimal improving transformation. $\qquad\square$

3 application condition

Definition 3.1 (**extended overlap**). *Let G and $C_0 \subset C_1$ be graphs and C' an overlap of G and C_0 with overlap morphisms $p : G \hookrightarrow C'$ and $q : C_0 \hookrightarrow C'$ An overlap C'' of G and C_1 is called the* extended overlap *of C' with C'' if $C' \subset C''$ and a morphism $q' : C_1 \hookrightarrow C''$ with $q'_{|C_0} = q$ exists.*

Definition 3.2 (**overlap shift**). *Let $r = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ be a plain rule, C a graph and C' an overlap of C and L with morphisms $p : L \hookrightarrow C'$, $k : K \hookrightarrow C'$, $c : C \hookrightarrow C'$ and the partial morphism $q : R \hookrightarrow C'$. We define*

$$D := \{e \in C' \mid (\exists e' \in L : p(e') = e$$
$$\vee \; \exists e' \in R : q(e') = e) \tag{3.1}$$
$$\wedge \; \exists e' \in C : c(e') = e\}$$

Let $r = L \overset{l}{\hookleftarrow} K' \overset{r}{\hookrightarrow} R$ be the rule with

$$K' := K \cup D$$

The graph H derived by the transformation $G \Rightarrow_{r,p} H$ is called the overlap shifted graph of C'.

Definition 3.3. Let $r = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ be a plain rule and c a constraint in ANF. The application condition ap_k of the condition at layer k of c is defined as:

1. If $d = \exists(a_k : C_k \hookrightarrow C_{k+1}, e)$ is the condition at layer k of c, with $C_k \subseteq C \subseteq C_{k+1}$, let P' be the overlap shifted graph of P with C:

$$\mathrm{ap}(k, C) := \bigvee_{P \in \mathrm{ol}(L, C_{k+1})} \neg \exists(L \hookrightarrow P, \textit{true}) \wedge \mathrm{Left}(\exists(R \hookrightarrow P', \textit{true}), r)$$

2. If $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ is the condition at layer k of c:

   (a) If $e = \textsf{false}$, for all $C_{k+1} \subseteq C$:

$$
\begin{aligned}
\mathrm{ap}(k, C) := &\bigvee_{P \in \mathrm{ol}(L, C_{k+1})} \mathrm{del}(P, C) \wedge \\
&\left(\bigwedge_{P \in \mathrm{ol}(L, C_{k+1})} \mathrm{ex}(P, C') \right) \wedge \\
&\left(\bigwedge_{P \in \mathrm{ol}(R, C_{k+1})} \mathrm{rem}(P, C') \right)
\end{aligned}
\tag{3.2}
$$

   (b) If $e = \exists(a_{k+1} : C_{k+1} \hookrightarrow C_{k+2}, f)$ with $C_{k+1} \subseteq C \subseteq C_{k+2}$:

$$
\begin{aligned}
\mathrm{ap}(k, C') := &\left(\bigvee_{P \in \mathrm{ol}(L, C_{k+1})} \mathrm{nex}(P, C') \wedge (\mathrm{rep}(P, C') \vee \mathrm{del}(P, C')) \right) \wedge \\
&\left(\bigwedge_{P \in \mathrm{ol}(L, C_{k+1})} \mathrm{ex}(P, C') \right) \wedge \\
&\left(\bigwedge_{P \in \mathrm{ol}(R, C_{k+1})} \mathrm{rem}(P, C') \right)
\end{aligned}
\tag{3.3}
$$

with

1. Let Q be the extended overlap of P with C'.

$$\mathrm{nex}(P, C') := \exists(a : L \hookrightarrow P, \neg \exists(b : P \hookrightarrow Q, \textit{true}))$$

2. Let Q be the extended overlap of P with C', Q' the overlap shifted graph of Q and P' the overlap shifted graph of P.

$$\mathrm{rep}(P, C') := \mathrm{Left}(\forall(a : R \hookrightarrow P', \exists(b : P' \hookrightarrow Q', \textit{true})), r)$$

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3. Let $i_1 : L \hookrightarrow P$ and $i_2 : C_k \hookrightarrow P$ be the overlap morphisms of P:

$$\mathrm{del}(P, C) := \begin{cases} \exists(L \hookrightarrow P, \textit{true}) & , \text{ if } i_1(L \setminus K) \cap i_2(C_k \setminus C_{k-1}) \neq \emptyset \\ \textit{false} & , \text{ otherwise} \end{cases}$$

4. Let $i' : L \hookrightarrow P$ and $i_j : C_j \hookrightarrow P$ be the inclusion morphisms for all $j \leq k$, let E be the set of all existentially bound graphs C_j with $j \leq k$:

$$\mathrm{ex}(P, C) := \begin{cases} \neg\exists(L \hookrightarrow P, \textit{true}) & , \text{ if } \bigcup_{C_j \in E}\big(i_j(C_j \setminus C_{j-1}) \cap i'(L \setminus K)\big) \neq \emptyset \\ \textit{true} & , \text{ otherwise} \end{cases}$$

5. Let $i' : R \hookrightarrow P$ and $i_j : C_j \hookrightarrow P$ be the inclusion morphisms for all $j \leq k$, let U be the set of all universally bound graphs C_j with $j \leq k$:

$$\mathrm{rem}(P, C) := \begin{cases} \mathrm{Left}(\neg\exists(R \hookrightarrow P, \textit{true}), r) & , \text{ if } \bigcup_{C_j \in U}\big(i_j(C_j \setminus C_{j-1}) \cap i'(R \setminus K)\big) \neq \emptyset \\ \textit{true} & , \text{ otherwise} \end{cases}$$

Lemma 3.4. *Let G be a graph, c a constraint in ANF, such that $G \models_k c$ and r a plain rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(b : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the condition at layer k of c. Then, every application of r, equipped with $\mathrm{ap}(k, C')$ with $C' \notin \mathcal{B}_{k,\textit{true}}$, such that a $C \in \mathcal{B}_{k,\textit{true}}$ with $C \subset C'$ exists is minimal consistency improving.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 3.5. *Let G be a graph, c a constraint in ANF, such that $G \models_k c$, and $r = L \xleftarrow{l} K \xrightarrow{r} R$ a plain rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(b : C_{k+1} \hookrightarrow C_{k+2}, e))$ be the condition at layer k of c and $\mathrm{ap}(k, C')$ with $C' \notin \mathcal{B}_{k,\textit{true}}$, such that a $C \in \mathcal{B}_{k,\textit{true}}$ with $C \subset C'$ exists. The following simplifications apply:*

1. *Let $P \in \mathrm{ol}(L, C_{k+1})$. If an injective morphism $p : P \hookrightarrow G$ does not exist, $\mathrm{nex}(P, C')$, $\mathrm{rep}(P, C')$, $\mathrm{del}(P, C')$ and $\mathrm{ex}(P, C')$ can be replaced by \textit{false}.*

2. *If $(L \setminus K) \cap C_{k+1} = \emptyset$, every $\mathrm{del}(P, C')$ can be replaced by \textit{false}.*

3. *If $(R \setminus K) \cap C' = \emptyset$, every $\mathrm{rep}(P, C')$ can be replaced by \textit{false}.*

4. *If 1. and 2. apply, $\mathrm{ap}(k, C')$ can be replaced by \textit{false}.*

5. *If $(R \setminus K) \cap C_{k+1} = \emptyset$, every $\mathrm{rem}(P, C')$ can be replaced by \textit{true}.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.1 potentially minimal improving rules

Definition 3.6 (potentially minimal improving rule). *Let a plain rule $r = L \xleftarrow{l} K \xrightarrow{r} R$ be given.*

Let E be the set of all existentially bound graphs and U be the set of all universally bound graphs C_j of c with $j \leq k$. The rule r is called potentially minimal improving *at layer k, if*

$$(L \setminus K) \cap \bigcup_{C_j \in E} C_j \setminus C_{j-1} = \emptyset$$

and

$$(R \setminus K) \cap \bigcup_{C_j \in U} C_j \setminus C_{j-1} = \emptyset$$

and either

$$(L \setminus K) \cap (C_{k+1} \setminus C_k) \neq \emptyset$$

or

$$(R \setminus K) \cap (C_{k+2} \setminus C') \neq \emptyset$$

Definition 3.7 (appl. conditions for potentially minimal improving rules). *Let a graph G, a constraint c, with $G \models \mathrm{part}(k, c, C')$ and $G \not\models_j c$ for all $j > k$, and a potentially minimal improving rule $r = L \xleftarrow{l} K \xrightarrow{r} R$ be given.*

$$\mathrm{ap}(k, C) := \bigvee_{P \in \mathrm{ol}(L, C_{k+1})} \exists(L \hookrightarrow P, \neg\exists(P \hookrightarrow Q, \textit{true}))$$

$$\mathrm{ap}(k, C) := \bigvee_{P \in \mathrm{ol}(L, C_{k+1})} \exists(L \hookrightarrow P, \neg\exists(P \hookrightarrow Q, \textit{true})) \wedge \mathrm{Left}(\exists(R \hookrightarrow Q', \textit{true}), r)$$

Definition 3.8. *Let a constraint c, and a plain rule $r = L \xleftarrow{l} K \xrightarrow{r} R$ be given.*

Let E be the set of all existentially bound graphs and U be the set of all universally bound graphs C_j of c with $j \leq k$. The rule r is called potentially minimal improving *at layer k with $C_k \subseteq C \subseteq C_{k+1}$, if*

$$(L \setminus K) \cap \bigcup_{C_j \in E} C_j \setminus C_{j-1} = \emptyset$$

and

$$(R \setminus K) \cap \bigcup_{C_j \in U} C_j \setminus C_{j-1} = \emptyset$$

and either 1 or 2 applies.

1.

$$L \subseteq C_k$$

with $(L \setminus K) \cap (C_k \setminus C_{k-1}) \neq \emptyset$

2.

$$C' \subseteq R$$

with $(R \setminus K) \cap (C_{k+1} \setminus C') \neq \emptyset$ and $C_{k-1} \subseteq C \subseteq C' \subseteq C_k$. If *1* applies, r is called a deleting potentially improving rule. If *2* applies, r is called a inserting potentially improving rule.

Definition 3.9 (appl. conditions for potentially minimal improving rules). *Let a constraint c in EANF and a potentially minimal improving rule $r = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ at layer k with C be given.*

$$\mathrm{ap}(j, C) := \begin{cases} \exists\big(L \hookrightarrow C_k, \neg\exists(C_k \hookrightarrow C, \textsf{true})\big) & , \text{ if } j = k \\ \textsf{false} & , \text{ if } j \neq k \end{cases}$$

Lemma 3.10. *Let a graph G, a constraint c with $G \models \mathrm{part}(k, c, C')$ and $G \not\models_j c$ for all $j > k$ and a potentially minimal improving rule $r = L \overset{l}{\hookleftarrow} K \overset{r}{\hookrightarrow} R$ at layer k with C' be given. Then, the rule r, equipped with $\mathrm{ap}(k, C)$ is a minimal consistency improving rule.*

Proof. □

Definition 3.11 (repairing rule set). *Let a graph G, a constraint c with $G \not\models c$ and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is called a repairing rule set if a sequence*

$$G = G_0 \Rightarrow_{r_0} \ldots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \ldots, n-1\}$ and $H \models c$.

Lemma 3.12. *Let a graph G, a constraint c in EANF with $G \models \mathrm{part}(k, c, C')$ and $G \not\models_j c$ for all $j > k$ and a set of rules $\mathcal{R}$ be given. Then, $\mathcal{R}$ is a repairing set if either 1 or 2 apply*

1. *A rule $r \in \mathcal{R}$ exists, such that r is a deleting potentially minimal improving rule at layer $j < k$.*

2. *Let k' be the biggest number, such that a deleting potentially improving rule for k' exists in $\mathcal{R}$. Then, for every $j = 2i + k$ with $i \in \{0, \ldots, (k' - k)/2 - 1\}$ a inserting potentially improving rule exists.*

References

[1] C. Sandmann and A. Habel. Rule-based graph repair. *arXiv preprint arXiv:1912.09610*, 2019.