

Rule-based Graph Repair using Minimally Restricted Consistency-Improving Transformations

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Abstract

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1 Preliminaries

Definition 1.1 (subgraph). Let G_1 and G_2 be graphs. The graph G_2 is called a subgraph of G_1 if an injective morphism $f : G_2 \rightarrow G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective.

Definition 1.2 (minimal uppergraph). Let G_1 and G_2 be graphs with $G_1 \subseteq G_2$. A graph C is called a minimal uppergraph of G_1 w.r.t G_2 , if $G_1 \subset C \subseteq G_2$ and no graph $C' \subset C$ with $G_1 \subset C' \subseteq G_2$ exists. The set of minimal uppergraphs of G_1 w.r.t. G_2 is denoted by $\mathcal{U}(G_1, G_2)$. If $G_1 = G_2$, we set $\mathcal{U}(G_1, G_2) = \{G_1\}$.

Definition 1.3 (overlap). Let G and G' be graphs. A graph H is called an overlap of G and G' if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by $\text{ol}(G, G')$.

Definition 1.4 (overlap at morphism). Let C, G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an overlap of G and C' at p if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by $\text{ol}_p(G, C')$.

Definition 1.5 (partial morphism). Let $f : G_1 \rightarrow G_2$ and $g : G_3 \rightarrow G_4$ be morphisms. The morphism g is called a partial morphism of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.

Definition 1.6 (nested graph condition). A graph condition over a graph C_0 is inductively defined as follows:

- *true* is a graph condition over every graph.
- $\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1 .
- $\neg d$ is a graph condition over C_0 if d is a graph condition over C_0 .
- $d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0 .

Conditions over the empty graph \emptyset are called constraints. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies *true*. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies d . An injective morphism satisfies $\neg d$ if it does not satisfy d , it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \wedge d_2$ if it satisfies d_1 or d_2 . A graph G satisfies a constraint c , $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c . We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg\exists(a : C_0 \hookrightarrow C_1, \neg d)$.

The nesting level nl of a condition is defined as $\text{nl}(\text{true} = 0$ and $\text{nl}(\exists(a : P \rightarrow Q, d)) := \text{nl}(d) + 1$.

Definition 1.7 (alternating quantifier normal form (ANF)). A graph condition c is in alternating normal form (ANF) if it is of the form

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \dots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 partial consistency improving

2.1 extended alternating quantifier normal form

Definition 2.1 (extended alternating quantifier normal form). A condition c is in extended alternating quantifier normal form (EANF) if it is in ANF, universally bound and ends with a condition of the form $\exists(a_k : C_k \hookrightarrow C_{k+1}, e)$ with $e \in \{\text{true}, \text{false}\}$.

Lemma 2.2. Any constraint in ANF can be transformed into an equivalent constraint in EANF.

Proof. Let c be a constraint in ANF. If c is universally bound and ends with a condition of the form $\exists(a_k : C_k \hookrightarrow C_{k+1}, e)$ with $e \in \{\text{true}, \text{false}\}$, c is already in EANF. We construct the equivalent constraint in EANF by two steps and show that, after each step, the derived constraint is equivalent to c .

1. If $c = \exists(a_1 : \emptyset \hookrightarrow C_0, e)$ is existentially bound, we show that c is equivalent to $d := \forall(a_0 : \emptyset \hookrightarrow \emptyset, c)$. Let G be a graph.
 - “ \implies ”: Let $p : \emptyset \hookrightarrow G$ be a morphism with $p \models c$, therefore a morphism $q : C_0 \rightarrow G$ with $q \models e$ and $p = q \circ a_0$ exists. Then, $p \models d$, since p is the only morphism from \emptyset to G and $p = p \circ a_1$ and $p \models c$.
 - “ \impliedby ”: Let $p : \emptyset \hookrightarrow G$ be a morphism with $p \models d$, therefore all morphisms $q : \emptyset \hookrightarrow G$ with $p = q \circ a_0$ satisfy c . With $p = p \circ a_0$, $p \models c$ follows immediately.
2. If c ends with a condition of the form $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ with $e \in \{\text{true}, \text{false}\}$. We show that d is equivalent to $d' = \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(a_{k+1} : C_{k+1} \hookrightarrow C_{k+1}, e))$. Let G be a graph.
 - (a) If $e = \text{true}$. Let $p : C_k \hookrightarrow G$ be a morphism, we show that $p \models d$ and $p \models d'$. Since every morphism satisfies true , a morphism $q : C_k \hookrightarrow G$ with $p = a_k \circ q$ and $p \not\models e$ cannot exist, therefore $p \models d$. The morphism a_{k+1} has to be the identity on C_{k+1} and therefore for every $q : C_k \hookrightarrow G$ with $p = a_k \circ q$ it holds that $q = a_{k+1} \circ q$ and $q \models e$. It follows that $p \models d'$.
 - (b) If $e = \text{false}$:
 “ \implies ”: Let $p : C_k \hookrightarrow G$ be a morphism with $p \models d$. Since $e = \text{false}$, no morphism $q : C_{k+1} \hookrightarrow G$ with $p = a_k \circ q$ exists. Therefore, no morphism $q : C_{k+1} \hookrightarrow G$ with $p = a_k \circ q$ and $q \not\models \exists(a_{k+1} : C_{k+1} \hookrightarrow C_{k+1}, e)$ exists. It follows, that $p \models d'$.

“ \Leftarrow ”: Let $p : C_k \hookrightarrow G$ be a morphism with $p \models d'$. Since no morphism satisfies false , no morphism $q : C_{k+1} \hookrightarrow G$ satisfies $\exists(a_{k+1} : C_{k+1} \hookrightarrow C_{k+1}, e)$. Hence, there does not exist a morphism $q : C_{k+1} \hookrightarrow G$ with $p = q \circ a_k$ and $p \models d$ follows.

□

2.2 conditions up to layer

Definition 2.3 (Layer of a subcondition). Let c be a condition and d a subcondition of c . The layer of d is defined as $\text{lay}(d) := \text{nl}(c) - \text{nl}(d) - 1$.

Definition 2.4 (substitution at layer). Let $c = Q(a : C_0 \hookrightarrow C_1, d)$ be a condition in ANF, such that the subcondition of c with layer $0 \leq k \leq \text{nl}(c)$ is a condition over C_k . Let e be a condition over C_k . The substitution in c at layer k with e , $\text{sub}(k, c, e)$, is recursively defined as:

1. If $k = 0$:

$$\text{sub}(0, c, e) := e$$

2. If $k > 0$:

$$\text{sub}(k, c, e) := Q(a : C_0 \hookrightarrow C_1, \text{sub}(k-1, d, e))$$

Definition 2.5 (Condition up to layer). Let c be a condition in ANF and d be the subcondition of c at layer $0 \leq k \leq \text{nl}(c)$. The condition up to layer k of c , $\text{cond}(k, c)$, is defined as

$$\text{cond}(k, c) := \begin{cases} \text{sub}(k, c, \text{true}) & , \text{if } k = 0 \vee d \text{ is existentially bound} \\ \text{sub}(k, c, \text{false}) & , \text{if } d \text{ is universally bound.} \end{cases}$$

Definition 2.6 (Satisfaction up to layer). Let G be a graph and c be a condition over C_0 . A morphism $p : C_0 \hookrightarrow G$ satisfies c up to layer k , $p \models_k c$, if

$$p \models \text{cond}(k, c).$$

A graph G satisfies a constraint c up to layer k , $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies $\text{cond}(k, c)$. The biggest k with $G \models_k c$ such that no $j > k$ with $G \models_j c$ exists is denoted by c_{\max} .

Lemma 2.7. Let G be a graph $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, then for any condition f over C_k it holds that

$$p \models \text{sub}(k, c, f).$$

Proof. Let k be the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound, let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be this subcondition. Let $q : G_{k-1} \rightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$. This must exist, since $p \models_k c$ and k is the smallest number such that $p \models_k v$ and the subcondition of c with layer k is universally bound.

Therefore, there does not exist a morphism $q' : C_k \rightarrow G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \rightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$. \square

Lemma 2.8. *Let G be a graph, $p : C_0 \rightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\text{lay}(d) = k$ is universally bound,*

$$p \models_k c \implies p \models c.$$

Proof. Follows immediately by using lemma 2.7 and setting f to the subcondition of c with layer $k + 1$. \square

Lemma 2.9. *Let c be a condition in ANF over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$. Let $d = Q(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k + 2$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that*

$$p \models \text{sub}(k+1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f))$$

with f being a \overline{Q} bound condition over C' .

Proof. If $p \models c$, we can choose $C' = C_{k+2}$ and $f = e$.

If $p \not\models c$, there does not exists a j with $p \models_j c$ and the subcondition of c with layer j is universally bound and $Q = \exists$ follows immediately. We choose $C' = C_{k+1}$ and $f = \text{true}$. Let $q : C_k \rightarrow G$ with $p = q \circ a_k \circ \dots \circ a_1$ and $q \circ \dots \circ a_\ell$ satisfying the condition up to $\ell - k$ of the subcondition of c at layer ℓ for all $0 \leq \ell \leq k$. This morphism must exists since $p \models_k c$ and $p \not\models c$. Let $q' : C_{k+1} \rightarrow G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a'_{k+2} has to be the identity and therefore $q' = q' \circ a'_{k+2}$. It follows that $q' \models \exists(a'_{k+2} : C_{k+1} \hookrightarrow C', \text{true})$ and therefore $p \models \text{sub}(k+1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f))$. \square

Definition 2.10 (partial condition). *Let c be a condition in ANF over C_0 . Let d be the subcondition of c at layer $k + 1$. The partial condition of c at layer k with C' , $\text{part}(k, c, C')$ is defined as:*

1. *If d is universally bound, let $e = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$ be the subcondition of c at layer $k + 2$ with $C_{k+1} \subseteq C' \subseteq C_{k+2}$:*

$$\text{part}(k, c, C') := \text{sub}(k+2, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true}))$$

2. *If $d = \exists(a : C_k \hookrightarrow C_{k+1}, f)$ is existentially bound with $C_k \subseteq C' \subseteq C_{k+1}$:*

$$\text{part}(k, c, C') := \text{sub}(k+1, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true}))$$

Definition 2.11 (biggest partially satisfying condition). Let G be a graph, c a condition over C_0 and $p : C_0 \rightarrow G$ a morphism with $p \models_k c$.

A partial condition $c = \text{part}(c_{\max}, c, C')$ with $p \models c$ is a biggest partially satisfying condition if there does not exist a graph $C' \subset C''$ with $p \models \text{part}(c_{\max}, c, C'')$. The graph C' is called a biggest partially satisfying graph.

The set of biggest partially satisfying conditions of c is denoted by \mathcal{P}_c^G .

The set of all biggest partially satisfying graphs is denoted by \mathcal{G}_c^G .

2.3 minimal consistency improving

Definition 2.12 (number of violations). Let G be a graph and c a constraint in EANF. The number of violations $\text{nvc}(j, G)$ at layer j in G is defined as:

1. If $j < c_{\max}$:

$$\text{nvc}(j, G) := 0$$

2. If $j = c_{\max}$, let $d = \forall(a_k : C_j \rightarrow C_{j+1}, e)$ be the subcondition of c at layer $j + 1$.

$$\text{nvc}(j, G) := \sum_{C \in \mathcal{G}_c} \sum_{C' \in \mathcal{U}(C, C_{j+1})} |\{q \mid q : C_{j+1} \rightarrow G \wedge q \not\models \text{part}(1, e, C')\}|$$

3. If $j > c_{\max}$:

$$\text{nvc}(j, G) := \infty$$

Definition 2.13 (minimal consistency improving). Let a graph G , a rule r and a constraint c in ANF be given.

A transformation $G \implies_{r,m} H$ is called minimal consistency improving, if

$$\text{nvc}(k, H) < \text{nvc}(k, G)$$

for any $0 \leq k \leq \text{nl}(c)$. A rule r is called minimal consistency improving, if all of its applications to graphs G with $G \not\models c$ are.

Lemma 2.14. Let a graph G , a morphism $p : C_0 \rightarrow G$ and a constraint c in ANF over C_0 with $p \models_k c$ be given. Then, $p \models_j c$ for all $j < k$ such that the subcondition of c at layer j is existentially bound.

Proof. 1. The subcondition of c at layer k is existentially bound: If an $j < k$ with $p \models_j c$ exists such that the subcondition of c at layer j is universally bound, let j_1 be the smallest of these. With lemma 2.7 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that the subcondition of c at layer ℓ is existentially bound and let $d = \exists(a_\ell : C_\ell \rightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of the subcondition of c at layer ℓ . Since $\ell < j_1$, a morphism $q : C_\ell \rightarrow G$ with $q \models d$ must exists and therefore a morphism $q' : C_{\ell+1} \rightarrow G$ with $q = q' \circ a_\ell$ must exists. It follows that $q \models \exists(a_\ell : C_\ell \rightarrow C_{\ell+1}, \text{true})$ and with that $p \models_\ell c$.

2. The subcondition of c at layer k is universally bound: With lemma 2.7 follows that $p \models_{k+1} c$. Since c is in ANF, 1. can be applied to $k+1$.

□

Theorem 2.1. *Let a graph G , a rule r and a constraint c in ANF be given. Let $k < \text{nl}(c)$ be the biggest number, such that $G \models_k c$. A transformation $G \Rightarrow_{r,m} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.*

Proof. No $\ell > k$ with $G \models_\ell c$ exists and $G \models_k c$. Hence, $\text{nvc}(k, G) > 0$ and $\text{nvc}(k, G) \neq \infty$. Since $j > k$, $\text{nvc}(k, H) = 0$ and it follows immediately that the transformation is minimal consistency improving. □

Definition 2.15 (direct minimal consistency improving). *Let G be a graph, r a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be the condition at layer $k = \text{c}_{\max} + 1 \leq \text{nl}(c)$ of c and*

$$\mathbf{G} := \bigcup_{C' \in \mathcal{G}_c^G} \mathcal{U}(C', C_{k+1})$$

be the set of all minimal upper-graphs of all biggest partially satisfying graphs. A transformation $t : G \Rightarrow_{r,m} H$ is called direct minimal consistency improving if $G \models_{k-1} c$ and equations (2.1), (2.2) and (2.3) hold.

Every occurrence of C_k in G that satisfies part(1, e , C') for any $C' \in \mathbf{G}$ still satisfies part(1, e , C') in H .

$$\begin{aligned} \forall p : C_k \hookrightarrow G \left(\bigwedge_{C' \in \mathbf{G}} (p \models \text{part}(1, e, C') \wedge \text{tr}_t \circ p \text{ is total}) \right. \\ \left. \implies \text{tr}_t \circ p \models \text{part}(1, e, C') \right) \end{aligned} \quad (2.1)$$

Every new inserted occurrence of C_k by t satisfies part(1, e , C') for all $C' \in \mathbf{G}$.

$$\forall p' : C_k \hookrightarrow H \left(\neg \exists p : C_k \hookrightarrow G (p' = \text{tr}_t \circ p) \implies \left(\bigwedge_{C' \in \mathbf{G}} p' \models \text{part}(1, e, C') \right) \right) \quad (2.2)$$

At least one occurrence of C_k in G that does not satisfy part(1, e , C'), for any $C' \in \mathbf{G}$, either has been destroyed by t or satisfies part(1, e , C') in H .

$$\begin{aligned} \exists p : C_k \hookrightarrow G \left(\bigvee_{C' \in \mathbf{G}} (p \not\models \text{part}(1, e, C') \wedge \text{tr}_t \circ p \text{ is not total} \right. \\ \left. \vee (\text{tr}_t \circ p \text{ is total} \wedge \text{tr}_t \circ p \models \text{part}(1, e, C')) \right) \end{aligned} \quad (2.3)$$

Lemma 2.16. *Let a graph G , a constraint c and a direct minimal improving transformation $t : G \Rightarrow_{r,m} H$ w.r.t. c be given. Then, t is also a minimal improving transformation.*

Proof. Let G be a graph with $k = c_{\max}$ and $G \models \text{part}(k, c, C)$ with $\text{part}(k, c, C) \in \mathcal{P}_c^G$. Let d be the subcondition of c at layer $k + 1$.

1. We show that equations (2.1) and (2.2) imply that $\text{nvc}(k, H) \leq \text{nvc}(k, G)$. Assume that $\text{nvc}(k, H) > \text{nvc}(k, G)$. Therefore, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \text{part}(1, d, C')$ for any $C' \in \mathcal{U}(C, C_{k+1})$ exists, such that either 1a or 1b is satisfied.
 - (a) There does exist a morphism $q' : C_k \hookrightarrow G$ with $q' \models \text{part}(1, d, C')$ and $p = \text{tr}_t \circ q'$.
 - (b) There does not exist a morphism $q : C_k \hookrightarrow G$, such that $p = \text{tr}_t \circ q$.

This is a contradiction, if 1a is satisfied, q' does not satisfy equation (2.1) and if 1b is satisfied q does not satisfy equation (2.2).

2. Since (2.3) is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \text{part}(1, d, C')$, such that either $\text{tr} \circ p$ is total and $p \models \text{part}(1, d, C')$ or $\text{tr} \circ p$ is not total exists, for any $C' \in \mathbf{G}$. In both cases the following holds

$$p \in \{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\} \wedge \\ \text{tr} \circ p \notin \{q \mid q : C_{k+2} \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}.$$

With that and 1 it follows that

$$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| < |\{q \mid q : C_{k+2} \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}|.$$

With 1 and 2 follows that $\text{nvc}(k, G) < \text{nvc}(k, H)$ and therefore t is a minimal improving transformation. \square

3 application condition

Definition 3.1 (extended overlap). Let G and $C_0 \subseteq C_1$ be graphs. Let C be an overlap of C_0 and G with the overlap morphism: $q : C_0 \hookrightarrow C$.

Let $p = L \leftarrow K \hookrightarrow R$ be a plain rule with $K = C_0$, $L = K$ and $R = C_1$. The graph H , derived by the transformation

$$C \Longrightarrow_{p,q} H$$

is called the extended overlap of C with C_1 . The extended overlap of an overlap C with an graph C_1 is denoted by $\text{eol}(C, C_1)$.

Lemma 3.2. Let graphs G , $C_0 \subset C_1$ and an overlap C of G and C_0 be given. Then, $\text{eol}(C, C_1)$ is an overlap of G and C_1 and $C \subset \text{eol}(C, C_1)$.

Proof. The graph $\text{eol}(C, C_1)$ is constructed by the transformation $C \Longrightarrow_{p,q} \text{eol}(C, C_1)$ with $p = C_0 \leftarrow C_0 \hookrightarrow C_1$. Since the comatch $n : C_1 \hookrightarrow \text{eol}(C, C_1)$ exists, $\text{eol}(C, C_1)$ is an overlap of G and C_1 . Because $L = K$, the transformation does not delete any elements and $C \subset \text{eol}(C, C_1)$ follows. \square

Definition 3.3 (overlap shift). Let $r = L \hookrightarrow K \hookrightarrow R$ be a plain rule, C a graph and C' an overlap of C and L with morphisms $p : L \hookrightarrow C'$, $k : K \hookrightarrow C'$, $c : C \hookrightarrow C'$ and the partial morphism $q : R \hookrightarrow C'$. We define

$$\begin{aligned} D := \{e \in C' &| (\exists e' \in L : p(e') = e) \\ &\vee \exists e' \in R : q(e') = e) \\ &\wedge \exists e' \in C : c(e') = e\} \end{aligned} \tag{3.1}$$

Let $r = L \hookrightarrow K' \hookrightarrow R$ be the rule with

$$K' := K \cup D$$

The graph H derived by the transformation $G \Rightarrow_{r,p} H$ is called the overlap shifted graph of C' . The overlap shifted graph of an graph C is denoted by $\text{ols}(C)$.

Definition 3.4. Let $r = L \hookrightarrow K \hookrightarrow R$ be a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer k with $k = 2i - 1$ for an $i \in \mathbb{N}$. The application condition ap_k of the condition at layer k of c with $C' \in \mathcal{U}(C_k, C_{k+1})$ is defined as:

$$\begin{aligned} \text{ap}(k, C') := & \left(\bigvee_{P \in \text{ol}(L, C_k)} \text{nex}(P, C') \wedge (\text{rep}(P, C') \vee \text{del}(P, C')) \right) \wedge \\ & \left(\bigwedge_{P \in \text{ol}(L, C_k)} \text{ex}(P, C') \right) \wedge \\ & \left(\bigwedge_{P \in \text{ol}(R, C_{k+1})} \text{rem}(P, C') \right) \end{aligned} \tag{3.2}$$

with

1.

$$\text{nex}(P, C') := \exists(a : L \hookrightarrow P, \neg \exists(b : P \hookrightarrow \text{eol}(P, C'), \text{true}))$$

2.

$$\text{rep}(P, C') := \text{Left}(\forall(a : R \hookrightarrow \text{ols}(P), \exists(b : \text{ols}(P) \hookrightarrow \text{ols}(\text{eol}(P, C')), \text{true}), r)$$

3. Let $i_1 : L \hookrightarrow P$ and $i_2 : C_k \hookrightarrow P$ be the overlap morphisms of P :

$$\text{del}(P, C') := \begin{cases} \exists(L \hookrightarrow P, \text{true}) & , \text{ if } i_1(L \setminus K) \cap i_2(C_k \setminus C_{k-1}) \neq \emptyset \\ \text{false} & , \text{ otherwise} \end{cases}$$

4. Let $i' : L \hookrightarrow P$ and $i_j : C_j \hookrightarrow P$ be the inclusion morphisms for all $j \leq k$, let E be the set of all existentially bound graphs C_j with $j \leq k$:

$$\text{ex}(P, C') := \begin{cases} \neg \exists(L \hookrightarrow P, \text{true}) & , \text{ if } \bigcup_{C_j \in E} (i_j(C_j \setminus C_{j-1}) \cap i'(L \setminus K)) \neq \emptyset \\ \text{true} & , \text{ otherwise} \end{cases}$$

5. Let $i' : R \hookrightarrow P$ and $i_j : C_j \hookrightarrow P$ be the inclusion morphisms for all $j \leq k$, let U be the set of all universally bound graphs C_j with $j \leq k$:

$$\text{rem}(P, C') := \begin{cases} \text{Left}(\neg \exists(R \hookrightarrow P, \text{true}), r) & , \text{ if } \bigcup_{C_j \in U} (i_j(C_j \setminus C_{j-1}) \cap i'(R \setminus K)) \neq \emptyset \\ \text{true} & , \text{ otherwise} \end{cases}$$

Lemma 3.5. Let G be a graph, c a constraint in EANF, with $G \not\models c$, and r a plain rule. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer c_{\max} . Then, every application of $(r, \text{ap}(c_{\max}, C'))$ with

$$C' \in \bigcap_{C \in \mathcal{G}_c^G} \mathcal{U}(C, C_{k+1})$$

is direct minimal consistency improving.

Proof.

□

Lemma 3.6. Let G be a graph, c a constraint in EANF, with $c_{\max} < \text{nl}(c)$, and $r = L \hookrightarrow K \hookrightarrow R$ a plain rule. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer $c_{\max} + 1$ and $\text{ap}(c_{\max}, C')$ the application condition constructed by definition 3.4 with $C' \in \mathcal{U}(C, C_{k+1})$ for any $C \in \mathcal{G}_c^G$. The following simplifications can be applied:

1. Let $P \in \text{ol}(L, C_k)$. If an injective morphism $p : P \hookrightarrow G$ does not exist, $\text{nex}(P, C')$ can be replaced by **false**.
2. If $(L \setminus K) \cap C_k = \emptyset$, every $\text{del}(P, C')$ can be replaced by **false** and $\text{ex}(P, C')$ can be replaced by **true**.
3. If $(R \setminus K) \cap C' = \emptyset$, every $\text{rep}(P, C')$ can be replaced by **false**.
4. If 2. and 3. apply, $\text{ap}(k, C')$ can be replaced by **false**.
5. If $(R \setminus K) \cap C_{k+1} = \emptyset$, every $\text{rem}(P, C')$ can be replaced by **true**.

Proof. Let $m : L \hookrightarrow G$ be the match of the application of r . We show the simplifications by showing that the replaced parts of $\text{ap}(c_{\max}, C')$ will be evaluated to the values they have been replaced with, if the required conditions are met. Note that $C_j \subseteq C_{k'}$ for all $0 \leq j \leq k'$ and $k' \in \{0, \dots, \text{nl}(c)\}$.

- Proof of 1.: Let $P \in \text{ol}(L, C_k)$ with $a : L \hookrightarrow P$, such that no morphism $p : P \hookrightarrow G$ exists. Therefore, no morphism $q : P \hookrightarrow G$ with $m = p \circ a$ exists and $\text{nex}(P, C')$ will be evaluated to **false**.
- Proof of 2.: Let $(L \setminus K) \cap C_k = \emptyset$ and $P \in \text{ol}(L, C_k)$ with $i' : L \hookrightarrow P$ and $i_j : C_j \hookrightarrow P$ for all $j \leq k$. It follows that $i'(L \setminus K) \cap i_k(C_k \setminus C_{k-1}) = \emptyset$ and with that $\text{del}(P, C') = \text{false}$ for every $P \in \text{ol}(L, C_k)$. Similar it follows that

$$\bigcup_{j \leq k} i_j(C_j \setminus C_{j-1}) \cap i'(L \setminus K) = \emptyset$$

and with that $\text{ex}(P, C') = \text{true}$ for every $P \in \text{ol}(L, C_k)$.

- Proof of 3.: work in progress.
- Proof of 4.: Let the simplifications 2. and 3. apply. Every $\text{del}(P, C')$ and every $\text{rep}(P, C')$ have been replaced by `false`. Therefore, the expression

$$\bigvee_{P \in \text{ol}(L, C_k)} \text{nex}(P, C') \wedge (\text{rep}(P, C') \vee \text{del}(P, C'))$$

will be evaluated to `false` and with that $\text{ap}(k, C')$ will also be evaluated to `false`.

- Proof of 5.: Let $(R \setminus K) \cap C_{k+1} = \emptyset$ and $P \in \text{ol}(R, C_{k+1})$ with the morphisms $i' : R \hookrightarrow P$ and $i_j : C_j \hookrightarrow P$ for all $j \leq k$. It follows that

$$\bigcup_{j \leq k} i'(R \setminus K) \cap i_j(C_j \setminus C_{j-1}) = \emptyset.$$

Therefore $\text{rem}(P, C') = \text{true}$ for all $P \in \text{ol}(R, C_{k+1})$.

□

3.1 potentially minimal improving rules

Definition 3.7 (potentially minimal improving rule). Let a constraint c and a plain rule $r = L \leftrightarrow K \leftrightarrow R$ be given. The rule r is called potentially minimal improving w.r.t c at layer k with $C_k \subseteq C \subseteq C_{k+1}$, if

$$(L \setminus K) \cap (C_{k-1} \cup (C_{k+1} \setminus C_k)) = \emptyset \quad (3.3)$$

and

$$(R \setminus K) \cap C_k = \emptyset \quad (3.4)$$

and either 1. or 2. applies.

1. The rule deletes edges and no node of $C_k \setminus C_{k-1}$:

$$L \subseteq C_k \wedge (L \setminus K) \cap (E_{C_k} \setminus E_{C_{k-1}}) \neq \emptyset \wedge (L \setminus K) \cap (V_{C_k} \setminus V_{C_{k-1}}) = \emptyset \quad (3.5)$$

2. The rule creates an instance of an upper-graph of C :

$$C' \subseteq R \quad \text{with} \quad (R \setminus K) \cap (C_{k+1} \setminus C') \neq \emptyset \quad \text{for any} \quad C' \in \mathcal{U}(C, C_{k+1}) \quad (3.6)$$

If 1 applies, r is called a deleting potentially improving rule. If 2 applies, r is called an inserting potentially improving rule.

Definition 3.8 (appl. conditions for potentially minimal improving rules). Let a constraint c in EANF and a potentially minimal improving rule $r = L \leftrightarrow K \leftrightarrow R$ w.r.t c at layer k with $C_k \subseteq C \subseteq C_{k+1}$ be given. We define the application condition for r as:

$$\text{ap}_{\text{pi}}(j, C) := \begin{cases} \exists(a_0 : L \hookrightarrow C_k, \neg \exists(a_1 : C_k \hookrightarrow C, \text{true})) & , \text{ if } j = k \\ \text{false} & , \text{ if } j \neq k. \end{cases}$$

Theorem 3.1. Let a graph G , a constraint c in EANF, with $G \models \text{part}(c_{\max}, c, C)$ and $\text{part}(c_{\max}, c, C) \in \mathcal{P}_c^G$, and a potentially minimal improving rule $r = L \leftrightarrow K \leftrightarrow R$ at layer c_{\max} with $C_{c_{\max}+1} \subseteq C \subseteq C_{c_{\max}+2}$ be given. Then, the rule $r' = (r, \text{appi}(k, C))$ is a direct minimal consistency improving rule.

Proof. Let $t : G \Rightarrow_{r', m} H$ be a transformation, $k = c_{\max} + 1$ and e be the subcondition of c at layer $k + 1$. We show that t is a direct minimal consistency improving transformation and with that, the statement follows. Firstly, we show that equation (2.1) is satisfied. Let $p : C_k \hookrightarrow G$ be a morphism. If r is a deleting minimal improving rule, either 1. or 2. applies and if r is a inserting and not an deleting minimal improving rule, only 2. applies, because c cannot destroy any occurrence of C_k .

1. If $p(C_k) \cap m(L \setminus K) \neq \emptyset$, $\text{tr}_t op$ is not total, since at least one element of $p(C_k)$ has been deleted by t and p does satisfy $\bigwedge_{C' \in \mathbf{G}} (p \models \text{part}(1, e, C') \wedge \text{tr}_t op \text{ is total}) \implies \text{tr}_t op \models \text{part}(1, e, C')$.
2. If $p(C_k \cap L) \cap m(L \setminus K) = \emptyset$, $\text{tr}_t op$ is total. Because (3.3) holds, t does not delete any elements of C_{k+1} and therefore $p \models \text{part}(1, e, C') \implies \text{tr}_t op \models \text{part}(1, e, C')$ for all $C_k \subseteq C' \subseteq C_{k+1}$.

With 1. and 2. follows that (2.1) is satisfied.

Secondly, we show that equation (2.2) is satisfied. Let $p' : C_k \hookrightarrow H$ be a morphism. Because (3.4) is satisfied, t does not create any elements of C_k and therefore, there must exist an morphism $p : C_k \hookrightarrow G$ with $\text{tr}_t op = p'$. It follows that (2.2) is satisfied.

Lastly, we show that equation (2.3) is satisfied. Since $\text{appi}(k, C)$ is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \exists(C_k \hookrightarrow C, \text{true}) = \text{part}(1, e, C)$ and $p(C_k) \cap m(L) \neq \emptyset$ must exist. If r is an deleting rule, t deletes at least one element of C_k , it follows that $\text{tr}_t op$ is not total and therefore (2.3) is satisfied. If r is an inserting rule and not an deleting one, no element of C_k is deleted by t and therefore $\text{tr}_t op$ is total. Because (3.6) holds, $\text{tr}_t op \models \exists(C_k \hookrightarrow C', \text{true}) = \text{part}(1, e, C')$ for an $C' \in \mathcal{U}(C, C_{k+1})$ and with that (2.3) is satisfied. \square

Definition 3.9 (repairing rule set). Let a constraint c in EANF and a set of rules \mathcal{R} be given. Then, \mathcal{R} is called a repairing rule set for c at layer k if for all graphs G with $k = c_{\max}$ a sequence

$$G = G_0 \Rightarrow_{r_0} \dots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \dots, n - 1\}$ and $H \models c$.

Corollary 3.10. Let a constraint c in EANF and a set of rules \mathcal{R} be given. If \mathcal{R} is a repairing rule set for c at layer k , \mathcal{R} is a repairing rule set w.r.t c at layer j for all $k < j \leq \text{nl}(c)$.

Corollary 3.11. Let a constraint c in EANF and a repairing rule set \mathcal{R} for c at layer 0 be given. Then, for all graphs G , a sequence

$$G = G_0 \Rightarrow_{r_0} \dots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $k \in \{0, \dots, n - 1\}$ and $H \models c$.

Theorem 3.2. Let a constraint c in EANF and a set of rules \mathcal{R} be given. Then, \mathcal{R} is a repairing set for c at layer $k \leq \text{nl}(c)$ if either 1. or 2. applies.

1. For any universally bound graph C_j at layer $j \leq k$ of c , \mathcal{R} contains $(r, \text{appi}(j, C_{j+1}))$ and r is a deleting potentially minimal improving rule at layer j with C_{j+1} .

2. Either 2a applies for all $j \in \{k + 2, \dots, \text{nl}(c)\}$ or 2b applies.

- (a) Let C_j be the existentially bound graph of c at layer j . A set of graphs

$$C_{j-1} = C'_0 \subset \dots \subset C'_n = C_j$$

exists, such that \mathcal{R} contains $(r_\ell, \text{appi}(j, C'_\ell))$ and $r_\ell = L_\ell \leftrightarrow K_\ell \hookrightarrow R_\ell$ with $L_\ell \subset C_{\ell-1}$ is an inserting potentially minimal improving rule at layer j with C_ℓ , for all $\ell \in \{0, \dots, n\}$.

- (b) For an universally bound graph C'_ℓ at layer $\ell > k$, \mathcal{R} contains $(r, \text{appi}(j, C_\ell))$, r is a deleting potentially minimal improving rule at layer ℓ and 2a applies for all $j \in \{k + 1, \dots, j - 1\}$.

Proof. Let a constraint c in EANF, a rule set \mathcal{R} and a graph G with $k = c_{\max}$ and $c_{\max} < \text{nl}(c)$ be given. We show that a sequence $G = C'_0 \Rightarrow \dots \Rightarrow C'_n = H$ with rules of \mathcal{R} exists, such that $H \models c$ if 1. or 2. of theorem 3.2 is satisfied.

1. Assume that 1 of theorem 3.2 holds. holds and let $(r, \text{appi}(j, C_{j+1})) \in \mathcal{R}$, such that $r = L \leftrightarrow K \hookrightarrow R$ is a deleting potentially minimal improving rule at layer $j \leq k$ with C_{j+1} and C_j is a universally bound graph of c . Let $q : C_j \hookrightarrow G$ be a morphism such that $q \not\models \exists(C_j \hookrightarrow C_{j+1}, \text{true})$. Since $L \subseteq C_j$ we can construct a morphism $m_1 : L \hookrightarrow G$ with $m_1(L) = q(L)$. It holds that $m_1 = q \circ a$, therefore $m_1 \models \text{appi}(j, C_{j+1})$. Since r only deletes edges a transformation $t : G = G_0 \Rightarrow_{r, m_1} G_1$ exists and $\text{tr}_t \circ p$ is not total. Because r does not insert any elements of C_j

$$|\{q : C_k \hookrightarrow G_0 \mid q \not\models d\}| < |\{q : C_k \hookrightarrow G_1 \mid q \not\models d\}|$$

with $d = \exists(C_j \hookrightarrow C_{j+1}, \text{true})$. By iteratively applying this construction, we generate a finite sequence of transformations

$$G = G_0 \Rightarrow_{r, m_1} G_1 \Rightarrow_{r, m_2} \dots \Rightarrow_{r, m_n} G_n = H$$

such that $|\{q : C_k \hookrightarrow G_n \mid q \not\models d\}| = 0$ and therefore $H \models_j c$. With lemma 2.8, $H \models c$ follows.

2. Assume that 2. of theorem 3.2 holds. Let $k < j < \text{nl}(c)$, such that $(r, \text{appi}(j, C_{j+1})) \in \mathcal{R}$ and r is a deleting potentially minimal improving rule at layer j . If no such j exists, we set $j = \text{nl}(c) + 1$.

- Let $k < \ell < j$, such that the subcondition of c at layer ℓ is existentially bound and let C_ℓ be the bound graph. There exists a set of graphs $C_{\ell-1} = C'_0 \subset \dots \subset C'_n = C_\ell$ such that $(r_i, \text{appi}(i, C'_i)) \in \mathcal{R}$ and $r_i = L_i \hookrightarrow K_i \hookrightarrow R_i$ is an inserting potentially minimal improving rule at layer ℓ with C'_i for all $i \in \{0, \dots, n\}$. Let $q : C_{\ell-1} \hookrightarrow G$ be a morphism, such that $q \not\models \exists(C_{\ell-1} \hookrightarrow C_1, \text{true})$. Since $L_1 \subset C_{\ell-1}$, we can construct a morphism $m : L_1 \hookrightarrow G$ with $m(L_1) = q(L_1)$.

□

References

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