

Rule-based Graph Repair using Minimally Restricted Consistency-Improving Transformations

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Abstract

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1 Preliminaries

Definition 1.1 (subgraph). Let G_1 and G_2 be graphs. The graph G_2 is called a subgraph of G_1 if an injective morphism $f : G_2 \rightarrow G_1$ exists. We use the notation $G_2 \subseteq G_1$ if G_2 is a subgraph of G_1 and $G_2 \subset G_1$ if f is not bijective.

Definition 1.2 (uppergraph). Let G_1 and G_2 be graphs with $G_1 \subseteq G_2$. A graph C is called an upper-graph of G_1 w.r.t G_2 , if $G_1 \subset C \subseteq G_2$. The set of uppergraphs of G_1 w.r.t. G_2 is denoted by $\mathcal{U}(G_1, G_2)$. If $G_1 = G_2$, we set $\mathcal{U}(G_1, G_2) = \{G_1\}$.

Definition 1.3 (overlap). Let G and G' be graphs. A graph H is called an overlap of G and G' if morphisms $p : G \hookrightarrow H$ and $p' : G' \hookrightarrow H$ such that p and p' are jointly surjective. The set of all overlaps of G and G' is denoted by $\text{ol}(G, G')$.

Definition 1.4 (overlap at morphism). Let C, G and C' with $C \subset C'$ be graphs and $p : C \hookrightarrow G$ a morphism. A graph H is called an overlap of G and C' at p if a morphism $p' : C' \hookrightarrow H$ with $p'|_C = p$ exists. The set of all overlaps of G and C' at p is denoted by $\text{ol}_p(G, C')$.

Definition 1.5 (partial morphism). Let $f : G_1 \rightarrow G_2$ and $g : G_3 \rightarrow G_4$ be morphisms. The morphism g is called a partial morphism of f if $G_3 \subseteq G_1$, $G_4 \subseteq G_2$ and $f|_{G_3} = g$.

Definition 1.6 (nested graph condition). A graph condition over a graph C_0 is inductively defined as follows:

- *true* is a graph condition over every graph.
- $\exists(a : C_0 \hookrightarrow C_1, d)$ is a graph condition over C_0 if a is a injective graph morphism and d is a graph condition over C_1 .
- $\neg d$ is a graph condition over C_0 if d is a graph condition over C_0 .
- $d_1 \wedge d_2$ and $d_1 \vee d_2$ are graph conditions over C_0 if d_1 and d_2 are graph conditions over C_0 .

Conditions over the empty graph \emptyset are called constraints. Every injective morphism $p : C_0 \hookrightarrow G$ satisfies *true*. An injective morphism p satisfies $\exists(a : C_0 \hookrightarrow C_1, d)$ if there exists an injective morphism $q : C_1 \hookrightarrow G$ such that $q \circ a = p$ and q satisfies c . An injective morphism satisfies $\neg d$ if it does not satisfy d , it satisfies $d_1 \wedge d_2$ if it satisfies d_1 and d_2 and it satisfies $d_1 \vee d_2$ if it satisfies d_1 or d_2 . A graph G satisfies a constraint c , $G \models c$, if $p : \emptyset \hookrightarrow G$ satisfies c . We use the abbreviation $\forall(a : C_0 \hookrightarrow C_1, d) := \neg \exists(a : C_0 \hookrightarrow C_1, \neg d)$.

The nesting level nl of a condition is defined as $\text{nl}(\text{true}) = 0$ and $\text{nl}(\exists(a : P \rightarrow Q, d)) := \text{nl}(d) + 1$.

Definition 1.7 (alternating quantifier normal form (ANF)[1]). A graph condition c is in alternating normal form (ANF) if it is of the form

$$c = Q(a_1 : C_0 \hookrightarrow C_1, \overline{Q}(a_2 : C_1 \hookrightarrow C_2, Q(a_3 : C_2 \hookrightarrow C_3, \overline{Q}(a_4 : C_3 \hookrightarrow C_4, \dots))))$$

with $Q \in \{\exists, \forall\}$ and $\overline{Q} = \exists$ if $Q = \forall$, $\overline{Q} = \forall$ if $Q = \exists$.

2 partial consistency improving

2.1 extended alternating quantifier normal form

Definition 2.1 (extended alternating quantifier normal form). *A condition c is in extended alternating quantifier normal form (EANF) if it is in ANF, universally bound and ends with a condition of the form $\exists(a_k : C_k \hookrightarrow C_{k+1}, e)$ with $e \in \{\text{true}, \text{false}\}$.*

Note that, given a condition c in EANF, every subcondition of c at layer $1 \leq k \leq \text{nl}(c)$ is universally bound, if k is an odd number and existentially bound, if k is an even number. Additionally, $\text{nl}(c)$ is always an even number.

Lemma 2.2. *Any condition in ANF can be transformed into an equivalent condition in EANF.*

Proof. Let c be a condition in ANF. If c is universally bound and ends with a condition of the form $\exists(a_k : C_k \hookrightarrow C_{k+1}, e)$ with $e \in \{\text{true}, \text{false}\}$, c is already in EANF. We construct the equivalent condition in EANF by two steps and show that, after each step, the constructed condition is equivalent to c .

1. If $c = \exists(a_1 : C_0 \hookrightarrow C_1, e)$ is existentially bound, we show that c is equivalent to $d := \forall(\text{id}_{C_0} : C_0 \hookrightarrow C_0, c)$. Let G be a graph.

“ \implies ”: Let $p : C_0 \hookrightarrow G$ be a morphism with $p \models c$, therefore a morphism $q : C_0 \rightarrow G$ with $q \models e$ and $p = q \circ a_1$ exists. Then, $p \models d$, since p is the only morphism from C_0 to G with $p = p \circ \text{id}_{C_0}$ and $p \models c$.

“ \impliedby ”: Let $p : C_0 \hookrightarrow G$ be a morphism with $p \models d$, therefore all morphisms $q : C_0 \hookrightarrow G$ with $p = q \circ \text{id}_{C_0}$ satisfy c . Since $p = p \circ \text{id}_{C_0}$, $p \models c$ follows immediately.

2. If c ends with a condition of the form $d := \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ with $e \in \{\text{true}, \text{false}\}$, we show that d is equivalent to $d' := \forall(a_k : C_k \hookrightarrow C_{k+1}, \exists(\text{id}_{C_{k+1}} : C_{k+1} \hookrightarrow C_{k+1}, e))$. Let G be a graph.

- (a) If $e = \text{true}$. Let $p : C_k \hookrightarrow G$ be a morphism, we show that $p \models d$ and $p \models d'$. Since every morphism satisfies true , a morphism $q : C_{k+1} \hookrightarrow G$ with $p = q \circ a_k$ and $p \not\models \text{true}$ cannot exist, therefore $p \models d$. The morphism $\text{id}_{C_{k+1}}$ is the identity on C_{k+1} and therefore for every $q : C_{k+1} \hookrightarrow G$ with $p = q \circ a_k$ it holds that $q = q \circ \text{id}_{C_{k+1}}$ and therefore $q \models \exists(\text{id}_{C_{k+1}} : C_{k+1} \hookrightarrow C_{k+1}, \text{true})$. It follows that $p \models d'$.

- (b) If $e = \text{false}$:

“ \implies ”: Let $p : C_k \hookrightarrow G$ be a morphism with $p \models d$. Since $e = \text{false}$, no morphism $q : C_{k+1} \hookrightarrow G$ with $p = q \circ a_k$ exists. Therefore, no morphism $q : C_{k+1} \hookrightarrow G$ with $p = q \circ a_k$ and $q \not\models \exists(\text{id}_{C_{k+1}} : C_{k+1} \hookrightarrow C_{k+1}, \text{false})$ exists. It follows that $p \models d'$.

“ \impliedby ”: Let $p : C_k \hookrightarrow G$ be a morphism with $p \models d'$. Since no morphism satisfies false , no morphism $q : C_{k+1} \hookrightarrow G$ satisfies $\exists(\text{id}_{C_{k+1}} : C_{k+1} \hookrightarrow C_{k+1}, \text{false})$.

Hence, there does not exist a morphism $q : C_{k+1} \hookrightarrow G$ with $p = q \circ a_k$ and $p \models d$ follows.

□

2.2 conditions up to layer

Definition 2.3 (Layer of a subcondition). Let c be a condition and d a subcondition of c . The layer of d is defined as $\text{lay}(d) := \text{nl}(c) - \text{nl}(d) - 1$.

Definition 2.4 (substitution at layer). Let $c = Q(a : C_0 \hookrightarrow C_1, d)$ be a condition in ANF, such that the subcondition of c with layer $0 \leq k \leq \text{nl}(c)$ is a condition over C_k . Let e be a condition over C_k . The substitution in c at layer k with e , $\text{sub}(k, c, e)$, is recursively defined as:

1. If $k = 0$:

$$\text{sub}(0, c, e) := e$$

2. If $k > 0$:

$$\text{sub}(k, c, e) := Q(a : C_0 \hookrightarrow C_1, \text{sub}(k-1, d, e))$$

Definition 2.5 (Condition up to layer). Let c be a condition in ANF and d be the subcondition of c at layer $0 \leq k \leq \text{nl}(c)$. The condition up to layer k of c , $\text{cond}(k, c)$, is defined as

$$\text{cond}(k, c) := \begin{cases} \text{sub}(k, c, \text{true}) & , \text{ if } k = 0 \vee d \text{ is existentially bound} \\ \text{sub}(k, c, \text{false}) & , \text{ if } d \text{ is universally bound.} \end{cases}$$

Definition 2.6 (Satisfaction up to layer). Let G be a graph and c be a condition over C_0 . A morphism $p : C_0 \hookrightarrow G$ satisfies c up to layer k , $p \models_k c$, if

$$p \models \text{cond}(k, c).$$

A graph G satisfies a constraint c up to layer k , $G \models_k c$, if $q : \emptyset \hookrightarrow G$ satisfies $\text{cond}(k, c)$. The biggest k with $G \models_k c$ such that no $j > k$ with $G \models_j c$ exists is denoted by c_{\max} .

Lemma 2.7. Let G be a graph $p : C_0 \hookrightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition $d = Q(a_k : C_{k-1} \hookrightarrow C_k, e)$ of c at layer k is universally bound, then for any condition f over C_k it holds that

$$p \models \text{sub}(k, c, f).$$

Proof. Let k be the smallest number such that $p \models_k c$ and the subcondition of c with layer k is universally bound, let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be this subcondition. Let $q : G_{k-1} \rightarrow G$ be a morphism such that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, \text{false})$. This must exist,

since $p \models_k c$ and k is the smallest number such that $p \models_k v$ and the subcondition of c with layer k is universally bound.

Therefore, there does not exist a morphism $q' : C_k \rightarrow G$ with $q = q' \circ a_k$. Hence, for every condition f over C_k a morphism $q' : C_k \rightarrow G$ with $q \not\models f$ and $q = q' \circ a_k$ cannot exist. It follows immediately that $q \models \forall(a_k : C_{k-1} \hookrightarrow C_k, f)$. \square

Lemma 2.8. *Let G be a graph, $p : C_0 \rightarrow G$ a morphism and c a condition over C_0 in ANF with $p \models_k c$. If the subcondition d of c with $\text{lay}(d) = k$ is universally bound,*

$$p \models_k c \implies p \models c.$$

Proof. Follows immediately by using lemma 2.7 and setting f to the subcondition of c with layer $k + 1$. \square

Lemma 2.9. *Let c be a condition in ANF over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$. Let $d = Q(a_{k+2} : C_{k+1} \hookrightarrow C_{k+2}, e)$ be the subcondition of c with layer $k + 2$. There does exist a graph $C_{k+1} \subseteq C' \subseteq C_{k+2}$ such that*

$$p \models \text{sub}(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f))$$

with f being a \bar{Q} bound condition over C' .

Proof. If $p \models c$, we can choose $C' = C_{k+2}$ and $f = e$.

If $p \not\models c$, there does not exist a j with $p \models_j c$ and the subcondition of c with layer j is universally bound and $Q = \exists$ follows immediately. We choose $C' = C_{k+1}$ and $f = \text{true}$. Let $q : C_k \rightarrow G$ with $p = q \circ a_k \circ \dots \circ a_1$ and $q \circ \dots \circ a_\ell$ satisfying the condition up to $\ell - k$ of the subcondition of c at layer ℓ for all $0 \leq \ell \leq k$. This morphism must exist since $p \models_k c$ and $p \not\models c$. Let $q' : C_{k+1} \rightarrow G$ be a morphism with $q = q' \circ a_{k+1}$. Since $C' = C_{k+1}$, the morphism a'_{k+2} has to be the identity and therefore $q' = q' \circ a'_{k+2}$. It follows that $q' \models \exists(a'_{k+2} : C_{k+1} \hookrightarrow C', \text{true})$ and therefore $p \models \text{sub}(k + 1, c, Q(a_{k+2} : C_{k+1} \hookrightarrow C', f))$. \square

Definition 2.10 (partial condition). *Let c be a condition in ANF over C_0 . Let d be the subcondition of c at layer $k + 1$. The partial condition of c at layer k with C' , $\text{part}(k, c, C')$ is defined as:*

1. *If d is universally bound, let $e = \exists(a : C_{k+1} \hookrightarrow C_{k+2}, f)$ be the subcondition of c at layer $k + 2$ with $C_{k+1} \subseteq C' \subseteq C_{k+2}$:*

$$\text{part}(k, c, C') := \text{sub}(k + 2, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true}))$$

2. *If $d = \exists(a : C_k \hookrightarrow C_{k+1}, f)$ is existentially bound with $C_k \subseteq C' \subseteq C_{k+1}$:*

$$\text{part}(k, c, C') := \text{sub}(k + 1, c, \exists(a : C_{k+1} \hookrightarrow C', \text{true}))$$

Definition 2.11 (biggest partially satisfying condition). Let G be a graph, c a condition over C_0 and $p : C_0 \hookrightarrow G$ a morphism with $p \models_k c$.

A partial condition $c = \text{part}(c_{\max}, c, C')$ with $p \models c$ is a biggest partially satisfying condition if there does not exist a graph $C' \subset C''$ with $p \models \text{part}(c_{\max}, c, C'')$. The graph C' is called a biggest partially satisfying graph.

The set of biggest partially satisfying conditions of c is denoted by \mathcal{P}_c^G .

The set of all biggest partially satisfying graphs is denoted by \mathcal{G}_c^G .

2.3 minimal consistency improving

Definition 2.12 (number of violations). Let G be a graph and c a constraint in EANF. The number of violations $\text{nvc}(j, G)$ at layer j in G is defined as:

1. If $j < c_{\max} + 1$:

$$\text{nvc}(j, G) := 0$$

2. If $j = c_{\max} + 1$, let $d = \forall(a_k : C_j \hookrightarrow C_{j+1}, e)$ be the subcondition of c at layer $j + 1$.

$$\text{nvc}(j, G) := \sum_{C' \in \mathcal{U}(C_j, C_{j+1})} |\{q \mid q : C_{j+1} \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}|$$

3. If $j > c_{\max}$:

$$\text{nvc}(j, G) := \infty$$

Definition 2.13 (minimal consistency improving). Let a graph G , a rule r and a constraint c in ANF be given.

A transformation $G \Rightarrow_{r,m} H$ is called minimal consistency improving w.r.t c , if

$$\text{nvc}(k, H) < \text{nvc}(k, G)$$

for any $0 \leq k \leq \text{nl}(c)$. A rule r is called minimal consistency improving, if all of its applications are.

Lemma 2.14. Let a graph G , a morphism $p : C_0 \rightarrow G$ and a constraint c in ANF over C_0 with $p \models_k c$ be given. Then, $p \models_j c$ for all $j < k$ such that the subcondition of c at layer j is existentially bound.

Proof. 1. The subcondition of c at layer k is existentially bound: If an $j < k$ with $p \models_j c$ exists such that the subcondition of c at layer j is universally bound, let j_1 be the smallest of these. With lemma 2.7 follows that $p \models_{j_2} c$ for all $j_1 < j_2$. Let $\ell < j_1$, such that the subcondition of c at layer ℓ is existentially bound and let $d = \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, e)$ be the condition up to layer $j_1 - \ell$ of the subcondition of c at layer ℓ . Since $\ell < j_1$, a morphism $q : C_\ell \rightarrow G$ with $q \models d$ must exist and therefore a morphism $q' : C_{\ell+1} \rightarrow G$ with $q = q' \circ a_k$ must exist. It follows that $q \models \exists(a_\ell : C_\ell \hookrightarrow C_{\ell+1}, \text{true})$ and with that $p \models_\ell c$.

2. The subcondition of c at layer k is universally bound: With lemma 2.7 follows that $p \models_{k+1} c$. Since c is in ANF, 1. can be applied to $k+1$. \square

Theorem 2.1. *Let a graph G , a rule r and a constraint c in ANF be given. Let $k < \text{nl}(c)$ be the biggest number, such that $G \models_k c$. A transformation $G \Rightarrow_{r,m} H$ is minimal consistency improving if $G \models_j c$ and $k < j$.*

Proof. No $\ell > k$ with $G \models_\ell c$ exists and $G \models_k c$. Hence, $\text{nvc}(k, G) > 0$ and $\text{nvc}(k, G) \neq \infty$. Since $j > k$, $\text{nvc}(k, H) = 0$ and it follows immediately that the transformation is minimal consistency improving. \square

Theorem 2.2. *Let a constraint c in EANF be given. Every sequence of minimal consistency improving transformation w.r.t c is finite.*

Proof. Let G_0 be a graph and

$$G_0 \Rightarrow_{\rho_1} G_1 \Rightarrow_{\rho_2} G_2 \Rightarrow_{\rho_3} \dots$$

be a sequence of minimal consistency improving transformations w.r.t c . We assume that $c_{\max}^{G_0} < \text{nl}(c)$, otherwise $\text{nvc}(j, G_0) = 0$ for all $j \in \{0, \dots, \text{nl}(c)\}$ and no transformation $G_0 \Rightarrow H$ is minimal consistency improving.

We show that after at most $j := \text{nvc}(c_{\max}^{G_0} + 1, G_0)$ transformations $G_j \models_{c_{\max}^{G_0} + 2} c$ holds. Note that j has to be finite, since G_0 contains only a finite number of occurrences of C_{j+1} . After each transformation it holds that $\text{nvc}(c_{\max}^{G_{i+1}} + 1, G_{i+1}) \leq \text{nvc}(c_{\max}^{G_i} + 1, G_i) - 1$. Therefore, after j transformations $\text{nvc}(c_{\max}^{G_0} + 1, G_j) \leq \text{nvc}(c_{\max}^{G_0} + 1, G_0) - j = 0$ holds and with that $G_j \models_{c_{\max}^{G_0} + 2} c$. By iteratively applying this, it follows that after a finite number of transformations a graph G_k with $G_k \models c$ exists. Since no minimal consistency improving transformation $G_k \Rightarrow G_{k+1}$ exists, the sequence has to be finite. \square

Definition 2.15 (direct minimal consistency improving). *Let G be a graph, ρ a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, e)$ be the condition at layer $k = c_{\max}^G + 1 \leq \text{nl}(c)$ of c . A transformation $t : G \Rightarrow_{\rho,m} H$ is called direct minimal consistency improving if the following equations hold.*

Every occurrence of C_k in G that satisfies $\text{part}(1, e, C')$ for any $C' \in \mathcal{U}(C_k, C_{k+1})$ still satisfies $\text{part}(1, e, C')$ in H .

$$\begin{aligned} \forall p : C_k \hookrightarrow G \Big(\bigwedge_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \models \text{part}(1, e, C') \wedge \text{tr}_t \circ p \text{ is total}) \\ \implies \text{tr}_t \circ p \models \text{part}(1, e, C') \Big) \end{aligned} \quad (2.1)$$

Every new inserted occurrence of C_k by t satisfies $\text{part}(1, e, C_{k+1})$.

$$\forall p' : C_k \hookrightarrow H \big(\neg \exists p : C_k \hookrightarrow G (p' = \text{tr}_t \circ p) \implies p' \models \text{part}(1, e, C_{k+1}) \big) \quad (2.2)$$

At least one occurrence of C_k in G that does not satisfy $\text{part}(1, e, C')$, for any $C' \in \mathcal{U}(C_k, C_{k+1})$, either has been destroyed by t or satisfies $\text{part}(1, e, C')$ in H .

$$\exists p : C_k \hookrightarrow G \left(\bigvee_{C' \in \mathcal{U}(C_k, C_{k+1})} (p \not\models \text{part}(1, e, C') \wedge (\text{tr}_t \circ p \text{ is not total} \vee (\text{tr}_t \circ p \text{ is total} \wedge \text{tr}_t \circ p \models \text{part}(1, e, C')))) \right) \quad (2.3)$$

No occurrence of a universally bound graph C_j with $j < k$ gets inserted.

$$\bigwedge_{\substack{i < k \\ i \text{ even}}} \forall p : C_i \hookrightarrow H (\exists p' : C_i \hookrightarrow G (p' = \text{tr}_t \circ p)) \quad (2.4)$$

No occurrence of an existentially bound graph C_j with $j < k$ gets inserted.

$$\bigwedge_{\substack{i < k \\ i \text{ odd}}} \forall p : C_i \hookrightarrow G (\text{tr}_t \circ p \text{ is total}) \quad (2.5)$$

Lemma 2.16. Let a transformation $t : G \Rightarrow H$ and a constraint c in EANF be given, such that (2.4) and (2.5) of definition 2.15 are satisfied. Then,

$$H \models_{c_{\max}^G} c.$$

Proof. Assume that $G \not\models_{c_{\max}^G} c$. Then, either a new occurrence of an universally bound graph of c has been inserted or an occurrence of an existentially bound graph of c has been destroyed. Therefore, the following holds:

$$\exists p : C_i \hookrightarrow G (\neg \exists p' : C_i \hookrightarrow G (p' = \text{tr}_t \circ p)) \vee \exists p : C_j \hookrightarrow G (\text{tr}_t \circ p \text{ is not total})$$

with $i, j < k$, i being odd and j being even. It follows immediately that either (2.4) or (2.5) is not satisfied. This is a contradiction. \square

Lemma 2.17. Let a graph G , a constraint c and a direct minimal improving transformation $t : G \Rightarrow_{r,m} H$ w.r.t. c be given. Then, t is also a minimal improving transformation.

Proof. Let G be a graph with $k = c_{\max}$ and $G \models \text{part}(k, c, C)$ with $\text{part}(k, c, C) \in \mathcal{P}_c^G$. Let d be the subcondition of c at layer $k + 1$.

1. We show that equations (2.1) and (2.2) imply that $\text{nvc}(k, H) \leq \text{nvc}(k, G)$. Assume that $\text{nvc}(k, H) > \text{nvc}(k, G)$. Therefore, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \text{part}(1, d, C')$ for any $C' \in \mathcal{U}(C, C_{k+1})$ exists, such that either 1a or 1b is satisfied.
 - (a) There does exist a morphism $q' : C_k \hookrightarrow G$ with $q' \models \text{part}(1, d, C')$ and $p = \text{tr}_t \circ q'$.
 - (b) There does not exist a morphism $q : C_k \hookrightarrow G$, such that $p = \text{tr}_t \circ q$.

This is a contradiction, if [1a](#) is satisfied, q' does not satisfy equation [\(2.1\)](#) and if [1b](#) is satisfied q does not satisfy equation [\(2.2\)](#).

2. Since [\(2.3\)](#) is satisfied, a morphism $p : C_k \hookrightarrow G$ with $p \not\models \text{part}(1, d, C')$, such that either $\text{tr} \circ p$ is total and $p \models \text{part}(1, d, C')$ or $\text{tr} \circ p$ is not total exists, for any $C' \in \mathbf{G}$. In both cases the following holds

$$p \in \{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\} \wedge \\ \text{tr} \circ p \notin \{q \mid q : C_{k+2} \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}.$$

With that and [1](#) it follows that

$$|\{q \mid q : C_k \hookrightarrow G \wedge q \not\models \text{part}(1, e, C')\}| < |\{q \mid q : C_{k+2} \hookrightarrow H \wedge q \not\models \text{part}(1, e, C')\}|.$$

With [1](#) and [2](#) follows that $\text{nvc}(k, G) < \text{nvc}(k, H)$ and therefore t is a minimal improving transformation. \square

2.4 Comparison with other consistency concepts

Lemma 2.18. *Let a constraint c in ANF, graphs G and H with $G \not\models c$, and a transformation $t : G \Rightarrow H$ be given. Then,*

$$t \text{ is } c\text{-guaranteeing} \implies t \text{ is minimal consistency improving w.r.t } c$$

,

$$t \text{ is } c\text{-guaranteeing} \not\Rightarrow t \text{ is direct minimal consistency improving w.r.t } c$$

and

$$t \text{ is minimal consistency improving w.r.t } c \not\Rightarrow t \text{ is } c\text{-guaranteeing}$$

Proof. Let c' be the equivalent constraint in EANF.

1. Let t be c -guaranteeing transformation, then $H \models c$. Since $G \not\models c$, $G \not\models c'$ follows and $\text{nvc}(c_{\max}^G + 1, G) > 0$ and $\text{nvc}(c_{\max}^G + 1, H) = 0$. Therefore t is minimal consistency improving.
2. Let a constraint $c = \exists(a_0 : \emptyset \hookrightarrow C_0, \forall(a_1 : C_0 \hookrightarrow C_1, \exists(a_2 : C_1 \hookrightarrow C_2, \text{true})))$ and a graph $G = C' \dot{\cup} C''$ with $C' = C_2 \setminus \{e\}$ for an $e \in E_{C_2 \setminus C_1}$ and the occurrences $p_1 : C_1 \hookrightarrow G$ and $p_2 : C_1 \hookrightarrow G$ be given. Let $t : G \Rightarrow H$ be a transformation with $H = C_2 \dot{\cup} C'''$ and $C''' = C' \setminus \{e'\}$ for an $e' \in E_{C_2 \setminus C_1}$, such that $\text{tr}_t \circ p_1$ and $\text{tr}_t \circ p_2$ are total. Then, t is c -guaranteeing and not direct minimal consistency improving, since $p_i \models \exists(a'_2 : C_1 \hookrightarrow C', \text{true})$ for $i = 1, 2$ and either $\text{tr}_t \circ p_1 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \text{true})$ or $\text{tr}_t \circ p_2 \not\models \exists(a'_2 : C_1 \hookrightarrow C', \text{true})$.
3. Let t be a minimal consistency improving transformation w.r.t c' , such that $\text{nvc}(c_{\max}^G + 1, H) > \text{nvc}(c_{\max}^G + 1, G) > 1$ and $H \not\models_{c_{\max}^G + 2} c'$. Then, $H \not\models c'$ and t is not a guaranteeing transformation.

□

Corollary 2.19. *Let c be an existentially bound constraint in ANF. Then, a transformation t is c -guaranteeing if and only if it is consistency improving. With lemma 2.18 follows:*

t is consistency guaranteeing w.r.t $c \implies t$ is minimal consistency improving w.r.t c and

t is minimal consistency improving w.r.t $c \not\Rightarrow t$ is consistency improving w.r.t c

Lemma 2.20. *let a universally bound constraint c with $\text{nl}(c) = 1$ in ANF, graphs G and H with $G \not\models c$, and a transformation $t : G \implies H$ be given. Then,*

t is consistency improving \iff t is minimal consistency improving

Proof. Let $c = \forall(a : \emptyset \hookrightarrow C, \text{false})$. Then, the equivalent constraint in EANF is $c' = \forall(a : \emptyset \hookrightarrow C, \exists(\text{id} : C \hookrightarrow C, \text{false}))$. Since $\mathcal{U}(C, C) = \{C\}$, $\text{nvc}(1, G)$ is the number of occurrences of C in G . This is exactly the definition of the number of violations for consistency improving transformations and

t is minimal consistency improving $\iff t$ is consistency improving

follows immediately. □

Lemma 2.21. *Let a universally bound constraint c with $\text{nl}(c) \geq 2$ in ANF, graphs G and H with $G \not\models c$ and a transformation $t : G \implies H$ be given. Then,*

t is direct consistency improving $\not\Rightarrow t$ is minimal consistency improving

and

t is direct minimal consistency improving $\not\Rightarrow t$ is consistency sustaining

Proof. 1. Let $c = \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \text{true}))$ be a constraint. Let $V_{C_0} = V_{C_1}$ and $|E_{C_1}| - |E_{C_0}| = 2$. Let $G = C' \dot{\cup} C'$ with $C' = C_1 \setminus \{e\}$ with $e \in E_{C_1} \setminus E_{C_0}$ and the occurrences $p_1 : C_0 \hookrightarrow G$ and $p_2 : C_0 \hookrightarrow G$ be given. It follows that $\text{nvc}(1, G) = 2$. Let $t : G \implies H$ be a transformation, such that $H = C_0$. Then, t is a direct consistency improving transformation, since H contains only one occurrence of C_0 not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1, \text{true})$. But, t is not minimal consistency improving since $\text{nvc}(1, H) = 3$.

2. Let $c := \forall(a_0 : \emptyset \hookrightarrow C_0, \exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \text{true}))))$ be a constraint.

Let a graph $G = C_0$ with the morphism $q : C_0 \hookrightarrow G$ and a transformation $t : G \implies H$ with $H := C_2 \dot{\cup} C_2$ be given, such that $\text{tr}_t \circ q$ is total. Then t is a direct minimal consistency improving transformation but not a consistency sustaining one, since H contains more occurrences of C_0 not satisfying $\exists(a_1 : C_0 \hookrightarrow C_1, \forall(a_2 : C_1 \hookrightarrow C_2, \exists(a_3 : C_2 \hookrightarrow C_3, \text{true})))$ than G .

□

3 application condition

Definition 3.1 (extended overlaps). Let G and $C_0 \subseteq C_1$ with the inclusion $i : C_0 \hookrightarrow C_1$ be graphs. Let C be an overlap of C_0 and G with the inclusion $q : C_0 \hookrightarrow C$. The set of extended overlaps of C with i , $\text{eol}(C, i)$, is the set of all overlaps C' of G and C_1 , such that $C \subseteq C'$ and $q \models \exists(i : C_0 \hookrightarrow C_1, \text{true})$.

Definition 3.2 (overlap shift). Let $\rho = L \leftarrow K \hookrightarrow R$ be a plain rule, C a graph and C' an overlap of C and L with morphisms $p : L \hookrightarrow C'$, $k : K \hookrightarrow C'$, $c : C \hookrightarrow C'$ and the partial morphism $q : R \hookrightarrow C'$. We define

$$\begin{aligned} D := \{e \in C' \mid & (\exists e' \in L : p(e') = e \\ & \vee \exists e' \in R : q(e') = e) \\ & \wedge \exists e' \in C : c(e') = e\} \end{aligned} \quad (3.1)$$

Let $r = L \leftarrow K' \hookrightarrow R$ be the rule with

$$K' := K \cup D$$

The graph H derived by the transformation $G \Rightarrow_{r,p} H$ is called the overlap shifted graph of C' with C and ρ . The overlap shifted graph of an graph C is denoted by $\text{ols}_\rho(C, C')$.

Definition 3.3 (application condition). Let $\rho = L \leftarrow K \hookrightarrow R$ be a plain rule and c a constraint in EANF. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer k with k being an even number.

The application condition ap_k of c at layer k with $C' \in \mathcal{U}(C_k, C_{k+1})$ and $i : C_k \hookrightarrow C'$ is defined as:

$$\text{ap}(k, C') := \left(\bigvee_{P \in \text{ol}(L, C_k)} \text{nex}(P, C') \wedge \text{rep}(P, C') \right) \wedge \text{nin} \wedge \text{rem} \wedge \text{nwo} \quad (3.2)$$

with

1.

$$\text{nex}(P, C') := \begin{cases} \exists(i_L : L \hookrightarrow P, \text{true}) & , \text{ if } C_k = C_{k+1} \\ \bigwedge_{Q \in \text{eol}(P, i)} \exists(i_L : L \hookrightarrow P, \neg \exists(i_P : P \hookrightarrow Q, \text{true})) & , \text{ otherwise} \end{cases}$$

2. Let \mathbf{P} be the set of all overlaps of R and C' , such that $i_R(R \setminus K) \cap i_{C'}(C') \neq \emptyset$:

(a) If $C_k = C_{k+1}$:

$$\text{rep}(P, C') := \begin{cases} \text{true} & , \text{ if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \text{false} & , \text{ otherwise} \end{cases}$$

(b) Otherwise:

$$\text{rep}(P, C') := \begin{cases} \text{true} & , \text{ if } i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset \\ \bigvee_{P' \in \mathbf{P}} \text{Left}(\exists(i_R : R \hookrightarrow P', \text{true}), \rho) & , \text{ otherwise} \end{cases}$$

3. Let E be the set of all graphs C_j of c with $j \leq k$ and j being odd and let \mathbf{P}_{C_j} be the set all overlaps of L and C_j with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.

$$\text{rem} := \bigwedge_{C \in E} \bigwedge_{C' \in \mathbf{P}_{C_j}} \neg \exists(i_L : L \hookrightarrow C', \text{true})$$

4. Let U be the set of all graphs C_j of c with $j \leq k$ and j being even and let \mathbf{P}_{C_j} be the set of all overlaps of R and C_j with $i_R(R \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$.

$$\text{nin} := \bigwedge_{C \in U} \bigwedge_{C' \in \mathbf{P}_{C_j}} \text{Left}(\neg \exists(i_R : R \hookrightarrow C', \text{true}), \rho)$$

5. Let E be the set of all overlaps of L and C_k , such that each $P \in E$ is also an overlap of L and $C'' \in \mathcal{U}(C_k, C_{k+1})$, $i_{C_k} \models \exists(a'_k : C_k \hookrightarrow C'', \text{true})$ and $i_L(L \setminus K) \cap i_{C''}(C'' \setminus C_k) \neq \emptyset$.

$$\text{nwo} := \bigwedge_{P \in E} \neg \exists(i_L : L \hookrightarrow P, \text{true})$$

Lemma 3.4. Let a graph G , a constraint c in EANF, with $G \not\models c$, and a plain rule ρ be given. Then, the rule $\rho'(\rho, \text{ap}(c_{\max}^G, C))$ for a graph $C \in \mathcal{U}(C_{c_{\max}^G}, C_{c_{\max}^G+1}^G)$ is a minimal consistency improving rule.

Proof. Let $t : G \Rightarrow_{\rho', m} H$ be a transformation and $k = c_{\max}^G + 1$. We show, by contradiction, that this transformation is direct minimal consistency improving by showing that (2.1), (2.2), (2.3), (2.4) and (2.5) are satisfied. With that, it follows that ρ' is a minimal consistency improving rule. Let $d = \forall(a_k : C_k \hookrightarrow C_{k+1}, e)$ be the subcondition of c at layer k .

1. Assume that (2.1) does not hold. Then, there does exist an morphism $p : C_k \hookrightarrow G$, such that $p \models \text{part}(1, e, C')$, $\text{tr}_t \circ p$ is total and $\text{tr}_t \circ p \not\models \text{part}(1, e, C')$ for a graph $C' \in \mathcal{U}(C_k, C_{k+1})$. There has to exist an overlap P of L and C' such that $i_{C_k} \models \exists(a_k^p : C_k \hookrightarrow C', \text{true})$ and $m \models \exists(i_L : L \hookrightarrow P, \text{true})$. Then, nwo and with that $\text{ap}(c_{\max}^G, C)$ is not satisfied.
2. Assume that (2.2) does not hold. Then, a morphism $p' : C_k \hookrightarrow H$ with $p' \not\models \text{part}(1, e, C_{k+1})$ exists, such that there does not exist a morphism $p : C_k \hookrightarrow G$ with $\text{tr}_t \circ p = p'$. Then, an overlap P of R and C_k with $i_R(R \setminus K) \cap i_R(C_k) \neq \emptyset$ exists, such that $m \models \text{Left}(\exists(i_R : R \hookrightarrow P, \text{true}), \rho)$. Then, $m \not\models \text{ap}(c_{\max}^G, C)$.

3. Assume that (2.3) does not hold. Then, there does not exist a morphism $p : C_k \hookrightarrow G$ with $p \not\models \text{part}(1, e, C)$, such that $\text{tr}_t \circ p$ is not total or $\text{tr}_t \circ p \models \text{part}(1, e, C)$ and $\text{tr}_t \circ p$ is total. Then, no overlap P of L and C_k with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ exists, such that $m \models \text{nex}(P, C)$. Also, no overlap P of C and R with $i_R(R \setminus K) \cap i_C(C) \neq \emptyset$ and $m \models \text{nex}(P, C)$ exists, such that $m \models \text{Left}(\exists(i_R : R \hookrightarrow P, \text{true}), \rho)$ and therefore $\text{rep}(P, C) = \text{false}$. It follows that for all $P \in \text{ol}(L, C_k)$ it holds that $\text{nex}(P, C) \wedge \text{rep}(P, C) = \text{false}$ and with that $m \not\models \text{ap}(c_{\max}^G, C)$.
4. Assume that (2.4) does not hold. Then, there does exist a morphism $p : C_j \hookrightarrow G$ with $j < k$ and i being even, such that no morphism $p' : C_j \hookrightarrow G$ with $\text{tr}_t \circ p' = p$ exists. Then, an overlap P of C_j and R with $i_R(R \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$ exists, such that $m \models \text{Left}(\exists(i_R : R \hookrightarrow P, \text{true}), \rho)$. It follows that $m \not\models \text{rem}$ and with that $m \not\models \text{ap}(c_{\max}^G, C)$.
5. Assume that (2.5) does not hold. Then, there does exist and morphism $p : C_j \hookrightarrow G$ with $j < k$ and j being odd, such that $\text{tr}_t \circ p$ is not total. Then, an overlap P of C_j and L with $i_L(L \setminus K) \cap i_{C_j}(C_j) \neq \emptyset$ exists, such that $m \models \exists(i_L : L \hookrightarrow P, \text{true})$. It follows that $m \not\models \text{rem}$ and with that $m \not\models \text{ap}(c_{\max}^G, C)$.

In total follows that if $m \models \text{ap}(c_{\max}^G, C)$, then t is a direct minimal improving transformation. \square

Lemma 3.5. *Let G be a graph, c a constraint in EANF, with $c_{\max}^G < \text{nl}(c)$, and $\rho = L \xleftarrow{c} K \xrightarrow{c} R$ a plain rule. Let $d = \forall(a_k : C_{k-1} \hookrightarrow C_k, \exists(b : C_k \hookrightarrow C_{k+1}, e))$ be the subcondition of c at layer $c_{\max} + 1$ and $\text{ap}(c_{\max}, C')$ the application condition constructed by definition 3.3 with $C' \in \mathcal{U}(C, C_{k+1})$ for any $C \in \mathcal{G}_c^G$. If*

$$((R \setminus K) \cap C_{k+1}) \cup ((L \setminus K) \cap C_{k+1}) = \emptyset$$

$\text{ap}(c_{\max}, C')$ can be replaced by false.

Proof. There does not exist an overlap P of C_k and L with $i_L(L \setminus K) \cap i_{C_k}(C_k) \neq \emptyset$ and $\text{rep}(P, C')$ will be equal to false, if $C_k = C_{k+1}$, or equal to $\bigvee_{P' \in \mathbf{P}} \text{Left}(\exists(i_R : R \hookrightarrow P', \text{true}), \rho)$. Since the set \mathbf{P} has to be empty, this expression can be replaced by false. It follows that $\text{rep}(P, C') = \text{false}$ for all $P \in \text{ol}(L, C_k)$ and therefore $\text{ap}(c_{\max}, C')$ will always be evaluated to false. \square

3.1 potentially minimal improving rules

Definition 3.6 (basic improving rule). *Let a constraint c and a plain rule $\rho = L \xleftarrow{c} K \xrightarrow{c} R$ be given. The rule ρ is called basic improving w.r.t c at layer k with $C_k \subset P \subseteq C_{k+1}$, $p : C_k \hookrightarrow P$ being the inclusion, and $k \in \{1, 3, \dots, \text{nl}(c)\}$, if*

$$(L \setminus K) \cap (C_{k-1} \cup (C_{k+1} \setminus C_k)) = \emptyset \tag{3.3}$$

and

$$(R \setminus K) \cap C_k = \emptyset \tag{3.4}$$

and either 1. or 2. applies.

1. The rule ρ deletes elements of $C_k \setminus C_{k-1}$:

$$L \subseteq C_k \wedge L \setminus K \neq \emptyset \quad (3.5)$$

Then, ρ is called a deleting basic improving rule.

2. The rule ρ creates an instance of P :

$$L = C_k \wedge P \subseteq R \quad (3.6)$$

and p is a partial morphism of r . Then, ρ is called an inserting basic improving rule.

Definition 3.7 (application conditions for basic improving rules). Let a constraint c in EANF and a basic improving rule $\rho = L \xleftarrow{l} K \xrightarrow{r} R$ w.r.t c at layer k with $C_k \subseteq P \subseteq C_{k+1}$ be given. We define the application condition for r as:

1. If ρ is a deleting potentially minimal improving rule:

$$\text{appi}(j, P) := \begin{cases} \exists(b : L \hookrightarrow C_k, \neg \exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, \text{true})) & , \text{ if } j = k \\ \text{false} & , \text{ if } j \neq k. \end{cases}$$

with

$$b(e) := \begin{cases} e & , \text{ if } e \notin C_{k-1} \\ a_k(e) & , \text{ otherwise} \end{cases}$$

2. If ρ is an inserting potentially minimal improving rule:

$$\text{appi}(j, P) := \begin{cases} \neg \exists(b : L \hookrightarrow P, \text{true}) & , \text{ if } j = k \\ \text{false} & , \text{ if } j \neq k \end{cases}$$

with b being a partial morphism of a_{k+1} .

Theorem 3.1. Let a graph G , a constraint c in EANF, with $G \models \text{part}(c_{\max}, c, C)$ and $\text{part}(c_{\max}, c, C) \in \mathcal{P}_c^G$, and a potentially minimal improving rule $\rho = L \xleftarrow{l} K \xrightarrow{r} R$ at layer c_{\max} with $C_{c_{\max}+1} \subseteq P \subseteq C_{c_{\max}+2}$ be given. Then, $\rho' = (\rho, \text{appi}(c_{\max}+1, P))$ is a direct minimal consistency improving rule.

Proof. Let $t : G \Rightarrow_{\rho', m} H$ be a transformation, $k = c_{\max} + 1$ and e be the subcondition of c at layer $k+1$. We show that t is a direct minimal consistency improving transformation. Firstly, we show that equation (2.1) is satisfied. Let $p : C_k \hookrightarrow G$ be a morphism. If ρ is a deleting potentially minimal improving rule, either 1. or 2. applies, if ρ is an inserting and not a deleting potentially minimal improving rule, only 2. applies, because ρ cannot destroy any occurrences of C_k in G .

1. If $p(C_k) \cap m(L \setminus K) \neq \emptyset$, $\text{tr}_t \circ p$ is not total, since at least one element of $p(C_k)$ has been deleted by t and p does satisfy $\bigwedge_{C' \in \mathbf{G}} (p \models \text{part}(1, e, C') \wedge \text{tr}_t \circ p \text{ is total}) \implies \text{tr}_t \circ p \models \text{part}(1, e, C')$.
2. If $p(C_k) \cap m(L \setminus K) = \emptyset$, $\text{tr}_t \circ p$ is total, since no element of $p(C_k)$ has been deleted by t . Because (3.3) holds, t does not delete any elements of $C_{k+1} \setminus C_k$ and therefore $p \models \text{part}(1, e, C') \implies \text{tr}_t \circ p \models \text{part}(1, e, C')$ for all $C_k \subseteq C' \subseteq C_{k+1}$.

With 1. and 2. follows that (2.1) is satisfied.

Secondly, we show that equation (2.2) is satisfied. Let $p' : C_k \hookrightarrow H$ be a morphism. Because (3.4) is satisfied, t does not create any elements of C_k and therefore, there must exist an morphism $p : C_k \hookrightarrow G$ with $\text{tr}_t \circ p = p'$. It follows that (2.2) is satisfied.

Lastly, we show that (2.3) is satisfied. We consider the cases that firstly, ρ is a deleting minimal potentially improving rule and secondly, that ρ is an inserting and not a deleting minimal potentially improving rule.

1. If ρ is a deleting potentially minimal improving rule, the condition $\text{appi}(k, P) = \exists(b : L \hookrightarrow C_k, \neg \exists(a_{k+2} : C_k \hookrightarrow C_{k+1}, \text{true}))$ is satisfied by m . Therefore a morphism $p : C_k \hookrightarrow G$ with $p \not\models \neg \exists(a_{k+1} : C_k \hookrightarrow C_{k+1}, \text{true}) = \text{part}(1, e, C_{k+1})$ and $m = p \circ b$ must exist. Since ρ is a deleting rule, at least one element of $p(C_k)$ has been deleted by t and therefore $\text{tr}_t \circ p$ is not total. It follows that (2.3) is satisfied.
2. If ρ is an inserting and not a deleting potentially minimal improving rule, $\text{appi}(k, P) = \neg \exists(b : L \hookrightarrow P, \text{true})$ is satisfied by m . Because $L = C_k$, $m \models \neg \exists(b : C_k \hookrightarrow P, \text{true}) = \text{part}(1, e, P)$. Since (3.6) is satisfied, $\text{tr} \circ p$ is total and $\text{tr}_t \circ p \models \text{part}(1, e, P)$. Therefore, (2.3) is satisfied. In total follows that t is a direct minimal consistency improving transformation and therefore r' is direct minimal consistency improving rule.

□

Definition 3.8 (repairing rule set). Let a constraint c in EANF and a set of rules \mathcal{R} be given. Then, \mathcal{R} is called a repairing rule set for c at layer k if for all graphs G with $k = c_{\max}$ a sequence

$$G = G_0 \Rightarrow_{r_0} \dots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \dots, n-1\}$ and $H \models_{k+2} c$.

Corollary 3.9. Let a constraint c in EANF and a set of rules \mathcal{R} be given. If \mathcal{R} is a repairing rule set for c at layer k , \mathcal{R} is a repairing rule set w.r.t c at layer j for all $k < j \leq \text{nl}(c)$.

Corollary 3.10. Let a constraint c in EANF and a repairing rule set \mathcal{R} for c at layer k , for all $k \in \{1, 3, \dots, \text{nl}(c)\}$, be given. Then, for all graphs G , a sequence

$$G = G_0 \Rightarrow_{r_0} \dots \Rightarrow_{r_{n-1}} G_n = H$$

exists, such that $r_j \in \mathcal{R}$ for all $j \in \{0, \dots, n-1\}$ and $H \models c$.

Definition 3.11 (decomposition of a graph). Let graphs $G_0 \subset G_1$ be given.
A decomposition of G_1 with G_0 is a minimal set

$$\mathbf{P} \subseteq \{P_v \mid v \in V_{G_1 \setminus G_0}\}$$

of subgraphs of G_1 , such that every element of G_1 is contained in at least one $P \in \mathbf{P}$ and every P_v is constructed in the following way: $G_0 \subset P_v$, $v \in P$ and for all nodes $v' \in P \setminus C_k$ it holds that P contains all edges $e \in E_{G_1 \setminus G_0}$ and all nodes $u \in V_{G_1}$ such that either $\text{tar}(e) = v' \wedge \text{src}(e) = u$ or $\text{src}(e) = u \wedge \text{tar}(e) = v'$ holds.

Lemma 3.12. Let graphs $G_0 \subset G_1$ and a decomposition \mathbf{P} of G_1 with G_0 be given. Then, for each pair $P, P' \in \mathbf{P}$ with $P \neq P'$ the following holds:

$$(P \setminus C_k) \cap (P' \setminus C_k) = \emptyset$$

Proof. Assume that $(P \setminus C_k) \cap (P' \setminus C_k) \neq \emptyset$, therefore a node $v \in G_1 \setminus G_0$ with $v \in P \cap P'$ exists. By the construction of P and P' it follows that $P = P_v$ and $P' = P_v$ and therefore $P = P'$. This is a contradiction. \square

Lemma 3.13. Let graphs $G_0 \subset G_1$ and a decomposition \mathbf{P} of G_1 with G_0 be given. Then,

$$G_1 = \bigcup_{P \in \mathbf{P}} P.$$

Proof. Let $H := \bigcup_{P \in \mathbf{P}} P$. Firstly, we show that $H \subseteq G_1$. Since every $P \in \mathbf{P}$ is a subgraph of G_1 it follows that $V_H \subseteq V_{G_1}$ and $E_H \subseteq E_{G_1}$.

Secondly, we show that $G_1 \subseteq H$. Let $u \in V_{G_1}$ be a node, if $u \in V_{G_0}$, then u is contained in every $P \in \mathbf{P}$ and therefore $u \in V_H$. Otherwise, if $u \notin V_{G_0}$, then u has to be, by the definition of \mathbf{P} , contained in at least one $P \in \mathbf{P}$ and $V_{G_1} \subseteq V_H$ follows. Let $e \in E_{G_1}$ be an edge. If $e \in E_{G_0}$, then e is contained in every $P \in \mathbf{P}$ and $e \in E_H$. Otherwise, if $e \notin E_{G_0}$, by the definition of \mathbf{P} , e has to be contained in at least one $P \in \mathbf{P}$. It follows that $e \in E_H$ and with that $E_{G_1} \subseteq E_H$. \square

Theorem 3.2. Let a constraint c in EANF and a set of rules \mathcal{R} be given. Then, \mathcal{R} is a repairing set of c at layer $k \leq \text{nl}(c)$ if either 1 or 2 applies.

1. For any universally bound graph C_j at layer $j \leq k$ of c , $(\rho, \text{ap}_{\text{pi}}(j, C_{j+1})) \in \mathcal{R}$ and ρ is a deleting potentially minimal improving rule at layer j with C_{j+1} , such that ρ only deletes edges of C_j .
2. A decomposition \mathbf{P} of C_k with C_{k-1} exists, such that for each $P \in \mathbf{P}$ a rule $(\rho, \text{ap}_{\text{pi}}(k, P)) \in \mathcal{R}$ exists, such that ρ is an inserting basic improving rule at layer k with P .

Proof. Let a constraint c in EANF, a rule set \mathcal{R} and a graph G with $k = c_{\max}$ and $c_{\max} < \text{nl}(c)$ be given. We show that a sequence $G = C'_0 \Rightarrow \dots \Rightarrow C'_n = H$ with rules of \mathcal{R} exists, such that $H \models_{k+2} c$ if 1. or 2. of theorem 3.2 is satisfied.

1. Assume that 1. of theorem 3.2 holds. Let $(\rho, \text{appi}(j, C_{j+1})) \in \mathcal{R}$, such that $\rho = L \xleftarrow{l} K \xrightarrow{r} R$ is a deleting potentially minimal improving rule at layer $j \leq k$ with C_{j+1} and C_j is a universally bound graph of c . Then, $\text{appi}(j, C_{j+1}) = \exists(b : L \hookrightarrow C_j, \neg \exists(a_j : C_j \hookrightarrow C_{j+1}, \text{true}))$. Let $q : C_j \hookrightarrow G$ be a morphism such that $q \not\models \exists(a_j : C_j \hookrightarrow C_{j+1}, \text{true})$. Since $L \subseteq C_j$, we can construct a morphism $m_1 : L \hookrightarrow G$ with $m_1 = q \circ b$ and therefore $m_1 \models \text{appi}(j, C_{j+1})$. Since r only deletes edges, a transformation $t : G = G_0 \Rightarrow_{r, m_1} G_1$ exists and $\text{tr}_t \circ p$ is not total. Because r does not insert any elements of C_j :

$$|\{q : C_k \hookrightarrow G_0 \mid q \not\models d\}| < |\{q : C_k \hookrightarrow G_1 \mid q \not\models d\}|$$

with $d = \exists(a_j : C_j \hookrightarrow C_{j+1}, \text{true})$. By iteratively applying this construction, we can generate a finite sequence of transformations

$$G = G_0 \Rightarrow_{r, m_1} G_1 \Rightarrow_{r, m_2} \dots \Rightarrow_{r, m_n} G_n = H$$

such that $|\{q : C_k \hookrightarrow G_n \mid q \not\models d\}| = 0$ and therefore $H \models_j c$. With lemma 2.8, $H \models_{k+2} c$ and $H \models c$ follows.

2. Assume that 2. of theorem 3.2 holds. Let $\rho_0 = (\rho, \text{appi}(k, P)) \in \mathcal{R}$, such that ρ is an inserting basic improving rule of c at layer k with $P \in \mathbf{P}$. Then, $\text{appi}(k, P) = \neg \exists(b : L \hookrightarrow P, \text{true})$. Let $q_0 : C_k \hookrightarrow G$ be a morphism, such that $q_0 \not\models \exists(a'_k : C_k \hookrightarrow P, \text{true})$ with a'_k being a partial morphism of a_k . Since $L = C_k$, we set $m_0 : C_k \hookrightarrow G$ with $m_0 = q_0$. It follows that $m_0 \models \neg \exists(a'_k : C_k \hookrightarrow P, \text{true}) = \text{appi}(k, P)$. Because r does not delete any elements, a transformation $t_0 : G \Rightarrow_{r_0, m_0} G_1$ exists and $\text{tr}_{t_0} \circ q \models \exists(a'_k : C_k \hookrightarrow P, \text{true})$. We set $q_1 = \text{tr}_{t_0} \circ q_0$ and apply the same method to q_1 .

By iteratively applying this, we can construct a finite sequence of transformations

$$G \Rightarrow_{r_0, m_0} G_0 \Rightarrow_{r_1, m_1} \dots \Rightarrow_{r_n, m_n} G_n$$

such that $m_i = \text{tr}_{t_{i-1}} \circ \dots \circ \text{tr}_{t_0} \circ m_0$ and $q \models \exists(b_i : C_k \hookrightarrow P_i, \text{true})$ for all $P_i \in \mathbf{P}$ with $q = \text{tr}_{t_n} \circ q_n$. Let $p_i : P_i \hookrightarrow G_n$ be the morphism, such that $q = p_i \circ b_i$.

Now, we can construct a morphism $p : C_{k+1} \hookrightarrow G$ with

$$p(e) := \begin{cases} p_1(e) & , \text{if } e \in P_1 \\ \vdots & \\ p_j(e) & , \text{if } e \in P_j. \end{cases}$$

Let $e \in C_k$, because $q(e) = p_i \circ b_i(e)$ and $q(e) = p_\ell \circ b_\ell(e)$ and b_i and b_ℓ are both partial morphisms of a_k , it follows that $b_i(e) = b_\ell(e)$ and therefore $p_i(e) = p_\ell(e)$. Because $(P_i \cap P_\ell) \setminus C_k = \emptyset$ for all $i \neq \ell$, p is a morphism and by the definition of p it follows that $q = p \circ a_k$ and therefore $q \models \exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$.

By iteratively applying this whole construction to all occurrences of C_k that do not satisfy $\exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$ the derived graph H does not contain any occurrences of C_k not satisfying $\exists(a_k : C_k \hookrightarrow C_{k+1}, \text{true})$ and therefore $H \models_{k+2} c$.

□

Corollary 3.14. *If a set of rule \mathcal{R} is a repairing set of c at layer $k \leq \text{nl}(c)$ and 1. of theorem 3.2 applies, then \mathcal{R} is a repairing set of c at layer j for all $k \leq j \leq \text{nl}(c)$.*

References

- [1] C. Sandmann and A. Habel. [Rule-based graph repair](#). *arXiv preprint arXiv:1912.09610*, 2019.