Noncrossing Bonds in Ferrers Graphs

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Preliminaries: The Bond Lattice

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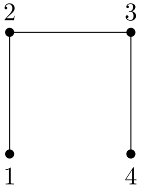
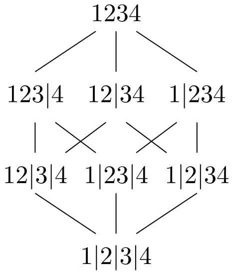


G



Π_G

Preliminaries: Noncrossing Bond Posets

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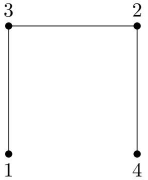

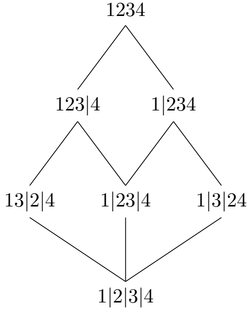

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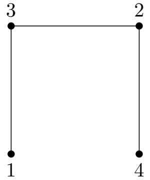
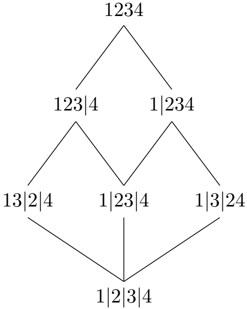




Studied systematically by Farmer–Hallam–Smyth (2020); some special cases were studied independently in my dissertation (2020).

EL-Shellability

$\mathcal{E}(P) = \{\text{edges of Hasse diagram of } P\}$

$\Lambda : \mathcal{E}(P) \to \mathbb{N}$

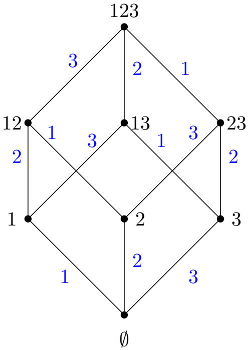
A chain $c \subseteq \mathcal{E}(P)$ is <u>increasing</u> wrt Λ if the labels **strictly** increase along c. A chain is <u>decreasing</u> wrt Λ if the labels **weakly** decrease.

Definition

Λ is an **EL-labeling** if, for all $x \leq_P y$, the interval $[x, y]$ has a <u>unique</u> increasing maximal chain whose Λ-word is lex-least among those of all maximal chains.

If P has an EL-labeling, we say that it's **EL-shellable**.

EL-Shellability



Poset topology (Stanley + Björner + Philip Hall + . . .):

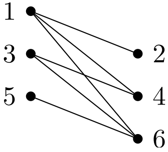
If P has an EL-labelling,

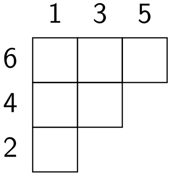
$$\mu_P(x,y) = (-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)} \cdot \#\{\text{decreasing maximal chains in } [x,y]\}.$$

A Nice Family of Labeled Graphs

V = finite set of positive integers s.t. $\min(V)$ is odd and $\max(V)$ is even

Γ_V = graph with vertex set V and edge $\{2i-1, 2j\}$ whenever $2i-1 < 2j$





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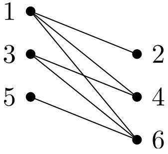

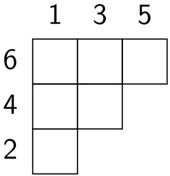

Γ_V appear naturally in the study of a certain family of hyperplane arrangements.

The Γ_V are also exactly the Ferrers graphs (each V encodes a Ferrers diagram)

A Nice Family of Bond Lattices

Theorem (L–Wachs)

A partition π of V belongs to Π_{Γ_V} iff $\min(B_i)$ is odd and $\max(B_i)$ is even for all nontrivial blocks B_i of π.

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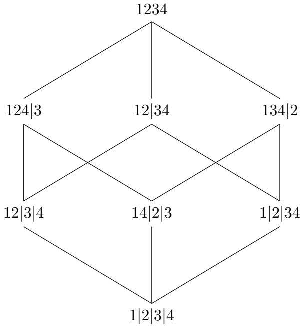
All bond lattices are EL-shellable (Björner characterizes all geometric lattices in terms of EL-labelings), so Π_{Γ_V} is EL-shellable.



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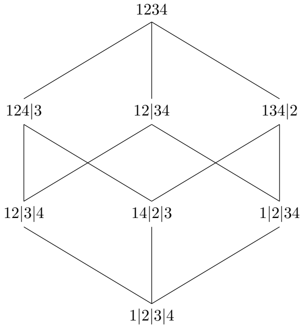

What about the noncrossing bond lattices?

Let $\mathrm{NC}^*_V := \mathrm{NC}_{\Gamma_V}$.

Result of Björner gives conditions for EL-labelings to be inherited by subposets; L.–Wachs frequently use a “good” edge-order of Γ_V that gives an EL-labeling of Π_{Γ_V}.

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NC_V^* *is EL-shellable for all V, using the same EL-labeling from L.–Wachs.*

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NC^*_V *is EL-shellable for all* V*, using the same EL-labeling from L.–Wachs.*

Farmer–Hallam–Smyth give several sufficient conditions for NC_G to be EL-shellable; NC^*_V satisfies some, but not all of them (so shellability also follows from their results).

Decreasing Chains in NC^*_V

Some more notation: Any V consists of alternating runs of odd and even integers, e.g.,

$$V_1 = \{1, 3, 5, 6, 8, 10, 11, 13, 14, 16\}.$$

$\mathrm{tp}(V) =$ vector whose entries count the lengths of these runs.

$$\mathrm{tp}(V_1) = (3, 3, 2, 2).$$

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Proposition (L.)

NC^*_V *depends only on* $\mathrm{tp}(V)$.

Decreasing Chains in NC^*_V: the Key Recurrence

Fix a representative $P_{\mathfrak{a}}$ among all NC^*_V with $\mathrm{tp}(V) = \mathfrak{a}$.

$N_{\mathfrak{a}}$ = number of decreasing maximal chains in $P_{\mathfrak{a}}$.

If $\mathfrak{a} = (a_1, b_1, \ldots, a_t, b_t)$, write

$$\begin{aligned} \downarrow \mathfrak{a} &= (a_1 - 1, b_1, \ldots, a_t, b_t) \\ \mathfrak{a} \downarrow &= (a_1, b_1, \ldots, a_t, b_t - 1). \end{aligned}$$

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Theorem (L.)

We have

$$N_\mathfrak{a} = N_{\downarrow \mathfrak{a}} + N_{\mathfrak{a} \downarrow} + \sum_{k=1}^{t-1} N_{(a_1, \ldots, b_k)} N_{(a_{k+1}, \ldots, b_t)},$$

with $N_{(1,1)} = 1$ *and* $N_\mathfrak{a} = 0$ *if any entry of* $\mathfrak{a}$ *is* ≤ 0.

This is a familiar recurrence!

Corollary

We have

$$N_{\underbrace{(1,1,\ldots,1)}_{2n}} = C_{n-1},$$

the $(n-1)$*st Catalan number.*

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Recall, $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

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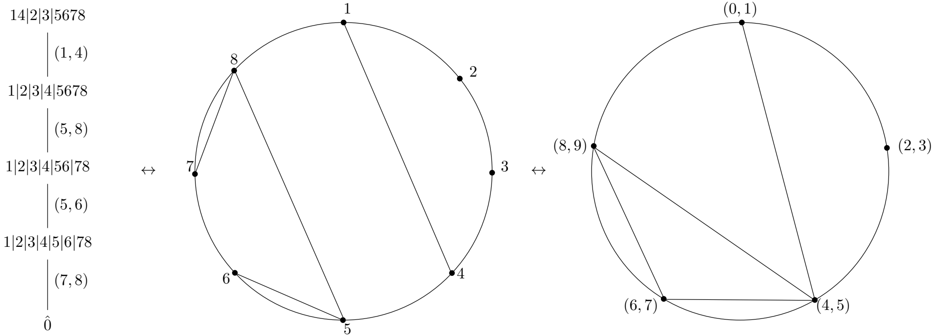
Can also construct a bijection between decreasing saturated chains starting at $\hat{0}$ and partial triangulations of the $(n+1)$-gon.

Decreasing Chains in NC_V^*



Other $N_{\mathfrak{a}}$

Have also solved the recurrence for $\mathfrak{a} = (1, b, 1, b, \ldots, 1, b)$ (equivalently for $(a, 1, a, 1, \ldots, a, 1)$).

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Proof by generating functions and Lagrange inversion; no bijective proof yet.

Also open: give a good characterization of corresponding decorated triangulations of the $(n+1)$-gon.

$(a, b, a, b, \dots, a, b)$

Natural next step: compute $N_{(a,b,\dots,a,b)}$.

Recurrence is **much** harder to solve. Numerical evidence suggests:

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Theorem (Special case of Bandrier–Wallner)

Suppose $a < b$, and let
$A_n(k) := \#\{\text{Dyck walks under } y = \frac{a}{b}x + \frac{k}{b} \text{ ending at } (bn-1, an-1)\}$.
Then

$$\sum_{k=1}^{a} A_n(k) = \frac{1}{n}\binom{(a+b)n-2}{an-1}.$$

Question for the Audience

Can we find a bijection?

Future Questions

- Dyck walks under PL curves?
- Other types?
- Wreath product groups?
- FHS describe "noncrossing NBC" sets that compute the char poly for some NC bond posets; can we count NCNBC sets for NC^*_V?
- Char poly of $\Pi_{\Gamma_{2n}}$ refines the median Genocchi numbers and has the ordinary Genocchi numbers as its constant term. So why do the <u>Catalan</u> numbers appear in char poly of NC^*_{2n}?
- We have an inclusion $\mathrm{NC}_G \hookrightarrow \Pi_G$; induces an inclusion of order complexes (no matter what labeling of G we pick). Can we analyze topology of NC_G in general (e.g. with Quillen's fiber theorem)?