Taming Infinity One Chunk at a Time: Concisely Represented Strategies in One-Counter MDPs

Michal Ajdarów[2] **James C. A. Main**[1]
Petr Novotný[2] Mickael Randour[1]

[1]F.R.S.-FNRS and UMONS – Université de Mons, Belgium

[2]Masaryk University, Brno, Czech Republic

Highlights 2025

Talk overview

- We study **synthesis** and **verification** in **Markov decision processes**.
- Small controllers, i.e., **concisely represented strategies** are desirable for practical synthesis.
- In **infinite-state systems**, such representations are necessary.

Contributions

We focus on **one-counter Markov decision processes** (OC-MDPs).

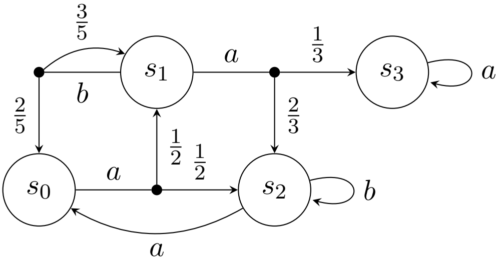
1. We focus on **interval strategies**, a subclass of strategies that make decisions based on the current state and counter value.
2. We provide **PSPACE** algorithms for the verification and synthesis of interval strategies.

Table of contents

Markov decision processes

Markov decision process (MDP) $\mathcal{M}$

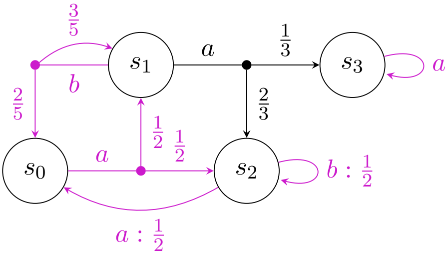
- **Finite or countable** state space S.
- **Finite** action space A.
- **Randomised** transition function $\delta\colon S \times A \to \mathcal{D}(S)$.

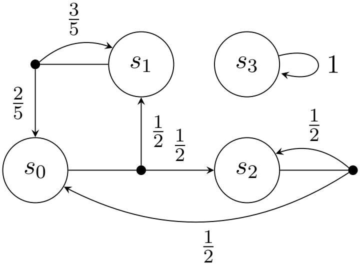


Plays are sequences in $(SA)^\omega$ coherent with transitions.
$\rightsquigarrow$ **Example**: $s_0 a s_1 b s_1 \ldots$

Strategies and induced Markov chains

- A **strategy** is a function $\sigma\colon (SA)^*S \to \mathcal{D}(A)$.
- σ is **memoryless** if its choices depend only on the **current state**.
- We view memoryless strategies as functions $S \to \mathcal{D}(A)$.
- A memoryless strategy σ induces a **Markov chain** over S.

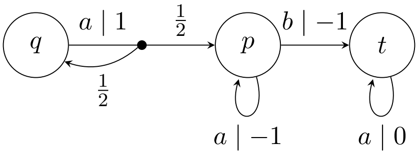




One-counter Markov decision processes

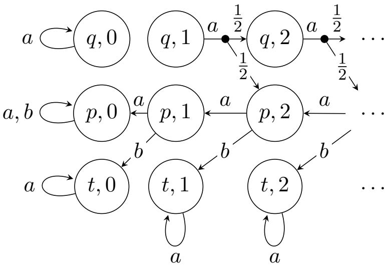
One-counter MDP (OC-MDP) $\mathcal{Q}$

- **Finite** MDP (Q, A, δ).
- **Weight function** $w\colon Q \times A \to \{-1, 0, 1\}$.



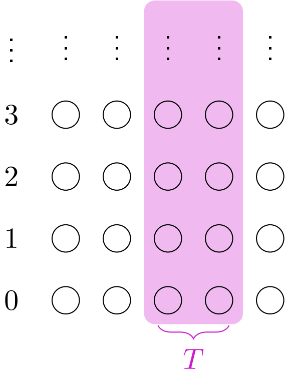
MDP $\mathcal{M}^{\leq\infty}(\mathcal{Q})$ induced by $\mathcal{Q}$

- **Countable** MDP over $S = Q \times \mathbb{N}$.
- State transitions via δ.
- Counter updates via w.

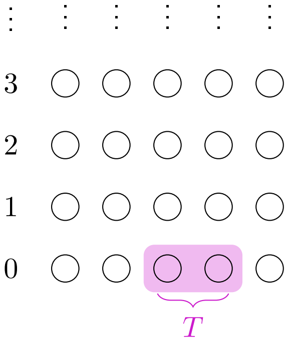


Objectives

- We study variants of **reachability objectives**.
- Let $T \subseteq Q$ **be a target**.



State reachability $\mathsf{Reach}(T)$



Selective termination $\mathsf{Term}(T)$

Interval strategies

We study subclasses of **memoryless strategies** of $\mathcal{M}^{\leq\infty}(\mathcal{Q})$.

Open-ended interval strategies (OEIS)

σ is an OEIS if $\exists\, k_0 \in \mathbb{N}$ s.t. $\forall\, q \in Q$ and $\forall\, k \geq k_0$, $\sigma(q,k) = \sigma(q,k_0)$.

$\mathbb{N}_0$	1	2	$\cdots$	$k_0 - 1$	k_0	$k_0 + 1$	$\cdots$
Q	σ_1	σ_2	$\cdots$	σ_{k_0-1}	σ_{k_0}	σ_{k_0}	$\cdots$

constant

Group counter values in intervals

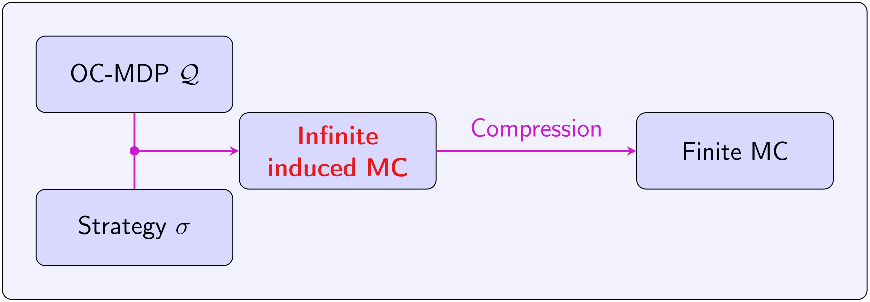
Inter.	I_1	I_2	$\cdots$	$I_d = [\![k_0, \infty]\!]$
Q	τ_1	τ_2	$\cdots$	$\tau_d = \sigma_{k_0}$

$\rightsquigarrow$ **Finite partition** of $\mathbb{N}_0$ into **intervals**

Interval strategy verification problem

Interval strategy verification problem

Decide whether $\mathbb{P}^{\sigma}_{\mathcal{M}^{\leq\infty}(\mathcal{Q}),s_{\mathsf{init}}}(\Omega) \geq \theta$ given an **interval strategy** σ, **an objective** $\Omega \in \{\mathsf{Reach}(T), \mathsf{Term}(T)\}$, a **threshold** $\theta \in \mathbb{Q} \cap [0,1]$ and an **initial configuration** $s_{\mathsf{init}} \in Q \times \mathbb{N}$.



Solving the verification problem

Main ideas

- **Transition probabilities** of compressed Markov chains are represented as **least solutions of equation systems**.
- Algorithm that checks the validity of a **universal formula in the theory of the reals** parameterised by the **strategy probabilities**.

Unbounded counter	Bounded counter
co-ETR	$\mathsf{P}^{\mathsf{PosSLP}}$
Square-root-sum-hard [EWY10][1]	Square-root-sum-hard

[1] Etessami et al., "Quasi-Birth-Death Processes, Tree-Like QBDs, Probabilistic 1-Counter Automata, and Pushdown Systems", Perform. Evaluation 2010.

Synthesis of interval strategies

We have also studied the **synthesis** of **structurally-constrained interval strategies**.

Parameterised interval strategy synthesis problem

Given **parameters** d **and** $n \in \mathbb{N}_0$, does there exists an interval partition $\mathcal{I}$ of $\mathbb{N}$ and an OEIS σ such that

1. $|\mathcal{I}| \leq d$ and all bounded $I \in \mathcal{I}$ satisfy $|I| \leq n$;
2. σ is **based on** $\mathcal{I}$ and
3. $\mathbb{P}^{\sigma}_{s_{\mathsf{init}}}(\mathsf{Term}(T)) \geq \theta$.

Unbounded counter	**Bounded counter**
PSPACE	$\mathsf{NP}^{\mathsf{ETR}}$
Square-root-sum-hard and NP-hard	

References I

[EWY10] Kousha Etessami, Dominik Wojtczak, and Mihalis Yannakakis. "Quasi-Birth-Death Processes, Tree-Like QBDs, Probabilistic 1-Counter Automata, and Pushdown Systems". In: *Perform. Evaluation* 67.9 (2010), pp. 837–857. DOI: 10.1016/J.PEVA.2009.12.009. URL: https://doi.org/10.1016/j.peva.2009.12.009.

[KEM06] Antonín Kucera, Javier Esparza, and Richard Mayr. "Model Checking Probabilistic Pushdown Automata". In: *Log. Methods Comput. Sci.* 2.1 (2006). DOI: 10.2168/LMCS-2(1:2)2006. URL: https://doi.org/10.2168/LMCS-2(1:2)2006.

[PB24] Jakob Piribauer and Christel Baier. "Positivity-hardness results on Markov decision processes". In: *TheoretiCS* 3 (2024). DOI: 10.46298/THEORETICS.24.9. URL: https://doi.org/10.46298/theoretics.24.9.

References II

[Tiw92] Prasoon Tiwari. "A problem that is easier to solve on the unit-cost algebraic RAM". In: *J. Complex.* 8.4 (1992), pp. 393–397. DOI: 10.1016/0885-064X(92)90003-T. URL: https://doi.org/10.1016/0885-064X(92)90003-T.