

## 7.3 Greeks

Definition 91 (Delta): <sup>Partial</sup> Derivative w.r.t the value of the underlying asset. [Black-Scholes:  $\frac{\partial V}{\partial S} = \Phi(d_+) =: \Delta$ ]

Definition 92 (Gamma): Second derivative w.r.t. the underlying. [Black-Scholes:  $\frac{\partial^2 V}{\partial S^2} = \frac{\Phi'(d_+)}{S_0 \sigma \sqrt{T}} =: \Gamma$ ]

Definition 93 (Vega): The partial derivative to "an option price"  
[Black-Scholes:  $\frac{\partial V}{\partial \sigma} = S_0 \sqrt{T} \Phi'(d_+)$ ]

→ see Script for Theta  $\left(\frac{\partial V}{\partial T}\right)$ , Rho  $\left(\frac{\partial V}{\partial r}\right)$ .

## 7.4. Hedging in Discrete Time: Delta and Delta-Gamma Hedging.

Recall the situation in continuous time

$$\begin{aligned} dV(t) &= \frac{\partial V(t)}{\partial t} dt + \sum_{i=0}^{\infty} \frac{\partial V}{\partial S_i} dS_i + \frac{1}{2} \sum_{i,j=0}^{\infty} \frac{\partial^2 V(t)}{\partial S_i \partial S_j} dS_i dS_j \\ &= \sum_{i=0}^{\infty} \frac{\partial V}{\partial S_i} dS_i + \underbrace{\left( \frac{\partial V(t)}{\partial t} dt + \frac{1}{2} \sum_{i,j=0}^{\infty} \frac{\partial^2 V}{\partial S_i \partial S_j} \overbrace{dS_i dS_j}^{\gamma_{ij} dt} \right)}_{=0} \end{aligned}$$

In a time discrete setup we have for  $\Delta V(t) \stackrel{=0}{=} V(t+\Delta t) - V(t)$

$$\begin{aligned} \Delta V(t) &= \frac{\partial V(t)}{\partial t} \Delta t + \sum_{i=0}^{\infty} \frac{\partial V}{\partial S_i} \Delta S_i + \frac{1}{2} \sum_{i,j=0}^{\infty} \frac{\partial^2 V}{\partial S_i \partial S_j} \Delta S_i \Delta S_j + \text{h.o.t.} \\ &\stackrel{\Delta \Pi}{=} \underbrace{\sum_{i=0}^{\infty} \frac{\partial V}{\partial S_i} \Delta S_i}_{\Delta \Pi} + \frac{1}{2} \sum_{i,j=0}^{\infty} \frac{\partial^2 V}{\partial S_i \partial S_j} \underbrace{(\Delta S_i \Delta S_j - \gamma_{ij} \Delta t)}_{\neq 0} + \underbrace{\text{h.o.t.}}_{\sim} \end{aligned}$$

Compare this to the replication portfolio:

$$\Delta \Pi(t) = \sum_{i=0}^n \phi_i(t) \Delta S_i(t)$$

To ensure that the replication portfolio remains self-financing we have to reformulate the cond. in discrete time:

$$\sum_{i=0}^n (\phi_i(t_k) - \phi_i(t_{k-1})) S_i(t_k) = 0$$

For example, we may ensure ~~this~~ via

$$(*) \quad \phi_0(t_k) = \phi_0(t_{k-1}) - \frac{1}{S_0(t_k)} \sum_{i=1}^n (\phi_i(t_k) - \phi_i(t_{k-1})) S_i(t_k)$$

#### 7.4.1 Delta Hedging:

We choose  $\phi_1, \dots, \phi_n$  according to the delta hedge, i.e.

$$\phi_i = \frac{\partial V}{\partial S_i} \quad \text{for } i=1, \dots, n$$

and then choose  $\phi_0$  according to ~~(\*)~~.

7.4.2.1 Example: Time-Discrete Delta Hedge under a Black-Scholes Model

Given a time discretization  $0 = t_0 < t_1 < \dots$ :

For the replication portfolio  $\phi_0(t)N(t) + \phi_1(t) \cdot S(t)$

We have 
$$\phi_1(t_k) = \frac{\partial V(t_k)}{\partial S(t_k)} = \Phi\left(\frac{1}{\sigma\sqrt{T-t_k}}\left[\log\frac{S(t_k)}{K} + r(T-t_k) + \frac{\sigma^2}{2}(T-t_k)\right]\right)$$
  
( $k=0, \dots$ )

$$\phi_0(t_k) = \frac{1}{N(t_k)} \left[ \phi_0(t_{k-1}) \cdot N(t_k) + \phi_1(t_{k-1}) S(t_k) - \phi_1(t_k) S(t_k) \right]$$
  
( $k=1, \dots$ )

At time  $t_0=0$  the portfolio is set up according to the option value

$$\phi_0(t_0) = \frac{1}{N(t_0)} \left( V(t_0) - \phi_1(t_0) S(t_0) \right)$$

```

1  Java - finmath lib/src/main/java6/net/finmath/monte Carlo/assetderivativevaluation/AssetModelMonteCarloSimulationInterface.java - Eclipse - /Users/fries/Documents/Development
2  AssetModelMonteCarloSimulationInterface.java  EuropeanOption.java  BlackScholesDeltaHedgedPortfolio.java
3
4  public interface AssetModelMonteCarloSimulationInterface extends MonteCarloSimulationInterface {
5
6      /**
7       * Returns the number of asset price processes.
8       * @return The number of asset price processes
9       */
10     int
11     getNumberOfAssets();
12
13     /**
14      * Returns the random variable representing the asset's value at a given time for a given asset.
15      * @param timeIndex Index of simulation time
16      * @param assetIndex Index of the asset (0 for a single asset model)
17      * @return The asset process as seen on simulation time
18      * @throws net.finmath.exception.CalculationException Thrown if the valuation fails, specific cause may be available via the <code>cause()</code>
19      */
20     RandomVariableInterface getAssetValue(int timeIndex, int assetIndex) throws CalculationException;
21
22     /**
23      * Returns the random variable representing the asset's value at a given time for a given asset.
24      * @param time Simulation time
25      * @param assetIndex Index of the asset (0 for a single asset model)
26      * @return The asset process as seen on simulation time
27      * @throws net.finmath.exception.CalculationException Thrown if the valuation fails, specific cause may be available via the <code>cause()</code>
28      */
29     RandomVariableInterface getAssetValue(double time, int assetIndex) throws CalculationException;
30
31     /**
32      * Returns the numeraire associated with the valuation measure used by this model.
33      * @param timeIndex The time index (associated with this model's time discretization).
34      * @return The numeraire associated with the valuation measure used by this model.
35      * @throws CalculationException Thrown if calculation of numeraire fails.
36      */
37     RandomVariableInterface getNumeraire(int timeIndex) throws CalculationException;
38
39     /**
40      * Returns the numeraire associated with the valuation measure used by this model.
41      * @param time The time for which the numeraire is requested.
42      * @return The numeraire associated with the valuation measure used by this model.
43      * @throws CalculationException Thrown if calculation of numeraire fails.
44      */
45 }

```

Abstraction of the model and the way it is numerically simulated.

←  $w_j \mapsto S_j(t_k, w_j)$

←  $N(t_k)$

```

29 public class EuropeanOption extends AbstractAssetMonteCarloProduct {
30
31     private final double maturity;
32     private final double strike;
33     private final Integer underlyingIndex;
34     private final String nameOfUnderlying;
35
36     /**
37      * Construct a product representing an European option on an asset S (where S the asset with index 0 from the model - single asset case).
38      * @param maturity The maturity T in the option payoff max(S(T)-K,0)
39      * @param strike The strike K in the option payoff max(S(T)-K,0).
40      * @param underlyingIndex The index of the underlying to be fetched from the model.
41      */
42     public EuropeanOption(double maturity, double strike, int underlyingIndex) {
43         super();
44         this.maturity = maturity;
45         this.strike = strike;
46         this.underlyingIndex = underlyingIndex;
47         this.nameOfUnderlying = null; // Use underlyingIndex
48     }
49
50     /**
51      * Construct a product representing an European option on an asset S (where S the asset with index 0 from the model - single asset case).
52      * @param maturity The maturity T in the option payoff max(S(T)-K,0)
53      * @param strike The strike K in the option payoff max(S(T)-K,0).
54      */
55     public EuropeanOption(double maturity, double strike) {
56         this(maturity, strike, 0);
57     }
58
59     /**
60      * This method returns the value random variable of the product within the specified model, evaluated at a given evaluationTime.
61      * Note: For a lattice this is often the value conditional to evaluationTime, for a Monte-Carlo simulation this is the (sum of) value dis
62      * Cashflows prior evaluationTime are not considered.
63      *
64      * @param evaluationTime The time on which this products value should be observed.
65      * @param model The model used to price the product.
66      * @return The random variable representing the value of the product discounted to evaluation time
67      * @throws net.finmath.exception.CalculationException Thrown if the valuation fails, specific cause may be available via the <code>cause()
68      */
69     @Override
70     public RandomVariableInterface getValue(double evaluationTime, AssetModelMonteCarloSimulationInterface model) throws CalculationException
71     // Get underlying and numeraire
72

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73
74     RandomVariableInterface underlyingAtMaturity = model.getAssetValue(maturity, underlyingIndex);
75
76     // The payoff: values = max(underlying - strike, 0) = V(T) = max(S(T)-K,0)
77     RandomVariableInterface values = underlyingAtMaturity.sub(strike).floor(0.0);
78
79     // Discounting...
80     RandomVariableInterface numeraireAtMaturity = model.getNumeraire(maturity);
81     RandomVariableInterface monteCarloWeights = model.getMonteCarloWeights(maturity);
82     values = values.div(numeraireAtMaturity).mult(monteCarloWeights);
83
84     // ...to evaluation time.
85     RandomVariableInterface numeraireAtEvalTime = model.getNumeraire(evaluationTime);
86     RandomVariableInterface monteCarloProbabilitiesAtEvalTime = model.getMonteCarloWeights(evaluationTime);
87     values = values.mult(numeraireAtEvalTime).div(monteCarloProbabilitiesAtEvalTime);
88
89     return values;
90 }
91
92

```

```

19 20 public class BlackScholesDeltaHedgedPortfolio extends AbstractAssetMonteCarloProduct {
21 22 // Properties of the European option we wish to replicate
23 private final double maturity;
24 private final double strike;
25
26 // Model assumptions for the hedge
27 private final double riskFreeRate;
28 private final double volatility;
29
30 /**
31  * Construction of a delta hedge portfolio assuming a Black-Scholes model.
32  *
33  * @param maturity Maturity of the option we wish to replicate.
34  * @param strike Strike of the option we wish to replicate.
35  * @param riskFreeRate Model riskFreeRate assumption for our delta hedge.
36  * @param volatility Model volatility assumption for our delta hedge.
37  */
38 public BlackScholesDeltaHedgedPortfolio(double maturity, double strike, double riskFreeRate, double volatility) {
39     super();
40     this.maturity = maturity;
41     this.strike = strike;
42     this.riskFreeRate = riskFreeRate;
43     this.volatility = volatility;
44 }
45
46 @Override
47 public RandomVariableInterface getValue(double evaluationTime, AssetModelMonteCarloSimulationInterface model) throws CalculationException
48
49 // Ask the model for its discretization
50 int timeIndexEvaluationTime = model.getTimeIndex(evaluationTime);
51
52 /*
53  * Going forward in time we monitor the hedge portfolio on each path.
54  */
55
56 // Initialize the portfolio to zero stocks and as much cash as the Black-Scholes Model predicts we need.
57 RandomVariableInterface underlyingToday = model.getAssetValue(0.0, 0.0);
58 RandomVariableInterface numeraireToday = model.getNumeraire(0.0);
59
60 RandomVariableInterface valueOfOptionAccordingBlackScholes = AnalyticFormulas.blackScholesGeneralizedOptionValue(
61     underlyingToday, mult(Math.exp(riskFreeRate * (maturity - 0.0))),
62     model.getRandomVariableForConstant(volatility),

```

```

63     maturity - 0.0,
64     strike,
65     model.getRandomVariableForConstant(Math.exp(-riskFreeRate * (maturity - 0.0)));
66
67 // We store the composition of the hedge portfolio (depending on the path)
68 RandomVariableInterface amountOfNumeraireAsset = valueOfOptionAccordingBlackScholes.div(numeraireToday);
69 RandomVariableInterface amountOfUnderlyingAsset = model.getRandomVariableForConstant(0.0);
70
71 for(int timeIndex = 0; timeIndex < timeIndexEvaluationTime; timeIndex++) {
72     // Get value of underlying and numeraire assets
73     RandomVariableInterface underlyingAtTimeIndex = model.getAssetValue(timeIndex, 0);
74     RandomVariableInterface numeraireAtTimeIndex = model.getNumeraire(timeIndex);
75
76     // Delta of option to replicate
77     RandomVariableInterface delta = AnalyticFormulas.blackScholesOptionDelta(
78         underlyingAtTimeIndex,
79         model.getRandomVariableForConstant(riskFreeRate),
80         model.getRandomVariableForConstant(volatility),
81         maturity - model.getTime(timeIndex), // remaining time
82         strike);
83
84     /*
85     * Change the portfolio according to the trading strategy
86     */
87
88     // Determine the delta hedge
89     RandomVariableInterface newNumberOfStocks = delta;
90     RandomVariableInterface stocksToBuy = newNumberOfStocks.sub(amountOfUnderlyingAsset);

```



Handwritten annotations on the first screenshot:

- From  $t_{k-1}$  to  $t_k$  (vertical text on the left).
- $\Delta = \frac{\partial V}{\partial S} \leadsto \phi_1(t_{k-1})$  (large bracketed equation on the right).
- self-financing cond. (red text on the right).
- $\phi_0(t_k)$  and  $\phi_1(t_k)$  (green text below the main equation).

```

60 RandomVariableInterface valueOfOptionAccordingBlackScholes = AnalyticFormulas.blackScholesGeneralizedOptionValue(
61     underlyingToday.mult(Math.exp(riskFreeRate * (maturity - 0.0))),
62     model.getRandomVariableForConstant(volatility),
63     maturity - 0.0,
64     strike,
65     model.getRandomVariableForConstant(Math.exp(-riskFreeRate * (maturity - 0.0))));
66
67 // We store the composition of the hedge portfolio (depending on the path)
68 RandomVariableInterface amountOfNumeraireAsset = valueOfOptionAccordingBlackScholes.div(numeraireToday);
69 RandomVariableInterface amountOfUnderlyingAsset = model.getRandomVariableForConstant(0.0);
70
71 for(int timeIndex = 0; timeIndex < timeIndexEvaluationTime; timeIndex++) {
72     // Get value of underlying and numeraire assets
73     RandomVariableInterface underlyingAtTimeIndex = model.getAssetValue(timeIndex,0);
74     RandomVariableInterface numeraireAtTimeIndex = model.getNumeraire(timeIndex);
75
76     // Delta of option to replicate
77     RandomVariableInterface delta = AnalyticFormulas.blackScholesOptionDelta(
78         underlyingAtTimeIndex,
79         model.getRandomVariableForConstant(riskFreeRate),
80         model.getRandomVariableForConstant(volatility),
81         maturity-model.getTime(timeIndex), // remaining time
82         strike);
83
84     /*
85     * Change the portfolio according to the trading strategy
86     */
87
88     // Determine the delta hedge
89     RandomVariableInterface newNumberOfStocks = delta;
90     RandomVariableInterface stocksToBuy = newNumberOfStocks.sub(amountOfUnderlyingAsset);
91
92     // Ensure self financing
93     RandomVariableInterface numeraireAssetsToSell = stocksToBuy.mult(underlyingAtTimeIndex).div(numeraireAtTimeIndex);
94     RandomVariableInterface newNumberOfNumeraireAsset = amountOfNumeraireAsset.sub(numeraireAssetsToSell);
95
96     // Update portfolio
97     amountOfNumeraireAsset = newNumberOfNumeraireAsset;
98     amountOfUnderlyingAsset = newNumberOfStocks;
99 }
100
101 /*
102 * At maturity, calculate the value of the replication portfolio
103 */

```

Handwritten annotations on the second screenshot:

- $T = \text{"evaluationTime"}$  (green text on the right).
- $S(T)$  and  $N(T)$  (green text on the right).
- $\Pi(\tau) = \phi_0(\tau)N(T) + \phi_1(\tau)\Delta$  (green text on the right).

```

72 // Get value of underlying and numeraire assets
73 RandomVariableInterface underlyingAtTimeIndex = model.getAssetValue(timeIndex,0);
74 RandomVariableInterface numeraireAtTimeIndex = model.getNumeraire(timeIndex);
75
76 // Delta of option to replicate
77 RandomVariableInterface delta = AnalyticFormulas.blackScholesOptionDelta(
78     underlyingAtTimeIndex,
79     model.getRandomVariableForConstant(riskFreeRate),
80     model.getRandomVariableForConstant(volatility),
81     maturity-model.getTime(timeIndex), // remaining time
82     strike);
83
84 /*
85 * Change the portfolio according to the trading strategy
86 */
87
88 // Determine the delta hedge
89 RandomVariableInterface newNumberOfStocks = delta;
90 RandomVariableInterface stocksToBuy = newNumberOfStocks.sub(amountOfUnderlyingAsset);
91
92 // Ensure self financing
93 RandomVariableInterface numeraireAssetsToSell = stocksToBuy.mult(underlyingAtTimeIndex).div(numeraireAtTimeIndex);
94 RandomVariableInterface newNumberOfNumeraireAsset = amountOfNumeraireAsset.sub(numeraireAssetsToSell);
95
96 // Update portfolio
97 amountOfNumeraireAsset = newNumberOfNumeraireAsset;
98 amountOfUnderlyingAsset = newNumberOfStocks;
99 }
100
101 /*
102 * At maturity, calculate the value of the replication portfolio
103 */
104
105 // Get value of underlying and numeraire assets
106 RandomVariableInterface underlyingAtEvaluationTime = model.getAssetValue(evaluationTime,0);
107 RandomVariableInterface numeraireAtEvaluationTime = model.getNumeraire(evaluationTime);
108
109 RandomVariableInterface portfolioValue = amountOfNumeraireAsset.mult(numeraireAtEvaluationTime)
110     .add(amountOfUnderlyingAsset.mult(underlyingAtEvaluationTime));
111
112 return portfolioValue;
113 }
114 }
115

```







