Approaches

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1 The OU-OU model

We model the three-dimensional log-returns $R = (R^1, R^2, R^3)$ as

$$\mathrm{d}R_t = \left[\mu - \frac{1}{2}\operatorname{diag}\left(\Sigma_t \Sigma_t^\top\right)\right]\mathrm{d}t + \Sigma_t \mathrm{d}B_t, \quad R_0 = r_0, \tag{1}$$

where $\mu \in \mathbb{R}^3$ is the drift, $\sigma_t \in \mathbb{R}^3$ is the volatility, and $B = (B^1, B^2, B^3)$ is a standard Brownian motion in $\mathbb{R}^3$. We assume that the volatility is given by and most importantly with $X := \log(\Sigma)$ the log-volatility process, we have

$$\begin{aligned} \mathrm{d}Y_t &= -\beta Y_t \mathrm{d}t + \Sigma_Y \mathrm{d}W_t^1, \quad X_0 = x_0, \\ \mathrm{d}X_t &= \lambda\left(Y_t - X_t\right)\mathrm{d}t + \Sigma_X \mathrm{d}W_t^2, \quad X_0 = x_0, \end{aligned}$$

and $X, Y \in \mathbb{R}^{3\times 3}$ are 3×3-matrix-valued processes, and W^1, W^2 are independent standard Brownian motions in $\mathbb{R}^3$.

The parameters $\beta, \lambda > 0$ are the mean-reversion rates, and $\Sigma_Y, \Sigma_X \in \mathcal{S}_{++}^3$ are the covariance matrices of the OU-OU model. The Brownian motions W^1, W^2 are independent standard Brownian motions in $\mathbb{R}^3$.

Example 1.1. In the case of a single asset, we have $R = (R^1)$ and $\Sigma = \sigma > 0$. The parameters of the OU-OU model in the reference are e.g.

$$\Sigma_X = 20, \quad \Sigma_Y = 0.625, \quad \lambda = 210, \quad \beta = 2.5.$$

Fitting the model

We can obtain the log-volatility process X by using

$$\langle R_t \rangle_t = \int_0^t \Sigma_s \Sigma_s^\top \mathrm{d}s, \tag{2}$$

together with the approximation

$$\langle R_t \rangle_t \approx \sum_{n=0}^{N-1} \left(R_{t_{n+1}} - R_{t_n}\right)\left(R_{t_{n+1}} - R_{t_n}\right)^\top, \tag{3}$$

for the covariation of the log-returns R. Hence we can estimate the log-volatility process X by

1.1 As discrete-time model

Discretizing the OU-OU model, on an equidistant grid $t_n = n\Delta t$ with $n \in \mathbb{N}$ and $\Delta t > 0$, we obtain

$$\begin{aligned} R_{t_{n+1}} &= R_{t_n} + \left[\mu - \frac{1}{2}\operatorname{diag}\left(\Sigma_{t_n}\Sigma_{t_n}^\top\right)\right] + \Sigma_{t_n} Z_n, && (4) \\ Y_{t_{n+1}} &= (1 - \beta Y_{t_n}) + \Sigma_Y Z_n^1, && (5) \\ X_{t_{n+1}} &= X_{t_n} + \lambda\left(Y_{t_n} - X_{t_n}\right) + \Sigma_X Z_n^2, && (6) \end{aligned}$$

where in the application the parameters correspond to the 1-hourly estimation of the model.

2 A GARCH model

In $GARCH(1,1)$, the volatility process is given by

$$\begin{aligned} r_t &= \mu + \sigma_t Z_t, \\ \Sigma_t^2 &= \omega + \alpha r_{t-1} r_{t-1}^T + \beta \Sigma_{t-1} \Sigma_{t-1}^T, \end{aligned}$$

where we test $Z \sim \mathcal{N}(0,1)$ or $Z \sim t_1$.