1 Problem

Prove that

$$\int_0^1 \frac{\ln t \, \mathrm{d}t}{1+t} = -\frac{\pi^2}{12}.$$

2 Solution

2.1 Step 1.

Consider the function

$$F(x) = \int_0^1 \frac{x \ln t \, \mathrm{d}t}{x+t}.$$

Taking derivative in x, we find that

$$F'(x) = -\frac{\ln(1+x)}{x},$$

and hence

$$F(x) = -\int_0^x \frac{\ln(1+t)\,\mathrm{d}t}{t}. \tag{1}$$

Proof. Indeed,

$$F'(x) = \int_0^1 \frac{\ln t \, \mathrm{d}t}{x+t} - \int_0^1 \frac{x \ln t \, \mathrm{d}t}{(x+t)^2} = \int_0^1 \frac{t \ln t \, \mathrm{d}t}{(x+t)^2}.$$

Integrating here by parts, we find

$$\int_0^1 \frac{t \ln t \, \mathrm{d}t}{(x+t)^2} = \int_0^1 \partial_t \left(\frac{-1}{x+t} \right) \cdot t \ln t \, \mathrm{d}t = -\frac{t \ln t}{(x+t)}\Big|_{t=0}^{t=1} + \int_0^1 \frac{(\ln t + 1)\mathrm{d}t}{x+t} = .$$

where in the second integral we integrated by parts.
End of proof. $\triangle$

2.2 Step 2.

Expanding the logarithm into series

$$\ln(1+t) = \sum_{j=0}^{\infty} \frac{(-1)^{j-1} t^j}{j}$$