



National Research University Higher School of Economics

# Youthful Passion Fruit

Alexandr Nekrasov, Iurii Pustovalov, Alexey Vasilyev

August 10, 2024

# Contest (1)

template.cpp

42 lines

```
#ifndef LOCAL
#define _GLIBCXX_DEBUG
#endif
#include <bits/stdc++.h>
using namespace std;

using ll = long long;
using ld = long double;
using ull = unsigned long long;

#define pbc push_back
#define mp make_pair
#define all(v) (v).begin(), (v).end()
#define vin(v) for (auto &el : a) cin >> el

mt19937 rnd(chrono::steady_clock::now().
    time_since_epoch().count());

template <typename T1, typename T2> inline void chkmin(
    T1 &x, const T2 &y) {
    if (y < x) {
        x = y;
    }
}

template <typename T1, typename T2> inline void chkmax(
    T1 &x, const T2 &y) {
    if (x < y) {
        x = y;
    }
}

void solve() {

}

signed main() {
    cin.tie(0)->sync_with_stdio(0);
    cout.precision(20), cout.setf(ios::fixed);
    int t = 1;
    // cin >> t;
    while (t--) {
        solve();
    }
}
```

genfolders.sh

6 lines

```
chmod +x bld*
for f in {A..Z}
do
    mkdir $f
    cp main.cpp bld* $f
done
```

bld

1 lines

```
g++ -std=c++17 -g -DLOCAL -fsanitize=address,bounds,
    undefined -o $1 $1.cpp

bldf
```

bldf

1 lines

```
g++ -std=c++17 -g -O2 -o $1 $1.cpp

hash.sh
```

```
# Hashes a file, ignoring all whitespace and comments.
# Use for verifying that code was correctly typed.
cpp -dD -P -fpreprocessed | tr -d '[:space:]' | md5sum |
    cut -c-6
```

# C++ (2)

GpHashtable.cpp

**Description:** Hash map with mostly the same API as unordered\_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

e44914, 11 lines

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;

const int RANDOM =
    chrono::high_resolution_clock::now().
        time_since_epoch().count();
struct hasher {
    int operator()(int x) const { return x ^ RANDOM; }
};

gp_hash_table<int, int, hasher> table;
```

OrderedSet.cpp

**Description:** A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change null\_type.

**Time:**  $\mathcal{O}(\log(n))$

dff260, 37 lines

```
<bits/extc++.h>, <bits/stdc++.h>
using namespace __gnu_pbds;
using namespace std;

template <typename T>
using ordered_set =
    tree<T, null_type, less<>, rb_tree_tag,
        tree_order_statistics_node_update>;

int main() {
    ordered_set<int> X;
    X.insert(1);
    X.insert(2);
    X.insert(4);
    X.insert(8);
    X.insert(16);

    assert(*X.find_by_order(1) == 2);
    assert(*X.find_by_order(2) == 4);
    assert(*X.find_by_order(4) == 16);
    assert(X.find_by_order(6) ==X.end());

    assert(X.order_of_key(-5) == 0);
    assert(X.order_of_key(1) == 0);
    assert(X.order_of_key(3) == 2);
    assert(X.order_of_key(4) == 2);
    assert(X.order_of_key(400) == 5);
    // std::cout << *X.find_by_order(1) << std::endl;
    // 2
    // std::cout << *X.find_by_order(2) << std::endl;
    // 4
    // std::cout << *X.find_by_order(4) << std::endl;
    // 16
```

```
// std::cout << (end(X) == X.find_by_order(6)) <<
// std::endl; // true

// std::cout << X.order_of_key(-5) << std::endl;
// 0
// std::cout << X.order_of_key(1) << std::endl;
// 0
// std::cout << X.order_of_key(3) << std::endl;
// 2
// std::cout << X.order_of_key(4) << std::endl;
// 2
// std::cout << X.order_of_key(400) << std::endl;
// 5
return 0;
}
```

# Strings (3)

## Manacher.cpp

Description: Manacher algorithm

Time:  $\mathcal{O}(n)$

a6ddfb, 27 lines

```
vector<int> manacherOdd(string s) {
    int n = s.size();
    vector<int> d1(n);
    int l = 0, r = -1;
    for (int i = 0; i < n; ++i) {
        int k = i > r ? 1 : min(d1[l + r - i], r - i + 1);
        while (i + k < n && i - k >= 0 && s[i + k] == s[i - k])
            ++k;
        d1[i] = k;
        if (i + k - 1 > r)
            l = i - k + 1, r = i + k - 1;
    }
}

vector<int> manacherEven(string s) {
    int n = s.size();
    vector<int> d2(n);
    l = 0, r = -1;
    for (int i = 0; i < n; ++i) {
        int k = i > r ? 0 : min(d2[l + r - i + 1], r - i + 1);
        while (i + k < n && i - k - 1 >= 0 && s[i + k] == s[i - k - 1])
            ++k;
        d2[i] = k;
        if (i + k - 1 > r)
            l = i - k, r = i + k - 1;
    }
}
```

## AhoCorasick.cpp

Description: Build aho-corasick automaton.

Time:  $\mathcal{O}(n)$

ae5fc2, 19 lines

```
int go(int v, char c);

int get_link(int v) {
    if (t[v].link == -1)
        if (v == 0 || t[v].p == 0)
            t[v].link = 0;
        else
            t[v].link = go(get_link(t[v].p), t[v].pch);
    return t[v].link;
}

int go(int v, char c) {
    if (t[v].go[c] == -1)
        if (t[v].next[c] != -1)
            t[v].go[c] = t[v].next[c];
        else
```

```
        t[v].go[c] = v == 0 ? 0 : go(get_link(v), c);
    }
    return t[v].go[c];
}
```

## SuffixArray.cpp

Description: Build suffix array

Time:  $\mathcal{O}(n \log(n))$

5bd011, 47 lines

```
vector<int> buildSuffixArray(string &s) {
    // Remove, if you want to sort cyclic shifts
    s += (char)(1);
    int n = s.size();
    vector<int> a(n);
    iota(all(a), 0);
    stable_sort(all(a), [&](int i, int j) { return s[i] < s[j]; });
    vector<int> c(n);
    int cc = 0;
    for (int i = 0; i < n; i++) {
        if (i == 0 || s[a[i]] != s[a[i - 1]]) {
            c[a[i]] = cc++;
        } else {
            c[a[i]] = c[a[i - 1]];
        }
    }
    for (int L = 1; L < n; L *= 2) {
        vector<int> cnt(n);
        for (auto i : c) {
            cnt[i]++;
        }
        vector<int> pref(n);
        for (int i = 1; i < n; i++) {
            pref[i] = pref[i - 1] + cnt[i - 1];
        }
        vector<int> na(n);
        for (int i = 0; i < n; i++) {
            int pos = (a[i] - L + n) % n;
            na[pref[c[pos]]++] = pos;
        }
        a = na;
        vector<int> nc(n);
        cc = 0;
        for (int i = 0; i < n; i++) {
            if (i == 0 || c[a[i]] != c[a[i - 1]] ||
                c[(a[i] + L) % n] != c[(a[i - 1] + L) % n]) {
                nc[a[i]] = cc++;
            } else {
                nc[a[i]] = nc[a[i - 1]];
            }
        }
        c = nc;
    }
    a.erase(a.begin());
    s.pop_back();
    return a;
}
```

## Lcp.cpp

**Description:** lcp array**Time:**  $\mathcal{O}(n)$ 

1cc27c, 43 lines

```
vector<int> perm;
vector<int> buildLCP(string &s, vector<int> &a) {
    int n = s.size();
    vector<int> ra(n);
    for (int i = 0; i < n; i++) {
        ra[a[i]] = i;
    }
    vector<int> lcp(n - 1);
    int cur = 0;
    for (int i = 0; i < n; i++) {
        cur--;
        chkmax(cur, 0);
        if (ra[i] == n - 1) {
            cur = 0;
            continue;
        }
        int j = a[ra[i] + 1];
        while (s[i + cur] == s[j + cur]) cur++;
        lcp[ra[i]] = cur;
    }
    perm.resize(a.size());
    for (int i = 0; i < a.size(); ++i) perm[a[i]] = i;
    return lcp;
}

int cntr[MAXN];
int spt[MAXN][lgg];
void build(vector<int> &a) {
    for (int i = 0; i < a.size(); ++i) {
        spt[i][0] = a[i];
    }
    for (int i = 2; i < MAXN; ++i) cntr[i] = cntr[i / 2] + 1;
    for (int h = 1; (1 << (h - 1)) < a.size(); ++h) {
        for (int i = 0; i + (1 << (h - 1)) < a.size(); ++i) {
            spt[i][h] = min(spt[i][h - 1], spt[i + (1 << (h - 1))][h - 1]);
        }
    }
}

int getLCP(int l, int r) {
    l = perm[l], r = perm[r];
    if (l > r) swap(l, r);
    int xx = cntr[r - l];
    return min(spt[l][xx], spt[r - (1 << xx)][xx]);
}
```

## Eertree.cpp

**Description:** Creates Eertree of string str**Time:**  $\mathcal{O}(n)$ 

7924c8, 40 lines

```
struct eertree {
    int len[MAXN], suffLink[MAXN];
    int to[MAXN][26];
    int numV, v;
```

```
void addLetter(int n, string &str) {
    while (str[n - len[v] - 1] != str[n])
        v = suffLink[v];
    int u = suffLink[v];
    while (str[n - len[u] - 1] != str[n])
        u = suffLink[u];
    int u_ = to[u][str[n] - 'a'];
    int v_ = to[v][str[n] - 'a'];
    if (v_ == -1) {
        v_ = to[v][str[n] - 'a'] = numV;
        len[numV++] = len[v] + 2;
        suffLink[v_] = u_;
    }
    v = v_;
}

void init() {
    len[0] = -1;
    len[1] = 0;
    suffLink[1] = 0;
    suffLink[0] = 0;
    numV = 2;
    for (int i = 0; i < 26; ++i) {
        to[0][i] = numV++;
        suffLink[numV - 1] = 1;
        len[numV - 1] = 1;
    }
    v = 0;
}

void init(int sz) {
    for (int i = 0; i < sz; ++i) {
        len[i] = suffLink[i] = 0;
        for (int j = 0; j < 26; ++j)
            to[i][j] = -1;
    }
}
```

};

## SuffixAutomaton.cpp

**Description:** Build suffix automaton.**Time:**  $\mathcal{O}(n)$ 

662a10, 45 lines

```
struct state {
    int len, link;
    map<char, int> next;
};

const int MAXLEN = 100000;
state st[MAXLEN * 2];
int sz, last;

void sa_init() {
    sz = last = 0;
    st[0].len = 0;
    st[0].link = -1;
    ++sz;
    /*
    // if you want to build an automaton for different
    strings:
```

```
for (int i=0; i<MAXLEN*2; ++i)
    st[i].next.clear();
*/
}

void sa_extend(char c) {
    int cur = sz++;
    st[cur].len = st[last].len + 1;
    int p;
    for (p = last; p != -1 && !st[p].next.count(c); p =
        st[p].link)
        st[p].next[c] = cur;
    if (p == -1)
        st[cur].link = 0;
    else {
        int q = st[p].next[c];
        if (st[p].len + 1 == st[q].len)
            st[cur].link = q;
        else {
            int clone = sz++;
            st[clone].len = st[p].len + 1;
            st[clone].next = st[q].next;
            st[clone].link = st[q].link;
            for (; p != -1 && st[p].next[c] == q; p =
                st[p].link)
                st[p].next[c] = clone;
            st[q].link = st[cur].link = clone;
        }
    }
    last = cur;
}
```

## PrefixZ.cpp

**Description:** Calculates Prefix,Z-functions**Time:**  $\mathcal{O}(n)$ 

1c4e93, 25 lines

```
vector<int> pf(string s) {
    int k = 0;
    vector<int> p(s.size());
    for (int i = 1; i < s.size(); ++i) {
        while (k && s[i] != s[k])
            k = p[k - 1];
        k += (s[i] == s[k]);
        p[i] = k;
    }
    return p;
}

vector<int> zf(string s) {
    int n = s.size();
    vector<int> z(n, 0);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r)
            z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
            ++z[i];
        if (i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
}
```

```

    z[0] = n;
    return z;
}

```

## MinShift.cpp

**Description:** Calculates min-cyclic-shift of s, Duval decomposition

**Time:**  $\mathcal{O}(n)$

3f0fb9, 20 lines

```

string minshift(string s) {
    int i = 0, ans = 0;
    s += s;
    int n = s.size();
    while (i < n / 2) {
        ans = i;
        int j = i + 1, k = i;
        while (j < n && s[k] <= s[j]) {
            if (s[k] < s[j])
                k = i;
            else
                ++k;
            ++j;
        }
        while (i <= k) {
            i += j - k;
        }
    }
    return s.substr(ans, n / 2);
}

```

## SA-IS.cpp

**Description:** Build suffix array

**Time:**  $\mathcal{O}(n)$

f90ffe, 87 lines

```

void induced_sort(vector<int> &vec, int LIM, vector<int>
    &sa, vector<bool> &sl,
    vector<int> &fx) {
    vector<int> l(LIM), r(LIM);
    for (int c : vec) {
        if (c + 1 < LIM) {
            ++l[c + 1];
        }
        ++r[c];
    }
    partial_sum(all(l), l.begin());
    partial_sum(all(r), r.begin());
    fill(all(sa), -1);
    for (int i = fx.size() - 1; i >= 0; --i) {
        sa[--r[vec[fx[i]]]] = fx[i];
    }
    for (int i : sa) {
        if (i >= 1 && sl[i - 1]) {
            sa[l[vec[i - 1]]++] = i - 1;
        }
    }
    fill(all(r), 0);
    for (int c : vec) ++r[c];
    partial_sum(all(r), r.begin());
    for (int k = sa.size() - 1, i = sa[k]; k >= 1; --k,
        i = sa[k])

```

```

        if (i >= 1 && !sl[i - 1]) sa[--r[vec[i - 1]]] =
            i - 1;
    }
    vector<int> SA_IS(vector<int> &vec, int LIM) {
        const int n = vec.size();
        vector<int> sa(n), fx;
        vector<bool> sl(n);
        sl[n - 1] = false;
        for (int i = n - 2; i >= 0; --i) {
            sl[i] = (vec[i] > vec[i + 1] || (vec[i] == vec[i + 1] && sl[i + 1]));
            if (sl[i] && !sl[i + 1]) {
                fx.pbc(i + 1);
            }
        }
        reverse(all(fx));
        induced_sort(vec, LIM, sa, sl, fx);
        vector<int> nfx(fx.size()), lmv(fx.size());
        for (int i = 0, k = 0; i < n; ++i) {
            if (!sl[sa[i]] && sa[i] >= 1 && sl[sa[i] - 1]) {
                {
                    nfx[k++] = sa[i];
                }
            }
        }
        int cur = 0;
        sa[n - 1] = cur;
        for (int k = 1; k < nfx.size(); ++k) {
            int i = nfx[k - 1], j = nfx[k];
            if (vec[i] != vec[j]) {
                sa[j] = ++cur;
                continue;
            }
            bool flag = false;
            for (int a = i + 1, b = j + 1;; ++a, ++b) {
                if (vec[a] != vec[b]) {
                    flag = true;
                    break;
                }
                if ((!sl[a] && sl[a - 1]) || (!sl[b] && sl[b - 1])) {
                    flag = !((!sl[a] && sl[a - 1]) && (!sl[b] && sl[b - 1]));
                    break;
                }
            }
            sa[j] = (flag ? ++cur : cur);
        }
        for (int i = 0; i < fx.size(); ++i) {
            lmv[i] = sa[fx[i]];
        }
        if (cur + 1 < (int)fx.size()) {
            auto lms = SA_IS(lmv, cur + 1);
            for (int i = 0; i < fx.size(); ++i) {
                nfx[i] = fx[lms[i]];
            }
        }
        induced_sort(vec, LIM, sa, sl, nfx);
        return sa;
    }
}

```

```

}
template <typename T>
vector<int> suffix_array(T &s, const int LIM = 128) {
    vector<int> vec(s.size() + 1);
    copy(all(s), begin(vec));
    vec.back() = (char)(1);
    auto ret = SA_IS(vec, LIM);
    ret.erase(ret.begin());
    return ret;
}

```

# Graph (4)

## Hungarian.cpp

**Description:** Hungarian algorithm

**Time:**  $\mathcal{O}(n^3)$

5afee5, 41 lines

```
int n, m;
vector<vector<int>> a;
vector<int> u(n + 1), v(m + 1), p(m + 1), way(m + 1);
for (int i = 1; i <= n; ++i) {
    p[0] = i;
    int j0 = 0;
    vector<int> minv(m + 1, INF);
    vector<char> used(m + 1, false);
    do {
        used[j0] = true;
        int i0 = p[j0], delta = INF, j1;
        for (int j = 1; j <= m; ++j)
            if (!used[j]) {
                int cur = a[i0][j] - u[i0] - v[j];
                if (cur < minv[j])
                    minv[j] = cur, way[j] = j0;
                if (minv[j] < delta)
                    delta = minv[j], j1 = j;
            }
        for (int j = 0; j <= m; ++j)
            if (used[j])
                u[p[j]] += delta, v[j] -= delta;
            else
                minv[j] -= delta;
        j0 = j1;
    } while (p[j0] != 0);
    do {
        int j1 = way[j0];
        p[j0] = p[j1];
        j0 = j1;
    } while (j0);

    // matching
    vector<int> ans(n + 1);
    for (int j = 1; j <= m; ++j) {
        ans[p[j]] = j;
    }

    // cost
    int cost = -v[0];
}
```

## BlossomShrinking.cpp

**Description:** Maximum matching in general graph

**Time:**  $\mathcal{O}(n^3)$

23839d, 118 lines

```
struct Edge {
    int u, v;
};
const int N = 510;
int n, m;
vector<int> g[N];
```

```
vector<Edge> perfectMatching;
int match[N], par[N], base[N];
bool used[N], blossom[N], lcaUsed[N];
int lca(int u, int v) {
    fill(lcaUsed, lcaUsed + n, false);
    while (u != -1) {
        u = base[u];
        lcaUsed[u] = true;
        if (match[u] == -1)
            break;
        u = par[match[u]];
    }
    while (v != -1) {
        v = base[v];
        if (lcaUsed[v])
            return v;
        v = par[match[v]];
    }
    assert(false);
    return -1;
}
void markPath(int v, int myBase, int children) {
    while (base[v] != myBase) {
        blossom[v] = blossom[match[v]] = true;
        par[v] = children;
        children = match[v];
        v = par[match[v]];
    }
}
int findPath(int root) {
    iota(base, base + n, 0);
    fill(par, par + n, -1);
    fill(used, used + n, false);
    queue<int> q;
    q.push(root);
    used[root] = true;
    while (!q.empty()) {
        int v = q.front();
        q.pop();
        for (auto to : g[v]) {
            if (match[v] == to)
                continue;
            if (base[v] == base[to])
                continue;
            if (to == root || (match[to] != -1 && par[match[to]] != -1)) {
                fill(blossom, blossom + n, false);
                int myBase = lca(to, v);
                markPath(v, myBase, to);
                markPath(to, myBase, v);
                for (int u = 0; u < n; ++u) {
                    if (!blossom[base[u]])
                        continue;
                    base[u] = myBase;
                    if (used[u])
                        continue;
                    used[u] = true;
                    q.push(u);
                }
            }
        }
    }
}
```

```
    }
    } else if (par[to] == -1) {
        par[to] = v;
        if (match[to] == -1) {
            return to;
        }
        used[match[to]] = true;
        q.push(match[to]);
    }
}
return -1;
}
void blossomShrinking() {
    fill(match, match + n, -1);
    for (int v = 0; v < n; ++v) {
        if (match[v] != -1)
            continue;
        int nxt = findPath(v);
        while (nxt != -1) {
            int parV = par[nxt];
            int parParV = match[parV];
            match[nxt] = parV;
            match[parV] = nxt;
            nxt = parParV;
        }
    }
    for (int v = 0; v < n; ++v) {
        if (match[v] != -1 && v < match[v]) {
            perfectMatching.push_back({v, match[v]});
        }
    }
}
signed main() {
    cin >> n;
    int u, v;
    set<pair<int, int>> edges;
    while (cin >> u >> v) {
        --u;
        --v;
        if (u > v)
            swap(u, v);
        if (edges.count({u, v}))
            continue;
        edges.insert({u, v});
        g[u].push_back(v);
        g[v].push_back(u);
    }
    blossomShrinking();
    cout << perfectMatching.size() * 2 << '\n';
    for (auto i : perfectMatching) {
        cout << i.u + 1 << " " << i.v + 1 << "\n";
    }
    return 0;
}
```

## Lct.cpp

**Description:** link-cut tree**Time:**  $\mathcal{O}(n \log(n))$ 

3d8a3f, 136 lines

```
#include <bits/stdc++.h>
using namespace std;

const int MAXN = 1e5 + 228;

struct node {
    node *ch[2];
    node *p;
    bool rev;
    int sz;

    node() {
        ch[0] = ch[1] = p = NULL;
        rev = false;
        sz = 1;
    }
};

int getsz(node *n) { return (n == NULL) ? 0 : n->sz; }

void pull(node *n) { n->sz = getsz(n->ch[0]) + getsz(n->ch[1]) + 1; }

void push(node *n) {
    if (n->rev) {
        if (n->ch[0]) {
            n->ch[0]->rev ^= 1;
        }
        if (n->ch[1]) {
            n->ch[1]->rev ^= 1;
        }
        swap(n->ch[0], n->ch[1]);
        n->rev = 0;
    }
}

bool isRoot(node *n) {
    return n->p == NULL || (n->p->ch[0] != n && n->p->ch[1] != n);
}

int chnum(node *n) { return n->p->ch[1] == n; }

void attach(node *n, node *p, int num) {
    if (n != NULL)
        n->p = p;
    if (p != NULL)
        p->ch[num] = n;
}

void rotate(node *n) {
    int num = chnum(n);
    node *p = n->p;
    node *b = n->ch[1 - num];
    n->p = p->p;
    if (!isRoot(p)) {
        p->p->ch[chnum(p)] = n;
    }
    attach(p, n, 1 - num);
    attach(b, p, num);
    pull(p);
    pull(n);
}

void splay(node *n) {
    node *nn = n;
    int top = 0;
    qq[top++] = nn;
    while (!isRoot(nn)) {
        nn = nn->p;
        qq[top++] = nn;
    }
    while (top) {
        push(qq[--top]);
    }
    while (!isRoot(n)) {
        if (!isRoot(n->p)) {
            if (chnum(n) == chnum(n->p)) {
                rotate(n->p);
            } else {
                rotate(n);
            }
        }
        rotate(n);
    }
}

void expose(node *n) {
    splay(n);
    n->ch[1] = NULL;
    pull(n);
    while (n->p != NULL) {
        splay(n->p);
        attach(n, n->p, 1);
        pull(n->p);
        splay(n);
    }
}

void makeRoot(node *n) {
    expose(n);
    n->rev ^= 1;
}

node *nodes[MAXN];

int main() {
    int n;
    cin >> n;
    for (int i = 0; i <= n; i++) {
        nodes[i] = new node();
    }
}
```

```
if (!isRoot(p)) {
    p->p->ch[chnum(p)] = n;
}
attach(p, n, 1 - num);
attach(b, p, num);
pull(p);
pull(n);
}

node *qq[MAXN];

void splay(node *n) {
    node *nn = n;
    int top = 0;
    qq[top++] = nn;
    while (!isRoot(nn)) {
        nn = nn->p;
        qq[top++] = nn;
    }
    while (top) {
        push(qq[--top]);
    }
    while (!isRoot(n)) {
        if (!isRoot(n->p)) {
            if (chnum(n) == chnum(n->p)) {
                rotate(n->p);
            } else {
                rotate(n);
            }
        }
        rotate(n);
    }
}

void expose(node *n) {
    splay(n);
    n->ch[1] = NULL;
    pull(n);
    while (n->p != NULL) {
        splay(n->p);
        attach(n, n->p, 1);
        pull(n->p);
        splay(n);
    }
}

void makeRoot(node *n) {
    expose(n);
    n->rev ^= 1;
}

node *nodes[MAXN];

int main() {
    int n;
    cin >> n;
    for (int i = 0; i <= n; i++) {
        nodes[i] = new node();
    }
}
```

```

}
int q;
cin >> q;
while (q--) {
    string s;
    cin >> s;
    int u, v;
    cin >> u >> v;
    makeRoot(nodes[u]);
    makeRoot(nodes[v]);
    if (s == "get") {
        if (isRoot(nodes[u]) && u != v) {
            cout << "-1" << endl;
        } else {
            cout << getsz(nodes[v]) - 1 << endl;
        }
    } else if (s == "link") {
        nodes[v]->p = nodes[u];
    } else {
        push(nodes[v]);
        nodes[v]->ch[1] = NULL;
        nodes[u]->p = NULL;
    }
}
}
}
```

## MaxFlow.cpp

**Description:** Dinic**Time:**  $\mathcal{O}(n^2m)$ 

1c1bc8, 72 lines

```
struct MaxFlow {
    const int inf = 1e9 + 20;
    struct edge {
        int a, b, cap;
    };
    int n;
    vector<edge> e;
    vector<vector<int>> g;
    MaxFlow() {}
    int s, t;
    vector<int> d, ptr;
    void init(int n_, int s_, int t_) {
        s = s_, t = t_, n = n_;
        g.resize(n);
        ptr.resize(n);
    }
    void addedge(int a, int b, int cap) {
        g[a].pb(e.size());
        e.pb({a, b, cap});
        g[b].pb(e.size());
        e.pb({b, a, 0});
    }
    bool bfs() {
        d.assign(n, inf);
        d[s] = 0;
        queue<int> q;
        q.push(s);
        while (q.size()) {
            int u = q.front(); q.pop();
            for (int i = ptr[u]; i < g[u].size(); i++) {
                int v = g[u][i];
                if (d[v] == inf && e[i].cap > 0) {
                    d[v] = d[u] + 1;
                    q.push(v);
                }
            }
            ptr[u] = g[u].size();
        }
        return d[t] != inf;
    }
    int dfs(int u, int f) {
        if (u == t) return f;
        for (int i = ptr[u]; i < g[u].size(); i++) {
            int v = g[u][i];
            if (d[v] == d[u] + 1 && e[i].cap > 0) {
                int f2 = dfs(v, min(f, e[i].cap));
                e[i].cap -= f2;
                e[e[i].a].cap += f2;
                return f2;
            }
        }
        return 0;
    }
    int maxflow() {
        int flow = 0;
        while (bfs()) {
            flow += dfs(s, inf);
        }
        return flow;
    }
};
```

```

    int v = q.front();
    q.pop();
    for (int i : g[v]) {
        if (e[i].cap > 0) {
            int b = e[i].b;
            if (d[b] > d[v] + 1) {
                d[b] = d[v] + 1;
                q.push(b);
            }
        }
    }
}
return d[t] != inf;
}
int dfs(int v, int flow) {
    if (v == t) return flow;
    if (!flow) return 0;
    int sum = 0;
    for (; ptr[v] < g[v].size(); ++ptr[v]) {
        int b = e[g[v][ptr[v]]].b;
        int cap = e[g[v][ptr[v]]].cap;
        if (cap <= 0) continue;
        if (d[b] != d[v] + 1) continue;
        int x = dfs(b, min(flow, cap));
        int id = g[v][ptr[v]];
        e[id].cap -= x;
        e[id ^ 1].cap += x;
        flow -= x;
        sum += x;
    }
    return sum;
}
int dinic() {
    int ans = 0;
    while (1) {
        if (!bfs()) break;
        ptr.assign(n, 0);
        int x = dfs(s, inf);
        if (!x) break;
        ans += x;
    }
    return ans;
}
};

```

## MCMF.cpp

Description: Min cost

Time:  $\mathcal{O}(?)$

32340a, 61 lines

```

struct MCMF {
    struct edge {
        int a, b, cap, cost;
    };
    vector<edge> e;
    vector<vector<int>> g;
    int s, t;
    int n;
    void init(int N, int S, int T) {

```

```

        s = S, t = T, n = N;
        g.resize(N);
        e.clear();
    }
    void addedge(int a, int b, int cap, int cost) {
        g[a].pbc(e.size());
        e.pbc({a, b, cap, cost});
        g[b].pbc(e.size());
        e.pbc({b, a, 0, -cost});
    }
    int getcost(int k) {
        int flow = 0;
        int cost = 0;
        while (flow < k) {
            vector<int> d(n, INF);
            vector<int> pr(n);
            d[s] = 0;
            queue<int> q;
            q.push(s);
            while (q.size()) {
                int v = q.front();
                q.pop();
                for (int i : g[v]) {
                    int u = e[i].b;
                    if (e[i].cap && d[u] > d[v] + e[i].cost) {
                        d[u] = d[v] + e[i].cost;
                        q.push(u);
                        pr[u] = i;
                    }
                }
            }
            if (d[t] == INF) return INF;
            int gf = k - flow;
            int v = t;
            while (v != s) {
                int id = pr[v];
                chkmin(gf, e[id].cap);
                v = e[id].a;
            }
            v = t;
            while (v != s) {
                int id = pr[v];
                e[id].cap -= gf;
                e[id ^ 1].cap += gf;
                cost += e[id].cost * gf;
                v = e[id].a;
            }
            flow += gf;
        }
        return cost;
    }
};

```

## GlobalMincut.cpp

Description: Global min cut

Time:  $\mathcal{O}(n^3)$

7b8a6b, 35 lines

```

const int MAXN = 500;
int n, g[MAXN][MAXN];
int best_cost = 1000000000;
vector<int> best_cut;
void mincut() {
    vector<int> v[MAXN];
    for (int i = 0; i < n; ++i)
        v[i].assign(1, i);
    int w[MAXN];
    bool exist[MAXN], in_a[MAXN];
    memset(exist, true, sizeof exist);
    for (int ph = 0; ph < n - 1; ++ph) {
        memset(in_a, false, sizeof in_a);
        memset(w, 0, sizeof w);
        for (int it = 0, prev; it < n - ph; ++it) {
            int sel = -1;
            for (int i = 0; i < n; ++i)
                if (exist[i] && !in_a[i] && (sel == -1 || w[i] > w[sel]))
                    sel = i;
            if (it == n - ph - 1) {
                if (w[sel] < best_cost)
                    best_cost = w[sel], best_cut = v[sel];
                v[prev].insert(v[prev].end(), v[sel].begin(), v[sel].end());
                for (int i = 0; i < n; ++i)
                    g[prev][i] = g[i][prev] += g[sel][i];
                exist[sel] = false;
            } else {
                in_a[sel] = true;
                for (int i = 0; i < n; ++i)
                    w[i] += g[sel][i];
                prev = sel;
            }
        }
    }
}

```



# Geometry (5)

## Point.cpp

Description: struct Point 80dfd5, 80 lines

```
const ld EPS = 1e-7;

ld sq(ld x) {
    return x * x;
}

int sign(ld x) {
    if (x < -EPS) {
        return -1;
    }
    if (x > EPS) {
        return 1;
    }
    return 0;
}

#define vec point
struct point { // % - cross, * - dot
    ld x, y;
    auto operator<=>(const point&) const = default;
};
ld operator*(const point &a, const point &b) {
    return a.x * b.x + a.y * b.y;
}
ld operator%(const point &a, const point &b) {
    return a.x * b.y - a.y * b.x;
}
point operator-(const point &a, const point &b) {
    return {a.x - b.x, a.y - b.y};
}
point operator+(const point &a, const point &b) {
    return {a.x + b.x, a.y + b.y};
}
point operator*(const point &a, ld b) {
    return {a.x * b, a.y * b};
}
point operator/(const point &a, ld b) {
    return {a.x / b, a.y / b};
}
bool operator<(const point &a, const point &b) {
    if (sign(a.y - b.y) != 0) {
        return a.y < b.y;
    } else if (sign(a.x - b.x) != 0) {
        return a.x < b.x;
    }
    return 0;
}
ld len2(const point &a) {
    return sq(a.x) + sq(a.y);
}
ld len(const point &a) {
    return sqrt(len2(a));
}
```

```
point norm(point a) {
    return a / len(a);
}
int half(point a) {
    return (sign(a.y) == -1 || (sign(a.y) == 0 && a.x < 0));
}
point ort(point a) {
    return {-a.y, a.x};
}
point turn(point a, ld ang) {
    return {a.x * cos(ang) - a.y * sin(ang), a.x * sin(ang) + a.y * cos(ang)};
}
ld getAngle(point &a, point &b) {
    return atan2(a % b, a * b);
}
bool cmpHalf(const point &a, const point &b) {
    if (half(a) != half(b)) {
        return half(b);
    } else {
        int sgn = sign(a % b);
        if (!sgn) {
            return len2(a) < len2(b);
        } else {
            return sgn == 1;
        }
    }
}
```

## Line.cpp

Description: struct Line 887306, 26 lines

```
struct line {
    ld a, b, c;
    void norm() {
        // for half planes
        ld d = len({a, b});
        assert(sign(d) > 0);
        a /= d;
        b /= d;
        c /= d;
    }
    ld eval(point p) const { return a * p.x + b * p.y + c; }
    bool isIn(point p) const { return sign(eval(p)) >= 0; }
    bool operator==(const line &other) const {
        return sign(a * other.b - b * other.a) == 0 &&
            sign(a * other.c - c * other.a) == 0 &&
            sign(b * other.c - c * other.b) == 0;
    }
};
line getln(point a, point b) {
    line res;
    res.a = a.y - b.y;
    res.b = b.x - a.x;
    res.c = -(res.a * a.x + res.b * a.y);
```

```
res.norm();
return res;
}

Intersections.cpp
Description: Geometry intersections 1178c1, 75 lines

bool isCrossed(ld lx, ld rx, ld ly, ld ry) {
    if (lx > rx)
        swap(lx, rx);
    if (ly > ry)
        swap(ly, ry);
    return sign(min(rx, ry) - max(lx, ly)) >= 0;
}

// if two segments [a, b] and [c, d] has AT LEAST one
// common point -> true
bool intersects(const point &a, const point &b, const point &c, const point &d) {
    if (!isCrossed(a.x, b.x, c.x, d.x))
        return false;
    if (!isCrossed(a.y, b.y, c.y, d.y))
        return false;
    if (sign((b - a) % (c - a)) * sign((b - a) % (d - a)) == 1) return 0;
    if (sign((d - c) % (a - c)) * sign((d - c) % (b - c)) == 1) return 0;
    return 1;
}

//intersecting lines
bool intersect(line &l, line &m, point &I) {
    ld d = l.b * m.a - m.b * l.a;
    if (sign(d) == 0) {
        return false;
    }
    ld dx = m.b * l.c - m.c * l.b;
    ld dy = m.c * l.a - l.c * m.a;
    I = {dx / d, dy / d};
    return true;
}

//intersecting circles
int intersect(point o1, ld r1, point o2, ld r2, point &i1, point &i2) {
    if (r1 < r2) {
        swap(o1, o2);
        swap(r1, r2);
    }
    if (sign(r1 - r2) == 0 && len2(o2 - o1) < EPS) {
        return 3;
    }
    ld ln = len(o1 - o2);
    if (sign(ln - r1 - r2) == 1 || sign(r1 - ln - r2) == 1) {
        return 0;
    }
    ld d = (sq(r1) - sq(r2) + sq(ln)) / 2 / ln;
    vec v = norm(o2 - o1);
    point a = o1 + v * d;
```

```

    if (sign(ln - r1 - r2) == 0 || sign(ln + r2 - r1)
        == 0) {
        i1 = a;
        return 1;
    }
    v = ort(v) * sqrt(sq(r1) - sq(d));
    i1 = a + v;
    i2 = a - v;
    return 2;
}
//intersecting line and circle, line should be normed
int intersect(point &o, ld r, line &l, point &i1, point
    &i2) {
    ld len = abs(l.eval(o));
    int sgn = sign(len - r);
    if (sgn == 1) {
        return 0;
    }
    vec v = norm(vec{l.a, l.b}) * len;
    if (sign(l.eval(o + v)) != 0) {
        v = vec{0, 0} - v;
    }
    point a = o + v;
    if (sgn == 0) {
        i1 = a;
        return 1;
    }
    v = norm({-l.b, l.a}) * sqrt(sq(r) - sq(len));
    i1 = a + v;
    i2 = a - v;
    return 2;
}

```

## Tangents.cpp

Description: Tangents to circles.

c73373, 43 lines

```

// tangents from point to circle
int tangents(point &o, ld r, point &p, point &i1, point
    &i2) {
    ld ln = len(o - p);
    int sgn = sign(ln - r);
    if (sgn == -1) {
        return 0;
    } else if (sgn == 0) {
        i1 = p;
        return 1;
    } else {
        ld x = sq(r) / ln;
        vec v = norm(p - o) * x;
        point a = o + v;
        v = ort(norm(p - o)) * sqrt(sq(r) - sq(x));
        i1 = a + v;
        i2 = a - v;
        return 2;
    }
}

```

```

void _tangents(point c, ld r1, ld r2, vector<line> &ans
    ) {
    ld r = r2 - r1;
    ld z = sq(c.x) + sq(c.y);
    ld d = z - sq(r);
    if (sign(d) == -1)
        return;
    d = sqrt(abs(d));
    line l;
    l.a = (c.x * r + c.y * d) / z;
    l.b = (c.y * r - c.x * d) / z;
    l.c = r1;
    ans.push_back(l);
}
// tangents between two circles
vector<line> tangents(point o1, ld r1, point o2, ld r2)
    {
    vector<line> ans;
    for (int i = -1; i <= 1; i += 2)
        for (int j = -1; j <= 1; j += 2)
            _tangents(o2 - o1, r1 * i, r2 * j, ans);
    for (int i = 0; i < (int)ans.size(); ++i)
        ans[i].c -= ans[i].a * o1.x + ans[i].b * o1.y;
    return ans;
}

```

## Hull.cpp

Description: Polygon functions

fd58f7, 16 lines

```

vector<point> hull(vector<point> p, bool need_all=false
    ) {
    sort(begin(p), end(p));
    p.erase(unique(begin(p), end(p)), end(p));
    int n = p.size(), k = 0;
    if (n <= 2) return p;
    vector<point> ch(2 * n);
    ld th = need_all ? -EPS : +EPS; // 0 : 1 if int
    for (int i = 0; i < n; ch[k++] = p[i++]) {
        while (k >= 2 && (ch[k - 1] - ch[k - 2]) % (p[i] -
            ch[k - 1]) < th) --k;
    }
    for (int i = n - 2, t = k + 1; i >= 0; ch[k++] = p[i
        --]) {
        while (k >= t && (ch[k - 1] - ch[k - 2]) % (p[i] -
            ch[k - 1]) < th) --k;
    }
    ch.resize(k - 1);
    return ch;
}

```

## IsInPolygon.cpp

Description: Is in polygon functions

f17b31, 65 lines

```

bool isOnSegment(point &a, point &b, point &x) {
    if (sign(len2(a - b)) == 0) {
        return sign(len(a - x)) == 0;
    }
    return sign((b - a) % (x - a)) == 0 && sign((b - x)
        * (a - x)) <= 0;
}

```

```

// optional (slower, but works better if there are
    some precision
// problems) return sign((b - a).len() - (x - a).
    len() - (x - b).len())
// == 0;
}

```

```

int isIn(vector<point> &p, point &a) {
    int n = p.size();
    // depends on limitations(2*MAXC + 228)
    point b = a + point{2e9 + 228, 1};
    int cnt = 0;
    for (int i = 0; i < n; ++i) {
        point x = p[i];
        point y = p[i + 1 < n ? i + 1 : 0];
        if (isOnSegment(x, y, a)) {
            // depends on the problem statement
            return 1;
        }
        cnt += intersects(x, y, a, b);
    }
    return 2 * (cnt % 2 == 1);
    /*optional (atan2 is VERY SLOW)!
    ld ans = 0;
    int n = p.size();
    for (int i = 0; i < n; ++i) {
        Point x = p[i];
        Point y = p[i + 1 < n ? i + 1 : 0];
        if (isOnSegment(x, y, a)) {
            // depends on the problem statement
            return true;
        }
        x = x - a;
        y = y - a;
        ans += atan2(x ^ y, x * y);
    }
    return abs(ans) > 1;*/
}

```

```

bool isInTriangle(point &a, point &b, point &c, point &
    x) {
    return sign((b - a) % (x - a)) >= 0 && sign((c - b)
        % (x - b)) >= 0 &&
        sign((a - c) % (x - c)) >= 0;
}

```

// points should be in the counterclockwise order

```

bool isInConvex(vector<point> &p, point &a) {
    int n = p.size();
    assert(n >= 3);
    // assert(isConvex(p));
    // assert(isCounterclockwise(p));
    if (sign((p[1] - p[0]) % (a - p[0])) < 0)
        return 0;
    if (sign((p[n - 1] - p[0]) % (a - p[0])) > 0)
        return 0;
    int pos = lower_bound(p.begin() + 2, p.end(), a,

```

```

        [&](point a, point b) -> bool
        {
            return sign((a - p[0]) %
                (b - p[0])) > 0;
        }) -
        p.begin();
    assert(pos > 1 && pos < n);
    return isInTriangle(p[0], p[pos - 1], p[pos], a);
}

```

## Diameter.cpp

**Description:** Rotating calipers.

**Time:**  $\mathcal{O}(n)$

0f341c, 21 lines

```

1d diameter(vector<point> p) {
    p = hull(p);
    int n = p.size();
    if (n <= 1) {
        return 0;
    }
    if (n == 2) {
        return len(p[0] - p[1]);
    }
    1d ans = 0;
    int i = 0, j = 1;
    while (i < n) {
        while (sign((p[(i + 1) % n] - p[i]) % (p[(j +
            1) % n] - p[j])) >= 0) {
            chkmax(ans, len(p[i] - p[j]));
            j = (j + 1) % n;
        }
        chkmax(ans, len(p[i] - p[j]));
        ++i;
    }
    return ans;
}

```

## TangentsAlex.cpp

**Description:** Find both tangents to the convex polygon.  
(Zakaldovany algos mozhet sgonyat za pivom tak zhe).

**Time:**  $\mathcal{O}(\log(n))$

2eeea8, 17 lines

```

pair<int, int> tangents_alex(vector<point> &p, point &a)
{
    int n = p.size();
    int l = __lg(n);
    auto findWithSign = [&](int val) {
        int i = 0;
        for (int k = 1; k >= 0; --k) {
            int i1 = (i - (1 << k) + n) % n;
            int i2 = (i + (1 << k)) % n;
            if (sign((p[i1] - a) % (p[i] - a)) == val)
                i = i1;
            if (sign((p[i2] - a) % (p[i] - a)) == val)
                i = i2;
        }
        return i;
    };
    return {findWithSign(1), findWithSign(-1)};
}

```

## IsHpiEmpty.cpp

**Description:** Determines is half plane intersectinos.

**Time:**  $\mathcal{O}(n)$  (expected)

20460a, 85 lines

```

// //all lines must be normed!!!!!!!!!!!!!! , sign > 0
// bool isHpiEmpty(vector<line> lines) {
//     // return hpi(lines).empty();
//     // // overflow/precision problems?
//     // shuffle(all(lines), rnd);
//     // const ld C = 1e9;
//     // point ans = {C, C};
//     // vector<point> box = {{-C, -C}, {C, -C}, {C, C},
//     {-C, C}};
//     for (int i = 0; i < 4; ++i)
//         lines.push_back(getln(box[i], box[(i + 1) %
//         4]));
//     int n = lines.size();
//     for (int i = n - 4; i >= 0; --i) {
//         if (lines[i].isIn(ans))
//             continue;
//         point up{0, C + 1}, down{0, -C - 1}, pi{-
//         lines[i].b, lines[i].a};
//         for (int j = i + 1; j < n; ++j) {
//             if (lines[j] == lines[i])
//                 continue;
//             point p, pj = {-lines[j].b, lines[j].a};
//             if (!intersect(lines[i], lines[j], p)) {
//                 if (sign(pi * pj) != -1)
//                     continue;
//                 if (sign(lines[i].c + lines[j].c) *
//                     (!sign(pi.y) ? sign(pi.x) :
//                     -1) ==
//                     -1)
//                     return true;
//             } else {
//                 if ((!sign(pi.y) ? sign(pi.x) : sign
//                 (pi.y)) * (sign(pi % pj)) ==
//                 1)
//                     chkmin(up, p);
//                 else
//                     chkmax(down, p);
//             }
//         }
//         if ((ans = up) < down)
//             return true;
//     }
//     for (int i = 0; i < n; ++i) {
//         assert(lines[i].eval(ans) < EPS);
//     }
//     return false;
// }

```

//all lines must be normed!!!!!!!!!!!!!! , sign > 0

```

bool isHpiEmpty(vector<line> lines) {
    // return hpi(lines).empty();
    // // overflow/precision problems?

```

```

shuffle(all(lines), rnd);
const ld C = 1e9;
point ans{C, C};
vector<point> box = {{-C, -C}, {C, -C}, {C, C}, {-C
    , C}};
for (int i = 0; i < 4; ++i)
    lines.push_back(getln(box[i], box[(i + 1) % 4])
        );
int n = lines.size();
for (int i = n - 4; i >= 0; --i) {
    if (lines[i].isIn(ans))
        continue;
    point up{0, C + 1}, down{0, -C - 1}, pi = {
        lines[i].b, -lines[i].a};
    for (int j = i + 1; j < n; ++j) {
        if (lines[j] == lines[i])
            continue;
        point p, pj = {lines[j].b, -lines[j].a};
        if (!intersect(lines[i], lines[j], p)) {
            if (sign(pi * pj) != -1)
                continue;
            if (-sign(lines[i].c + lines[j].c) *
                (!sign(pi.y) ? -sign(pi.x) :
                -1) ==
                -1)
                return true;
        } else {
            if ((!sign(pi.y) ? sign(-pi.x) : sign(-
            pi.y)) * (sign(pi % pj)) ==
            1)
                chkmin(up, p);
            else
                chkmax(down, p);
        }
    }
    if ((ans = up) < down)
        return true;
}
// for (int i = 0; i < n; ++i) {
//     assert(lines[i].eval(ans) < EPS);
// }
return false;
}

```

## HalfPlaneIntersection.cpp

**Description:** Find the intersection of the half planes.

**Time:**  $\mathcal{O}(n \log(n))$

fdf28f, 62 lines

```

vec getPoint(line l) { return {-l.b, l.a}; }

```

```

bool bad(line a, line b, line c) {
    point x;
    assert(intersect(b, c, x) == 1);
    return a.eval(x) < 0;
}

```

```

// Do not forget about the bounding box
vector<point> hpi(vector<line> lines) {

```

```

sort(all(lines), [](line a1, line b1) -> bool {
    point a = getPoint(a1);
    point b = getPoint(b1);
    if (half(a) != half(b)) {
        return half(a) < half(b);
    }
    return a % b > 0;
});

vector<pair<line, int>> st;
for (int it = 0; it < 2; it++) {
    for (int i = 0; i < (int)lines.size(); i++) {
        bool flag = false;
        while (!st.empty()) {
            if (len(getPoint(st.back().first) -
                getPoint(lines[i])) < EPS) {
                if (lines[i].c >= st.back().first.c) {
                    flag = true;
                    break;
                } else {
                    st.pop_back();
                }
            } else if (getPoint(st.back().first) %
                getPoint(lines[i]) < EPS / 2) {
                return {};
            } else if (st.size() >= 2 &&
                bad(st[st.size() - 2].first,
                    st[st.size() - 1].first,
                    lines[i])) {
                st.pop_back();
            } else {
                break;
            }
        }
        if (!flag)
            st.push_back({lines[i], i});
    }
}

```

```

vector<int> en(lines.size(), -1);
vector<point> ans;
for (int i = 0; i < (int)st.size(); i++) {
    if (en[st[i].second] == -1) {
        en[st[i].second] = i;
        continue;
    }
    for (int j = en[st[i].second]; j < i; j++) {
        point I;
        assert(intersect(st[j].first, st[j + 1].
            first, I) == 1);
        ans.push_back(I);
    }
    break;
}
return ans;
}

```

## CHT.cpp

Description: CHT for minimum, k is decreasing, works for equal slopes b09420, 34 lines

```

struct line {
    int k, b;
    int eval(int x) {
        return k * x + b;
    }
};

struct part {
    line a;
    double x;
};

double intersection(line a, line b) {
    return (a.b - b.b) / (b.k - a.k);
}

struct ConvexHullMin {
    vector<part> st;
    void add(line a) {
        if (!st.empty() && st.back().a.k == a.k) {
            if (st.back().a.b > a.b) st.pop_back();
            else return;
        }
        while (st.size() > 1 && intersection(st[st.size()
            - 2].a, a) <= st[st.size() - 2].x) st.
            pop_back();
        if (!st.empty()) st.back().x = intersection(st.
            back().a, a);
        st.push_back({a, INF});
    }

    int get_val(int x) {
        int l = -1, r = (int)st.size() - 1;
        while (r - l > 1) {
            int m = (l + r) / 2;
            if (st[m].x < x) l = m;
            else r = m;
        }
        return st[r].a.eval(x);
    }
};

```

## DynamicCHT.cpp

Description: Dynamic CHT for maximum 8a0777, 30 lines

```

struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const {
        return Q ? p < o.p : k < o.k;
    }
};

struct LineContainer : multiset<Line> {
    const ll inf = LLONG_MAX;
    ll div(ll a, ll b) {
        return a / b - ((a ^ b) < 0 && a % b);
    }

    bool isect(iterator x, iterator y) {
        if (y == end()) { x->p = inf; return false; }
        if (x->k == y->k) x->p = x->m > y->m ? inf : -
            inf;

```

```

        else x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }

    void add(ll k, ll m) {
        auto z = insert({k, m, 0}); y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y =
            erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    }

    ll query(ll x) {
        assert(!empty());
        Q = 1; auto l = *lower_bound({0, 0, x}); Q = 0;
        return l.k * x + l.m;
    }
};

```

# Math (6)

## BerlekampMassey.cpp

Description: Find the shortest linear-feedback shift register

Time:  $\mathcal{O}(n^2)$

505033, 36 lines

```
vector<int> berlekamp_massey(vector<int> x) {
    vector<int> ls, cur;
    int lf = 0, d = 0;
    for (int i = 0; i < x.size(); ++i) {
        ll t = 0;
        for (int j = 0; j < cur.size(); ++j) {
            t = (t + 1ll * x[i - j - 1] * cur[j]) % MOD;
        }
        if ((t - x[i]) % MOD == 0)
            continue;
        if (cur.empty()) {
            cur.resize(i + 1);
            lf = i;
            d = (t - x[i]) % MOD;
            continue;
        }
        ll k = -(x[i] - t) * pw(d, MOD - 2) % MOD;
        vector<int> c(i - lf - 1);
        c.push_back(k);
        for (auto &j : ls)
            c.push_back(-j * k % MOD);
        if (c.size() < cur.size())
            c.resize(cur.size());
        for (int j = 0; j < cur.size(); ++j) {
            c[j] = (c[j] + cur[j]) % MOD;
        }
        if (i - lf + (int)ls.size() >= (int)cur.size()) {
            tie(ls, lf, d) = make_tuple(cur, i, (t - x[i]) % MOD);
        }
        cur = c;
    }
    for (auto &i : cur)
        i = (i % MOD + MOD) % MOD;
    return cur;
}
// for a_i = 2 * a_{i-1} + a_{i-1} returns {2, 1}
```

## GoncharFedor.cpp

Description: Calculating number of points  $x, y \geq 0, Ax + By \leq C$

Time:  $\mathcal{O}(\log(C))$

0ef10e, 11 lines

```
ll solve_triangle(ll A, ll B, ll C) { // x,y >=0, Ax+By <=C
    if (C < 0)
        return 0;
    if (A > B)
        swap(A, B);
    ll p = C / B;
    ll k = B / A;
```

```
    ll d = (C - p * B) / A;
    return solve_triangle(B - k * A, A, C - A * (k * p + d + 1)) +
        (p + 1) * (d + 1) + k * p * (p + 1) / 2;
}
```

## CRT.cpp

Description: CRT for arbitrary modulus

28309e, 25 lines

```
int extgcd(int a, int b, int &x, int &y) { // define int ll
    if (a == 0) {
        x = 0, y = 1;
        return b;
    }
    int x1, y1;
    int g = extgcd(b % a, a, x1, y1);
    x = y1 - x1 * (b / a);
    y = x1;
    return g;
}
int lcm(int a, int b) { return a / __gcd(a, b) * b; }
int crt(int mod1, int mod2, int rem1, int rem2) {
    int r = (rem2 - (rem1 % mod2) + mod2) % mod2;
    int x, y;
    int g = extgcd(mod1, mod2, x, y);
    if (r % g) return -1;
    x %= mod2;
    if (x < 0) x += mod2;
    int ans = (x * (r / g)) % mod2;
    ans = ans * mod1 + rem1;
    assert(ans % mod1 == rem1);
    assert(ans % mod2 == rem2);
    return ans % lcm(mod1, mod2);
}
```

## Fastmod.cpp

Description: Fast multiplication by modulo(in [0;2b))

38ea39, 7 lines

```
struct FastMod {
    ull b, m;
    FastMod(ull b) : b(b), m(-1ULL / b) {}
    ull reduce(ull a) { // a % b + (0 or b)
        return a - (ull)((__uint128_t(m) * a) >> 64) * b;
    }
};
```

## PrimalityTest.cpp

Description: Checking primality of p

Time:  $\mathcal{O}(\log(C))$

ad2714, 32 lines

```
const int iters = 8; // can change
bool isprime(ll p) {
    if (p == 1 || p == 4)
        return 0;
    if (p == 2 || p == 3)
        return 1;
    for (int it = 0; it < iters; ++it) {
```

```
        ll a = rnd() % (p - 2) + 2;
        ll nw = p - 1;
        while (nw % 2 == 0)
            nw /= 2;
        ll x = binpow(a, nw, p); // int128
        if (x == 1)
            continue;
        ll last = x;
        nw *= 2;
        while (nw <= p - 1) {
            x = (__int128_t)x * x % p;
            if (x == 1) {
                if (last != p - 1) {
                    return 0;
                }
                break;
            }
            last = x;
            nw *= 2;
        }
        if (x != 1)
            return 0;
    }
    return 1;
}
```

## XorConvolution.cpp

Description: Calculating xor-convolution of 2 vectors modulo smth

Time:  $\mathcal{O}(n \log(n))$

454afd, 23 lines

```
void fwht(vector<int> &a) {
    int n = a.size();
    for (int l = 1; l < n; l <= 1) {
        for (int i = 0; i < n; i += 2 * l) {
            for (int j = 0; j < l; ++j) {
                int u = a[i + j], v = a[i + j + l];
                a[i + j] = add(u, v), a[i + j + l] = sub(u, v);
            }
        }
    }
}
// https://judge.yosupo.jp/problem/bitwise_xor_convolution
vector<int> xorconvo(vector<int> a, vector<int> b) {
    int n = 1;
    while (n < max(a.size(), b.size()))
        n *= 2;
    a.resize(n), b.resize(n);
    fwht(a), fwht(b);
    int in = inv(n);
    for (int i = 0; i < n; ++i)
        a[i] = mul(a[i], mul(b[i], in));
    fwht(a);
    return a;
}
```

## Factorization.cpp

**Description:** Factorizing a number real quick**Time:**  $\mathcal{O}(n^{\frac{1}{4}})$ 

f0d7c6, 51 lines

```

11 gcd(11 a, 11 b) {
    while (b)
        a %= b, swap(a, b);
    return a;
}

11 f(11 a, 11 n) { return ((__int128_t)a * a % n + 1) % n; }

vector<11> factorize(11 n) {
    if (n <= 1e6) { // can add primality check for speed?
        vector<11> res;
        for (11 i = 2; i * i <= n; ++i) {
            while (n % i == 0) {
                res.pbc(i);
                n /= i;
            }
        }
        if (n != 1)
            res.pbc(n);
        return res;
    }
    11 x = rnd() % (n - 1) + 1;
    11 y = x;
    11 tries = 10 * sqrt(sqrt(n));
    const int C = 60;
    for (11 i = 0; i < tries; i += C) {
        11 xs = x;
        11 ys = y;
        11 m = 1;
        for (int k = 0; k < C; ++k) {
            x = f(x, n);
            y = f(f(y, n), n);
            m = ((__int128_t)m * abs(x - y) % n);
        }
        if (gcd(n, m) == 1)
            continue;
        x = xs, y = ys;
        for (int k = 0; k < C; ++k) {
            x = f(x, n);
            y = f(f(y, n), n);
            11 res = gcd(n, abs(x - y));
            if (res != 1 && res != n) {
                vector<11> v1 = factorize(res), v2 = factorize(n / res);
                for (auto j : v2)
                    v1.pbc(j);
                return v1;
            }
        }
    }
    return {n};
}

```

## NTT.cpp

**Description:** Calculating FFT modulo MOD**Time:**  $\mathcal{O}(n \log(n))$ 

3e2f3a, 226 lines

// DONT FORGET TO CALL initNTT() AND CHECK MAXLOG

```

const int MOD = 998244353;
const int G = 3;
const int MAXLOG = 23;
int W[1 << MAXLOG];
bool nttinit = false;
vector<int> pws;

int add(int a, int b) {
    a += b;
    if (a >= MOD) {
        return a - MOD;
    }
    return a;
}

int sub(int a, int b) {
    a -= b;
    if (a < 0) {
        return a + MOD;
    }
    return a;
}

int mul(int a, int b) {
    return (11) a * b % MOD;
}

int power(int a, int n) {
    int ans = 1;
    while (n) {
        if (n & 1) {
            ans = mul(ans, a);
        }
        a = mul(a, a);
        n >>= 1;
    }
    return ans;
}

int inv(int a) {
    return power(a, MOD - 2);
}

void initNTT() {
    assert((MOD - 1) % (1 << MAXLOG) == 0);
    pws.push_back(power(G, (MOD - 1) / (1 << MAXLOG)));
    for (int i = 0; i < MAXLOG - 1; ++i) {
        pws.push_back(mul(pws.back(), pws.back()));
    }
    assert(pws.back() == MOD - 1);
    W[0] = 1;
    for (int i = 1; i < (1 << MAXLOG); ++i) {
        W[i] = mul(W[i - 1], pws[0]);
    }
}

```

```

}
}

void ntt(int n, vector<int>& a, bool rev) {
    if (!nttinit) {
        initNTT();
        nttinit = 1;
    }
    int lg = log2(n);
    vector<int> rv(n);
    for (int i = 1; i < n; ++i) {
        rv[i] = (rv[i >> 1] >> 1) ^ ((i & 1) << (lg - 1));
        if (rv[i] > i) swap(a[i], a[rv[i]]);
    }
    int num = MAXLOG - 1;
    for (int len = 1; len < n; len *= 2) {
        for (int i = 0; i < n; i += 2 * len) {
            for (int j = 0; j < len; ++j) {
                int u = a[i + j], v = mul(W[j << num], a[i + j + len]);
                a[i + j] = add(u, v);
                a[i + j + len] = sub(u, v);
            }
            --num;
        }
    }
    if (rev) {
        int rev_n = power(n, MOD - 2);
        for (int i = 0; i < n; ++i) a[i] = mul(a[i], rev_n);
        reverse(a.begin() + 1, a.end());
    }
}

vector<int> conv(vector<int> a, vector<int> b) {
    int lg = 0;
    while ((1 << lg) < a.size() + b.size() + 1)
        ++lg;
    int n = 1 << lg;
    assert(a.size() + b.size() <= n + 1);
    a.resize(n);
    b.resize(n);
    ntt(n, a, false);
    ntt(n, b, false);
    for (int i = 0; i < n; ++i) {
        a[i] = mul(a[i], b[i]);
    }
    ntt(n, a, true);
    while (a.size() > 1 && a.back() == 0) {
        a.pop_back();
    }
    return a;
}

vector<int> add(vector<int> a, vector<int> b) {
    a.resize(max(a.size(), b.size()));
    for (int i = 0; i < (int) b.size(); ++i) {

```

```

        a[i] = add(a[i], b[i]);
    }
    return a;
}

vector<int> sub(vector<int> a, vector<int> b) {
    a.resize(max(a.size(), b.size()));
    for (int i = 0; i < (int) b.size(); ++i) {
        a[i] = sub(a[i], b[i]);
    }
    return a;
}

vector<int> inv(const vector<int> &a, int need) {
    vector<int> b = {inv(a[0])};
    while ((int) b.size() < need) {
        vector<int> a1 = a;
        int m = b.size();
        a1.resize(min((int) a1.size(), 2 * m));
        b = conv(b, sub({2}, conv(a1, b)));
        b.resize(2 * m);
    }
    b.resize(need);
    return b;
}

vector<int> div(vector<int> a, vector<int> b) {
    if (count(all(a), 0) == a.size()) {
        return {0};
    }
    assert(a.back() != 0 && b.back() != 0);
    int n = a.size() - 1;
    int m = b.size() - 1;
    if (n < m) {
        return {0};
    }
    reverse(all(a));
    reverse(all(b));
    a.resize(n - m + 1);
    b.resize(n - m + 1);
    vector<int> c = inv(b, b.size());
    vector<int> q = conv(a, c);
    q.resize(n - m + 1);
    reverse(all(q));
    return q;
}

vector<int> mod(vector<int> a, vector<int> b) {
    auto res = sub(a, conv(b, div(a, b)));
    while (res.size() > 1 && res.back() == 0) {
        res.pop_back();
    }
    return res;
}

vector<int> multipoint(vector<int> a, vector<int> x) {
    int n = x.size();
    vector<vector<int>> tree(2 * n);

```

```

    for (int i = 0; i < n; ++i) {
        tree[i + n] = {x[i], MOD - 1};
    }
    for (int i = n - 1; i; --i) {
        tree[i] = conv(tree[2 * i], tree[2 * i + 1]);
    }
    tree[1] = mod(a, tree[1]);
    for (int i = 2; i < 2 * n; ++i) {
        tree[i] = mod(tree[i >> 1], tree[i]);
    }
    vector<int> res(n);
    for (int i = 0; i < n; ++i) {
        res[i] = tree[i + n][0];
    }
    return res;
}

vector<int> deriv(vector<int> a) {
    for (int i = 1; i < (int) a.size(); ++i) {
        a[i - 1] = mul(i, a[i]);
    }
    a.back() = 0;
    if (a.size() > 1) {
        a.pop_back();
    }
    return a;
}

vector<int> integ(vector<int> a) {
    a.push_back(0);
    for (int i = (int) a.size() - 1; i; --i) {
        a[i] = mul(a[i - 1], inv(i));
    }
    a[0] = 0;
    return a;
}

vector<int> log(vector<int> a, int n) {
    assert(a[0] == 1);
    auto res = integ(conv(deriv(a), inv(a, n)));
    res.resize(n);
    return res;
}

vector<int> exp(vector<int> a, int need) {
    assert(a[0] == 0);
    vector<int> b = {1};
    while ((int) b.size() < need) {
        vector<int> a1 = a;
        int m = b.size();
        a1.resize(min((int) a1.size(), 2 * m));
        a1[0] = add(a1[0], 1);
        b = conv(b, sub(a1, log(b, 2 * m)));
        b.resize(2 * m);
    }
    b.resize(need);
    return b;
}

```

## FFT.cpp

Description: Calculating product of two polynomials

Time:  $O(n \log(n))$ 

3adba5, 67 lines

```

const ld PI = acos(-1);
using cd = complex<ld>;
const int MAXLOG = 19, N = (1 << MAXLOG), MAXN = (1 <<
    MAXLOG) + 228;
int rev[MAXN];
cd w[MAXN];
bool fftInit = false;

void initFFT() {
    for (int i = 0; i < N; ++i) {
        w[i] = cd(cos(2 * PI * i / N), sin(2 * PI * i /
            N));
    }
    rev[0] = 0;
    for (int i = 1; i < N; ++i) {
        rev[i] = (rev[i >> 1] >> 1) ^ ((i & 1) << (
            MAXLOG - 1));
    }
}

void FFT(int n, vector<cd>& a, bool rv = false) {
    if (!fftInit) {
        initFFT();
        fftInit = 1;
    }
    int LOG = ceil(log2(n));
    for (int i = 0; i < n; ++i) {
        if (i < (rev[i] >> (MAXLOG - LOG))) {
            swap(a[i], a[(rev[i] >> (MAXLOG - LOG))]);
        }
    }
    for (int lvl = 0; lvl < LOG; ++lvl) {
        int len = 1 << lvl;
        for (int st = 0; st < n; st += len * 2) {
            for (int i = 0; i < len; ++i) {
                cd x = a[st + i], y = a[st + len + i] *
                    w[i << (MAXLOG - 1 - lvl)];
                a[st + i] = x + y;
                a[st + i + len] = x - y;
            }
        }
    }
    if (rv) {
        reverse(a.begin() + 1, a.end());
        for (auto& i : a) i /= n;
    }
}

vector<ll> mul(vector<ll> a, vector<ll> b) {
    int xd = max(a.size(), b.size()) * 2;
    int cur = 1;
    while (cur < xd) {
        cur *= 2;
    }
    a.resize(cur);

```

```

b.resize(cur);
vector<cd> ma(cur), mb(cur);
for (int i = 0; i < cur; ++i) {
    ma[i] += a[i];
    mb[i] += b[i];
}
FFT(cur, ma);
FFT(cur, mb);
for (int i = 0; i < cur; ++i) ma[i] *= mb[i];
FFT(cur, ma, true);
vector<ll> ans(cur);
for (int i = 0; i < cur; ++i) {
    ans[i] = (ll)(ma[i].real() + 0.5);
}
return ans;
}

```

## AndConvolution.cpp

**Description:** Calculating and-convolution modulo smth

**Time:**  $\mathcal{O}(n \log(n))$

5dedf4, 24 lines

```

void conv(vector<int> &a, bool x) {
    int n = a.size();
    for (int j = 0; (1 << j) < n; ++j) {
        for (int i = 0; i < n; ++i) {
            if (!(i & (1 << j))) {
                if (x)
                    a[i] = add(a[i], a[i | (1 << j)]);
                else
                    a[i] = sub(a[i], a[i | (1 << j)]);
            }
        }
    }
}

vector<int> andcon(vector<int> a, vector<int> b) {
    int n = 1;
    while (n < max(a.size(), b.size()))
        n *= 2;
    a.resize(n), b.resize(n);
    conv(a, 1), conv(b, 1);
    for (int i = 0; i < n; ++i)
        a[i] = mul(a[i], b[i]);
    conv(a, 0);
    return a;
}

```

## Simplex.cpp

**Description:** Simplex

**Time:** exponential XD(ok for 200-300 variables/bounds)

4dda3c, 99 lines

```

/* solver for linear programs of the form
maximize  $c^T x$ , subject to  $A x \leq b$ ,  $x \geq 0$ 
outputs target function for optimal solution and
the solution by reference
if unbounded above : returns inf, if infeasible :
returns -inf
create Simplex_Steep <ld> LP(A, b, c), then call LP.
Solve (x)
*/

```

```

template <typename DOUBLE>
struct Simplex_Steep {
    using VD = vector<DOUBLE>;
    using VVD = vector<VD>;
    using VI = vector<int>;
    DOUBLE EPS = 1e-12;
    int m, n;
    VI B, N;
    VVD D;
    Simplex_Steep(const VVD &A, const VD &b, const VD &
        c)
        : m(b.size()), n(c.size()), B(m), N(n + 1), D(m
            + 2, VD(n + 2)) {
        for (int i = 0; i < m; i++)
            for (int j = 0; j < n; j++) D[i][j] = A[i][
                j];
        for (int i = 0; i < m; i++) {
            B[i] = n + i;
            D[i][n] = -1;
            D[i][n + 1] = b[i];
        }
        for (int j = 0; j < n; j++) {
            N[j] = j;
            D[m][j] = -c[j];
        }
        N[n] = -1;
        D[m + 1][n] = 1;
    }
    void Pivot(int r, int s) {
        for (int i = 0; i < m + 2; i++)
            if (i != r)
                for (int j = 0; j < n + 2; j++)
                    if (j != s) D[i][j] -= D[r][j] * D[
                        i][s] / D[r][s];
        for (int j = 0; j < n + 2; j++)
            if (j != s) D[r][j] /= D[r][s];
        for (int i = 0; i < m + 2; i++)
            if (i != r) D[i][s] /= -D[r][s];
        D[r][s] = 1.0 / D[r][s];
        swap(B[r], N[s]);
    }
    bool Simplex(int phase) {
        int x = m + (int)(phase == 1);
        while (true) {
            int s = -1;
            DOUBLE c_val = -1;
            for (int j = 0; j <= n; j++) {
                if (phase == 2 && N[j] == -1) continue;
                DOUBLE norm_sq = 0;
                for (int k = 0; k <= m; k++) norm_sq +=
                    D[k][j] * D[k][j];
                norm_sq = max(norm_sq, EPS);
                DOUBLE c_val_j = D[x][j] / sqrt1(
                    norm_sq);
                if (s == -1 || c_val_j < c_val ||
                    (c_val == c_val_j && N[j] < N[s]))
                    {
                        s = j;

```

```

                        c_val = c_val_j;
                    }
            }
            if (D[x][s] >= -EPS) return true;
            int r = -1;
            for (int i = 0; i < m; i++) {
                if (D[i][s] <= EPS) continue;
                if (r == -1 || D[i][n + 1] / D[i][s] <
                    D[r][n + 1] / D[r][s] ||
                    (D[i][n + 1] / D[i][s] == D[r][n +
                        1] / D[r][s] &&
                     B[i] < B[r]))
                    r = i;
            }
            if (r == -1) return false;
            Pivot(r, s);
        }
    }
    DOUBLE Solve(VD &x) {
        int r = 0;
        for (int i = 1; i < m; i++)
            if (D[i][n + 1] < D[r][n + 1]) r = i;
        if (D[r][n + 1] <= -EPS) {
            Pivot(r, n);
            if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
                return -numeric_limits<DOUBLE>::
                    infinity();
            for (int i = 0; i < m; i++)
                if (B[i] == -1) {
                    int s = -1;
                    for (int j = 0; j <= n; j++)
                        if (s == -1 || D[i][j] < D[i][s]
                            ||
                            (D[i][j] == D[i][s] && N[j]
                                < N[s]))
                            s = j;
                    Pivot(i, s);
                }
        }
        if (!Simplex(2)) return numeric_limits<DOUBLE>
            >::infinity();
        x = VD(n);
        for (int i = 0; i < m; i++)
            if (B[i] < n) x[B[i]] = D[i][n + 1];
        return D[m][n + 1];
    }
};

```



## 6.1 Fun things

$$ClassesCount = \frac{1}{|G|} \sum_{\pi \in G} I(\pi)$$
$$ClassesCount = \frac{1}{|G|} \sum_{\pi \in G} k^{C(\pi)}$$

Stirling 2kind - count of partitions of n objects into k nonempty sets:

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$
$$S(n, k) = \sum_{j=0}^{n-1} \binom{n-1}{j} S(j, k - 1)$$
$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k+j} \binom{k}{j} j^n$$
$$n! \approx \sqrt{2n\pi} \left(\frac{n}{e}\right)^n$$

$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i}$ ,  $n_i, k_i$  - digits of  $n, k$  in p-adic system

$$\int_a^b f(x)dx \approx \frac{b-a}{6} (f(a) + 4f(\frac{a+b}{2}) + f(b))$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \, O(loglog)$$
$$G(n) = n \oplus (n \ggg 1)$$
$$g(n) = \sum_{d|n} f(d) \Rightarrow f(n) = \sum_{d|n} g(d) \mu(\frac{n}{d})$$
$$\sum_{d|n} \mu(d) = [n = 1], \mu(1) = 1, \mu(p) = -1, \mu(p^k) = 0$$
$$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\operatorname{tg}(a \pm b) = \frac{\operatorname{tg} a \pm \operatorname{tg} b}{1 \mp \operatorname{tg} a \operatorname{tg} b}$$
$$\operatorname{ctg}(a \pm b) = \frac{\operatorname{ctg} a \operatorname{ctg} b \mp 1}{\operatorname{ctg} b \pm \operatorname{ctg} a}$$
$$\sin \frac{a}{2} = \pm \sqrt{\frac{1 - \cos a}{2}}$$
$$\cos \frac{a}{2} = \pm \sqrt{\frac{1 + \cos a}{2}}$$
$$\operatorname{tg} \frac{a}{2} = \frac{\sin a}{1 - \cos a} = \frac{1 - \cos a}{\sin a}$$
$$\sin a \sin b = \frac{\cos(a - b) - \cos(a + b)}{2}$$
$$\sin a \cos b = \frac{\sin(a - b) + \sin(a + b)}{2}$$
$$\cos a \cos b = \frac{\cos(a - b) + \cos(a + b)}{2}$$

1 jan 2000 - saturday, 1 jan 1900 - monday, 14 apr 1961 - friday

Bell numbers: 0:1, 1:1, 2:2, 3:5, 4:15, 5:52, 6:203, 7:877, 8:4140, 9:21147, 10:115975, 11:678570, 12:4213597, 13:27644437, 14:190899322, 15:1382958545, 16:10480142147, 17:82864869804, 18:682076806159, 19:5832742205057, 20:51724158235372, 21:474869816156751, 22:4506715738447323, 23:44152005855084346

Fibonacci: 45:1134903170. 46:1836311903(max int), 91: 4660046610375530309

Highly composite numbers:

$\leq 1000 : d(840) = 32, \leq 10^4 : d(9240) = 64, \leq 10^5 :$

$d(83160) = 128, \leq 10^6 : d(720720) = 240, \leq 10^7 :$

$d(8648640) = 448, \leq 10^8 : d(91891800) = 768, \leq 10^9 :$

$d(931170240) = 1344, \leq 10^{11} : d(97772875200) = 4032, \leq$

$10^{15} : d(866421317361600) = 26880, \leq 10^{18} :$

$d(897612484786617600) = 103680$

# Table of Basic Integrals (7)

## Basic Forms

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1 \quad (7.1)$$

$$\int \frac{1}{x} dx = \ln |x| \quad (7.2)$$

$$\int u dv = uv - \int v du \quad (7.3)$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| \quad (7.4)$$

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \quad (7.5)$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, \quad n \neq -1 \quad (7.6)$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \quad (7.7)$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x \quad (7.8)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (7.9)$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln |a^2+x^2| \quad (7.10)$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} \quad (7.11)$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln |a^2+x^2| \quad (7.12)$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (7.13)$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \quad a \neq b \quad (7.14)$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln |a+x| \quad (7.15)$$

$$\int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a} \ln |ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (7.16)$$

## Integrals with Roots

$$\int \sqrt{x-a} \, dx = \frac{2}{3} (x-a)^{3/2} \quad (7.17)$$

$$\int \frac{1}{\sqrt{x \pm a}} \, dx = 2\sqrt{x \pm a} \quad (7.18)$$

$$\int \frac{1}{\sqrt{a-x}} \, dx = -2\sqrt{a-x} \quad (7.19)$$

$$\int x\sqrt{x-a} \, dx = \begin{cases} \frac{2a}{3} (x-a)^{3/2} + \frac{2}{5} (x-a)^{5/2}, & \text{or} \\ \frac{2}{3} x(x-a)^{3/2} - \frac{4}{15} (x-a)^{5/2}, & \text{or} \\ \frac{2}{15} (2a+3x)(x-a)^{3/2} \end{cases} \quad (7.20)$$

$$\int \sqrt{ax+b} \, dx = \left( \frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{ax+b} \quad (7.21)$$

$$\int (ax+b)^{3/2} \, dx = \frac{2}{5a} (ax+b)^{5/2} \quad (7.22)$$

$$\int \frac{x}{\sqrt{x \pm a}} \, dx = \frac{2}{3} (x \mp 2a) \sqrt{x \pm a} \quad (7.23)$$

$$\int \sqrt{\frac{x}{a-x}} \, dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a} \quad (7.24)$$

$$\int \sqrt{\frac{x}{a+x}} \, dx = \sqrt{x(a+x)} - a \ln [\sqrt{x} + \sqrt{x+a}] \quad (7.25)$$

$$\int x\sqrt{ax+b} \, dx = \frac{2}{15a^2} (-2b^2 + abx + 3a^2x^2) \sqrt{ax+b} \quad (7.26)$$

$$\int \sqrt{x(ax+b)} \, dx = \frac{1}{4a^{3/2}} \left[ (2ax+b) \sqrt{ax(ax+b)} - b^2 \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \right] \quad (7.27)$$

## Integrals of Rational Functions

$$\int \sqrt{x^3(ax+b)} \, dx = \left[ \frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3} \right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left| a\sqrt{x} + \sqrt{a(ax+b)} \right| \quad (7.28)$$

$$\int \sqrt{x^2 \pm a^2} \, dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (7.29)$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \quad (7.30)$$

$$\int x\sqrt{x^2 \pm a^2} \, dx = \frac{1}{3} (x^2 \pm a^2)^{3/2} \quad (7.31)$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (7.32)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} \quad (7.33)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} \, dx = \sqrt{x^2 \pm a^2} \quad (7.34)$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} \, dx = -\sqrt{a^2 - x^2} \quad (7.35)$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} \, dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \quad (7.36)$$

$$\int \sqrt{ax^2 + bx + c} \, dx = \frac{b + 2ax}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (7.37)$$

$$\int x\sqrt{ax^2 + bx + c} \, dx = \frac{1}{48a^{5/2}} \left( 2\sqrt{a}\sqrt{ax^2 + bx + c} (-3b^2 + 2abx + 8a(c + ax^2)) + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| \right) \quad (7.38)$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} \, dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (7.39)$$

$$\int \frac{x}{\sqrt{ax^2 + bx + c}} \, dx = \frac{1}{a} \sqrt{ax^2 + bx + c} - \frac{b}{2a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \quad (7.40)$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 + x^2}} \quad (7.41)$$

## Integrals with Logarithms

$$\int \ln ax \, dx = x \ln ax - x \quad (7.42)$$

$$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} \quad (7.43)$$

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{x^3}{9} \quad (7.44)$$

$$\int x^n \ln x \, dx = x^{n+1} \left( \frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right), \quad n \neq -1 \quad (7.45)$$

$$\int \frac{\ln ax}{x} \, dx = \frac{1}{2} (\ln ax)^2 \quad (7.46)$$

$$\int \frac{\ln x}{x^2} \, dx = -\frac{1}{x} - \frac{\ln x}{x} \quad (7.47)$$

$$\int \ln(ax + b) \, dx = \left( x + \frac{b}{a} \right) \ln(ax + b) - x, a \neq 0 \quad (7.48)$$

$$\int \ln(x^2 + a^2) \, dx = x \ln(x^2 + a^2) + 2a \tan^{-1} \frac{x}{a} - 2x \quad (7.49)$$

$$\int \ln(x^2 - a^2) \, dx = x \ln(x^2 - a^2) + a \ln \frac{x+a}{x-a} - 2x \quad (7.50)$$

$$\int \ln(ax^2 + bx + c) \, dx = \frac{1}{a} \sqrt{4ac - b^2} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} - 2x + \left( \frac{b}{2a} + x \right) \ln(ax^2 + bx + c) \quad (7.51)$$

$$\int x \ln(ax + b) \, dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2} \left( x^2 - \frac{b^2}{a^2} \right) \ln(ax + b) \quad (7.52)$$

$$\int x \ln(a^2 - b^2x^2) \, dx = -\frac{1}{2}x^2 + \frac{1}{2} \left( x^2 - \frac{a^2}{b^2} \right) \ln(a^2 - b^2x^2) \quad (7.53)$$

$$\int (\ln x)^2 dx = 2x - 2x \ln x + x(\ln x)^2 \quad (7.54)$$

$$\int (\ln x)^3 dx = -6x + x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x \quad (7.55)$$

$$\int x(\ln x)^2 dx = \frac{x^2}{4} + \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x \quad (7.56)$$

$$\int x^2(\ln x)^2 dx = \frac{2x^3}{27} + \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x \quad (7.57)$$

## Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a}e^{ax} \quad (7.58)$$

$$\int \sqrt{x}e^{ax} dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\operatorname{erf}(i\sqrt{ax}), \text{ where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (7.59)$$

$$\int xe^x dx = (x-1)e^x \quad (7.60)$$

$$\int xe^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \quad (7.61)$$

$$\int x^2e^x dx = (x^2 - 2x + 2)e^x \quad (7.62)$$

$$\int x^2e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} \quad (7.63)$$

$$\int x^3e^x dx = (x^3 - 3x^2 + 6x - 6)e^x \quad (7.64)$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx \quad (7.65)$$

$$\int x^n e^{ax} dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \text{ where } \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \quad (7.66)$$

$$\int e^{ax^2} dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\operatorname{erf}(ix\sqrt{a}) \quad (7.67)$$

$$\int e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\operatorname{erf}(x\sqrt{a}) \quad (7.68)$$

$$\int xe^{-ax^2} dx = -\frac{1}{2a}e^{-ax^2} \quad (7.69)$$

$$\int x^2e^{-ax^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\operatorname{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \quad (7.70)$$

## Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a} \cos ax \quad (7.71)$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \quad (7.72)$$

$$\int \sin^3 ax dx = -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a} \quad (7.73)$$

$$\int \sin^n ax dx = -\frac{1}{a} \cos ax {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \quad (7.74)$$

$$\int \cos ax dx = \frac{1}{a} \sin ax \quad (7.75)$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \quad (7.76)$$

$$\int \cos^3 ax dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a} \quad (7.77)$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)} \cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \quad (7.78)$$

$$\int \cos x \sin x dx = \frac{1}{2} \sin^2 x + c_1 = -\frac{1}{2} \cos^2 x + c_2 = -\frac{1}{4} \cos 2x + c_3 \quad (7.79)$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \quad (7.80)$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \quad (7.81)$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x \quad (7.82)$$

$$\int \cos^2 ax \sin bx \, dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \quad (7.83)$$

$$\int \cos^2 ax \sin ax \, dx = -\frac{1}{3a} \cos^3 ax \quad (7.84)$$

$$\int \sin^2 ax \cos^2 bx \, dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \quad (7.85)$$

$$\int \sin^2 ax \cos^2 ax \, dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \quad (7.86)$$

$$\int \tan ax \, dx = -\frac{1}{a} \ln \cos ax \quad (7.87)$$

$$\int \tan^2 ax \, dx = -x + \frac{1}{a} \tan ax \quad (7.88)$$

$$\int \tan^n ax \, dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \quad (7.89)$$

$$\int \tan^3 ax \, dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax \quad (7.90)$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| = 2 \tanh^{-1} \left( \tan \frac{x}{2} \right) \quad (7.91)$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax \quad (7.92)$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| \quad (7.93)$$

$$\int \sec x \tan x \, dx = \sec x \quad (7.94)$$

$$\int \sec^2 x \tan x \, dx = \frac{1}{2} \sec^2 x \quad (7.95)$$

$$\int \sec^n x \tan x \, dx = \frac{1}{n} \sec^n x, n \neq 0 \quad (7.96)$$

$$\int \csc x \, dx = \ln \left| \tan \frac{x}{2} \right| = \ln |\csc x - \cot x| + C \quad (7.97)$$

$$\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax \quad (7.98)$$

$$\int \csc^3 x \, dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln |\csc x - \cot x| \quad (7.99)$$

$$\int \csc^n x \cot x \, dx = -\frac{1}{n} \csc^n x, n \neq 0 \quad (7.100)$$

$$\int \sec x \csc x \, dx = \ln |\tan x| \quad (7.101)$$

## Products of Trigonometric Functions and Monomials

$$\int x \cos x \, dx = \cos x + x \sin x \quad (7.102)$$

$$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \quad (7.103)$$

$$\int x^2 \cos x \, dx = 2x \cos x + (x^2 - 2) \sin x \quad (7.104)$$

$$\int x^2 \cos ax \, dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \quad (7.105)$$

$$\int x^n \cos x \, dx = -\frac{1}{2} (i)^{n+1} [\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)] \quad (7.106)$$

$$\int x^n \cos ax \, dx = \frac{1}{2} (ia)^{1-n} [(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, iax)] \quad (7.107)$$

$$\int x \sin x \, dx = -x \cos x + \sin x \quad (7.108)$$

$$\int x \sin ax \, dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \quad (7.109)$$

$$\int x^2 \sin x \, dx = (2 - x^2) \cos x + 2x \sin x \quad (7.110)$$

$$\int x^2 \sin ax \, dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \quad (7.111)$$

$$\int x^n \sin x \, dx = -\frac{1}{2}(i)^n [\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)] \quad (7.112)$$

$$\int x \cos^2 x \, dx = \frac{x^2}{4} + \frac{1}{8} \cos 2x + \frac{1}{4} x \sin 2x \quad (7.113)$$

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{1}{8} \cos 2x - \frac{1}{4} x \sin 2x \quad (7.114)$$

$$\int x \tan^2 x \, dx = -\frac{x^2}{2} + \ln \cos x + x \tan x \quad (7.115)$$

$$\int x \sec^2 x \, dx = \ln \cos x + x \tan x \quad (7.116)$$

## Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) \quad (7.117)$$

$$\int e^{bx} \sin ax \, dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) \quad (7.118)$$

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) \quad (7.119)$$

$$\int e^{bx} \cos ax \, dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) \quad (7.120)$$

$$\int x e^x \sin x \, dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) \quad (7.121)$$

$$\int x e^x \cos x \, dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) \quad (7.122)$$

## Integrals of Hyperbolic Functions

$$\int \cosh ax \, dx = \frac{1}{a} \sinh ax \quad (7.123)$$

$$\int e^{ax} \cosh bx \, dx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases} \quad (7.124)$$

$$\int \sinh ax \, dx = \frac{1}{a} \cosh ax \quad (7.125)$$

$$\int e^{ax} \sinh bx \, dx = \begin{cases} \frac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases} \quad (7.126)$$

$$\int \tanh ax \, dx = \frac{1}{a} \ln \cosh ax \quad (7.127)$$

$$\int e^{ax} \tanh bx \, dx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[ 1 + \frac{a}{2b}, 1, 2 + \frac{a}{2b}, -e^{2bx} \right] \\ \quad - \frac{1}{a} e^{ax} {}_2F_1 \left[ 1, \frac{a}{2b}, 1 + \frac{a}{2b}, -e^{2bx} \right] & a \neq b \\ \frac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \quad (7.128)$$

$$\int \cos ax \cosh bx \, dx = \frac{1}{a^2 + b^2} [a \sin ax \cosh bx + b \cos ax \sinh bx] \quad (7.129)$$

$$\int \cos ax \sinh bx \, dx = \frac{1}{a^2 + b^2} [b \cos ax \cosh bx + a \sin ax \sinh bx] \quad (7.130)$$

$$\int \sin ax \cosh bx \, dx = \frac{1}{a^2 + b^2} [-a \cos ax \cosh bx + b \sin ax \sinh bx] \quad (7.131)$$

$$\int \sin ax \sinh bx \, dx = \frac{1}{a^2 + b^2} [b \cosh bx \sin ax - a \cos ax \sinh bx] \quad (7.132)$$

$$\int \sinh ax \cosh ax \, dx = \frac{1}{4a} [-2ax + \sinh 2ax] \quad (7.133)$$

$$\int \sinh ax \cosh bx \, dx = \frac{1}{b^2 - a^2} [b \cosh bx \sinh ax - a \cosh ax \sinh bx] \quad (7.134)$$

| Problem | Status | Comment | Iurii | Alex | Leha |
|---------|--------|---------|-------|------|------|
| A - 1   |        |         |       |      |      |
| B - 2   |        |         |       |      |      |
| C - 3   |        |         |       |      |      |
| D - 4   |        |         |       |      |      |
| E - 5   |        |         |       |      |      |
| F - 6   |        |         |       |      |      |
| G - 7   |        |         |       |      |      |
| H - 8   |        |         |       |      |      |
| I - 9   |        |         |       |      |      |
| J - 10  |        |         |       |      |      |
| K - 11  |        |         |       |      |      |
| L - 12  |        |         |       |      |      |
| M - 13  |        |         |       |      |      |
| N - 14  |        |         |       |      |      |
| O - 15  |        |         |       |      |      |