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1 notation and definitions (skippable)

Notation and definition are standard, but this section still exists for completeness.

2 $\mathbb{R}/\mathbb{Z}$ is compact

Suppose $\{U_\lambda \,|\, \lambda \in \Lambda\}$ is an open cover of $\frac{\mathbb{R}}{\mathbb{Z}}$. Define $V_\lambda = \pi_\sim^{-1}(U_\lambda)$. Note that $\{V_\lambda \cap [0,1] \,|\, \lambda \in \Lambda\}$ is an open cover of $[0,1]$ with subspace topology. Hence, there exists an open cover $\{V_1 \cap [0,1], \ldots, V_n \cap [0,1]\}$ of $[0,1]$. Note that saturation of $[0,1]$ is $\mathbb{R}$ and V_λ are $\sim$ - saturated. Hence, $\{V_1, \ldots, V_n\}$ is an open cover of $\mathbb{R}$. Lastly, $\bigcup_{i=1}^n \pi_\sim(V_{\lambda_i}) = \bigcup_{i=1}^n U_{\lambda_i} = \pi_\sim(\bigcup_{i=1}^n V_{\lambda_i}) = \pi_\sim(\mathbb{R}) = \frac{\mathbb{R}}{\mathbb{Z}}$.

Hence, we found a finite subcover.

3 $\mathbb{R}/\mathbb{Z}$ is connected

For any space X and an equivalence relation $\sim$ on X, X is connected $\implies$ $X/_\sim$ is connected.

Proof: Let $X/_\sim$ be not connected, i.e. there exists a continuous map $f : X/_\sim \to \{0,1\}$. But $\pi_\sim$ is continuous, hence $f \circ \pi_\sim : X \to \{0,1\}$ is continuous, which contradicts connectedness of $\mathbb{R}$.

4 $\mathbb{R}/\mathbb{Z}$ is locally connected