Probability Theory Exercises

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1. Exercise

   (i) Construct the smallest and largest possible σ-algebra in a given set Ω.

   (ii) Construct all possible σ-algebras in a set of magnitude 4: $\Omega_4 := \{a, b, c, d\}$.

   (iii) **Sketch** what a σ-algebra in the set $\Omega_5 := [0, 5]\,(\Omega_5 \subset \mathbb{R})$ would conceptually look like.

   (iv) Compare the σ-algebras constructed or sketched in (ii) und (iii). What do you notice?

2. Exercise
   Let $\mathscr{A}$ und $\mathscr{B}$ be two σ-algebras in Ω with $\mathscr{A} \subset \mathscr{B}$.

   (i) Show that $\mathscr{A}$-measurability of $f : \Omega \to \mathbb{R}$ implies $\mathscr{B}$-measurability.

   (ii) Show that an indicator function $\mathbb{1}_M : \Omega \to \mathbb{R}$ is $\mathscr{A}$-measurable exactly when $M \in \mathscr{A}$.

   (iii) Let $f, g : \Omega \to \mathbb{R}$ be $\mathscr{A}$-measurable functions. Show that the sets

   $$\{f < g\}, \{f \leq g\}, \{f = g\} \text{ and } \{f \neq g\}$$

   are elements of $\mathscr{A}$.[1]
   Hint: The $\mathscr{A}$-measurability of a function f is equivalent to:

   $$\begin{aligned} &\forall \alpha \in \mathbb{R} : \{f \leq \alpha\} \in \mathscr{A} \iff \forall \alpha \in \mathbb{R} : \{f \geq \alpha\} \in \mathscr{A} \\ \iff &\forall \alpha \in \mathbb{R} : \{f > \alpha\} \in \mathscr{A} \iff \forall \alpha \in \mathbb{R} : \{f < \alpha\} \in \mathscr{A} \end{aligned}$$

3. Exercise
   Given $f, g : \Omega \to \mathbb{R}$ $\mathscr{A}$-measurable functions. Show:

   (i) $\alpha + \beta \cdot g$ is $\mathscr{A}$-measurable ($\alpha, \beta \in \mathbb{R}$).

   (ii) $f + g$ is $\mathscr{A}$-measurable.

   (iii) f^2 is $\mathscr{A}$-measurable.

   (iv) $f \cdot g$ is $\mathscr{A}$-measurable.

[1] With the usual notation utilised in the field of statisitcs: $\{f < g\} = \{\omega \in \Omega \mid f(\omega) < g(\omega)\}$

4. Exercise

Let $X : \Omega \to [0, +\infty]$ be a random variable with *Exponential* distribution, implying:

$$F_X(z) = P(\{X \leq z\}) = \int_0^z \lambda \cdot \exp(-\lambda \cdot t)\,\mathrm{d}t = 1 - \exp(-\lambda \cdot z)$$

(i) Show that X posesses a density.

(ii) Derive the distribution function of the random variable $Y := 1 - \exp(-c \cdot X)$. What special case is constructed by setting c equal to λ?

(iii) Calculate, insofar either exists, expected value and variance of X.

(iv) Show that, for all $s, t \geq 0$

$$P(X \leq s + t \mid X > s) = P(X \leq t)$$

Interpret the statement.

5. Exercise

Describe the „elements“(or rather components) of a random variable and its associated probability space. Evaluate and explicitly describe the necessity of the often assumed measurability property.

Hint: The concept of „events“und „results“should be considered as well.