# Probability Theory Solutions

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1. Exercise

   (i) $\mathscr{A}_{min} = \{\emptyset, \Omega\}$, $\mathscr{A}_{max} = \mathcal{P}(\Omega)$, insofar existent.

   (ii) $\mathscr{A}_1 = \{\emptyset, \Omega_4\}$
   $\mathscr{A}_2 = \{\emptyset, \Omega_4, \{a\}, \{b, c, d\}\}$
   $\mathscr{A}_3 = \{\emptyset, \Omega_4, \{b\}, \{a, c, d\}\}$
   $\mathscr{A}_4 = \{\emptyset, \Omega_4, \{c\}, \{a, b, d\}\}$
   $\mathscr{A}_5 = \{\emptyset, \Omega_4, \{d\}, \{a, b, c\}\}$
   $\mathscr{A}_6 = \{\emptyset, \Omega_4, \{a, b\}, \{c, d\}\}$
   $\mathscr{A}_7 = \{\emptyset, \Omega_4, \{a, c\}, \{b, d\}\}$
   $\mathscr{A}_8 = \{\emptyset, \Omega_4, \{a, d\}, \{b, c\}\}$
   $\mathscr{A}_9 = \{\emptyset, \Omega_4, \{a\}, \{b\}\{c, d\}, \{b, c, d\}\{a, c, d\}, \{a, b\}\}$
   $\mathscr{A}_{10} = \{\emptyset, \Omega_4, \{a\}, \{c\}\{b, d\}, \{b, c, d\}\{a, b, d\}, \{a, c\}\}$
   $\mathscr{A}_{11} = \{\emptyset, \Omega_4, \{a\}, \{d\}\{b, c\}, \{b, c, d\}\{a, b, c\}, \{a, d\}\}$
   $\mathscr{A}_{12} = \{\emptyset, \Omega_4, \{b\}, \{c\}\{a, d\}, \{a, c, d\}\{a, b, d\}, \{b, c\}\}$
   $\mathscr{A}_{13} = \{\emptyset, \Omega_4, \{b\}, \{d\}\{a, c\}, \{a, c, d\}\{a, b, c\}, \{b, b\}\}$
   $\mathscr{A}_{14} = \{\emptyset, \Omega_4, \{c\}, \{d\}\{a, b\}, \{a, b, d\}\{a, b, c\}, \{c, d\}\}$
   $\mathscr{A}_{15} = \mathcal{P}(\Omega_4)$

   (iii) Simply sketch what an element of the $\sigma$-algebra would look like and which objects are implied to be elements of it due to this. Refer to the definition of a $\sigma$-algebra if the construction is unclear.

   (iv) We were able to explicitly construct the algebra in (ii) due to the finite number of elements in $\Omega_4$. $\Omega_5$ does not only contain an infinite number of elements, but also uncountably many. As such, the explicit notation of the $\sigma$-algebra becomes impossible.

2. Exercise

   Note that we may speak of measurability when actually meaning $\mathscr{A}$-measurability. Since the context implies which $\sigma$-algebra underlies our notion of measurability, this is done in order to make the solution somewhat less dense in notation and therefore, hopefully, more readable.

   (i) Trivially follows from the definition: $f$ is $\mathscr{A}$-measurable, therefore:

   $$\{f \leq \alpha\} \in \mathscr{A}$$

   Since $\mathscr{A} \subset \mathscr{B}$, $\mathscr{B}$-measurability follows:

   $$\{f \leq \alpha\} \in \mathscr{B}$$

(ii) It holds (Consider the reasons why!):

$$\{\mathbb{1}_M \leq \alpha\} = \begin{cases} \Omega & \text{if } 1 \leq \alpha \\ M^{\mathrm{C}} & \text{if } 0 \leq \alpha < 1 \\ \emptyset & \text{if } \alpha < 0 \end{cases}$$

Due to $M^{\mathrm{C}} \in \mathscr{A} \iff M \in \mathscr{A}$, all preimages lie in $\mathscr{A}$, if and only if $M \in \mathscr{A}$ holds.

(iii) Solution becomes very convoluted without use of the hint. This tells us that $\{f < r\}$ and $\{r < g\}$ and therefore the sets $\{f < r\} \cap \{r < g\}$ lie in $\mathscr{A}$. From the properties of $\sigma$-algebras, it then follows that the following sets also lie in $\mathscr{A}$:

$$\bigcup_{r \in \mathbb{Q}} \{f < r\} \cap \{r < g\} = \{f < g\}$$

With $\{f < g\}$ the complement $\{f \geq g\}$ too lies in $\mathscr{A}$. Analogously it is possible to show that $\{f \leq g\}$ lies in $\mathscr{A}$. Since $\{f = g\} = \{f \leq g\} \cap \{f \geq g\}$, the set $\{f = g\}$ also lies in $\mathscr{A}$.

3. Exercise

(i) For $\beta = 0$ the claim is trivial!? We therefore assume $\beta \neq 0$. With $r$ being an arbitrary real number, it holds:

$$\{\alpha + \beta \cdot g \leq r\} = \begin{cases} \{g < \frac{r-\alpha}{\beta}\} & \text{if } \beta > 0 \\ \{g > \frac{r-\alpha}{\beta}\} & \text{if } \beta < 0 \\ \{\alpha < r\} & \text{if } \beta = 0 \end{cases}$$

In each case an element of $\mathscr{A}$ is specified. In the first two cases based on the measurability of $g$, in the third case since either $\{\alpha < r\} = \emptyset$ or $\{\alpha < r\} = \Omega$ holds.

(ii) Setting $\beta = -1$ and following along the proof in (i), it follows that the function $\alpha - g$ is measurable for all $\alpha \in \mathbb{R}$. Due to the measurability of $f$, based on Exercise 2 (iii) it then follows that for all $\alpha \in \mathbb{R}$ the set

$$\{f \leq \alpha - g\} = \{f + g \leq \alpha\}$$

lies within the $\sigma$-Algebra $\mathscr{A}$. Hence the claim follows.

(iii) It holds:

$$\{f^2 \geq \alpha\} = \begin{cases} \Omega & \text{if } \alpha \leq 0 \\ \{f \geq \sqrt{\alpha}\} \cup \{f \leq -\sqrt{\alpha}\} & \text{if } \alpha > 0 \end{cases}$$

Since the sets $\{f \geq \sqrt{\alpha}\}$ and $\{f \leq -\sqrt{\alpha}\}$ lie in $\mathscr{A}$ due to the measurability of $f$, $\{f^2 \geq \alpha\}$ too lies in $\mathscr{A}$.

(iv) Since according to parts (i) and (ii) $\frac{1}{2}(f+g)$ and $\frac{1}{2}(f-g)$ are measurable, the claim follows from part (iii) by the following representation (Check the equality yourself!)

$$f \cdot g = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$$

4. Exercise

(i) Basically solved within the exercise text. Check the definition of a density and take a look at the given distribution function.

(ii) For $z \leq 0$ it holds that $\{Y \leq z\} = \emptyset$ and therefore $F_Y(z) = 0$.
For $z \geq 1$ it holds that $\{Y \leq z\} = \Omega$ and therefore $F_Y(z) = 1$.
For $0 < z < 1$ it holds that

$$\begin{aligned}
F_Y(z) &= P(\{Y \leq z\}) \\
&= P(\{1 - \exp(-c \cdot X) \leq z\}) \\
&= P(\{X \leq -\frac{\ln(1-z)}{c}\}) \\
&= 1 - \exp(\lambda \cdot \frac{\ln(1-z)}{c}) \\
&= 1 - (1-z)^{\frac{\lambda}{c}}
\end{aligned}$$

For $c = \lambda$ a uniform distribution is constructed.

(iii) Partial integration, i.e. integration by parts, immediately results in $\mathbb{E}(X) = \frac{1}{\lambda}$. Analogously, twofold partial integration results in $\mathrm{Var}(X) = \frac{1}{\lambda^2}$. Values only exist for $\lambda > 0$.

(iv) Memorylessness is described by the property.

$$\begin{aligned}
P(X \leq s+t \mid X > s) &= \frac{P(\{X \leq s+t\} \cap \{X > s\})}{P(\{X > s\})} \\
&= \frac{P(\{s < X \leq s+t\})}{1 - F_X(s)} \\
&= \frac{F_X(s+t) - F_X(s)}{1 - F_X(s)} \\
&= 1 - \exp(-\lambda \cdot t) \\
&= F_X(t) = P(X \leq t)
\end{aligned}$$

5. Aufgabe
Simply consult the discussed theory. Necessary components supposed to be explained: sample space, event vs. result, $\sigma$-algebra, probability measure, domain and image of the random variable, measurability.