Math Camp

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Multivariate Optimization

Optimizing multivariate functions

- Parameters $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ such that $f(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y})$ is maximized
- Policy $\boldsymbol{x} \in \Re^n$ that maximizes $U(\boldsymbol{x})$
- Weights $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_K)$ such that a weighted average of forecasts $\boldsymbol{f} = (f_1, f_2, \ldots, f_k)$ have minimum loss

$$\min_{\boldsymbol{\pi}} \quad = \quad -(\sum_{j=1}^{K} \pi_j f_j - y)^2$$

Today we'll describe analytic and computational approaches to optimization

- Analytic recipe for optimization
- Computational optimization
    - Multivariate Newton-Raphson
    - BFGS
    - Approximate Optimization: k-means

Multivariate Optimization

Definition

Let $\boldsymbol{x} \in \Re^n$ and let $\delta > 0$. Define a neighborhood of $\boldsymbol{x}$, $B(\boldsymbol{x}, \delta)$, as the set of points such that,

$$B(\boldsymbol{x}, \delta) \quad = \quad \{\boldsymbol{y} \in \Re^n : ||\boldsymbol{x} - \boldsymbol{y}|| < \delta\}$$

Definition

Suppose $f : X \rightarrow \Re$ with $X \subset \Re^n$. A vector $\boldsymbol{x}^* \in X$ is a global maximum if , for all other $\boldsymbol{x} \in X$

$$f(\boldsymbol{x}^*) \quad > \quad f(\boldsymbol{x})$$

A vector $\boldsymbol{x}^{local}$ is a local maximum if there is a neighborhood around $\boldsymbol{x}^{local}$, $Q \subset X$ such that, for all $x \in Q$,

$$f(\boldsymbol{x}^{local}) \quad > \quad f(\boldsymbol{x})$$

Multivariate Optimization

Definition

A set $X \subset R^n$ is compact if it is closed and bounded

Theorem

Multivariate Extreme Value Theorem Suppose $f : X \rightarrow \Re$ be continuous and $X \subset \Re^n$ and X compact. Then f takes on its maximum and minimum values on X.

We're going to come up with the multivariate equivalent of the first order and second order conditions now

Gradient

Definition

Suppose $f : X \rightarrow \Re^n$ with $X \subset \Re^1$ is a differentiable function. Define the gradient vector of f at $\boldsymbol{x}_0$, $\nabla f(\boldsymbol{x}_0)$ as,

$$\nabla f(\boldsymbol{x}_0) = \left(\frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}, \frac{\partial f(\boldsymbol{x}_0)}{\partial x_2}, \frac{\partial f(\boldsymbol{x}_0)}{\partial x_3}, \ldots, \frac{\partial f(\boldsymbol{x}_0)}{\partial x_n}\right)$$

Gradient First Order Condition

Theorem

Suppose $f : X \rightarrow \Re^1$, $X \subset \Re^n$. Suppose $\boldsymbol{a} \in X$ is a local extremum. Then,

$$\begin{aligned} \nabla f(\boldsymbol{a}) &= \mathbf{0} \\ &= (0, 0, \ldots, 0) \end{aligned}$$

- Proof (intuition): same as one dimensional case (left-hand, right hand), just do it dimension by dimension
- Critical Values:
  1) Maximum
  2) Minimum
  3) Saddle point
- Second Derivative Test!

Second Order Conditions: Hessian

Definition

Suppose $f : X \rightarrow \Re^1$, $X \subset \Re^n$, with f a twice differentiable function. We will define the Hessian matrix as the matrix of second derivatives at $\boldsymbol{x}^* \in X$,

$$\boldsymbol{H}(f)(\boldsymbol{x}^*) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\boldsymbol{x}^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{x}^*) & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\boldsymbol{x}^*) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\boldsymbol{x}^*) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\boldsymbol{x}^*) & \ldots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\boldsymbol{x}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\boldsymbol{x}^*) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\boldsymbol{x}^*) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\boldsymbol{x}^*) \end{pmatrix}$$

General test $\rightsquigarrow$ Two Dimensional Test $\rightsquigarrow$ Example

Hessians

Definition

Consider $n \times n$ matrix $\boldsymbol{A}$. If, for all $\boldsymbol{x} \in \Re^n$ where $\boldsymbol{x} \neq 0$:

$$\begin{aligned} \boldsymbol{x}^{'}\boldsymbol{A}\boldsymbol{x} &> 0 \; \boldsymbol{A} \text{ is positive definite} \\ \boldsymbol{x}^{'}\boldsymbol{A}\boldsymbol{x} &< 0 \; \boldsymbol{A} \text{ is negative definite} \end{aligned}$$

If $\boldsymbol{x}^{'}\boldsymbol{A}\boldsymbol{x} > 0$ for some $\boldsymbol{x}$ and $\boldsymbol{x}^{'}\boldsymbol{A}\boldsymbol{x} < 0$ for other $\boldsymbol{x}$, then we say $\boldsymbol{A}$ is indefinite

Approximating functions and second order conditions

Theorem

Taylor's Theorem *Suppose* $f : \Re \rightarrow \Re$, $f(x)$ *is infinitely differentiable function. Then, the taylor expansion of* $f(x)$ *around* a *is given by*

$$
\begin{aligned}
f(x) &= f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + \ldots \\
f(x) &= \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n
\end{aligned}
$$

Example Function

Suppose $a = 0$ and $f(x) = e^x$. Then,

$$\begin{aligned} f^{'}(x) &= e^x \\ f^{''}(x) &= e^x \\ \vdots &\vdots \vdots \\ f^n(x) &= e^x \end{aligned}$$

This implies

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots + \frac{x^n}{n!} + \ldots$$

Multivariate Taylor's Theorem

Theorem

Suppose $f : \Re^n \to \Re$ is a three-times continously differentiable function, then around $\boldsymbol{a} \in \Re^n$,

$$f(\boldsymbol{x}) \quad = \quad f(\boldsymbol{a}) + \nabla f(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})^{'}\mathbf{H}(f)(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}) + R(\boldsymbol{a}, \boldsymbol{x})$$

where $\frac{R(\boldsymbol{x},\boldsymbol{a})}{||\boldsymbol{x}-\boldsymbol{a}||^2} \to 0$ *as* $\boldsymbol{x} \to \boldsymbol{a}$

Intuition for Quadratic Form

Suppose $\boldsymbol{x}^*$ is some critical value,

$$f(\boldsymbol{x}) \quad = \quad f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) + (\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^*)\mathbf{H}(f)(x^*)(\boldsymbol{x} - \boldsymbol{x}^*) + R(\boldsymbol{x}^*, \boldsymbol{x}$$

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \quad = \quad 0(\boldsymbol{x} - \boldsymbol{x}^*) + (\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^*)\mathbf{H}(f)(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) + R(\boldsymbol{x}^*, \boldsymbol{x})$$

For $\boldsymbol{x}$ near $\boldsymbol{x}^*$, $R(\boldsymbol{x}^*, \boldsymbol{x}) \approx 0$

$\boldsymbol{H}(f)(\boldsymbol{x}^*)$ positive definite $\rightarrow f(\boldsymbol{x}) > f(\boldsymbol{x}^*) \rightarrow$ local minimum
$\boldsymbol{H}(f)(\boldsymbol{x}^*)$ negative definite $\rightarrow f(\boldsymbol{x}) < f(\boldsymbol{x}^*) \rightarrow$ local maximum

Theorem

Second Derivative Test

- If $\boldsymbol{H}(f)(\boldsymbol{a})$ is *positive definite* then $\boldsymbol{a}$ is a local minimum
- If $\boldsymbol{H}(f)(\boldsymbol{a})$ is *negative definite* then $\boldsymbol{a}$ is a local maximum
- If $\boldsymbol{H}(f)(\boldsymbol{a})$ is *indefinite* then $\boldsymbol{a}$ is a saddle point

Second Derivative Test

Many ways to assess definiteness $\rightsquigarrow$ use determinant

Theorem

Two Dimensional, Second Derivative Test. Suppose $f : X \rightarrow \Re$ with $X \subset \Re^2$ and f twice differentiable. Write the Hessian of f at a critical value $\boldsymbol{a}$,

$$\boldsymbol{H}(f)(\boldsymbol{a}) \quad = \quad \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Then, we can conduct the second derivative test as:

- $AC - B^2 > 0$ and $A > 0 \rightsquigarrow$ *positive definite* $\rightsquigarrow$ $\boldsymbol{a}$ *is a local minimum*
- $AC - B^2 > 0$ and $A < 0 \rightsquigarrow$ *negative definite* $\rightsquigarrow$ $\boldsymbol{a}$ *is a local maximum*
- $AC - B^2 < 0 \rightsquigarrow$ *indefinite* $\rightsquigarrow$ *saddle point*
- $AC - B^2 = 0$ *inconclusive*

Multivariate Recipe

1) Calculate gradient
2) Set equal to zero, solve system of equations
3) Calculate Hessian
4) Assess Hessian at critical values
5) Boundary values? (if relevant)

Example 1: A Simple Optimization Problem

Suppose $f : \Re^2 \rightarrow \Re$ with

$$f(x_1, x_2) \quad = \quad 3(x_1 + 2)^2 + 4(x_2 + 4)^2$$

Calculate gradient

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= (6x_1 + 12, 8x_2 + 32) \\ \mathbf{0} &= (6x_1^* + 12, 8x_2^* + 32) \end{aligned}$$

We now solve the system of equations to yield $x_1^* = -2$ and $x_2^* = -4$

Example 1: A Simple Optimization Problem

$$\mathbf{H}(f)(\boldsymbol{x}^*) = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}$$

$\det(\mathbf{H}(f)(\boldsymbol{x}^*)) = 48$ and $6 > 0$ so $\mathbf{H}(f)(\boldsymbol{x}^*)$ is positive definite. local minimum

Example 2: Two Dimensional Ideal Points

Suppose legislators are considering legislation $\boldsymbol{x} \in \Re^2$. And suppose legislator i has utility function $U_i : \Re^2 \rightarrow \Re$,

$$U(\boldsymbol{x})_i = -(x_1 - \mu_1)^2 - (x_2 - \mu_2)^2$$

What is legislator i's optimal policy?
$\nabla f(\boldsymbol{x}) = (-2(x_1 - \mu_1), -2(x_2 - \mu_2))$
$\nabla f(\boldsymbol{x}) = \mathbf{0}$

$$\begin{aligned} -2(x_1^{*} - \mu_1) &= 0 \\ -2(x_2^{*} - \mu_2) &= 0 \end{aligned}$$

Solving yields $x_1^{*} = \mu_1$ and $x_2^{*} = \mu_2$.

Example 2: Two Dimensional Ideal Points

$$U(\boldsymbol{x})_i = -(x_1 - \mu_1)^2 - (x_2 - \mu_2)^2$$

Call $\boldsymbol{\mu} = (\mu_1, \mu_2)$
The Hessian at the critical value is

$$\begin{aligned} \boldsymbol{H}(f)(\boldsymbol{\mu}) &= \begin{pmatrix} \frac{\partial^2 U_i}{\partial x_1 \partial x_1}(\boldsymbol{\mu}) & \frac{\partial^2 U_i}{\partial x_1 \partial x_2}(\boldsymbol{\mu}) \\ \frac{\partial^2 U_i}{\partial x_2 \partial x_1}(\boldsymbol{\mu}) & \frac{\partial^2 U_i}{\partial x_2 \partial x_2}(\boldsymbol{\mu}) \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

So, $-2 * -2 - 0 = 4 > 0$ and $-2 < 0 \rightsquigarrow$ negative definite, maximum
$\boldsymbol{\mu} = (\mu_1, \mu_2)$ are legislator i's two dimensional ideal point.

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Our task:

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Our task:

- Obtain likelihood (summary estimator)

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- Derive maximum likelihood estimators for μ and σ^2

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Our task:

- Obtain likelihood (summary estimator)
- Derive maximum likelihood estimators for μ and σ^2
- Characterize sampling distribution

Example 3: Maximum Likelihood Estimation, Normal Distribution

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$$L(\mu, \sigma^2 | \boldsymbol{Y}) \quad \propto \quad \prod_{i=1}^{n} f(Y_i | \mu, \sigma^2)$$

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\begin{aligned}
L(\mu, \sigma^2 | \boldsymbol{Y}) & \propto \prod_{i=1}^{n} f(Y_i | \mu, \sigma^2) \\
& \propto \prod_{i=1}^{N} \frac{\exp[-\frac{(Y_i - \mu)^2}{2\sigma^2}]}{\sqrt{2\pi\sigma^2}}
\end{aligned}
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$$
l(\mu, \sigma^2 | \boldsymbol{Y}) = -\sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{2\sigma^2} - \frac{n}{2} log(2\pi) - \frac{n}{2} \log(\sigma^2) + {\color{red} c}
$$

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& = -\sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{2\sigma^2} - \frac{n}{2}\log(\sigma^2) + c^{'}
\end{aligned}
$$

Example 3: Log-Likelihood Plot

- In R, drew 10,000 realizations from

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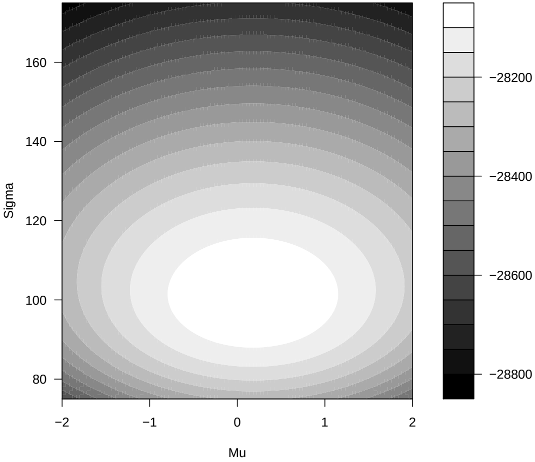
$$Y_i \sim \text{Normal}(0.25, 100)$$

- Used realized values y_i evaluate $l(\mu, \sigma^2|\boldsymbol{y})$

Example 3: Log-Likelihood Plot



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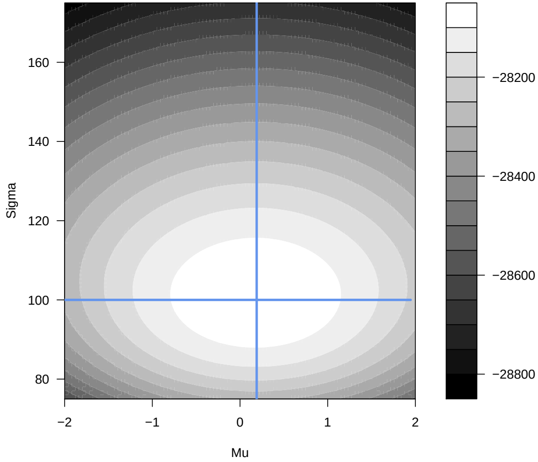

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Let's find $\widehat{\mu}$ and $\widehat{\sigma}^2$ that maximizes log-likelihood.

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\frac{\partial l(\mu, \sigma^2) | \boldsymbol{Y})}{\partial \mu} &= \sum_{i=1}^{n} \frac{2(Y_i - \mu)}{2\sigma^2}
\end{aligned}
$$

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\frac{\partial l(\mu, \sigma^2)|\boldsymbol{Y})}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i - \mu)^2
\end{aligned}
$$

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\begin{aligned}
0 &= -\sum_{i=1}^{n} \frac{2(Y_i - \widehat{\mu})}{2\widehat{\sigma}^2} \\
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Solving for $\widehat{\mu}$ and $\widehat{\sigma}^2$ yields,

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\widehat{\mu} = \frac{\sum_{i=1}^{n} Y_i}{n}
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\end{aligned}
$$

Solving for $\widehat{\mu}$ and $\widehat{\sigma}^2$ yields,

$$
\begin{aligned}
\widehat{\mu} &= \frac{\sum_{i=1}^{n} Y_i}{n} \\
\widehat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2
\end{aligned}
$$

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$$\mathbf{H}(f)(\widehat{\mu}, \widehat{\sigma}^2) = \begin{pmatrix} \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{Y})}{\partial \mu^2} & \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{Y})}{\partial \sigma^2 \partial \mu} \\ \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{Y})}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 l(\mu,\sigma^2|\boldsymbol{Y})}{\partial^2 \sigma^2} \end{pmatrix}$$

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Taking derivatives and evaluating at MLE's yields,

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Taking derivatives and evaluating at MLE's yields,

$$\mathbf{H}(f)(\widehat{\mu}, \widehat{\sigma}^2) = \begin{pmatrix} \frac{-n}{\widehat{\sigma}^2} & 0 \\ 0 & \frac{-n}{(\widehat{\sigma}^2)^2} \end{pmatrix}$$

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Taking derivatives and evaluating at MLE's yields,

$$\mathbf{H}(f)(\widehat{\mu}, \widehat{\sigma}^2) = \begin{pmatrix} \frac{-n}{\widehat{\sigma}^2} & 0 \\ 0 & \frac{-n}{(\widehat{\sigma}^2)^2} \end{pmatrix}$$

$\det(\mathbf{H}(f)(\widehat{\mu}, \widehat{\sigma}^2)) = n^2/\widehat{\sigma}^5$ and $-n/\widehat{\sigma}^2 < 0 \rightsquigarrow$ maximum

Computational Optimization

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Computational solutions: simplify. Trade offs

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Analytic solutions: often hard.
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- Newton-Raphson: expensive
- BFGS: less expensive
- EM-like optimization: solve intractable problems, parallelizable

Multivariate Newton Raphson

Suppose $f : \Re^n \rightarrow \Re$. Suppose we have guess $\boldsymbol{x}_t$.

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$$\boldsymbol{x}_{t+1} \quad = \quad \boldsymbol{x}_t - \mathbf{H}(f)(\boldsymbol{x}_t)^{-1} \nabla f(\boldsymbol{x}_t)$$

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Derivation (intuition):

Multivariate Newton Raphson

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Derivation (intuition): Approximate function with tangent plane.

Multivariate Newton Raphson

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   $\boldsymbol{r}_j = (r_{1j}, r_{2j}, \ldots, r_{Nj})$

   $\boldsymbol{r} = (\boldsymbol{r}_1^{'}, \boldsymbol{r}_2^{'}, \ldots, \boldsymbol{r}_K^{'})$ ($N \times K$ matrix)

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Notation. Representation of document i:

$$\boldsymbol{y}_i = (y_{i1}, y_{i2}, \ldots, y_{iM})$$

Specifying the Method

1) Assume Euclidean distance between objects.
2) Objective function

$$f(\boldsymbol{r}, \boldsymbol{\mu}, \boldsymbol{y}) = \sum_{i=1}^{N} \sum_{j=1}^{K} r_{ij} \left(\sum_{m=1}^{M} (y_{im} - \mu_{km})^2 \right)$$

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Two observations:

- If $K = N$ $f(r^*, \boldsymbol{\mu}^*, \boldsymbol{y}) = 0$ (Minimum)
    - Each observation in own cluster
    - $\boldsymbol{\mu}_i = \boldsymbol{y}_i$
- If $K = 1$, $f(r^*, \boldsymbol{\mu}^*, \boldsymbol{y}) = N \times \sigma^2$
    - Each observation in one cluster
    - Center: average of documents

Specifying the Method

1) Assume Euclidean distance between objects
2) Objective function
3) Algorithm for optimization

Iterative algorithm, Each Iteration t

- Conditional on $\boldsymbol{\mu}^{t-1}$ (from previous iteration), choose $\boldsymbol{r}^t$
- Conditional on $\boldsymbol{r}^t$, choose $\boldsymbol{\mu}^t$

Repeat until convergence, measured as change in f.

$$\text{Change} \quad = \quad f(\boldsymbol{\mu}^t, \boldsymbol{r}^t, \boldsymbol{y}) - f(\boldsymbol{\mu}^{t-1}, \boldsymbol{r}^{t-1}, \boldsymbol{y})$$

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$$f(\boldsymbol{r}, \boldsymbol{\mu}, \boldsymbol{y}) = \sum_{i=1}^{N}\sum_{j=1}^{K} r_{ij}\left(\sum_{m=1}^{M}(y_{im} - \mu_{km})^2\right)$$

Algorithm for estimation:
Begin: initialize $\boldsymbol{\mu}_1^{t-1}, \boldsymbol{\mu}_2^{t-1}, \ldots, \boldsymbol{\mu}_K^{t-1}$
Choose $\boldsymbol{r}^t$

$$r_{ij}^t = \begin{cases} 1 \text{ if } j = \arg\min_k \sum_{m=1}^{M}(y_{im} - \mu_{km})^2 \\ 0 \text{ otherwise }, \end{cases} .$$

In words: Assign each document $\boldsymbol{y}_i$ to the closest center $\boldsymbol{\mu}_k$

$$f(\boldsymbol{r}, \boldsymbol{\mu}, \boldsymbol{y}) = \sum_{i=1}^{N}\sum_{j=1}^{K} r_{ij}\left(\sum_{m=1}^{M}(y_{im} - \mu_{km})^2\right)$$

Conditional on $\boldsymbol{r}^t$, choose $\boldsymbol{\mu}^t$
Let's focus on $\boldsymbol{\mu}_k$

$$f(\boldsymbol{r}, \boldsymbol{\mu}_k, \boldsymbol{y})_k = \sum_{i=1}^{N} r_{ik}\left(\sum_{m=1}^{M}(y_{im} - \mu_{km})^2\right)$$

Focus on just μ_{km}

$$f(\boldsymbol{r}, \mu_{km}, \boldsymbol{y})_{km} = \sum_{i=1}^{N} r_{ik}(y_{im} - \mu_{km})^2$$

Quadratic: take derivative, set equal to zero (second derivative test works)

$$\begin{aligned}
\frac{\partial f(\boldsymbol{r}, \mu_{km}, \boldsymbol{y})_{km}}{\partial \mu_{km}} &= -2\sum_{i=1}^{N} r_{ik}(y_{im} - \mu_{km}) \\
2\sum_{i=1}^{N} r_{ik}(y_{im} - \mu_{km}^{t}) &= 0 \\
\sum_{i=1}^{N} r_{ik} y_{im} - \mu_{km}^{t} \sum_{i=1}^{N} r_{ik} &= 0 \\
\frac{\sum_{i=1}^{N} r_{ik} y_{im}}{\sum_{i=1}^{N} r_{ik}} &= \mu_{km}^{t}
\end{aligned}$$

$$\boldsymbol{\mu}_k^t = \frac{\sum_{i=1}^N r_{ik}\boldsymbol{y}_i}{\sum_{i=1}^N r_{ik}}$$

In words:

- $\boldsymbol{\mu}_k^t$ is the average of documents assigned to the k^{th} cluster

Algorithm, In Words

- Conditional on center estimates, assign documents to closest cluster centers
- Conditional on document assignments, cluster centers are averages of documents assigned to the cluster

Expectation-Maximization (EM) [connection guarantees convergence]

- Estimation of $r \rightsquigarrow$ Expectation step (data augmentation)
- Estimation of $\boldsymbol{\mu}_k \rightsquigarrow$ Maximization Step

Visual Example

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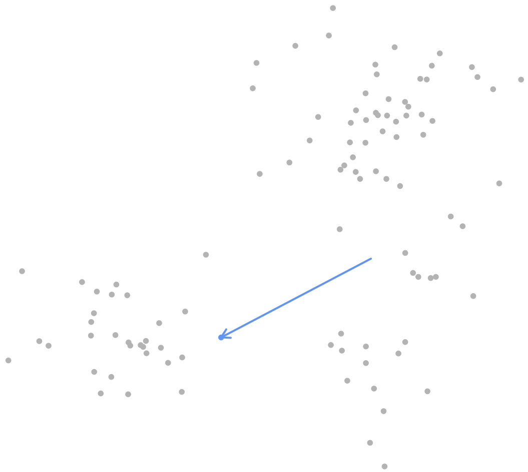

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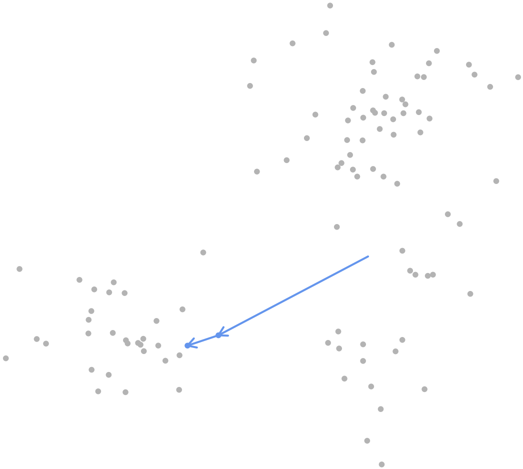

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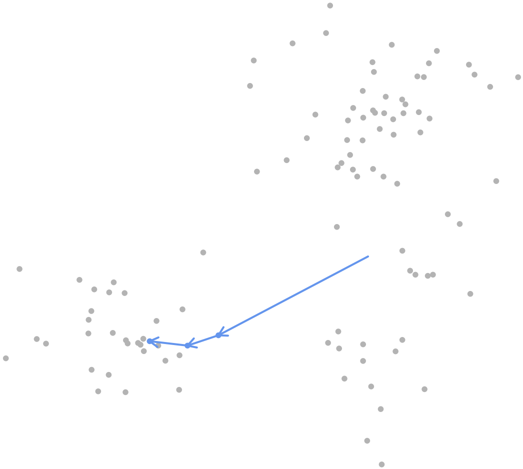

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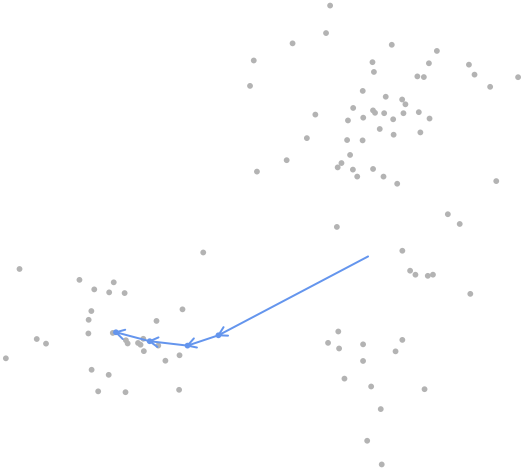

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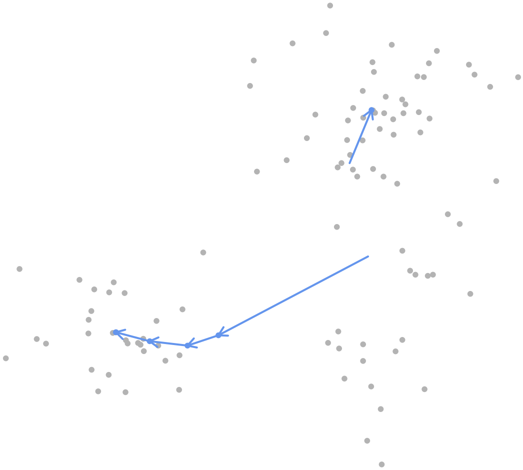

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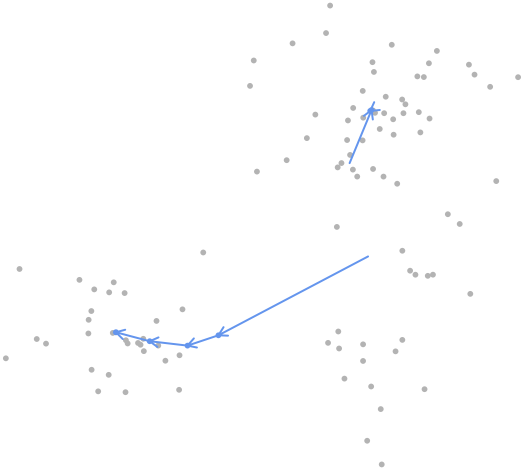

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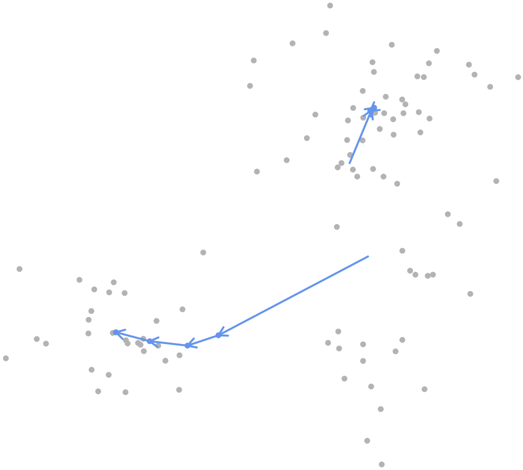

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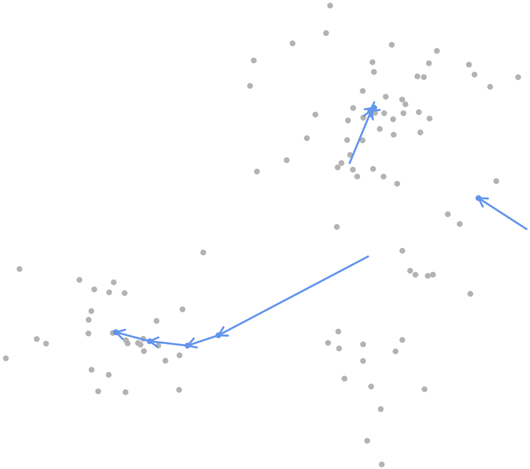

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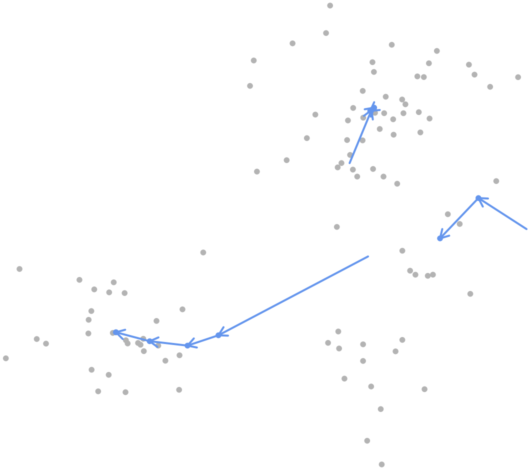

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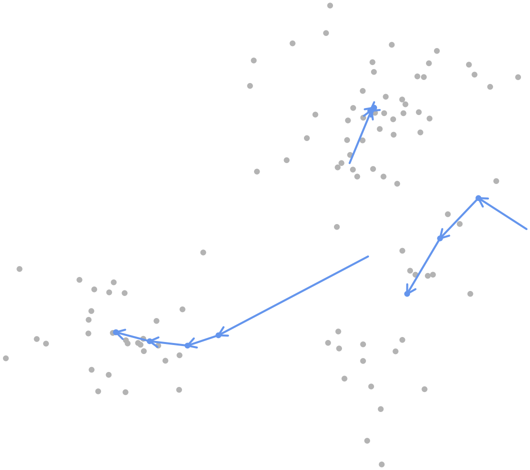

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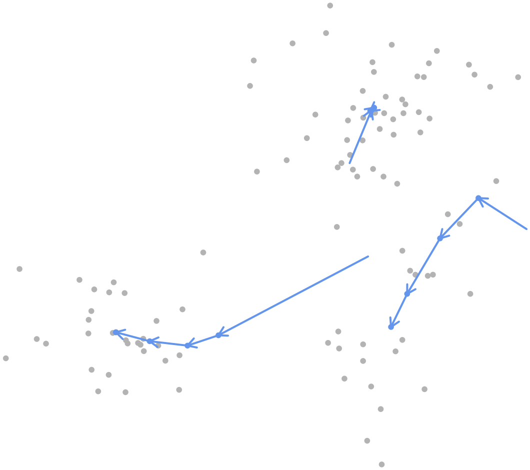

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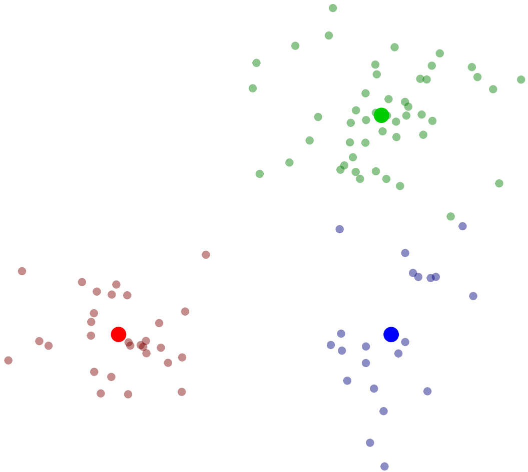

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- Randomly select most fit, then combine

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- Sample a new subset, perform optimization, combine with previous sample
- Converges on local extrema (given regulatory conditions)

Genetic Optimization:

- Evaluate fitness of solutions
- Randomly select most fit, then combine
- Can converge to global maximum, but might require extensive run time

Where We Are Going

- Done with math component
- Start probability tomorrow