Math Camp

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Department of Political Science
University of Chicago

August 29th, 2017

Lab this afternoon!

130-300pm

Convergence

Big idea today is convergence

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- Sequence $\rightarrow$ converge on some number

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- Function $\rightarrow$ limit (use to calculate derivatives)

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- Continuity $\rightarrow$ a function doesn't jump (converge on itself)

Convergence

Big idea today is convergence

- Sequence $\rightarrow$ converge on some number
- Function $\rightarrow$ limit (use to calculate derivatives)
- Continuity $\rightarrow$ a function doesn't jump (converge on itself)
- Derivatives $\rightarrow$ limits that measure a function's properties

Sequence: Definition + Examples

Definition

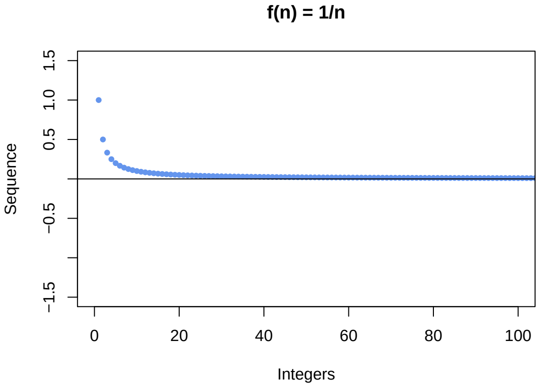
A sequence is a function whose domain is the set of positive integers

We'll write a sequence as,

$$\{a_n\}_{n=1}^{\infty} = (a_1, a_2, \ldots, a_N, \ldots)$$

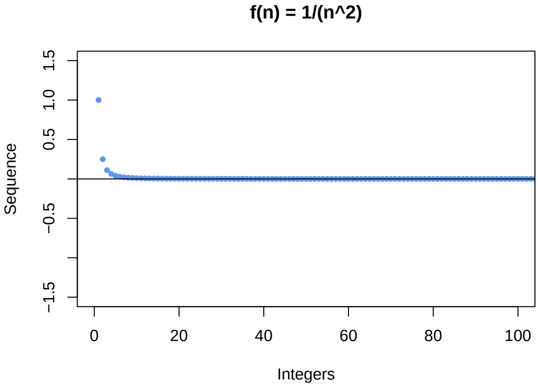
Sequence: Definition + Examples

$$\left\{ \frac{1}{n} \right\} = (1, 1/2, 1/3, 1/4, \ldots, 1/N, \ldots,)$$



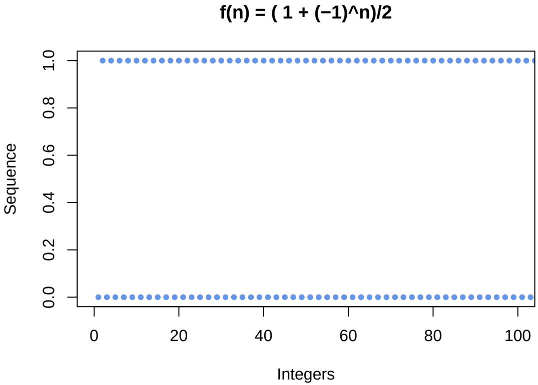
Sequence: Definition + Examples

$$\left\{\frac{1}{n^2}\right\} = (1, 1/4, 1/9, 1/16, \ldots, 1/N^2, \ldots,)$$



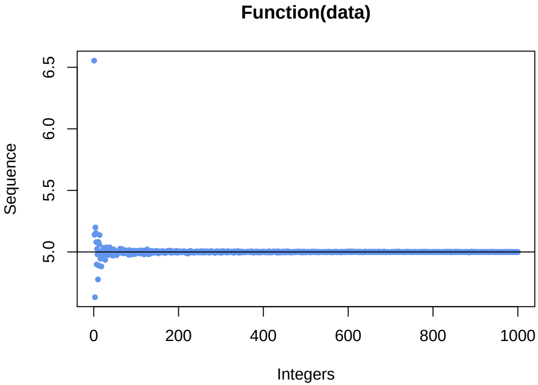
Sequence: Definition + Examples

$$\left\{ \frac{1 + (-1)^n}{2} \right\} \quad = \quad (0, 1, 0, 1, \ldots, 0, 1, 0, 1 \ldots,)$$



Sequence: Definition + Examples

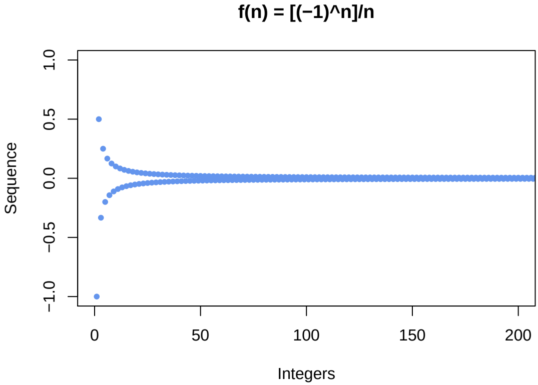
$$
\begin{aligned}
\{\theta\}_{n=1}^{\infty} &= (\theta_1, \theta_2, \ldots, \theta_n, \ldots) \\
\theta_n &= f(\text{n responses (vote choice) })
\end{aligned}
$$



Sequence: Convergence

Consider the sequence

$$\left\{ \frac{(-1)^n}{n} \right\} = \left(-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \frac{-1}{7}, \frac{1}{8}, \ldots\right)$$



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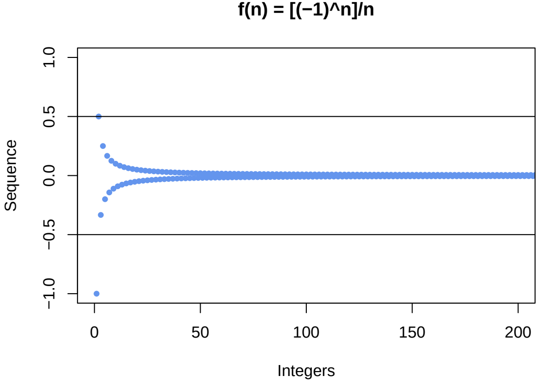

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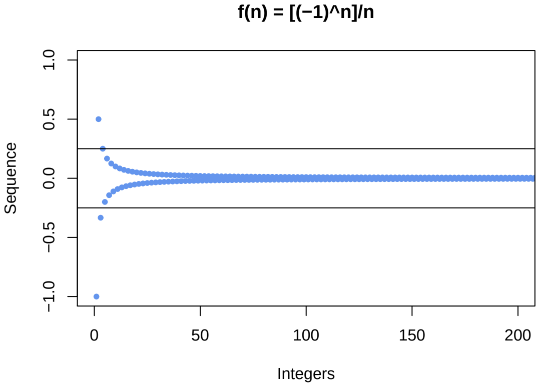

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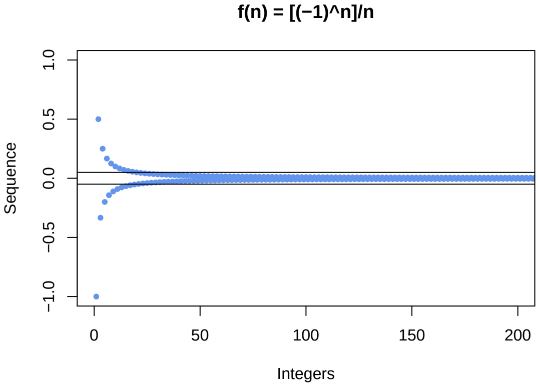

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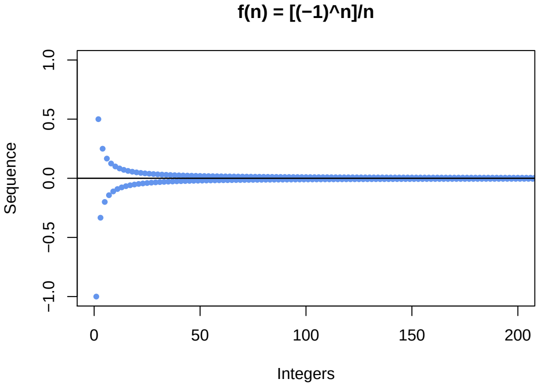

Sequence: Convergence definition

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *converges to a real number* A *if for each* $\epsilon > 0$ *there is a positive integer* N *such that for all* $n \geq N$ *we have* $|a_n - A| < \epsilon$

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2) $\epsilon > 0$ is some arbitrary real-valued number.

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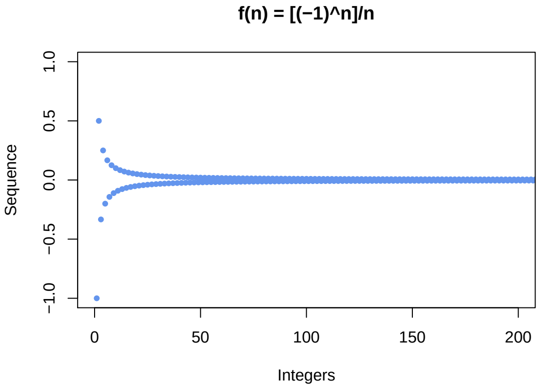
2) $\epsilon > 0$ is some arbitrary real-valued number. Think about this as our error tolerance. Notice $\epsilon > 0$.
3) As we will see the N will depend upon ϵ
4) Implies the sequence never gets further than ϵ away from A

Sequence: Convergence definition



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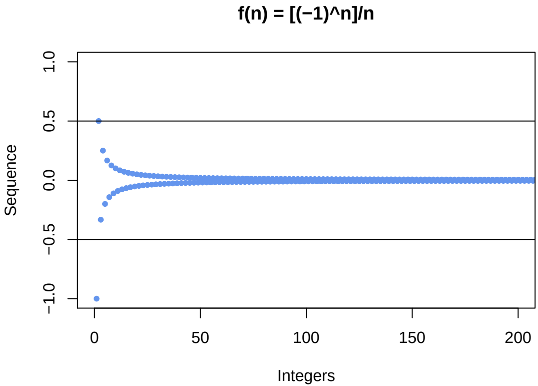

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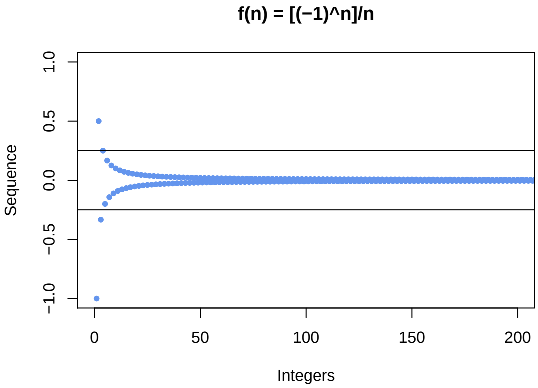

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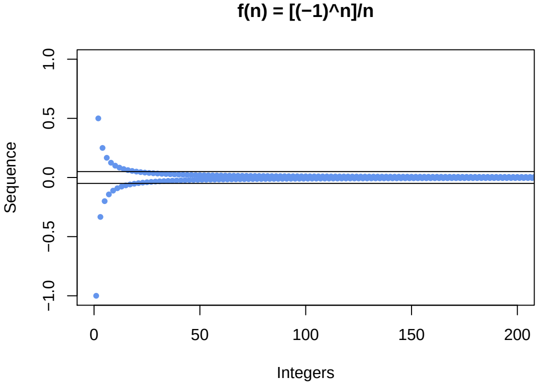

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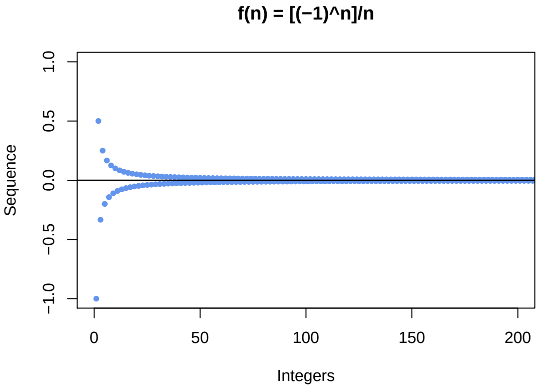

Sequence: Proof of Convergence

Theorem

$\left\{\frac{1}{n}\right\}$ *converges to 0*

Proof.

We need to show that for ϵ there is some N_{ϵ} such that, for all $n \geq N_{\epsilon}$ $|\frac{1}{n} - 0| < \epsilon$. Without loss of generality (WLOG) select an ϵ. Then,

$$\begin{aligned} |\frac{1}{N_{\epsilon}} - 0| &< \epsilon \\ \frac{1}{N_{\epsilon}} &< \epsilon \\ \frac{1}{\epsilon} &< N_{\epsilon} \end{aligned}$$

For each epsilon, then, any $N_{\epsilon} > \frac{1}{\epsilon}$ will suffice. □

Sequence: Divergence + Bounded

Definition

If a sequence, $\{a_n\}$ converges we'll call it convergent. If it doesn't we'll call it divergent. If there is some number M such that, for all n $|a_n| < M$, then we'll call it bounded

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$$\left\{\frac{1 + (-1)^n}{2}\right\} = (0, 1, 0, 1, \ldots, 0, 1, 0, 1 \ldots,)$$

- All convergent sequences are bounded
- If a sequence is constant, $\{C\}$ it converges to C. proof?

Algebra of Sequences

How do we add, multiply, and divide sequences?

Theorem

Suppose $\{a_n\}$ *converges to* A *and* $\{b_n\}$ *converges to* B. *Then,*

- $\{a_n + b_n\}$ *converges to* $A + B$
- $\{a_n b_n\}$ *converges to* $A \times B$.
- *Suppose* $b_n \neq 0 \; \forall \; n$ *and* $B \neq 0$. *Then* $\left\{\frac{a_n}{b_n}\right\}$ *converges to* $\frac{A}{B}$.

Working Together

- Consider the sequence $\left\{\frac{1}{n}\right\}$—what does it converge to?
- Consider the sequence $\left\{\frac{1}{2n}\right\}$ what does it converge to?

Challenge Questions

- What does $\left\{3 + \frac{1}{n}\right\}$ converge to?
- What about $\left\{(3 + \frac{1}{n})(100 + \frac{1}{n^4})\right\}$?
- Finally, $\left\{\frac{300 + \frac{1}{n}}{100 + \frac{1}{n^4}}\right\}$?

Work smarter, not harder

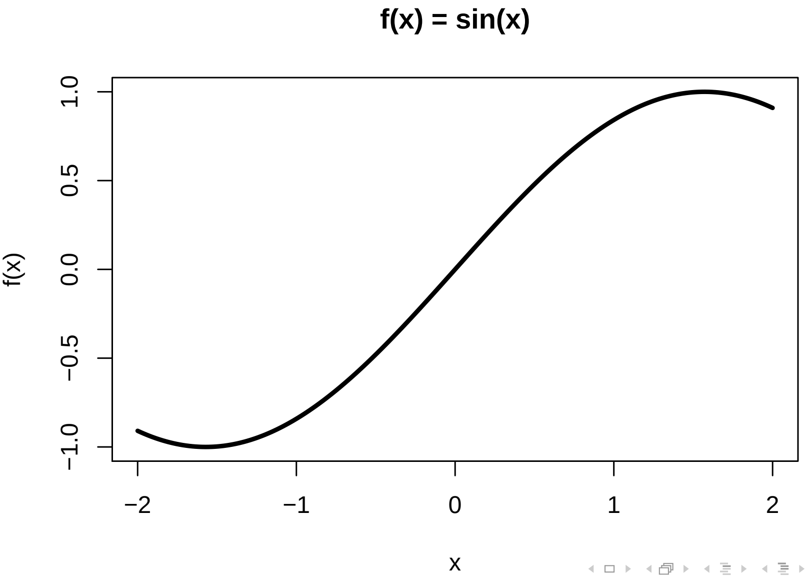
Divide into teams, let's reconvene in about 10 minutes.

Sequences$\rightsquigarrow$ Limits of Functions

Calculus/Real Analysis: study of functions on the real line.
Limit of a function: how does a function behave as it gets close to a particular point?

- Derivatives
- Asymptotics
- Game Theory

Limits of Functions



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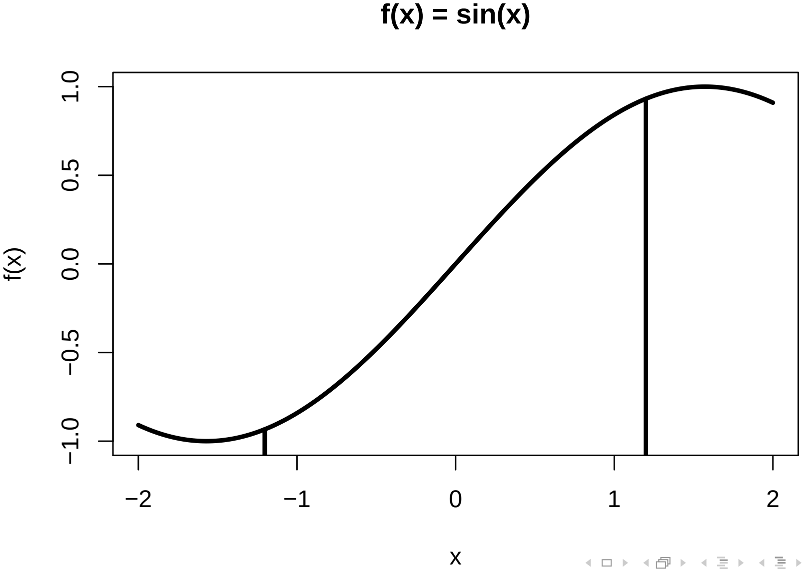

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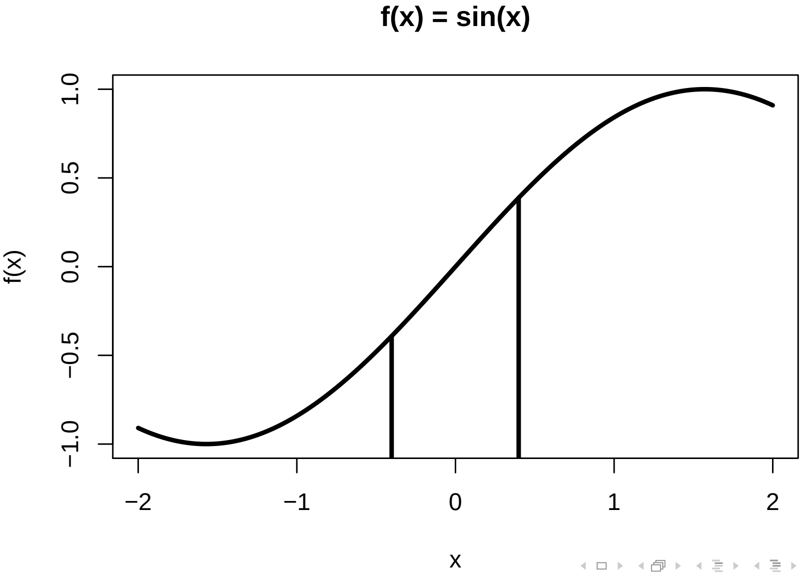

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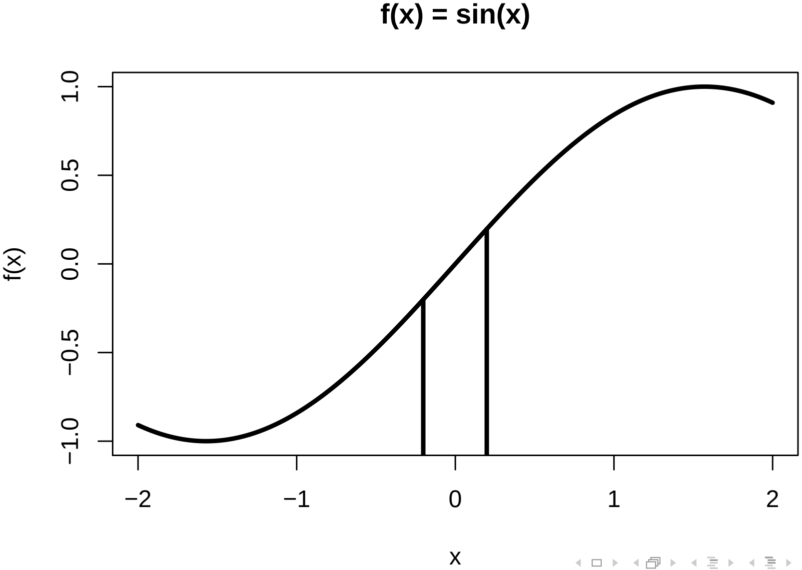

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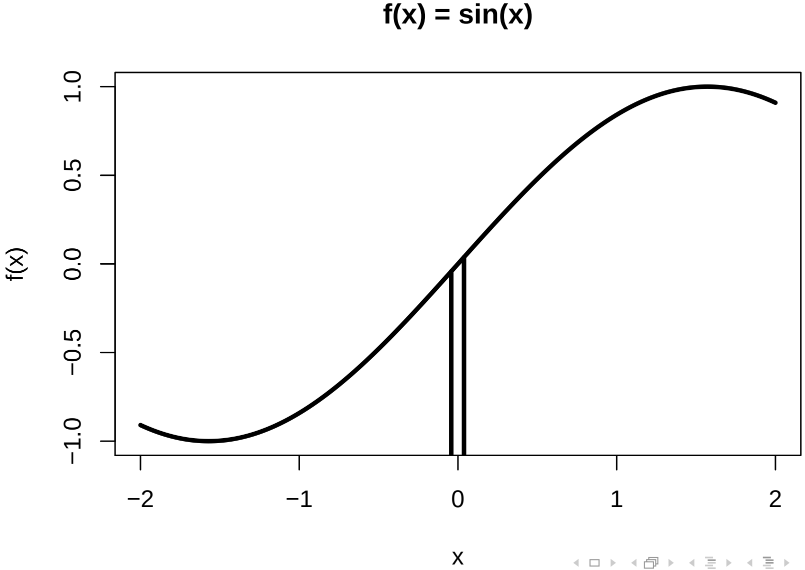

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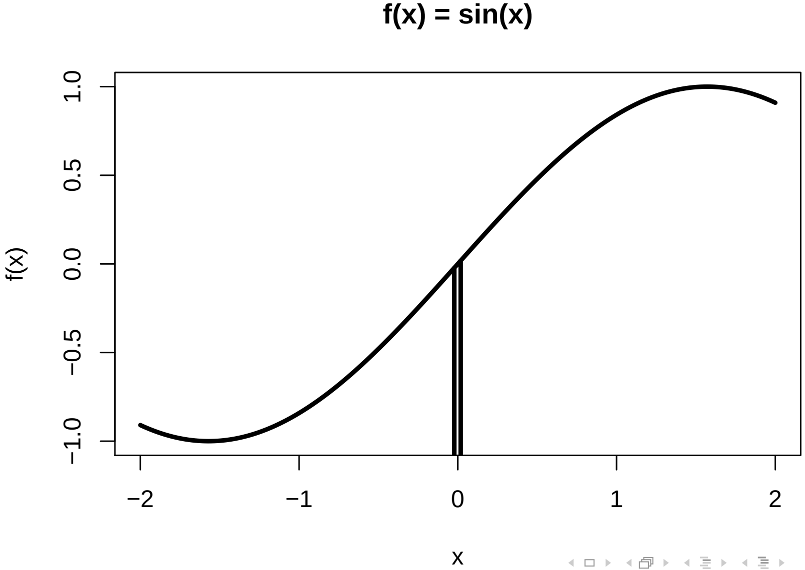

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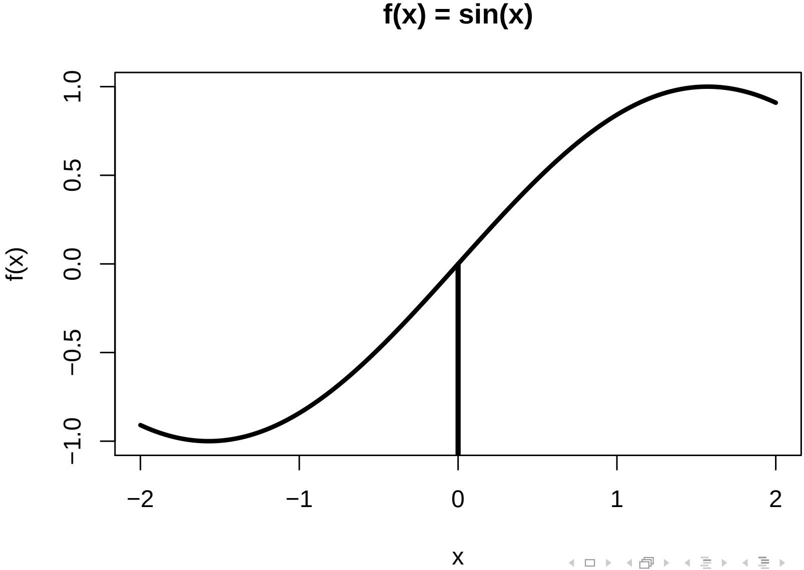


Precise Definition of Limits of Functions

Definition

Suppose $f : \Re \rightarrow \Re$. *We say that* f *has a limit* L *at* x_0 *if, for each* $\epsilon > 0$, *there is a* $\delta > 0$ *such that* $|x - x_0| < \delta$ *implies that* $|f(x) - L| < \epsilon$.

- Limits are about the behavior of functions at points. Here x_0.
- As with sequences, we let ϵ define an error rate
- δ defines an area around x_0 where $f(x)$ is going to be within our error rate

Precise Definition of Limit: Example

Theorem

The function $f(x) = x + 1$ has a limit of 1 at $x_0 = 0$.

Proof.

WLOG choose $\epsilon > 0$. We want to show that there is δ_ϵ such that, $|x - x_0| < \delta_\epsilon$ implies $|f(x) - 1| < \epsilon$. In other words,

$$\begin{aligned} |x| < \delta_\epsilon \quad &\text{implies} \quad |(x+1) - 1| < \epsilon \\ |x| < \delta_\epsilon \quad &\text{implies} \quad |x| < \epsilon \end{aligned}$$

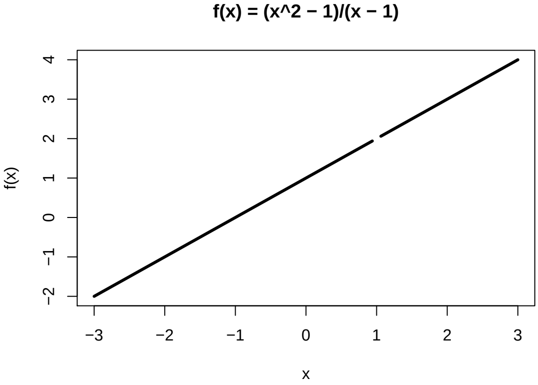
But if $\delta_\epsilon = \epsilon$ then this holds, we are done. □

Precise Definition of Limit: Example

A function can have a limit of L at x_0 even if $f(x_0) \neq L$(!)

Theorem

The function $f(x) = \frac{x^2-1}{x-1}$ *has a limit of* 2 *at* $x_0 = 1$.



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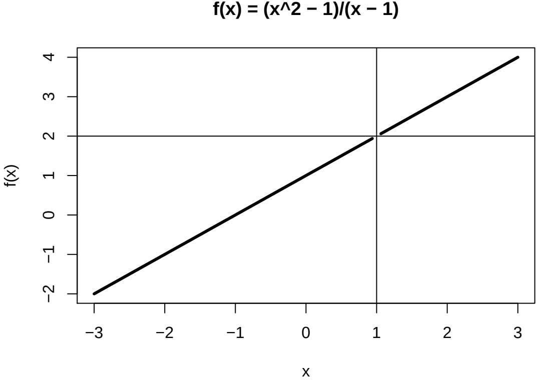


Precise Definition of Limit: Example

Proof.

For all $x \neq 1$,

$$\begin{aligned} \frac{x^2 - 1}{x - 1} &= \frac{(x+1)(x-1)}{x-1} \\ &= x + 1 \end{aligned}$$

Choose $\epsilon > 0$ and set $x_0 = 1$. Then, we're looking for δ_ϵ such that

$$|x - 1| < \delta_\epsilon \quad \text{implies} \quad |(x+1) - 2| < \epsilon$$

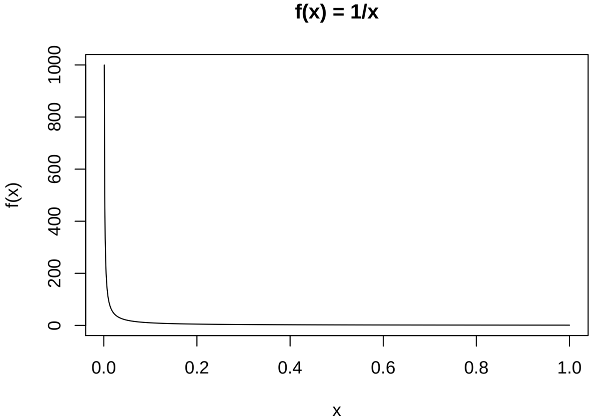
Again, if $\delta_\epsilon = \epsilon$, then this is satisfied.

□

Not all Functions have Limits!

Theorem

Consider $f : (0,1) \rightarrow \Re$, $f(x) = 1/x$. $f(x)$ *does not have a limit at* $x_0 = 0$



Proof.

Choose $\epsilon > 0$. We need to show that there does not exist δ such that

$$|x| < \delta \quad \text{implies} \quad \left|\frac{1}{x} - L\right| < \epsilon$$

But, there is a problem. Because

$$\begin{aligned} \frac{1}{x} - L &< \epsilon \\ \frac{1}{x} &< \epsilon + L \\ x &> \frac{1}{L + \epsilon} \end{aligned}$$

This implies that there can't be a δ, because x has to be bigger than $\frac{1}{L+\epsilon}$. $\square$

Intuitive Definition of Limit

Definition

If a function f tends to L at point x_0 we say is has a limit L at x_0 we commonly write,

$$\lim_{x \rightarrow x_0} f(x) \quad = \quad L$$

Definition

If a function f tends to L at point x_0 as we approach from the right, then we write

$$\lim_{x \rightarrow x_0^+} f(x) \quad = \quad L$$

and call this a right hand limit
If a function f tends to L at point x_0 as we approach from the left, then we write

$$\lim_{x \rightarrow x_0^-} f(x) \quad = \quad L$$

and call this a left-hand limit

Regression discontinuity designs

Left-hand, Right-hand, and Limits

Theorem

The $\lim_{x \to x_0} f(x)$ *exists if and only if* $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$

Left-hand, Right-hand, and Limits

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- Intuition that $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) \Rightarrow \lim_{x\to x_0} f(x)$. If they are equal we can take the smallest δ and we can guarantee proof.

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- Intuition that $\lim_{x \rightarrow x_0} f(x) \Rightarrow \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x)$. Absolute value is symmetric—so we must be converging from each side. (contradiction could work too!)

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- We can also appeal to sequences to prove this stuff

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- We can also appeal to sequences to prove this stuff

Trick: we'll show limits don't exist by showing $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$

Finding Limits

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Justin: Limits are used constantly in political science. And getting comfortable with this notation (by seeing it many times) is important

Student: fine. How am I going to find the limit? I can't do a $\delta - \epsilon$ proof yet.

Justin: yes, those take time. For this class, graphing will be critical.

Algebra of Limits

Theorem

Suppose $f : \Re \rightarrow \Re$ *and* $g : \Re \rightarrow \Re$ *with limits* A *and* B *at* x_0. *Then,*

$$
\begin{aligned}
i.)\ \lim_{x \rightarrow x_0} (f(x) + g(x)) &= \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x) = A + B \\
ii.)\ \lim_{x \rightarrow x_0} f(x) g(x) &= \lim_{x \rightarrow x_0} f(x) \lim_{x \rightarrow x_0} g(x) = AB
\end{aligned}
$$

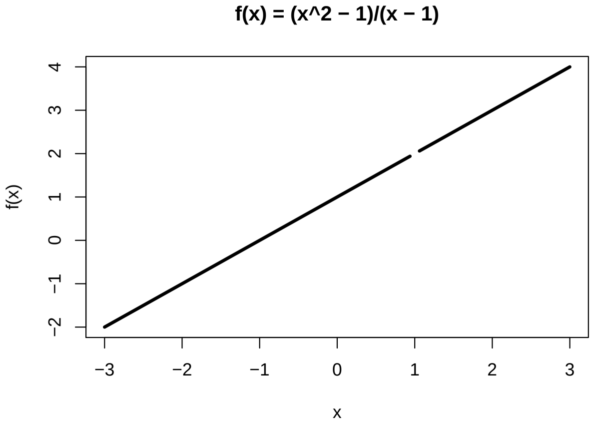
Suppose $g(x) \neq 0$ *for all* $x \in \Re$ *and* $B \neq 0$ *then* $\frac{f(x)}{g(x)}$ *has a limit at* x_0 *and*

$$
\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{A}{B}
$$

Challenge Problems

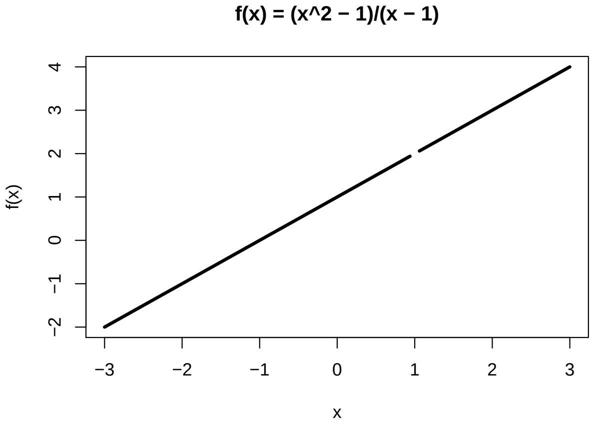
Suppose $\lim_{x \rightarrow x_0} f(x) = a$. Find $\lim_{x \rightarrow x_0} \frac{f(x)^3 + f(x)^2}{f(x)}$

Continuity



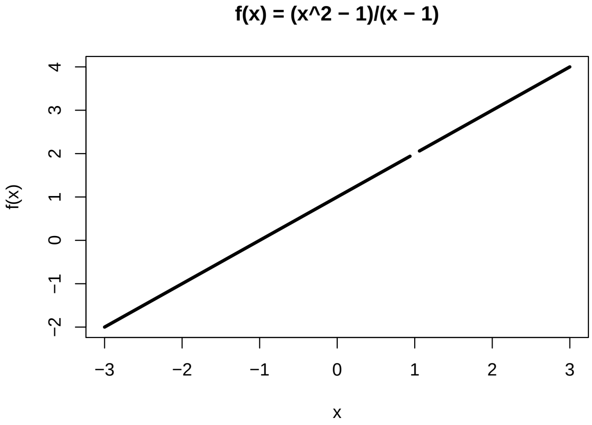
- Limit exists at 1

Continuity



- Limit exists at 1
- But hole in function

Continuity



- Limit exists at 1
- But hole in function
- Fails the pencil test, discontinuous at 1

Continuity, Rigorous Definition

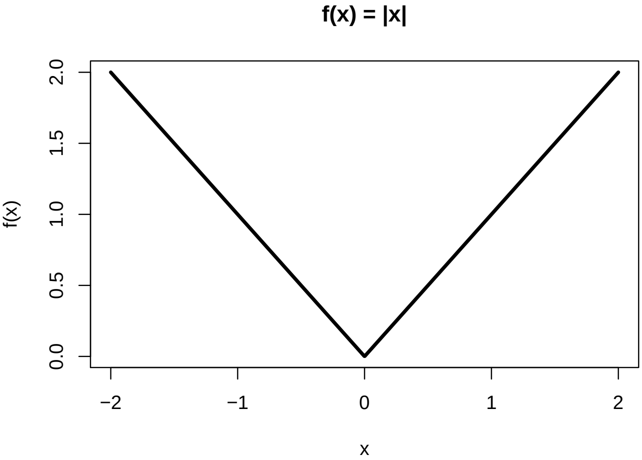
Definition

Suppose $f : \Re \rightarrow \Re$ *and consider* $x_0 \in \Re$. *We will say* f *is continuous at* x_0 *if for each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that if,*

$$\begin{aligned} |x - x_0| &< \delta \text{ for all } x \in \Re \text{ then} \\ |f(x) - f(x_0)| &< \epsilon \end{aligned}$$

- Previously $f(x_0)$ was replaced with L.
- Now: $f(x)$ has to converge on itself at x_0.
- Continuity is more restrictive than limit

Examples



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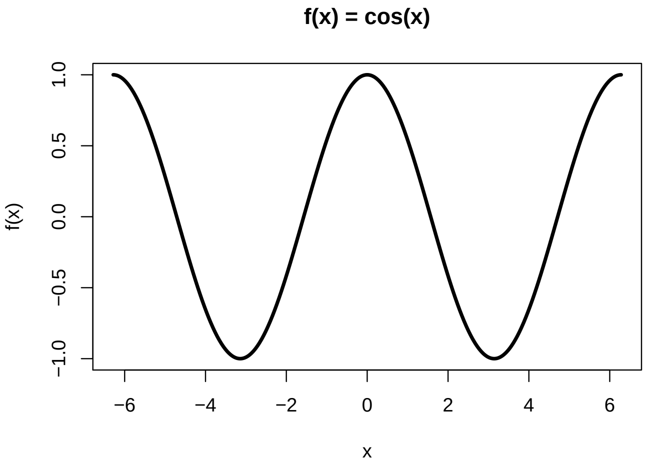
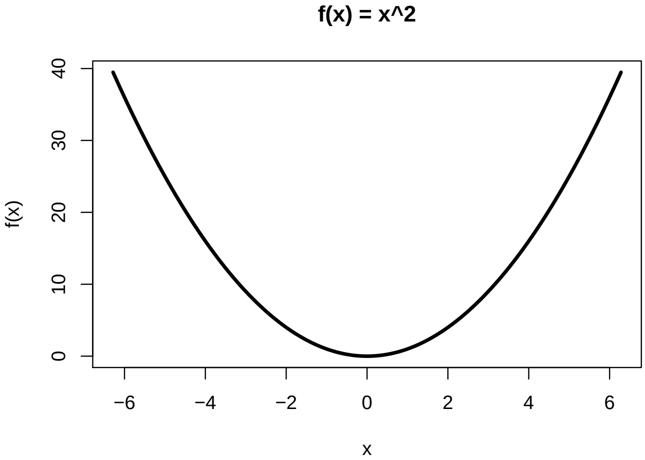
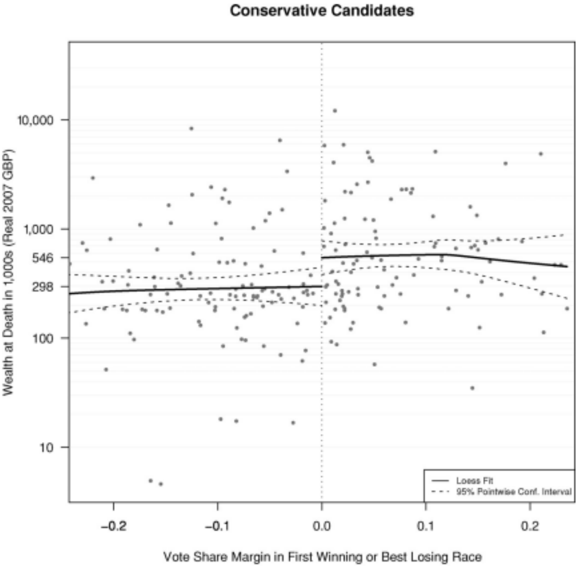


Examples



Examples



Conservative Candidates
Wealth at Death in 1,000s (Real 2007 GBP)
10,000
1,000
546
298
100
10
−0.2
−0.1
0.0
0.1
0.2
Vote Share Margin in First Winning or Best Losing Race
Loess Fit
95% Pointwise Conf. Interval

Continuity and Limits

Theorem

Let $f : \Re \rightarrow \Re$ with $x_0 \in \Re$. Then f is continuous at x_0 if and only if f has a limit at x_0 and that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Proof.

$(\Rightarrow)$. Suppose f is continuous at x_0. This implies that for each $\epsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. This is the definition of a limit, with $L = f(x_0)$.
$(\Leftarrow)$. Suppose f has a limit at x_0 and that limit is $f(x_0)$. This implies that for each $\epsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. But this is the definition of continuity. $\square$

Algebra of Continuous Functions

Theorem

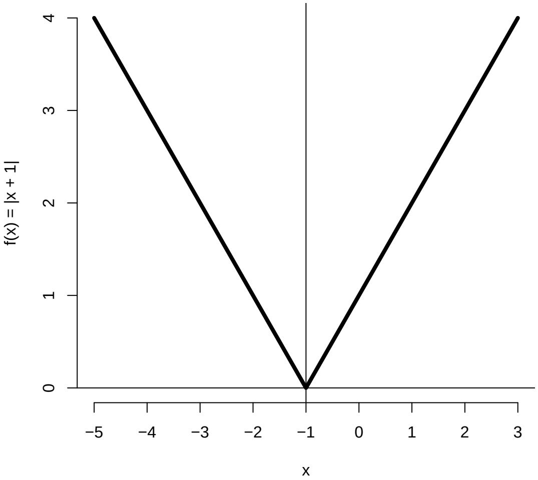
Suppose $f : \Re \to \Re$ and $g : \Re \to \Re$ are continuous at x_0. Then,

i.) $f(x) + g(x)$ *is continuous at* x_0

ii.) $f(x)g(x)$ *is continuous at* x_0

iii. *if* $g(x_0) \neq 0$, *then* $\frac{f(x)}{g(x)}$ *is continuous at* x_0

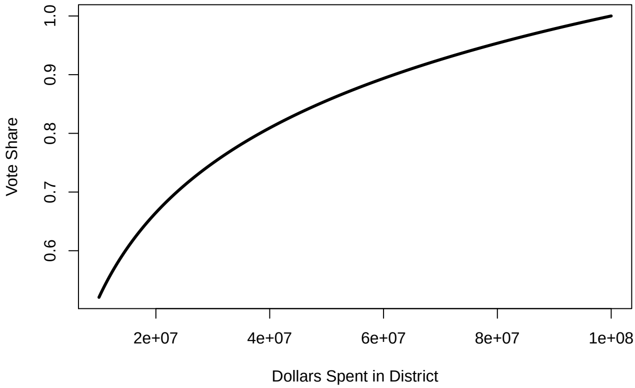
Use theorem about limits to prove continuous theorems.


f(x) = |x + 1|
0
1
2
3
4
−5
−4
−3
−2
−1
0
1
2
3
x

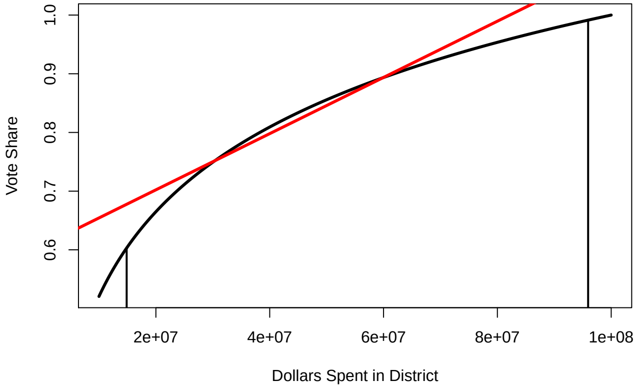
How Functions Change

- Derivatives—Rates of change in functions
- Foundational across a lot of work in Poli Sci.
- A special limit
- Cover three broad ideas
  - Geometric interpretation/intuition
  - Formulas/Algebra derivatives
  - Famous theorems

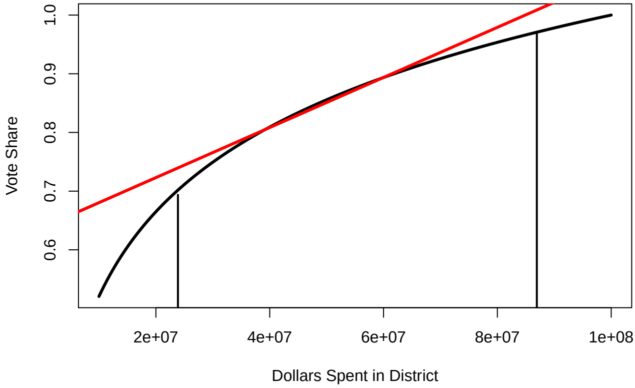
Rates of Change in a Function



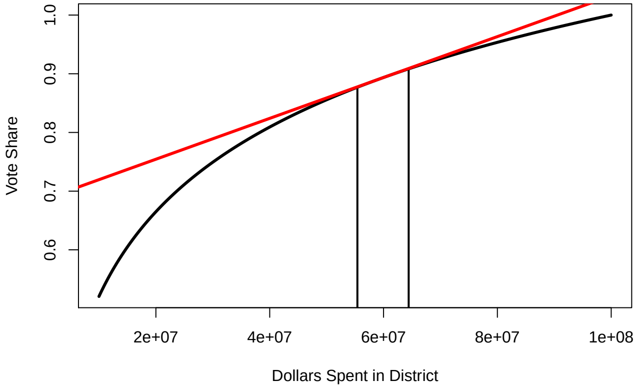
Rates of Change in a Function



Rates of Change in a Function



Rates of Change in a Function



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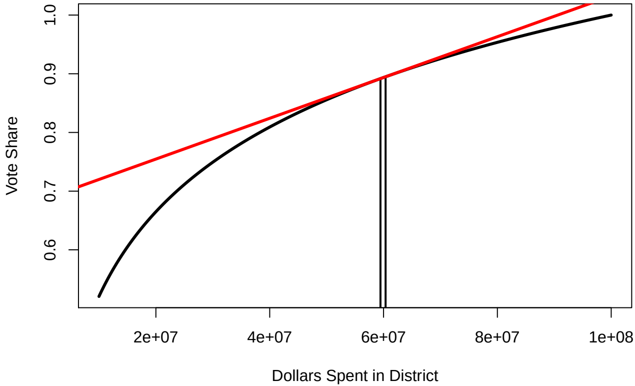

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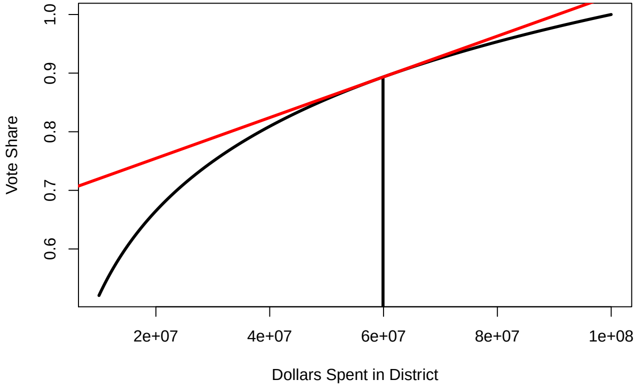

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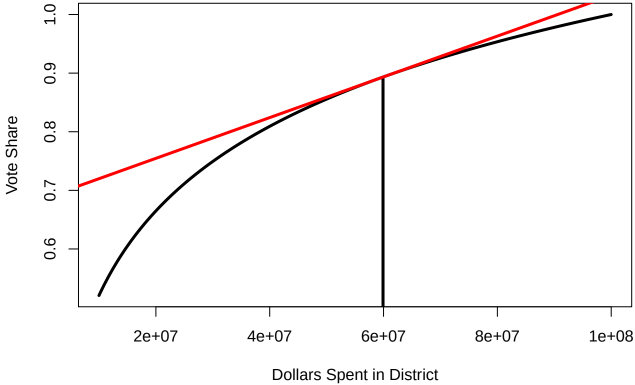

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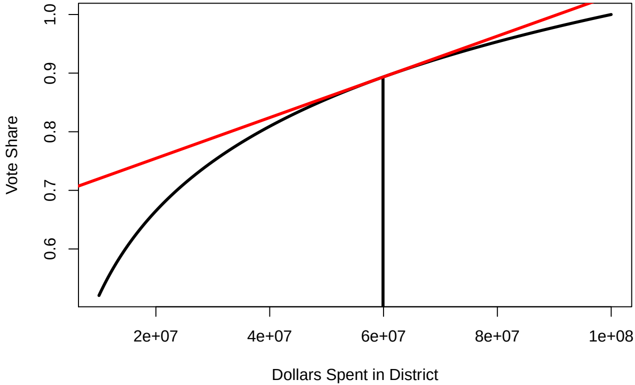
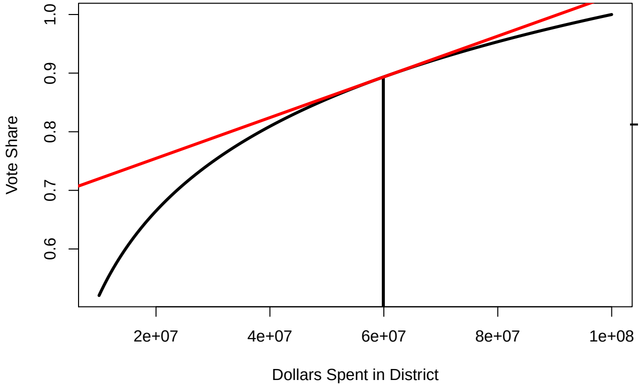


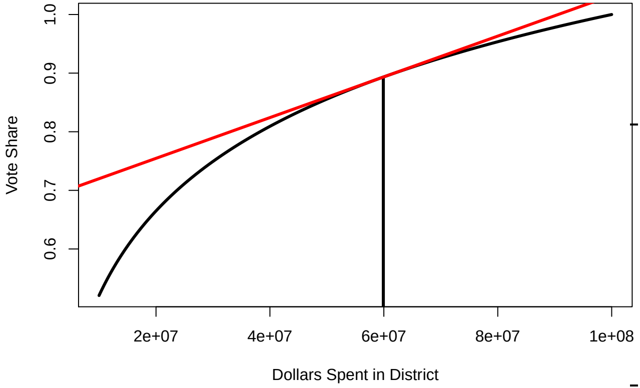
- Rate of Change $\rightsquigarrow$ Return on Vote Share/$ Invested

Rates of Change in a Function



- Rate of Change $\rightsquigarrow$ Return on Vote Share/$ Invested
- Instantaneous rate of change $\rightsquigarrow$ Increase in vote share in response to infinitesimally small increase in spending

Rates of Change in a Function



- Rate of Change $\rightsquigarrow$ Return on Vote Share/$ Invested
- Instantaneous rate of change $\rightsquigarrow$ Increase in vote share in response to infinitesimally small increase in spending
- Limit

Derivative Definition

Suppose $f : \Re \rightarrow \Re$.

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$$R(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

Definition

Derivative Definition

Suppose $f : \Re \rightarrow \Re$. Measure rate of change at a point x_0 with a function $R(x)$,

$$R(x) \quad = \quad \frac{f(x) - f(x_0)}{x - x_0}$$

- $R(x)$ defines the rate of change.

Definition

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Let $f : \Re \rightarrow \Re$. If the limit

$$\begin{aligned} \lim_{x \rightarrow x_0} R(x) &= \frac{f(x) - f(x_0)}{x - x_0} \\ &= f^{'}(x_0) \end{aligned}$$

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Definition

Let $f : \Re \rightarrow \Re$. If the limit

$$\lim_{x \rightarrow x_0} R(x) = \frac{f(x) - f(x_0)}{x - x_0} = f^{'}(x_0)$$

exists then we say that f is differentiable at x_0. If $f^{'}(x_0)$ exists for all $x \in$ Domain, then we say that f is differentiable.

Derivative Examples

- Suppose $f(x) = x^2$ and consider $x_0 = 1$. Then,

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- Suppose $f(x) = x^2$ and consider $x_0 = 1$. Then,

$$\begin{aligned} \lim_{x \rightarrow 1} R(x) &= \lim_{x \rightarrow 1} \frac{x^2 - 1^2}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1} \end{aligned}$$

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$\lim_{x\to 0^-} R(x) = -1$, but $\lim_{x\to 0^+} R(x) = 1$. So, not differentiable at 0.

Continuity and Derivatives

- $f(x) = |x|$ is continuous but not differentiable. This is because the change is too abrupt.
- Suggests differentiability is a stronger condition

Theorem

Let $f : \Re \rightarrow \Re$ be differentiable at x_0. Then f is continuous at x_0.

Continuity and Derivatives

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This proof is all in the setup. Realize that,

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0)$$

Continuity and Derivatives

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Let $f : \Re \rightarrow \Re$ be differentiable at x_0. Then f is continuous at x_0.

Proof.

This proof is all in the setup. Realize that,

$$\begin{aligned} f(x) &= \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0) \\ &= R(x)(x - x_0) + f(x_0) \end{aligned}$$

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If $f(x)$ is continuous at x_0 then, $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} [R(x)(x - x_0) + f(x_0)]$$

Continuity and Derivatives

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Let $f : \Re \to \Re$ be differentiable at x_0. Then f is continuous at x_0.

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This proof is all in the setup. Realize that,

$$
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&= R(x)(x - x_0) + f(x_0)
\end{aligned}
$$

If $f(x)$ is continuous at x_0 then, $\lim_{x \to x_0} f(x) = f(x_0)$.

$$
\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} [R(x)(x - x_0) + f(x_0)] \\
&= \left(\lim_{x \to x_0} R(x)\right) \left(\lim_{x \to x_0} (x - x_0)\right) + \lim_{x \to x_0} f(x_0)
\end{aligned}
$$

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$$\begin{aligned} \lim_{x \rightarrow x_0} f(x) &= \lim_{x \rightarrow x_0} [R(x)(x - x_0) + f(x_0)] \\ &= \left(\lim_{x \rightarrow x_0} R(x)\right)\left(\lim_{x \rightarrow x_0} (x - x_0)\right) + \lim_{x \rightarrow x_0} f(x_0) \\ &= f^{'}(x_0)0 + f(x_0) = f(x_0) \end{aligned}$$

What goes wrong?

Consider the following piecewise function:

$$\begin{aligned} f(x) &= x^2 \text{ for all } x \in \Re \setminus 0 \\ f(x) &= 1000 \text{ for } x = 0 \end{aligned}$$

Consider derivative at 0. Then,

$$\begin{aligned} \lim_{x \to 0} R(x) &= \lim_{x \to 0} \frac{f(x) - 1000}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2}{x} - \lim_{x \to 0} \frac{1000}{x} \end{aligned}$$

$\lim_{x \to 0} \frac{1000}{x}$ diverges, so the limit doesn't exist.

Calculating Derivatives

- Rarely will we take limit to calculate derivative.
- Rather, rely on rules and properties of derivatives
- Important: do not forget core intuition

Strategy:

- Algebra theorems
- Some specific derivatives
- Work on problems

Some Derivative Rules

Suppose a is some constant, $f(x)$ and $g(x)$ are functions

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Suppose a is some constant, $f(x)$ and $g(x)$ are functions

$$f(x) = x \quad ; \quad f^{'}(x) = 1$$
$$f(x) = ax^k \quad ; \quad f^{'}(x) = (a)(k)x^{k-1}$$

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Suppose a is some constant, $f(x)$ and $g(x)$ are functions

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f(x) = ax^{k} \quad &; \quad f^{'}(x) = (a)(k)x^{k-1} \\
f(x) = e^{x} \quad &; \quad f^{'}(x) = e^{x} \\
f(x) = \sin(x) \quad &; \quad f^{'}(x) = \cos(x)
\end{aligned}$$

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f(x) = e^{x} \quad &; \quad f^{'}(x) = e^{x} \\
f(x) = \sin(x) \quad &; \quad f^{'}(x) = \cos(x) \\
f(x) = \cos(x) \quad &; \quad f^{'}(x) = -\sin(x)
\end{aligned}
$$

Algebra of Derivatives

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Suppose $f : \Re \rightarrow \Re$ *and* $g : \Re \rightarrow \Re$ *and both are differentiable at* $x_0 \in \Re$*. Then,*

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$$h^{'}(x_0) = f^{'}(x_0) + g^{'}(x_0)$$

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$$h^{'}(x_0) \quad = \quad f^{'}(x_0) + g^{'}(x_0)$$

ii) $h(x) = f(x)g(x)$ *is differentiable at* x_0 *and*

Algebra of Derivatives

Theorem

Suppose $f : \Re \rightarrow \Re$ *and* $g : \Re \rightarrow \Re$ *and both are differentiable at* $x_0 \in \Re$*. Then,*

i) $h(x) = f(x) + g(x)$ *is differentiable at* x_0 *and*

$$h^{'}(x_0) = f^{'}(x_0) + g^{'}(x_0)$$

ii) $h(x) = f(x)g(x)$ *is differentiable at* x_0 *and*

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iii) $h(x) = \frac{f(x)}{g(x)}$ *with* $g(x) \neq 0$ *then,*

$$h^{'}(x_0) = \frac{f^{'}(x_0)g(x_0) - g^{'}(x_0)f(x_0)}{g(x_0)^2}$$

Challenge Problems

Differentiate the following functions and evaluate at the specified value

1) $f(x) = x^3 + 5x^2 + 4x$, at $x_0 = 2$
2) $f(x) = sin(x)x^3$ at $x_0 = y$
3) $f(x) = \frac{e^x}{x^3}$ at $x = 2$
4) $g(x) = \log(x)x^3$ at $x = x_0$
5) Suppose $f(x) = x^2$ and $g(x) = x^3$. Find all x such that $f^{'}(x) > g^{'}(x)$.

Proving Property of Derivatives

Theorem

Suppose $f(x) = x^k$ and k is a positive integer. If $k = 0$ then $f^{'}(x) = 0$. If $k > 0$, then, $f^{'}(x) = kx^{k-1}$.

Proof.

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$$f^{'}(\tilde{x}) = \lim_{x \rightarrow \tilde{x}} \frac{x - \tilde{x}}{x - \tilde{x}}$$

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\begin{aligned}
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& = 1 = 1x^0
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$$\begin{aligned} g^{'}(x) = f(x)x^{'} + f^{'}(x)x &= x^r 1 + rx^{r-1}x \\ &= x^r + rx^r = (r+1)x^r \end{aligned}$$

Chain Rule

Common to have functions in functions

$$\begin{aligned} f(x) &= \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}} \\ &= \frac{f(g(x))}{\sqrt{2\pi}} \end{aligned}$$

To deal with this, we use the chain rule

Theorem

Suppose $g : \Re \rightarrow \Re$ and $f : \Re \rightarrow \Re$. Suppose both $f(x)$ and $g(x)$ are differentiable at x_0. Define $h(x) = g(f(x))$. Then,

$$h^{'}(x_0) = g^{'}(f(x_0))f^{'}(x_0)$$

Examples of Chain Rule in Action

- $h(x) = e^{2x}$. $g(x) = e^x$. $f(x) = 2x$. So $h(x) = g(f(x)) = g(2x) = e^{2x}$. Taking derivatives, we have

$$h^{'}(x) \quad = \quad g^{'}(f(x))f^{'}(x) = e^{2x}2$$

- $h(x) = \log(\cos(x))$. $g(x) = log(x)$. $f(x) = \cos(x)$. $h(x) = g(f(x)) = g(\cos(x)) = log(\cos(x))$

$$h^{'}(x) \quad = \quad g^{'}(f(x))f^{'}(x) = \frac{-1}{\cos(x)}\sin(x) = -\tan(x)$$

Derivatives and Properties of Functions

Derivatives reveal an immense amount about functions

- Often use to optimize a function (tomorrow)
- But also reveal average rates of change
- Or crucial properties of functions

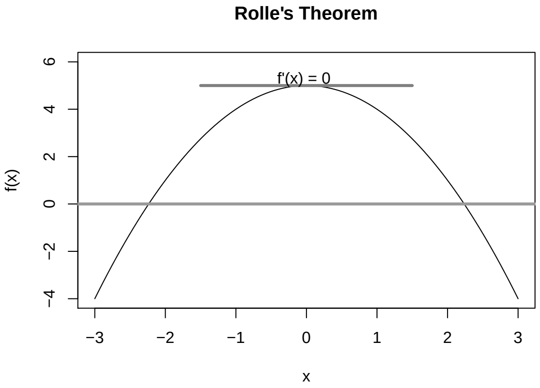
Goal: introduce ideas. Hopefully make them less shocking when you see them in work

Relative Maxima, Minima and Derivatives

Theorem

Suppose $f : [a, b] \rightarrow \Re$. Suppose f has a relative maxima or minima on (a, b) and call that $c \in (a, b)$. Then $f^{'}(c) = 0$.

Intuition:



Relative Maxima, Minima and Derivatives

Theorem

Rolle's Theorem Suppose $f : [a, b] \rightarrow \Re$ *and* f *is continuous on* $[a, b]$ *and differentiable on* (a, b)*. Then if* $f(a) = f(b) = 0$*, there is* $c \in (a, b)$ *such that* $f^{'}(c) = 0$*.*

Proof Intuition Consider (WLOG) a relative maximum c. Consider the left-hand and right-hand limits

$$\begin{aligned}
\lim_{x \rightarrow c^{-}} \frac{f(x) - f(c)}{x - c} & \geq 0 \\
\lim_{x \rightarrow c^{+}} \frac{f(x) - f(c)}{x - c} & \leq 0
\end{aligned}$$

Theorem

Rolle's Theorem *Suppose* $f : [a, b] \rightarrow \Re$ *and* f *is continuous on* $[a, b]$ *and differentiable on* (a, b). *Then if* $f(a) = f(b) = 0$, *there is* $c \in (a, b)$ *such that* $f^{'}(c) = 0$.

But we also know that

$$\begin{aligned}\lim_{x \rightarrow c^{-}} \frac{f(x) - f(c)}{x - c} &= f^{'}(c) \\ \lim_{x \rightarrow c^{+}} \frac{f(x) - f(c)}{x - c} &= f^{'}(c)\end{aligned}$$

The only way, then, that $\lim_{x \rightarrow c^{-}} \frac{f(x)-f(c)}{x-c} = \lim_{x \rightarrow c^{+}} \frac{f(x)-f(c)}{x-c}$ is if $f^{'}(c) = 0$.

What Goes Up Must Come Down

Theorem

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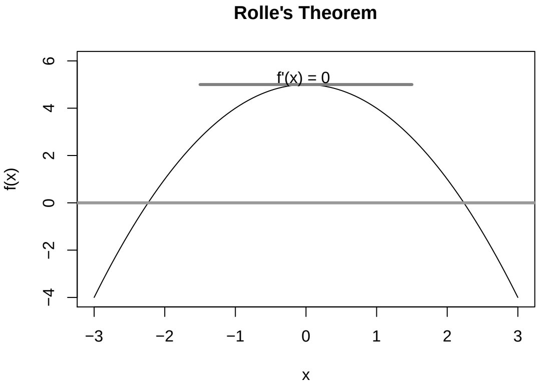


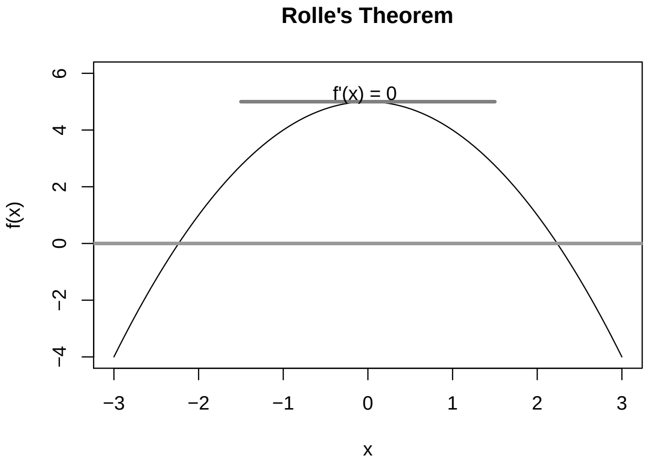
Mean Value Theorem

Theorem

If $f : [a, b] \rightarrow \Re$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem, Rotated



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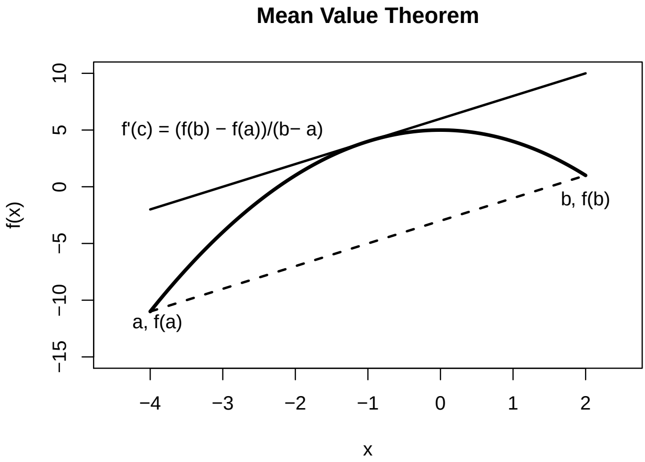

Why You Should Care

1) This will come up in a formal theory article. You'll at least know where to look
2) It allows us to say lots of powerful stuff about functions

Powerful Applications of Mean Value Theorem

Theorem

Suppose that $f : [a, b] \rightarrow \Re$ is continuous on $[a, b]$ and differentiable on (a, b). Then,

i) *If $f^{'}(x) \neq 0$ for all $x \in (a, b)$ then f is 1-1*
ii) *If $f^{'}(x) = 0$ then $f(x)$ is constant*
iii) *If $f^{'}(x) > 0$ for all $x \in (a, b)$ then then f is strictly increasing*
iv) *If $f^{'}(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing*

Let's prove these in turn

- Why—because they are just about applying ideas

If $f^{'}(x) \neq 0$ for all $x \in (a, b)$ then f is 1-1

By way of contradiction, suppose that f is not 1-1. Then there is $x, y \in (a, b)$ such that $f(x) = f(y)$. Then,

$$f^{'}(c) \quad = \quad \frac{f(x) - f(y)}{x - y} = \frac{0}{x - y} = 0$$

If $f^{'}(x) \neq 0$ for all $x \in (a, b)$ then f is 1-1

If $f^{'}(x) \neq 0$ for all $x \in (a, b)$ then f is 1-1

$f^{'} \neq 0$ for all x!

If $f^{'}(x) = 0$ then $f(x)$ is constant

By way of contradiction, suppose that there is $x, y \in (a, b)$ such that $f(x) \neq f(y)$. But then,

$$f^{'}(c) \quad = \quad \frac{f(x) - f(y)}{x - y} \neq 0$$

contradiction

If $f^{'}(x) > 0$ for all $x \in (a, b)$ then then f is strictly increasing

By way of contradiction, suppose that there is $x, y \in (a, b)$ with $y < x$ but $f(y) > f(x)$. But then,

$$f^{'}(c) = \frac{f(x) - f(y)}{x - y} < 0$$

contradiction

Bonus: proof for strictly decreasing

Approximating functions and second order conditions

Theorem

Taylor's Theorem *Suppose* $f : \Re \rightarrow \Re$, $f(x)$ *is infinitely differentiable function. Then, the taylor expansion of* $f(x)$ *around* a *is given by*

$$
\begin{aligned}
f(x) &= f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + \ldots \\
f(x) &= \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n
\end{aligned}
$$

Example Function

Suppose $a = 0$ and $f(x) = e^x$. Then,

$$\begin{aligned} f^{'}(x) &= e^x \\ f^{''}(x) &= e^x \\ \vdots & \vdots \vdots \\ f^{n}(x) &= e^x \end{aligned}$$

This implies

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots + \frac{x^n}{n!} + \ldots$$

Wrap up

Lots of territory.
What are your questions?

This Week

Lab Tonight!