

# Math Camp

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# Multivariable Calculus

Functions of many variables:

- 1) Policies may be multidimensional (policy provision and pork buy off)
- 2) Countries may invest in offensive and defensive resources for fighting wars
- 3) Ethnicity and resources could affect investment

Today:

- 0) Determinant
- 0) Eigenvector/Diagonalization
- 1) Multivariate functions
- 2) Partial Derivatives, Gradients, Jacobians, and Hessians
- 3) Total Derivative, Implicit Differentiation, Implicit Function Theorem
- 4) Multivariate Integration

# Determinant

Suppose we have a **square** ( $n \times n$ ) matrix  $A$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A determinant is a function that assigns a number to square matrices

# Determinant

Facts needed to define determinant :

## Definition

A *permutation* of the set of integers  $\{1, 2, \dots, J\}$  is an arrangement of these integers in some order without omissions or repetition.

For example, consider  $\{1, 2, 3, 4\}$

$\{3, 2, 1, 4\}$

$\{4, 3, 2, 1\}$

If we have  $J$  integers then there are  $J!$  permutations

# Determinant

## Definition

An *inversion* occurs when a larger integer occurs before a smaller integer in a permutation

*Even permutation: total inversions are even*

*Odd permutation: total inversions are odd*

## Count the inversions

$\{3, 2, 1\}$

$\{1, 2, 3\}$

$\{3, 1, 2\}$

$\{2, 1, 3\}$

$\{1, 3, 2\}$

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*For a square  $n \times n$  matrix  $A$ , we will call an **elementary product** an  $n$  element long product, with no two components coming from the same row or column. We will call a **signed** elementary product one that multiplies odd permutations of the column numbers by  $-1$ .*

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There are  $n!$  elementary products



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*Suppose  $A$  is an  $n \times n$  matrix. Define the determinant function  $\det(A)$  to be the sum of signed elementary products from  $A$ . Call  $\det(A)$  the **determinant** of  $A$*

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R Code!



# An Introduction to Eigenvectors, Values, and Diagonalization

## Definition

Suppose  $\mathbf{A}$  is an  $N \times N$  matrix and  $\lambda$  is a scalar.

If

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Then  $\mathbf{x}$  is an *eigenvector* and  $\lambda$  is the associated *eigenvalue*

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# An Introduction to Eigenvectors, Values, and Diagonalization

## Theorem

Suppose  $\mathbf{A}$  is an *invertible*  $N \times N$  matrix and further suppose that  $\mathbf{A}$  has  $N$  distinct eigenvalues and  $N$  linearly independent eigenvectors. Then we can write  $\mathbf{A}$  as,

$$\mathbf{A} = \mathbf{W} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix} \mathbf{W}^{-1}$$

where  $\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N)$  is an  $N \times N$  matrix with the  $N$  eigenvectors as column vectors.



Proof:  
Note

$$\begin{aligned}\mathbf{A}\mathbf{W} &= (\lambda_1 \mathbf{w}_1 \quad \lambda_2 \mathbf{w}_2 \quad \dots \quad \lambda_N \mathbf{w}_N) \\ &= \mathbf{W}\mathbf{\Lambda} \\ \mathbf{A} &= \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}\end{aligned}$$

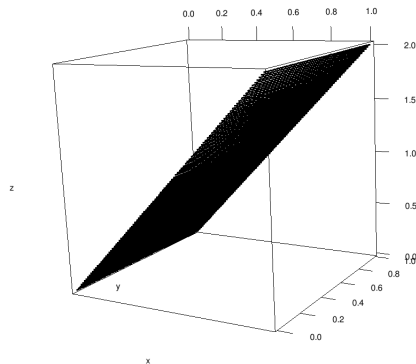
# Examples of Diagonalization

Suppose  $\mathbf{A}$  is an  $N \times N$  invertible matrix with eigenvalues  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  and eigenvectors  $\mathbf{W}$ . Calculate  $\mathbf{A}\mathbf{A} = \mathbf{A}^2$

$$\begin{aligned}\mathbf{A}\mathbf{A} &= \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}\mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1} \\ &= \mathbf{W} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix} \mathbf{W}^{-1} \\ &= \mathbf{W} \begin{pmatrix} \lambda_1^2 & 0 & \dots & 0 \\ 0 & \lambda_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N^2 \end{pmatrix} \mathbf{W}^{-1}\end{aligned}$$

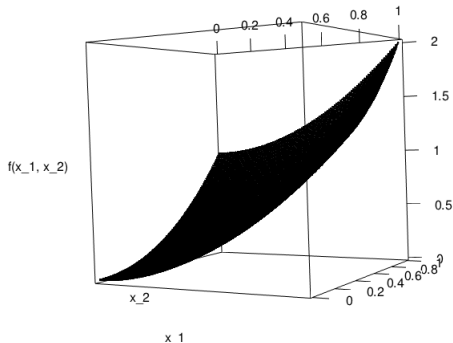
# Multivariate Functions

$$f(x_1, x_2) = x_1 + x_2$$



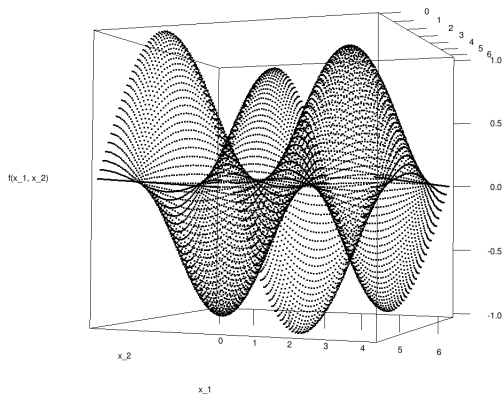
# Multivariate Functions

$$f(x_1, x_2) = x_1^2 + x_2^2$$



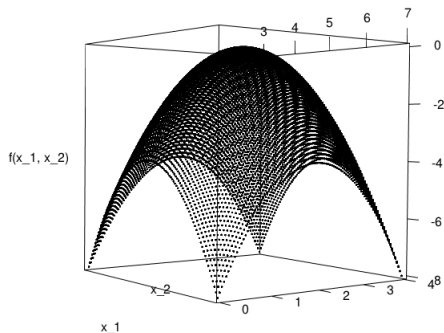
# Multivariate Functions

$$f(x_1, x_2) = \sin(x_1) \cos(x_2)$$



# Multivariate Functions

$$f(x_1, x_2) = -(x - 5)^2 - (y - 2)^2$$



# Multivariate Functions

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

# Multivariate Functions

$$\begin{aligned} f(\mathbf{x}) &= f(x_1, x_2, \dots, x_N) \\ &= x_1 + x_2 + \dots + x_N \\ &= \sum_{i=1}^N x_i \end{aligned}$$



# Multivariate Functions

## Definition

Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ . We will call  $f$  a *multivariate* function. We will commonly write,

$$f(\mathbf{x}) = f(x_1, x_2, x_3, \dots, x_n)$$

-  $\mathbb{R}^n = \mathbb{R} \underbrace{\times}_{\text{cartesian}} \mathbb{R} \times \mathbb{R} \times \dots \mathbb{R}$

- The function we consider will take  $n$  inputs and output a single number (that lives in  $\mathbb{R}^1$ , or the real line)

# Example 1

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

Evaluate at  $\mathbf{x} = (x_1, x_2, x_3) = (2, 3, 2)$

$$\begin{aligned} f(2, 3, 2) &= 2 + 3 + 2 \\ &= 7 \end{aligned}$$

# Example 1

$$f(x_1, x_2) = x_1 + x_2 + x_1 x_2$$

Evaluate at  $\mathbf{w} = (w_1, w_2) = (1, 2)$

$$\begin{aligned} f(w_1, w_2) &= w_1 + w_2 + w_1 w_2 \\ &= 1 + 2 + 1 \times 2 \\ &= 5 \end{aligned}$$

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$$\begin{aligned} f(T, x_2) &= \Pr(\text{Vote}_i = 1 | T, x_2) \\ &= \beta_0 + \beta_1 T + \beta_2 x_2 \end{aligned}$$

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# Regression Models and Randomized Treatments

Often we administer randomized experiments:

The most recent wave of interest began with **voter mobilization**, and wonder if individual  $i$  turns out to vote,  $\text{Vote}_i$

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$$f(T = 1, x_2) - f(T = 0, x_2) = \beta_0 + \beta_1 1 + \beta_2 x_2 - (\beta_0 + \beta_1 0 + \beta_2 x_2)$$



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# Multivariate Derivative

## Definition

Suppose  $f : X \rightarrow \mathbb{R}^1$ , where  $X \subset \mathbb{R}^n$ .  $f(\mathbf{x}) = f(x_1, x_2, \dots, x_N)$ . If the limit,

$$\begin{aligned}\frac{\partial}{\partial x_i} f(\mathbf{x}_0) &= \frac{\partial}{\partial x_i} f(x_{01}, x_{02}, \dots, x_{0i}, x_{0i+1}, \dots, x_{0N}) \\ &= \lim_{h \rightarrow 0} \frac{f(x_{01}, x_{02}, \dots, x_{0i} + h, \dots, x_{0N}) - f(x_{01}, x_{02}, \dots, x_{0i}, \dots, x_{0N})}{h}\end{aligned}$$

exists then we call this the partial derivative of  $f$  with respect to  $x_i$  at the value  $\mathbf{x}_0 = (x_{01}, x_{02}, \dots, x_{0N})$ .

# Rules for Taking Partial Derivatives

Partial Derivative:  $\frac{\partial f(\mathbf{x})}{\partial x_i}$

- Treat each instance of  $x_i$  as a **variable** that we would differentiate before
- Treat each instance of  $\mathbf{x}_{-i} = (x_1, x_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  as a **constant**

# Example Partial Derivatives

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2) \\ &= x_1 + x_2\end{aligned}$$

Partial derivative, with respect to  $x_1$  at  $(x_{01}, x_{02})$

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_1} \Big|_{(x_{01}, x_{02})} &= 1 + 0 \Big|_{x_{01}, x_{02}} \\ &= 1\end{aligned}$$

# Example Partial Derivatives

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, x_3) \\ &= x_1^2 \log(x_1) + x_2 x_1 x_3 + x_3^2\end{aligned}$$

# Example Partial Derivatives

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, x_3) \\&= x_1^2 \log(x_1) + x_2 x_1 x_3 + x_3^2\end{aligned}$$

What is the partial derivative with respect to  $x_1$ ?  
 $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$ .

Evaluated at

$$\begin{aligned}\frac{\partial f(\mathbf{x})}{\partial x_1} \Big|_{\mathbf{x}_0} &= 2x_1 \log(x_1) + x_1^2 \frac{1}{x_1} + x_2 x_3 \Big|_{\mathbf{x}_0} \\&= 2x_{01} \log(x_{01}) + x_{01} + x_{02} x_{03}\end{aligned}$$

# Example Partial Derivatives

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, x_3) \\ &= x_1^2 \log(x_1) + x_2 x_1 x_3 + x_3^2\end{aligned}$$

What is the partial derivative with respect to  $x_1$ ?  $x_2$ ?      Evaluated at  $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$ .

$$\begin{aligned}\frac{\partial f(\mathbf{x})}{\partial x_2} \Big|_{\mathbf{x}_0} &= x_1 x_3 \Big|_{\mathbf{x}_0} \\ &= x_{01} x_{03}\end{aligned}$$



# Example Partial Derivatives

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What is the partial derivative with respect to  $x_1$ ?  $x_2$ ?  $x_3$ ? Evaluated at  $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$ .

$$\begin{aligned}\frac{\partial f(\mathbf{x})}{\partial x_3} \Big|_{\mathbf{x}_0} &= x_1 x_2 + 2x_3 \Big|_{\mathbf{x}_0} \\ &= x_{01} x_{02} + 2x_{03}\end{aligned}$$

# Rate of Change, Linear Regression

Suppose we regress **Approval**<sub>*i*</sub> rate for Obama in month *i* on Employ<sub>*i*</sub> and Gas<sub>*i*</sub>. We obtain the following model:

$$\text{Approval}_i = 0.8 - 0.5\text{Employ}_i - 0.25\text{Gas}_i$$

We are modeling  $\text{Approval}_i = f(\text{Employ}_i, \text{Gas}_i)$ . What is partial derivative with respect to employment?

$$\frac{\partial f(\text{Employ}_i, \text{Gas}_i)}{\partial \text{Employ}_i} = -0.5$$

# Gradient

## Definition

Suppose  $f : X \rightarrow \mathbb{R}^1$  with  $X \subset \mathbb{R}^n$  is a differentiable function. Define the gradient vector of  $f$  at  $\mathbf{x}_0$ ,  $\nabla f(\mathbf{x}_0)$  as,

$$\nabla f(\mathbf{x}_0) = \left( \frac{\partial f(\mathbf{x}_0)}{\partial x_1}, \frac{\partial f(\mathbf{x}_0)}{\partial x_2}, \frac{\partial f(\mathbf{x}_0)}{\partial x_3}, \dots, \frac{\partial f(\mathbf{x}_0)}{\partial x_n} \right)$$

- The gradient points in the direction that the function is **increasing** in the fastest direction
- We'll use this to do optimization (both analytic and computational)

# Example Gradient Calculation

Suppose

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, \dots, x_n) \\&= x_1^2 + x_2^2 + \dots + x_n^2 \\&= \sum_{i=1}^n x_i^2\end{aligned}$$

Then  $\nabla f(\mathbf{x}^*)$  is

$$\nabla f(\mathbf{x}^*) = (2x_1^*, 2x_2^*, \dots, 2x_n^*)$$

So if  $\mathbf{x}^* = (3, 3, \dots, 3)$  then

$$\begin{aligned}\nabla f(\mathbf{x}^*) &= (2 * 3, 2 * 3, \dots, 2 * 3) \\&= (6, 6, \dots, 6)\end{aligned}$$

# Second Partial Derivative

## Definition

Suppose  $f : X \rightarrow \mathbb{R}$  where  $X \subset \mathbb{R}^n$  and suppose that  $\frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i}$  exists. Then we define,

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i} \equiv \frac{\partial}{\partial x_j} \left( \frac{\partial f(\mathbf{x})}{\partial x_i} \right)$$

- Second derivative could be with respect to  $x_i$  or with some other variable  $x_j$
- Nagging question: does order matter?

# Second Partial Derivative: Order Doesn't Matter

## Theorem

*Young's Theorem* Let  $f : X \rightarrow \mathbb{R}$  with  $X \subset \mathbb{R}^n$  be a twice differentiable function on all of  $X$ . Then for any  $i, j$ , at all  $\mathbf{x}^* \in X$ ,

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}^*) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{x}^*)$$

# Second Order Partial Derivates

$$f(\mathbf{x}) = x_1^2 x_2^2$$

Then,

$$\frac{\partial^2}{\partial x_1 \partial x_1} f(\mathbf{x}) = 2x_2^2$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} f(\mathbf{x}) = 4x_1 x_2$$

$$\frac{\partial^2}{\partial x_2 \partial x_2} f(\mathbf{x}) = 2x_1^2$$

# Hessians

## Definition

Suppose  $f : X \rightarrow \mathbb{R}^1$ ,  $X \subset \mathbb{R}^n$ , with  $f$  a twice differentiable function. We will define the **Hessian** matrix as the matrix of second derivatives at  $\mathbf{x}^* \in X$ ,

$$\mathbf{H}(f)(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}^*) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}^*) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}^*) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Hessians are **symmetric**
- They describe **curvature** of a function (think, how bended)
- Will be the basis for second derivative test for multivariate optimization



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Suppose  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , with

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$$H(f)(\mathbf{x}) = \begin{pmatrix} 2x_2^2 x_3^2 & 4x_1 x_2 x_3^2 & 4x_1 x_2^2 x_3 \\ 4x_1 x_2 x_3^2 & 2x_1^2 x_3^2 & 4x_1^2 x_2 x_3 \\ 4x_1 x_2^2 x_3 & 4x_1^2 x_2 x_3 & 2x_1^2 x_2^2 \end{pmatrix}$$

# Functions with Multidimensional Codomains

## Definition

Suppose  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . We will call  $f$  a *multivariate* function. We will commonly write,

$$f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

# Example Functions

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}^2$ ,

$$f(t) = (t^2, \sqrt{t})$$

# Example Functions

Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined as

$$f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$



# Example Functions

Suppose we have some policy  $\mathbf{x} \in \Re^M$ . Suppose we have  $N$  legislators where legislator  $i$  has utility

$$U_i(\mathbf{x}) = \sum_{j=1}^M -(x_j - \mu_{ij})^2$$

We can describe the utility of all legislators to the proposal as

$$f(\mathbf{x}) = \begin{pmatrix} \sum_{j=1}^M -(x_j - \mu_{1j})^2 \\ \sum_{j=1}^M -(x_j - \mu_{2j})^2 \\ \vdots \\ \sum_{j=1}^M -(x_j - \mu_{Nj})^2 \end{pmatrix}$$

# Jacobian

## Definition

Suppose  $f : X \rightarrow \mathbb{R}^n$ , where  $X \subset \mathbb{R}^m$ , with  $f$  a differentiable function. Define the **Jacobian** of  $f$  at  $\mathbf{x}$  as

$$\mathbf{J}(f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

# Example of Jacobian

$$f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

$$\mathbf{J}(f)(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

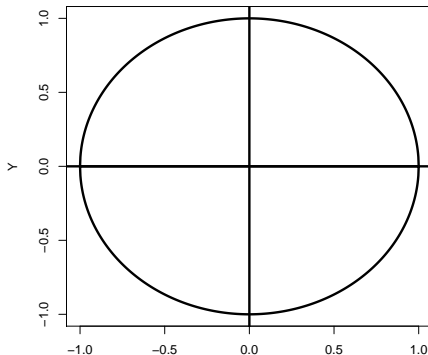
# Implicit Functions and Differentiation

We have defined functions **explicitly**

$$Y = f(x)$$

We might also have an **implicit** function:

$$1 = x^2 + y^2$$



# Implicit Function Theorem (From Avi Acharya's Notes)

## Definition

Suppose  $X \subset \mathbb{R}^m$  and  $Y \subset \mathbb{R}$ . Let  $f : X \cup Y \rightarrow \mathbb{R}$  be a differentiable function (with continuous partial derivatives). Let  $(\mathbf{x}^*, y^*) \in X \cup Y$  such that

$$\begin{aligned}\frac{\partial f(\mathbf{x}^*, y^*)}{\partial y} &\neq 0 \\ f(\mathbf{x}^*, y^*) &= 0\end{aligned}$$

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Then there exists  $B \subset \mathbb{R}^n$  such that there is a differentiable function  $g : B \rightarrow \mathbb{R}$  such that  $\mathbf{x}^* \in B$  then  $g(\mathbf{x}^*) = y^*$  and  $f(\mathbf{x}, g(\mathbf{x})) = 0$ . The derivative of  $g$  for  $\mathbf{x} \in B$  is given by

$$\frac{\partial g}{\partial x_j} = -\frac{\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial y}}$$

# Example 1: Implicit Function Theorem

Suppose that the equation is

$$1 = x^2 + y^2$$

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$$\frac{\partial f}{\partial x} = 2x$$

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$$\begin{aligned} \frac{\partial g(x)}{\partial x} \Big|_{x_0} &= -\frac{\partial f / \partial x}{\partial f / \partial y} \\ &= -\frac{2x_0}{2y} = -\frac{x_0}{\sqrt{1-x_0^2}} \text{ if } y > 0 \end{aligned}$$

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As  $x \rightarrow 1$  or  $x \rightarrow -1$  this derivative diverges  
The intuition from the Implicit Function Theorem is that any function  $g(x) = y$  there would need an “infinite” slope.

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$$\frac{\partial f(x, y) / \partial x}{\partial f(x, y) / \partial y} = \frac{2x}{-1} = - \frac{\partial y}{\partial x}$$

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In this example, the negative sign is “moving things to the other side”. In general, the negative sign will capture that we want to measure the **compensatory** behavior of the function: how  $y$  moves in response to some  $x_i$  **along a level curve**

## Example 2: Implicit Function Theorem (From Jim Fearon)

Suppose there  $n$  individuals, each individual  $i$  earns pre-tax income  $y_i > 0$ .

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Suppose:

$$U_i(t, y_i) = y_i(1 - t^2) + t\bar{y}$$

## Example 2: Implicit Function Theorem (From Jim Fearon)

An individual's optimal tax rate is:

$$\begin{aligned}\frac{\partial U_i(t, y_i)}{\partial t} &= -2y_i t + \bar{y} \\ 0 &= -2y_i t^* + \bar{y} \\ \frac{\bar{y}}{2y_i} &= t_i^*\end{aligned}$$

Checking the second derivative:

$$\frac{\partial^2 U_i(t, y_i)}{\partial^2 t} = -2y_i$$

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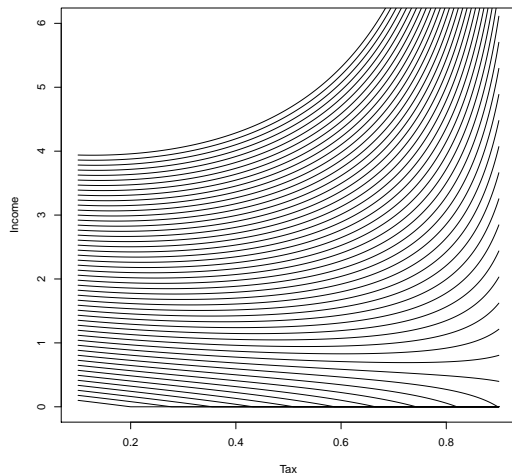
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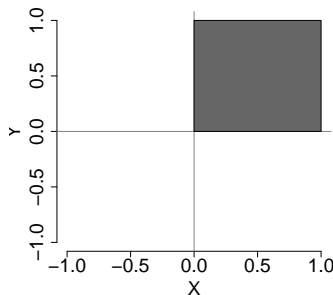
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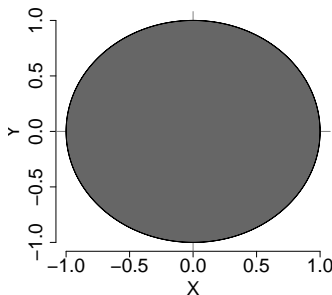
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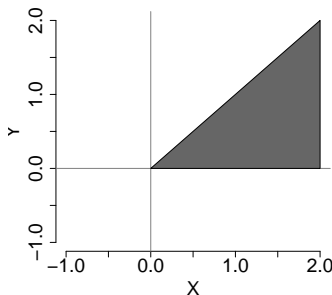
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How do calculate the area under the function over these regions?

# Multivariate Integration

## Definition

*Suppose  $f : X \rightarrow \mathbb{R}$  where  $X \subset \mathbb{R}^n$ . We will say that  $f$  is integrable over  $A \subset X$  if we are able to calculate its area with refined partitions of  $A$  and we will write the integral  $I = \int_A f(\mathbf{x}) d\mathbf{A}$*

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## Theorem

**Fubini's Theorem** Suppose  $A = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$  and that  $f : A \rightarrow \mathbb{R}$  is **integrable**. Then

$$\int_A f(\mathbf{x}) d\mathbf{A} = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\mathbf{x}) dx_1 dx_2 \dots dx_{n-1} dx_n$$

# Multivariate Integration Recipe

$$\int_A f(\mathbf{x}) d\mathbf{A} = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\mathbf{x}) dx_1 dx_2 \dots dx_{n-1} dx_n$$

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- 2) Work inside to out, **iterating**
- 3) At the last step, we should arrive at a number



**Intuition:** Three Dimensional Jello Molds, a discussion

# Multivariate Uniform Distribution

Suppose  $f : [0, 1] \times [0, 1] \rightarrow \mathfrak{R}$  and  $f(x_1, x_2) = 1$  for all  $x_1, x_2 \in [0, 1] \times [0, 1]$ . What is  $\int_0^1 \int_0^1 f(x) dx_1 dx_2$ ?

$$\begin{aligned} \int_0^1 \int_0^1 f(x) dx_1 dx_2 &= \int_0^1 \int_0^1 1 dx_1 dx_2 \\ &= \int_0^1 x_1 \Big|_0^1 dx_2 \\ &= \int_0^1 (1 - 0) dx_2 \\ &= \int_0^1 1 dx_2 \\ &= x_2 \Big|_0^1 \\ &= 1 \end{aligned}$$

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Suppose  $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  and that

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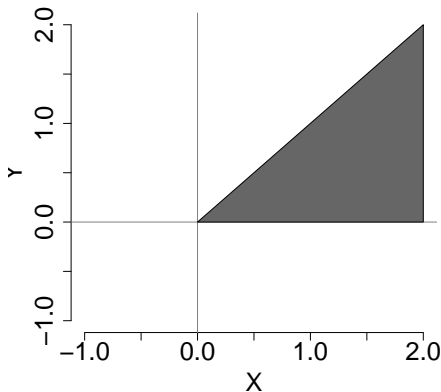
# Challenge Problems

- 1) Find  $\int_0^1 \int_0^1 x_1 + x_2 dx_1 dx_2$
- 2) Demonstrate that

$$\int_0^b \int_0^a x_1 - 3x_2 dx_1 dx_2 = \int_0^a \int_0^b x_1 - 3x_2 dx_2 dx_1$$

# More Complicated Bounds of Integration

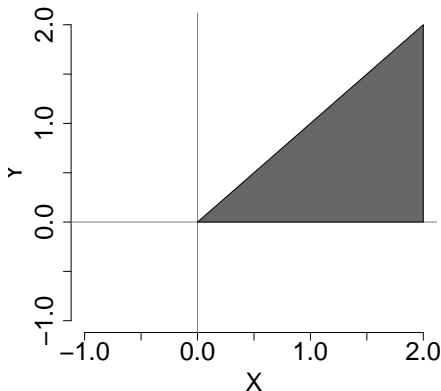
So far, we have integrated over **rectangles**. But often, we are interested in more complicated regions



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## Example 4: More Complicated Regions

Suppose  $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ ,  $f(x_1, x_2) = x_1 + x_2$ . Find area of function where  $x_1 < x_2$ .

**Trick:** we need to determine bound. If  $x_1 < x_2$ ,  $x_1$  can take on any value from 0 to  $x_2$

$$\begin{aligned}\iint_{x_1 < x_2} f(\mathbf{x}) &= \int_0^1 \int_0^{x_2} x_1 + x_2 dx_1 dx_2 \\&= \int_0^1 x_2 x_1 \Big|_0^{x_2} dx_2 + \int_0^1 \frac{x_1^2}{2} \Big|_0^{x_2} \\&= \int_0^1 x_2^2 dx_2 + \int_0^1 \frac{x_2^2}{2} \\&= \frac{x_2^3}{3} \Big|_0^1 + \frac{x_2^3}{6} \Big|_0^1 \\&= \frac{1}{3} + \frac{1}{6} \\&= \frac{3}{6} = \frac{1}{2}\end{aligned}$$

Consider the same function and let's switch the bounds.

$$\begin{aligned}\iint_{x_1 < x_2} f(\mathbf{x}) &= \int_0^1 \int_{x_1}^1 x_1 + x_2 dx_2 dx_1 \\&= \int_0^1 x_1 x_2 \Big|_{x_1}^1 + \int_0^1 \frac{x_2^2}{2} \Big|_{x_1}^1 dx_1 \\&= \int_0^1 x_1 - x_1^2 + \int_0^1 \frac{1}{2} - \frac{x_1^2}{2} dx_1 \\&= \frac{x_1^2}{2} \Big|_0^1 - \frac{x_1^3}{3} \Big|_0^1 + \frac{x_1}{2} \Big|_0^1 - \frac{x_1^3}{6} \Big|_0^1 \\&= \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{6} \\&= 1 - \frac{3}{6} \\&= \frac{1}{2}\end{aligned}$$



## Example 5: More Complicated Regions

Suppose  $f[0, 1] \times [0, 1] \rightarrow \mathbb{R}$ ,  $f(x_1, x_2) = 1$ . What is the area of  $x_1 + x_2 < 1$ ? Where is  $x_1 + x_2 < 1$ ? Where,  $x_1 < 1 - x_2$

$$\begin{aligned}\iint_{x_1+x_2<1} f(\mathbf{x}) d\mathbf{x} &= \int_0^1 \int_0^{1-x_2} 1 dx_1 dx_2 \\ &= \int_0^1 x_1 \Big|_0^{1-x_2} dx_2 \\ &= \int_0^1 (1 - x_2) dx_2 \\ &= x_2 \Big|_0^1 - \frac{x_2^2}{2} \Big|_0^1 \\ &= 1 - \left(\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$