Math Camp

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Where We've Been, Where We're Going

Calculus: Analyze behavior of functions on real line

- Convergence
- Differentiation
- Integration

Linear Algebra

- Data stored in matrices
- Higher dimensional spaces
  - complex world, condition on many factors
  - flood of big data, store in many dimensions
- Linear Algebra:
  - Algebra of matrices
  - Geometry of high dimensional space
  - Calculus (multivariable) in many dimensions

Very important for regression(!!!!)

Points + Vectors

- A point in $\Re^1$
    - 1
    - π
    - e
- An ordered pair in $\Re^2 = \Re \times \Re$
    - $(1, 2)$
    - $(0, 0)$
    - (π, e)
- An ordered triple in $\Re^3 = \Re \times \Re \times \Re$
    - $(3.1, 4.5, 6.11132)$

$\vdots$

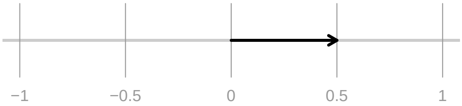
- An ordered n-tuple in $\Re^n = \Re \times \Re \times \ldots \times \Re$
    - $(a_1, a_2, \ldots, a_n)$

Points and Vectors

Definition

A point $\boldsymbol{x} \in \Re^n$ *is an ordered n-tuple,* $(x_1, x_2, \ldots, x_n)$. *The vector* $\boldsymbol{x} \in \Re^n$ *is the arrow pointing from the origin* $(0, 0, \ldots, 0)$ *to* $\boldsymbol{x}$.

One Dimensional Example



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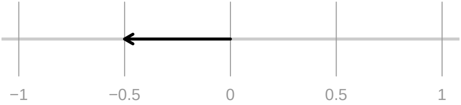

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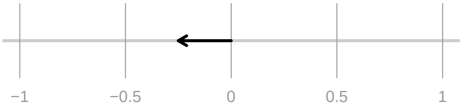
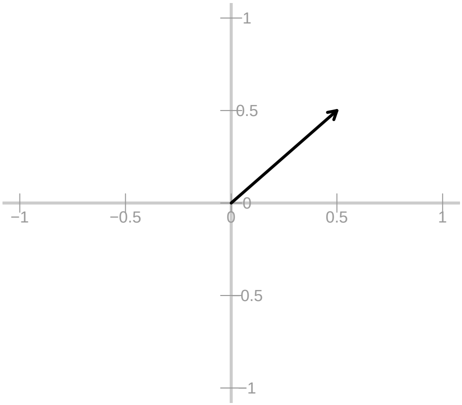


Two Dimensional Example



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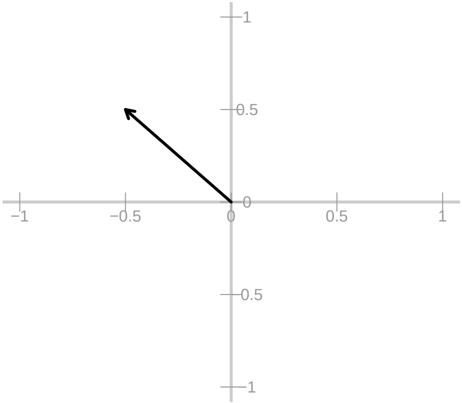

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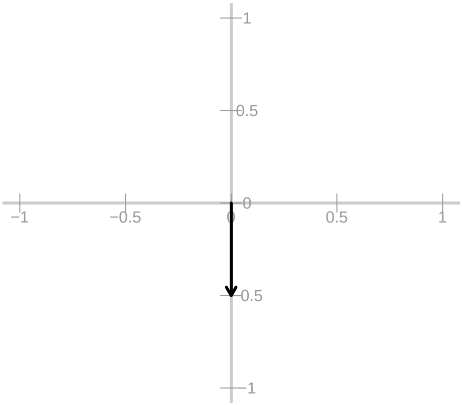

Three Dimensional Example

- (Latitude, Longitude, Elevation)
- (1, 2, 3)
- (0, 1, 0)

N-Dimensional Example

- Individual campaign donation records

$$\boldsymbol{x} = (1000, 0, 10, 50, 15, 4, 0, 0, 0, \ldots, 2400000000)$$

- Counties have proportion of vote for Obama

$$\boldsymbol{y} = (0.8, 0.5, 0.6, \ldots, 0.2)$$

- Run experiment, assess feeling thermometer of elected official

$$\boldsymbol{t} = (0, 100, 50, 70, 80, \ldots, 100)$$

Arithmetic with Vectors

Definition

Suppose $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are vectors* $\boldsymbol{u}, \boldsymbol{v} \in \Re^n$,

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3, \ldots, u_n) \\ \boldsymbol{v} &= (v_1, v_2, v_3, \ldots, v_n) \end{aligned}$$

We will say $\boldsymbol{u} = \boldsymbol{v}$ *if* $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$
Define the sum of $\boldsymbol{u} + \boldsymbol{v}$ *as*

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n)$$

Suppose $k \in \Re$. *We will call* k *a scalar.*
Define $k\boldsymbol{u}$ *as the scalar product*

$$k\boldsymbol{u} = (ku_1, ku_2, \ldots, ku_n)$$

Examples

Suppose:

$$\begin{aligned} \boldsymbol{u} &= (1, 2, 3, 4, 5) \\ \boldsymbol{v} &= (1, 1, 1, 1, 1) \\ k &= 2 \end{aligned}$$

Then,

$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= (1 + 1, 2 + 1, 3 + 1, 4 + 1, 5 + 1) = (2, 3, 4, 5, 6) \\ k\boldsymbol{u} &= (2 \times 1, 2 \times 2, 2 \times 3, 2 \times 4, 2 \times 5) = (2, 4, 6, 8, 10) \\ k\boldsymbol{v} &= (2 \times 1, 2 \times 1, 2 \times 1, 2 \times 1, 2 \times 1) = (2, 2, 2, 2, 2) \end{aligned}$$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ *and* k *and* l *are scalars.*

a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ and k and l are scalars.

a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

Proof.

$$
\begin{aligned}
\boldsymbol{u} + \boldsymbol{v} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \\
&= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n) \\
&= \boldsymbol{v} + \boldsymbol{u}
\end{aligned}
$$

□

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ and k and l are scalars.

b) $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ and k and l are scalars.

b) $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$

Proof.

$$\begin{aligned}
\boldsymbol{u} + \boldsymbol{0} &= (u_1 + 0, u_2 + 0, \ldots, u_n + 0) \\
&= (0 + u_1, 0 + u_2, \ldots, 0 + u_n) = \boldsymbol{0} + \boldsymbol{u} \\
&= (u_1, u_2, \ldots, u_n) \\
&= \boldsymbol{u}
\end{aligned}$$

□

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ and k and l are scalars.

c) $(l + k)\boldsymbol{u} = l(\boldsymbol{u}) + k(\boldsymbol{u})$

Proof.

How can we show this? □

Challenge Proofs

- Show that $1\boldsymbol{u} = \boldsymbol{u}$
- Show that $\boldsymbol{u} + -1\boldsymbol{u} = \mathbf{0}$

Inner Product

Definition

Suppose $\boldsymbol{u} \in \Re^n$ and $\boldsymbol{v} \in \Re^n$ then define $\boldsymbol{u} \cdot \boldsymbol{v}$,

$$
\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{v} &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \\
&= \sum_{i=1}^{N} u_i v_i
\end{aligned}
$$

Examples

Suppose $\boldsymbol{u} = (1, 2, 3)$ and $\boldsymbol{v} = (2, 3, 1)$. Then,

$$
\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{v} &= 1 \times 2 + 2 \times 3 + 3 \times 1 \\
&= 2 + 6 + 3 \\
&= 11
\end{aligned}
$$

Suppose $\boldsymbol{y} = (y_1, y_2, \ldots, y_N)$ and $\mathbf{1} = (1, 1, 1, \ldots, 1)$. Then,

$$
\begin{aligned}
\boldsymbol{y} \cdot \mathbf{1} &= y_1 + y_2 + \ldots + y_n \\
&= \sum_{i=1}^{n} y_i
\end{aligned}
$$

R Code

Create a vector in R

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
```

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Create a vector in R

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vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
```

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Create a vector in R

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vec[5]<- 5
x1<- c(2, 2, 3, 2)
```

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Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
x1<- c(2, 2, 3, 2)
x2<- c(5, 3, 1, 3)
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
x1<- c(2, 2, 3, 2)
x2<- c(5, 3, 1, 3)
add <- x1 + x2
```

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add
[1] 7 5 4 5
```

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```

```
scalar<- 10 *x1
```

R Code

Create a vector in R

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[1] 7 5 4 5
```

```
scalar<- 10 *x1
scalar
[1] 20 20 30 20
```

R Code

Create a vector in R

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x2<- c(5, 3, 1, 3)
add <- x1 + x2
add
[1] 7 5 4 5
```

```
scalar<- 10 *x1
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[1] 20 20 30 20
output<- x1 %*% x2
```

R Code

Create a vector in R

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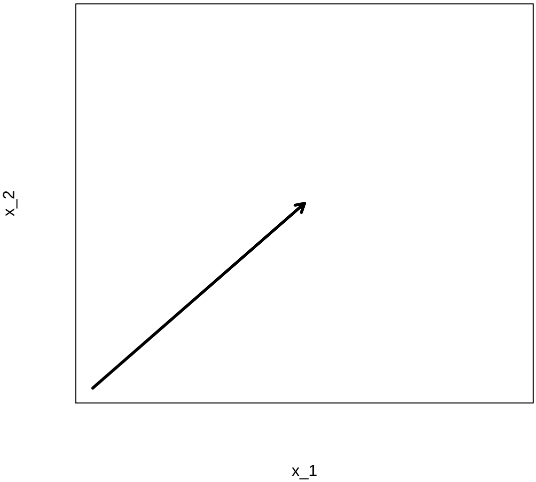
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
x1<- c(2, 2, 3, 2)
x2<- c(5, 3, 1, 3)
add <- x1 + x2
add
[1] 7 5 4 5
```

```
scalar<- 10 *x1
scalar
[1] 20 20 30 20
output<- x1 %*% x2
output
[,1]
[1,] 25
```

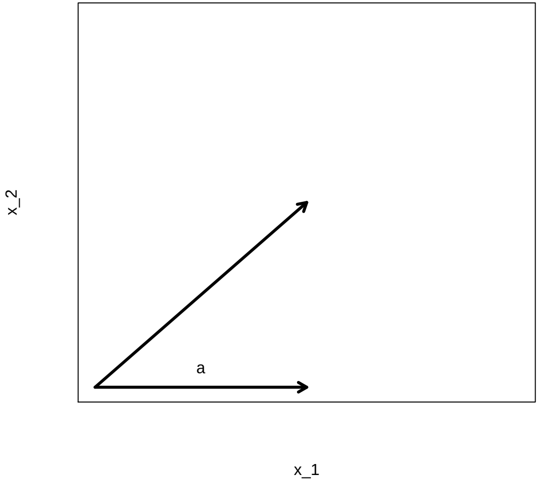
Challenge Problems

- Suppose $\boldsymbol{v} = (1, 4, 1, 4)$ and $\boldsymbol{w} = (4, 1, 4, 1)$. Calculate: $\boldsymbol{v} \cdot \boldsymbol{w}$
- Prove $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$
- Super hard: prove $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$.

Vector Length

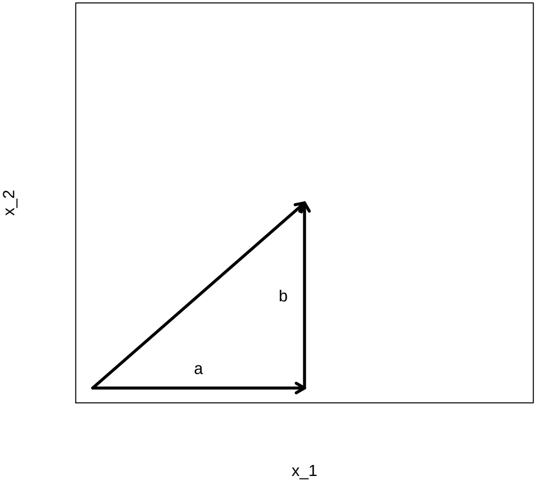


Vector Length



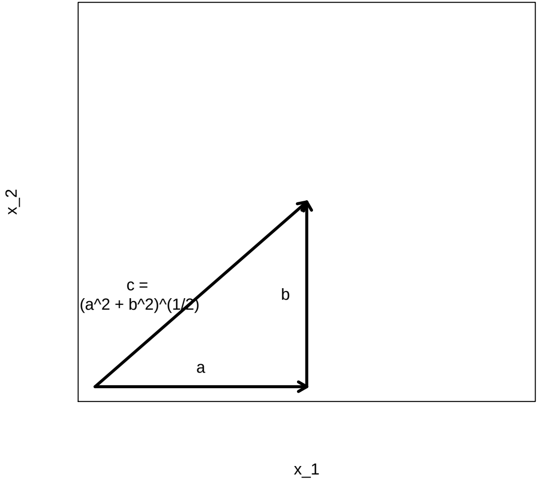
- Pythogorean Theorem: Side with length a

Vector Length



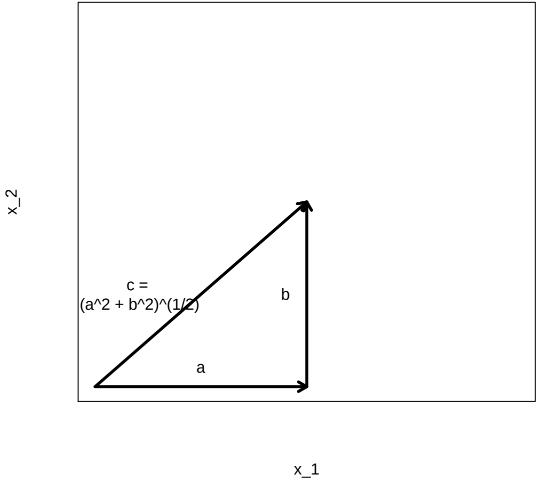
- Pythogorean Theorem: Side with length a
- Side with length b and right triangle

Vector Length



- Pythogorean Theorem: Side with length a
- Side with length b and right triangle
- $c = \sqrt{a^2 + b^2}$

Vector Length



- Pythogorean Theorem: Side with length a
- Side with length b and right triangle
- $c = \sqrt{a^2 + b^2}$
- This is generally true

Vector Length

Definition

Suppose $\boldsymbol{v} \in \Re^n$*. Then, we will define its length as*

$$\begin{aligned} ||\boldsymbol{v}|| &= (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \\ &= (v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2)^{1/2} \end{aligned}$$

Calculating a Length

Example 1: suppose $\boldsymbol{x} = (1, 1, 1)$.

$$\begin{aligned} ||\boldsymbol{x}|| &= (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2} \\ &= (1 + 1 + 1)^{1/2} \\ &= \sqrt{3} \end{aligned}$$

Example 2: R code for length function

```
len.vec<- function(x) {
p1< - sqrt(x% * %x)
return(p1)
}
x <- c(1,1,1)
len.vec(x)
[,1]
[1,] 1.732051
```

Coding Problem

Let's calculate the length of some vectors

- Write a function to assess the length of a vector.
- Use it to calculate the length of:
    - `y<- c(10, 20, 30, 40)`
    - `x<- seq(1, 1000*pi, len=1000)`

Texts in Space

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$$\text{Doc1} \quad = \quad (1, 1, 3, \ldots, 5)$$

Texts in Space

$$\begin{aligned} \text{Doc1} &= (1, 1, 3, \ldots, 5) \\ \text{Doc2} &= (2, 0, 0, \ldots, 1) \end{aligned}$$

Texts in Space

$$\begin{aligned}
\text{Doc1} &= (1, 1, 3, \ldots, 5) \\
\text{Doc2} &= (2, 0, 0, \ldots, 1) \\
\textbf{Doc1}, \textbf{Doc2} &\in \Re^{M}
\end{aligned}$$

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Provides many operations that will be useful

Texts in Space

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Provides many operations that will be useful
Inner Product between documents:

Texts in Space

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Provides many operations that will be useful

Inner Product between documents:

$$\textbf{Doc1} \cdot \textbf{Doc2} = (1, 1, 3, \ldots, 5)^{'}(2, 0, 0, \ldots, 1)$$

Texts in Space

$$
\begin{aligned}
\text{Doc1} &= (1, 1, 3, \ldots, 5) \\
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\textbf{Doc1}, \textbf{Doc2} &\in \Re^M
\end{aligned}
$$

Provides many operations that will be useful

Inner Product between documents:

$$
\begin{aligned}
\textbf{Doc1} \cdot \textbf{Doc2} &= (1, 1, 3, \ldots, 5)^{'}(2, 0, 0, \ldots, 1) \\
&= 1 \times 2 + 1 \times 0 + 3 \times 0 + \ldots + 5 \times 1
\end{aligned}
$$

Texts in Space

$$
\begin{aligned}
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Provides many operations that will be useful

Inner Product between documents:

$$
\begin{aligned}
\textbf{Doc1} \cdot \textbf{Doc2} &= (1, 1, 3, \ldots, 5)^{'} (2, 0, 0, \ldots, 1) \\
&= 1 \times 2 + 1 \times 0 + 3 \times 0 + \ldots + 5 \times 1 \\
&= 7
\end{aligned}
$$

Length of document:

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$$
\begin{aligned}
||\mathbf{Doc1}|| &\equiv \sqrt{\mathbf{Doc1} \cdot \mathbf{Doc1}} \\
&= \sqrt{(1, 1, 3, \ldots, 5)'(1, 1, 3, \ldots, 5)} \\
&= \sqrt{1^2 + 1^2 + 3^2 + 5^2} \\
&= 6
\end{aligned}
$$

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Cosine of the angle between documents:

Length of document:

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&= \sqrt{(1, 1, 3, \ldots, 5)'(1, 1, 3, \ldots, 5)} \\
&= \sqrt{1^2 + 1^2 + 3^2 + 5^2} \\
&= 6
\end{aligned}$$

Cosine of the angle between documents:

$$\begin{aligned}
\cos\theta &\equiv \left(\frac{\mathbf{Doc1}}{||\mathbf{Doc1}||}\right) \cdot \left(\frac{\mathbf{Doc2}}{||\mathbf{Doc2}||}\right) \\
&= \frac{7}{6 \times 2.24} \\
&= 0.52
\end{aligned}$$

Measuring Similarity

Documents in space $\rightarrow$ measure similarity/dissimilarity

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What properties should similarity measure have?

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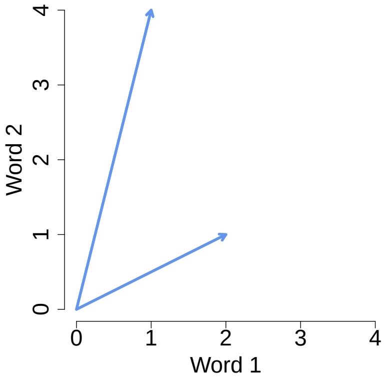
Documents in space $\rightarrow$ measure similarity/dissimilarity

What properties should similarity measure have?

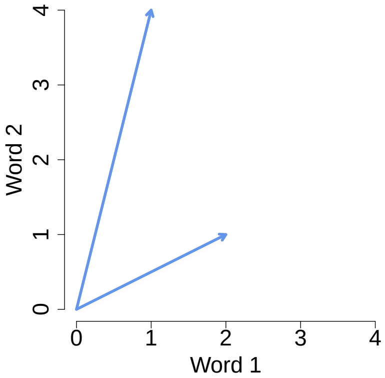
- Maximum: document with itself
- Minimum: documents have no words in common (orthogonal)
- Increasing when more of same words used
- ? $s(a, b) = s(b, a)$.

Measuring Similarity



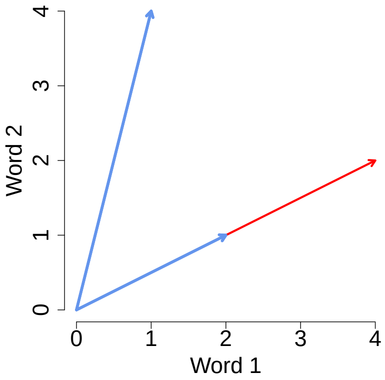
Measure 1: Inner product

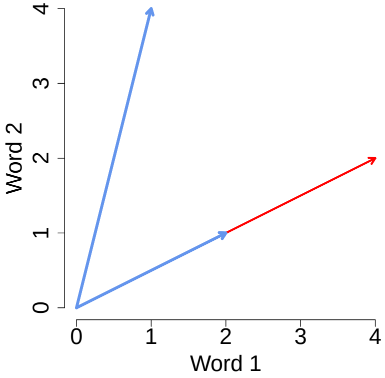
Measuring Similarity



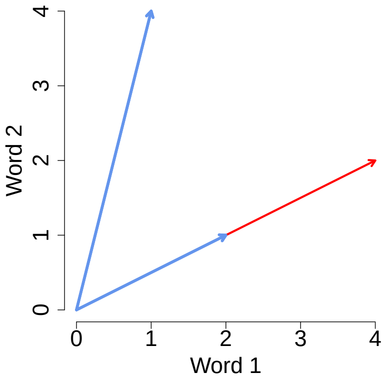
Measure 1: Inner product

$$(2,1)^{'} \cdot (1,4) \quad = \quad 6$$


Word 2
0
1
2
3
4
Word 1
0
1
2
3
4

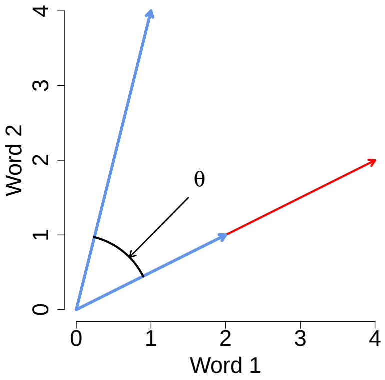


Problem(?): length dependent



Problem(?): length dependent

$$(4,2)^{'}(1,4) \quad = \quad 12$$



Problem(?): length dependent

$$
\begin{aligned}
(4,2)^{'}(1,4) &= 12 \\
a \cdot b &= ||a|| \times ||b|| \times \cos\theta
\end{aligned}
$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

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$$\cos\theta = \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right)$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

$$\begin{aligned} \cos\theta &= \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right) \\ \frac{(4,2)}{||(4,2)||} &= (0.89, 0.45) \end{aligned}$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

$$
\begin{aligned}
\cos\theta &= \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right) \\
\frac{(4,2)}{||(4,2)||} &= (0.89, 0.45) \\
\frac{(2,1)}{||(2,1)||} &= (0.89, 0.45)
\end{aligned}
$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

$$
\begin{aligned}
\cos\theta &= \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right) \\
\frac{(4,2)}{||(4,2)||} &= (0.89, 0.45) \\
\frac{(2,1)}{||(2,1)||} &= (0.89, 0.45) \\
\frac{(1,4)}{||(1,4)||} &= (0.24, 0.97)
\end{aligned}
$$

Cosine Similarity

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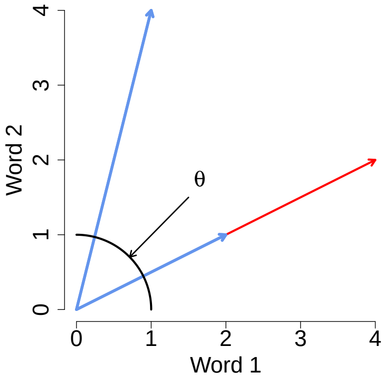
\frac{(1,4)}{||(1,4)||} &= (0.24, 0.97) \\
(0.89, 0.45)^{'}(0.24, 0.97) &= 0.65
\end{aligned}
$$

Cosine Similarity



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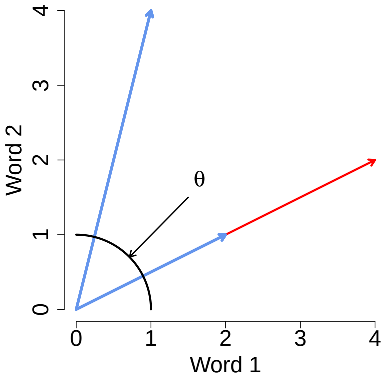
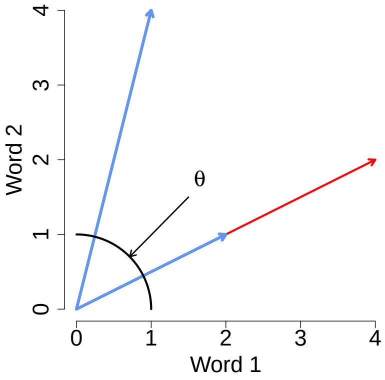
Cosine Similarity



$\cos\theta$: removes document length from similarity measure
Project onto Hypersphere

Cosine Similarity

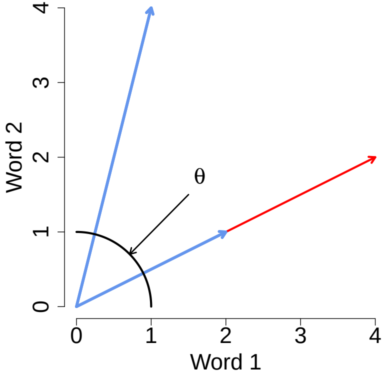


$\cos\theta$: removes document length from similarity measure

Project onto Hypersphere

$\cos\theta \rightarrow$ Inverse distance on Hypersphere

Cosine Similarity



$\cos\theta$: removes document length from similarity measure

Project onto Hypersphere

$\cos\theta \rightarrow$ Inverse distance on Hypersphere

von Mises Fisher distribution : distribution on sphere surface

Matrices

Definition

A *Matrix* *is a rectangular array of numbers*

$$\boldsymbol{A} \quad = \quad \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

If $\boldsymbol{A}$ *has* m *rows* n *columns we will say that* $\boldsymbol{A}$ *is an* $m \times n$ *matrix. Suppose* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are* $m \times n$ *matrices. Then* $\boldsymbol{X} = \boldsymbol{Y}$ *if* $x_{ij} = y_{ij}$ *for all* i *and* j

Simple Examples

$$\boldsymbol{I} \quad = \quad \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

If $\boldsymbol{I}$ is an $n \times n$ matrix we will call an identity matrix.

Simple Examples

$$\boldsymbol{X} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

$\boldsymbol{X}$ is an 2×3 matrix

Matrix Algebra

Definition

Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $m \times n$ matrices. Then define

$$
\begin{aligned}
\boldsymbol{X} + \boldsymbol{Y} &= \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} & \ldots & y_{1n} \\ y_{21} & y_{22} & \ldots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \ldots & y_{mn} \end{pmatrix} \\
&= \begin{pmatrix} x_{11} + y_{11} & x_{12} + y_{12} & \ldots & x_{1n} + y_{1n} \\ x_{21} + y_{21} & x_{22} + y_{22} & \ldots & x_{2n} + y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} + y_{m1} & x_{m2} + y_{m2} & \ldots & x_{mn} + y_{mn} \end{pmatrix}
\end{aligned}
$$

Matrix Algebra

Definition

Suppose $\boldsymbol{X}$ is an $m \times n$ matrix and $k \in \Re$. Then,

$$k\boldsymbol{X} = \begin{pmatrix} kx_{11} & kx_{12} & \ldots & kx_{1n} \\ kx_{21} & kx_{22} & \ldots & kx_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ kx_{m1} & kx_{m2} & \ldots & kx_{mn} \end{pmatrix}$$

Prove theorems about this tonight

R Code

Using matrix command `mat1<- matrix(NA, nrow=3, ncol=2) ## Creating matrix`

```
mat1[1,1]<- 1
mat1[1,2]<- 2
mat1[2,1]<- 1
mat1[2,2]<- 4
mat1[3,1]<- exp(1)
mat1[3,2]<- 4
```

R Code

Using rbind

```
r1<- c(1, 2)
r2<- c(1, 4)
r3<- c(exp(1) , 4)
mat1<- rbind(r1, r2, r3)
```

R Code

Using cbind

```
c1<- c(1, 1, exp(1) )
c2<- c(2, 4, 4)
```

R Code

```
dim(mat1)
[1] 3 2
mat1 + mat1
[,1] [,2]
[1,] 2.000000 4
[2,] 2.000000 8
[3,] 5.436564 8
```

R Code

What if the matrices are of different dimension

```
mat1<- matrix(1, nrow=3, ncol=2)
mat2<- matrix(2, nrow=10, ncol=3)
mat1 + mat2
Error in mat1 + mat2 :  non-conformable arrays
```

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

Matrix Transpose

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\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}
$$

$$
\boldsymbol{X}^{'} = \begin{pmatrix} x_{11} & & & \\ x_{12} & & & \\ \vdots & & & \\ x_{1n} & & & \end{pmatrix}
$$

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \\ \end{pmatrix}$$

$$\boldsymbol{X}^{'} = \begin{pmatrix} x_{11} & x_{21} & & \\ x_{12} & x_{22} & & \\ \vdots & \vdots & & \\ x_{1n} & x_{2n} & & \\ \end{pmatrix}$$

Matrix Transpose

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$$

$$
\boldsymbol{X}^{'} = \begin{pmatrix} x_{11} & x_{21} & \dots \\ x_{12} & x_{22} & \dots \\ \vdots & \vdots & \ddots \\ x_{1n} & x_{2n} & \dots \end{pmatrix}
$$

Matrix Transpose

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\boldsymbol{X} &= \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix} \\
\boldsymbol{X}^{'} &= \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{m1} \\ x_{12} & x_{22} & \ldots & x_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{mn} \end{pmatrix}
\end{aligned}
$$

Matrix Transpose

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$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \\ \end{pmatrix}$$

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If $\boldsymbol{X}$ is an $m \times n$ then $\boldsymbol{X}^{'}$ is $n \times m$.

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

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If $\boldsymbol{X}$ is an $m \times n$ then $\boldsymbol{X}^{'}$ is $n \times m$.
If $\boldsymbol{X} = \boldsymbol{X}^{'}$ then we say $\boldsymbol{X}$ is symmetric.

Matrix Transpose

Example 1: $\boldsymbol{X} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ then $\boldsymbol{X}^{'} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$

In R

```
mat1<- matrix(c(1, 2, 3), nrow=3, ncol=2)
mat2<- t(mat1)
dim(mat1)
3 2
dim(mat2)
2 3
```

Matrix Multiplication

How do we multiply matrices?

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Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

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Suppose we have two matrices

Matrix Multiplication

How do we multiply matrices?

Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

$$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Matrix Multiplication

How do we multiply matrices?

Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

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$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{Y}$$

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We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{aligned}$$

Matrix Multiplication

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We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 1 + 1 \times 3 & \\ & \end{pmatrix}
\end{aligned}
$$

Matrix Multiplication

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Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

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&= \begin{pmatrix} 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ & \end{pmatrix}
\end{aligned}$$

Matrix Multiplication

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Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

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&= \begin{pmatrix} 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ 1 \times 1 + 1 \times 3 & \\ & \end{pmatrix}
\end{aligned}$$

Matrix Multiplication

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Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

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Matrix Multiplication

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Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix} \end{aligned}$$

Matrix Multiplication

Definition

Suppose $\boldsymbol{X}$ *is an* $m \times n$ *matrix and* $\boldsymbol{Y}$ *is an* $n \times k$ *matrix. Then define the matrix* $\boldsymbol{A}$ *an* $m \times k$ *matrix that obtains from multiplying* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *as,*

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\
&= \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \ldots & y_{1k} \\ y_{21} & y_{22} & \ldots & y_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \ldots & y_{nk} \end{pmatrix} \\
&= \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} + \ldots + x_{1n}y_{n1} & \ldots & x_{11}y_{1k} + x_{12}y_{2k} + \ldots + x_{1n}y_{nk} \\ \vdots & \ddots & \vdots \\ x_{m1}y_{11} + x_{m2}y_{21} + \ldots + x_{mn}y_{n1} & \ldots & x_{m1}y_{11} + x_{m2}y_{12} + \ldots + x_{mn}y_{nk} \end{pmatrix}
\end{aligned}
$$

Definition

Suppose $\boldsymbol{X}$ *is an* $m \times n$ *matrix and* $\boldsymbol{Y}$ *is an* $n \times k$ *matrix. Write the* row *vectors of* $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{pmatrix}$ *and* $\boldsymbol{Y}$ *as column vector* $\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \dots & \boldsymbol{y}_k \end{pmatrix}$.

Then the $m \times k$ *matrix* $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{Y}$ *can be written as*

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{x}_1 \cdot \boldsymbol{y}_1 & \boldsymbol{x}_1 \cdot \boldsymbol{y}_2 & \dots & \boldsymbol{x}_1 \cdot \boldsymbol{y}_k \\ \boldsymbol{x}_2 \cdot \boldsymbol{y}_1 & \boldsymbol{x}_2 \cdot \boldsymbol{y}_2 & \dots & \boldsymbol{x}_2 \cdot \boldsymbol{y}_k \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}_m \cdot \boldsymbol{y}_1 & \boldsymbol{x}_m \cdot \boldsymbol{y}_2 & \dots & \boldsymbol{x}_m \cdot \boldsymbol{y}_k \end{pmatrix}$$

Matrix Multiplication

Let's work on an example together!

$$\boldsymbol{X} = \begin{pmatrix} 1 & 4 & 5 \\ 10 & 2 & 3 \end{pmatrix}$$

$$\boldsymbol{Y} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 3 & 5 \end{pmatrix}$$ What is $\boldsymbol{X}\boldsymbol{Y}$?

Matrix Multiplication

Let's work on an example together!

$\boldsymbol{X} = \begin{pmatrix} 1 & 4 & 5 \\ 10 & 2 & 3 \end{pmatrix}$

$\boldsymbol{Y} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 3 & 5 \end{pmatrix}$ What is $\boldsymbol{XY}$?

Not all matrices can be multiplied.

Matrix $\boldsymbol{AB}$ exists only if the number of columns in $\boldsymbol{A}$ = number of rows in $\boldsymbol{B}$. If $\boldsymbol{AB}$ exists we will say the matrices are conformable

Matrix Multiplication with a Vector

Suppose $\boldsymbol{X} = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 5 & 1 & 2 \\ 3 & 5 & 3 & 4 \end{pmatrix}$ a 3×4 matrix and that $\boldsymbol{v} = \begin{pmatrix} 3 \\ 3 \\ 4 \\ 10 \end{pmatrix}$ a 4×1 matrix (or a column vector) what is
$\boldsymbol{X}\boldsymbol{v}$?
What is $\boldsymbol{X}^{'}\boldsymbol{v}$?

Algebraic Properties

Suppose $\boldsymbol{X}$ is an $m \times n$ matrix and $\boldsymbol{Y}$ is an $n \times k$ matrix. Suppose that

$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ as the identity matrix and that $k \in Re$.

- $\boldsymbol{X}\boldsymbol{Y} \neq \boldsymbol{Y}\boldsymbol{X}$ in general !!!! (but it could)
- $\boldsymbol{X}\boldsymbol{I} = \boldsymbol{X}$ (let's talk it out!)
- $(\boldsymbol{X}^{'})^{'} = \boldsymbol{X}$
- $(\boldsymbol{X}\boldsymbol{Y})^{'} = \boldsymbol{Y}^{'}\boldsymbol{X}^{'}$
- $(k\boldsymbol{X})^{'} = k\boldsymbol{X}^{'}$
- $(\boldsymbol{X} + \boldsymbol{Y})^{'} = \boldsymbol{X}^{'} + \boldsymbol{Y}^{'}$

Examples, Implenting in `R`

R and matrix multiplication

```
X<- matrix(NA, nrow=2, ncol=3)
Y<- matrix(NA, nrow=3, ncol=2)
X[1,]<- c(1, 4, 5)
X[2,]<- c(10, 2, 3)
Y[1,]<- c(2, 3)
Y[2,]<- c(1, 5)
Y[3,]<- c(3, 5)
```

```
A<- X%*%Y
> A
[,1] [,2]
[1,] 21 48
[2,] 31 55
```

Matrix Inversion

Big topic: suppose $\boldsymbol{X}$ is an $n \times n$ matrix. We want to find the matrix X^{-1} such that

$$\boldsymbol{X}^{-1}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix.
Why?

- Regression
- Solving systems of equations
- Will provide intuition about "colinearity", "fixed effects", "treatment designs" and what we can learn as social scientists

Calculate $\rightsquigarrow$ Properties of Inverses $\rightsquigarrow$ when do inverses exist $\rightsquigarrow$ Application to regression analysis

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 0x_2 + x_3 &= 5
\end{aligned}
\tag{0.1}
$$

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_2 + 0x_3 &= 0 \\
0x_1 + x_2 + x_3 &= 0 \\
x_1 + 0x_2 + x_3 &= 0
\end{aligned}
$$

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 5x_2 + 0x_3 &= 5 \\
0x_1 + 0x_2 + 3x_3 &= 6
\end{aligned}
$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
$$\boldsymbol{x} = (x_1, x_2, x_3)$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
$\boldsymbol{x} = (x_1, x_2, x_3)$
$\boldsymbol{b} = (0, 5, 6)$

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 5x_2 + 0x_3 &= 5 \\
0x_1 + 0x_2 + 3x_3 &= 6
\end{aligned}
$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\boldsymbol{x} = (x_1, x_2, x_3)$

$\boldsymbol{b} = (0, 5, 6)$

The system of equations are now,

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
$\boldsymbol{x} = (x_1, x_2, x_3)$
$\boldsymbol{b} = (0, 5, 6)$
The system of equations are now,

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 5x_2 + 0x_3 &= 5 \\
0x_1 + 0x_2 + 3x_3 &= 6
\end{aligned}$$

$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
$\boldsymbol{x} = (x_1, x_2, x_3)$
$\boldsymbol{b} = (0, 5, 6)$
The system of equations are now,

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$\boldsymbol{A}^{-1}$ exists if and only if $\boldsymbol{Ax} = \boldsymbol{b}$ has only one solution.

Matrix Inversion, Definition

Definition

Suppose $\boldsymbol{X}$ is an $n \times n$ matrix. We will call $\boldsymbol{X}^{-1}$ the inverse of $\boldsymbol{X}$ if

$$\boldsymbol{X}^{-1}\boldsymbol{X} \;=\; \boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$$

If $\boldsymbol{X}^{-1}$ exists then $\boldsymbol{X}$ is invertible. If $\boldsymbol{X}^{-1}$ does not exist, then we will say $\boldsymbol{X}$ is singular.

Matrix Inversion

You'll never invert a matrix by hand.

We're going to use R

```
X<- matrix(NA, nrow=3, ncol=3)
X[1,]<- c(2, 3, 4)
X[2,]<- c(3, 1, 3)
X[3,]<- c(2, 4, 2)
X.inv<- solve(X)
> X.inv
[,1] [,2] [,3]
[1,] -0.5 0.5 0.25
[2,] 0.0 -0.2 0.30
[3,] 0.5 -0.1 -0.35
X.inv%*%X
[,1] [,2] [,3]
[1,] 1 0.000000e+00 -2.220446e-16
[2,] 0 1.000000e+00 0.000000e+00
[3,] 0 -2.220446e-16 1.000000e+00
```

Matrix Inversion

1) Calculate Inverses
2) Properties of Inverses

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Theorem

The inverse of matrix $\boldsymbol{X}$, $\boldsymbol{X}^{-1}$, is unique

Matrix Inversion

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2) Properties of Inverses

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Proof.

By way of contradiction, suppose not. Then there are at least two matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ such that $\boldsymbol{AX} = \boldsymbol{I}$ and $\boldsymbol{CX} = \boldsymbol{I}$

This implies that,

$$\begin{aligned} \boldsymbol{AXC} &= (\boldsymbol{AX})C \\ &= \boldsymbol{IC} \\ &= \boldsymbol{C} \end{aligned}$$

Matrix Inversion

But it also implies that

$$\begin{aligned} \boldsymbol{AXC} &= \boldsymbol{A}(\boldsymbol{XC}) \\ &= \boldsymbol{A}(\boldsymbol{I}) \\ &= \boldsymbol{A} \end{aligned}$$

So $\boldsymbol{C} = \boldsymbol{AXC} = \boldsymbol{A}$ or $\boldsymbol{C} = \boldsymbol{A}$ but this contradicts our assumption that there are two unique inverses.

Matrix Inversion

Theorem

Suppose $\boldsymbol{A}$ *has inverse* $\boldsymbol{A}^{-1}$ *and* $\boldsymbol{B}$ *has inverse* $\boldsymbol{B}^{-1}$*. Then,*

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Matrix Inversion

Theorem

Suppose $\boldsymbol{A}$ has inverse $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ has inverse $\boldsymbol{B}^{-1}$. Then,

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Proof.

We need to show that $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) = (\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{I}$.

$$\begin{aligned}
(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) &= \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} \\
&= \boldsymbol{B}^{-1}\boldsymbol{I}\boldsymbol{B} \\
&= \boldsymbol{B}^{-1}\boldsymbol{B} \\
&= \boldsymbol{I}
\end{aligned}$$

Matrix Inversion

$$\begin{aligned}(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) &= \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} \\ &= \boldsymbol{AIA}^{-1} \\ &= \boldsymbol{AA}^{-1} \\ &= \boldsymbol{I}\end{aligned}$$

So $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Challenge Inversion Proofs

- Show that $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
- Show that $(k\boldsymbol{A})^{-1} = \frac{1}{k}\boldsymbol{A}^{-1}$

Matrix Inversion

1) How to Calculate an Inverse
2) Inversion properties
3) When do inverses exist?

Linear Independence: not repeated information in matrix will be the key (for both inversion and regressions)

Matrix Inversion: Existence

Definition

Suppose we have a set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$
And consider the system of equations

$$k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2 + \ldots + k_r \boldsymbol{v}_r = \mathbf{0}$$

If the only solution is $k_1 = 0, k_2 = 0, k_3 = 0, \ldots, k_r = 0$ *then we say that the set is linearly independent. If there are other solutions, then the set is linearly dependent.*

Matrix Inversion: Existence

Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$
Can we write this as a combination of other vectors?

Matrix Inversion: Existence

Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$
Can we write this as a combination of other vectors? no!

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Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$
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Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$, $\boldsymbol{v}_4 = (1, 2, 3)$.
Can we write this as a combination of other vectors?

Matrix Inversion: Existence

Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$
Can we write this as a combination of other vectors? no!
Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$, $\boldsymbol{v}_4 = (1, 2, 3)$.
Can we write this as a combination of other vectors?

$$\boldsymbol{v}_4 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2 + 3\boldsymbol{v}_3$$

Matrix Inversion: Existence

Theorem

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_K \in \Re^n$. If $K > n$ then the set is linearly dependent

Matrix Inversion: Existence

Theorem

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_K \in \Re^n$. If $K > n$ then the set is linearly dependent

- $\boldsymbol{v}_1 = (v_{11}, v_{21}, \ldots, v_{n1})$

Matrix Inversion: Existence

Theorem

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_K \in \Re^n$. *If* $K > n$ *then the set is linearly dependent*

- $\boldsymbol{v}_1 = (v_{11}, v_{21}, \ldots, v_{n1})$
- Says that if there are more vectors in the set than elements in each vector, one must be linearly dependent

Matrix Inversion: Existence

We care because of the following theorem

Theorem

Suppose $\boldsymbol{X}$ *is an* $n \times n$ *matrix. Recall we can write this matrix as* $\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix}$.

Then $\boldsymbol{X}$ *has an inverse if and only if* $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ *is linearly independent*

If this is true, we say $\boldsymbol{X}$ has full rank

Linear Regression

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$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$$

Linear Regression

In methods classes you learn about linear regression. For each i (individual) we observe covariates $x_{i1}, x_{i2}, \ldots, x_{ik}$ and independent variable Y_i. Then,

$$\begin{array}{rcl}
Y_1 & = & \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \ldots + \beta_k x_{1k} \\
Y_2 & = & \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \ldots + \beta_k x_{2k} \\
\vdots & \vdots & \vdots \\
Y_i & = & \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} \\
\vdots & \vdots & \vdots \\
Y_n & = & \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + \ldots + \beta_k x_{nk}
\end{array}$$

Linear Regression

- Define $\boldsymbol{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{ik})$
- Define $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix}$
- Define $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)$
- Define $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$.

Then we can write

$$\boldsymbol{Y} = \boldsymbol{X}\beta$$

Linear Regression

$$\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{X}\beta \\
\boldsymbol{X}^{'}\boldsymbol{Y} &= \boldsymbol{X}^{'}\boldsymbol{X}\beta \\
(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} &= (\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{X}\beta \\
(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} &= \beta
\end{aligned}$$

Big question: is $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ invertible?
We'll investigate in homework!

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose $\boldsymbol{A}$ *is an* $N \times N$ *matrix and* λ *is a scalar.*
If

$$\boldsymbol{A}\boldsymbol{x} \quad = \quad \lambda \boldsymbol{x}$$

Then $\boldsymbol{x}$ *is an eigenvector and* λ *is the associated eigenvalue*

An Introduction to Eigenvectors, Values, and Diagonalization

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- $\boldsymbol{A}$ stretches the eigenvector $\boldsymbol{x}$

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- $\boldsymbol{A}$ stretches $\boldsymbol{x}$ by λ

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- To find eigenvectors/values: (`eigen` in `R`)

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- $\boldsymbol{A}$ stretches $\boldsymbol{x}$ by λ
- To find eigenvectors/values: (`eigen` in `R`)
  - Find λ that solves $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$
  - Find vectors in null space of:

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

Definition

Suppose $\boldsymbol{X}$ *is an* $N \times J$ *matrix. Then* $\boldsymbol{X}$ *can be written as:*

$$\boldsymbol{X} = \underbrace{\boldsymbol{U}}_{N \times N} \underbrace{\boldsymbol{S}}_{N \times J} \underbrace{\boldsymbol{V}^{'}}_{J \times J}$$

Where:

$$\begin{aligned} \boldsymbol{U}^{'}\boldsymbol{U} &= \boldsymbol{I}_N \\ \boldsymbol{V}^{'}\boldsymbol{V} &= \boldsymbol{V}\boldsymbol{V}^{'} = \boldsymbol{I}_J \end{aligned}$$

$\boldsymbol{S}$ *contains* $\min(N, J)$ *singular values,* $\sqrt{\lambda_j} \geq 0$ *down the diagonal and then 0's for the remaining entries*