Chapter 1

$\mathscr{D}_X$-modules and the Riemann-Hilbert Correspondence

Chapter 2

The Behaviour of $\mathscr{A}_X^R$-Modules

Mention BVWZ

The classical approximation of the roots of the b-polynomial due to Kashiwara (1976) relies on a quotient module $\int \mathscr{M}/\mathscr{D}_X u$ being holonomic. This is no longer true in the multivariate case but a refined assumption, called relative holonomicity, due to Maisonobe (2016) still holds. This refinement works with $\mathscr{D}_X \times \mathbb{C}[s]$-modules whence one gets characteristic varieties inside $T^*X \times \mathbb{C}^p$.

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2.1 Modules over $\mathscr{A}_X^R$

Maybe also mention the example Robin put on the whiteboard? Possibly in the main body?

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2).

Proposition 2.1.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\operatorname{gr}^{rel}\mathscr{M}$ is coherent over $\operatorname{gr}^{rel}\mathscr{A}_X^R$.*

Proposition 2.1.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\operatorname{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\operatorname{Ch}^{rel}\mathscr{M}$.*

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\operatorname{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

2.2 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.

more introduction

Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties. A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := (\mathcal{O}_Y \times \mathcal{O}_R) \otimes_{\mu^{-1}(\mathcal{O}_X \times \mathcal{O}_R)} \mu^{-1}\mathscr{A}_X^R$.

Definition 2.2.1. *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be surpressed whenever there is no ambiguity.*

To compute $\int^j \mathscr{M}$ a resolution for $\mathscr{A}_{Y\to X}$ is required.

Definition 2.2.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map δ given by*

$$\begin{aligned} m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto & \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k \\ & - \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \end{aligned}$$

Lemma 2.2.1. *The complex of $(\mathscr{A}_X, f^{-1}\mathscr{A}_Y)$-bimodules $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}^R$ is a resolution of $\mathscr{A}_{Y\to X}^R$ as a bimodule by locally free left $\mathscr{A}_Y^R$-modules.*

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Proof. This will be analogous to the case of $\mathscr{D}_Y$-modules in (Sabbah, 2011, p33). Also see `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` □

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 2.2.2. *It holds that $\int = \boldsymbol{R}\mu_*(- \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$.*

Lemma 2.2.3. *Construction of global section in $\int^j \mathscr{M}$.*

Theorem 2.2.4. *Long exact sequence*

Functorial Properties of the Direct Image

Theorem 2.2.5. *Let $\mu : Z \to Y$ and $\nu : Y \to X$ be morphisms of smooth algebraic varieties. If μ is proper then $\int_{\nu\circ\mu} = \int_\nu \int_\mu$.*

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` □

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $\boldsymbol{D}_{qc}^{b,r}(\mathscr{A}_Y^R)$ the full subcategory of $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ consisting of those complexes of right $\mathscr{A}_Y^R$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\mathrm{Spec}\, R}$. Similarly for $\boldsymbol{D}_{coh}^{b,r}(\mathscr{A}_Y^R)$ with the cohomology being coherent $\mathscr{A}_Y^R$-modules.

Theorem 2.2.6. *Let $\mu : X \to Y$ be a morphism of nonsingular algebraic varieties. Then the direct image $\int$ takes $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_Y^R)$ into $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_X^R)$. Moreover, when μ is proper the direct image takes $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_Y^R)$ into $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_X^R)$.*

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture18.pdf` □

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j \mathscr{M}$ in terms of $\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara Kashiwara (1976).

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The proof we consider here is adapted from a proof by Malgrange (1985) and we follow the exposition by Sabbah (2011).

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

Theorem 2.2.7. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

Coherent should suffice in algebraic case I think, Sabbah needs assumptions to guarantee the existence of a global filtration in analytic case.

$$\mathrm{Ch}^{rel}\Big(\int^j \mathscr{M}\Big) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

The first step is to note that a similar inclusion is easy for the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ define $\int^j \mathcal{M} := \mathscr{H}^j\left(\boldsymbol{R}\mu_*(\boldsymbol{L}(T^*\mu)^*\mathcal{M})\right)$. Note that $(T^*\mu)^*$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by the tensor product $- \otimes_{f^{-1}\mathcal{O}_X \times \mathcal{O}_R} \mathrm{gr}^{rel}\mathscr{A}_X^R$. Hence, looking at the supports, the following result is immediate.

Lemma 2.2.8. *For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ it holds that*

$$\operatorname{Supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}\operatorname{Supp}\mathcal{M}\right).$$

Applying this to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to understand the difference between $\mathrm{gr}^{rel}\int^j \mathscr{M}$ and $\int^j \mathrm{gr}^{rel}\mathscr{M}$. This may be done using relative Rees modules.

Definition 2.2.3. *Let z be a new variable. The relative Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}^{rel}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

The following obvious isomorphisms of filtered modules allow us to view the relative Rees module as a way to interpolate between $\mathscr{M}$ and $\mathrm{gr}^{rel}\mathscr{M}$

$$\frac{\mathcal{R}^{rel}\mathscr{M}}{(z-1)\mathcal{R}^{rel}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}^{rel}\mathscr{M}}{z\mathcal{R}^{rel}\mathscr{M}} = \mathrm{gr}^{rel}\mathscr{M}.$$

Conversely, the second formula may be used to produce a filtered $\mathscr{A}_Y^R$-module from any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

One can define a direct image of graded $\mathcal{R}\mathscr{A}_Y^R$-modules similarly to the $\mathscr{A}_Y^R$-module direct image and these are coherent graded $\mathcal{R}\mathscr{A}_X^R$-modules similarly to theorem 2.2.6.

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Lemma 2.2.9. *For sufficiently large $\ell \gg 1$ the kernel of z^ℓ in $\int^j \mathcal{R}^{rel}\mathscr{M}$ is locally stationary. The quotient of $\int^j \mathcal{R}^{rel}\mathscr{M}$ by it's z-torsion is the $\mathcal{R}\mathscr{A}_X^R$-coherent module associated to a good filtration on $\int^j \mathscr{M}$.*

Proof. The statement that the kernel locally satbilises and that the quotient is coherent follow from $\int \mathcal{R}^{rel}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}_X^R$.

For it's association to $\int^j \mathscr{M}$ consider the short exact sequence $0 \to \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z-1} \mathcal{R}^{rel}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$0 \to \int^0 \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z-1} \int^0 \mathcal{R}^{rel}\mathscr{M} \to \int^0 \mathscr{M} \to \int^1 \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z-1} \cdots.$$

Since $\int^{j+1} \mathcal{R}^{rel}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module one has that $z-1$ is injective whence it follows that $\int^j \mathcal{R}^{rel}\mathscr{M}/(z-1)\int^j \mathcal{R}^{rel}\mathscr{M} \cong \int^j \mathscr{M}$ as desired. $\square$

Corollary 2.2.10. *For all $\ell \geq 1$ it holds that $\int^j \mathcal{R}^{rel}\mathscr{M}/z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}^{rel}\mathscr{M}/z^\ell\mathcal{R}^{rel}\mathscr{M})$. Further, if ℓ is sufficiently large then $\mathcal{R}^{rel}\int^j \mathscr{M}/z^\ell\mathcal{R}^{rel}\int^j \mathscr{M}$ is a quotient of $\int^j \mathcal{R}^{rel}\mathscr{M}/z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}$.*

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