Chapter 1

The Behaviour of $\mathscr{A}_X^R$-Modules

[1]

The classical approximation of the roots of the b-polynomial due to **?** relies on a quotient module $\int \mathscr{M}/\mathscr{D}_X u$ being holonomic. This is no longer true in the multivariate case but a refined assumption, called relative holonomicity, due to **?** still holds. This refinement works with $\mathscr{D}_X \times \mathbb{C}[s]$-modules whence one gets characteristic varieties inside $T^*X \times \mathbb{C}^p$[2].

[3]

1.1 Modules over $\mathscr{A}_X^R$

Basic Definitions and Properties

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. It will also be convenient to use the abbreviation $\mathcal{O}_X^R := \mathcal{O}_{X \times \operatorname{Spec} R}$.

The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration. The associated graded objects are denoted by gr^{rel}. Denote $\pi : T^*X \times \operatorname{Spec} R \to X \times \operatorname{Spec} R$ for the projection map. As in the case of $\mathscr{D}_X$-modules in chapter 1 [4] one can view $\pi^{-1}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$ as a subsheaf of $\mathcal{O}_{T^*X}^R$ and for any $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module $\mathcal{M}$ there is a corresponding module on $T^*X \times \operatorname{Spec} R$ defined by $\mathcal{O}_{T^*X}^R \otimes_{\pi^{-1}\mathrm{gr}^{rel}\mathscr{A}_X^R} \pi^{-1}\mathcal{M}$. By abuse of notation the corresponding module on $T^*X \times \operatorname{Spec} R$ is still denoted with $\mathcal{M}$ and we adopt the perspective that $\mathrm{gr}^{rel}\mathscr{A}_X^R$-modules always live on $T^*X \times \operatorname{Spec} R$ unless explicitly mentioned otherwise.

Similarly to the case of $\mathscr{D}_X$ in the first chapter that [5] it holds that $\mathscr{A}_X^R$ is the sheaf of rings generated by $\mathcal{O}_X^R$ and Θ_X inside of $\mathcal{E}nd_{\underline{\mathbb{C}}}(\mathcal{O}_X^R)$. Giving a left $\mathscr{A}_X^R$-module is equivalent

[1]Note: Mention BVWZ
[2]Note: p?
[3]Note: Maybe also mention the example Robin put on the whiteboard? Possibly in the main body?
[4]Note: cite
[5]Note: Cite when C1 is written

to giving a $\mathcal{O}_X^R$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X. Similarly, giving a right $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (**?**, Chapter 2). [6]

Proposition 1.1.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\mathrm{gr}^{rel}\mathscr{M}$ is coherent over $\mathrm{gr}^{rel}\mathscr{A}_X^R$. Such a filtration is called a good filtration.*

Proposition 1.1.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\mathrm{Ch}^{rel}\mathscr{M}$.*

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

Basic Operations

For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ and left $\mathscr{D}_X$-module $\mathscr{N}$ the tensor product $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ comes equipped with a right $\mathscr{A}_X^R$-module structure defined by

$$f \cdot (m \otimes n) = mf \otimes n; \qquad \xi \cdot (m \otimes n) = m\xi \otimes n - m \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X. The same definition applies for a $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X^R} \mathscr{N}$ whenever $\mathscr{N}$ is a left $\mathscr{A}_X^R$-module.

Similarly, given a left $\mathscr{D}_X$-module $\mathscr{L}$ and a left $\mathscr{A}_X^R$-module $\mathscr{N}$ a left $\mathscr{A}_X^R$-module structure on $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{L}$ is defined by

$$f \cdot (\ell \otimes n) = \ell \otimes fn; \qquad \xi \cdot (\ell \otimes n) = \xi\ell \otimes n + \ell \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

Lemma 1.1.3. *Let $\mathscr{M}, \mathscr{N}$ be right and left $\mathscr{A}_X^R$-modules respectively and let $\mathscr{L}$ be a left $\mathscr{D}_X$-module. Then there is a isomorphism of left $\mathscr{A}_X^R$-modules*

$$(\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}) \otimes_{\mathcal{O}_X^R} \mathscr{N} \cong \mathscr{M} \otimes_{\mathcal{O}_X^R} (\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}).$$

Proof. This is immediate by checking that the obvious bijection conserves the $\mathscr{A}_X^R$-module structure. Note that the only nontrivial check is the action of a section ξ from Θ_X. $\square$

Lemma 1.1.4. *Let $\mathscr{N}$ be a left $\mathscr{A}_X^R$-module which is locally free as a $\mathcal{O}_X^R$-module. Consider $\mathscr{A}_X^R$ as a right $\mathscr{A}_X^R$-module, then $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$ is locally free as a right $\mathscr{A}_X^R$-module.*

[6] Note: Probably cite C1 instead

Proof. Consider local coordinates $x_1, \ldots, x_n$ on X and a local $\mathcal{O}_X^R$-basis $\{n_\beta\}_\beta$ for $\mathscr{N}$. Then $\{1 \otimes n_\beta\}_\beta$ will be a local $\mathscr{A}_X^R$-basis for $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

To see that this generates the $\mathscr{A}_X^R$-module note that $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ is a $\mathcal{O}_X^R$-basis set when α runs over all multi-indices in $\mathbb{Z}_{\geq 0}^n$. These sections can be recovered using the $\mathscr{A}_X^R$-action on the proposed generating set by induction on $|\alpha|$. Indeed, $\xi^\alpha \cdot (1 \otimes n_\beta)$ equals $\xi^\alpha \otimes n_\beta$ up to a element in the $\mathcal{O}_X^R$-span of $\{\xi^\gamma \otimes n_\beta\}_{|\gamma|<|\alpha|}$.

For the freedom, suppose there is a local $\mathscr{A}_X^R$-relation $\sum_\beta P_\beta \cdot 1 \otimes n_\beta = 0$ with some P_β nonzero. This is of the form $\sum_{\alpha,\beta} f_{\alpha,\beta}\xi^\alpha \cdot 1 \otimes n_\beta = 0$ with the $f_{\alpha,\beta}$ sections of $\mathcal{O}_X^R$ not all equal to zero. Pick some multi-index $\mu \in \mathbb{Z}_{\geq 0}^n$ and of maximal degree such that $f_{\mu,\beta}$ is non-zero for some β. Then, rewriting $\sum_{\alpha,\beta} f_\alpha \xi^\alpha \cdot 1 \otimes n_\beta = 0$ in terms of the $\mathcal{O}_X^R$-basis $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ one finds a non-zero coeffiencient at $\xi^\eta \otimes n_\beta$ for some β which is a contradiction. $\square$

1.2 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.[7]

Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties, by abuse of notation we will also denote μ for the induced map from $Y \times \operatorname{Spec} R$ to $X \times \operatorname{Spec} R$.

A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}\mathscr{A}_X^R$. Here, the right $\mu^{-1}\mathscr{A}_X^R$-module structure is just the action on the second component and definitions like **??** are used to define the left $\mathscr{A}_Y^R$-module structure. To be precise

$$f \cdot (g \otimes \mu^{-1}h_X) = fg \otimes \mu^{-1}h_X; \qquad \xi \cdot (g \otimes \mu^{-1}h_X) = \xi g \otimes \mu^{-1}h_X + g \otimes T\mu(\xi)\mu^{-1}h_X$$

for any sections f of $\mathcal{O}_Y^R$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Definition 1.2.1. *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be surpressed whenever there is no ambiguity.*

To compute the direct image $\int^j \mathscr{M}$ a resolution for the transfer bimodule $\mathscr{A}_{Y\to X}$ is required.

Definition 1.2.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\operatorname{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\operatorname{Sp}_Y^{-k}(\mathscr{M}) =$*

[7]Note: more introduction

$\mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map δ given by

$$m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k$$
$$- \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k$$

The following lemma and it's proof are a generalisation of exercise 1.20 in **?** to the relative case.

Lemma 1.2.1. *The relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_X^R$ as left $\mathscr{A}_X^R$-module.*

Proof. Define a filtration on $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ by the complexes $F_k \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ which have term $F_{k-\ell}\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot ℓ. This filtration induces the complexes $\mathrm{gr}_k^{rel} \mathrm{Sp}_X^\bullet(\mathscr{A}_Y^R)$ with term $\mathrm{gr}_{k-\ell}^{rel} \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot ℓ.

In local coordinates $x_1, \ldots, x_n$ one finds that $\mathrm{gr}^{rel}\mathrm{Sp}_Y^\bullet := \oplus_k \mathrm{gr}^{rel}{}_k \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is the Koszul complex of $\mathcal{O}_Y^R[\xi_1, \ldots, \xi_n] = \mathrm{gr}^{rel}\mathscr{A}_Y^R$ with respect to $\xi_1, \ldots, \xi_n$.[8] Since $\xi_1, \ldots, \xi_n$ form a regular sequence a standard result on Koszul complexes[9] yields that $\mathrm{gr}^{rel}\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_Y^R$ as $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module.

On the other hand, it is immediate that $F_0 \mathrm{Sp}^\bullet(\mathscr{A}_Y^R) = \mathrm{gr}_0^{rel} \mathrm{Sp}^\bullet(\mathscr{A}_Y^R)$ is $\mathcal{O}_Y^R$ viewed as a complex. Hence, there is no contribution to $\mathrm{gr}^{rel} \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ from the terms of $k > 0$. That is to say that $\mathrm{gr}_k^{rel} \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is quasi-isomorphic to the zero complex for $k > 0$. Hence, $F_0 \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \hookrightarrow \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a quasi-isomorphism by the exactness of the direct limit. [10] It follows that $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a resolution of $\mathcal{O}_X^R$. That the terms of $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ are locally free follows from **??** after some minor adjustments in the statement and proof. □

Define the transfer Spencer complex as the complex of $(\mathscr{A}_Y^R, f^{-1}\mathscr{A}_X)$-bimodules given by $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X}^R$. The following lemma and it's proof are direct generalisation of exercise 3.4 in **?** to the relative case.

Lemma 1.2.2. *The transfer Spencer complex $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R)$ is a resolution of $\mathscr{A}_{Y\to X}^R$ as a bimodule by locally free left $\mathscr{A}_Y^R$-modules.*

Proof. To see that the terms of the complex are locally free recall from **??** the following isomorphisms of left $\mathscr{A}_Y^R$-modules

$$(\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X} \cong \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y^R} (\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}).$$

[11] Note that $\mathscr{A}_{Y\to X}^R$ is a locally free $\mathcal{O}_Y^R$-module since it is the pullback of a locally free module on $X \times \mathrm{Spec}\, R$. Combined with the fact that $\wedge^\ell \Theta$ is a locally free $\mathcal{O}_Y$-module this yields that $\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}$ is a locally free $\mathcal{O}_Y^R$-module. Hence **??** is applicable and yields that the terms of the transfer Spencer complex are locally free $\mathscr{A}_Y^R$-modules.

That the transfer Spencer complex is a resolution of $\mathscr{A}_{Y\to X}^R$ follows from **??** by using that $\mathscr{A}_{Y\to X}^R$ is a locally free and hence flat over $\mathcal{O}_Y^R$. □

[8] Note: Should I explain what a Koszul complex is?
[9] Note: Give reference to some book
[10] Note: Would be nice to give a reference, proof may be found on stackexchange
[11] Note: May be possible to remove this step from the proof and removing need for minor adjustment of previous proof.

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 1.2.3. *It holds that* $\int = \boldsymbol{R}\mu_*(- \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R))$.

A strategy one can employ in proving theorems on some space X is by first solving them on a nicer space Y equipped with a map $Y \to X$. This can then be related to the problem on X by use of the direct image. For this purpose it is useful that any global section of $\mathcal{M}$ induces a global section of the direct image. This is usually done in the language of left modules but for us it is more natural to work with right $\mathscr{A}_Y^R$-modules.

Lemma 1.2.4. *Let* $\mathscr{M}$ *be a right* $\mathscr{A}_Y^R$*-module. Then any global section* $m \in \Gamma(Y, \mathscr{M})$ *induces a global section of* $\int^0 \mathscr{M})$.

Proof. By the Leray spectral sequence there is a functorial isomorphism

$$\mathbb{H}^\bullet(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)) \cong \mathbb{H}^\bullet(X, \boldsymbol{R}\mu_*(\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R))).$$

In particular it follows that $\mathbb{H}^0(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)) \cong \Gamma(X, \int^0 \mathscr{M})$. The Čech spectral sequence now induces the desired global section in the direct image based on the section $m \otimes 1$ of $\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^0_{Y\to X}(\mathscr{A}_Y^R)$. ☐

Theorem 1.2.5. *Long exact sequence*

Functorial Properties of the Direct Image

Theorem 1.2.6. *Let* $\mu : Z \to Y$ *and* $\nu : Y \to X$ *be morphisms of smooth algebraic varieties. If* μ *is proper then* $\int_{\nu\circ\mu} = \int_\nu \int_\mu$.

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` ☐

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_Y^R)$ the full subcategory of $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ consisting of those complexes of right $\mathscr{A}_Y^R$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\mathrm{Spec}\, R}$. Similarly for $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_Y^R)$ with the cohomology being coherent $\mathscr{A}_Y^R$-modules.

Theorem 1.2.7. *Let* $\mu : X \to Y$ *be a morphism of nonsingular algebraic varieties. Then the direct image* $\int$ *takes* $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_Y^R)$ *into* $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_X^R)$*. Moreover, when* μ *is proper the direct image takes* $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_Y^R)$ *into* $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_X^R)$.

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture18.pdf` ☐

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j \mathscr{M}$ in terms of $\mathrm{Ch}^{rel}\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara.

The original proof by **?** uses the theory of microlocal differential operators. The idea of the following proof is due to **?** in a K-theoretic context. We follow the exposition of **?** and replace it with the corresponding relative notions.

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

Theorem 1.2.8. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

$$\mathrm{Ch}^{rel}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

Note that the statement is local so, after replacing X by some affine open, it may be assumed that $X \times \operatorname{Spec} R$ and $Y \times \operatorname{Spec} R$ are compact. The first step is to note that a similar inclusion is easy for the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. The direct image functor on $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules $\mathcal{M}$ is defined by $\int^j \mathcal{M} := \boldsymbol{R}^j\widetilde{\mu}_*(\boldsymbol{L}(T^*\mu)^*\mathcal{M})$. Here, $(T^*\mu)^*(-)$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by $- \otimes_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R$. Looking at the supports the following result is immediate.

Lemma 1.2.9. *For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ it holds that*

$$\operatorname{supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1} \operatorname{supp} \mathcal{M}\right).$$

Applying this lemma to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to show that $\operatorname{supp} \mathrm{gr}^{rel}\int^j \mathscr{M} \subseteq \operatorname{supp} \int^j \mathrm{gr}^{rel}\mathscr{M}$. This is proved in **??**. The main technical ingredient in the proof is the Rees modules associated to a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$.

Definition 1.2.3. *Let z be a new variable. The Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

Given a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it follows that $\mathcal{R}\mathscr{M}$ is a coherent $\mathcal{R}\mathscr{A}_Y^R$-module similarly to **??**. The following isomorphisms of filtered modules on $Y \times \operatorname{Spec} R$ are essential. They mean that the Rees module can be viewed as a parametrisation of various relevant modules.

$$\frac{\mathcal{R}\mathscr{M}}{(z-1)\mathcal{R}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}\mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z^\ell\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}.$$

Here $\mathrm{gr}^{rel}_{[\ell]}$ takes a filtered object and returns $\oplus_k F_k/F_{k-\ell}$. The first formula may be be used to find a corresponding filtered $\mathscr{A}_Y^R$-module for any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

The jth direct image of a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is the sheaf of $\mathcal{R}\mathscr{A}_X^R$-modules on $X \times \operatorname{Spec} R$ defined by $\int^j \mathcal{M} = \boldsymbol{R}^j \mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X})$. Here the filtration on $\mathscr{A}^R_{Y\to X}$ is defined by $F_i\mathscr{A}^R_{Y\to X} = \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}F_i\mathscr{A}_X^R$. The direct image may be restricted to the category of graded Rees modules in which case it returns a graded Rees module. Coherence is preserved similarly to **??**.

Recall that a $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules on $Y \times \operatorname{Spec} R$ could be be viewed as a sheaf on $T^*Y \times \operatorname{Spec} R$ and is already equipped with a direct image. The Rees module viewpoint agrees with the earlier definition by the following lemma.

Lemma 1.2.10. *Consider a filterd $\mathscr{A}_Y^R$-module $\mathscr{M}$. Then viewing $\int^j \mathcal{R}\mathscr{M}/z\mathcal{R}\mathscr{M}$ with it's $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module structure as a sheaf on $T^*X \times \operatorname{Spec} R$ recovers the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module direct image $\int^j \mathrm{gr}^{rel}\mathscr{M}$. Viewing $\int^j \mathcal{R}\mathscr{M}/(z-1)\mathscr{M}$ as a $\mathscr{A}_X^R$-module recovers $\int^j \mathscr{M}$.*

Proof. We give the proof for $\int^j \mathrm{gr}^{rel}\mathscr{M}$, the proof for $\int^j \mathscr{M}$ is similar but easier.

Consider the following Cartesian square

$$\begin{array}{ccccc}
\mu^*T^*X \times \operatorname{Spec} R & \xrightarrow{T^*\mu} & T^*Y \times \operatorname{Spec} R & \xrightarrow{\pi_Y} & Y \times \operatorname{Spec} R \\
\downarrow{\scriptstyle \widetilde{\mu}} & & & & \downarrow{\scriptstyle \mu} \\
T^*X \times \operatorname{Spec} R & & \xrightarrow{\qquad\quad \pi_X \qquad\quad} & & X \times \operatorname{Spec} R.
\end{array}$$

Since π_X is flat the derived version of the flat base change theorem yields that [12]

$$\boldsymbol{L}\pi_X^*\boldsymbol{R}\mu_*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}) = \boldsymbol{R}\widetilde{\mu}_*\boldsymbol{L}(T^*\mu \circ \pi_Y)^*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}).$$

Since π_X is flat it follows that $\mathscr{H}^j\boldsymbol{L}\pi_X^*(-) = \pi_X^*\mathscr{H}^j(-)$[13]. It now suffices to show that the right hand side is $\int \mathrm{gr}^{rel}\mathscr{M}$.

Since π_Y is flat it holds that $\boldsymbol{L}(T^*\mu \circ \pi_Y)^* = \boldsymbol{L}(T^*\mu)^* \circ \boldsymbol{L}\pi_Y^*$[14]. We show that $\boldsymbol{L}\pi_Y^*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}) \cong \mathrm{gr}^{rel}\mathscr{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \widetilde{\mu}^*\,\mathrm{gr}^{rel}\mathscr{A}_X^R$ from which the result follows immediately.

Let $\mathcal{F}^\bullet$ denote a bimodule resolution for $\mathcal{R}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathcal{R}\mathscr{A}_Y^R$-modules. Then $(\mathcal{R}\mathscr{A}_Y^R/z\mathcal{R}\mathscr{A}_Y^R) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{F}^\bullet$ is a bimodule resolution for $\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. Now $\boldsymbol{L}\pi_Y^*$ just means applying $\pi^{-1}(-) \otimes \mathcal{O}_{T^*Y}$ to the terms of this free resolution. Due to flatness this yields a free resolution in $\pi^*\,\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules of $\pi^*\,\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X}$. Since $\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X} = \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}\,\mathrm{gr}^{rel}\mathscr{A}_X^R$ and $\pi^*\mu^* = \widetilde{\mu}^*\pi^*$ the desired equality follows. [15] $\square$

It turns out that one can directly compare $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ and $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ when ℓ is large. Some care is required since since $\int^j \mathcal{R}\mathscr{M}$ may have $\mathbb{C}[z]$-torsion.

---

[12]Note: Check in detail that the theorem is applicable and has this conclusion due to flatness
[13]Note: $\mathscr{H}^j\boldsymbol{L}\pi_X^*(-) = \pi_X^*\mathscr{H}^j(-)$?
[14]Note: $\boldsymbol{L}(T^*\mu \circ \pi_Y)^* = \boldsymbol{L}(T^*\mu)^* \circ \boldsymbol{L}\pi_Y^*$?
[15]Note: Write out more

Lemma 1.2.11. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, for sufficiently large ℓ, the kernel of z^ℓ in $\int^j \mathcal{R}\mathscr{M}$ stabilises. For such ℓ the quotient $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is the $\mathcal{R}\mathscr{A}_X^R$-coherent module associated to a good filtration on $\int^j \mathscr{M}$.*

Proof. By $\int \mathcal{R}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}_X^R$ it follows that $\ker z^\ell$ locally stabilises. This is sufficient since $X \times \operatorname{Spec} R$ is assumed to be compact.

Now consider the short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z-1} \mathcal{R}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z-1} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z-1} \cdots .$$

Since $\int^{j+1} \mathcal{R}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module it follows that $z-1$ is injective whence $\int^j \mathcal{R}\mathscr{M}/(z-1)\int^j \mathcal{R}\mathscr{M} \cong \int^j \mathscr{M}$. This yields the desired result using that $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is $\mathbb{C}[z]$-torsion free and the isomorphism

$$\frac{\int^j \mathcal{R}\mathscr{M}}{(z-1)\int^j \mathcal{R}\mathscr{M}} \cong \frac{\int^j \mathcal{R}\mathscr{M}/\ker z^\ell}{(z-1)(\int^j \mathcal{R}\mathscr{M}/\ker z^\ell)}.$$

$\square$

From now on we equip $\int^j \mathscr{M}$ with the good filtation inherited from the Rees module's direct image.

Lemma 1.2.12. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, if ℓ is sufficiently large, $\mathrm{gr}_{[\ell]}^{rel}\int^j \mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}_{[\ell]}^{rel}\mathscr{M}$.*

Proof. The short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \mathcal{R}\mathscr{M} \to \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \cdots .$$

Hence, $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M})$ and it remains to show that $\mathcal{R}\int^j \mathscr{M}/z^\ell\mathcal{R}\int^j \mathscr{M}$ is a quotient of $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$.

Let ℓ be sufficiently large so that **??** yields a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong \mathcal{R}\int^j \mathscr{M}$. The map z^ℓ induces a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong z^\ell \int^j \mathcal{R}\mathscr{M}$. Therefore $z^\ell \int^j \mathcal{R}\mathscr{M}/z^{2\ell} \int^j \mathcal{R}\mathscr{M} \cong \mathcal{R}\int^j \mathscr{M}/z^\ell\mathcal{R}\int^j \mathscr{M}$. The desired quotient follows by applying the map $m \mapsto z^\ell m$ on $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$. $\square$

The main remaining task is to relate these results to the desired case of $\ell = 1$.

Definition 1.2.4. *For any $\ell \geq 1$ the G-filtration on a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is defined by the decreasing sequence of $\mathrm{gr}_{[\ell]}^{rel}\mathscr{A}_Y^R$-submodules $G_j\mathcal{M} := z^j\mathcal{M}$.*

Lemma 1.2.13. *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ there is the a isomorphism of $\mathrm{gr}\,\mathscr{A}_Y^R$-modules*

$$\mathrm{gr}^G \mathrm{gr}_{[\ell]}^{rel}\mathscr{M} \cong (\mathrm{gr}^{rel}\mathscr{M})^\ell.$$

Proof. This follows from directly from the fact that $G_j \operatorname{gr}_{[\ell]}^{rel} \mathscr{M} = \oplus_k F_{k-j}\mathscr{M}/F_{k-\ell}\mathscr{M}$. □

Lemma 1.2.14. *Consider a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$. Then one has a isomorphism $\operatorname{gr}^G \int \mathcal{M} \cong \int \operatorname{gr}^G \mathcal{M}$ in $\boldsymbol{D}^{b,r}(\operatorname{gr}^{rel}\mathscr{A}_X^R)$.*

Proof. Writing out the direct image functors the desired result is a isomorphism

$$\operatorname{gr}^G \boldsymbol{R}\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R) \cong \boldsymbol{R}\mu_*(\operatorname{gr}^G \mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \operatorname{gr}^{rel}\mathscr{A}_X^R).$$

The proof of the commutation proceeds in two steps corresponding to the two derived functors.

Let $\mathscr{F}^\bullet$ be a bimodule resolution for $\mathcal{R}\mathscr{A}_{Y\to X}^R$ by locally free left $\mathcal{R}\mathscr{A}_Y^R$-modules. There is a G-filtration on this complex given by $z^j(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) = (z^j\mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. By the flatness of locally free sheaves and the short exact sequence $0 \to \oplus_j z^j \mathcal{M} \to \oplus_j z^{j-1}\mathcal{M} \to \operatorname{gr}^G \mathcal{M} \to 0$ it follows that $\operatorname{gr}^G(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) \cong (\operatorname{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. Further, by the argument in the proof of **??** the complex of $\operatorname{gr}^G \mathscr{A}_Y^R$-modules $(\operatorname{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ can be viewed as a representative of $(\operatorname{gr}^G \mathcal{M}) \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \operatorname{gr}^{rel}\mathscr{A}_X^R$.[16]

Denote $\mathcal{G}(-)$ for the functor which takes a sheaf complex and returns its Godement resolution. Flabby sheaves are acyclic for μ_* so the Godement resolution may be used to compute $\boldsymbol{R}\mu_*$. Moreover, since the terms of a Godement resolution are essentially direct sums of formal products of stalks, it is immediate that $z^i\mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(z^i\mathcal{N}^\bullet)$ and that $\operatorname{gr}^G \mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(\operatorname{gr}^G \mathcal{N}^\bullet)$ for any complex of right $\mu^{-1}\mathcal{R}\mathscr{A}_X^R$-modules $\mathcal{N}^\bullet$. Applying μ_* to these equalities and setting $\mathcal{N}^\bullet = \mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ yields the desired result. □

Proposition 1.2.15. *For a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it holds that*

$$\operatorname{supp} \operatorname{gr}^{rel} \int^j \mathscr{M} \subseteq \operatorname{supp} \int^j \operatorname{gr}^{rel} \mathscr{M}.$$

Proof. Let $\ell \geq 0$ be sufficiently large so that **??** holds, that is to say that $\operatorname{gr}_{[\ell]}^{rel} \int^j \mathscr{M}$ is a subquotient of $\int^j \operatorname{gr}_{[\ell]}^{rel} \mathscr{M}$. By **??** it holds that $\operatorname{gr}^G \operatorname{gr}_{[\ell]}^{rel} \int^j \mathscr{M} \cong (\operatorname{gr}^{rel} \int \mathscr{M})^\ell$. Since $\operatorname{gr}_{[\ell]}^{rel} \int^j \mathscr{M}$ is a subquotient of $\int \operatorname{gr}_{[\ell]}^{rel} \mathscr{M}$ it remains to show that the support of $\operatorname{gr}^G \int^j \operatorname{gr}_{[\ell]}^{rel} \mathscr{M}$ is a subset of the support of $\int^j \operatorname{gr} \mathscr{M}$.

This can be established with the spectral sequence associated of the G-filtered complex $\int \operatorname{gr}_{[\ell]}^{rel} \mathscr{M}$. Since the G-filtration is finite on $\operatorname{gr}_{[\ell]}^{rel} \mathscr{A}_X^R$-modules the associated spectral sequence abbuts by general results[17]. To be precise the associated spectral sequence with terms $E^2_{pq} = \mathscr{H}^{p+q} \operatorname{gr}^G \int \operatorname{gr}_{[\ell]}^{rel} \mathscr{M}$ abuts to $\operatorname{gr}^G \int \mathscr{M}$. By **??** and **??** it holds that $E^2_{pq} \cong (\int^{p+q} \operatorname{gr} \mathscr{M})^\ell$. [18] It follows that $\operatorname{supp} \operatorname{gr}^G \int^j \operatorname{gr}_{[\ell]}^{rel} \mathscr{M}$ is a subset of the support of $\int \operatorname{gr} \mathscr{M}$ which completes the proof. □

[16]Note: Check after lemma is entirely proven

[17]Note: Found spectral sequence result online, add good reference.

[18]Note: Or E^1? Seems to depend on preference but should actually matter somewhat for the differentials.