

Chapter 1

\mathcal{D}_X -modules and the Riemann-Hilbert Correspondence

Chapter 2

The Behaviour of \mathcal{A}_X^R -Modules

Mention BVWZ

The classical approximation of the roots of the b -polynomial due to Kashiwara (1976) relies on a quotient module $\int \mathcal{M}/\mathcal{D}_X u$ being holonomic. This is no longer true in the multivariate case but a refined assumption, called relative holonomicity, due to Maisonobe (2016) still holds. This refinement works with $\mathcal{D}_X \times \mathbb{C}[s]$ -modules whence one gets characteristic varieties inside $T^*X \times \mathbb{C}^p$.

p?

2.1 Modules over \mathcal{A}_X^R

Let X be a smooth complex irreducible algebraic variety of dimension n and denote \mathcal{D}_X for it's sheaf of rings of algebraic differential operators. For a regular commutative \mathbb{C} -algebra integral domain R we define a sheaf of rings on $X \times \text{Spec } R$ by

$$\mathcal{A}_X^R = \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R; \quad \mathcal{A}_X = \mathcal{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\text{Spec } R}$. The order filtration $F_p \mathcal{D}_X$ extends to a filtration $F_p \mathcal{A}_X^R = F_p \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R$ on \mathcal{A}_X^R which is called the relative filtration.

The proof of the following results proceeds precisely like the case of \mathcal{D}_X -modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2).

Proposition 2.1.1. *A quasi-coherent \mathcal{A}_X^R -module \mathcal{M} is coherent if and only if it admits a filtration such that $\text{gr}^{\text{rel}} \mathcal{M}$ is coherent over $\text{gr}^{\text{rel}} \mathcal{A}_X^R$.*

Proposition 2.1.2. *Let \mathcal{M} be a coherent \mathcal{A}_X^R -module, then the support of $\text{gr}^{\text{rel}} \mathcal{M}$ in $T^*X \times \text{Spec } R$ is independent of the chosen filtration. It is called the characteristic variety of \mathcal{M} and denoted $\text{Ch}^{\text{rel}} \mathcal{M}$.*

A coherent \mathcal{A}_X^R -module \mathcal{M} is said to be relative holonomic over R if $\text{Ch}^{\text{rel}} \mathcal{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \text{Spec } R$.

2.2 Direct Image Functor for \mathcal{A}_X^R -modules

In this section we state the natural generalisation of the direct image functor for \mathcal{D}_X -modules to the relative case of \mathcal{A}_X^R -modules. As with \mathcal{D} -modules this is the most natural for right-modules.

more introduction

Maybe also mention the example Robin put on the whiteboard? Possibly in the main body?

Transfer Modules and \mathcal{A}_Y^R -module Direct Image

Let $\mu : Y \rightarrow X$ be some morphism of smooth algebraic varieties. A-priori it is not even clear what \mathcal{A}_X^R -module should correspond to \mathcal{A}_Y^R since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathcal{A}_Y^R, \mu^{-1}\mathcal{A}_X^R)$ -bimodule $\mathcal{A}_{Y \rightarrow X}^R := (\mathcal{O}_Y \times \mathcal{O}_R) \otimes_{\mu^{-1}(\mathcal{O}_X \times \mathcal{O}_R)} \mu^{-1}\mathcal{A}_X^R$.

Definition 2.2.1. The direct image functor \int_μ from $\mathbf{D}^{b,r}(\mathcal{A}_Y^R)$ to $\mathbf{D}^{b,r}(\mathcal{A}_X^R)$ is defined to be $\mathbf{R}\mu_*(- \otimes_{\mathcal{A}_Y^R}^L \mathcal{A}_{Y \rightarrow X}^R)$. For any \mathcal{A}_Y^R module \mathcal{M} the j -th direct image is the \mathcal{A}_X^R -modules $\int_\mu^j \mathcal{M} = \mathcal{H}^j \int_\mu \mathcal{M}$. The subscript μ will be suppressed whenever there is no ambiguity.

To compute $\int_\mu^j \mathcal{M}$ a resolution for $\mathcal{A}_{Y \rightarrow X}$ is required.

Definition 2.2.2. Let \mathcal{M} be a right \mathcal{A}_Y^R -module, the relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathcal{M})$ is a complex of right \mathcal{A}_Y^R -modules, concentrated in negative degrees, with $\mathrm{Sp}_Y^{-k}(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right- \mathcal{A}_Y^R -linear map δ given by

$$\begin{aligned} m \otimes \xi_1 \wedge \cdots \wedge \xi_k &\mapsto \sum_{i < j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k \\ &\quad - \sum_{i=1}^k (-1)^i m \xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \end{aligned}$$

Lemma 2.2.1. The complex of $(\mathcal{A}_X, f^{-1}\mathcal{A}_Y)$ -bimodules $\mathrm{Sp}_{Y \rightarrow X}^\bullet(\mathcal{A}_Y^R) := \mathrm{Sp}_Y^\bullet(\mathcal{A}_Y^R) \otimes_{\mathcal{O}_Y} \mathcal{A}_{Y \rightarrow X}^R$ is a resolution of $\mathcal{A}_{Y \rightarrow X}^R$ as a bimodule by locally free left \mathcal{A}_Y^R -modules.

Proof. This will be analogous to the case of \mathcal{D}_Y -modules in (Sabbah, 2011, p33). Also see <http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf> \square

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 2.2.2. It holds that $\int = \mathbf{R}\mu_*(- \otimes_{\mathcal{A}_Y^R} \mathrm{Sp}_{Y \rightarrow X}^\bullet(\mathcal{A}_Y^R))$.

Lemma 2.2.3. Construction of global section in $\int^j \mathcal{M}$.

Theorem 2.2.4. Long exact sequence

Functorial Properties of the Direct Image

Theorem 2.2.5. Let $\mu : Z \rightarrow Y$ and $\nu : Y \rightarrow X$ be morphisms of smooth algebraic varieties. If μ is proper then $\int_{\nu \circ \mu} = \int_\nu \int_\mu$.

Proof. See <http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf> \square

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \rightarrow Y \times X$ and $\pi : Y \times X \rightarrow X$.

Denote by $\mathbf{D}_{qc}^{b,r}(\mathcal{A}_Y^R)$ the full subcategory of $\mathbf{D}^{b,r}(\mathcal{A}_Y^R)$ consisting of those complexes of right \mathcal{A}_Y^R -modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\mathrm{Spec} R}$. Similarly for $\mathbf{D}_{coh}^{b,r}(\mathcal{A}_Y^R)$ with the cohomology being coherent \mathcal{A}_Y^R -modules.

Theorem 2.2.6. *Let $\mu : X \rightarrow Y$ be a morphism of nonsingular algebraic varieties. Then the direct image \int takes $D_{qc}^{b,r}(\mathcal{A}_Y^R)$ into $D_{qc}^{b,r}(\mathcal{A}_X^R)$. Moreover, when μ is proper the direct image takes $D_{coh}^{b,r}(\mathcal{A}_Y^R)$ into $D_{coh}^{b,r}(\mathcal{A}_X^R)$.*

Proof. See <http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture18.pdf> □

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \rightarrow X$ be a proper morphism of smooth algebraic varieties. Given a coherent \mathcal{A}_X^R -module \mathcal{M} with relative characteristic variety $\text{Ch}^{rel} \mathcal{M}$. We desire to estimate $\text{Ch}^{rel} \int^j \mathcal{M}$ in terms of \mathcal{M} . Such an estimate in the non-relative case is known due to Kashiwara (1976).

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The proof we consider here is adapted from a proof by Malgrange (1985) and we follow the exposition by Sabbah (2011).

Consider the following cotangent diagram

$$\begin{array}{ccc} & \mu^* T^* X \times \text{Spec } R & \\ T^* \mu \swarrow & & \searrow \tilde{\mu} \\ T^* Y \times \text{Spec } R & & T^* X \times \text{Spec } R \end{array}$$

where the maps $T^* \mu$ and $\tilde{\mu}$ act on the first component.

Theorem 2.2.7. *Let \mathcal{M} be a coherent \mathcal{A}_Y^R -module. Then, for any $j \geq 0$, we have*

$$\text{Ch}^{rel} \left(\int^j \mathcal{M} \right) \subseteq \tilde{\mu} \left((T^* \mu)^{-1} (\text{Ch}^{rel} \mathcal{M}) \right).$$

The first step is to note that a similar inclusion is easy for the $\text{gr}^{rel} \mathcal{A}_Y^R$ -modules. For any $\text{gr}^{rel} \mathcal{A}_Y^R$ -module \mathcal{M} define $\int^j \mathcal{M} := \mathcal{H}^j(\mathbf{R}\mu_*(\mathbf{L}(T^* \mu)^* \mathcal{M}))$. Note that $(T^* \mu)^*$ produces a sheaf on $\mu^* T^* X \times \text{Spec } R$ by the tensor product $- \otimes_{f^{-1} \mathcal{O}_X \times \mathcal{O}_R} \text{gr}^{rel} \mathcal{A}_X^R$. Hence, looking at the supports, the following result is immediate.

Lemma 2.2.8. *For any $\text{gr}^{rel} \mathcal{A}_Y^R$ -module \mathcal{M} it holds that*

$$\text{Supp} \int^j \mathcal{M} \subseteq \tilde{\mu} \left((T^* \mu)^{-1} \text{Supp } \mathcal{M} \right).$$

Applying this to $\text{gr}^{rel} \mathcal{M}$ it remains to understand the difference between $\text{gr}^{rel} \int^j \mathcal{M}$ and $\int^j \text{gr}^{rel} \mathcal{M}$. This may be done using relative Rees modules.

Definition 2.2.3. *Let z be a new variable. The relative Rees sheaf of rings $\mathcal{R}\mathcal{A}_Y^R$ is defined as the subsheaf $\bigoplus_p F_p \mathcal{A}_Y^R z^p$ of $\mathcal{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered \mathcal{A}_Y^R -module \mathcal{M} gives rise to a $\mathcal{R}\mathcal{A}_Y$ -module $\mathcal{R}^{rel} \mathcal{M} := \bigoplus_p F_p \mathcal{M} z^p$.*

Coherent should suffice in algebraic case I think, Sabbah needs assumptions to guarantee the existence of a global filtration in analytic case.

The following obvious isomorphisms of filtered modules allow us to view the relative Rees module as a way to interpolate between \mathcal{M} and $\text{gr}^{\text{rel}}\mathcal{M}$

$$\frac{\mathcal{R}^{\text{rel}}\mathcal{M}}{(z-1)\mathcal{R}^{\text{rel}}\mathcal{M}} \cong \mathcal{M}; \quad \frac{\mathcal{R}^{\text{rel}}\mathcal{M}}{z\mathcal{R}^{\text{rel}}\mathcal{M}} = \text{gr}^{\text{rel}}\mathcal{M}.$$

Conversely, the second formula may be used to produce a filtered \mathcal{A}_Y^R -module from any graded $\mathcal{R}\mathcal{A}_Y^R$ -module without $\mathbb{C}[z]$ -torsion.

One can define $\int^j \mathcal{R}^{\text{rel}}\mathcal{M}$ similarly to the \mathcal{A}_Y^R -module direct image and these are coherent $\mathcal{R}\mathcal{A}_X^R$ -modules similarly to theorem 2.2.6. The following is immediate from $\int \mathcal{R}^{\text{rel}}\mathcal{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathcal{A}_X^R$.

Lemma 2.2.9. *For sufficiently large $\ell \gg 1$ the kernel of z^ℓ in $\int^j \mathcal{R}^{\text{rel}}\mathcal{M}$ is locally stationary. The quotient of $\int^j \mathcal{R}^{\text{rel}}\mathcal{M}$ by it's z -torsion is a $\mathcal{R}\mathcal{A}_X^R$ -coherent module.*

This Rees module is associated with some good filtration on $\int^j \mathcal{M}$. (Why?)

Corollary 2.2.10. *For all $\ell \geq 1$ it holds that $\int^j \mathcal{R}^{\text{rel}}\mathcal{M}/z^\ell \int^j \mathcal{R}^{\text{rel}}\mathcal{M}$ is a submodule of $\int^j (\mathcal{R}^{\text{rel}}\mathcal{M}/z^\ell \mathcal{R}^{\text{rel}}\mathcal{M})$. Further, if ℓ is sufficiently large then $\mathcal{R}^{\text{rel}}\int^j \mathcal{M}/z^\ell \mathcal{R}^{\text{rel}}\int^j \mathcal{M}$ is a quotient of $\int^j \mathcal{R}^{\text{rel}}\mathcal{M}/z^\ell \int^j \mathcal{R}^{\text{rel}}\mathcal{M}$.*

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