Chapter 1

$\mathscr{D}_X$-modules and the Riemann-Hilbert Correspondence

Chapter 2

The Behaviour of $\mathscr{A}_X^R$-Modules

Mention BVWZ

The classical approximation of the roots of the b-polynomial due to Kashiwara (1976) relies on a quotient module $\int \mathscr{M}/\mathscr{D}_X u$ being holonomic. This is no longer true in the multivariate case but a refined assumption, called relative holonomicity, due to Maisonobe (2016) still holds. This refinement works with $\mathscr{D}_X \times \mathbb{C}[s]$-modules whence one gets characteristic varieties inside $T^*X \times \mathbb{C}^p$. p?

2.1 Modules over $\mathscr{A}_X^R$

Maybe also mention the example Robin put on the whiteboard? Possibly in the main body?

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2).

Proposition 2.1.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\operatorname{gr}^{rel}\mathscr{M}$ is coherent over $\operatorname{gr}^{rel}\mathscr{A}_X^R$.*

Proposition 2.1.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\operatorname{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\operatorname{Ch}^{rel}\mathscr{M}$.*

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\operatorname{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

2.2 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules. more introduction

Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties. A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := (\mathcal{O}_Y \times \mathcal{O}_R) \otimes_{\mu^{-1}(\mathcal{O}_X \times \mathcal{O}_R)} \mu^{-1}\mathscr{A}_X^R$.

Definition 2.2.1. *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be surpressed whenever there is no ambiguity.*

To compute $\int^j \mathscr{M}$ a resolution for $\mathscr{A}_{Y\to X}$ is required.

Definition 2.2.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map δ given by*

$$m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k$$
$$- \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k$$

Lemma 2.2.1. *The complex of $(\mathscr{A}_X, f^{-1}\mathscr{A}_Y)$-bimodules $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}^R$ is a resolution of $\mathscr{A}_{Y\to X}^R$ as a bimodule by locally free left $\mathscr{A}_Y^R$-modules.*

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Proof. This will be analogous to the case of $\mathscr{D}_Y$-modules in (Sabbah, 2011, p33). Also see `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` □

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 2.2.2. *It holds that $\int = \boldsymbol{R}\mu_*(- \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$.*

Lemma 2.2.3. *Construction of global section in $\int^j \mathscr{M}$.*

Theorem 2.2.4. *Long exact sequence*

Functorial Properties of the Direct Image

Theorem 2.2.5. *Let $\mu : Z \to Y$ and $\nu : Y \to X$ be morphisms of smooth algebraic varieties. If μ is proper then $\int_{\nu\circ\mu} = \int_\nu \int_\mu$.*

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` □

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $\boldsymbol{D}_{qc}^{b,r}(\mathscr{A}_Y^R)$ the full subcategory of $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ consisting of those complexes of right $\mathscr{A}_Y^R$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\mathrm{Spec}\, R}$. Similarly for $\boldsymbol{D}_{coh}^{b,r}(\mathscr{A}_Y^R)$ with the cohomology being coherent $\mathscr{A}_Y^R$-modules.

Theorem 2.2.6. *Let $\mu : X \to Y$ be a morphism of nonsingular algebraic varieties. Then the direct image $\int$ takes $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_Y^R)$ into $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_X^R)$. Moreover, when μ is proper the direct image takes $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_Y^R)$ into $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_X^R)$.*

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture18.pdf` □

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j \mathscr{M}$ in terms of $\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara Kashiwara (1976).

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The idea of the following proof is due to Malgrange (1985) in a K-theoretic context and we follow the exposition by Sabbah (2011).

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

Theorem 2.2.7. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

$$\mathrm{Ch}^{rel}\Big(\int^j \mathscr{M}\Big) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

Note that the statement is local so, after replacing X by some affine open, we may assume that $X \times \operatorname{Spec} R$ and $Y \times \operatorname{Spec} R$ are compact.

The first step is to note that a similar inclusion is easy for the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ define $\int^j \mathcal{M} := \mathscr{H}^j\left(\boldsymbol{R}\mu_*(\boldsymbol{L}(T^*\mu)^*\mathcal{M})\right)$. Note that $(T^*\mu)^*$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by the tensor product $- \otimes_{f^{-1}\mathcal{O}_X \times \mathcal{O}_R} \mathrm{gr}^{rel}\mathscr{A}_X^R$. Hence, looking at the supports, the following result is immediate.

Lemma 2.2.8. *For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ it holds that*

$$\operatorname{Supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1} \operatorname{Supp} \mathcal{M}\right).$$

Applying this to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to understand the difference between $\mathrm{gr}^{rel}\int^j \mathscr{M}$ and $\int^j \mathrm{gr}^{rel}\mathscr{M}$. This may be done using relative Rees modules.

Definition 2.2.3. *Let z be a new variable. The relative Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}^{rel}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

The following obvious isomorphisms of filtered modules allow us to view the relative Rees module as a parametrisation of gradings with different steps.

$$\frac{\mathcal{R}^{rel}\mathscr{M}}{(z-1)\mathcal{R}^{rel}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}^{rel}\mathscr{M}}{z\mathcal{R}^{rel}\mathscr{M}} = \mathrm{gr}^{rel}\mathscr{M}; \qquad \frac{\mathcal{R}^{rel}\mathscr{M}}{z^\ell\mathcal{R}^{rel}\mathscr{M}} = \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}.$$

Here $\mathrm{gr}^{rel}_{[\ell]}$ takes a filtered object and returns the graded object $\oplus_k F_k/F_{k-\ell}$. In particular $\mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ is a graded $\mathrm{gr}_{[\ell]}\mathscr{A}_X^Y$-module. The first step is to understand the interaction between direct images and $\mathrm{gr}^{rel}_{[\ell]}$ for large ℓ. The first formula may be also be used to find the corresponding filtered $\mathscr{A}_Y^R$-module for any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

One can define a direct image of $\mathcal{R}\mathscr{A}_Y^R$-modules similarly to the $\mathscr{A}_Y^R$-module direct image and this preserves are coherence and being graded similarly to theorem 2.2.6.

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Lemma 2.2.9. *For sufficiently large ℓ the kernel of z^ℓ in $\int^j \mathcal{R}^{rel}\mathscr{M}$ stabilises. For such ℓ the quotient $\int^j \mathcal{R}^{rel}\mathscr{M} / \ker z^\ell$ is the $\mathcal{R}\mathscr{A}_X^R$-coherent module associated to a good filtration on $\int^j \mathscr{M}$.*

Proof. By $\int \mathcal{R}^{rel}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}_X^R$ one gets that $\ker z^\ell$ locally stabilises. This is sufficient due to $X \times \operatorname{Spec} R$ being compact.

Now consider the short exact sequence $0 \to \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z-1} \mathcal{R}^{rel}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z-1} \int^j \mathcal{R}^{rel}\mathscr{M} \to \int^j \mathscr{M} \to \int^{j+1} \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z-1} \cdots.$$

Since $\int^{j+1} \mathcal{R}^{rel}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module one has that $z-1$ is injective whence it follows that $\int^j \mathcal{R}^{rel}\mathscr{M}/(z-1)\int^j \mathcal{R}^{rel}\mathscr{M} \cong \int^j \mathscr{M}$. This yields the desired result using that $\int^j \mathcal{R}^{rel}\mathscr{M} / \ker z^\ell$ is $\mathbb{C}[z]$-torsion free and the isomorphism

$$\frac{\int^j \mathcal{R}^{rel}\mathscr{M}}{(z-1)\int^j \mathcal{R}^{rel}\mathscr{M}} \cong \frac{\int^j \mathcal{R}^{rel}\mathscr{M} / \ker z^\ell}{(z-1)(\int^j \mathcal{R}^{rel}\mathscr{M} / \ker z^\ell)}.$$

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From now on we equip $\int^j \mathscr{M}$ with the good filtation inherited from the Rees module's direct image. By the formula relating Rees modules and $\mathrm{gr}^{rel}_{[\ell]}$ the direct image of Rees modules induces a direct image of $\mathrm{gr}_{[\ell]}\mathscr{A}_Y^R$-modules.

Lemma 2.2.10. *If ℓ is sufficiently large then $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$.*

Proof. The goal is to establish the equivalent statement for Rees modules. The short exact sequence $0 \to \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z^\ell} \mathcal{R}^{rel}\mathscr{M} \to \mathcal{R}^{rel}\mathscr{M}/z^\ell\mathcal{R}^{rel}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z^\ell} \int^j \mathcal{R}^{rel}\mathscr{M} \to \int^j \mathcal{R}^{rel}\mathscr{M}/z^\ell\mathcal{R}^{rel}\mathscr{M} \to \int^{j+1} \mathcal{R}^{rel}\mathscr{M} \xrightarrow{z^\ell} \cdots.$$

Hence, $\int^j \mathcal{R}^{rel}\mathscr{M}/z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}^{rel}\mathscr{M}/z^\ell\mathcal{R}^{rel}\mathscr{M})$ and it remains to show that $\mathcal{R}^{rel}\int^j \mathscr{M}/z^\ell\mathcal{R}^{rel}\int^j \mathscr{M}$ is a quotient of $\int^j \mathcal{R}^{rel}\mathscr{M}/z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}$.

Let ℓ be sufficiently large so that the foregoing lemma yields a isomorphism $\int^j \mathcal{R}^{rel}\mathscr{M}/\ker z^\ell \cong \mathcal{R}^{rel}\int^j \mathscr{M}$. The map z^ℓ induces a isomorphism $\int^j \mathcal{R}^{rel}\mathscr{M}/\ker z^\ell \cong z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}$. Therefore $z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}/z^{2\ell} \int^j \mathcal{R}^{rel}\mathscr{M} \cong \mathcal{R}^{rel}\int^j \mathscr{M}/z^\ell\mathcal{R}^{rel}\int^j \mathscr{M}$. The desired quotient follows by applying the map $m \mapsto z^\ell m$ on $\int^j \mathcal{R}^{rel}\mathscr{M}/z^\ell \int^j \mathcal{R}^{rel}\mathscr{M}$. □

The main remaining task is to relate these results to the desired case of $\ell = 1$.

Definition 2.2.4. *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ the G-filtration on $\mathrm{gr}_{[\ell]}^{rel}\mathscr{M}$ is defined by*

$$G_j\mathscr{M} = \bigoplus_k \frac{F_{k+j-\ell}\mathscr{M}}{F_{k-\ell}\mathscr{M}}.$$

Lemma 2.2.11. *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ one has the following isomorphism of $\mathrm{gr}\,\mathscr{A}_Y^R$-modules*

$$\mathrm{gr}^G \mathrm{gr}_{[\ell]}^{rel}\mathscr{M} \cong (\mathrm{gr}^{rel}\mathscr{M})^\ell.$$

Proof. This is immediate by a direct computation. □

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