Introduction

The purpose of this thesis is provide a improved estimate for the zero locust of Bernstein-Sato ideals.

Historically, this problem arose when trying to extend a function to the entire complex plane. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be some fixed positive polynomial and $g \in \mathcal{C}_c^\infty$ be some test function. Gelfand asked if it is possible to find a meromorphic extension of the function

$$\Gamma_g(s) = \int_{\mathbb{R}^n} g f^s dx; \qquad \mathrm{Re}(s) > 0$$

to the entire complex plane. A proof by Bernstein relies on the existence of a differential operator $P(x, \partial, s)$ such that

$$P(x, \partial, s) f^{s+1} = b(s) f^s$$

for some polynomial $b(s) \in \mathbb{C}[s]$. The polynomial of minimal degree is called the Bernstein-Sato polynomial. In particular, this method shows that any pole of Γ_g is a root of $b(s)$ up a shift with a negative integer.

The relation between the roots of $b(s)$ and the poles of functions like Γ_g is the topic of a notable open problem called the monodromy conjecture. Futher, the roots of $b(s)$ give some insight into the behaviour of the ill-defined function f^{-1}. Suppose that $f(0) = 0$ and consider f as a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let $t \in \mathbb{C}^\times$ be close to 0 and consider the intersection $X_t := f^{-1}(t) \cap B_\varepsilon$ where B_ε is some small ball centered at 0. Twisting t around the origin induces a diffeomorphism M of X_t which is called the monodromy. In particular, one gets linear endomorphisms M^* on the vector spaces $H^j(X_t, \mathbb{C})$ the eigenvalues of this action are called the eigenvalues of monodromy. The monodromy theorem states that the eigenvalues of monodromy are precisely the set $\exp(Z(b))$ where $Z(b)$ denotes the collection of roots of the Bernstein-Sato polynomial.

This shows that the roots of the Bernstein-Sato polynomial are a worthy topic of study. Estimation of the roots of the Bernstein-Sato polynomial has been done in terms of data from a resolution of singularities. The resolution reduces the problem to the case where f is a monomial in which case the Bernstein-Sato polynomial can be computed explicitly. The main difficulty is to connect the easier problem to the original problem. This relies on the sheaf-theoretic framework of $\mathscr{D}_X$-modules and their direct images. Here $\mathscr{D}_X$ denotes the sheaf of differential operators on a space X.

The original estimate due to Kashiwara establishes that the roots of $b(s)$ are negative rational numbers. A lower bound for the distance between the largest root and 0 was established by Lichtin. This refined estimate uses a similar methodology to Kashiwara but replaced f^s with the distribution $f^s dx$.

In this thesis a multivariate generalization of the problem is considered. Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a function germ with coordinate functions $f_1, \dots, f_p$ be polynomials and define variables $s_1, \dots, s_p$. It is then still known that there exists a differential operator $P(x, \partial, s)$ such that

$$P(x, \partial, s) f_1^{s_1+1} \cdots f_p^{s_p+1} = b(s) f_1^{s_1} \cdots f_p^{s_p}$$

for some polynomial $b(s) \in \mathbb{C}[s_1, \dots, s_p]$. The collection of all possible polynomial $b(s)$ form an ideal of $\mathbb{C}[s_1, \dots, s_p]$ which is called the Bernstein-Sato ideal and is denoted B_F. The roots of Bernstein-Sato polynomials are generalised by the zero locust $Z(B_F)$.

There are generalisations of the monodromy conjecture and monodromy theorem to the multivariate situation involving $Z(B_F)$. Kashiwara's estimate for the roots of $b(s)$ has been generalised to a estimation of the Bernstein-Sato ideal by Budur et al. This thesis also generalises the refined estimate due to Lichtin.

There are two main steps. Firstly, one must check that the properties of the direct images of $\mathscr{D}_X$-modules generalise to $\mathscr{D}_X\langle s_1, \dots, s_p\rangle$-modules. Then, a inductive argument is used to reduce the number of s_i to one. Homological algebra is used to control error terms of the induction process.

The first chapter in this thesis gives a overview of various known results for $\mathscr{D}_X$-modules. This builds up to the Riemann-Hilbert correspondence which is a broad generalisation of the the equivalence between systems of differential equations and their solutions. The second chapter generalises notions for $\mathscr{D}_X$-modules to $\mathscr{D}_X\langle s_1, \dots, s_p\rangle$-modules and contains the proof for the improved estimate of $Z(B_F)$.

Contents

Chapter 1

Categorical Preliminaries

This chapter contains some categorical preliminaries on the topic of derived category theory and spectral sequences.

Derived category theory allows to measure the lack of exactness in a functor $F : \mathcal{A} \to \mathcal{B}$ by encoding error-terms in derived functors $R^iF : \mathcal{A} \to \mathcal{B}$. For instance the non-exactness of the tensor product may be measured by $\mathcal{T}or$-functors.

Spectral sequences were historically developed by Leray in to compute the cohomology of the pushforward of a sheaf. There is some overlap between derived category theory and spectral sequences. In particular the Grothendieck spectral sequence allows one to compute the derived functor of some composition $F \circ G$ based on the derived functors of F and G individually. This theorem is a essential technical ingredient in the proofs of chapter 3.

The discussion of derived category theory in this chapter is a summarisation of the relevant parts of (Dimca, 2004, Chapters 1, 2 and 5). The section on spectral sequences is based on Weibel (1995).

1.1 Spectral Sequences

Fix a abelian category $\mathcal{A}$. Given a double complex E_{**} one can define a total complex with terms $\mathrm{Tot}(E)_n = \oplus_{i+j=n} E_{ij}$. The motivating question behind spectral sequences is how the homology of the total complex may be computed.

Definition 1.1.1. *A homology spectral sequence starting at the a-th sheet consists of families of objects $E^r_{\bullet\bullet} = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ for $r \geq a$ and maps $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ such that*

(i) The maps d^r_{pq} are differentials in the sense that $d^r \circ d^r = 0$.

(ii) The $(r+1)$-st sheet is the homology of the r-th sheet $E^{r+1}_{pq} \cong \ker(d^r_{pq})/\operatorname{Im}(d^r_{p+r,q-r+1})$.

In particular, one can show that a double complex gives rise to a spectral sequence. The E^0-sheet is simply the double complex with it's vertical maps. The E^1-sheet is the homology of the E^0-sheet with horizontal maps induced by those of the double complex. The differentials on the higher sheets may be constructed by diagram chasing.

Definition 1.1.2. *A homology spectral sequence is said to be bounded if for each n there are finitely many terms $E^\bullet_{pq}$ with $p + q = n$.*

Provided boundedness, there is for each pq a value r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$. This stable value is denoted E^∞_{pq}.

Definition 1.1.3. *A bounded spectral sequence is said to converge to a family of object H_* if any H_n admits a finite filtration*

$$0 = F_s H_n \subseteq \cdots F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

such that $E^\infty_{pq} \cong F_p H_{p+q}/F_{p-1} H_{p+q}$.

The orignal double complex in the motivating problem can also be recovered from the total complex using the filtration $F_m \operatorname{Tot}(E)_n = \oplus_{p+q=n, p<m} E_{pq}$.

Definition 1.1.4. *A filtration of a chain complex $C_\bullet$ is a family of subcomplexes $\{F_m C_\bullet\}_{m\in\mathbb{Z}}$. The filtration is said to be exhaustive if $C_\bullet = \cup_m F_m C_\bullet$.*

Proposition 1.1.1. *A filtration of a chain complex $C_\bullet$ detemines a spectral sequence starting with $E^0_{pq} = F_p C_{p+q}/F_{p-1} C_{p+q}$ and $E^1_{pq} = H_{p+q} E^0_{p\bullet}$.*

Definition 1.1.5. *A filtration on a chain complex $C_\bullet$ is said to be bounded if, for each n, there are integers $s < t$ such that $F_s C_n = 0$ and $F_t C_n = C_n$.*

Proposition 1.1.2. *Let $C_\bullet$ be a chain complex with a bounded filtration. Then the associated spectral sequence is bounded and converges to $H_*(C_\bullet)$.*

1.2 Derived Categories

The category $C(\mathcal{A})$ contains full subcategories $C^*(\mathcal{A})$ with $* \in \{+, -, b\}$ denoting that the complexes in $\mathcal{A}$ are bounded below, above or bounded on both sides respectively. For example $C^+(\mathcal{A})$ may contain complexes of the form $\cdots \to 0 \to \cdots X^{-1} \to X^0 \to \cdots$. For a complex $X^\bullet$ and $k \in \mathbb{Z}$ one has a shifted complex $X^\bullet[k]$ with $(X^\bullet[k])^s = X^{k+s}$. Further, denote $\operatorname{Hom}^k(X^\bullet, Y^\bullet) := \operatorname{Hom}(X^\bullet, Y^\bullet[k])$ which are the chain maps that change the grading by k.

Definition 1.2.1. *Two complex morphisms $u, v : X^\bullet \to Y^\bullet$ are called homotopic if there exists $h \in \operatorname{Hom}^{-1}(X^\bullet, Y^\bullet)$ such that $u - v = d_Y h + h d_X$.*

Definition 1.2.2. *A morphism $u : X^\bullet \to Y^\bullet$ of complexes in $C^*(\mathcal{A})$ is called a quasi-isomorphism if the induced morphism in cohomology $H^k(u) : H^k(X^\bullet) \to H^k(Y^\bullet)$ is a isomorphism for all k. This may be denoted $u \sim v$.*

The derived category $D^*(\mathcal{A})$ is defined as the category obtained from $C^*(\mathcal{A})$ by localising with respect to the multiplicative system formed by the quasi-isomorphisms. This means that $D^*(\mathcal{A})$ has the same objects as $C^*(\mathcal{A})$ but the quasi-isomorphisms of $C^*(\mathcal{A})$ have been turned into isomorphisms. This definition can be made more concrete provided the category has enough injectives.

Definition 1.2.3. *A abelian category $\mathcal{A}$ has enough injectives if for any object X in $\mathcal{A}$ there is an exact sequence $0 \to X \to I$ in $\mathcal{A}$ with I injective.*

Definition 1.2.4. *Let $\mathcal{A}$ be a abelian category. The homotopical category of complexes $K^*(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $C^*(\mathcal{A})$ and as morphisms*

$$\mathrm{Hom}_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet) := \mathrm{Hom}_{C^*(\mathcal{A})}(X^\bullet, Y^\bullet)/\sim .$$

Observe that two homotopic maps induce the same morphism in cohomology. It follows that there is a well-defined functor $p^*_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$.

Proposition 1.2.1. *Let $\mathcal{A}$ be a abelian category with enough injectives and denote $I(\mathcal{A})$ for the full subcategory from the injective objects. Then the natural functor*

$$p^*_{\mathcal{A}} : K^+(I(\mathcal{A})) \to D^+(\mathcal{A})$$

is a equivalence of categories.

By passing to the opposite categories one gets a similar theorem in categories with enough projectives for $D^-(\mathcal{A})$.

1.3 Triangulated Categories

The categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ remain additive but may fail to be exact. In particular, the notion of short exact sequences no longer makes sense. Instead, $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ may be viewed as triangulated categories which is to say that they come equipped with a notion of exact triangles.

Definition 1.3.1. *Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. The mapping cone of u is the complex in $C^*(\mathscr{A})$ given by*

$$C^\bullet_u := Y^\bullet \oplus (X^\bullet[1])$$

with $d_u(y, x) = (dy + u(x), -dx)$.

The concept of a mapping cone originated in a construction from algebraic topology which explains the name. Observe that the mapping cone gives rise to a triangle

$$T_u : X^\bullet \xrightarrow{u} Y^\bullet \to C^\bullet_u \to X^\bullet[1]$$

which may be denoted more intuitively as

$$\begin{tikzcd} X^\bullet \arrow[rr, "u"] & & Y^\bullet \arrow[dl, "q"] \\ & C^\bullet_u \arrow[ul, "+1"] & \end{tikzcd}$$

The triangles T_u may be used to encode short exact sequences.

Proposition 1.3.1. *Given a short exact sequence in $C^*(\mathcal{A})$*

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

there exists a quasi-isomorphism $m : C^\bullet_u \to Z^\bullet$ with $m \circ q = v$.

This shows that a short exact sequence induces a triangle isomorphic to a standard triangle T_u in $D^*(\mathcal{A})$. Further evidence that the triangles T_u behave like short exact sequences is given by the following result.

Proposition 1.3.2. *Let $u : X^\bullet \to Y^\bullet$ be a morphism in $C^*(\mathcal{A})$.*

(i) The composition of any two consecutive maps in T_u is homotopic to 0.

(ii) The triangle T_u induces a long exact sequence in cohomology

$$\cdots \to H^k(X^\bullet) \xrightarrow{u} H^k(Y^\bullet) \to H^k(C_u^\bullet) \xrightarrow{\delta} H^{k+1}(X^\bullet) \to \cdots$$

where the connecting morphism δ comes from the map $C_u^\bullet \to X^\bullet[1]$.

Further investigation of the properties of T_u gives rise to the concept of a triangulated category. These definitions and properties are pleasant in their own right so we go into some detail.

The distinguish triangles $\mathcal{T}$ in $K^*(\mathcal{A})$ or $D^*(\mathcal{A})$ are the family of triangles which are isomorphic to a triangle of the form T_u. Observe that these categories have a shift functor T given by $TX^\bullet = X^\bullet[1]$.

Definition 1.3.2. *An additive category $\mathcal{D}$ equipped with a self-equivalence T and family of distinguished triangles $\mathcal{T}$ is called a triangulated category if the following axioms are satisfied.*

(Tr1) Any triangle isomorphic to a distinguish triangle is distinguished. For any object X the triangle $X \to X \to 0 \to TX$ is distinguish where the first morphism is the identity. Any morphism $u : X \to Y$ is part of some distinguished triangle $X \xrightarrow{u} Y \to Z \to TX$.

(Tr2) A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is distinguished.

(Tr3) A commutative diagram of the following from whose rows are distinguished triangles gives rise to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow & & \downarrow & & & & \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \end{array}$$

(Tr4) For any triple of distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{x} A \longrightarrow TX$$

$$Y \xrightarrow{v} Z \longrightarrow B \xrightarrow{y} TY$$

$$X \xrightarrow{vu} Z \longrightarrow C \longrightarrow TX$$

there is a distinguished triangle

$$A \xrightarrow{a} C \xrightarrow{b} B \xrightarrow{(Tx)y} TA$$

such that (id_X, v, a) and (u, id_Z, b) are morphisms of triangles.

Proposition 1.3.3. *Let $\mathcal{A}$ be a abelian category. Then $K^*(\mathscr{A})$ and $D^*(\mathscr{A})$ are triangulated categories.*

A triangle $X \to Y \to Z \to TX$ will also be denoted $X \to Y \to Z \xrightarrow{+1} X$ and $T^m X$ may be denoted with $X[m]$. Now the data of the final axiom can be organised as follows. Correspondingly, $(Tr4)$ is also referred to as the octahedral axiom.

$$\begin{tikzcd}
& C \arrow[dr] \arrow[ddl] & \\
A \arrow[ur] \arrow[d] & & B \arrow[ll] \arrow[ddl] \\
X \arrow[rr, "v \circ u"] \arrow[dr, "u"'] & & Z \arrow[u] \arrow[uul] \\
& Y \arrow[uul] \arrow[ur, "v"'] &
\end{tikzcd}$$

Definition 1.3.3. *Let $\mathcal{D}$ be a triangulated category. A subcategory $\mathcal{C}$ of $\mathcal{D}$ is said to be stable under extensions if any distinguished triangle in $\mathcal{D}$ with two vertices in $\mathcal{C}$ also has it's third vertex in $\mathcal{D}$.*

Definition 1.3.4. *Let $\mathcal{C}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{C}$ is a triangulated subcategory if $\mathcal{C}$ is stable under extensions and $T\mathcal{C} \subseteq \mathcal{C}$.*

Definition 1.3.5. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ a abelian category. An additive functor $F : \mathcal{D} \to \mathcal{A}$ is a cohomological functor if for any distinguished triangle in $\mathcal{D}$*

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced sequence $F(X) \to F(Y) \to F(Z)$ is a exact in $\mathcal{A}$. If F is a cohomological functor one sets $F^i = F \circ T^i$.

The family of functors F^i is conservative if for any distinguished triangle

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced long sequence

$$\cdots \to F^i(X) \to F^i(Y) \to F^i(Z) \to F^{i+1}(X) \to \cdots$$

is exact.

The key example for the above definition is given by the cohomological functor $H^0 : K^*(\mathscr{A}) \to \mathscr{A}$ and the conservative system of functors H^k.

Definition 1.3.6. *Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories. A functor $F : \mathcal{D} \to \mathcal{D}'$ is called a functor of triangulated categories if it is compatible with the shift functor and transforms distinguished triangles in $\mathcal{D}$ into distinguished triangles of $\mathcal{D}'$.*

1.4 Derived Functors

Given abelian categories $\mathcal{A}, \mathcal{B}$ and a functor of triangulated categories $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ one may wonder if there is a natural lift to the derived categories.

Definition 1.4.1. *Let F be as above. The right derived functor of F is a initial couple (R^*F, ξ_F) consisting of a functor of triangulated categories $F^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : p_B \circ F \to R^*F \circ p^*_{\mathcal{A}}$. By initial it is mean that for any other such couple (G, ζ) there is a unique natural transformation $\eta : R^*F \to G$ such that $\zeta = (\eta \circ p^*_{\mathcal{A}}) \circ \xi_F$.*

The dual notion is a left derived functor. This is a final couple (L^*F, ξ_F) consisting of a functor of triangulated categories $F^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : L^*F \circ p^*_{\mathcal{A}} \to p_B \circ F$. It is clear that, if a derived functor exists, it is unique up to unique isomorphism.

There are general theorems on the existence of derived functors which may be found in (Dimca, 2004, Chapter 1). The following will be sufficient for our applications.

Theorem 1.4.1. *Consider a functor of triangulated categories $F : K^+(\mathcal{A}) \to K(\mathcal{B})$. If $\mathcal{A}$ has enough injectives and F is additive then the right derived functor R^+F exists.*

By dualising, a similar theorem applies to $F : K^-(\mathcal{A}) \to K(\mathcal{B})$ for the existence of L^-F in categories with enough projectives.

The main use of derived functors is to fix a lack of exactness in F. Recall from proposition 1.3.1 that a short-exact sequence in $C^+(\mathcal{A})$ induces a distinguished triangle in $D^+(\mathcal{A})$. Applying R^+F to the distinguished triangle returns a distinguished triangle by R^+F being a functor of triangulated categories. Further, there is a associated long exact sequence. The higher-order terms measure measures to what degree the original functor failed to be exact.

Definition 1.4.2. *Let $F : K^*(\mathcal{A}) \to K(B)$ be a functor of triangulated categories such that R^*F exists. For any $n \in \mathbb{Z}$ one defines $R^nF : \mathcal{A} \to \mathcal{B}$ to be the composition*

$$\mathscr{A} \xrightarrow{\iota} D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

where ι sends a object to the chain complex with a single non-trivial term. Similarly, one defines $\mathbb{R}^nF : D^*(\mathcal{A}) \to \mathcal{B}$ as the composition

$$D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}.$$

Proposition 1.4.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose that the derived functor R^*F of the induced functor $F : K^*(\mathscr{A}) \to K(\mathcal{B})$ exists. Then, for any short exact sequence in $\mathcal{A}$*

$$0 \to X \to Y \to Z \to 0$$

there is a long exact sequence in $\mathcal{B}$

$$\cdots R^iF(X) \to R^iF(Y) \to R^iF(Z) \to R^{i+1}F(X) \to \cdots .$$

In the situation of **??** the derived functor can be computed explicitly. Pick some object $X^\bullet$ in $D^+(\mathcal{A})$. By proposition 1.2.1 there is a quasi-isomorphism $X^\bullet \to I^\bullet$ for some complex of injective objects $I^\bullet$. Then one has explicitly

$$R^+F(X^\bullet) \cong p_{\mathcal{B}} \circ F(I^\bullet).$$

Further, if F is exact one has that $F(I^\bullet)$ is quasi-isomorphic to $F(X^\bullet)$ whence $R^+F(X^\bullet)$ is $p_{\mathcal{B}} \circ F(X^\bullet)$.

In practice, it is often difficult to find a concrete injective resolution.

Definition 1.4.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A object X in $\mathcal{A}$ is F-acyclic if $R^iF(X) = 0$ for all $i \geq 1$.*

Computation derived functors can also be done using F-acyclic resolutions. One can show that injective objects are F-acyclic for any left-exact functor. Hence, this generalises the earlier computations.

Proposition 1.4.3. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive functors between abelian categories with enough injective objects. Suppose that F is left-exact and that G transforms injective objects into F-acyclic objects, then there is an isomorphism*

$$R^+(F \circ G) = R^+F \circ R^+G.$$

Theorem 1.4.4 (Grothendieck Spectral Sequence)**.** *Let F, G be as in the previous proposition. Then, for any object X of $\mathcal{A}$, there is a spectral sequence*

$$E_2^{pq} = R^pF(R^qG(X))$$

converging to $R^{p+q}(F \circ G)(X)$.

We conclude this section by considering a few important examples of derived functors which will be used later on. Let X a topological space and $\mathcal{A}_X$ a sheaf of rings which may not be commutative. Denote $C^{*,l}(\mathcal{A}_X)$ for the corresponding category of complexes of left $\mathcal{A}_X$-modules, $C^{*,r}(\mathcal{A}_X)$ for right $\mathcal{A}_X$-modules and $C^{*,lr}(\mathcal{A}_X)$ for bimodules.

Using theorem 1.4.1 one can establish that the global sections functor $\Gamma(X, \bar{)}$ on $C^{*,*}(\mathcal{A}_X)$ has a derived functor $R^+\Gamma(X, \bar{)}$. The cohomology of a sheaf of modules is given by the functors $H^k(X, -) := R^k\Gamma(X, -)$ and the hypercohomology of a complex of modules is given by the functors $\mathbb{H}^k(X, -) := \mathbb{R}^k\Gamma(X, -)$. The cohomology sheaf of a complex $\mathscr{M}^\bullet$ is the sheaf associated to the presheaf $U \mapsto \mathbb{H}^k(U, \mathscr{M}^\bullet)$ and is denoted $\mathscr{H}^k(\mathscr{M}^\bullet)$.

Cohomology measures the failure of sections to be global. Correspondingly, acyclic objects are given by sheaves which have no such failure.

Definition 1.4.4. *A sheaf of $\mathcal{A}_X$-modules $\mathscr{F}$ is called flabby if for any open $U \subseteq X$ the restriction morphism $\rho_U^X : \mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.*

Proposition 1.4.5. *If $\mathscr{F}$ is flabby then $\mathscr{F}$ is $\Gamma(X, \bar{)}$-acyclic.*

The hypercohomology of a sheaf complex can be computed using flabby resolutions. Concrete flabby resolutions may be found using the Godement resolution. A sheaf $\mathscr{M}$ gives rise to a flabby sheaf $\mathscr{F}$ by the formal product of stalks. The same argument applies to the cokernel of $\mathscr{M} \to \mathscr{F}$ and iterating the argument yields a flabby resolution for $\mathscr{M}$. For a sheaf complex $\mathscr{M}^\bullet$ the flabby resolutions of the $\mathscr{M}^j$ produce a double complex $\mathscr{F}^{\bullet,\bullet}$. The total complex of $\mathscr{F}^{\bullet,\bullet}$ yields a flabby resolution of $\mathscr{M}^\bullet$.

A classical example of a non-exact functor is given by the tensor product. This may be considered as a bifunctor

$$\otimes_R : C^{-,l}(\mathcal{A}_X) \times C^{-,r}(\mathcal{A}_X) \to C^{-,lr}(\mathcal{A}_X)$$

where $(\mathscr{M}^\bullet \otimes \mathscr{N}^\bullet)^n = \oplus_{i+j=n} \mathscr{M}^i \otimes \mathscr{N}^j$ with differentials defined at $\mathscr{M}^i \otimes \mathscr{N}^j$ by $d(m \otimes n) = dm \otimes n + (-1)^i m \otimes dn$. The category of $\mathcal{A}_X$-modules admits locally free resolutions. In particular, it has enough projective objects. Essentially by the remark after theorem 1.4.1 it is then possible to construct a derived left-derived functor

$$\otimes_R^L : D^{-,l}(\mathcal{A}_X) \times D^{-,r}(\mathcal{A}_X) \to D^{-,lr}(\mathcal{A}_X).$$

This yields $\mathcal{T}or$-sheaves $\mathcal{T}or_k^{\mathcal{A}_X}(X^\bullet, Y^\bullet) = \mathscr{H}^{-k}(X^\bullet \otimes_{\mathcal{A}_X}^L Y^\bullet)$.

A similar procedure applies to the $\mathcal{H}om_{\mathcal{A}_X}$-bifunctor which is defined by $\mathcal{H}om^n_{\mathcal{A}_X}(\mathscr{M}^\bullet, \mathscr{N}^\bullet) = \prod_{j \in Z} \mathcal{H}om_{\mathcal{A}_X}(\mathscr{M}^j, \mathscr{N}^{n+j})$ with the differentials on $\mathcal{H}om^n_{\mathcal{A}_X}(M^\bullet, N^\bullet)$ given by $d\varphi = d_N \circ \varphi - (-1)^n \varphi \circ d_M$. There is a induced derived bifunctor

$$R\mathcal{H}om^\bullet_{\mathcal{A}_X}(-,-) :: D^{-,l}(\mathcal{A}_X)^{opp} \times D^{+,lr}(\mathcal{A}_X) \to D^r(\mathcal{A}_X).$$

This yields the $\mathcal{E}xt$-sheaves $\mathcal{E}xt^n_{\mathcal{A}_X}(M^\bullet, N^\bullet) = \mathscr{H}^n R\mathcal{H}om^\bullet_{\mathcal{A}_X}(M^\bullet, N^\bullet)$.

1.5 t-structures

A generalisation of positive and negatively supported complexes is given by the concept of a t-structure.

Definition 1.5.1. *A t-structure on a triangulated category $\mathcal{D}$ consists of two strictly full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ such that, setting $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, the following properties hold.*

(i) It holds that $D^{\leq 0}$ is a subcategory of $D^{\leq 1}$ and $D^{\geq 1}$ is a subcategory of $D^{\geq 0}$.

(ii) For any objects X in $\mathcal{D}^{\leq 0}$ and Y of $\mathcal{D}^{\geq 1}$ it holds that $\mathrm{Hom}(X, Y) = 0$.

(iii) For any object X of $\mathcal{D}$ there is a distinguished triangle

$$A \to X \to B \xrightarrow{+1} A$$

with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

Definition 1.5.2. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then $\mathcal{D} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure.*

In the motivating case of $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ the heart of the t-structure recovers the original abelian category $\mathcal{A}$.

Proposition 1.5.1. *The heart $\mathcal{D}$ of a t-structure is an abelian category which is stable by extensions.*

Observe that $D^*(\mathcal{A})$ comes equipped with a truncation functors $\tau_{\leq m} : D^*(\mathcal{A}) \to D^-(\mathcal{A})$ which sends a complex $X^\bullet$ to

$$\tau_{\leq m} X^\bullet : \cdots \to X^{m-1} \to \ker d \to 0 \to 0 \to \cdots$$

and similarly a truncation functor $\tau_{\geq m}$ is defined by

$$\tau_{\geq m} X^\bullet : \cdots \to 0 \to 0 \to \operatorname{coim} d \to A^{m+1} \to \cdots .$$

This generalises to t-structures.

Proposition 1.5.2. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then the inclusion of $\mathcal{D}^{\leq}$ in $\mathcal{D}$ has a right adjoint functor $\tau_{\leq n}$. Similarly, the inclusion of $\mathcal{D}^{\geq n}$ in $\mathcal{D}$ has a left adjoint $\tau_{\geq n}$.*

Observe that in the example of $D^*(\mathcal{A})$ one has that $\tau_{\geq 0}\tau_{\leq 0} X^\bullet$ is the complex with a single entry $H^0(X^\bullet)$. This generalises to t-structures by viewing ${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0}$ as a functor from $\mathcal{D}$ to it's heart $\mathcal{C}$. Further let ${}^tH^i :=^t H^0 \circ T^i$.

Definition 1.5.3. *A t-structure is said to be non-degenerated if $\cap \mathcal{D}^{\leq n} = \cap \mathcal{D}^{\geq n} = \mathrm{Null}$ where* Null *denotes the family of objects which are isomorphic to the zero object in $\mathcal{D}$.*

Proposition 1.5.3. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then ${}^tH^0 : \mathcal{D} \to \mathcal{C}$ is a cohomological functor.*

Proposition 1.5.4. *Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then the system of functors ${}^tH^i$ is conservative and $X \in \mathcal{D}^{\leq 0}$ if and only if ${}^tH^i(X) = 0$ for $i > 0$. Similarly $X \in \mathcal{D}^{\geq 0}$ if and only if ${}^tH^i(X) = 0$ for $i < 0$.*

Definition 1.5.4. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures. A functor of triangulate categories $F : \mathcal{D}_1 \to \mathcal{D}_2$ is called left or right t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ or $F(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$ respectively. The functor F is called t-exact if it is left and right t-exact.*

Definition 1.5.5. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. The perverse functor ${}^pF$ associated to F is the induced functor on the hearts ${}^pF =^t H^0 \circ F \circ j_1$ where j_1 denotes the inclusion functor $j_1 : \mathcal{C}_1 \to \mathcal{C}_2$.*

Chapter 2

$\mathscr{D}_X$-modules and the Riemann-Hilbert Correspondence

The subject of this chapter are modules over rings of differential operators. Throughout X can be a smooth algebraic variety or a complex manifold. The ring of differential operators $\mathscr{D}_X$ will be defined formally in the next section. For the purpose of this section it's sufficient to note that local sections of $\mathscr{D}_X$ are of the form $\sum c_\alpha \partial^\alpha$ with c_α local sections of $\mathcal{O}_X$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

A $\mathscr{D}_X$-modules gives a canonical description of systems of differential equations. Consider a system of differential equations

$$\sum_{j=1}^{k} P_{ij}(x, \partial) f_j = 0; \qquad i = 1, \ldots, m$$

with unknown functions f_j of $\mathcal{O}_X$ and differential operators P_{ij}. The functions f_j are somewhat arbitrary in the description of this system. For instance, take $g_j = \lambda_j f_j$ for certain non-zero functions λ_j. There is then a associated system of equations for g_j. A solution of the g_j-system corresponds uniquely to a solution of the f_j-system.

Consider the cokernel $\mathscr{M}$ of the map

$$P : \mathscr{D}_X^k \to \mathscr{D}_X^m : (Q_1, \ldots, Q_k) \mapsto \left(\ldots, \sum_{j=1}^{k} Q_j P_{ij}, \ldots \right).$$

This map is left $\mathscr{D}_X$-linear so $\mathscr{M}$ is a left $\mathscr{D}_X$-module. Note that it is necessary to distinguish between left and right modules because differential operators form a non-commutative ring. Direct verification shows that the solutions of the system of differential equations are encoded in $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$. More generally, for any $\mathscr{D}_X$-module $\mathscr{N}$ the solutions in $\mathscr{N}$ are encoded by $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$. This shows that $\mathscr{D}_X$-modules provide a canonical description of differential equations.

2.1 $\mathscr{D}_X$-modules

From now on let X be a smooth algebraic variety over $\mathbb{C}$ and denote $n = \dim X$. The properties discussed in this section are common knowledge in the field of $\mathscr{D}_X$-modules. For detailed references see Bjork (1979), Kashiwara (2003) or Hotta and Tanisaki (2007).

Definition 2.1.1. *The sheaf of differential operators $\mathscr{D}_X$ is the subsheaf of rings in $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the vector fields Θ_X.*

Filtrations

Observe that $\mathscr{D}_X$ is a sheaf of non-commutative rings. Given local coordinates $x_1, \ldots, x_n$ on X it holds that

$$\partial_i x_j - x_j \partial_i = \delta_{ij}$$

where δ denotes the Kronecker delta.

This non-commutativity exits the typical domain of algebraic geometry. This can be resolved by consideration of a graded structure. The essential observation here is that differential operators commute up to a element of lower order.

Definition 2.1.2. *The order filtration on $\mathscr{D}_X$ is defined inductively to be given by the sheaves of $\mathcal{O}_X$-submodules $F_i\mathscr{D}_X$ such that $F_0\mathscr{D}_X = \mathcal{O}_X$ and $[F_i\mathscr{D}_X, F_i\mathscr{D}_X] \subseteq F_{i-1}\mathscr{D}_X$.*

The term $F_i\mathscr{D}_X$ in the order filtration can be described as containing all differential operators of order less than or equal to i. Indeed, given local coordinates $x_1, \ldots, x_n$ one can show that $F_i\mathscr{D}_X$ is the $\mathcal{O}_X$-module locally generated by $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where α is a multi-index with $|\alpha| \leq n$. The following observations are immediate.

Lemma 2.1.1. *The $F_i\mathscr{D}_X$ are coherent $\mathcal{O}_X$-modules and form a exhaustive filtration. This is to say that $\cup_{i\geq 0} F_i\mathscr{D}_X = \mathscr{D}_X$ and that for any $i, j \geq 0$ it holds that $F_i\mathscr{D}_X \cdot F_j\mathscr{D}_X \subseteq F_{i+j}\mathscr{D}_X$.*

There is a similar notion of filtrations on $\mathscr{D}_X$-modules $\mathscr{M}$. Without any harm let's assume that $\mathscr{M}$ is a left $\mathscr{D}_X$-module, the case for right modules is analogous. A filtration consists of $\mathcal{O}_X$-submodules $F_i\mathscr{M}$ of $\mathscr{M}$ such that $\cup_i F_i\mathscr{M} = \mathscr{M}$ and $F_i\mathscr{D}_X \cdot F_j\mathscr{M} \subseteq F_{i+j}\mathscr{M}$.

Stepping over to the graded object has the advantage that $\operatorname{gr}\mathscr{D}_X$ is commutative by definition of the order filtration whence the classical methods of algebraic geometry are applicable. The symplectic structure of T^*X captures part of the non-commutativity. Indeed, given two differential operators P, Q. Pick local coordinates $x_1, \ldots, x_n$ and decompose $P = \sum_\alpha p_\alpha \partial^\alpha$ and $Q = \sum_\beta q_\beta \partial^\beta$. Let m_1, m_2 be the maximal values of $|\alpha|$ and $|\beta|$ with non-zero coefficients. Then the induced elements of P and Q in $\operatorname{gr}\mathscr{D}_X$ are of the form $p = \sum_{|\alpha|=m_1} p_\alpha \xi^\alpha$ and $q = \sum_{|\beta|=m_2} q_\beta \xi^\beta$ where ξ_i is the induced element of ∂_i. On the other hand, the induced element of $PQ - QP$ is $\sum_{i=1}^n \frac{\partial p}{\partial \xi_i}\frac{\partial q}{\partial x_i} - \frac{\partial q}{\partial \xi_i}\frac{\partial p}{\partial x_i}$. This is precisely $\{p, q\}$ where $\{-,-\}$ is the Poisson bracket.

One can view $\operatorname{gr}\mathscr{D}_X$ as a subsheaf of $\mathcal{O}_{T^*X}$. Denote π for the projection of $T^*X \to X$. Then any $\operatorname{gr}\mathscr{D}_X$-module $\mathcal{M}$ has a corresponding module on T^*X defined by $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\operatorname{gr}\mathscr{D}_X} \mathcal{M}$. By abuse of notation this module is still denoted $\mathcal{M}$ and it will always be implicitly assumed that $\operatorname{gr}\mathscr{D}_X$-modules live on T^*X unless it is explicitly mentioned otherwise. In particular, for a filtration of the $\mathscr{D}_X$-module $\mathscr{M}$ the graded object $\operatorname{gr}\mathscr{M} = \oplus_i F_i\mathscr{M}/F_{i-1}\mathscr{M}$ is a $\operatorname{gr}\mathscr{D}_X$-module.

Proposition 2.1.2. *A $\mathscr{D}_X$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\operatorname{gr}\mathscr{M}$ is a coherent $\operatorname{gr}\mathscr{D}_X$-module. Such a filtration is called a good filtration.*

Proof. A proof of this result may be found in (Hotta and Tanisaki, 2007, Chapter 2). □

Proposition 2.1.3. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then the support of* $\mathrm{gr}^{rel}\mathscr{M}$ *in* T^*X *is a independent of the chosen good filtration. It is called the characteristic variety of* $\mathscr{M}$ *and denoted* $\mathrm{Ch}\,\mathscr{M}$.

Proof. A proof of this result may be found in (Hotta and Tanisaki, 2007, Chapter 2). ☐

Proposition 2.1.4. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then* $\mathrm{Ch}\,\mathscr{M}$ *is a homogeneous and isotropic closed subset of* T^*X.

Proof. These results may be found in (Kashiwara, 2003, Chapter 2). ☐

Proposition 2.1.5. *Consider a short exact sequence of coherent $\mathscr{D}_X$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

then it holds that

$$\mathrm{Ch}\,\mathscr{M}_2 = \mathrm{Ch}\,\mathscr{M}_1 \cup \mathrm{Ch}\,\mathscr{M}_3.$$

Proof. A good filtration on $\mathscr{M}_2$ induces good filtrations on $\mathscr{M}_1$ and $\mathscr{M}_3$ and one has a short exact sequence

$$0 \to \mathrm{gr}\,\mathscr{M}_1 \to \mathrm{gr}\,\mathscr{M}_2 \to \mathrm{gr}\,\mathscr{M}_3 \to 0$$

whence the result follows. ☐

Holonomicity

A particularly nice class of $\mathscr{D}_X$-modules are given by maximally overdetermined systems of differential equations. This is to say that there are many relations for $\mathscr{M}$ or equivalently that $\mathrm{Ch}\,\mathscr{M}$ is small. Observe that by isotropic part of proposition 2.1.4 it follows that $\dim \mathrm{Ch}\,\mathscr{M} \geq n$.

Definition 2.1.3. *A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called holonomic if* $\dim \mathrm{Ch}\,\mathscr{M} = n$.

For technical purposes it is mostly important that holonomic modules have finiteness properties.

Proposition 2.1.6. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then, for any $x \in X$, the stalk $\mathscr{M}_x$ is a $(\mathscr{D}_X)_x$-module of finite length.*

Proof. This result may be found in (Kashiwara, 2003, Chapter 4). ☐

Recall from the introduction that $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ encodes the solutions in $\mathscr{N}$ of a system of differential equations.

Proposition 2.1.7. *Let $\mathscr{M}, \mathscr{N}$ be holonomic $\mathscr{D}_X$-modules. Then, for any $x \in X$, the stalk $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{M})_x$ is a finite-dimensional vector space over $\mathbb{C}$.*

Proof. This result may be found in (Kashiwara, 2003, Chapter 4). ☐

Corollary 2.1.8. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ is $\mathbb{C}$-algebraic. This is to say that for any $\varphi \in \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ there exists some polynomial b with coefficients in $\mathbb{C}$ such that $b(\varphi) = 0$.*

2.2 Regular singularities

This section is based on (Kashiwara, 2003, Chapter 5). Let $X = \mathbb{C}$ considered with it's analytical topology and consider a ordinary differential operator $P(x, \partial) = \sum_{k=0}^{m} a_k(x)\partial^k$. Suppose that $a_m(x) \neq 0$ for any $x \neq 0$. Then $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$ is locally of the form $\mathcal{O}_X^m$ as a $\mathscr{D}_X$-module near any point $x \neq 0$. In particular the solutions $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ form a locally constant sheaf of rank m outside of 0. The solutions near zero may be more subtle due to monodromy.

Observe that $\mathrm{Ch}\,\mathscr{M} \subseteq \{(x, \xi) : x\xi = 0\}$. Hence, for any filtration on $\mathscr{M}$ there exists some $N > 0$ such that $(x\xi)^N \operatorname{gr} \mathscr{M} = 0$.

Proposition 2.2.1. *The following conditions are equivalent.*

1. *There exists a filtration on $\mathscr{M}$ such that $x\xi \operatorname{gr} \mathscr{M} = 0$.*

2. *The equation $P(x, \partial)u$ has m linearly independent solutions of the form $x^\lambda \sum_{j=0}^{s} u_j \log(x)^j$ near 0 for some $s \geq 0$, $\lambda \in \mathbb{C}$ and holomorphic u_j if and only P has a regular singularity in 0.*

If these two equivalent conditions are satisfied one calls 0 a regular singularity of $\mathscr{M}$. This has the following generalisation to higher dimensions.

Definition 2.2.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a complex manifold X with characteristic variety determined by some ideal sheaf $\mathcal{I}$. Then $\mathscr{M}$ is called regular holonomic if it it admits a filtration such that $\mathcal{I} \operatorname{gr}(\mathscr{M}) = 0$.*

It appears that these definitions should generalise directly to the algebraic situation. However, this has unintended consequences for the Riemann-Hilbert correspondence which states that a system of differential equations should be equivalent to the system of solutions. Concretely, the systems of differential equations are encoded in regular holonomic $\mathscr{D}_X$-modules $\mathscr{M}$.

For a example, let $X = \mathbb{C}$ as before and consider the regular holonomic $\mathscr{D}_X$-modules $\mathcal{O}_X$ and $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X(\partial - 1)$. These are analytically isomorphic by the map which sends f to $f\exp(x)$. In particular the Riemann-Hilbert correspondence shows that they have isomorphic systems of solutions. However, $\mathcal{O}_X$ and $\mathscr{M}$ are not algebraically isomorphic. This seems to suggest that the equivalence between differential equations and their systems of solutions would not hold in the algebraic case. The problem is that $\mathscr{M}$ is not regular at infinity.

The adjusted definition proceeds in two steps which we sketch. The precise details may be found in (Borel, 1987, Chapter 7). Firstly, one induces a module corresponding to $\mathscr{M}$ on the smooth completion $\overline{X}$ of the smooth algebraic variety X. Hereafter, the old definition may be applied on the analytification of $\overline{X}$ by use of the GAGA principle. The smooth completion ensures that the regularity also holds at infinity.

2.3 Perverse Sheaves

The Riemann-Hilbert correspondence states in great generality that there is a equivalence between a system of differential equations and the system of solutions. Philosophically, this is a significant result because it yields a connection between the algebraic/analytic world of differential equations and the topological world associated to their solutions.

2.4 Riemann-Hilbert Correspondence

2.5 Monodromy

R

Chapter 3

Relative Holonomic Modules

3.1 Introduction

Fix a smooth complex variety X and a morphism $F : X \to \mathbb{C}^p : x \mapsto (f_1(x), \dots, f_p(x))$. Denote D for the divisor defined by $\prod f_i$ and let $\mu : Y \to X$ be a resolution of singularities for (X, D). This means that μ is a projective morphism which is a isomorphism over the complement of D and such that $\mu^* D = \sum_{i=1}^r a_i E_i$ is in normal crossings form. The behaviour of μ over D is measured by the relative canonical divisor $K_{Y/X} = \sum_{i=1}^r k_i E_i$ which is locally defined by the Jacobian of μ. Write $G : Y \to \mathbb{C}^p$ for the lift of F to Y. Introducing new variables $s_1, \dots, s_p$ we abbreviate $F^s = f_1^{s_1} \cdots f_p^{s_p}$ and similarly for G^s.

The local Bernstein-Sato Ideal $B_{F,x}$ of the function germ of F at some point $x \in X$ consists of all polynomials $b(s_1, \dots, s_n)$ such that there exists some local partial differential operator $P \in \mathscr{D}_{X,x} \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_n]$ with the following equality in the stalk at x

$$b(s_1, \dots, s_n) F^s = P \cdot F^{s+1}.$$

The global Bernstein-Sato Ideal B_F of F is the intersection of all local Bernstein-Sato Ideals.[1]

The goal of this chapter is to estimate the zero locust $Z(B_F) \subseteq \mathbb{C}^p$. This zero locust generalises the roots of the Bernstein-Sato polynomial in the monovariate case. The classical approximation of the roots of the b-polynomial is due to Kashiwara (1976) and this estimation was further refined by Lichtin (1989). The idea in both proofs is that it is easy to explicitly compute the Bernstein-Sato polynomial for monomials and that one can reduce to this case by use of the resolution of singularities. The main non-trivial step in these arguments is to translate the solution upstairs to a solution on X. This makes use of the direct image of $\mathscr{D}_X$-modules. The essential insight in the refined estimate due to Lichtin is that the direct image of $\mathscr{D}_X$-modules is more natural for right $\mathscr{D}_X$-modules than left $\mathscr{D}_X$-modules.

The estimate by Kashiwara has been generalised to the multivariate situation in Budur et al. (2020). The main challenge in such a multivariate generalisation is that the classical proof relies on modules of the type $\mathscr{D}_X f^s / \mathscr{D}_X f^{s+1}$ being holonomic. This is no longer the case for the multivariate generalisation $\mathscr{D}_X[s_1, \dots, s_n] f^s / \mathscr{D}_X[s_1, \dots, s_n] f^{s+1}$. The notion of relative holonomicity, due to Maisonobe (2016), still holds.

[1]Note: Restate more generally with $+a$ when proof is done.

In this chapter we generalise the refined estimate by Lichtin (1989) to the multivariate situation. The main new ingredient is a induction argument which reduces the problem to the monovariate case where relative holonomicity becomes ordinary holonomicity. This induction is similar to the arguments in Budur et al. (2019).

Theorem 3.1.1. *With notation as above every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{>0}$.

[2]

3.2 Relative Notions

Modules over $\mathscr{A}_X^R$

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. It will also be convenient to use the abbreviation $\mathcal{O}_X^R := \mathcal{O}_{X \times \operatorname{Spec} R}$.

The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration. The associated graded objects are denoted by gr^{rel}. Denote $\pi : T^*X \times \operatorname{Spec} R \to X \times \operatorname{Spec} R$ for the projection map. As in the case of $\mathscr{D}_X$-modules in chapter 1 [3] one can view $\pi^{-1}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$ as a subsheaf of $\mathcal{O}_{T^*X}^R$ and for any $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module $\mathcal{M}$ there is a corresponding module on $T^*X \times \operatorname{Spec} R$ defined by $\mathcal{O}_{T^*X}^R \otimes_{\pi^{-1}\mathrm{gr}^{rel}\mathscr{A}_X^R} \pi^{-1}\mathcal{M}$. By abuse of notation the corresponding module on $T^*X \times \operatorname{Spec} R$ is still denoted with $\mathcal{M}$ and we adopt the perspective that $\mathrm{gr}^{rel}\mathscr{A}_X^R$-modules always live on $T^*X \times \operatorname{Spec} R$ unless explicitly mentioned otherwise.

Similarly to the case of $\mathscr{D}_X$ in the first chapter that [4] it holds that $\mathscr{A}_X^R$ is the sheaf of rings generated by $\mathcal{O}_X^R$ and Θ_X inside of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X^R)$. Giving a left $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X^R$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X. Similarly, giving a right $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2). [5]

Proposition 3.2.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\mathrm{gr}^{rel}\mathscr{M}$ is coherent over $\mathrm{gr}^{rel}\mathscr{A}_X^R$. Such a filtration is called a good filtration.*

[2]Note: Should also give a overview of the results that are already known here.
[3]Note: cite
[4]Note: Cite when C1 is written
[5]Note: Probably cite C1 instead

Proposition 3.2.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen good filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\mathrm{Ch}^{rel}\mathscr{M}$.*

Lemma 3.2.3. *Consider a short exact sequence of coherent $\mathscr{A}_X^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

then it holds that

$$\mathrm{Ch}^{rel}\mathscr{M}_2 = \mathrm{Ch}^{rel}\mathscr{M}_1 \cup \mathrm{Ch}^{rel}\mathscr{M}_3.$$

Basic Operations

For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ and left $\mathscr{D}_X$-module $\mathscr{N}$ the tensor product $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ comes equipped with a right $\mathscr{A}_X^R$-module structure defined by

$$f \cdot (m \otimes n) = mf \otimes n; \qquad \xi \cdot (m \otimes n) = m\xi \otimes n - m \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X. Putting multiplication by f on the other side of the tensor product this definition is also applicable for a right $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ if $\mathscr{M}$ is a right $\mathscr{D}_X$-module and $\mathscr{N}$ is a left $\mathscr{A}_X^R$-module. If both are $\mathscr{A}_X^R$-modules there is a right $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

Similarly, given a left $\mathscr{D}_X$-module $\mathscr{L}$ and a left $\mathscr{A}_X^R$-module $\mathscr{N}$ a left $\mathscr{A}_X^R$-module structure on $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}$ is defined by

$$f \cdot (\ell \otimes n) = \ell \otimes fn; \qquad \xi \cdot (\ell \otimes n) = \xi\ell \otimes n + \ell \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

Lemma 3.2.4. *Let $\mathscr{M}, \mathscr{N}$ be right and left $\mathscr{A}_X^R$-modules respectively and let $\mathscr{L}$ be a left $\mathscr{D}_X$-module. Then there is a isomorphism of left $\mathscr{A}_X^R$-modules*

$$(\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}) \otimes_{\mathcal{O}_X^R} \mathscr{N} \cong \mathscr{M} \otimes_{\mathcal{O}_X^R} (\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}).$$

Proof. This is immediate by checking that the obvious bijection conserves the $\mathscr{A}_X^R$-module structure. Note that the only non-trivial check is the action of a section ξ from Θ_X. □

Lemma 3.2.5. *Let $\mathscr{N}$ be a left $\mathscr{A}_X^R$-module which is locally free as a $\mathcal{O}_X^R$-module. Consider $\mathscr{A}_X^R$ as a right $\mathscr{A}_X^R$-module, then $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$ is locally free as a right $\mathscr{A}_X^R$-module.*

Proof. Consider local coordinates $x_1, \ldots, x_n$ on X and a local $\mathcal{O}_X^R$-basis $\{n_\beta\}_\beta$ for $\mathscr{N}$. Then $\{1 \otimes n_\beta\}_\beta$ will be a local $\mathscr{A}_X^R$-basis for $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

To see that this generates the $\mathscr{A}_X^R$-module note that $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ is a $\mathcal{O}_X^R$-basis set when α runs over all multi-indices in $\mathbb{Z}^n_{\geq 0}$. These sections can be recovered using the $\mathscr{A}_X^R$-action on the proposed generating set by induction on $|\alpha|$. Indeed, $\xi^\alpha \cdot (1 \otimes n_\beta)$ equals $\xi^\alpha \otimes n_\beta$ up to a element in the $\mathcal{O}_X^R$-span of $\{\xi^\gamma \otimes n_\beta\}_{|\gamma|<|\alpha|}$.

For the freedom, suppose there is a local $\mathscr{A}_X^R$-relation $\sum_\beta P_\beta \cdot 1 \otimes n_\beta = 0$ with some P_β non-zero. This is of the form $\sum_{\alpha,\beta} f_{\alpha,\beta}\xi^\alpha \cdot 1 \otimes n_\beta = 0$ with the $f_{\alpha,\beta}$ sections of $\mathcal{O}_X^R$ not all equal to zero. Pick some multi-index $\mu \in \mathbb{Z}^n_{\geq 0}$ and of maximal degree such that $f_{\mu,\beta}$ is non-zero for some β. Then, rewriting $\sum_{\alpha,\beta} f_\alpha \xi^\alpha \cdot 1 \otimes n_\beta = 0$ in terms of the $\mathcal{O}_X^R$-basis $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ one finds a non-zero coefficient at $\xi^\eta \otimes n_\beta$ for some β which is a contradiction. □

Lemma 3.2.6. *The functor $\Omega_X \otimes_{\mathcal{O}_X} -$ which takes a left $\mathscr{A}_X^R$-modules and returns a right $\mathscr{A}_X^R$-module is an equivalence of categories with pseudoinverse $\mathscr{H}om_{\mathcal{O}_X}(\Omega_X, -)$.*

Proof. For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ the left $\mathscr{A}_X^R$-module structure on $\mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathscr{M})$ is defined by

$$(f \cdot \varphi)(\omega) = \varphi(\omega) \cdot f; \qquad (\xi \cdot \varphi)(\omega) = \varphi(\omega \cdot \xi) - \varphi(\omega) \cdot \xi.$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

For any left $\mathscr{A}_X^R$-module $\mathscr{M}$ there is a natural isomorphism of $\mathcal{O}_X^R$-modules $\Omega_X \otimes_{\mathcal{O}_X} \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathscr{M}) \cong \mathscr{M}$ by sending $\omega \otimes \varphi$ to $\varphi(\omega)$. Similarly for any right $\mathscr{A}_X^R$-module $\mathscr{M}$ the isomorphism $\mathscr{M} \cong \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \Omega_X \otimes \mathscr{M})$ associates to a section m of $\mathscr{M}$ the morphism $\omega \mapsto \omega \otimes m$. A direct computation verifies these isomorphisms commute with the $\mathscr{A}_X^R$-module structure. $\square$

Relative Holonomicity

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

The following lemma and it's proof may be found in Maisonobe (2016).

Lemma 3.2.7. *Let $\mathscr{M}$ be a finitely generated $\mathscr{A}_Y^R$-module. Suppose that $\mathrm{Ch}^{rel}\mathscr{M} \subseteq \Lambda \times \operatorname{Spec} R$ for some, not necessarily irreducible, conic Lagrangian subvariety $\Lambda \subseteq T^*X$. Then $\mathscr{M}$ is relative holonomic over R.*

The Bernstein-Sato ideal may be defined more generally for any $\mathscr{A}_X^R$-module $\mathscr{M}$ as $B_{\mathscr{M}} := \mathrm{Ann}_R \mathscr{M}$. To see how this generalises B_F one considers $\mathscr{A}_X^R F^s$ as a $\mathscr{A}_X^R\langle t\rangle$-module. Here t is a new variable which commutes with sections of $\mathscr{D}_X$ and satisfies $ts_i - s_i t = 1$ for any $i = 1, \dots, n$. The $\mathscr{A}_X^R\langle t\rangle$-module structure on $\mathscr{A}_X^R F^s$ is then defined by extending $tF^s = F^{s+1}$. From this point of view $B_F = B_{\mathscr{A}_X^R F^s / t\mathscr{A}_X^R F^s}$.

The Bernstein-Sato ideal may be recovered from the characteristic variety. The following result is due to Maisonobe (2016) in the analytical case and Budur et al. (2019) in the algebraic case.

Proposition 3.2.8. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$ module. Then $Z(B_{\mathscr{M}}) = \pi_2(\mathrm{Ch}^{rel}(\mathscr{M}))$ where $\pi_2 : T^*X \times \operatorname{Spec} R \to \operatorname{Spec} R$ is the projection on the second coordinate.*

3.3 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.[6]

[6] Note: more introduction

Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties, by abuse of notation we will also denote μ for the induced map from $Y \times \operatorname{Spec} R$ to $X \times \operatorname{Spec} R$.

A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}\mathscr{A}_X^R$. Here, the right $\mu^{-1}\mathscr{A}_X^R$-module structure is just the action on the second component and definitions like section 3.2 are used to define the left $\mathscr{A}_Y^R$-module structure. To be precise

$$f \cdot (g \otimes \mu^{-1}h_X) = fg \otimes \mu^{-1}h_X; \qquad \xi \cdot (g \otimes \mu^{-1}h_X) = \xi g \otimes \mu^{-1}h_X + g \otimes T\mu(\xi)\mu^{-1}h_X$$

for any sections f of $\mathcal{O}_Y^R$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Definition 3.3.1. *The direct image functor $\int_\mu$ from $D^{b,r}(\mathscr{A}_Y^R)$ to $D^{b,r}(\mathscr{A}_X^R)$ is defined to be $R\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be suppressed whenever there is no ambiguity.*

To compute the direct image $\int^j \mathscr{M}$ a resolution for the transfer bimodule $\mathscr{A}_{Y\to X}$ is required.

Definition 3.3.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\operatorname{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\operatorname{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k\Theta_Y$ and as differential the right $\mathscr{A}_Y^R$-linear map δ given by*

$$\begin{aligned} m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto & \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \hat{\xi}_j \wedge \cdots \wedge \xi_k \\ & - \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_k \end{aligned}$$

The following lemma and it's proof are a generalisation of exercise 1.20 in Sabbah (2011) to the relative case.

Lemma 3.3.1. *The relative Spencer complex $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_X^R$ as left $\mathscr{A}_X^R$-module.*

Proof. Define a filtration on $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ by the complexes $F_k \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ which have term $F_{k-\ell}\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell\Theta_Y$ in spot ℓ. This filtration induces the complexes $\operatorname{gr}_k^{rel} \operatorname{Sp}_X^\bullet(\mathscr{A}_Y^R)$ with term $\operatorname{gr}_{k-\ell}^{rel} \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell\Theta_Y$ in spot ℓ.

In local coordinates $x_1, \dots, x_n$ one finds that $\operatorname{gr}^{rel}\operatorname{Sp}_Y^\bullet := \oplus_k \operatorname{gr}^{rel}{}_k \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is the Koszul complex of $\mathcal{O}_Y^R[\xi_1, \dots, \xi_n] = \operatorname{gr}^{rel}\mathscr{A}_Y^R$ with respect to $\xi_1, \dots, \xi_n$.[7] Since $\xi_1, \dots, \xi_n$ form a regular sequence a standard result on Koszul complexes[8] yields that $\operatorname{gr}^{rel}\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_Y^R$ as $\operatorname{gr}^{rel}\mathscr{A}_Y^R$-module.

On the other hand, it is immediate that $F_0 \operatorname{Sp}^\bullet(\mathscr{A}_Y^R) = \operatorname{gr}_0^{rel} \operatorname{Sp}^\bullet(\mathscr{A}_Y^R)$ is $\mathcal{O}_Y^R$ viewed as a complex. Hence, there is no contribution to $\operatorname{gr}^{rel} \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ from the terms of $k > 0$. That is to say that $\operatorname{gr}_k^{rel} \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is quasi-isomorphic to the zero complex for $k > 0$.

[7] Note: Should I explain what a Koszul complex is?

[8] Note: Give reference to some book

Hence, $F_0 \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R) \hookrightarrow \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a quasi-isomorphism by the exactness of the direct limit. [9] It follows that $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a resolution of $\mathcal{O}_X^R$. That the terms of $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ are locally free follows from lemma 3.2.5 after some minor adjustments in the statement and proof. □

Define the transfer Spencer complex as the complex of $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X)$-bimodules given by $\operatorname{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X}^R$. The following lemma and it's proof are direct generalisation of exercise 3.4 in Sabbah (2011) to the relative case.

Lemma 3.3.2. *The transfer Spencer complex $\operatorname{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R)$ is a resolution of $\mathscr{A}_{Y\to X}^R$ as a bimodule by locally free left $\mathscr{A}_Y^R$-modules.*

Proof. To see that the terms of the complex are locally free recall from lemma 3.2.4 the following isomorphisms of left $\mathscr{A}_Y^R$-modules

$$(\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X} \cong \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y^R} (\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}).$$

[10] Note that $\mathscr{A}_{Y\to X}^R$ is a locally free $\mathcal{O}_Y^R$-module since it is the pullback of a locally free module on $X \times \operatorname{Spec} R$. Combined with the fact that $\wedge^\ell\Theta$ is a locally free $\mathcal{O}_Y$-module this yields that $\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}$ is a locally free $\mathcal{O}_Y^R$-module. Hence lemma 3.2.4 is applicable and yields that the terms of the transfer Spencer complex are locally free $\mathscr{A}_Y^R$-modules.

That the transfer Spencer complex is a resolution of $\mathscr{A}_{Y\to X}^R$ follows from lemma 3.3.1 by using that $\mathscr{A}_{Y\to X}^R$ is a locally free and hence flat over $\mathcal{O}_Y^R$. □

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 3.3.3. *It holds that $\int = R\mu_*(- \otimes_{\mathscr{A}_Y^R} \operatorname{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$.*

Proposition 3.3.4. *For any short exact sequence of $\mathscr{A}_X^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

there is a long exact sequence in direct images

$$0 \to \int^0 \mathscr{M}_1 \to \int^0 \mathscr{M}_2 \to \int^0 \mathscr{M}_3 \to \int^1 \mathscr{M}_1 \to \cdots.$$

[11] A strategy one can employ in proving theorems on some space X is by first solving them on a nicer space Y equipped with a map $Y \to X$. This can then be related to the problem on X by use of the direct image. For this purpose it is useful that any global section of $\mathscr{M}$ induces a global section of the direct image. This is usually done in the language of left modules but for us it is more natural to work with right $\mathscr{A}_Y^R$-modules.

Lemma 3.3.5. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module. Then any global section $m \in \Gamma(Y, \mathscr{M})$ induces a global section of $\int^0 \mathscr{M})$.*

[9] Note: Would be nice to give a reference, proof may be found on stackexchange

[10] Note: May be possible to remove this step from the proof and removing need for minor adjustment of previous proof.

[11] Note: Provide reference

Proof. By the Leray spectral sequence there is a functorial isomorphism

$$\mathbb{H}^\bullet(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)) \cong \mathbb{H}^\bullet(X, R\mu_*(\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R))).$$

In particular it follows that $\mathbb{H}^0(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)) \cong \Gamma(X, \int^0 \mathscr{M})$. The Čech spectral sequence now induces the desired global section in the direct image based on the section $m \otimes 1$ of $\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^0_{Y\to X}(\mathscr{A}_Y^R)$. □

Functorial Properties of the Direct Image

The following properties are well-known for $\mathscr{D}_X$-modules. We only sketch the main steps of the proofs so that it is clear that no new problems for $\mathscr{A}_X^R$-modules and refer to chapters 6 and 9 in Borel (1987) for the details.

Theorem 3.3.6. *Let $\mu : Z \to Y$ and $\nu : Y \to X$ be morphisms of smooth algebraic varieties. There is a isomorphism of functors $\int_{\nu\circ\mu} = \int_\nu \int_\mu$.*

Proof. By definition of the direct image we must compare the following functors

$$\int_\nu \int_\mu = R\nu_*(R\mu_*(- \otimes^L_{\mathscr{A}_Z^R} \mathscr{A}^R_{Z\to Y}) \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}^R_{Y\to X})$$

$$\int_{\nu\circ\mu} = R(\nu\circ\mu)_*(- \otimes^L_{\mathscr{A}_Z^R} \mathscr{A}^R_{Z\to X})$$

Note that $R(\nu\circ\mu)_* = R\nu_* \circ R\mu_*$. A calculation expanding the definitions shows that

$$\mathscr{A}^R_{Z\to X} \cong \mathscr{A}^R_{Z\to Y} \otimes^L_{\mu^{-1}\mathscr{A}_Y^R} f^{-1}\mathscr{A}^R_{Y\to X}.$$

The result now follows by applying the general observation that for any $\mathscr{F}^\bullet \in D^{b,r}(\mu^{-1}\mathscr{A}_Y^R)$ and $\mathscr{G}^\bullet \in D^{b,r}(\mathscr{A}_Y^R)$ there is a isomorphism

$$(R\mu_*\mathscr{F}^\bullet) \otimes^L_{\mathscr{A}_Y^R} \mathscr{G}^\bullet \cong R\mu_*(\mathscr{F}^\bullet \otimes^L_{\mu^{-1}\mathscr{A}_Y^R} \mu^{-1}\mathscr{G}^\bullet).$$

□

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $D^{b,r}_{qc}(\mathscr{A}_Y^R)$ the full subcategory of $D^{b,r}(\mathscr{A}_Y^R)$ consisting of those complexes of right $\mathscr{A}_Y^R$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\mathrm{Spec}\,R}$. Similarly for $D^{b,r}_{coh}(\mathscr{A}_Y^R)$ with the cohomology being coherent $\mathscr{A}_Y^R$-modules.

Theorem 3.3.7. *Let $\mu : Y \to X$ be a morphism of smooth algebraic varieties. Then the direct image $\int$ takes $D^{b,r}_{qc}(\mathscr{A}_Y^R)$ into $D^{b,r}_{qc}(\mathscr{A}_X^R)$. Moreover, when μ is proper the direct image takes $D^{b,r}_{coh}(\mathscr{A}_Y^R)$ into $D^{b,r}_{coh}(\mathscr{A}_X^R)$.*

Proof. This proof boils down to a reduction to a similar theorem for $\mathcal{O}_X^R$-modules.

One can show that every object in $D^{b,r}_{coh}(\mathscr{A}_Y^R)$ can be represented by a complex with terms $\mathscr{F}^p \otimes_{\mathcal{O}_Y^R} \mathscr{A}_Y^R$ where $\mathscr{F}^p$ is a coherent $\mathcal{O}_Y^R$-modules. This ultimately reduces to computation of the direct image to a spectral sequence with

$$E_1^{p,q} = (R^q\mu_*\mathscr{F}^p) \otimes_{\mathcal{O}_X^R} \mathscr{A}_X^R.$$

The properness of μ yields that $R^q\mu_*\mathscr{F}^p$ is coherent as a $\mathcal{O}_X^R$-module. In particular the $E_1^{p,q}$-terms are coherent as $\mathscr{A}_X^R$-modules which implies that the same holds for the cohomology sheaves of the direct image. □

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j\mathscr{M}$ in terms of $\mathrm{Ch}^{rel}\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara.

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The idea of the following proof is due to Malgrange (1985) in a K-theoretic context. We follow the exposition of Sabbah (2011) and replace it with the corresponding relative notions.

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

The first step is to note that the behaviour of $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules is easy to understand. The direct image functor on $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules $\mathcal{M}$ is defined by $\int^j \mathcal{M} := R^j\widetilde{\mu}_*(L(T^*\mu)^*\mathcal{M})$. Here, $(T^*\mu)^*(-)$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by $- \otimes_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R$. Looking at the supports the following result is immediate.

Lemma 3.3.8. *For any* $\mathrm{gr}^{rel}\mathscr{A}_Y^R$*-module* $\mathcal{M}$ *it holds that*

$$\operatorname{supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1} \operatorname{supp} \mathcal{M}\right).$$

Applying this lemma to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to show that $\operatorname{supp} \mathrm{gr}^{rel}\int^j\mathscr{M} \subseteq \operatorname{supp} \int^j \mathrm{gr}^{rel}\mathscr{M}$. This is proved in proposition 3.3.14. The main technical ingredient in the proof is the Rees modules associated to a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$.

Definition 3.3.3. *Let z be a new variable. The Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

Given a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it follows that $\mathcal{R}\mathscr{M}$ is a coherent $\mathcal{R}\mathscr{A}_Y^R$-module similarly to proposition 3.2.1. The following isomorphisms of filtered modules on $Y \times \operatorname{Spec} R$ are essential. They mean that the Rees module can be viewed as a parametrisation of various relevant modules.

$$\frac{\mathcal{R}\mathscr{M}}{(z-1)\mathcal{R}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}\mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z^\ell\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}.$$

Here $\mathrm{gr}^{rel}_{[\ell]}$ takes a filtered object and returns $\oplus_k F_k/F_{k-\ell}$. The first formula may be be used to find a corresponding filtered $\mathscr{A}_Y^R$-module for any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

The j-th direct image of a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is the sheaf of $\mathcal{R}\mathscr{A}_X^R$-modules on $X \times \operatorname{Spec} R$ defined by $\int^j \mathcal{M} = R^j\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R)$. Here the filtration on $\mathscr{A}_{Y\to X}^R$ is defined by

$F_i\mathscr{A}^R_{Y\to X} = \mathcal{O}^R_Y \otimes_{\mu^{-1}\mathcal{O}^R_X} \mu^{-1}F_i\mathscr{A}^R_X$. The direct image may be restricted to the category of graded Rees modules in which case it returns a graded Rees module. Coherence is preserved similarly to theorem 3.3.7.

Recall that a $\mathrm{gr}^{rel}\mathscr{A}^R_Y$-modules on $Y \times \operatorname{Spec} R$ could be be viewed as a sheaf on $T^*Y \times \operatorname{Spec} R$ and is already equipped with a direct image. The Rees module viewpoint agrees with the earlier definition by the following lemma.

Lemma 3.3.9. *Consider a filtered $\mathscr{A}^R_Y$-module $\mathscr{M}$. Then viewing $\int^j \mathcal{R}\mathscr{M}/z\mathcal{R}\mathscr{M}$ with it's $\mathrm{gr}^{rel}\mathscr{A}^R_X$-module structure as a sheaf on $T^*X \times \operatorname{Spec} R$ recovers the $\mathrm{gr}^{rel}\mathscr{A}^R_Y$-module direct image $\int^j \mathrm{gr}^{rel}\mathscr{M}$. Viewing $\int^j \mathcal{R}\mathscr{M}/(z-1)\mathscr{M}$ as a $\mathscr{A}^R_X$-module recovers $\int^j\mathscr{M}$.*

Proof. We give the proof for $\int^j \mathrm{gr}^{rel}\mathscr{M}$, the proof for $\int^j\mathscr{M}$ is similar but easier. Consider the following Cartesian square

$$\begin{tikzcd}
\mu^*T^*X \times \operatorname{Spec} R \arrow[r, "T^*\mu"] \arrow[d, "\widetilde{\mu}"] & T^*Y \times \operatorname{Spec} R \arrow[r, "\pi_Y"] & Y \times \operatorname{Spec} R \arrow[d, "\mu"] \\
T^*X \times \operatorname{Spec} R \arrow[rr, "\pi_X"] & & X \times \operatorname{Spec} R.
\end{tikzcd}$$

The derived version of the flat base change theorem is applicable, see proposition 3.1.0 in chapter 4 of Berthelot et al. (2006), and yields that

$$L\pi_X^* R\mu_*\left(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}^R_Y} \mathcal{R}\mathscr{A}^R_{Y\to X}\right) \cong R\widetilde{\mu}_* L(T^*\mu \circ \pi_Y)^*\left(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}^R_Y} \mathcal{R}\mathscr{A}^R_{Y\to X}\right).$$

Since π_X is flat it follows that $\mathscr{H}^j L\pi_X^*(R\mu_*-) = \pi_X^*(R^j\mu_*-)$. It now suffices to show that the right hand of the isomorphism side is $\int \mathrm{gr}^{rel}\mathscr{M}$. We show that $L\pi_Y^*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}^R_Y} \mathcal{R}\mathscr{A}^R_{Y\to X}) \cong \mathrm{gr}^{rel}\mathscr{M} \otimes^L_{\mu^{-1}\mathcal{O}^R_X} \widetilde{\mu}^* \mathrm{gr}^{rel}\mathscr{A}^R_X$ whence the result follows since $L(T^*\mu \circ \pi_Y)^* = L(T^*\mu)^* \circ L\pi_Y^*$ by π_Y being flat.

Let $\mathcal{F}^\bullet$ denote a bimodule resolution for $\mathcal{R}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathcal{R}\mathscr{A}^R_Y$-modules. Then $(\mathcal{R}\mathscr{A}^R_Y/z\mathcal{R}\mathscr{A}^R_Y) \otimes_{\mathcal{R}\mathscr{A}^R_Y} \mathcal{F}^\bullet$ is a bimodule resolution for $\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathrm{gr}^{rel}\mathscr{A}^R_Y$-modules. Now $L\pi_Y^*$ just means applying $\pi^{-1}(-) \otimes \mathcal{O}_{T^*Y}$ to the terms of this free resolution. Due to flatness this yields a free resolution in $\pi^*\mathrm{gr}^{rel}\mathscr{A}^R_Y$-modules of $\pi^*\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X}$. Since $\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X} = \mathcal{O}^R_Y \otimes_{\mu^{-1}\mathcal{O}^R_X} \mu^{-1}\mathrm{gr}^{rel}\mathscr{A}^R_X$ and $\pi^*\mu^* = \widetilde{\mu}^*\pi^*$ the desired equality follows. □

It turns out that one can directly compare $\mathrm{gr}^{rel}_{[\ell]}\int^j\mathscr{M}$ and $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ when ℓ is large. Some care is required since since $\int^j \mathcal{R}\mathscr{M}$ may have $\mathbb{C}[z]$-torsion.

Lemma 3.3.10. *Consider a $\mathscr{A}^R_Y$-module $\mathscr{M}$ with a good filtration. Then, for sufficiently large ℓ, the kernel of z^ℓ in $\int^j \mathcal{R}\mathscr{M}$ stabilises. For such ℓ the quotient $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is the $\mathcal{R}\mathscr{A}^R_X$-coherent module associated to a good filtration on $\int^j\mathscr{M}$.*

Proof. By $\int \mathcal{R}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}^R_X$ it follows that $\ker z^\ell$ stabilises.

Now consider the short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z-1} \mathcal{R}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z-1} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z-1} \cdots.$$

Since $\int^{j+1} \mathcal{R}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module it follows that $z-1$ is injective whence $\int^j \mathcal{R}\mathscr{M}/(z-1)\int^j \mathcal{R}\mathscr{M} \cong \int^j \mathscr{M}$. This yields the desired result using that $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is $\mathbb{C}[z]$-torsion free and the isomorphism

$$\frac{\int^j \mathcal{R}\mathscr{M}}{(z-1)\int^j \mathcal{R}\mathscr{M}} \cong \frac{\int^j \mathcal{R}\mathscr{M}/\ker z^\ell}{(z-1)(\int^j \mathcal{R}\mathscr{M}/\ker z^\ell)}.$$

□

From now on we equip $\int^j \mathscr{M}$ with the good filtration inherited from the Rees module's direct image.

Lemma 3.3.11. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, if ℓ is sufficiently large, $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$.*

Proof. The short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \mathcal{R}\mathscr{M} \to \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \cdots.$$

Hence, $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M})$ and it remains to show that $\mathcal{R}\int^j \mathscr{M}/z^\ell\mathcal{R}\int^j \mathscr{M}$ is a quotient of $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$.

Let ℓ be sufficiently large so that lemma 3.3.10 yields a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong \mathcal{R}\int^j \mathscr{M}$. The map z^ℓ induces a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong z^\ell \int^j \mathcal{R}\mathscr{M}$. Therefore $z^\ell \int^j \mathcal{R}\mathscr{M}/z^{2\ell}\int^j \mathcal{R}\mathscr{M} \cong \mathcal{R}\int^j \mathscr{M}/z^\ell\mathcal{R}\int^j \mathscr{M}$. The desired quotient follows by applying the map $m \mapsto z^\ell m$ on $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$. □

The main remaining task is to relate these results to the desired case of $\ell = 1$.

Definition 3.3.4. *For any $\ell \geq 1$ the G-filtration on a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is defined by the decreasing sequence of $\mathrm{gr}^{rel}_{[\ell]}\mathscr{A}_Y^R$-submodules $G_j\mathcal{M} := z^j\mathcal{M}$.*

Lemma 3.3.12. *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ there is the a isomorphism of $\mathrm{gr}\,\mathscr{A}_Y^R$-modules*

$$\mathrm{gr}^G\,\mathrm{gr}^{rel}_{[\ell]}\mathscr{M} \cong (\mathrm{gr}^{rel}\mathscr{M})^\ell.$$

Proof. This follows from directly from the fact that $G_j\,\mathrm{gr}^{rel}_{[\ell]}\mathscr{M} = \oplus_k F_{k-j}\mathscr{M}/F_{k-\ell}\mathscr{M}$. □

Lemma 3.3.13. *Consider a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$. Then one has a isomorphism $\mathrm{gr}^G \int \mathcal{M} \cong \int \mathrm{gr}^G \mathcal{M}$ in $D^{b,r}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$.*

Proof. Writing out the direct image functors the desired result is a isomorphism

$$\mathrm{gr}^G R\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}) \cong R\mu_*(\mathrm{gr}^G \mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R).$$

The proof of the commutation proceeds in two steps corresponding to the two derived functors.

Let $\mathscr{F}^\bullet$ be a bimodule resolution for $\mathcal{R}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathcal{R}\mathscr{A}_Y^R$-modules. There is a G-filtration on this complex given by $z^j(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) = (z^j\mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. By the flatness of locally free sheaves and the short exact sequence $0 \to \oplus_j z^j \mathcal{M} \to \oplus_j z^{j-1}\mathcal{M} \to \operatorname{gr}^G \mathcal{M} \to 0$ it follows that $\operatorname{gr}^G(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) \cong (\operatorname{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. Similarly to the arguments in the proof of lemma 3.3.9 the complex of $\operatorname{gr}^G \mathscr{A}_Y^R$-modules $(\operatorname{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ can be viewed as a representative of $(\operatorname{gr}^G \mathcal{M}) \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \operatorname{gr}^{rel}\mathscr{A}_X^R$.

Denote $\mathcal{G}(-)$ for the functor which takes a sheaf complex and returns its Godement resolution. Flabby sheaves are acyclic for μ_* so the Godement resolution may be used to compute $R\mu_*$. Moreover, since the terms of a Godement resolution are essentially direct sums of formal products of stalks, it is immediate that $z^i\mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(z^i\mathcal{N}^\bullet)$ and that $\operatorname{gr}^G \mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(\operatorname{gr}^G \mathcal{N}^\bullet)$ for any complex of right $\mu^{-1}\mathcal{R}\mathscr{A}_X^R$-modules $\mathcal{N}^\bullet$. Applying μ_* to these equalities and setting $\mathcal{N}^\bullet = \mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ yields the desired result. □

Proposition 3.3.14. *For a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it holds that*

$$\operatorname{supp} \operatorname{gr}^{rel} \int^j \mathscr{M} \subseteq \operatorname{supp} \int^j \operatorname{gr}^{rel} \mathscr{M}.$$

Proof. Let $\ell \geq 0$ be sufficiently large so that lemma 3.3.11 holds, that is to say that $\operatorname{gr}^{rel}_{[\ell]} \int^j \mathscr{M}$ is a subquotient of $\int^j \operatorname{gr}^{rel}_{[\ell]} \mathscr{M}$. By lemma 3.3.12 it holds that $\operatorname{gr}^G \operatorname{gr}^{rel}_{[\ell]} \int^j \mathscr{M} \cong (\operatorname{gr}^{rel} \int \mathscr{M})^\ell$. Since $\operatorname{gr}^{rel}_{[\ell]} \int^j \mathscr{M}$ is a subquotient of $\int \operatorname{gr}^{rel}_{[\ell]} \mathscr{M}$ it remains to show that the support of $\operatorname{gr}^G \int^j \operatorname{gr}^{rel}_{[\ell]} \mathscr{M}$ is a subset of the support of $\int^j \operatorname{gr} \mathscr{M}$.

This can be established with the spectral sequence associated of the G-filtered complex $\int \operatorname{gr}^{rel}_{[\ell]} \mathscr{M}$. Since the G-filtration is finite on $\operatorname{gr}^{rel}_{[\ell]} \mathscr{A}_X^R$-modules the associated spectral sequence abuts by general results[12]. To be precise the associated spectral sequence with terms $E^2_{pq} = \mathscr{H}^{p+q} \operatorname{gr}^G \int \operatorname{gr}^{rel}_{[\ell]} \mathscr{M}$ abuts to $\operatorname{gr}^G \int \mathscr{M}$. By lemma 3.3.13 and lemma 3.3.12 it holds that $E^2_{pq} \cong (\int^{p+q} \operatorname{gr} \mathscr{M})^\ell$. [13] It follows that $\operatorname{supp} \operatorname{gr}^G \int^j \operatorname{gr}^{rel}_{[\ell]} \mathscr{M}$ is a subset of the support of $\int \operatorname{gr} \mathscr{M}$ which completes the proof. □

Theorem 3.3.15. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

$$\operatorname{Ch}^{rel}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\operatorname{Ch}^{rel}\mathscr{M})\right).$$

Proof. This is immediate from lemma 3.3.8 and proposition 3.3.14. □

Corollary 3.3.16. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$ the direct image $\int^j \mathscr{M}$ is a relative holonomic $\mathscr{A}_X^R$-module.*

Proof. This follows from theorem 3.3.15 and ([14]). □

3.4 Non-commutative Homological Notions

In this section we discuss homological notions associated to the $\mathcal{E}xt$-functor the noncommutative sheaf of rings $\mathscr{A}_X^R$. These notions are particularly well-behaved for relatively holonomic modules. The results are sheaf-theoretic rewording of the similar results in Budur et al. (2019) which are themselves derived from the appendices of Björk (1993). Throughout X is assumed to be a smooth variety of dimension n.

[12] Note: Found spectral sequence result online, add good reference.
[13] Note: Or E^1? Seems to depend on preference but should actually matter somewhat for the differentials.
[14] Note: isotropic is conserved Kashiwara (1976)

Definition 3.4.1. *For a non-zero coherent sheaf of $\mathscr{A}_X^R$-modules $\mathscr{M}$ the smallest integer $k \geq 0$ such that $\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \neq 0$ is called the grade of $\mathscr{M}$ and is denoted $j(\mathscr{M})$.*

The following proposition gives geometrical meaning to grades.

Proposition 3.4.1. *For coherent $\mathscr{A}_X^R$-modules $\mathscr{M}$ it holds that*

$$j(\mathscr{M}) + \dim \mathrm{Ch}^{rel}\mathscr{M} = 2n + \dim R$$

where $\dim R$ denotes the Krull dimension of the ring R.

Proof. This is lemma 3.2.2 in Budur et al. (2019). □

Corollary 3.4.2. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module. Then $\mathscr{M}$ has grade strictly greater than n if and only if $B_{\mathscr{M}}$ is non-zero.*

Proof. This is immediate from proposition 3.2.8 and proposition 3.4.1. □

Definition 3.4.2. *A non-zero coherent sheaf of $\mathscr{A}_X^R$-modules $\mathscr{M}$ is called j-pure if $j(\mathscr{N}) = j(\mathscr{M}) = j$ for every non-zero submodule $\mathscr{N}$.*

Lemma 3.4.3. *Let $\mathscr{M}$ be a non-zero coherent $\mathscr{A}_X$-module of grade j. Then $\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A})$ has grade greater than or equal to k for any $k \geq 0$ and $\mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$ is a j-pure $\mathscr{A}_X^R$-module.*

Moreover $\mathscr{M}$ is j-pure if and only if $\mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^R) = 0$ for every $k \neq j$.

Proof. This is lemma 4.3.5 in Budur et al. (2019). □

Lemma 3.4.4. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade j. Then $\mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$ is a relative holonomic $\mathscr{A}_X^R$-module and*

$$\mathrm{Ch}^{rel}\, \mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \subseteq \mathrm{Ch}^{rel}\mathscr{M}.$$

Proof. This is lemma 3.2.4 in Budur et al. (2019). □

Lemma 3.4.5. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module and let $I \subseteq R$ be an ideal. Then, for any $k \geq 0$, $\mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I})$ is a relative holonomic $\mathscr{A}_X^{R/I}$-module.*

Proof. Compute $\mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I})$ with a locally free $\mathscr{A}_X^R$-resolution of $\mathscr{A}_X^{R/I}$. Then lemma 3.2.3 and lemma 3.2.7 show that it is a relative holonomic $\mathscr{A}_X^R$-module. This means that it admits a relative filtration over $\mathscr{A}_X^R$ such that the graded object has a support of the form

$$\mathrm{supp}\,\mathrm{gr}^{rel}_{\mathscr{A}_X^R}\, \mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I}) = \bigcup \Lambda \times S_\Lambda$$

for Lagrangian subvarieties $\Lambda \subseteq T^*X \times \operatorname{Spec} R$ and algebraic varieties $S_\Lambda \subseteq \operatorname{Spec} R$.

On the other hand $\mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I})$ is also a coherent $\mathscr{A}_X^{R/I}$-module. The earlier filtration descends to a filtration over $\mathscr{A}_X^{R/I}$ and it holds that

$$\mathrm{supp}^{rel}_{\mathscr{A}_X^{R/I}}\, \mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I}) = (\mathrm{Id}_{T^*X} \times \Delta)^{-1}(\mathrm{supp}\,\mathrm{gr}^{rel}_{\mathscr{A}_X^R}\, \mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I}))$$

where $\Delta : \operatorname{Spec} R/I \to \operatorname{Spec} R$ is the closed embedding. This yields the desired result. □

Lemma 3.4.6. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which is $(n+k)$-pure for some $0 \leq k \leq \dim R$. If $b \in R$ is not contained in any minimal prime ideal containing $B_{\mathscr{M}}$ then multiplication by b induces injective automorphisms on $\mathscr{M}$ and $\mathcal{E}xt^{n+k}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$. Moreover, there exists a good filtration on $\mathscr{M}$ such that b induces a injection on $\mathrm{gr}^{rel}\mathscr{M}$.*

The proof of the following lemma is a slight modification on the proof of proposition 3.4.3 in Budur et al. (2019).

Lemma 3.4.7. *Let $\mathscr{M}$ be a non-zero relative holonomic $\mathscr{A}_X^R$-module of grade $j(\mathscr{M}) = n$ then, for any non-unit $b \in R$, it holds that $\mathscr{M} \otimes_R R/(b)$ is a non-zero relative holonomic $\mathscr{A}_X^{R/(\ell)}$-module of grade n.*

Proof. Applying lemma 3.4.5 with $k = 0$ yields that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic $\mathscr{A}_X^{R/(\ell)}$-module.

It remains to establish that $\mathscr{M} \otimes_R R/(b)$ is non-zero of grade n. By taking a free resolution of $\mathscr{M}$ one has that

$$R\mathcal{H}om_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \otimes^L_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)} \cong R\mathcal{H}om_{\mathscr{A}_X^{R/(b)}}(\mathscr{M} \otimes^L_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}, \mathscr{A}_X^{R/(b)})$$

where we note that $\mathscr{A}_X^{R/(b)}$ is a $\mathscr{A}_X^R$-bimodule so that both tensor products are well-defined. We compare the Grothendieck spectral sequences of both sides of this isomorphism.

The spectral sequence associated with the right-hand-side has E^2-sheet

$$E^2_{pq} = \mathcal{E}xt^p_{\mathscr{A}_X^{R/(b)}}(\mathcal{T}or^{\mathscr{A}_X^R}_{-q}(\mathscr{M}, \mathscr{A}_X^{R/(b)}), \mathscr{A}_X^{R/(b)}).$$

Recall from lemma 3.4.3 terms with $p > n$ have grade greater than n and due to **??** there are no non-zero terms with $p < n$. Hence, the only term with $p + q = n$ which could potentially have degree n is E^2_{n0}. If we can show that the total cohomology of degree n on the left-hand-side has grade n then it follows that $\mathcal{E}xt^n_{\mathscr{A}_X^{R/(b)}}(\mathscr{M} \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}), \mathscr{A}_X^{R/(b)}) \neq 0$.

The spectral sequence associated to the left-hand-side has E^2-sheet given by

$$E^2_{pq} = \mathcal{T}or^{\mathscr{A}_X^R}_{-q}(\mathcal{E}xt^p_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^{R/(b)}).$$

Note that there are no non-zero differentials which map into E^j_{0n} or for $j \geq 2$. Further, the differentials out of the E^j_{0n} land in $E^j_{-j(n+j)}$ which is a subquotient of $\mathcal{T}or^{\mathscr{A}_X^R}_j(\mathcal{E}xt^{n+j+1}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^{R/(b)})$. Since $\mathcal{E}xt^{n+j}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$ is relative holonomic of degree $n+j$ proposition 3.4.1 and a argument similarly to that which established that $\mathscr{M} \otimes_R R/(b)$ is relative holonomic yield that $E_{j(n+j+1)^j}$ has grade greater than or equal to $n + j - 1$. We will show that that E^2_{0n} has grade n. Then using proposition 3.4.1 and lemma 3.2.3 on the exact sequences

$$0 \to E^{j+1}_{0n} \to E^j_{0n} \to E_{j(n+j+1)^j}$$

it follows that E^j_{0n} has grade n for every $j \geq 2$. This shows that the total cohomology of degree n has grade n and concludes the proof.

Denote $\mathcal{E}^n := \mathcal{E}xt^n_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$, by lemma 3.4.3 it holds that $\mathcal{E}^n$ is a n-pure relative holonomic $\mathscr{A}_X^R$-module. By lemma 3.4.6 it follows that b induces injections on $\mathcal{E}^n$ and

$\mathrm{gr}^{rel}\,\mathcal{E}^n$ for some appropriate filtration. In particular the vertical maps in the following diagram are injective

$$\begin{tikzcd}
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] \arrow[d, "b"] & F_i\mathcal{E}^n \arrow[r] \arrow[d, "b"] & \mathrm{gr}_i^{rel}\,\mathcal{E}^n \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] & F_i\mathcal{E}^n \arrow[r] & \mathrm{gr}_i^{rel}\,\mathcal{E}^n \arrow[r] & 0
\end{tikzcd}$$

so the snake lemma yields a short exact sequence

$$0 \longrightarrow F_{i-1}\mathcal{E}^n \otimes_R R/(b) \longrightarrow F_i\mathcal{E}^n \otimes_R R/(b) \longrightarrow \mathrm{gr}_i^{rel}\,\mathcal{E}^n \otimes_R R/(b) \longrightarrow 0.$$

The injectivity of b on $\mathrm{gr}^{rel}\,\mathcal{E}^n$ implies that b is also injective on $\mathcal{E}^n/F_i\mathcal{E}^n$. A similar application of the snake lemma now yields a short exact sequence

$$0 \longrightarrow F_i\mathcal{E}^n \otimes_R R/(b) \longrightarrow \mathcal{E}^n \otimes_R R/(b) \longrightarrow (\mathcal{E}^n/F_i\mathcal{E}^n) \otimes_R R/(b) \longrightarrow 0.$$

A filtration on $\mathcal{E}^n \otimes_R R/(b)$ is induced by the image of $F_i\mathcal{E}^n$. By the injectivity of the short exact sequences one now has isomorphisms

$$F_i(\mathcal{E}^n \otimes_R R/(b)) \cong F_i\mathcal{E}^n/(F_i\mathcal{E}^n \cap b\mathcal{E}^n) \cong F_i\mathcal{E}^n/bF_i\mathcal{E}^n \cong (F_i\mathcal{E}^n) \otimes_R R/(b)$$

combined with the surjectivity of the first short exact sequence it follows that

$$\mathrm{gr}^{rel}(\mathcal{E}^n \otimes_R R/(b)) \cong \mathrm{gr}^{rel}\,\mathcal{E}^n \otimes_R R/(b).$$

It follows that

$$\mathrm{Ch}^{rel}(\mathcal{E}^n \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}) = (\mathrm{Id}_{T^*X} \times \Delta)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})$$

with $\Delta : \operatorname{Spec} R/(b) \to \operatorname{Spec} R$ the closed embedding as before. Since $\mathscr{M}$ has grade n this equality and proposition 3.4.1 imply that $\mathrm{Ch}^{rel}(\mathcal{E}^n \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)})$ has dimension $n + \dim R - 1$. In particular it follows that $\mathcal{E}^n \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}$ is non-zero and has grade n. This concludes the proof. $\square$

By lemma 3.4.3 the following definition gives a class of j-pure modules.

Definition 3.4.3. *A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be j-Cohen-Macaulay for some $j \geq 0$ if $\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) = 0$ for any $k \neq j$.*

The property of being j-pure is not stable when restricting to a subscheme of $\operatorname{Spec} R$. For the subclass of j-Cohen-Macaulay modules the restriction is more well-behaved.

Lemma 3.4.8. *Let $\mathscr{M}$ be a relative holonomic and $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^R$-module. Let $b \in R$ be non-vanishing on every irreducible component of $Z(B_{\mathscr{M}})$. Then it holds that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^{R/(b)}$-module or zero.*

Proof. This is shown in the proof of proposition 3.4.3 in Budur et al. (2019). This proof is similar to the proof of lemma 3.4.7 which was based on Budur et al. (2019). $\square$

Lemma 3.4.9. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade $n+k$. Then there exists a open* $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ *such that $\mathscr{M} \otimes_R R'$ is a relative holonomic and $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^{R'}$ module. Moreover it may be assumed that the complement of* $\operatorname{Spec} R'$ *in* $\operatorname{Spec} R$ *has codimension $> k$.*

Proof. This is established in the proof of lemma 3.5.2 in Budur et al. (2019). □

The following lemma is a generalisations of a result by Kashiwara (1976) to the relative case. The proof and statement are more involved than the original result by Kashiwara but follow the same line of thought.

Lemma 3.4.10. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which comes equipped with the structure of a $\mathscr{A}_X^R\langle t\rangle$-module. Suppose that $\mathscr{M}$ has grade $j(\mathscr{M}) = n+k$ with $k \geq 1$. Then there exists a open* $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ *such that $\mathscr{M}' = \mathscr{M} \otimes_R R'$ is a relative holonomic $\mathscr{A}_X^{R'}$-module with $t^N \mathscr{M}' = 0$ for N sufficiently large. Moreover, it may be assumed that* $\operatorname{Spec} R \setminus \operatorname{Spec} R'$ *has codimension strictly greater than k.*

Proof. The proof is split in two main parts. The first part it is establishes that $\mathcal{E}xt^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ stabilises and that the $t^i\mathscr{M}'$ are $(n+k)$-Cohen-Macauley. It follows that $t^i\mathscr{M}'$ stabilises and the final part of the proof deduces that the stable value is zero.

Note that $\mathscr{M}/t\mathscr{M}$ is a coherent sheaves over the Noetherian sheaf of rings $\mathscr{A}_X^R$. Hence, the kernel of the morphisms $\mathscr{M}/t\mathscr{M} \to t^i\mathscr{M}/t^{i+1}\mathscr{M}$ stabilise. Let $N \geq 0$ be sufficiently large so that these kernels are constant for $i \geq N$.

By use of lemma 3.4.9 it may be assumed that $\operatorname{Spec} R'$ is such that $t^i\mathscr{M}'$, $\mathscr{M}'/t^i\mathscr{M}'$, $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ and the kernels K_i of the morphisms $\mathscr{M}'/t\mathscr{M}' \to t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ are zero or $(n+k)$-Cohen-Macauley for any $i = 0, \ldots, N$. Since localisation is an exact functor the stabilisation of kernels for $i \geq N$ is still valid over $\operatorname{Spec} R'$. The first steps in this proof use the stabilisation to establish that the modules are actually $(n+k)$-Cohen-Macaulay for arbitrary $i \geq 0$. For notational simplicity we abbreviate $\mathcal{E}xt^k(\mathscr{M}') := \mathcal{E}xt^k_{\mathscr{A}_X^{R'}}(\mathscr{M}', \mathscr{A}_X^{R'})$.

The surjection $\mathscr{M}'/t\mathscr{M}' \twoheadrightarrow t^i\mathscr{M}'/t^{i+1}$ induces a long exact sequence

$$0 \to \mathcal{E}xt^{n+k}\left(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}\right) \to \mathcal{E}xt^{n+k}\left(\frac{\mathscr{M}'}{t\mathscr{M}'}\right) \to \mathcal{E}xt^{n+k}\left(K_i\right) \to \mathcal{E}xt^{n+k+1}\left(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}\right) \to \cdots$$

In particular there is a isomorphism $\mathcal{E}xt^{n+k+1}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}') \cong \mathcal{E}xt^{n+k}(K_i)/\operatorname{Im}\mathcal{E}xt^{n+k}(\mathscr{M}'/t\mathscr{M}')$ whose left-hand-side is known to vanish when $i \leq N$. Since the right-hand-side is constant for $i \geq N$ it follows that $\mathcal{E}xt^{n+k+1}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}') \cong 0$ for any $i \geq 0$. The higher order terms of the long exact sequence yield $\mathcal{E}xt^{n+k+j}(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}) \cong 0$ for $j > 1$. This shows that $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

The injection $t^{i+1}\mathscr{M}' \to t^i\mathscr{M}'$ induces exact sequences

$$\mathcal{E}xt^{n+k+j}\left(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}\right) \to \mathcal{E}xt^{n+k+j}\left(t^i\mathscr{M}'\right) \to \mathcal{E}xt^{n+k+j}\left(t^{i+1}\mathscr{M}'\right).$$

By induction on i it follows that $t^i\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$. Similarly the long exact sequence induced by the surjection $\mathscr{M}/t^{i+1}\mathscr{M}' \twoheadrightarrow \mathscr{M}'/t^i\mathscr{M}'$ yields that $\mathscr{M}/t^i\mathscr{M}$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

By the Cohen-Macauley results which have been established it follows that the morphisms $\mathcal{E}xt^{n+k}(t^i\mathscr{M}') \to \mathcal{E}xt^{n+k}(t^{i+1}\mathscr{M}')$ and $\mathcal{E}xt^{n+k}(\mathscr{M}') \to \mathcal{E}xt^{n+k}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ are surjective. Note that $\mathcal{E}xt^{n+k}(\mathscr{M})$ is a coherent sheaf over the Noetherian sheaf of rings $\mathscr{A}_X^{R'}$.

Hence the kernels of $\mathcal{E}xt^{n+k}(\mathscr{M}') \to \mathcal{E}xt^{n+k}(t^i\mathscr{M}')$ stabilise. After possibly increasing N it follows that the morphisms $\mathcal{E}xt^{n+k}(t^i\mathscr{M}') \to \mathcal{E}xt^{n+k}(t^{i+1}\mathscr{M}')$ are isomorphisms for $i \geq N$ which means that $\mathcal{E}xt^{n+k}(t^{i+1}\mathscr{M}')$ stabilises.

By $t^i\mathscr{M}'$ being $(n+k)$-Cohen-Macaulay it follows that $\mathcal{E}xt^{n+k}(\mathcal{E}xt^{n+k}(t^i\mathscr{M}')) \cong t^i\mathscr{M}'$.[15] It follows that $t^i\mathscr{M}'$ stabilises for $i \geq N$ and remains to show that this stable value is 0. If the stable value is non-zero then it is $(n+k)$-Cohen-Macaulay with $k > 1$. By corollary 3.4.2 it follows that there exists some non-zero $b(s_1, \ldots, s_p) \in B_{t^N\mathscr{M}'}$. Note that one has the commutation relation

$$tb(s_1, \ldots, s_p) = b(s_1 + 1, \ldots, s_p + 1)t.$$

Since $t^{N+1}\mathscr{M}' = t^N\mathscr{M}'$ it follows by iteration that $b(s_1 + n, \ldots, s_p + n) \in B_{t^N\mathscr{M}'}$ for any $n \geq 0$. This implies that $Z(B_{t^N\mathscr{M}'}) = 0$ which means that $t^N\mathscr{M}' = 0$.[16] □

3.5 Estimation of the Bernstein-Sato Zero Locust

This section contains the main result of this chapter, namely a proof of the improved estimate for the Bernstein-Sato zero locust which was announced in theorem 3.1.1. We use the same notation as section 3.1. This proof is similar to the method employed by Lichtin (1989) and Kashiwara (1976) but a new induction argument is required in the proof of lemma 3.5.6.

Recall that the global Bernstein-Sato Ideal is the intersection of all local ones. This means that the global Bernstein-Sato zero locust $Z(B_F)$ is the union of all local ones so it suffices to estimate $Z(B_{F,x})$. In particular, it may be assumed that X is affine and admits global coordinates $x_1, \ldots, x_n$.

Due to these global coordinates there is a $\mathcal{O}_X^R$-linear isomorphism between any left $\mathscr{A}_X^R$-module $\mathscr{N}$ and it's right version $\Omega_X \otimes_{\mathcal{O}_X} \mathscr{N}$. Concretely, any section u of $\mathscr{N}$ gives rise to the section $u^* := udx$. Further, for any operator P of $\mathscr{A}_X^R$ there is a adjoint P^* such that

$$(P \cdot u)^* = u^* \cdot P^*$$

for any section u of $\mathscr{N}$. For a vector field $\xi := \sum_i \xi_i \partial_i$ comparison of the definitions shows that $\xi^* := \sum_i \partial_i \xi_i$ satisfies this equality and this extends to $\mathscr{A}_X^R$ by iterating.

By this procedure the functional equation $PF^{s+1} = bF^s$ may equivalently be stated as the equation

$$F^{s+1}dx \cdot P^* = bF^s dx$$

in $\mathscr{A}_X F^s \otimes_{\mathcal{O}_X} \Omega_X$. The corresponding module $\mathscr{M}$ on Y will be the submodule of $\Omega_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_Y G^s$ spanned by $G^s\mu^*(dx)$. Observe that $\Omega_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_Y G^s$ is relative holonomic by **??** so the submodule $\mathscr{M}$ is certainly also relative holonomic. One can equip $\mathscr{M}$ with the structure of a $\mathscr{A}_Y\langle t\rangle$-module by the action $tG^s\mu^*(dx) = G^{s+1}\mu^*(dx)$.

The replacement of $\mathscr{A}_X F^s$ by it's right version $\mathscr{A}_X F^s \otimes_{\mathcal{O}_X} \Omega_X$ is convenient because the direct image functor is more natural for right modules. This will make it easier to transfer information along the resolution of singularities. Further, it explains why the relative canonical divisor occurs in the improved estimate of theorem 3.1.1. This is because $\mu^*(dx)$ gives a local equation for $K_{Y/X}$.

[15] Note: Provide reference, maybe include and prove earlier since this is easy from double complex

[16] Note: Geometrically obvious but may want to add formal argument

Lemma 3.5.1. *In the notation of section 3.1 a polynomial of the form* $\prod_{i=1}^{p}\prod_{j=1}^{N}(\mathrm{mult}_{E_i}(g_1)$ $s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + j)$ *belongs to the Bernstein-Sato ideal* $B_{\mathscr{M}/t\mathscr{M}}$ *if* $N \geq 0$ *is sufficiently large.*

Proof. This may be checked locally. Take a open $U \subseteq Y$ which is sufficiently small to admit local coordinates $z_1, \ldots, z_n$ where z_i determines E_i if $E_i \cap U \neq \emptyset$.

In these local coordinates $G^s = \prod_{i=1}^{p} u_i^{s_i} \prod_{i=1}^{n} z_i^{\sum_{j=1}^{p} M_{ij}s_j}$ and $\mu^*(dx) = v\prod_{i=1}^{n} z_i^{m_i} dz$ where $M_{ij} \leq \mathrm{mult}_{E_i}(g_j)$, $m_i \leq k_i$ and u_i, v are local units. For notational convenience set $u_i = 1$ and $s_i = 0$ for $i > p$. Denote $N_i = \sum_j M_{ij}$ and $\xi_i = \partial_i - \sum_{j=1}^{n} s_j u_j \partial_i(u_j)$ for any $i = 1, \ldots, p$. Let $P = v^{-1}(\prod_{i=1}^{p} u_i^{-1})\xi_1^{N_1} \cdots \xi_p^{N_p} v$ then

$$v\prod_{i=1}^{n} u_i^{s_j+1} z_i^{\sum_{j=1}^{p} M_{ij}(s_j+1)+m_i} dz \cdot P = bv\prod_{i=1}^{n} u_i^{s_j} z_i^{\sum_{j=1}^{p} M_{ij}s_j+m_i} dz$$

where

$$b = \prod_{i=1}^{p}(\sum_{j=1}^{p} M_{ij}s_j + m_i + N_i)(\sum_{j=1}^{p} M_{ij}s_j + m_i + N_i - 1)\cdots(\sum_{j=1}^{p} M_{ij}s_j + m_i + 1).$$

□

By lemma 3.3.5 the global section $G^s\mu^*(dx)$ of $\mathscr{M}$ gives rise to a global section u of $\int^0 \mathscr{M}$. Denote $\mathscr{U}$ for the right $\mathscr{A}_X$-module generated by u. From lemma 3.5.1 one gets a b-polynomial of a desirable form for $\int^0 \mathscr{M}/t\int^0 \mathscr{M}$. The main remaining difficulty in is to induce a b-polynomial for $\mathscr{U}/t\mathscr{U}$. This will exploit lemma 3.4.10 whence it is needed that $\int^0 \mathscr{M}/\mathscr{U}$ has grade at least $n+1$.

In what follows we want to consider the $\mathscr{A}_Y$-module $\mathscr{M}$ as a $\mathscr{D}_Y$-module. This could disturb coherence. To solve this one introduces new coordinates such that there are vector fields $\mathcal{S}_1, \ldots, \mathcal{S}_p$ which acts as $s_1, \ldots, s_p$ on the generator.

Note that there are finitely many codimension 1 components in $Z(B_F)$. Hence, there exist p independent linear polynomials $\sum_{i=1}^{p} d_{ij}s_i$ such that for any j there is no hyperplane parallel to $\sum_{i=1}^{p} d_{ij}s_i = 0$ in $Z(B_F)$. Moreover, it may be assumed that the d_{ij} are non-negative integers. Introduce new coordinates $z_{n+1}, \ldots, z_{n+p}$ and set $\mathscr{X} := X \times \mathbb{C}^p$ and $\mathscr{Y} := Y \times \mathbb{C}^p$. For any $j = 1, \ldots, p$ set $\widetilde{f}_j = f_j \prod_{i=1}^{p} z_{n+i}^{d_{ij}}$. Note that the induced map $\mathscr{Y} \to \mathscr{X}$ is a resolution of singularities for $\prod \widetilde{f}_i$ and that $\widetilde{g}_j = g_j \prod_{i=1}^{p} z_{n+i}^{d_{ij}}$ is the pullback of $\widetilde{f}_i$.

For any $i = 1, \ldots, p$ it holds that

$$\widetilde{G}^s\mu^*(dx) \cdot \partial_{n+i} = \sum_{j=1}^{p} d_{ij}s_j x_j^{-1}\widetilde{G}^s\mu^*(dx).$$

Since the linear polynomials are independent a appropriate $\mathbb{C}$-linear combination provides a vector field ζ_j with $\widetilde{G}^s\mu^*(dx)\cdot\zeta_j = s_j z_j^{-1}\widetilde{G}^s\mu^*(dx)$. Set $\mathcal{S}_j = \zeta_j z_j$ so that $\widetilde{G}^s\mu^*(dx)\cdot\mathcal{S}_j = s_j\widetilde{G}^s\mu^*(dx)$. This solves the coherence issue.

Lemma 3.5.2. *If* $b \in B_{\widetilde{F}}$ *for any* $x \in X \times \{0\}^p$ *then* $b \in B_F$.

Proof. Take local coordinates $x_1, \ldots, x_{n+p}$ near x and let P be in the stalk of $\mathscr{A}_{\mathscr{X}}$ at x such that $b\widetilde{F}^s = P\widetilde{F}^{s+1}$. Similarly to the above there is a $\mathbb{C}$-basis $\xi_1, \ldots, \xi_p$ for the span of $\partial_{n+1}, \ldots, \partial_{n+p}$ so that $\mathcal{S}_j := x_{n+j}\xi_j$ satisfies $\mathcal{S}_j \cdot \widetilde{F}^s = s_j \widetilde{F}^s$. Expand P as a polynomials in $\xi_1, \ldots, \xi_p$

$$P = \sum_\alpha P_\alpha \xi_1^{\alpha_1} \cdots \xi_p^{\alpha_p}$$

where the coefficients P_α live in a stalk of $\mathcal{O}_{\mathscr{X}} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{D}_{\mathscr{X}}$.

Let N be greater than the maximal value of $|\alpha|$ then

$$(x_{n+1} \cdots x_{n+p})^N b\widetilde{F}^s = \left(\sum_\alpha \prod_{i=1}^p (s_i+1)^{\alpha_i} \sum_\beta Q_{\alpha\beta} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \right) \widetilde{F}^{s+1}$$

where the P_α were expanded as polynomials in $\partial_1, \ldots, \partial_n$ with coefficients $Q_{\alpha\beta}$ from $\mathcal{O}_{\mathscr{X}}$. Observe that $\partial_1, \ldots, \partial_n$ act on the formal symbol $\widetilde{F}^{s+1}$ the same as they act on the formal symbol F^{s+1}.

Now consider this functional equation on the analytification of $\mathscr{X}$ and expand the $Q_{\alpha\beta}$ as power series at x. Identifying powers of $x_{n+1} \cdots x_{n+p}$ on both sides a functional equation with analytical coefficients for F^s follows. This establishes that $b \in B_{F,x}$ for any $x \in X$ provided analytical and algebraic Bernstein-Sato ideals are equal.[17] $\square$

Note that replacing F by $\widetilde{F}$ leaves theorem 3.1.1 unchanged up to hyperplanes parallel to $\sum_{i=1}^p d_{ij}s_i = 0$. These are not in $Z(B_F)$ by assumption so, by lemma 3.5.2, it remains to prove the theorem for $\widetilde{F}$. For notational simplicity we simply write F instead of $\widetilde{F}$. Further, we denote $m = n + p$ for the the dimension of $\mathscr{X}$ and $\mathscr{Y}$.

Let $\ell_1, \ldots, \ell_{p-1} \in \mathbb{C}[s]$ be degree one polynomials which will be fixed later. For any $i = 0 \ldots, p$ let L_i be the ideal of $\mathbb{C}[s]$ generated by $\ell_1, \ldots, \ell_i$. Assume that the ℓ_i are chosen sufficiently generically so that $Z(L_{p-1})$ is a line.

Lemma 3.5.3. *The $\mathscr{D}_{\mathscr{Y}}$-module $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is coherent and it's characteristic variety satisfies* $\mathrm{Ch}\, \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1} \subseteq V \cup W$ *where V is isotropic and W is a irreducible variety of dimension $m+1$ which dominates $\mathscr{Y}$.*

Proof. Recall that we ensured that $\mathscr{M}$ is a coherent $\mathscr{D}_{\mathscr{Y}}$-module. Hence, also $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ will be a coherent $\mathscr{D}_{\mathscr{Y}}$-module.

Take local coordinates $z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+p}$ on $\mathscr{Y}$ as in the proof of lemma 3.5.1. This is to say that locally

$$G^s \mu^*(dx) = v \prod_{i=1}^n u_i^{s_j} z_i^{\sum_{j=1}^p M_{ij}s_j + m_i} \prod_{i=1}^n z_{n+i}^{\sum_{j=1}^p d_{ij}s_j} dz.$$

Let s_0 denote a new variable so that $\mathbb{C}[s]/L_{p-1} \cong \mathbb{C}[s_0]$. Then $\mathscr{M} \otimes_{\mathbb{C}[s_0]} R/L_{p-1}$ may be viewed as the $\mathscr{D}_{\mathscr{Y}}$-module which is locally generated by a formal symbol

$$[G^s \mu^*(dx)] = v \prod_{i=1}^{2n} u_i^{A_i s_0 + a_i} z_i^{B_i s_0 + b_i} dz$$

[17] Note: Is this true?

where A_i, B_i, a_i, b_i are complex numbers and we set $u_{n+i} = 1$. Moreover, since the linear functions $\sum_{j=1}^{p} d_{ij} s_j$ on the final terms in $G^s \mu^*(dx)$ formed a basis for the linear polynomials there will be at least one B_{i+n} which is non-zero.

Denote $w = v \prod_{i=1}^{n} u_i^{a_i}$ and consider for any $j = 1, \ldots, n+p$ the operation of $w^{-1}\partial_j w z_j$ on the generator

$$[G^s \mu^*(dx)] \cdot w^{-1}\partial_j w = ((B_j s_0 + b_j) z_j^{-1} + \sum_{i=1}^{n} A_i s_0 u_i^{-1} \partial_j(u_i))[G^s \mu^*(dx)].$$

Recall that the $s_1, \ldots, s_n$ could be produced by acting with a vector field. Since s_0 is found with affine relations it follows that there exists some differential operator $\mathcal{S}_0$ of degree 1 such that $s_0[G^s \mu^*(dx)] = [G^s \mu^*(dx)] \cdot \mathcal{S}_0$. Now we get a well-defined surjection $\mathscr{D}_{\mathscr{Y}}/I \twoheadrightarrow \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ where I denotes the right ideal generated by $w^{-1}\partial_j w z_j - b_j - \mathcal{S} h_i$ with $h_j = B_j + z_j \sum_{i=1}^{n} A_i u_i^{-1} \partial_j(u_i)$ for $j = 1, \ldots, n+p$.

Note that $z_j \sum_{i=1}^{n} A_i u_i^{-1} \partial_j(u_i) = 0$ for $j > n$. Hence, the h_{n+j} are complex scalars and they are not all zero since there exists a non-zero B_{n+j}. After renumbering we now have that $h_1 \in \mathbb{C}^\times$. Denoting ζ_j, σ_0 for the elements of $\operatorname{gr} \mathscr{D}_{\mathscr{X}}$ which correspond to $\partial_j, \mathcal{S}_0$ respectively it holds that $\operatorname{gr} I$ contains $z_j \zeta_j - h_j \sigma_0$ for any $j = 1, \ldots, n+p$. Then also $h_1 z_j \zeta_j - h_j z_1 \zeta_1$ is in $\operatorname{gr} I$ for any $j = 2, \ldots, n+p$. This yields the desired bound for the characteristic variety. $\square$

Lemma 3.5.4. *If the $\mathscr{A}_{\mathscr{X}}$-module $\int^0 \mathscr{M}/U$ has grade m then the quotients $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ are relative holonomic $\mathscr{A}_{\mathscr{X}}^{R/L_i}$-modules of grade m.*

Proof. This follows by induction on $i = 0, \ldots, p$ using lemma 3.4.7 which is applicable by corollary 3.4.2. $\square$

Lemma 3.5.5. *Any polynomial $b \in \mathbb{C}[s]$ which is not in L_i induces a injective automorphisms on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$.*

Proof. Observe that $\mathscr{M}$ has a trivial Bernstein-Sato ideal so that it has degree m by corollary 3.4.2 By inductively applying lemma 3.4.7 it holds that $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ has degree m. In particular, it has a trivial Bernstein-Sato ideal.

Similarly to the proof of lemma 3.5.3 one can pick local coordinates z_i such that $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ is generated by some formal symbol $[G^s \mu^*(dx)]$. Further pick some isomorphism $\mathbb{C}[s]/L_i \cong \mathbb{C}[\widetilde{s}]$. By definition of the formal symbol ∂_i acts on $[G^s \mu^*(dx)]$ as a polynomial in $\widetilde{s}$ with rational functions of the z_i as coefficients.

If b is not injective it follows by clearing denominators that there is some non-zero polynomial $f = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \widetilde{s}^\beta$ with $[G^s \mu^*(dx)] f = 0$. Further, it can be assumed that the degree of f in z is zero. Indeed, if z_i occurs in f then one can find a non-zero polynomial g of lesser degree such that

$$[G^s \mu^*(dx)] f \partial_1 = [G^s \mu^*(dx)] \partial_1 f + [G^s \mu^*(dx)] g.$$

The left-hand-side of this equality vanishes and the term $[G^s \mu^*(dx)] \partial_1 f$ must also vanish since ∂_i acts as a rational function. This means that $[G^s \mu^*(dx)] g = 0$. Repeating this procedure it may be assumed that f is a non-zero polynomial in $\mathbb{C}[\widetilde{s}]$.

Since $\mathbb{C}[\widetilde{s}]$ commutes with $\mathscr{A}_Y^{\mathbb{C}[\widetilde{s}]}$ it follows that f is a non-zero Bernstein-Sato polynomial of $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ which is a contradiction. $\square$

Lemma 3.5.6. *The relative holonomic $\mathscr{A}_{\mathscr{X}}$-module $\int^0 \mathscr{M}/\mathscr{U}$ has grade $j(\int^0 \mathscr{M}/\mathscr{U}) \geq m+1$.*

Proof. Suppose that $\int^0 \mathscr{M}/\mathscr{U}$ has grade m. A contradiction will be derived by replacing $\int^0 \mathscr{M}/\mathscr{U}$ with a holonomic $\mathscr{D}_{\mathscr{X}}$-module. The first task is to understand how $\int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1})$ relates to $(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$.

Recall from lemma 3.5.5 that ℓ_i is injective on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$. This implies that $\ell_i \int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1} = \int^0 \ell_i \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$. The injective automorphisms of ℓ_i on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$ induces a long exact sequence of $\mathscr{A}_X^{\mathbb{C}[s]/L_{i-1}}$-modules

$$0 \to \int^0 \left(\mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \xrightarrow{\ell_i} \int^0 \left(\mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \to \int^0 \left(\mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_i}\right) \to \cdots$$

whence $(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/(\ell_i)$ is a submodule of $\int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i)$. The quotient is isomorphic to the kernel K_i of ℓ_i on $\int^1(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1})$.

Applying a tensor product with $\mathbb{C}[s]/L_{p-1}$ to the inclusion $(\int^0 \mathscr{M}/L_{i-1}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/(\ell_i) \hookrightarrow \int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i)$ yields a exact sequence

$$\cdots \to \mathcal{T}or_1^{\mathbb{C}[s]}\left(K_i, \frac{\mathbb{C}[s]}{L_{p-1}}\right) \to \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}} \to \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_i}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}}.$$

We claim that the ℓ_i can be chosen so that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}$-module of grade greater than or equal to $m+1$. Let's show how this suffices to finish the proof and prove this claim afterwards.

By corollary 3.4.2 the statement that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ has grade greater than m is equivalent to the existence of a non-zero polynomial $b_i \in \mathbb{C}[s]/L_{p-1}$ which annihilates $\mathcal{T}or_1^{\mathbb{C}[s]/L_i}(K_i, \mathbb{C}[s]/L_{p-1}))$. Denote $B = \prod_{i=1}^{p-1} b_i$ and note that the kernels of the automorphisms induced by B^N form a increasing sequence inside the coherent $\mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}$-module $(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$. Such a increasing sequence must stabilise for sufficiently large N. Then it follows that

$$\operatorname{Im} \mathcal{T}or_1^{\mathbb{C}[s]}\left(K_i, \frac{\mathbb{C}[s]}{L_{p-1}}\right) \bigcap B^N \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}} = 0.$$

We get injections

$$\cdots \hookrightarrow B^N \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}} \hookrightarrow B^N \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_i}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}} \hookrightarrow \cdots$$

Since μ is proper the Kashiwara estimate for $\mathscr{D}_{\mathscr{X}}$-modules[18] is applicable and lemma 3.5.3 yields that $\int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1})$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module with characteristic variety $\widetilde{\mu}\left((T^*\mu)^{-1}(V \cup W)\right)$. It follows that $B^N(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module and

$$\operatorname{Ch} B^N \left(\left(\int^0 \mathscr{M}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}}\right) \subseteq \mathcal{L} \cup \widetilde{\mu}\left((T^*\mu)^{-1}(V \cup W)\right).$$

[18]Note: Put in Chapter 1

Now observe that $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a quotient of $B^N(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ with support in the divisor D. Hence, $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module with

$$\mathrm{Ch}\left(B^N\left(\int^0 \mathscr{M}/\mathscr{U}\right) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}\right) \subseteq \mathcal{L} \cup \widetilde{\mu}\left((T^*\mu)^{-1}(V \cup W)\right) \cap (T^*\mathscr{X} \times_{\mathscr{X}} \operatorname{supp} D).$$

This means $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a holonomic $\mathscr{D}_{\mathscr{X}}$-module. Indeed, by ([19]) $\widetilde{\mu}((T^*\mu)^{-1}(V))$ remains isotropic and forms no obstruction to the characteristic variety being Lagrangian. Moreover, $\widetilde{\mu}((T^*\mu)^{-1}(W))$ is irreducible of dimension $m+1$ and dominates $\mathscr{X}$. Intersecting with $T^*\mathscr{X} \times_{\mathscr{X}} \operatorname{supp} D$ then yields a closed strict subset which necessarily has lower dimension. Hence, it follows that $\dim \mathrm{Ch}\, B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1} \leq m$. This means that $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is holonomic. By ([20]) the Bernstein-Sato ideal of holonomic modules is non-zero. But then also the Bernstein-Sato ideal of $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is non-zero. This contradicts lemma 3.5.4 and we conclude that the assumption the grade is m must have been wrong.

It remains to show that the ℓ_j can be chosen so that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$-module of grade greater than or equal to $m+1$. This means we must understand the $\mathcal{E}xt$-functor of a $\mathcal{T}or$. Hence, we consider the interaction between the derived $\mathcal{H}om$-functor and the derived tensor product.

By a taking a $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}$-free resolution of K_i one finds that

$$R\mathcal{H}om_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}}(K_i \otimes^L_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/Li}} \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}) \cong R\mathcal{H}om_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}) \otimes^L_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}} \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$$

where we note that $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$ is a $\mathscr{A}_X^{\mathbb{C}[s]/L_i}$-bimodule so that both tensor products are defined. We compare the Grothendieck spectral sequences of both sides.

The spectral sequence on the left-hand-side has terms

$$E^2_{rq} = \mathcal{E}xt^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}}(\mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

Since $\mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}$-module these terms are only non-zero for $r = m$ or $r = m+1$. In particular, the spectral sequence degenerates at E^2. Note that the statement that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ has grade greater than or equal to $m+1$ is equivalent to $E^2_{m,-1} = 0$.

The spectral sequence on the right-hand-side has terms

$$E^2_{rq} = \mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathcal{E}xt^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

The claim follows if we can ensure that that all terms with $r - q = m - 1$ vanish on $\mathscr{X} \times \operatorname{Spec} R$ for some open subset $\operatorname{Spec} R \subseteq \mathbb{C}^p$. Indeed, then by corollary 3.4.2 the terms have grade $m+1$ and it follows that the same must hold for the terms of the spectral sequence on the left hand side. Since $\mathcal{E}xt^m$ of a relative holonomic module is m-pure or zero this means that the $E^2_{m,-1}$-term in the left-hand-side spectral sequence vanishes.

[19]Note: Put somewhere in Chapter 1 (Kashiwara, 1976, proposition 4.9).

[20]Note: Bjork (1979) holonomic implies s is algebraic

The ℓ_i and the open $\operatorname{Spec} R$ are constructed by induction on i. For any i, j, k with $k \leq i$ denote $\mathcal{E}_{ik}^{n+j} := \mathcal{E}xt^{n+j}_{\mathscr{A}_{\mathscr{X}}^{R/L_k}}(K_k, \mathscr{A}_{\mathscr{X}}^{R/L_k}) \otimes_{\mathscr{A}_{\mathscr{X}}^{R/L_k}} \mathscr{A}_{\mathscr{X}}^{R/L_i}$. In every induction step it is ensured that

(i) $\mathcal{E}_{ii}^{n+j}$ is $(n+j)$-Cohen-Macaulay over $\mathscr{A}_X^{R/L_i}$ or zero for every $j \geq 0$.

(ii) $Z(L_i) \cap \operatorname{Spec} R \neq 0$.

(iii) ℓ_i induces a injection on $\mathcal{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$.

By abuse of notation L_i may also denote the ideal of R generated by $\ell_1, \dots, \ell_i$.

Take some arbitrary ℓ_1 for the base-case and use lemma 3.4.9 to find a open $\operatorname{Spec} R \subseteq \mathbb{C}^p$ such that $\mathcal{E}_{11}^{n+j}$ is $(n+j)$-Cohen-Macaulay for every $j \geq 0$. This only requires removing a strict closed subset of $\operatorname{Spec} \mathbb{C}[s]/L_1$ so $Z(L_1) \cap \operatorname{Spec} R = \operatorname{Spec} R/L_1$ is non-empty. The final property is vacuous for $i = 1$.

Now assume that $i > 1$ and that $\ell_1, \dots, \ell_{i-1}$ are already constructed. First let's ensure that ℓ_i induces a injection on $\mathcal{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$. By iterative application of lemma 3.4.8 it holds that $\mathcal{E}_{(i-1)k}^{n+j}$ is $(n+j)$-Cohen-Macaulay over $\mathscr{A}_{\mathscr{X}}^{L_{i-1}}$. Take ℓ_i so that the induced element of R/L_{i-1} is non-constant and does not vanish on any irreducible component of the Bernstein-Sato zero locust of $\mathcal{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$. Then, by lemma 3.4.6 the desired injectivity follows. As before, lemma 3.4.9 can be used to to find a open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathcal{E}_{ii}^{n+j}$ is $(n+j)$-Cohen-Macaulay for every $j \geq 0$ and $Z(L_i) \cap \operatorname{Spec} R' = \operatorname{Spec} R'/L_i$ is non-empty. Note that replacing $\operatorname{Spec} R$ by $\operatorname{Spec} R'$ will conserve the induction hypothesis. This concludes the inductive construction of the ℓ_i.

Applying injectivity of ℓ_i on $\mathcal{E}_{(i-1)k}^{n+j}$ with the free resolution $\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}} \to \mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}$ for $\mathscr{A}_{\mathscr{X}}^{R/L_i}$ yields that $\mathcal{T}or_m^{\mathscr{A}_{\mathscr{X}}^{R/L_i}}(\mathcal{E}_{(i-1)k}^{n+j}, \mathscr{A}_{\mathscr{X}}^{R/L_i}) = 0$ for all $m > 0$. By taking a $\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}$-free resolution of $\mathcal{E}_{(i-1)k}^{n+j}$ it follows that

$$\mathcal{E}_{(i-1)k}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}} \cong \mathcal{E}_{ik}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}.$$

Iterative application of the isomorphism yields $\mathcal{E}_{ii}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}} \cong \mathcal{E}_{(p-2)i}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_{p-2}}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}$. This means that

$$\mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathcal{E}xt^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}) \cong \mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathcal{E}^r_{(p-2)i}, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$$

The right right-hand-side of this isomorphism was already observed to vanish for any $-q > 0$ and the left-hand-side is precisely the E^2_{rq}-term of the spectral sequence. This establishes that the E^2_{rq}-terms with $r - q = m - 1$ vanish for $q > 0$. The remaining terms $E_{m-1,0}$ is zero regardless since it involves $\mathcal{E}xt^{m-1}$ of a relative holonomic module. This shows that the $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ are relative holonomic $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$-module of grade greater than or equal to $m + 1$ and concludes the proof. □

The following lemma and it's proof are similar to the monovariate case which may be found in (Bjork, 1979, p246).

Lemma 3.5.7. *There is a morphism right $\mathscr{A}_{\mathscr{X}}^R$-modules $\mathscr{U} \to \mathscr{A}_{\mathscr{X}}^R F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega_{\mathscr{X}}$ sending u to $F^s dx$.*

Proof. The resolution of singularities $\mathscr{Y} \to \mathscr{X}$ is a isomorphism on the complement of $\prod f_i = 0$. Hence, a isomorphism $\mathscr{U} = \int^0 \mathscr{M} \cong \mathscr{A}_{\mathscr{X}}^R F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega_{\mathscr{X}}$ holds outside of $\prod f_i = 0$.

Pick some open set $V \subseteq \mathscr{X}$ we must show that whenever $uP = 0$ in $\mathscr{U}(V)$ it follows that $(F^s dx)P = 0$. Due to the isomorphism it is certainly the case that $(F^s dx)P = 0$ outside of $\prod f_i = 0$. Hence, the support of the coherent sheaf of $\mathcal{O}_V^R$-modules $\mathcal{O}_V^R(F^s dx)P$ lies in $\prod f_i = 0$. The Nullstellen Satz now yields that $(\prod f_i)^N (F^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that $\prod f_i$ is a non-zero divisor of $(F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega_{\mathscr{X}})(V)$. Hence, it follows that $(F^s dx)P = 0$ on V as desired. □

Now all ingredients are in place for the proof of theorem 3.1.1.

Theorem 3.5.8. *With notation as in section 3.1 every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{\geq 0}$.

Proof. By lemma 3.5.6 the $\mathscr{A}_X$-module $\mathscr{M}/\mathscr{U}$ has grade greater than or equal to $m+1$. Hence lemma 3.4.10 provides $N \geq 1$ such that $t^N \mathscr{M}/\mathscr{U} = 0$ on a open $\mathscr{X} \times \operatorname{Spec} R$ for some open $\operatorname{Spec} R \subseteq \mathbb{C}^p$ with complement of codimension strictly greater than 1.

Let $b(s_1, \ldots, s_p)$ denote the Bernstein-Sato polynomial for $\mathscr{M}/t\mathscr{M}$ provided by lemma 3.5.1. Set $B := \prod_{i=0}^{N+1} b(s_1 + i, \ldots, s_p + i)$ then it follows that $B\mathscr{M} \subseteq t\mathscr{U}$ on $\mathscr{X} \times \operatorname{Spec} R$. In particular this means that B is in the Bernstein-Sato ideal of $\mathscr{U}/t\mathscr{U}$ over $\operatorname{Spec} R$. By the surjection of lemma 3.5.7 this means that $B \in B_F$ over $\operatorname{Spec} R$. This proves which proves the theorem because the complement of $\operatorname{Spec} R$ has codimension strictly greater than 1. □

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