Contents

Chapter 1

Categorical Preliminaries

This chapter contains some categorical preliminaries on the topic of derived category theory and spectral sequences.

Derived category theory allows one to measure the lack of exactness in a functor $F : \mathcal{A} \to \mathcal{B}$ by encoding error-terms in derived functors $R^iF : \mathcal{A} \to \mathcal{B}$. For instance the non-exactness of the tensor product may be measured by $\mathcal{T}or$-functors.

Spectral sequences were historically developed by Leray to compute the cohomology of the pushforeward of a sheaf. There is some overlap between derived category theory and spectral sequences. In particular the Grothendieck spectral sequence allows one to compute the derived functor of some composition $F \circ G$ based on the derived functors of F and G individually. This theorem is a essential technical ingredient in the proofs of chapter 3.

The discussion of derived category theory in this chapter summarises the relevant parts of chapters 1,2 and 5 of Dimca (2004). The section on spectral sequences is based on chapter 5 of Weibel (1995).

1.1 Spectral Sequences

Fix an abelian category $\mathcal{A}$. Denote $C(\mathcal{A})$ for the category with complexes of objects in $\mathcal{A}$

$$X^\bullet : \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots .$$

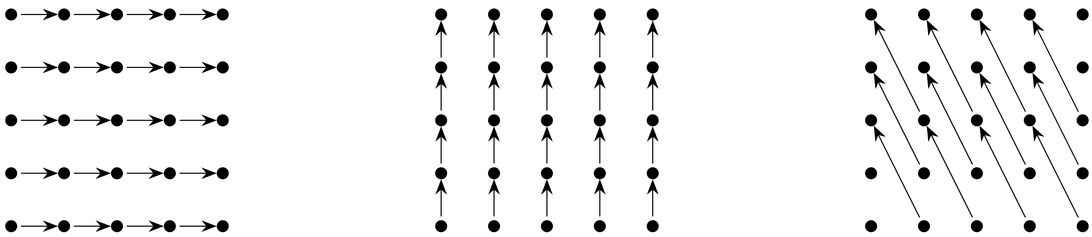
A double complex $E^{\bullet\bullet}$ gives rise to a total complex with terms $\mathrm{Tot}(E)^n = \oplus_{i+j=n} E^{ij}$. The motivating question behind spectral sequences is how the cohomology of the total complex may be computed.

First compute horizontal cohomology to get data E_1^{**}. By the commutativity of the double complex there are vertical differentials on E_1^{**} and one can compute the vertical cohomology to get E_2^{**}. Diagram chasing allows to construct higher-order differentials leading to the following notion.

Definition 1.1.1. *A cohomology spectral sequence starting at the a-th sheet consists of families of objects* $\{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ *for* $r \geq a$ *and maps* $d_{pq}^r : E_r^{pq} \to E_r^{p+r,q-r+1}$ *such that*

(i) The maps d_{pq}^r *are differentials in the sense that* $d^r \circ d^r = 0$.

(ii) The $(r+1)$*-st sheet is the cohomology of the* r*-th sheet* $E_{pq}^{r+1} \cong \ker(d_{pq}^r)/\operatorname{Im}(d_{p-r,q+r-1}^r)$.



E_0^{**} E_1^{**} E_2^{**}

Definition 1.1.2. *A cohomology spectral sequence is said to be bounded if for each n there are finitely many non-zero terms E_a^{pq} with $p+q=n$.*

In a bounded complex there is, for each choice of p, q, a value r_0 such that $E_r^{pq} = E_{r+1}^{pq}$ for all $r \geq r_0$. This stable value is denoted E_∞^{pq}.

Definition 1.1.3. *A bounded spectral sequence is said to converge to a family of objects H^* if any H^n admits a finite filtration*

$$0 = F^s H^n \subseteq \cdots F^p H^n \subseteq F^{p+1} H^n \subseteq \cdots \subseteq F^t H^n = H^n$$

such that $E_\infty^{pq} \cong F^p H^{p+q} / F^{p-1} H^{p+q}$.

Observe that H^* is not necessarily uniquely identified by a convergent spectral sequence. The total complex in the motivating problem comes equipped with two filtrations. A vertical filtration $F^m \operatorname{Tot}(E)^n = \oplus_{p+q=n, p<m} E^{pq}$ and a similar horizontal filtration.

Definition 1.1.4. *A filtration of a complex $C_\bullet$ is a family of subcomplexes $\{F^m C^\bullet\}_{m \in \mathbb{Z}}$. The filtration is said to be exhaustive if $C^\bullet = \cup_m F^m C^\bullet$.*

Proposition 1.1.5. *(Weibel, 1995, Theorem 5.4.1.) A exhaustive filtration of a complex $C^\bullet$ determines a spectral sequence starting with $E_0^{pq} = F^p C^{p+q} / F^{p-1} C^{p+q}$ and $E_1^{pq} = H^{p+q} E_0^{p\bullet}$.*

Definition 1.1.6. *A filtration on a complex $C^\bullet$ is said to be bounded if, for each n, there are integers $s < t$ such that $F^s C^n = 0$ and $F^t C^n = C^n$.*

The following proposition may be used in the motivating problem to recover the cohomology of the total complex up to extension. This is to say that we know that the total cohomology $H^* = H^* \operatorname{Tot}(E)^\bullet$ admits a filtration such that $E_\infty^{pq} \cong F^p H^{p+q} / F^{p-1} H^{p+q}$.

Proposition 1.1.7. *(Weibel, 1995, Theorem 5.51.) Let $C^\bullet$ be a complex with a exhaustive bounded filtration. Then, the associated spectral sequence is bounded and converges to $H^*(C^\bullet)$.*

1.2 Derived Categories

The category $C(\mathcal{A})$ contains full subcategories $C^*(\mathcal{A})$ with $* \in \{+, -, b\}$ denoting that the complexes in $\mathcal{A}$ are bounded below, above or bounded on both sides respectively. For example $C^+(\mathcal{A})$ may contain complexes of the form $\cdots \to 0 \to \cdots \to X^{-1} \to X^0 \to \cdots$. For a complex $X^\bullet$ and $k \in \mathbb{Z}$ one has a shifted complex $X^\bullet[k]$ with $(X^\bullet[k])^s = X^{k+s}$. Further, denote $\operatorname{Hom}^k(X^\bullet, Y^\bullet) := \operatorname{Hom}(X^\bullet, Y^\bullet[k])$ for the chain maps that change the grading by k.

Definition 1.2.1. *Two complex morphisms $u, v : X^\bullet \to Y^\bullet$ are called homotopic if there exists $h \in \mathrm{Hom}^{-1}(X^\bullet, Y^\bullet)$ such that $u - v = d_Y h + h d_X$. This may be denoted $u \sim v$.*

Definition 1.2.2. *A morphism $u : X^\bullet \to Y^\bullet$ of complexes in $C^*(\mathcal{A})$ is called a quasi-isomorphism if the induced morphism in cohomology $H^k(u) : H^k(X^\bullet) \to H^k(Y^\bullet)$ is a isomorphism for all k.*

The idea behind the following definition is to retain the same objects as $C^*(\mathcal{A})$ but turn quasi-isomorphisms into isomorphisms. The technicalities may be found in chapter 8 of Deligne (1977).

Definition 1.2.3. *The derived category $D^*(\mathcal{A})$ is defined as the category obtained from $C^*(\mathcal{A})$ by localising with respect to the multiplicative system formed by the quasi-isomorphisms.*

This definition can be made more concrete provided the category has enough injectives.

Definition 1.2.4. *A abelian category $\mathcal{A}$ has enough injectives if for any object X in $\mathcal{A}$ there is an exact sequence $0 \to X \to I$ in $\mathcal{A}$ with I injective.*

Definition 1.2.5. *Let $\mathcal{A}$ be a abelian category. The homotopical category of complexes $K^*(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $C^*(\mathcal{A})$ and as morphisms*

$$\mathrm{Hom}_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet) := \mathrm{Hom}_{C^*(\mathcal{A})}(X^\bullet, Y^\bullet)/\sim .$$

Observe that two homotopic maps induce the same morphism in cohomology. It follows that there is a well-defined functor $p^*_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$.

Proposition 1.2.6. *(Dimca, 2004, Proposition 1.3.10.) Let $\mathcal{A}$ be a abelian category with enough injectives and denote $I(\mathcal{A})$ for the full subcategory of injective objects. Then the natural functor*

$$p^+_{\mathcal{A}} : K^+(I(\mathcal{A})) \to D^+(\mathcal{A})$$

is a equivalence of categories.

By passing to the opposite categories one gets a similar theorem in categories with enough projectives for $D^-(\mathcal{A})$.

1.3 Triangulated Categories

The categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ remain additive but may fail to be exact. In particular, the notion of short exact sequences no longer makes sense. Instead, $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ may be viewed as triangulated categories which is to say that they come equipped with a notion of exact triangles.

Definition 1.3.1. *Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. The mapping cone of u is the complex in $C^*(\mathcal{A})$ given by*

$$C^\bullet_u := Y^\bullet \oplus (X^\bullet[1])$$

with $d_u(y, x) = (dy + u(x), -dx)$.

The concept of a mapping cone originated in a construction from algebraic topology which explains the name. Observe that the mapping cone gives rise to a triangle

$$T_u : X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{q} C_u^\bullet \to X^\bullet[1]$$

which may be denoted more intuitively as

$$\begin{tikzcd} X^\bullet \arrow[rr, "u"] & & Y^\bullet \arrow[dl, "q"] \\ & C_u^\bullet \arrow[ul, "+1"] & \end{tikzcd}$$

The triangles T_u may be used to encode short exact sequences.

Proposition 1.3.2. *(Dimca, 2004, Proposition 1.1.23.) Given a short exact sequence in $C^*(\mathcal{A})$*

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

there exists a quasi-isomorphism $m : C_u^\bullet \to Z^\bullet$ with $m \circ q = v$.

This shows that a short exact sequence induces a triangle isomorphic to a standard triangle T_u in $D^*(\mathcal{A})$. Further evidence that the triangles T_u behave like short exact sequences is given by the following result.

Proposition 1.3.3. *(Dimca, 2004, Lemma 1.1.20, Proposition 1.1.21) Let $u : X^\bullet \to Y^\bullet$ be a morphism in $C^*(\mathcal{A})$.*

(i) The composition of any two consecutive maps in T_u is homotopic to 0.

(ii) The triangle T_u induces a long exact sequence in cohomology

$$\cdots \to H^k(X^\bullet) \xrightarrow{u} H^k(Y^\bullet) \to H^k(C_u^\bullet) \xrightarrow{\delta} H^{k+1}(X^\bullet) \to \cdots$$

where the connecting morphism δ comes from the map $C_u^\bullet \to X^\bullet[1]$.

Further investigation of the properties of T_u gives rise to the concept of a triangulated category. These definitions and properties are pleasant in their own right so we go into some detail.

The distinguished triangles $\mathcal{T}$ in $K^*(\mathcal{A})$ or $D^*(\mathcal{A})$ are the family of triangles which are isomorphic to a triangle of the form T_u. Observe that these categories have a shift functor T given by $TX^\bullet = X^\bullet[1]$.

Definition 1.3.4. *An additive category $\mathcal{D}$ equipped with a self-equivalence T and a family of distinguished triangles $\mathcal{T}$ is called a triangulated category if the following axioms are satisfied.*

(Tr1) Any triangle isomorphic to a distinguish triangle is distinguished. For any object X the triangle $X \xrightarrow{\mathrm{Id}} X \to 0 \to TX$ is distinguished. Any morphism $u : X \to Y$ is part of some distinguished triangle $X \xrightarrow{u} Y \to Z \to TX$.

(Tr2) A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{Tu} TY$ is distinguished.

(Tr3) A commutative diagram of the following form whose rows are distinguished triangles gives rise to a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & TX \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & TA \end{tikzcd}$$

(Tr4) For any triple of distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{x} A \longrightarrow TX$$

$$Y \xrightarrow{v} Z \longrightarrow B \xrightarrow{y} TY$$

$$X \xrightarrow{v \circ u} Z \longrightarrow C \longrightarrow TX$$

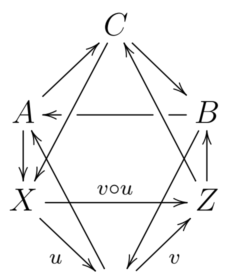
there is a distinguished triangle

$$A \xrightarrow{a} C \xrightarrow{b} B \xrightarrow{(Tx) \circ y} TA$$

such that (id_X, v, a) and (u, id_Z, b) are morphisms of triangles.

Proposition 1.3.5. *(Dimca, 2004, Proposition 1.2.4.) Let $\mathcal{A}$ be a abelian category. Then $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ are triangulated categories.*

A triangle $X \to Y \to Z \to TX$ will also be denoted $X \to Y \to Z \xrightarrow{+1} X$ and $T^m X$ may be denoted with $X[m]$. Now the data of the final axiom can be organised as follows. Correspondingly, $(Tr4)$ is also referred to as the octahedral axiom.



Definition 1.3.6. *Let $\mathcal{D}$ be a triangulated category. A subcategory $\mathcal{C}$ of $\mathcal{D}$ is said to be stable under extensions if any distinguished triangle in $\mathcal{D}$ with two vertices in $\mathcal{C}$ also has its third vertex in $\mathcal{D}$.*

Definition 1.3.7. *Let $\mathcal{C}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. One calls $\mathcal{C}$ is a triangulated subcategory if $\mathcal{C}$ is stable under extensions and $T\mathcal{C} \subseteq \mathcal{C}$.*

Definition 1.3.8. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ a abelian category. An additive functor $F : \mathcal{D} \to \mathcal{A}$ is a cohomological functor if for any distinguished triangle in $\mathcal{D}$*

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{A}$. If F is a cohomological functor one sets $F^i = F \circ T^i$.

The family of functors F^i is conservative if for any distinguished triangle

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced long sequence

$$\cdots \to F^i(X) \to F^i(Y) \to F^i(Z) \to F^{i+1}(X) \to \cdots$$

is exact.

The key example for the above definition is given by the cohomological functor $H^0 : K^*(\mathcal{A}) \to \mathcal{A}$ and the conservative system of functors H^i.

Definition 1.3.9. *Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories. A functor $F : \mathcal{D} \to \mathcal{D}'$ is called a functor of triangulated categories if it is compatible with the shift functor and transforms distinguished triangles in $\mathcal{D}$ into distinguished triangles of $\mathcal{D}'$.*

1.4 Derived Functors

Given abelian categories $\mathcal{A}, \mathcal{B}$ and a functor of triangulated categories $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ one may wonder if there is a natural lift to the derived categories.

Definition 1.4.1. *Let F be as above. The right derived functor of F is a initial couple (R^*F, ξ_F) consisting of a functor of triangulated categories $R^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : p_B \circ F \to R^*F \circ p^*_{\mathcal{A}}$. By initial it is meant that for any other such couple (G, ζ) there is a unique natural transformation $\eta : R^*F \to G$ such that $\zeta = (\eta \circ p^*_{\mathcal{A}}) \circ \xi_F$.*

The dual notion is a left derived functor. This is a final couple (L^*F, ξ_F) consisting of a functor of triangulated categories $L^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : L^*F \circ p^*_{\mathcal{A}} \to p_B \circ F$. It is clear that, if a derived functor exists, it is unique up to unique isomorphism.

There are general theorems on the existence of derived functors which may be found in chapter 1 of Dimca (2004). The following will be sufficient for our applications.

Theorem 1.4.2. *(Dimca, 2004, Remark 1.3.15.) Consider a functor of triangulated categories $F : K^+(\mathcal{A}) \to K(\mathcal{B})$. If $\mathcal{A}$ has enough injectives and F is additive then the right derived functor R^+F exists.*

By dualising, a similar theorem applies to $F : K^-(\mathcal{A}) \to K(\mathcal{B})$ for the existence of L^-F in categories with enough projectives.

The main use of derived functors is to fix a lack of exactness in F. Recall from proposition 1.3.2 that a short-exact sequence in $C^+(\mathcal{A})$ induces a distinguished triangle in $D^+(\mathcal{A})$. Since R^+F is a functor of triangulated categories it will transform this distinguished triangle into a distinguished triangle of $D(\mathcal{B})$. Further, there is a associated long exact sequence whose higher-order terms measure to what degree the original functor failed to be exact.

Definition 1.4.3. *Let $F : K^*(\mathcal{A}) \to K(B)$ be a functor of triangulated categories such that R^*F exists. For any $n \in \mathbb{Z}$ one defines $R^nF : \mathcal{A} \to \mathcal{B}$ to be the composition*

$$\mathcal{A} \xrightarrow{\iota} D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

where ι sends a object to the chain complex with a single non-trivial term. Similarly, one defines $\mathbb{R}^nF : D^*(\mathcal{A}) \to \mathcal{B}$ as the composition

$$D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}.$$

Proposition 1.4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories. Suppose that the derived functor R^*F of the induced functor $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ exists. Then, for any short exact sequence in $\mathcal{A}$*

$$0 \to X \to Y \to Z \to 0$$

there is a long exact sequence in $\mathcal{B}$

$$\cdots \to R^iF(X) \to R^iF(Y) \to R^iF(Z) \to R^{i+1}F(X) \to \cdots .$$

Proof. This is immediate by R^*F being a functor of triangulated categories and the fact that the cohomology H^i forms a conservative system of functors.

□

In the situation of theorem 1.4.2 the derived functor can be computed explicitly. Pick some object $X^\bullet$ in $D^+(\mathcal{A})$. By proposition 1.2.6 there is a quasi-isomorphism $X^\bullet \to I^\bullet$ for some complex of injective objects $I^\bullet$. Then one has explicitly

$$R^+F(X^\bullet) \cong p_{\mathcal{B}} \circ F(I^\bullet).$$

Further, if F is exact one has that $F(I^\bullet)$ is quasi-isomorphic to $F(X^\bullet)$ whence $R^+F(X^\bullet)$ is isomorphic to $p_{\mathcal{B}} \circ F(X^\bullet)$.

In practice, it is often difficult to find a concrete injective resolution.

Definition 1.4.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A object X in $\mathcal{A}$ is F-acyclic if $R^iF(X) = 0$ for all $i \geq 1$.*

Computation derived functors can also be done using F-acyclic resolutions. One can show that injective objects are F-acyclic for any left-exact functor. Hence, this generalises the earlier computations.

Proposition 1.4.6. *(Dimca, 2004, Theorem 1.3.18.) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive functors between abelian categories with enough injective objects. Suppose that F is left-exact and that G transforms injective objects into F-acyclic objects, then there is an isomorphism*

$$R^+(F \circ G) \cong R^+F \circ R^+G.$$

The following theorem is known as the Grothendieck spectral sequence and will be used extensively in chapter 3.

Theorem 1.4.7. *(Dimca, 2004, Theorem 1.3.19.) Let F, G be as in the previous proposition. Then, for any object X of $\mathcal{A}$, there is a spectral sequence*

$$E_2^{pq} = R^p F(R^q G(X))$$

converging to $R^{p+q}(F \circ G)(X)$.

Proof. A modification of the proof is discussed in detail in **??**.

The main idea is to consider the double complex $F(I^\bullet) \to J^{\bullet\bullet}$ provided by the dual of the following lemma. The total complex of $G(J^{\bullet\bullet})$ is equipped with two filtrations, a vertical and a horizontal filtration, comparing the spectral sequences from proposition 1.1.7 yields the desired result.

□

Proposition 1.4.8. *(Weibel, 1995, Lemma 5.7.2.) Suppose $\mathcal{A}$ admits enough projectives. Then, for any complex $X^\bullet$ there exists a lower half-plane double complex $P^{\bullet\bullet}$ of projective objects such that*

(i) There is a map $P^{0,\bullet} \to X^\bullet$ such that $P^{\bullet,q} \to X^q$ is a projective resolution for every p.

(ii) If $X^q = 0$ the corresponding column $P^{\bullet,q}$ is zero.

(iii) The horizontal cocycles, coboundaries and cohomology on $P^{\bullet,q}$ form projective resolutions for the q-th cocycles, coboundaries and cohomology of $X^\bullet$ respectively.

For later use on the structure of the Cartan-Eilenberg resolution we remark that the columns $P^{\bullet,q}$ are found as direct sums of projective resolutions for the boundaries and cohomology at level q and $q+1$.

We conclude this section by considering a few important examples of derived functors which will be used later on.

Let X be a topological space equipped with a sheaf of rings $\mathcal{R}_X$ which need-not be commutative. The corresponding categories of complexes of left or right modules are denoted $C^{*,\ell}(\mathcal{R}_X)$ and $C^{*,r}(\mathcal{R}_X)$ respectively. Similarly, the category of complexes of bimodules is denoted $C^{*,\ell r}(\mathcal{R}_X)$. Using theorem 1.4.2 one can establish that the global sections functor $\Gamma(X, -)$ on $C^{+,*}(\mathcal{R}_X)$ has a derived functor $R^+\Gamma(X, -)$. The cohomology of a sheaf of modules is given by the functors $H^k(X, -) := R^k\Gamma(X, -)$ and the hypercohomology of a complex of modules is given by the functors $\mathbb{H}^k(X, -) := \mathbb{R}^k\Gamma(X, -)$.

Cohomology measures the failure of sections to be global. Correspondingly, acyclic objects are given by sheaves which have no such failure.

Definition 1.4.9. *A sheaf of $\mathcal{R}_X$-modules $\mathscr{F}$ is called flabby if for any open $U \subseteq X$ the restriction morphism $\rho_U^X : \mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.*

Proposition 1.4.10. *(Dimca, 2004, Proposition 2.1.8.) If $\mathscr{F}$ is a flabby sheaf of $\mathcal{R}_X$-modules then $\mathscr{F}$ is $\Gamma(X, -)$-acyclic.*

Let $f : Y \to X$ be a continuous map between topological spaces. The direct image of a sheaf $\mathcal{S}$ on Y is the sheaf $f_*\mathcal{S}$ on X defined by

$$(f_*\mathcal{S})(U) = \mathcal{S}(f^{-1}(U)).$$

Suppose that Y, X are equipped with sheaves of rings $\mathcal{R}_Y, \mathcal{R}_X$ respectively and that $f_*\mathcal{R}_Y$ is a $\mathcal{R}_X$-algebra. Then the direct image yields a functor from the category of $\mathcal{R}_Y$-modules to the category of $\mathcal{R}_X$-modules. This is a left-exact functor so the derived functor may be computed by injective resolutions. One can verify that flabby sheaves are f_*-acyclic so that flabby resolutions may also be used in the computations.

A classical example of a non-exact functor is given by the tensor product. This may be considered as a bifunctor

$$\otimes_{\mathcal{R}_X} : C^{-,\ell}(\mathcal{R}_X) \times C^{-,r}(\mathcal{R}_X) \to C^{-,\ell r}(\mathcal{R}_X)$$

where $(\mathscr{M}^\bullet \otimes_{\mathcal{R}_X} \mathscr{N}^\bullet)^n = \oplus_{i+j=n} \mathscr{M}^i \otimes_{\mathcal{R}_X} \mathscr{N}^j$ with differentials defined at $\mathscr{M}^i \otimes_{\mathcal{R}_X} \mathscr{N}^j$ by $d(m \otimes n) = dm \otimes n + (-1)^i m \otimes dn$. If it exists, the left-derived functor is denoted

$$\otimes^L_{\mathcal{R}_X} : D^{-,\ell}(\mathcal{R}_X) \times D^{-,r}(\mathcal{R}_X) \to D^{-,\ell r}(\mathcal{R}_X).$$

This yields $\mathcal{T}or$-sheaves $\mathcal{T}or_k^{\mathcal{R}_X}(X^\bullet, Y^\bullet) = H^{-k}(X^\bullet \otimes^L_{\mathcal{R}_X} Y^\bullet)$.

A similar procedure applies to the $\mathcal{H}om_{\mathcal{R}_X}$-bifunctor which is defined by $\mathcal{H}om^n_{\mathcal{R}_X}(\mathscr{M}^\bullet, \mathscr{N}^\bullet) = \prod_{j \in Z} \mathcal{H}om_{\mathcal{R}_X}(\mathscr{M}^j, \mathscr{N}^{n+j})$ with the differentials on $\mathcal{H}om^n_{\mathcal{R}_X}(M^\bullet, N^\bullet)$ given by $d\varphi = d_N \circ \varphi - (-1)^n \varphi \circ d_M$. If the right-derived functor exists it is denoted

$$R\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-) : D^{-,\ell}(\mathcal{R}_X)^{opp} \times D^{+,\ell r}(\mathcal{R}_X) \to D^r(\mathcal{R}_X).$$

This yields the $\mathcal{E}xt$-sheaves $\mathcal{E}xt^n_{\mathcal{R}_X}(M^\bullet, N^\bullet) = R^n\mathcal{H}om^\bullet_{\mathcal{R}_X}(M^\bullet, N^\bullet)$.

1.5 t-structures

A generalisation of positive and negatively supported complexes is given by the concept of a t-structure.

Definition 1.5.1. *A t-structure on a triangulated category $\mathcal{D}$ consists of two strictly full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ such that, setting $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, the following properties hold.*

(i) It holds that $\mathcal{D}^{\leq 0}$ is a subcategory of $\mathcal{D}^{\leq 1}$ and $D^{\geq 1}$ is a subcategory of $\mathcal{D}^{\geq 0}$.

(ii) For any objects X in $\mathcal{D}^{\leq 0}$ and Y of $\mathcal{D}^{\geq 1}$ it holds that $\mathrm{Hom}(X, Y) = 0$.

(iii) For any object X of $\mathcal{D}$ there is a distinguished triangle

$$A \to X \to B \xrightarrow{+1} A$$

with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

Definition 1.5.2. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure.*

In the motivating case of $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ the heart of the t-structure recovers the original abelian category $\mathcal{A}$.

Proposition 1.5.3. *(Dimca, 2004, Proposition 5.1.2.) The heart $\mathcal{C}$ of a t-structure is an abelian category which is stable by extensions.*

Observe that $D^*(\mathcal{A})$ comes equipped with truncation functors $\tau_{\leq m} : D^*(\mathcal{A}) \to D^-(\mathcal{A})$ which sends a complex $X^\bullet$ to

$$\tau_{\leq m} X^\bullet : \cdots \to X^{m-1} \to \ker d \to 0 \to 0 \to \cdots$$

and similarly truncation functors $\tau_{\geq m} : D^*(\mathscr{A}) \to D^+(\mathscr{A})$ are defined by

$$\tau_{\geq m} X^\bullet : \cdots \to 0 \to 0 \to \operatorname{coim} d \to X^{m+1} \to \cdots .$$

This generalises to t-structures.

Proposition 1.5.4. *(Dimca, 2004, Proposition 5.1.4.) Let $\mathcal{D}$ be a triangulated category with a t-structure. Then the inclusion of $\mathcal{D}^{\leq n}$ in $\mathcal{D}$ has a right adjoint functor $\tau_{\leq n}$. Similarly, the inclusion of $\mathcal{D}^{\geq n}$ in $\mathcal{D}$ has a left adjoint $\tau_{\geq n}$.*

Observe that in the example of $D^*(\mathcal{A})$ one has that $\tau_{\geq 0}\tau_{\leq 0} X^\bullet$ is the complex with a single entry $H^0(X^\bullet)$. This generalises to t-structures by viewing ${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0}$ as a functor from $\mathcal{D}$ to it's heart $\mathcal{C}$. Further, let ${}^tH^i := {}^tH^0 \circ T^i$.

Definition 1.5.5. *A t-structure is said to be non-degenerated if $\cap \mathcal{D}^{\leq n} = \cap \mathcal{D}^{\geq n} = \mathrm{Null}$ where Null denotes the family of objects which are isomorphic to the zero object in $\mathcal{D}$.*

Proposition 1.5.6. *(Dimca, 2004, Proposition 5.1.6.) Let $\mathcal{D}$ be a triangulated category with a t-structure. Then ${}^tH^0 : \mathcal{D} \to \mathcal{C}$ is a cohomological functor.*

Proposition 1.5.7. *(Dimca, 2004, Proposition 5.1.7.) Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then the system of functors ${}^tH^i$ is conservative.*

Proposition 1.5.8. *(Dimca, 2004, Proposition 5.1.7.) Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then $X \in \mathcal{D}^{\leq 0}$ if and only if ${}^tH^i(X) = 0$ for $i > 0$. Similarly $X \in \mathcal{D}^{\geq 0}$ if and only if ${}^tH^i(X) = 0$ for $i < 0$.*

Definition 1.5.9. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures. A functor of triangulated categories $F : \mathcal{D}_1 \to \mathcal{D}_2$ is called left or right t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ or $F(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$ respectively. The functor F is called t-exact if it is left and right t-exact.*

Definition 1.5.10. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. The perverse functor ${}^pF$ associated to F is the induced functor on the hearts ${}^pF =^t H^0 \circ F \circ j_1$ where j_1 denotes the inclusion functor $j_1 : \mathcal{C}_1 \to \mathcal{D}_1$.*

Proposition 1.5.11. *(Dimca, 2004, Proposition 5.1.9.) Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a t-exact functor of triangulated categories. Then F sends the heart $\mathcal{C}_1$ into the heart $\mathcal{C}_2$ and the induced functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is naturally isomorphic to ${}^pF$.*

Chapter 2

$\mathscr{D}_X$-modules

The goal of this chapter is to summarise some of the results and definitions which are common knowledge in the field of $\mathscr{D}_X$-modules.

The basic definitions of $\mathscr{D}_X$-module theory are given in section 2.1. The theory builds up to the Riemann-Hilbert correspondence in section 2.2 which states in general terms that a system of differential equations is equivalent to it's solutions. This result is powerful because it yields a connection between algebraic geometry and topology. A particular instantiation of this correspondence is the connection between Bernstein-Sato polynomials and monodromy discussed in section 2.3. Finally, we include the estimation of the roots of Bernstein-Sato polynomials due to Kashiwara and Lichtin.

Detailed treatments of the theory of $\mathscr{D}_X$-modules may be found in Bjork (1979), Kashiwara (2003), Hotta and Tanisaki (2007) or Borel (1987).

2.1 Sheaf of Differential Operators

Let X denote a smooth algebraic variety or a complex manifold.

Definition 2.1.1. *The sheaf of differential operators $\mathscr{D}_X$ is the subsheaf of rings in $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the vector fields Θ_X.*

Observe that $\mathscr{D}_X$ is a sheaf of non-commutative rings. Given local coordinates $x_1, \ldots, x_n$ on X it holds that

$$\partial_i x_j - x_j \partial_i = \delta_{ij}$$

where δ denotes the Kronecker delta.

Lemma 2.1.2. *(Hotta and Tanisaki, 2007, Proposition 1.4.6., Theorem 4.1.2) For any $x \in X$ the stalk $\mathscr{D}_{X,x}$ is left and right Noetherian. Moreover, in the algebraic case $\mathscr{D}_X$ is a left and right Noetherian sheaf of rings.*

Giving a left $\mathscr{D}_X$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X$ and ξ of Θ_X. Similarly, giving a right $\mathscr{D}_X$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X$ and ξ of Θ_X.

Translation between left and right-modules is possible. Denote ω_X for the sheaf of top-level differential forms. Then ω_X comes equipped with the structure of a right $\mathscr{D}_X$-module where vector fields act by the Lie derivative.

For any left $\mathscr{D}_X$-module $\mathscr{M}$ a right $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X} \omega_X$ may be defined by

$$m \otimes \omega \cdot \xi = m \otimes \omega\xi - \xi m \otimes \omega.$$

For any right $\mathscr{D}_X$-module $\mathscr{M}$ a left $\mathscr{D}_X$-module structure on $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathscr{M})$ may be defined by

$$(\xi \cdot \varphi)(\omega) = \varphi(\omega \cdot \xi) - \varphi(\omega) \cdot \xi.$$

The following lemma follows by a direct computation.

Lemma 2.1.3. *The functor $- \otimes_{\mathcal{O}_X} \omega_X$ is a equivalence of categories with pseudoinverse $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, -)$.*

Example 2.1.4. *Consider the differential equation defining the square root function $f(z) = z^{-1/2}$*

$$zf'(z) - 1/2f(z) = 0.$$

The use of f to describe the differential equation is somewhat arbitrary. For instance take $g(z) = \exp(z)f(z)$ which satisfies the differential equation

$$2zg'(z) - (2z+1)g(z) = 0.$$

The statement that non-trivial solutions of the differential equation for $f(z)$ can not be global on $\mathbb{C}^\times$ is equivalent to the same statement for the solutions to the differential equation for $g(z)$.

Thus one is led to the following question. Is it possible to present a differential equation without having to make some arbitrary choice of function f to describe it? This can indeed be accomplished by use of $\mathscr{D}_X$-modules.

In the current example one should consider the analytic left $\mathscr{D}_\mathbb{C}$-module $\mathscr{M}$ which occurs as cokernel of the map

$$P : \mathscr{D}_\mathbb{C} \to \mathscr{D}_\mathbb{C} : Q \mapsto Q(z\partial_x - 1/2).$$

The solutions are then encoded in the sheaf $\mathcal{H}om_{\mathscr{D}_\mathbb{C}}(\mathscr{M}, \mathcal{O}_\mathbb{C})$.

Remark 2.1.5. *More generally than the foregoing example for a system of differential equations*

$$\sum_{j=1}^{k} P_{ij}(x, \partial) f_j = 0; \qquad i = 1, \ldots, m$$

with unknown functions f_j on X and differential operators $P_{ij} \in \mathscr{D}_X(X)$ one can consider the cokernel $\mathscr{M}$ of the map

$$P : \mathscr{D}_X^k \to \mathscr{D}_X^m : (Q_1, \ldots, Q_k) \mapsto \left(\ldots, \sum_{j=1}^{k} Q_j P_{ij}, \ldots \right).$$

The solutions in a left $\mathscr{D}_X$-module $\mathscr{N}$ are encoded by $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$.

Example 2.1.6. *This example is essential in the study of Bernstein-Sato equations. Let $f \in \mathcal{O}_X(X)$ be some global section and introduce a new variable s. The free $\mathbb{C}[x, f^{-1}]$-module $\mathbb{C}[x, f^{-1}]f^s$ is equipped with the structure of left $\mathscr{D}_X$-module by setting*

$$\xi f^s = s\xi(f)f^{-1}f^s$$

for any section ξ of Θ_X. One denotes $\mathscr{D}_X f^s$ for the $\mathscr{D}_X$-submodule generated by f^s.

Filtrations

The non-commutativity of the sheaf of differential operators exits the typical domain of algebraic geometry. This can be resolved by consideration of a graded structure. The essential observation is that differential operators commute up to an element of lower order.

Definition 2.1.7. *The order filtration on $\mathscr{D}_X$ is defined inductively to be given by the sheaves of $\mathcal{O}_X$-submodules $F_i\mathscr{D}_X$ such that $F_0\mathscr{D}_X = \mathcal{O}_X$ and $F_i\mathscr{D}_X$ is maximal with $[F_i\mathscr{D}_X, F_i\mathscr{D}_X] \subseteq F_{i-1}\mathscr{D}_X$.*

The term $F_i\mathscr{D}_X$ in the order filtration can be described as containing all differential operators of order less than or equal to i. Indeed, given local coordinates $x_1, \ldots, x_n$ one can see that $F_i\mathscr{D}_X$ is the $\mathcal{O}_X$-module which is locally generated by $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where α is a multi-index with $|\alpha| \leq i$. The following observations are immediate.

Lemma 2.1.8. *The $F_i\mathscr{D}_X$ are coherent $\mathcal{O}_X$-modules and form a exhaustive filtration. This is to say that $\cup_{i\geq 0} F_i\mathscr{D}_X = \mathscr{D}_X$ and that for any $i, j \geq 0$ it holds that $F_i\mathscr{D}_X \cdot F_j\mathscr{D}_X \subseteq F_{i+j}\mathscr{D}_X$.*

One denotes $\operatorname{gr}\mathscr{D}_X = \oplus_{i\geq 0} F^i\mathscr{D}_X / F_{i-1}\mathscr{D}_X$ for the induced graded sheaf of rings. Observe that $\operatorname{gr}\mathscr{D}_X$ is commutative by definition of the order filtration.

Let $\pi : T^*X \to X$ be the projection map. It is known that $\operatorname{gr}\mathscr{D}_X \cong \pi_*\mathcal{O}_{T^*X}$ (Hotta and Tanisaki, 2007, Section 2.1). The symplectic structure of T^*X captures part of the non-commutativity. Indeed, consider two differential operators P, Q. Pick local coordinates $x_1, \ldots, x_n$ and decompose $P = \sum_\alpha p_\alpha \partial^\alpha$ and $Q = \sum_\beta q_\beta \partial^\beta$. Let m_1, m_2 be the maximal values of $|\alpha|$ and $|\beta|$ with non-zero coefficients. Then the induced elements of P and Q in $\operatorname{gr}\mathscr{D}_X$ are of the form $p = \sum_{|\alpha|=m_1} p_\alpha \xi^\alpha$ and $q = \sum_{|\beta|=m_2} q_\beta \xi^\beta$ where ξ_i is the induced element of ∂_i. On the other hand, the induced element of $PQ - QP$ is $\sum_{i=1}^n \frac{\partial p}{\partial \xi_i}\frac{\partial q}{\partial x_i} - \frac{\partial q}{\partial \xi_i}\frac{\partial p}{\partial x_i}$. This is precisely $\{p, q\}$ where $\{-, -\}$ is the Poisson bracket.

There is a similar notion of filtrations on $\mathscr{D}_X$-modules $\mathscr{M}$. Assume that $\mathscr{M}$ is a left $\mathscr{D}_X$-module, the case for right modules is analogous. A filtration consists of a increasing sequence of quasi-coherent $\mathcal{O}_X$-submodules $F_i\mathscr{M}$ of $\mathscr{M}$ such that $\cup_i F_i\mathscr{M} = \mathscr{M}$ and $F_i\mathscr{D}_X \cdot F_j\mathscr{M} \subseteq F_{i+j}\mathscr{M}$ where i runs over $\mathbb{Z}_{\geq 0}$. The graded objects $\operatorname{gr}\mathscr{M} = \oplus_{i\geq 0} F_i\mathscr{M}/F_{i-1}\mathscr{M}$ are $\operatorname{gr}\mathscr{D}_X$-modules.

The $\operatorname{gr}\mathscr{D}_X$-module $\operatorname{gr}\mathscr{M}$ has a corresponding module on T^*X defined by $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\operatorname{gr}\mathscr{D}_X} \pi^{-1}\operatorname{gr}\mathscr{M}$. By abuse of notation this module is still denoted $\operatorname{gr}\mathscr{M}$ and it will always be implicitly assumed that $\operatorname{gr}\mathscr{D}_X$-modules live on T^*X unless it is explicitly mentioned otherwise.

Proposition 2.1.9. *(Hotta and Tanisaki, 2007, Theorem 2.1.3., Section 4.1) A $\mathscr{D}_X$-module $\mathscr{M}$ is coherent if and only if it locally admits a filtration such that $\operatorname{gr}\mathscr{M}$ is a coherent $\operatorname{gr}\mathscr{D}_X$-module. Such a filtration is called a good filtration. In the algebraic case the filtration can be taken globally.*

Proposition 2.1.10. *(Hotta and Tanisaki, 2007, Theorem 2.2.1.) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then the support of $\operatorname{gr}\mathscr{M}$ in T^*X is a independent of the chosen good filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\operatorname{Ch}\mathscr{M}$.*

Proposition 2.1.11. *(Hotta and Tanisaki, 2007, Theorem 2.3.1, 2) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then $\operatorname{Ch}\mathscr{M}$ is a homogeneous and involutive closed subset of T^*X.*

Remark 2.1.12. *The characteristic variety corresponds to the so-called method of characteristics in the classical study of partial differential equations. This method allows one to use the characteristic variety to determine qualitative properties such as the propagation of shock waves.*

Further relations between characteristic varieties and the properties of differential equations are made precise in the study of microlocal analysis.

Characteristic varieties behave well with respect to quotients and submodules.

Proposition 2.1.13. *Consider a short exact sequence of coherent $\mathscr{D}_X$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

then it holds that

$$\operatorname{Ch} \mathscr{M}_2 = \operatorname{Ch} \mathscr{M}_1 \cup \operatorname{Ch} \mathscr{M}_3.$$

Proof. A good filtration on $\mathscr{M}_2$ induces good filtrations on $\mathscr{M}_1$ and $\mathscr{M}_3$ and one has a short exact sequence

$$0 \to \operatorname{gr} \mathscr{M}_1 \to \operatorname{gr} \mathscr{M}_2 \to \operatorname{gr} \mathscr{M}_3 \to 0$$

whence the result follows.

□

The characteristic variety corresponding to the $\mathscr{D}_X$-module in Example 2.1.6 is understood and may provide some intuition for general characteristic varieties.

Proposition 2.1.14. *(Kashiwara, 1976, Theorem 5.3) The characteristic variety of the coherent $\mathscr{D}_X$-module $\mathscr{D}_X f^s$ is the closure of*

$$W_f = \{(x, sf^{-1}(x)df(x)); \qquad f(x) \neq 0, \quad s \in \mathbb{C}\}$$

in T^*X.

The following result follows from proposition 2.1.14 by establishing that the part of the closure of W_f above $f = 0$ is isotropic.

Proposition 2.1.15. *(Kashiwara, 1976, Proposition 5.6) One can write $\operatorname{Ch} \mathscr{D}_X f^s = \Lambda \cup W$ for some isotropic variety $\Lambda \subseteq T^*X$ and a irreducible $(n+1)$-dimensional variety W which dominates X.*

Direct Image

In this section we describe the direct image of $\mathscr{D}_Y$-modules. Let $\mu : Y \to X$ be a morphism of smooth algebraic varieties or complex manifolds.

A-priori, it is not even clear what $\mathscr{D}_X$-module should correspond to $\mathscr{D}_Y$. This issue may be resolved by use of the transfer $(\mathscr{D}_Y, \mu^{-1}\mathscr{D}_X)$-bimodule $\mathscr{D}_{Y\to X} := \mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\mathscr{D}_X$. Here, the right $\mu^{-1}\mathscr{D}_X$-module structure is just the action on the second component and the left $\mathscr{D}_Y$-module structure is defined by

$$f \cdot (g \otimes \mu^{-1}h) = fg \otimes \mu^{-1}h; \qquad \xi \cdot (g \otimes \mu^{-1}h) = \xi g \otimes \mu^{-1}h + g \otimes T\mu(\xi)\mu^{-1}h$$

for any sections f of $\mathcal{O}_Y$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Example 2.1.16. *If $\mu : \mathbb{C}^2 \to \mathbb{C} : (x, y) \mapsto x$ is the projection map then sections of $\mathscr{D}_{\mathbb{C}^2\to\mathbb{C}}$ may be identified with finite sums of the form $\sum_{j=0}^n f_j(x,y)\partial_x^j$ where $f_j(x,y)$ are sections of $\mathcal{O}_{\mathbb{C}^2}$. The left $\mathscr{D}_{\mathbb{C}^2}$-module structure is such that*

$$\partial_y \cdot f\partial_x^j = \partial_y(f)\partial_x^j; \qquad \partial_x \cdot f\partial_x^j = \partial_x(f)\partial_x^j + f\partial_x^{j+1}.$$

Example 2.1.17. *If $\mu : \mathbb{C} \to \mathbb{C}^2 : y \mapsto (0, y)$ is the inclusion then sections of $\mathscr{D}_{\mathbb{C}\to\mathbb{C}^2}$ may be identified with finite sums of the form $\sum f_{ij}(x,y)\partial_x^i\partial_y^j$ with $f_{ij}(x,y)$ sections of $\mathcal{O}_{\mathbb{C}^2}$ defined in a neighbourhood of the y-axis. The left $\mathscr{D}_{\mathbb{C}}$-module structure is such that*

$$\partial_y \cdot f\partial_x^i\partial_y^j = \partial_y(f)\partial_x^i\partial_y^j + f\partial_x^i\partial_y^{j+1}.$$

Definition 2.1.18. *The direct image functor $\int_\mu : D^{b,r}(\mathscr{D}_Y) \to D^{b,r}(\mathscr{D}_X)$ is given by $R\mu_*(-\otimes^L_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$. For any $\mathscr{D}_Y$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{D}_X$-modules $\int_\mu^j \mathscr{M} = H^j \int_\mu \mathscr{M}$. The subscript μ will be suppressed whenever there is no ambiguity.*

Remark 2.1.19. *A explicit free resolution for the transfer module is known. This involves the Spencer complex $\mathrm{Sp}^\bullet_Y(\mathscr{M})$ of a $\mathscr{D}_Y$-module $\mathscr{M}$ with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k\Theta_Y$. The details may be found in Sabbah (2011).*

Remark 2.1.20. *A direct image functor for left $\mathscr{D}_Y$-modules is induced as*

$$\int \mathscr{M} := R\mathcal{H}om_{\mathcal{O}_X}\left(\omega_X, \int(\mathscr{M} \otimes_{\mathcal{O}_Y} \omega_Y)\right).$$

The definition for the direct image functor is somewhat subtle due to passing through derived categories but many nice properties follow. Most notably, it is immediate from the derived definition that one gets a long exact sequence.

Proposition 2.1.21. *For any short exact sequence of $\mathscr{D}_Y$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

there is a long exact sequence of $\mathscr{D}_X$-modules

$$0 \to \int^0 \mathscr{M}_1 \to \int^0 \mathscr{M}_2 \to \int^0 \mathscr{M}_3 \to \int^1 \mathscr{M}_1 \to \cdots.$$

Proposition 2.1.22. *(Borel, 1987, Chapter VI, Section 5) Let $\mu : Z \to Y$ and $\nu : Y \to X$ be a morphisms of smooth varieties. Then there is a isomorphism of functors $\int_{\nu\circ\mu} \cong \int_\nu \int_\mu$.*

A similar theorem applies to complex manifolds provided μ is proper (Sabbah, 2011, Theorem 3.3.6). Denote $D^{*,*}_{coh}(\mathscr{D}_X)$ for the full subcategory of $D^{*,*}(\mathscr{D}_X)$ consisting of the objects with coherent cohomology. The coherence properties of the direct image in the analytic case require the following notion.

Definition 2.1.23. *A $\mathscr{D}_Y$-module $\mathscr{M}$ is said to be μ-good if there exists a open cover $\{V_j\}_{j\in J}$ of X such that $\mathscr{M}$ admits a good filtration on $\mu^{-1}(V_j)$ for any $j \in J$.*

Note that, by proposition 2.1.9, any coherent $\mathscr{D}_Y$-module on a algebraic variety is μ-good.

Theorem 2.1.24. *(Sabbah, 2011, Theorem 3.4.1.) Let $\mathscr{M}$ be a μ-good $\mathscr{D}_Y$-module and suppose that μ is proper on the support of $\mathscr{M}$. Then, $\int \mathscr{M}$ has coherent cohomology.*

2.2 Riemann-Hilbert Correspondence

This section concerns the Riemann-Hilbert correspondence which states that a system of differential equations is equivalent to it's system of solutions. The systems of differential equations are encoded in regular holonomic $\mathscr{D}_X$-modules. The solutions are given by perverse sheaves.

Holonomic Modules

A particularly nice class of $\mathscr{D}_X$-modules are given by maximally overdetermined systems of differential equations. This is to say that there are many relations for $\mathscr{M}$ or equivalently that $\operatorname{Ch}\mathscr{M}$ is small. Observe that the involutive part of proposition 2.1.11 implies that $\dim \operatorname{Ch}\mathscr{M} \geq n$ for any coherent $\mathscr{D}_X$-module $\mathscr{M}$.

Definition 2.2.1. *A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called holonomic if* $\dim \operatorname{Ch}\mathscr{M} = n$.

The full subcategory of $D^{*,*}(\mathscr{D}_X)$ consisting of complexes with holonomic cohomology is denoted $D_h^{*,*}(\mathscr{D}_X)$. For technical purposes it is mostly important that holonomic modules have finiteness properties.

Proposition 2.2.2. *(Kashiwara, 2003, Proposition 4.42) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then, for any $x \in X$, the stalk $\mathscr{M}_x$ is a $\mathscr{D}_{X,x}$-module of finite length.*

Proposition 2.2.3. *(Bjork, 1979, Chapter 5, Proposition 9.2) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ is $\mathbb{C}$-algebraic. This is to say that for any $\varphi \in \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ there exists some polynomial b with coefficients in $\mathbb{C}$ such that $b(\varphi) = 0$.*

Holonomicity and direct images

The goal of this section is to show that the direct image functor preserves holonomicity. Let $\mu : Y \to X$ be a proper morphism. Given a coherent right $\mathscr{D}_X$-module $\mathscr{M}$ with characteristic variety $\operatorname{Ch}\mathscr{M}$. We must estimate $\operatorname{Ch}\int^j\mathscr{M}$ in terms of $\operatorname{Ch}\mathscr{M}$.

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The idea of the following proof is due to Malgrange (1985) in a K-theoretic context. We follow the exposition of Sabbah (2011). For the sake of notational simplicity we assume that μ is a morphism of varieties, the analytic case is similar and may be found in *loc. cit.*

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y & & T^*X \end{tikzcd}$$

The first step is to note that the behaviour of $\operatorname{gr}\mathscr{D}_Y$-modules is easy to understand. The direct image of a right $\operatorname{gr}\mathscr{D}_Y$-module $\mathcal{M}$ is defined by $\int \mathcal{M} := R\widetilde{\mu}_*(L(T^*\mu)^*\mathcal{M})$. Looking at the supports the following result is immediate.

Lemma 2.2.4. *For any* $\operatorname{gr}\mathscr{D}_Y$*-module* $\mathcal{M}$ *it holds that*

$$\operatorname{supp}\int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}\operatorname{supp}\mathcal{M}\right).$$

Applying this lemma to $\operatorname{gr}\mathscr{M}$ it remains to show that $\operatorname{supp}\operatorname{gr}\int^j\mathscr{M} \subseteq \operatorname{supp}\int^j \operatorname{gr}\mathscr{M}$ for some good filtration on $\int^j\mathscr{M}$. This is proved in proposition 2.2.12. The main technical ingredient in the proof is the Rees module associated to a filtered $\mathscr{D}_Y$-module $\mathscr{M}$.

Definition 2.2.5. *Let* ρ *be a new variable. The Rees sheaf of rings* $\mathcal{R}\mathscr{D}_Y$ *is defined as the subsheaf* $\oplus_{j\geq 0}F_j\mathscr{D}_Y\rho^j$ *of* $\mathscr{D}_Y\otimes_{\mathbb{C}}\mathbb{C}[\rho]$*. Similarly, any filtered* $\mathscr{D}_Y$*-module* $\mathscr{M}$ *gives rise to a* $\mathcal{R}\mathscr{D}_Y$*-module* $\mathcal{R}\mathscr{M} := \oplus_{j\geq 0}F_j\mathscr{M}\rho^j$*.*

Given a coherent $\mathscr{D}_Y$-module $\mathscr{M}$ with a good filtration it follows that $\mathcal{R}\mathscr{M}$ is a coherent $\mathcal{R}\mathscr{D}_Y$-module.The following isomorphisms of filtered modules on Y are essential. They mean that the Rees module can be viewed as a parametrisation of various relevant modules.

$$\frac{\mathcal{R}\mathscr{M}}{(\rho-1)\mathcal{R}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}} \cong \operatorname{gr}\mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{\rho^\ell\mathcal{R}\mathscr{M}} \cong \operatorname{gr}_{[\ell]}\mathscr{M}.$$

Here $\operatorname{gr}_{[\ell]}\mathscr{M} = \oplus_k F_k\mathscr{M}/F_{k-\ell}\mathscr{M}$ is a $\operatorname{gr}_{[\ell]}\mathscr{D}_X$-module. The first formula may be be used to find a corresponding filtered $\mathscr{D}_Y$-module for any graded $\mathcal{R}\mathscr{D}_Y$-module without $\mathbb{C}[r]$-torsion.

The direct image of a $\mathcal{R}\mathscr{D}_Y$-module $\mathcal{M}$ is defined by $\int\mathcal{M} = R^j\mu_*(\mathcal{M}\otimes^L_{\mathcal{R}\mathscr{D}_Y}\mathcal{R}\mathscr{D}_{Y\to X})$. Here the filtration on $\mathscr{D}_{Y\to X}$ is defined by $F_i\mathscr{D}_{Y\to X} = \mathcal{O}_Y\otimes_{\mu^{-1}\mathcal{O}_X}\mu^{-1}F_i\mathscr{D}_X$. Coherence is preserved similarly to theorem 2.1.24.

The direct image may be restricted to the category of graded Rees modules. Note that we now have multiple notions of direct images for modules over $\mathscr{D}_Y$ and $\operatorname{gr}\mathscr{D}_Y$. The Rees module viewpoint agrees with the earlier definitions by the following lemma.

Lemma 2.2.6. *Consider a filtered right* $\mathscr{D}_Y$*-module* $\mathscr{M}$*. Then viewing* $\int^j\mathcal{R}\mathscr{M}/\rho\mathcal{R}\mathscr{M}$ *with it's* $\operatorname{gr}\mathscr{D}_X$*-module structure as a sheaf on* T^*X *recovers the* $\operatorname{gr}\mathscr{D}_Y$*-module direct image* $\int^j\operatorname{gr}\mathscr{M}$*. Viewing* $\int^j\mathcal{R}\mathscr{M}/(\rho-1)\mathscr{M}$ *as a* $\mathscr{D}_X$*-module recovers* $\int^j\mathscr{M}$*.*

Proof. We give the proof for $\int^j\operatorname{gr}\mathscr{M}$, the proof for $\int^j\mathscr{M}$ is similar but easier. Consider the following Cartesian square

$$\begin{tikzcd}
\mu^*T^*X \arrow[r, "T^*\mu"] \arrow[d, "\widetilde{\mu}"] & T^*Y \arrow[r, "\pi_Y"] & Y \arrow[d, "\mu"] \\
T^*X \arrow[rr, "\pi_X"] & & X.
\end{tikzcd}$$

The derived version of the flat base change theorem (Berthelot et al., 2006, Chapter 4, Proposition 3.1.0) yields that

$$L\pi_X^*R\mu_*\left(\frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}}\otimes^L_{\mathcal{R}\mathscr{D}_Y}\mathcal{R}\mathscr{D}_{Y\to X}\right) \cong R\widetilde{\mu}_*L(T^*\mu)^*L(\pi_Y)^*\left(\frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}}\otimes^L_{\mathcal{R}\mathscr{D}_Y}\mathcal{R}\mathscr{D}_{Y\to X}\right).$$

Since π_X is flat it follows that $H^jL\pi_X^*(R\mu_*-) = \pi_X^*(R^j\mu_*-)$.

It now suffices to show that the right hand of the isomorphism is $\int\operatorname{gr}\mathscr{M}$. First observe that, using remark 2.1.19, one can take a bimodule resolution $\mathscr{F}^\bullet$ for $\mathcal{R}\mathscr{D}_{Y\to X}$ by free left $\mathcal{R}\mathscr{D}_Y$-modules such that $\mathscr{F}^\bullet \otimes_{\mathcal{R}\mathscr{D}_Y} (\mathcal{R}\mathscr{D}_{Y\to X}/\rho\mathcal{R}\mathscr{D}_{Y\to X})$ is a bimodule resolution for $\operatorname{gr}\mathscr{D}_{Y\to X}$ by free left $\operatorname{gr}\mathscr{D}_Y$-modules. By use of this free resolution one sees that

$$\frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}} \otimes^L_{\mathcal{R}\mathscr{D}_Y} \mathcal{R}\mathscr{D}_{Y\to X} \cong \frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}} \otimes^L_{\mathcal{R}\mathscr{D}_Y} \frac{\mathcal{R}\mathscr{D}_{Y\to X}}{\rho\mathcal{R}\mathscr{D}_{Y\to X}}.$$

Note that $L\pi_Y^*$ involves taking a tensor product with $\mathcal{O}_{T^*Y}$ so that we get

$$L\pi_Y^*\left(\frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}} \otimes^L_{\mathcal{R}\mathscr{D}_Y} \mathcal{R}\mathscr{D}_{Y\to X}\right) \cong \operatorname{gr}\mathscr{M} \otimes^L_{\mathcal{O}_{T^*Y}} \left(\mathcal{O}_{T^*Y} \otimes_{(\mu\circ\pi_Y)^{-1}\mathcal{O}_X} (\mu\circ\pi_Y)^{-1}(\operatorname{gr}\mathscr{D}_X)\right).$$

Finally observe that $\mathcal{O}_{\mu^*T^*X} \cong \widetilde{\mu}^* \operatorname{gr}\mathscr{D}_X$. This means that the term $\operatorname{gr}\mathscr{M}$ is being tensored with in the above formula is already implicit in $L(T^*\mu)^*$. This is to say that

$$L(T^*\mu)^* L\pi_Y^*\left(\frac{\mathcal{R}\mathscr{M}}{\rho\mathcal{R}\mathscr{M}} \otimes^L_{\mathcal{R}\mathscr{D}_Y} \mathcal{R}\mathscr{D}_{Y\to X}\right) \cong L(T^*\mu)^*(\operatorname{gr}\mathscr{M})$$

which yields the desired result.

□

It turns out that one can directly compare $\operatorname{gr}_{[\ell]}\int^j\mathscr{M}$ and $\int^j \operatorname{gr}_{[\ell]}\mathscr{M}$ when ℓ is large. Some care is required since since $\int^j \mathcal{R}\mathscr{M}$ may have $\mathbb{C}[\rho]$-torsion.

Lemma 2.2.7. *Consider a $\mathscr{D}_Y$-module $\mathscr{M}$ with a good filtration. Then, for sufficiently large ℓ, the kernel of ρ^ℓ in $\int^j \mathcal{R}\mathscr{M}$ stabilises. For such ℓ the quotient $\int^j \mathcal{R}\mathscr{M}/\ker\rho^\ell$ is the $\mathcal{R}\mathscr{D}_X$-coherent module associated to a good filtration on $\int^j\mathscr{M}$.*

Proof. By $\int^j \mathcal{R}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{D}_X$ it follows that $\ker\rho^\ell$ stabilises.

Now consider the short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{\rho-1} \mathcal{R}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{\rho-1} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{\rho-1} \cdots.$$

Since $\int^{j+1}\mathcal{R}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{D}_X$-module it follows that $\rho-1$ is injective whence $\int^j \mathcal{R}\mathscr{M}/(\rho-1)\int^j \mathcal{R}\mathscr{M} \cong \int^j\mathscr{M}$. This yields the desired result using that $\int^j \mathcal{R}\mathscr{M}/\ker\rho^\ell$ is $\mathbb{C}[\rho]$-torsion free and the isomorphism

$$\frac{\int^j \mathcal{R}\mathscr{M}}{(\rho-1)\int^j \mathcal{R}\mathscr{M}} \cong \frac{\int^j \mathcal{R}\mathscr{M}/\ker\rho^\ell}{(\rho-1)(\int^j \mathcal{R}\mathscr{M}/\ker\rho^\ell)}.$$

□

From now on we equip $\int^j\mathscr{M}$ with the good filtration inherited from the Rees module's direct image.

Lemma 2.2.8. *Consider a $\mathscr{D}_Y$-module $\mathscr{M}$ with a good filtration. Then, if ℓ is sufficiently large, $\operatorname{gr}_{[\ell]}\int^j\mathscr{M}$ is a subquotient of $\int^j \operatorname{gr}_{[\ell]}\mathscr{M}$.*

Proof. The short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{\rho^\ell} \mathcal{R}\mathscr{M} \to \mathcal{R}\mathscr{M}/\rho^\ell\mathcal{R}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{\rho^\ell} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathcal{R}\mathscr{M}/\rho^\ell\mathcal{R}\mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{\rho^\ell} \cdots .$$

Hence, $\int^j \mathcal{R}\mathscr{M}/\rho^\ell \int^j \mathcal{R}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}\mathscr{M}/\rho^\ell\mathcal{R}\mathscr{M})$ and it remains to show that $\mathcal{R}\int^j\mathscr{M}/\rho^\ell\mathcal{R}\int^j\mathscr{M}$ is a quotient of $\int^j \mathcal{R}\mathscr{M}/\rho^\ell \int^j \mathcal{R}\mathscr{M}$.

Let ℓ be sufficiently large so that lemma 2.2.7 yields a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker\rho^\ell \cong \mathcal{R}\int^j \mathscr{M}$. The map ρ^ℓ induces a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker\rho^\ell \cong \rho^\ell \int^j \mathcal{R}\mathscr{M}$. Therefore $\rho^\ell \int^j \mathcal{R}\mathscr{M}/\rho^{2\ell} \int^j \mathcal{R}\mathscr{M} \cong \mathcal{R}\int^j\mathscr{M}/\rho^\ell\mathcal{R}\int^j\mathscr{M}$. The desired quotient follows by applying the map $m \mapsto \rho^\ell m$ on $\int^j \mathcal{R}\mathscr{M}/\rho^\ell \int^j \mathcal{R}\mathscr{M}$.

□

The main remaining task is to relate these results to the desired case of $\ell = 1$.

Definition 2.2.9. *For any $\ell \geq 1$ the G-filtration on a $\mathcal{R}\mathscr{D}_Y$-module $\mathcal{M}$ is defined by the decreasing sequence of $\mathrm{gr}_{[\ell]}\mathscr{D}_Y$-submodules $G_j\mathcal{M} := \rho^j\mathcal{M}$.*

Observe that $G_j\,\mathrm{gr}_{[\ell]}\mathscr{M} = \oplus_k F_{k-j}\mathscr{M}/F_{k-\ell}\mathscr{M}$ for any filtered $\mathscr{D}_Y$-module $\mathscr{M}$. Hence, the following lemma is immediate.

Lemma 2.2.10. *For any filtered $\mathscr{D}_Y$-module $\mathscr{M}$ and $\ell \geq 1$ there is an isomorphism*

$$\mathrm{gr}^G_j\,\mathrm{gr}_{[\ell]}\mathscr{M} \cong \mathrm{gr}\,\mathscr{M}$$

for any $j \geq 0$.

Lemma 2.2.11. *Consider a $\mathcal{R}\mathscr{D}_Y$-module $\mathcal{M}$. Then one has a isomorphism $\mathrm{gr}^G \int \mathcal{M} \cong \int \mathrm{gr}^G \mathcal{M}$ in $\mathscr{D}^{b,r}(\mathrm{gr}\,\mathscr{D}_X)$.*

Proof. Observe that the free resolution for $\mathcal{R}\mathscr{D}_{Y\to X}$ following from remark 2.1.19 can be made to respect the G-filtrations. It follows that $\mathrm{gr}^G(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{D}_Y} \mathcal{R}\mathscr{D}_{Y\to X}) \cong \mathrm{gr}^G \mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X} \mathrm{gr}\,\mathscr{D}_X$. Similarly, the terms of the Godement resolution are essentially direct sums of formal products over stalks and hence respect the G-filtration. We conclude that

$$\mathrm{gr}^G R\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{D}_Y} \mathcal{R}\mathscr{D}_{Y\to X}) \cong R\mu_*(\mathrm{gr}^G \mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X} \mathrm{gr}\,\mathscr{D}_X)$$

which is the desired isomorphism.

□

Proposition 2.2.12. *For a filtered $\mathscr{D}_Y$-module $\mathscr{M}$ with a good filtration it holds that*

$$\mathrm{supp}\,\mathrm{gr} \int^j \mathscr{M} \subseteq \mathrm{supp} \int^j \mathrm{gr}\,\mathscr{M}.$$

Proof. Let $\ell \geq 0$ be sufficiently large so that lemma 2.2.8 holds, that is to say that $\mathrm{gr}_{[\ell]}\int^j\mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}_{[\ell]}\mathscr{M}$. By lemma 2.2.10 it holds that $\mathrm{gr}^G\,\mathrm{gr}_{[\ell]}\int^j\mathscr{M} \cong (\mathrm{gr}\int^j\mathscr{M})^\ell$. Morover, by lemma 2.2.8 $\mathrm{gr}_{[\ell]}\int^j\mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}_{[\ell]}\mathscr{M}$. This shows that $\mathrm{supp}\,\mathrm{gr}\int \mathscr{M} \subseteq \mathrm{supp}\,\mathrm{gr}^G \int \mathrm{gr}_{[\ell]}\mathscr{M}$. Finally, observe that $\mathrm{supp}\,\mathrm{gr}^G \int \mathrm{gr}_{[\ell]}\mathscr{M} = \mathrm{supp}\int \mathrm{gr}\,\mathscr{M}$ since, by lemma 2.2.11, $\mathrm{gr}^G \int \mathrm{gr}_{[\ell]}\mathscr{M} \cong \int \mathrm{gr}^G\,\mathrm{gr}_{[\ell]}\mathscr{M} \cong \int(\mathrm{gr}\,\mathscr{M})^\ell$. This concludes the proof.

□

Theorem 2.2.13. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module. Then, for any $j \geq 0$, we have*

$$\operatorname{Ch}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\operatorname{Ch}\mathscr{M})\right).$$

Proof. This is immediate from lemma 2.2.4 and proposition 2.2.12. □

Corollary 2.2.14. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_Y$-module. Then, for any $j \geq 0$ the direct image $\int^j \mathscr{M}$ is a holonomic $\mathscr{D}_X$-module.*

Proof. This follows from theorem 2.2.13 and lemma 2.2.15. □

Lemma 2.2.15. *(Kashiwara, 1976, Proposition 4.9.) Let $\mu : Y \to X$ be a proper morphism and $V \subseteq T^*Y$ an isotropic subvariety. Then $\widetilde{\mu}((T^*\mu)^{-1}(\operatorname{Ch}\mathscr{M}))$ is also isotropic.*

Regular singularities

Let $X = \mathbb{C}$ considered as a complex manifold and consider a ordinary differential operator $P(x, \partial) = \sum_{k=0}^{m} a_k(x)\partial^k$. Suppose that $a_m(x) \neq 0$ for any $x \neq 0$. Then $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$ is locally isomorphic to $\mathcal{O}_X^m$ as a $\mathscr{D}_X$-module near any point $x \neq 0$. In particular the solutions $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ form a locally constant sheaf of rank m outside of 0. The solutions near zero may be more subtle due to monodromy.

Observe that $\operatorname{Ch}\mathscr{M} \subseteq \{(x, \xi) : x\xi = 0\}$. Hence, for any filtration on $\mathscr{M}$ there exists some $N > 0$ such that $(x\xi)^N \operatorname{gr}\mathscr{M} = 0$.

Proposition 2.2.16. *(Kashiwara, 2003, Section 5.1) The following conditions are equivalent.*

1. *There exists a filtration on $\mathscr{M}$ such that $x\xi \operatorname{gr}\mathscr{M} = 0$.*

2. *The equation $P(x, \partial)u = 0$ has m linearly independent solutions of the form $x^\lambda \sum_{j=0}^{s} u_j \log(x)^j$ near 0 for some $s \geq 0$, $\lambda \in \mathbb{C}$ and holomorphic u_j.*

If these two equivalent conditions are satisfied one calls 0 a regular singularity of $\mathscr{M}$. This has the following generalisation to higher dimensions.

Definition 2.2.17. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a complex manifold X with characteristic variety determined by some ideal sheaf $\mathcal{I}$. Then $\mathscr{M}$ is called regular holonomic if it it admits a filtration such that $\mathcal{I}\operatorname{gr}(\mathscr{M}) = 0$.*

Denote $D^{**}_{rh}(\mathscr{D}_X)$ for the full subcategory of $D^{**}(\mathscr{D}_X)$ consisting of complexes with regular holonomic cohomology.

It appears that these definitions should generalise directly to the algebraic situation. However, this has unintended consequences for the Riemann-Hilbert correspondence. For a example, let $X = \mathbb{C}$ as before and consider the regular holonomic $\mathscr{D}_X$-modules $\mathcal{O}_X$ and $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X(\partial - 1)$. These are analytically isomorphic by the map which sends $f(x)$ to $f(x)\exp(x)$. In particular the Riemann-Hilbert correspondence shows that they have isomorphic systems of solutions. However, $\mathcal{O}_X$ and $\mathscr{M}$ are not algebraically isomorphic. This seems to suggest that the equivalence between differential equations and their systems of solutions would not hold in the algebraic case. The problem is that $\mathscr{M}$ is not regular at infinity.

There are a number of equivalent definitions for regularity in the algebraic case. The following definition expresses that the analytic definition may be used provided one adds the points at infinity. This uses the analytification functor which is provided by the GAGA principle and respects holonomicity.

Definition 2.2.18. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a smooth variety X. Denote $\iota : X \to \overline{X}$ for the smooth completion of X. Then $\mathscr{M}$ is called regular if $(\int_\iota \mathscr{M})^{an}$ is regular holonomic on the complex manifold $\overline{X}^{an}$.*

Perverse Sheaves

Classically, the solutions to a differential equation on a vector bundle produces a local system. One can not expect local systems in the case of general $\mathscr{D}_X$-modules since their support could be a proper subvariety.

Definition 2.2.19. *Let X be a complex manifold. A stratification of X consists of a locally finite partition $X = \sqcup_{j\in J} X_j$ into connected locally closed subsets, called strata, such that*

(i) For any $j \in J$ the fronteer $\partial X_j = \overline{X}_j \setminus X_j$ is a union of strata.

(ii) For any $j \in J$ the spaces $\overline{X}_j$ and ∂X_j are closed complex analytic subspaces.

The same definition applies on algebraic varieties by replacing the analytic subspaces by subvarieties.

Definition 2.2.20. *A $\mathbb{C}_X$-module $\mathcal{F}$ is called a constructible sheaf on X if there exists a stratification $X = \sqcup_{\alpha\in A} X_\alpha$ such that $\mathcal{F}|_{X_\alpha}$ is a local system of finite rank on X_α for any $\alpha \in A$.*

Denote $D^b_c(X)$ for the full subcategory of $D^b(\mathbb{C}_X)$ consisting of complexes with constructible cohomology. Such complexes are called constructible.

For a constructible complex $\mathcal{F}^\bullet$ in $D^b_c(X)$ the supports and cosupports are defined dually by

$$\operatorname{supp}^m \mathcal{F}^\bullet = \operatorname{supp} H^m\mathcal{F}^\bullet; \qquad \operatorname{cosupp}^m \mathcal{F}^\bullet = \operatorname{supp}^{-m} \mathbb{D}\mathcal{F}^\bullet$$

where $\mathbb{D}\mathcal{F}^\bullet := R\mathcal{H}om_{\mathbb{C}}(\mathcal{F}^\bullet, \mathbb{C}_X)$. The support $\operatorname{supp}\mathcal{F}^\bullet$ is the closure of the union of the $\operatorname{supp}^m \mathcal{F}^\bullet$.

Theorem 2.2.21. *(Dimca, 2004, Theorem 4.1.5.) Let $\mathcal{F}^\bullet$ be a constructible complex on Y and consider a morphism $\mu : Y \to X$ which is proper on $\operatorname{supp}\mathcal{F}^\bullet$. Then $Rf_*(\mathcal{F}^\bullet)$ is constructible on X.*

Theorem 2.2.22. *(Dimca, 2004, Theorem 4.1.16) Let $\mathcal{F}^\bullet$ be a complex of $D^b(\mathbb{C}_X)$. Then $\mathcal{F}^\bullet$ is constructible if and only if the dual $\mathbb{D}\mathcal{F}^\bullet$ is constructible.*

Let $D^{\leq 0}(X)$ denote the full subcategory of $D^b_c(X)$ consisting of complexes with $\dim \operatorname{supp}^{-m} \mathcal{F}^\bullet < m$ and $\dim \operatorname{supp}^m \mathcal{F}^\bullet = 0$ for all $m \geq 0$. Dually $D^{\geq 0}(X)$ consists of complexes with $\dim \operatorname{cosupp}^{-m} \mathcal{F}^\bullet < m$ and $\dim \operatorname{cosupp}^m \mathcal{F}^\bullet = 0$ for all $m \geq 0$.

Proposition 2.2.23. *(Dimca, 2004, Proposition 5.1.12) The pair $(D^{\leq 0}(X), D^{\geq 0}(X))$ is a non-degenerated t-structure on the triangulated category $D^b_c(X)$.*

Definition 2.2.24. *The heart of the t-structure on $D^b_c(X)$ are called the perverse sheaves* $\operatorname{Perv}(X) = D^{\leq 0}(X) \cap D^{\geq 0}(X)$.

Observe that the objects in $\operatorname{Perv}(X)$ are not sheaves but complexes. The reason for the terminology perverse sheaves is that the functor $U \mapsto \operatorname{Perv}(U)$ has the gluing property of sheaves. More precisely, it is a stack. Perverse sheaves still capture the local systems.

Theorem 2.2.25. *(Dimca, 2004, Theorem 5.1.20) Let X be a complex manifold of dimension n. Then $\mathcal{L}[n]$ is a perverse sheaf on X for any local system $\mathcal{L}$ on X.*

The following are immediate from proposition 1.5.3 and proposition 1.5.8.

Proposition 2.2.26. *A constructible complex $\mathcal{F}^\bullet$ is a perverse sheaf if and only if ${}^pH(\mathcal{F}^\bullet) = 0$ for all $k \neq 0$.*

Proposition 2.2.27. *For any distinguished triangle in $D^b_c(X)$*

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \xrightarrow{+1}$$

it holds that if two terms are perverse sheaves then so is the third.

Riemann-Hilbert Correspondence

Recall from example 2.1.4 and remark 2.1.5 that $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ encodes the solutions of a system of differential equations. More generally, the solutions complex is the functor $\operatorname{Sol}(-) := R\mathcal{H}om_{\mathscr{D}_X}(-, \mathcal{O}_X)[n]$ from $D^{b,\ell}(\mathscr{D}_X)^{opp}$ to $D^b(\mathbb{C}_X)$. This is a contravariant functor. The contravariance may be fixed using the duality functor

$$\mathbb{D} = R\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X) \otimes^L_{\mathcal{O}_X} \omega_X^{-1}[n]$$

from $D^{b,*}(\mathscr{D}_X)^{opp}$ to $D^{b,*}(\mathscr{D}_X)$. The de Rham complex of $\mathscr{M}^\bullet$ is defined by

$$\operatorname{DR}(\mathscr{M}^\bullet) := \Omega^\bullet_X \otimes_{\mathscr{D}_X} \mathscr{M}^\bullet[n].$$

Proposition 2.2.28. *(Dimca, 2004, Theorem 5.3.1.) There is a natural isomorphism* $\operatorname{Sol}(-) \cong \operatorname{DR}(\mathbb{D}-)$.

Proposition 2.2.29. *(Dimca, 2004, Theorem 5.3.1.) For any holonomic complex $\mathscr{M}^\bullet$ in $D^{b,\ell}_h(\mathscr{D}_X)$ the complexes $\operatorname{Sol}(\mathscr{M}^\bullet)$ and $\operatorname{DR}(\mathscr{M}^\bullet)$ are constructible.*

We are finally ready to state the Riemann-Hilbert correspondence on the equivalence between differential equations and their solutions.

Theorem 2.2.30 (Riemann-Hilbert Correspondence)**.** *The de Rham functor $\operatorname{DR} : D^{b,\ell}_{rh}(\mathscr{D}_X) \to D^b_c(X)$ is a t-exact equivalence of categories and commutes with direct images.*

Corollary 2.2.31. *The de Rham functor is a equivalence of categories between the category of regular holonomic $\mathscr{D}_X$-modules and* $\operatorname{Perv}(X)$.

Proof. Follows from the Riemann-Hilbert correspondence and proposition 1.5.11. □

2.3 Interpretation and estimation of Bernstein-Sato polynomials

Philosophically, the Riemann-Hilbert correspondence states that there is a intimate connection between $\mathscr{D}_X$-modules and topology. The goal of this section is to investigate a particular instantiation of this connection, namely the connection between Bernstein-Sato polynomials and monodromy.

Further, we include Kashiwara and Lichtin's proof for the estimation of the roots of the Bernstein-Sato polynomial. This proof is a important framework for the generalisation in chapter 3.

We focus on the local analytic case. The algebraic case will be discussed in detail in the next chapter. Consider $\mathbb{C}^n$ as a complex manifold and take a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with $f(x) = 0$.

Bernstein-Sato polynomials

Definition 2.3.1. *Let s be a new variable. The local Bernstein-Sato polynomial $b_{f,0}(s) \in \mathbb{C}[s]$ is the monic polynomial of minimal degree such that there exists some differential operator $P(x, \partial, s)$ in $\mathscr{D}_{\mathbb{C}^n,0} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with*

$$P(x, \partial, s)f^{s+1} = b_{f,0}(s)f^s$$

in the stalk at x.

The fact that there always exists a Bernstein-Sato polynomial was proved by I.N. Bernstein, I.S. Gelfand and independently by Atiyah.

Remark 2.3.2. *Algorithms to compute the Bernstein-Sato polynomials are known due to Oaku (1997). These algorithms have been implemented in software packages such as SINGULAR. This package was used in the computation of the following examples.*

Example 2.3.3. *The monomial $f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ on $\mathbb{C}^n$ satisfies the Bernstein-Sato relation*

$$\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f^{s+1} = \prod_{i=1}^{n} (\alpha_i s + \alpha_i) \cdots (\alpha_i s + 1) f^s.$$

Example 2.3.4. *The hyperplane arrangement $f(x, y) = x(x+y)(x+2y)$ on $\mathbb{C}^2$ has local Bernstein-Sato polynomial*

$$b_{f,0} = (s + 2/3)(s + 1)^2(s + 4/3).$$

Example 2.3.5. *The cusp singularity $f(x, y) = x^2 - y^2$ on $\mathbb{C}^2$ has local Bernstein-Sato polynomial*

$$b_{f,0}(s) = (s + 5/6)(s + 1)(s + 7/6).$$

Example 2.3.6. *The cardoid $f(x, y) = (x^2 + y^2 + x)^2 - (x^2 + y^2)$ on $\mathbb{C}^2$ has local Bernstein-Sato polynomial*

$$b_{f,0}(s) = (s + 5/6)(s + 1)(s + 7/6).$$

Observe that this is the same local Bernstein-Sato polynomial as for the cusp in example 2.3.5. This is no coincidence, the analytic curve germs corresponding to the cusp and the cardoid at the origin are isomorphic and the Bernstein-Sato polynomial is an invariant of singularities.

Note that in all these examples the roots of the local Bernstein-Sato polynomial are negative rational numbers. This is a general fact due to Kashiwara (1976). The proof of this statement will be discussed in further on.

Let $Z(b_{f,0})$ denote the set of zeros of the Bernstein-Sato polynomial.

Proposition 2.3.7. *Whenever f is non-constant with $f(0) = 0$ it holds that $-1 \in Z(b_{f,0})$.*

Proof. Substitute $s = -1$ in the Bernstein-Sato equation

$$P(x, \partial, s)f^{s+1} = b_{f,0}(s)f^s$$

to get that $p = b_{f,0}(-1)f^{-1}$ for some analytic germ $p \in \mathcal{O}_{\mathbb{C}^n,0}$. In particular, p is a well-defined in 0 whereas f^{-1} has a pole in 0. This means that the equality is only possible if $b_{f,0}(-1) = 0$.

□

The roots of the Bernstein-Sato polynomial provide an invariant of singularities. In particular, these are trivial whenever there are no singularities.

Proposition 2.3.8. *(Igusa, 2007, Section 4.2) If f is non-singular in 0 then $Z(b_{f,0}) = \{-1\}$.*

Remark 2.3.9. *There are a number of connections between Bernstein-Sato polynomials and other invariants of singularities. We will soon discuss how $Z(b_{f,0})$ is connected to the topological invariant of the eigenvalues of monodromy. A invariant called the jumping numbers will be encountered in chapter 3.*

A open problem, called the monodromy conjecture, concerns the relation between the roots of Bernstein-Sato polynomials and the poles of a certain meromorphic function called the Zeta function. Another connection to the world of topology is was given by a conjecture of Yano (1982). This conjecture uses the topological invariant of Puiseux characteristics in the case of a plane curve. The conjecture was proved in full generality by Blanco (2019).

Monodromy

Theorem 2.3.10. *(Milnor (1968)) Let $B \subseteq \mathbb{C}^n$ be a small ball and pick $t \in \mathbb{C}^\times$ close to zero. The diffeomorphism class of $F_{f,0} := f^{-1}(t) \cap B$ is independent of the choice of t. This diffeomorphism class is called the Milnor fiber.*

Going over a loop around the origin in $\mathbb{C}^\times$ induces a well-defined endomorphism M^* on the singular cohomology $H^j(F_{f,0}, \mathbb{C})$ for every $j \in \mathbb{Z}$. This is called the monodromy action and only depends on the local singularity $(f, 0)$. In particular, this means that the eigenvalues of M^* on $H^j(F_{f,0}, \mathbb{C})$ are invariants of the singularity. If $\lambda \in \mathbb{C}$ is a eigenvalue of M^* on some $H^j(F_{f,0}, \mathbb{C})$ it is called a eigenvalue of monodromy. The following theorem is due to Malgrange and Kashiwara.

Theorem 2.3.11. *The set of eigenvalues of monodromy is equal to the set* $\exp(2\pi i\, Z(b_{f,0}))$.

Monodromy is a topological notion whereas the Bernstein-Sato polynomial is defined in terms of $\mathscr{D}_X$-modules. This suggest that the Riemann-Hilbert correspondence is involved. Indeed, the monodromy of the Milnor fiber can be encoded in a constructible complex so that the Riemann-Hilbert correspondence is applicable.

Take a small open ball $B \subseteq \mathbb{C}^n$ such that f is defined on B. Let $\widetilde{\mathbb{C}}^\times$ denote the universal cover of $\mathbb{C}^\times$ and consider the projection $p : B \times \widetilde{\mathbb{C}}^\times \to B$. Denote $\iota : f^{-1}(0) \to B$ for the inclusion map.

Definition 2.3.12. *Deligne's nearby cycle functor from* $D_c^b(B)$ *to* $D_c^b(f^{-1}(0))$ *is given by* $\psi_f := L\iota^* \circ Rp_* \circ Lp^*$.

Denote $\iota_0 : \{0\} \to f^{-1}(0)$ for the inclusion map. The following theorem is due to Deligne.

Theorem 2.3.13. *There is a isomorphism*

$$H^i(F_{f,0}, \mathbb{C}) \cong \mathbb{H}^i(L\iota_0^*(\psi_f \mathbb{C}_B))$$

and the monodromy action on the cohomology of the Milnor fiber corresponds with the action of the covering transformations $\widetilde{\mathbb{C}}^\times \to \widetilde{\mathbb{C}}^\times$ *on the nearby cycles.*

To describe the $\mathscr{D}$-theoretic counterpart of this constructible complex requires the technical notion of V-filtrations. The interested reader may find these concepts in Budur (2015).

Estimation of $Z(b_{f,0})$

The main idea employed in the estimation of $Z(b_{f,0})$ by Kashiwara (1976) is that one can reduce to the monomial case of example 2.3.3 by a resolution of singularities. Hereafter one can use the $\mathscr{D}$-module direct image functor to relate the result on the resolution to the desired result on the original space. Fix a small ball B on which f may be defined.

Definition 2.3.14. *Let* D *be a divisor on* B. *A strong log-resolution of* (B, D) *consists of a projective morphism* $\mu : Y \to B$ *with* Y *smooth such that* μ *is a isomorphism over the complement of* D *and* μ^*D *a simple normal crossings divisor.*

Let D be the divisor determined by f. By Hironaka's resolution of singularities one can find a strong log-resolution $\mu : Y \to B$ for (B, D). Let $g = f \circ \mu$ denote the pullback of f to Y and let $\operatorname{mult}_E(g)$ denotes the order of vanishing of g on some irreducible component E of μ^*D. Kashiwara was able to establish the following estimate by consideration of the direct image of the $\mathscr{D}_Y$-module $\mathscr{D}_Y g^s$.

Theorem 2.3.15. *(Kashiwara, 1976, Corollary 5.2) Every root of* $b_{f,0}(s)$ *is of the form* $-c/\operatorname{mult}_E(g)$ *with* $c \in \mathbb{Z}_{>0}$. *In particular* $Z(b_{f,0}) \subseteq \mathbb{Q}_{<0}$.

Combining this estimate with theorem 2.3.11 one gets the following theorem.

Theorem 2.3.16. *The eigenvalues of monodromy are roots of unity.*

Lichtin (1989) improved the estimate by similar computations for the right $\mathscr{D}_Y$-module $\mathscr{M}$ spanned by $g^s\mu^*(dx)$ inside $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. The advantage of this approach is that $\mu^*(dx)$ involves the local behaviour of μ. Denote k_E for the order of vanishing of the Jacobian $\operatorname{Jac}\mu$ on E, this is also the coefficients of the relative canonical divisor $K_{Y/B}$ on E.

Theorem 2.3.17. *Every root of $b_{f,0}(s)$ is of the form $-(k_E+c)/\operatorname{mult}_E(g)$ with $c \in \mathbb{Z}_{>0}$.*

We now provide the proof for this improved estimate following Lichtin and Kashiwara.

One can ensure that multiplication by s stays inside $\mathscr{D}_B f^s$ with the following trick. Introduce a new coordinate x_{n+1} and set $\widetilde{f} = x_{n+1}f$ on a small ball $\widetilde{B}$ of $\mathbb{C}^{n+1}$. Then $x_{n+1}\partial_{n+1}$ acts like s on $\widetilde{f}^s$. The induced map $\widetilde{Y} \to \widetilde{B}$ is a strong log resolution for the divisor determined by $\widetilde{f}$. Now suppose we can prove theorem 2.3.17 for $\widetilde{f}$. Then, the theorem also follows for f due to the following result.

Lemma 2.3.18. *The Bernstein-Sato polynomial $b_{f,0}(s)$ is a divisor of $b_{\widetilde{f},0}(s)$.*

Proof. Let P be in the stalk $\mathscr{D}_{\widetilde{B},0}$ such that $P\widetilde{f}^{s+1} = b_{\widetilde{f},0}(s)\widetilde{f}^s$. Expand $P = \sum_{j=1}^N P_j\partial_{n+1}^j$ with coefficients P_j in $\mathscr{D}_{B,0}$. Then

$$x_{n+1}^N b_{\widetilde{f},0}(s)\widetilde{f}^s = \left(\sum_{j=1}^k (s+1)^j \sum_\alpha Q_\alpha \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}\right)\widetilde{f}^{s+1}$$

where the P_j were expanded as polynomials in $\partial_1,\dots,\partial_n$ with coefficients Q_α in $\mathcal{O}_{B,0}$.

Observe that $\partial_1,\dots,\partial_n$ act on the formal symbol $\widetilde{f}^{s+1}$ the same as they act on the formal symbol f^{s+1}. Expand the Q_α as power series in $x_1,\dots,x_{n+1}$ and identify powers of x_{n+1} on both sides for the desired functional equation.

□

For notational simplicity we write f instead of $\widetilde{f}$ and B instead of $\widetilde{B}$ from here on. The dimension of B will be denoted by $m = n+1$.

Let t be a new variable. The sheaf of rings $\mathscr{D}_B\langle s,t\rangle$ is found from $\mathscr{D}_B$ by adjoining s and t subject to $ts - st = 1$ where s,t commute with $\mathscr{D}_B$. One can view $\mathscr{D}_B f^s$ as a $\mathscr{D}_B\langle s,t\rangle$-module by the action $tP(s)f^s = P(s+1)f^{s+1}$ for any differential operator P in $\mathscr{D}_B[s]$. In this notation the functional equation for $b_{f,0}$ means that $b_{f,0} \in \operatorname{Ann}_{\mathbb{C}[s]}(\mathscr{D}_B f^s/t\mathscr{D}_B f^s)_x$.

There is a $\mathcal{O}_B$-linear isomorphism between any left $\mathscr{D}_B$-module $\mathscr{N}$ and it's right version $\mathscr{N}\otimes_{\mathcal{O}_B}\omega_B$. Concretely, any section u of $\mathscr{N}$ gives rise to the section $u^* := udx$. Further, for any operator P of $\mathscr{D}_B$ there is a adjoint P^* such that

$$(P\cdot u)^* = u^*\cdot P^*$$

for any section u of $\mathscr{N}$. For a vector field $\xi := \sum_i \xi_i\partial_i$ comparison of the definitions shows that $\xi^* := -\sum_i \partial_i\xi_i$ satisfies this equality and this extends to $\mathscr{D}_B$ by iterating. By this procedure the functional equation $Pf^{s+1} = b(s)f^s$ may equivalently be stated as the equation

$$f^{s+1}dx\cdot P^* = b(s)f^s dx$$

in $\mathscr{D}_{B,0}f^s\otimes_{\mathcal{O}_{B,0}}\omega_{B,0}$. The corresponding module $\mathscr{M}$ on Y will be the submodule of $\mathscr{D}_Y g^s\otimes_{\mathcal{O}_Y}\omega_Y$ spanned by $g^s\mu^*(dx)$. Observe that $\mathscr{M}$ can be equipped with a $\mathscr{D}_Y\langle s,t\rangle$-module structure as before.

Lemma 2.3.19. *The polynomial* $b(s) = \prod_i \prod_{j=1}^{\mathrm{mult}_{E_i}(g)} (\mathrm{mult}_{E_i}(g)s + k_i + j)$ *annihilates* $\mathscr{M}/t\mathscr{M}$ *where* E_i *runs over the irreducible components of* $\mu^* D$.

Proof. This may be checked locally. If the chosen point is on none of divisors E_i of $\mu^* D$ then g is invertible so that $\mathscr{M}/t\mathscr{M}$ is trivial. Now suppose we are working near a point $y \in Y$ which is on E_i if and only if $i \in I$ with I non-empty. Then one can pick local coordinates y_i such that

$$g = \prod_{i \in I} y_i^{\mathrm{mult}_{E_i}(g)}; \qquad \mu^*(dx) = u \prod_{i \in I} y_i^{k_i} dy$$

where u is a local unit. Now set $P = u^{-1}(\prod_{i \in I} \partial_i^{\mathrm{mult}_{E_i}(g)})u$ to get

$$g^{s+1}\mu^*(dx) \cdot P^* = q(s) g^s \mu^*(dx)$$

where $q(s) = \prod_{i \in I} \prod_{j=1}^{\mathrm{mult}_{E_i}(g)} (\mathrm{mult}_{E_i}(g)s + k_i + j)$.

□

Observe that s, t can be viewed as $\mathscr{D}_Y$-linear injective endomorphisms on $\mathscr{M}$. The associated the long exact sequence of direct images yields a $\mathscr{D}_B\langle s, t\rangle$-module structure on the direct image $\int^0 \mathscr{M}$ where the functorial nature of the direct image is used to ensure that $ts - st = 1$. Similarly, the polynomial $b(s)$ provided by lemma 2.3.19 annihilates $\int^0 \mathscr{M}/t \int^0 \mathscr{M}$.

Consider the surjection $\mathscr{D}_Y \to \mathscr{M}$ induced by $1 \mapsto g^s \mu^*(dx)$. The associated long exact sequence includes a morphism $\int^0 \mathscr{D}_Y \to \int^0 \mathscr{M}$. Observe that $\int^0 \mathscr{D}_Y = R^0 \mu_*(\mathscr{D}_{Y \to B})$ contains a global section corresponding to the section 1 of $\mathscr{D}_{Y \to B}$. Let u be the image of this section in $\int^0 \mathscr{M}$ and denote $\mathscr{U}$ for the right $\mathscr{D}_B\langle s, t\rangle$-module generated by u.

Lemma 2.3.20. *There is a surjective morphism of right* $\mathscr{D}_B\langle s, t\rangle$*-modules* $\mathscr{U} \to \mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B$ *sending* u *to* $f^s dx$.

Proof. Pick some open set $V \subseteq B$. To show this yields a well-defined morphism of $\mathscr{D}_B$-modules it must be show that $(f^s dx)P = 0$ whenever $uP = 0$ in $\mathscr{U}(V)$.

The resolution of singularities $Y \to B$ is a isomorphism on the complement of the divisor D determined by f. Hence, $\mathscr{U}$, $\int^0 \mathscr{M}$ and $\mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B$ are isomorphic outside of D. It follows that the support of the coherent sheaf of $\mathcal{O}_V$-modules $\mathcal{O}_V(f^s dx)P$ lies in D. The Nullstellen Satz now yields that $f^N(f^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that f is a non-zero divisor of $(\mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B)(V)$. Therefore, $(f^s dx)P = 0$ on V as desired.

Finally, observe that $tu = fu$ so that this morphism of $\mathscr{D}_B$-modules also commutes with the actions by t and s.

□

Due to lemma 2.3.19 there is a suitable b-polynomial for $\int^0 \mathscr{M}$. By lemma 2.3.20 it remains to compare $\int^0 \mathscr{M}$ and $\mathscr{U}$.

Lemma 2.3.21. *The quotient* $\int^0 \mathscr{M}/\mathscr{U}$ *is a holonomic* $\mathscr{D}_B$*-module.*

Proof. By proposition 2.1.13 the characteristic variety of $\mathscr{M}$ is a subset of the characteristic variety of $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. This has the same characteristic variety as $\mathscr{D}_Y g^s$ using the $\mathcal{O}_Y$-linear isomorphism between $\mathscr{D}_Y g^s$ and $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. By proposition 2.1.15 it follows that $\operatorname{Ch} \mathscr{M} \subseteq W \cup \Lambda$ for some isotropic $\Lambda \subseteq T^*Y$ and a irreducible $(m+1)$-dimensional variety W which dominates Y.

Observe that $\mathscr{M}$ is certainly μ-good since it admits a global good filtration $F_i\mathscr{M} := F_i\mathscr{D}_Y \cdot g^s \mu^*(dx)$. Hence, theorem 2.2.13 is applicable and yields that

$$\operatorname{Ch} \int^0 \mathscr{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\Lambda \cup W)\right).$$

By lemma 2.2.15 the set $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda))$ is still isotropic and will not form any obstruction to $\int^0 \mathscr{M}/\mathscr{U}$ being holonomic. Further, observe that $\widetilde{\mu}((T^*\mu)^{-1}(W))$ remains a irreducible $(m+1)$-dimensional variety which dominates B. On the other hand μ is a isomorphism outside of D so $\int^0 \mathscr{M}/\mathscr{U}$ is only supported on D. Intersecting $\widetilde{\mu}((T^*\mu)^{-1}(W))$ with D yields a m-dimensional variety whence the desired result follows.

□

Proposition 2.3.22. *For sufficiently large N it holds that $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = 0$.*

Proof. The sequence $t^n \int^0 \mathscr{M}/\mathscr{U}$ forms a decreasing sequence of holonomic $\mathscr{D}_B$-modules. By proposition 2.2.2 the induced sequence of $\mathscr{D}_{B,0}$ modules in the stalk at 0 must stabilise. Let N be sufficiently large such that $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0$ attains the stable value.

Applying proposition 2.2.3 to the $\mathscr{D}_{B,0}$-linear endomorphism s produces a non-zero polynomial $q(s) \in \mathbb{C}[s]$ which annihilates $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0$. Let $q(s)$ be of minimal degree with this property. Observe that $q(s+1)t = tq(s)$ so, using the stabilisation, it follows that

$$q(s+1)t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = tq(s)t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = 0.$$

This means that $q(s) - q(s+1)$ also annihilates $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0$. By the minimality of the degree of $q(s)$ it follows that $q(s) - q(s+1) = 0$ which is to say that $q(s)$ is a non-zero constant. This means that $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = 0$ as desired.

□

Putting all these facts together yields the proof of theorem 2.3.17.

Proof. Let N be as in proposition 2.3.22 and denote $b(s)$ for the polynomial provided by lemma 2.3.19. Set $\Pi(s) = b(s+N+1)b(s+N)\cdots b(s)$ and observe that $\Pi(s)\mathscr{M}_0 \subseteq t^{N+1}\mathscr{M}_0 \subseteq t\mathscr{U}_0$. In particular this means that $\Pi(s) \in \operatorname{Ann}_{\mathbb{C}[s]} \mathscr{U}_0/t\mathscr{U}_0$.

The $\mathscr{D}_B\langle s, t\rangle$-linear surjection $\mathscr{U} \to \mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B$ from lemma 2.3.20 now implies that $b_{f,0}(s)$ divides $\Pi(s)$. This yields the desired estimate for $Z(b_{f,0})$.

□

Chapter 3

Estimation of Bernstein-Sato zero-loci

Let $F = (f_1, \dots, f_r) \in \mathbb{C}^n \to \mathbb{C}^r$ be a tuple of polynomials. Introduce new variables $s = (s_1, \dots, s_r)$ and fix a tuple of natural numbers $a = (a_1, \dots, a_r) \in \mathbb{Z}^r_{\geq a}$ such that the product $f_1^{a_1} \dots f_r^{a_r}$ is not invertible. The monovariate Bernstein-Sato polynomial in section 2.3 is generalised by the Bernstein-Sato ideal B_F^a which consists of all polynomials $b(s) \in \mathbb{C}[s]$ such that

$$b(s)F^s \in \mathscr{D}_{\mathbb{C}^n}[s]F^{s+a}$$

where $F^s = f_1^{s_1} \cdots f_r^{s_r}$. The main purpose of this chapter is to generalise the estimate by Lichtin and Kashiwara given in **??** to the zero locus of B_F^a.

?? provides an introduction to Bernstein-Sato zero loci and the available results on their estimation. It turns out that the $\mathscr{D}_X$-modules encoding the Bernstein-Sato relations are no longer holonomic due to the new variables s_i. This is essentially the problem which has been surmounted in Budur et al. (2021) and Budur et al. (2020) in order to prove a topological interpretation of $Z(B_F^a)$. We adapt the technique from these papers to suit our goal. The corresponding notions are introduced in section 3.2. The proof for the generalisation of Lichtin's estimate is given in section 3.3. This establishes an upper bound on the Bernstein-Sato zero locus. A number of lower bounds on the Bernstein-Sato zero locus in terms of jumping walls of mixed multiplier ideals are given in section 3.4.

3.1 Bernstein-Sato ideal

Let X be a smooth affine complex variety and consider a morphism $F : X \to \mathbb{C}^r$. As in Example 2.1.6 let $\mathscr{D}_X[s]F^{s+a}$ be the $\mathscr{D}_X[s]$-submodule of the free $\mathbb{C}[x, f^{-1}, s]$-module $\mathbb{C}[x, f^{-1}, s]F^{s+a}$ obtained by applying formally the operators in $\mathscr{D}_X[s]$ to the symbol F^{s+a} by using the usual derivation rules, where $f = f_1 \dots f_r$.

Definition 3.1.1. *The Bernstein-Sato zero locus associated to F and $a \in \mathbb{Z}^r_{\geq 0}$ is the collection of polynomials $b(s) \in \mathbb{C}[s]$ such that*

$$b(s)F^s \in \mathscr{D}_X[s]F^{s+a}.$$

When $r > 1$ the ring $\mathbb{C}[s]$ is not a principal ideal domain so B_F^a may not be principal. The zero locus of the ideal B_F^a is denoted

$$Z(B_F^a) \subseteq \mathbb{C}^r.$$

This construction extends easily to the case when $F : (X, x) \to (\mathbb{C}^r, 0)$ is the germ of a holomorphic map of complex manifolds. These give rise to the so-called local Bernstein-Sato ideals $B^a_{F,x}$. The global Bernstein-Sato ideal is known to be equal the intersection of all local $B^a_{F,x}$ for x in the zero locus of f Brianccon and Maisonobe (2002).

Example 3.1.2. *Consider the case where $X = \mathbb{C}^2$ with $f_1 = x^{l_1}y^{k_1}$ and $f_2 = x^{l_2}y^{k_2}$. Then a Bernstein-Sato relation for $a = (1, 1)$ may be found by acting with $\partial_x^{l_1+l_2}\partial_y^{k_1+k_2}$ on F^{s+1}. This shows that*

$$\prod_{i=1}^{l_1+l_2}(l_1s_1 + l_2s_2 + i)\prod_{j=1}^{k_1+k_2}(k_1s_1 + k_2s_2 + k) \in B^1_F.$$

In this case it is straightforward to see that B^1_F is actually the principal ideal generated by this element. Correspondingly, the Bernstein-Sato zero locus $Z(B^1_F)$ consists of a number of lines. Some lines coming from the x-part and others coming from the y-part.

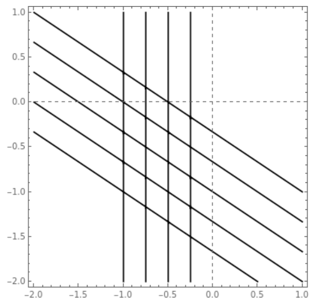


Figure 3.1: An example of the Bernstein-Sato zero-locus $Z(B^1_F)$ in the monomial case when $F = (x^2y^4, x^3)$. The dotted lines correspond the the axes.

In the case where $a = (1, \ldots, 1)$ the Bernstein-Sato ideal B^a_F can be computed using SINGULAR. This software package has been used to compute the examples in fig. 3.3 and fig. 3.2 which were visualised using Mathematica. A-priori the Bernstein-Sato zero locus could have been an arbitrary affine variety, however in these examples it always seems to consist of a number of hyperplanes. This may be explained by the following theorem.

Theorem 3.1.3. *((Budur et al., 2020, Theorem 1.1.1)) Let $F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ be a morphism of smooth complex affine irreducible algebraic varieties, or the germ at $x \in X$ of a holomorphic map on a complex manifold. Let $a \in \mathbb{Z}^r_{\geq 0}$ such that $\prod_{j=1}^r f_j^{a_j}$ is not invertible. Then:*

1. *Every irreducible component of $Z(B^a_F)$ of codimension 1 is a hyperplane of type $l_1s_1 + \ldots + l_rs_r + b = 0$ with $l_j \in \mathbb{Q}_{\geq 0}$, $b \in \mathbb{Q}_{>0}$, and for each such hyperplane there exists j with $a_j \neq 0$ such that $l_j > 0$.*

2. *Every irreducible component of $Z(B^a_F)$ of codimension > 1 can be translated by an element of $\mathbb{Z}^r$ inside a component of codimension 1.*

For $r = 1$ this is equivalent to the classical result that the roots of the Bernstein-Sato polynomial b_f are negative rational numbers, due to Kashiwara (1976). The first part without the strict positivity of l_j is due to Sabbah (1987) and Gyoja (1993). The second

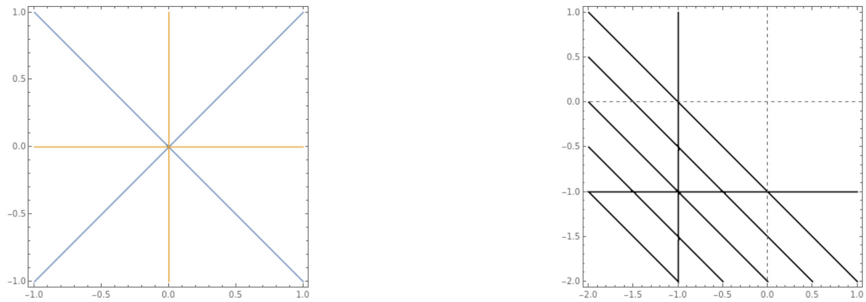


Figure 3.2: On the left: the zero loci determined by $f_1(x,y) = xy$ and $f_2(x,y) = (x+y)(x-y)$ respectively in orange and blue. On the right: the Bernstein-Sato zero-locus $Z(B_F^1)$.

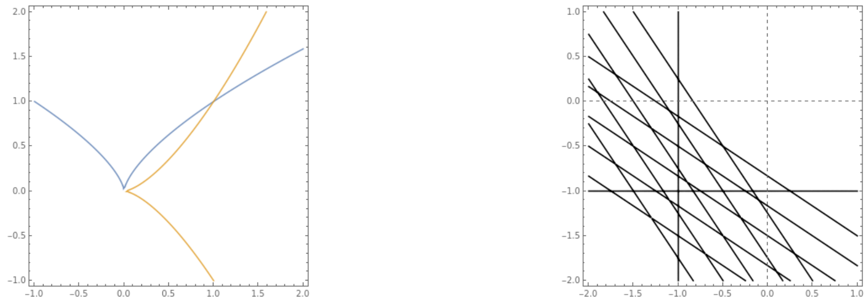


Figure 3.3: On the left: the zero loci determined by $f_1(x,y) = y^2-x^3$ and $f_2(x,y) = x^2-y^3$ respectively in orange and blue. On the right: the Bernstein-Sato zero-locus $Z(B_F^1)$.

part for the case $a = (1,\dots,1)$ is due to Maisonobe (2016), a completely different proof of which was given recently by van der Veer (2021).

The first goal of this chapter is to to further refine part (1) of the above theorem by generalising Lichtin's estimate from section 2.3. This means we provide a lower bound on the constant c in terms of numerical data from a resolution of singularities.

Let $\mu : Y \to X$ be a strong log resolution of f. The numerical data is given by the orders of vanishing $\mathrm{ord}_E(f_j) \in \mathbb{Z}_{\geq 0}$ of f_j along irreducible components E of μ^*D, and the orders of vanishing $k_E = \mathrm{ord}_E(\det \mathrm{Jac}(\mu)) \in \mathbb{Z}_{\geq 0}$ of the determinant of the Jacobian of μ, also equal to the coefficients of the relative canonical divisor K_μ of μ. We show:

Theorem 3.1.4. *Every irreducible component of $Z(B_F^a)$ of codimension 1 is a hyperplane of the form*

$$\mathrm{ord}_E(f_1)s_1 + \cdots + \mathrm{ord}_E(f_r)s_r + k_E + c = 0$$

with $c \in \mathbb{Z}_{>0}$.

Without the term k_E the statement was proven for $r \geq 1$ by (Budur et al., 2020, Lemma 4.4.6) generalising Kashiwara's estimate. The case $r = 1$ of Theorem 3.1.4 is due to Lichtin Lichtin (1989), a new proof of which was given by Dirks-Mustaţă **?**.

The second part of this chapter contains a number of lower bounds for the Bernstein-Sato zero locus. Firstly, we provide a multivariate generalisation for the fact that the Bernstein-Sato polynomial for $r = 1 = a$ always has the trivial root -1.

Proposition 3.1.5. *Let E be an irreducible component of D. Then $\sum_{j=1}^r \mathrm{ord}_E(f_j)s_j + c = 0$ determines an irreducible component of $Z(B_F^a)$ for $c = 1,\dots,\sum_{j=1}^r \mathrm{ord}_E(f_j)a_j$.*

Further, we generalise the fact that the jumping numbers of f in $[0, \mathrm{lct}(f)+1)$ are roots of $b_f(s)$ in the case $r = 1$ (**?**, Theorem 2). For any $\lambda \in \mathbb{R}^r_{\geq 0}$ the mixed multiplier ideal sheaf of F^λ is given by

$$\mathcal{J}(F^\lambda) = \mu_* \mathcal{O}_Y(K_\mu - \lfloor \sum_{j=1}^r \lambda_j \mu^* D_j \rfloor)$$

where D_i denotes the divisor determined by f_i and $\lfloor - \rfloor$ is the round-down of an $\mathbb{R}$-divisor. Associated to λ is the region

$$\mathcal{R}_F(\lambda) := \{\lambda' \in \mathbb{R}^r_{\geq 0} : \mathcal{J}(F^\lambda) \subseteq \mathcal{J}(F^{\lambda'})\}.$$

The jumping numbers from the case $r = 1$ are generalised by the jumping wall associated to λ which is the intersection of the boundary of $\mathcal{R}_F(\lambda)$ with $\mathbb{R}^r_{>0}$. By the definition of mixed multiplier ideals the facets of the jumping wall are cut out by hyperplanes of the form $\sum_{j=1}^r \mathrm{ord}_E(f_j) s_j = k_E + c$ with $c \in \mathbb{Z}_{>0}$ and E an irreducible component of $\mu^* D$.

Example 3.1.6. *Let $f_i(x, y) = \prod_{j=1}^{n_1} \ell_{1j}$ and $f_2(x, y) = \prod_{j=1}^{n_2} \ell_{2j}$ for linear polynomials ℓ_{ij} which go through the origin and all have different zero loci. Consider a blow-up at the origin. For any irreducible polynomial $p \in \mathbb{C}[x, y]$ denote E_p for the divisor determined by p and denote E_e for the exceptional divisor of the blowup. Then the global sections of $\mathcal{J}(F^s)$ are the polynomials $h \in \mathbb{C}[x, y]$ such that for all i, j*

$$\mathrm{ord}_{E_{\ell_{ij}}} h \geq \lfloor s_i \rfloor; \qquad \mathrm{ord}_{E_e} h \geq \lfloor n_1 s_1 + n_2 s_2 - 1 \rfloor.$$

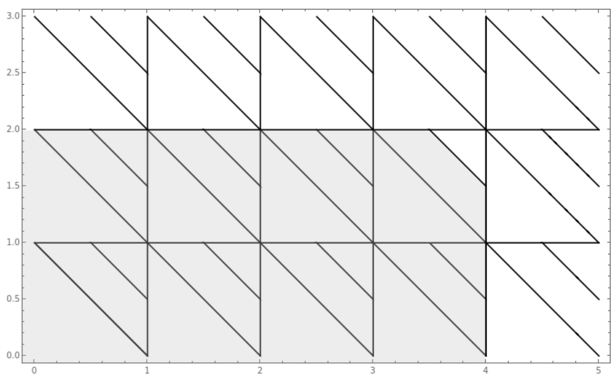


Figure 3.4: The jumping walls and a region $\mathcal{R}_F(\lambda)$ for $f_1(x, y) = xy$ and $f_2(x, y) = (x + y)(x - y)$.

The log-canonical threshold is generalised by the LCT-polytope

$$\mathrm{LCT}(F) := \bigcap_E \{\lambda \in \mathbb{R}^r_{\geq 0} : \sum_{j=1}^r \mathrm{ord}_E(f_j)\lambda_j \leq k_E + 1\}$$

where E runs over the irreducible components of $\mu^* D$. The suitable generalisation for the region $[0, \mathrm{lct}(f) + 1)$ is the KLT_a-region

$$\mathrm{KLT}_a(F) := \bigcap_E \{\lambda \in \mathbb{R}^r_{\geq 0} : \sum_{j=1}^r \mathrm{ord}_E(f_j)(\lambda_j - a_j) < k_E + 1\}.$$

A rephrasing in terms of being log-canonical or Kawamata log-terminal is provided in section 3.4. This rephrasing also shows that the regions are independent of the chosen resolution.

Theorem 3.1.7. *Suppose that $\sum_{j=1}^{r} \mathrm{ord}_E(f_j)s_j = k_E + c$ determines a facet of a jumping wall which intersects $\mathrm{KLT}_a(F)$. Then $\sum_{j=1}^{r} \mathrm{ord}_E(f_j)s_j + k_E + c = 0$ determines a component of $Z(B_F^a)$.*

This theorem was shown by **?** for $Z(B_F^1)$ when $f_1, \ldots, f_r$ are germs of plane curves. We employ the same method, which is essentially the one used in (Ein et al., 2004, Theorem B) based on an idea by Kollár (1997).

From Theorem 3.1.7 we deduce a generalisation for the fact that the largest root of the Bernstein-Sato polynomial $b_f(s)$ is equal to the log-canonical threshold when $r = 1$.

Corollary 3.1.8. *Suppose that $\sum_{j=1}^{r} \mathrm{ord}_E(f_j)s_j = k_E + 1$ determines a face of $\mathrm{LCT}(F)$. If $a_j \neq 0$ and $\mathrm{ord}_E(f_j) \neq 0$ for some j, then $\sum_j \mathrm{ord}_E(f_j)s_j + k_E + 1 = 0$ determines a component of $Z(B_F^a)$.*

Note that Theorem 3.1.4 implies the analogue of the maximality statement: the components of codimension one of $Z(B_F^a)$ originating from the LCT-polytope are closest to the origin with that slope.

Saito (2007) also introduced a real version of jumping numbers and proved that these produce roots of the Bernstein-Sato polynomial. Interesting about these real jumping numbers is that they do not have to agree with the complex jumping numbers. These results are of further interest due to applications in algebraic statistics.

Mixed multiplier ideals and their jumping walls will be defined on real algebraic manifolds in section 3.4. There are also the associated notions of a RKLT_a-region, RLCT-polytope and real Bernstein-Sato ideal $B_{\mathbb{R},F}^a$. We prove the following generalisation of Saito's result on the real jumping walls.

Theorem 3.1.9. *Suppose that $\sum_{j=1}^{r} \mathrm{ord}_E(g_j)s_j = k_E + c$ determines a facet of a real jumping wall which intersects $\mathrm{RKLT}_a(F)$. Then $\sum_{j=1}^{r} \mathrm{ord}_E(g_j)s_j + k_E + c = 0$ determines a component of $Z(B_{\mathbb{R},F}^a)$.*

3.2 $\mathscr{D}_X[s]$-modules

This section provides preliminaries on the theory of $\mathscr{D}_X[s]$-modules, such as direct images and homological properties.

Relative holonomic $\mathscr{D}$-modules

Let X be a smooth complex variety and let R be a regular commutative finitely generated $\mathbb{C}$-algebra integral domain. The sheaf of relative differential operators on X is defined by

$$\mathscr{D}_X^R := \mathscr{D}_X \otimes_{\mathbb{C}} R.$$

The order filtration $F_j\mathscr{D}_X$ on $\mathscr{D}_X$ extends to a filtration $F_j\mathscr{D}_X^R := F_j\mathscr{D}_X \otimes_{\mathbb{C}} R$ on $\mathscr{D}_X^R$. The graded objects for this filtration are denoted by gr^{rel}. Denote $\pi_{T^*X} : T^*X \to X$ for the projection map. Recall from section 2.1 that $\mathrm{gr}\,\mathscr{D}_X \cong \pi_*\mathcal{O}_{T^*X}$, it follows that

$\mathrm{gr}^{rel}\mathscr{D}_X^R \cong (\pi_{T^*X} \times \pi_{\operatorname{Spec} R})_* \mathcal{O}_{T^*X \times \operatorname{Spec} R}$ for $\pi_{\operatorname{Spec} R} : \operatorname{Spec} R \to \{\mathrm{pt}\}$ the projection onto a point.

Since $\mathscr{D}_X^R$ is a sheaf of non-commutative rings, one should distinguish between left and right $\mathscr{D}_X^R$-modules. We may also refer to a $\mathscr{D}_X^R$-module without specifying left or right if no confusion is possible. In these cases it is intended that the result holds in either case.

For any filtered $\mathscr{D}_X^R$-module $\mathscr{M}$ there is an associated sheaf of modules on $T^*X \times \operatorname{Spec} R$ given by $(\pi \times \mathrm{Id}_{\operatorname{Spec} R})^{-1}(\mathrm{gr}^{rel}\mathscr{M}) \otimes_{\pi^{-1}\mathrm{gr}^{rel}\mathscr{D}_X^R} \mathcal{O}_{T^*X \times \operatorname{Spec} R}$. From here on out we write $\mathrm{gr}^{rel}\mathscr{D}_X^R$ and $\mathrm{gr}^{rel}\mathscr{M}$ for the corresponding sheaves on $T^*X \times \operatorname{Spec} R$.

A filtration compatible with $F_\bullet\mathscr{D}_X^R$ on a $\mathscr{D}_X^R$-module $\mathscr{M}$ is said to be good if $\mathrm{gr}^{rel}\mathscr{M}$ is a coherent $\mathrm{gr}^{rel}\mathscr{D}_X^R$-module. A quasi-coherent $\mathscr{D}_X^R$-module $\mathscr{M}$ locally admits a good filtration if and only if it is coherent (Hotta and Tanisaki, 2007, Corollary D.1.2), in fact one can take this filtration to be global (Hotta and Tanisaki, 2007, Proof of Theorem 2.1.3). For a coherent $\mathscr{D}_X^R$-module $\mathscr{M}$ the support $\mathrm{Ch}^{rel}\mathscr{M}$ of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration (Hotta and Tanisaki, 2007, Lemma D.3.1.) and is called the relative characteristic variety. Equivalently, the relative characteristic variety is locally determined by the radical of the annihilator ideal of $\mathrm{gr}^{rel}\mathscr{M}$ in $\mathrm{gr}^{rel}\mathscr{D}_X^R$.

Proposition 3.2.1. *(Budur et al., 2021, Lemma 3.2.2) For any short exact sequence of coherent $\mathscr{D}_X^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

it holds that $\mathrm{Ch}^{rel}\mathscr{M}_2 = \mathrm{Ch}^{rel}\mathscr{M}_1 \cup \mathrm{Ch}^{rel}\mathscr{M}_3$.

Definition 3.2.2. *A coherent $\mathscr{D}_X^R$-module $\mathscr{M}$ is said to be relative holonomic if its characteristic variety is a finite union $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ where $\Lambda_w \subseteq T^*X$ are irreducible conic Lagrangian subvarieties and $S_w \subseteq \operatorname{Spec} R$ are irreducible subvarieties.*

Proposition 3.2.3. *(Budur et al., 2021, Lemma 3.2.4) Any subquotient of a relative holonomic module is relative holonomic.*

The functor which associates to a left $\mathscr{D}_X^R$-module $\mathscr{M}$ the right $\mathscr{D}_X^R$-module $\mathscr{M} \otimes_{\mathcal{O}_X} \omega_X$ is an equivalence of categories. The pseudoinverse associates $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathscr{M})$ to a given right-module $\mathscr{M}$.

Pick local coordinates $x_1, \ldots, x_n$ on X, these are regular functions such that is to say that $dx_1, \ldots, dx_n$ are a local basis for Ω_X^1. There is an induced section $dx := dx_1 \wedge \ldots \wedge dx_n$ for ω_X.

For any left $\mathscr{D}_X^R$-module $\mathscr{M}$ one now gets a locally defined $\mathcal{O}_X \otimes R$-linear isomorphism $\mathscr{M} \to \mathscr{M} \otimes_{\mathcal{O}_X} \omega_X$ associating to any section m the section $m^* = mdx$. This can be made to commute with the $\mathscr{D}_X^R$-module structure. This is to say that for any operator P of $\mathscr{D}_X^R$ there is an adjoint P^* such that

$$(P \cdot m)^* = m^* \cdot P^*.$$

Indeed, for a vector field $\xi = \sum \xi_i \partial_i$ this is satisfied by setting $\xi^* = -\sum \partial_i \xi_i$. Iteration then extends to differential operators of arbitrary order.

Direct image

Let $\mu : Y \to X$ be a morphism of varieties. The direct image functor $\int \mathscr{M} := R\mu_*(\mathscr{M} \otimes^L_{\mathscr{D}_Y} \mathscr{D}_{Y \to X})$ on right $\mathscr{D}_Y$-modules induces an $\mathscr{D}_Y^R$-module direct image functor. Indeed, consider a right $\mathscr{D}_Y^R$-module $\mathscr{M}$ and observe that multiplication by $r \in R$ is $\mathscr{D}_Y$-linear. By the functoriality of the $\mathscr{D}_Y$-module direct image it follows that there is an associated endomorphism on $\int \mathscr{M}$. This equips the direct image with a canonical structure as complex of $\mathscr{D}_X^R$-modules. For any $j \in \mathbb{Z}$ we still call $\int^j \mathscr{M} := H^j \int \mathscr{M}$ the j-th direct image.

Whenever μ is proper and $\mathscr{M}$ is coherent as $\mathscr{D}_Y^R$-module it holds that $\int^j \mathscr{M}$ is coherent over $\mathscr{D}_X^R$ for any j. The proof for this statement is identical to the absolute case (Hotta and Tanisaki, 2007, Theorem 2.5.1). The following proposition may be established identically to the absolute case which was established in corollary 2.2.14.

Proposition 3.2.4. *Suppose that μ is proper and let $\mathscr{M}$ be a relative holonomic right $\mathscr{D}_Y^R$-module. Then $\int^j \mathscr{M}$ is relative holonomic for any $j \in \mathbb{Z}$.*

Homological notions

In this section it is assumed that X is affine. We denote $n = \dim X$ and $r = \dim R$ for the Krull dimension of the ring R. For some results in this section the distinction between left and right modules is relevant. Such results have been stated in terms of right $\mathscr{D}_X^R$-modules, which is the case we will need. It should be clear that these results have obvious analogues for left $\mathscr{D}_X^R$-modules.

Grades

Definition 3.2.5. *Let $\mathscr{M}$ be a non-zero coherent $\mathscr{D}_X^R$-module. The smallest integer $j \geq 0$ such that $\mathcal{E}xt^j_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R) \neq 0$ is called the grade of $\mathscr{M}$ and is denoted $j(\mathscr{M})$. If $\mathscr{M} = 0$ then $j(\mathscr{M})$ is said to be infinite.*

Remark 3.2.6. *Observe that $\mathcal{E}xt$ localises. This is to say that for any localisation R' of R*

$$\mathcal{E}xt^j_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R) \otimes_R R' \cong \mathcal{E}xt^j_{\mathscr{D}_X^{R'}}(\mathscr{M} \otimes_R R', \mathscr{D}_X^{R'}).$$

Hence, $j(\mathscr{M}) \leq j(\mathscr{M} \otimes_R R')$.

Definition 3.2.7. *The Bernstein-Sato-ideal of a $\mathscr{D}_X^R$-module $\mathscr{M}$ is given by $B_{\mathscr{M}} := \operatorname{Ann}_R \mathscr{M}$.*

Proposition 3.2.8 ((Budur et al., 2021, Lemma 3.4.1))**.** *Let $\mathscr{M}$ be a relative holonomic $\mathscr{D}_X^R$-module then*

$$\dim \operatorname{Ch}^{rel} \mathscr{M} + j(\mathscr{M}) = 2n + r.$$

Remark 3.2.9. *Further (Budur et al., 2021, Lemma 3.2.2) states that $Z(B_{\mathscr{M}})$, the reduced closed subscheme defined by the radical ideal of $B_{\mathscr{M}}$ in $\operatorname{Spec} R$, is the projection of $\operatorname{Ch}^{rel}\mathscr{M}$ on $\operatorname{Spec} R$. Hence, $j(\mathscr{M}) = n + k$ for a relative holonomic module $\mathscr{M}$ if and only if $Z(B_{\mathscr{M}})$ has codimension k in $\operatorname{Spec} R$.*

Definition 3.2.10. *A non-zero coherent $\mathscr{D}_X^R$-module $\mathscr{M}$ is said to be j-pure if $j(\mathscr{N}) = j(\mathscr{M}) = j$ for every non-zero submodule $\mathscr{N}$.*

Proposition 3.2.11 ((Budur et al., 2021, Lemma 4.3.5.))**.** *Let $\mathscr{M}$ be a non-zero coherent right $\mathscr{D}_X^R$-module of grade j. Then:*

1. $\mathcal{E}xt^j_{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^R_X)$ *is a* j*-pure left* $\mathscr{D}^R_X$*-module and* $\mathcal{E}xt^k_{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^R_X)$ *has grade* $\geq k$ *for any* $k \neq j$*;*

2. $\mathscr{M}$ *is* j*-pure if and only if* $\mathcal{E}xt^k_{\mathscr{D}^R_X}(\mathcal{E}xt^k_{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^R_X), \mathscr{D}^R_X) = 0$ *for every* $k \neq j$*.*

Proposition 3.2.12 ((Budur et al., 2021, Lemma 3.4.2))**.** *Let* $\mathscr{M}$ *be a* j*-pure relative holonomic* $\mathscr{D}^R_X$*-module and suppose that* $b \in R$ *is not contained in any minimal prime ideal of* R *containing* $B_{\mathscr{M}}$*. Then there exists a good filtration on* $\mathscr{M}$ *such that multiplication by* b *induces injective endomorphisms on* $\mathscr{M}$ *and* $\mathrm{gr}^{rel}\mathscr{M}$*.*

Proposition 3.2.13 ((Budur et al., 2021, Lemma 3.2.4.))**.** *Let* $\mathscr{M}$ *be a relative holonomic right* $\mathscr{D}^R_X$*-module. Then, for any* $k \geq 0$*,* $\mathcal{E}xt^k_{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^R_X)$ *is a relative holonomic left* $\mathscr{D}^R_X$*-module.*

Proposition 3.2.14. *Let* $\mathscr{M}$ *be a relative holonomic right* $\mathscr{D}^R_X$*-module and let* $P \subseteq R$ *be a prime ideal with* $Z(P)$ *non-singular. Then, for any* $k \geq 0$*,* $\mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X)$ *is a relative holonomic right* $\mathscr{D}^{R/P}_X$*-module.*

Proof. The assumption on P is equivalent to assuming that R/P is a ring of the same type as R, namely a commutative regular finitely generated $\mathbb{C}$-algebra integral domain. Let $\mathcal{F}^\bullet$ be a resolution of $\mathscr{D}^{R/P}_X$ by free $\mathscr{D}^R_X$-bimodules of finite rank. Then $\mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X)$ is found from the cohomology of the complex $\mathscr{M} \otimes_{\mathscr{D}^R_X} \mathcal{F}^\bullet$. The entries of this complex remain relative holonomic so Proposition 3.2.3 shows that $\mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X)$ is a relative holonomic right $\mathscr{D}^R_X$-module.

This means that $\mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X)$ admits a good filtration such that

$$\operatorname{supp} \mathrm{gr}^{rel}_{\mathscr{D}^R_X} \mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X) = \cup_w \Lambda_w \times S_w$$

for Lagrangian subvarieties $\Lambda_w \subseteq T^*X \times \operatorname{Spec} R$ and algebraic varieties $S_w \subseteq \operatorname{Spec} R$. This filtration descends to a good filtration over $\mathscr{D}^{R/P}_X$ and it holds that

$$\operatorname{supp} \mathrm{gr}^{rel}_{\mathscr{D}^{R/P}_X} \mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X) = (\mathrm{Id}_{T^*X} \times \Delta)^{-1}(\operatorname{supp} \mathrm{gr}^{rel}_{\mathscr{D}^R_X} \mathcal{T}or_k^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X))$$

where $\Delta : \operatorname{Spec} R/P \to \operatorname{Spec} R$ is the closed embedding. This proves the desired result. $\square$

Proposition 3.2.15. *(Weibel, 1995, Corollary 5.8.4) For any coherent right* $\mathscr{D}^R_X$*-module* $\mathscr{M}$ *and prime ideal* $P \subseteq R$ *with* $Z(P) \subseteq \operatorname{Spec} R$ *non-singular there is a spectral sequence*

$$E_2^{pq} = \mathcal{E}xt^p_{\mathscr{D}^{R/P}_X}(\mathcal{T}or_{-q}^{\mathscr{D}^R_X}(\mathscr{M}, \mathscr{D}^{R/P}_X), \mathscr{D}^{R/P}_X)$$

converging to $H^{p+q}(R\mathcal{H}om_{\mathscr{D}^{R/P}_X}(\mathscr{M} \otimes^L_{\mathscr{D}^R_X} \mathscr{D}^{R/P}_X, \mathscr{D}^{R/P}_X))$*.*

Proof. This is simply the Grothendieck spectral sequence for this composition of derived functors. It is applicable since $- \otimes_{\mathscr{D}^R_X} \mathscr{D}^{R/P}_X$ transforms projective right $\mathscr{D}^R_X$-modules into projective right $\mathscr{D}^{R/P}_X$-modules and these are acyclic for $\mathcal{H}om_{\mathscr{D}^{R/P}_X}(-, \mathscr{D}^{R/P}_X)$. $\square$

Proposition 3.2.16. *For any coherent right $\mathscr{D}_X^R$-module $\mathscr{M}$ and right $\mathscr{D}_X^R$-module $\mathscr{N}$ there is a spectral sequence*

$$E_2^{pq} = \mathcal{T}or_{-p}^{\mathscr{D}_X^R}(\mathscr{N}, \mathcal{E}xt^q_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R))$$

converging to $H^{p+q}(\mathscr{N} \otimes^L_{\mathscr{D}_X^R} R\mathcal{H}om_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R))$.

Proof. The conditions of the Grothendieck spectral sequence (Weibel, 1995, Corollary 5.8.4) are not satisfied in this case. For instance, $\mathcal{H}om_{\mathscr{D}_X^R}(-, \mathscr{D}_X^R)$ does not necessarily transform projective objects into $(\mathscr{N} \otimes_{\mathscr{D}_X^R} -)$-acyclic objects. Still, the same proof applies with some minor modifications.

Take a resolution $\mathscr{F}^\bullet \to \mathscr{M}$ by free right $\mathscr{D}_X^R$-modules of finite rank. Let $\mathscr{P}^{\bullet\bullet} \to \mathcal{H}om_{\mathscr{D}_X^R}(\mathscr{F}^\bullet, \mathscr{D}_X^R)$ be the Cartan-Eilenberg resolution provided by proposition 1.4.8. It is known that any coherent left $\mathscr{D}_X^R$-module admits a projective resolution of length $\leq 2n+r$ (Budur et al., 2021, p6). Using this fact in the construction of the Cartan-Eilenberg resolution allows one to assume that $\mathscr{P}^{pq} = 0$ for $-p > 2n + r$.

Consider the double complex $\mathscr{N} \otimes_{\mathscr{D}_X^R} \mathscr{P}^{\bullet\bullet}$. The total cohomology of this double complex comes equipped with two filtrations one vertical and one horizontal. These filtrations are bounded due to the bounds on p so proposition 1.1.7 is applicable and yield two spectral sequences converging to the total cohomology. The first spectral sequence has

$$E_2^{pq} = H_h^p(H_v^q(\mathscr{N} \otimes_{\mathscr{D}_X^R} \mathscr{P}^{\bullet\bullet})).$$

Observe that $\mathcal{H}om_{\mathscr{D}_X^R}(\mathscr{F}^q, \mathscr{D}_X^R)$ is a free left module due to $\mathscr{F}^q$ being free of finite rank. In particular, these are acyclic for $(\mathscr{N} \otimes_{\mathscr{D}_X^R} -)$. Since $\mathscr{P}^{\bullet q} \to \mathcal{H}om_{\mathscr{D}_X^R}(\mathscr{F}^q, \mathscr{D}_X^R)$ is a quasi-isomorphism it follows that $E_2^{pq} = 0$ for any $p > 0$ and $E_2^{0q} = H^q(\mathscr{N} \otimes^L_{\mathscr{D}_X^R} R\mathcal{H}om^p_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R))$.

The other spectral sequence has

$$\begin{aligned} E_2^{pq} &= H_v^q(H_h^p(\mathscr{N} \otimes_{\mathscr{D}_X^R} \mathscr{P}^{\bullet\bullet})) \\ &= \mathcal{T}or_{-p}^{\mathscr{D}_X^R}(\mathscr{N}, \mathcal{E}xt^q_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R)) \end{aligned}$$

where it was used that the image and cohomology of the horizontal differentials are projective by the structure of the Cartan-Eilenberg resolution.

Both spectral sequences converge to the same total cohomology. Since the first spectral sequence collapses on the second sheet this means that $\mathcal{T}or_{-p}^{\mathscr{D}_X^R}(\mathscr{N}, \mathcal{E}xt^q_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R))$ converges to $H^{p+q}(\mathscr{N} \otimes^L_{\mathscr{D}_X^R} R\mathcal{H}om_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R))$. □

The proof of the following proposition is based on the proof of (Budur et al., 2021, Proposition 3.4.3). An alternative proof, based on the purifications, has been provided by van der Veer and will appear in a forthcoming paper.

Proposition 3.2.17. *Let $\mathscr{M}$ be a non-zero relative holonomic right $\mathscr{D}_X^R$-module of grade n. Then, for any $b \in R$ with $Z(b)$ non-singular and irreducible it holds that $\mathscr{M} \otimes_R R/(b)$ is a non-zero relative holonomic right $\mathscr{D}_X^{R/(b)}$-module of grade n.*

Proof. Applying Proposition 3.2.14 with $k = 0$ yields that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic right $\mathscr{D}_X^{R/(b)}$-module.

It remains to establish that $\mathscr{M} \otimes_R R/(b)$ is non-zero of grade n. By consideration of a resolution of $\mathscr{M}$ by free right modules of finite rank one has that

$$\mathscr{D}_X^{R/(b)} \otimes^L_{\mathscr{D}_X^R} R\mathcal{H}om_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R) \cong R\mathcal{H}om_{\mathscr{D}_X^{R/(b)}}(\mathscr{M} \otimes^L_{\mathscr{D}_X^R} \mathscr{D}_X^{R/(b)}, \mathscr{D}_X^{R/(b)})$$

where we note that $\mathscr{D}_X^{R/(b)}$ is a $\mathscr{D}_X^R$-bimodule so that both tensor products are well-defined. We compare the spectral sequences of both sides of this isomorphism.

The spectral sequence associated with the right-hand-side has E_2-sheet

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathscr{D}_X^{R/(b)}}(\mathcal{T}or^{\mathscr{D}_X^R}_{-q}(\mathscr{M}, \mathscr{D}_X^{R/(b)}), \mathscr{D}_X^{R/(b)}).$$

Recall from Proposition 3.2.11 that terms with $p > n$ have grade greater than n and due to Proposition 3.2.8 there are no non-zero terms with $p < n$. Hence, the only term with $p + q = n$ which could potentially have grade n is $E_2^{n,0}$. It now suffices to show that the total cohomology of degree n on the left-hand-side has grade n. Indeed, both spectral sequences must have the same limit and due to Proposition 3.2.8 and Proposition 3.2.1 the total cohomology of degree n can only have grade n if some $E_\infty^{p,q}$-term with $p + q = n$ has grade n. Again by Proposition 3.2.8 and Proposition 3.2.1 this is only possible if some $E_2^{p,q}$-term has grade n.

The spectral sequence associated to the left-hand-side has E_2-sheet given by

$$E_2^{p,q} = \mathcal{T}or^{\mathscr{D}_X^R}_{-p}(\mathscr{D}_X^{R/(b)}, \mathcal{E}xt^q_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R)).$$

Note that there are no non-zero differentials out of $E_j^{0,n}$ for $j \geq 2$. Further, the differentials into $E_j^{0,n}$ stem from $E_j^{-j,(n+j-1)}$ which is a subquotient of $\mathcal{T}or_j^{\mathscr{D}_X^R}(\mathscr{D}_X^{R/(b)}, \mathcal{E}xt^{n+j-1}_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R))$. Observe that $\mathscr{D}_X^R \xrightarrow{b} \mathscr{D}_X^R$ yields a free resolution for $\mathscr{D}_X^{R/(b)}$. It follows that $E_j^{-j,(n+j-1)} = 0$ for $j \geq 2$ whence $E_j^{0,n} = E_2^{0,n}$ for all $j \geq 2$. It remains to show that that $E_2^{0,n}$ has grade n.

Denote $\mathcal{E}^n := \mathcal{E}xt^n_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R)$, by Proposition 3.2.11 $\mathcal{E}^n$ is a n-pure relative holonomic left $\mathscr{D}_X^R$-module. By Remark 3.2.9 we have that $B_{\mathcal{E}^n} = 0$, in particular b does not belong to any minimal prime ideal containing $B_{\mathcal{E}^n}$. Hence, by Proposition 3.2.12, $\mathcal{E}^n$ admits a good filtration such that b induces injections on $\mathcal{E}^n$ and $\mathrm{gr}^{rel}\,\mathcal{E}^n$. In particular the vertical maps in the following diagram are injective

$$\begin{tikzcd}
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] \arrow[d, "b"] & F_i\mathcal{E}^n \arrow[r] \arrow[d, "b"] & \mathrm{gr}_i^{rel}\,\mathcal{E}^n \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] & F_i\mathcal{E}^n \arrow[r] & \mathrm{gr}_i^{rel}\,\mathcal{E}^n \arrow[r] & 0
\end{tikzcd}$$

so the snake lemma yields a short exact sequence

$$0 \longrightarrow R/(b) \otimes_R F_{i-1}\mathcal{E}^n \longrightarrow R/(b) \otimes_R F_i\mathcal{E}^n \longrightarrow R/(b) \otimes_R \mathrm{gr}_i^{rel}\,\mathcal{E}^n \longrightarrow 0.$$

The injectivity of b on $\mathrm{gr}^{rel}\,\mathcal{E}^n$ implies that b is also injective on $\mathcal{E}^n/F_i\mathcal{E}^n$. A similar application of the snake lemma now yields a short exact sequence

$$0 \longrightarrow R/(b) \otimes_R F_i\mathcal{E}^n \longrightarrow R/(b) \otimes_R \mathcal{E}^n \longrightarrow R/(b) \otimes_R (\mathcal{E}^n/F_i\mathcal{E}^n) \longrightarrow 0.$$

A filtration on $R/(b) \otimes_R \mathcal{E}^n$ is induced by the image of $F_i\mathcal{E}^n$. By the injectivity of the second short exact sequence one now has isomorphisms

$$F_i(R/(b) \otimes_R \mathcal{E}^n) \cong F_i\mathcal{E}^n/(F_i\mathcal{E}^n \cap b\mathcal{E}^n) \cong F_i\mathcal{E}^n/bF_i\mathcal{E}^n \cong R/(b) \otimes_R (F_i\mathcal{E}^n)$$

combined with the surjectivity of the first short exact sequence it follows that

$$\mathrm{gr}^{rel}(R/(b) \otimes_R \mathcal{E}^n) \cong R/(b) \otimes_R \mathrm{gr}^{rel}\,\mathcal{E}^n.$$

It follows that

$$\mathrm{Ch}^{rel}(\mathscr{D}_X^{R/(b)} \otimes_{\mathscr{D}_X^R} \mathcal{E}^n) = (\mathrm{Id}_{T^*X} \times \Delta)^{-1}(\mathrm{Ch}^{rel}\mathcal{E})$$

with $\Delta : \operatorname{Spec} R/(b) \to \operatorname{Spec} R$ the closed embedding as before. Since $\mathcal{E}^n$ has grade n this equality and Proposition 3.2.8 imply that $\mathrm{Ch}^{rel}(\mathscr{D}_X^{R/(b)} \otimes_{\mathscr{D}_X^R} \mathcal{E}^n)$ has dimension $n + r - 1$. In particular it follows that $\mathscr{D}_X^{R/(b)} \otimes_{\mathscr{D}_X^R} \mathcal{E}^n$ is non-zero and has grade n. This concludes the proof.

□

Relative Cohen-Macaulay $\mathscr{D}$-modules

Definition 3.2.18 (Budur et al. (2021))**.** *A coherent $\mathscr{D}_X^R$-module $\mathscr{M}$ is said to be j-Cohen-Macaulay for some $j \geq 0$ if $\mathcal{E}xt^k_{\mathscr{D}_X^R}(\mathscr{M}, \mathscr{D}_X^R) = 0$ for any $k \neq j$.*

Remark 3.2.19. *If $\mathscr{M}$ is j-Cohen-Macaulay then, by Proposition 3.2.11, it is j-pure.*

Proposition 3.2.20. *(Budur et al., 2021, Proof of Proposition 3.4.3) Let $\mathscr{M}$ be a relative holonomic and $(n+k)$-Cohen-Macaulay right $\mathscr{D}_X^R$-module. Let $b \in R$ be a prime element such that $Z(b)$ is non-singular and does not contain any irreducible component of $Z(B_{\mathscr{M}})$. Then it holds that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic and $(n+k)$-Cohen-Macaulay right $\mathscr{D}_X^{R/(b)}$-module or zero.*

Proposition 3.2.21. *(Budur et al., 2021, Proof of Proposition 3.5.2) Let $\mathscr{M}$ be a relative holonomic right $\mathscr{D}_X^R$-module of grade $n+k$ and $V \subseteq \operatorname{Spec} R$ be a variety with irreducible components of codimension $\leq k$. Then there exists an affine open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathscr{M} \otimes_R R'$ is a relative holonomic and $(n+k)$-Cohen-Macaulay right $\mathscr{D}_X^{R'}$-module. Moreover, it may be assumed that $\operatorname{Spec} R'$ intersects every irreducible component of V.*

Similarly, if $\mathscr{M}$ is relative holonomic of grade $> n+k$ and $V \subseteq \operatorname{Spec} R$ is a variety with irreducible components of codimension $\leq k$ then there exists an affine open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathscr{M} \otimes_R R' = 0$ and $\operatorname{Spec} R'$ intersects every irreducible component of V.

3.3 Upper bound for the Bernstein-Sato zero locus

Since B_F^a is the intersection of all local $B_{F,x}^a$ it is sufficient to establish Theorem 3.1.4 locally. Hence, without loss of generality, we may assume that X is affine and admits local coordinates $x_1, \ldots, x_n$. Recall from the introduction that $G = F \circ \mu$ and denote g_j for the coordinate functions of G.

Translation to right-module

By the translation between left and right modules in Section 3.2 the functional equation $PF^{s+a} = bF^s$ may be restated as the equation

$$F^{s+a}dx \cdot P^* = bF^s dx$$

in $\mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_X} \omega_X$. Define $\mathscr{M}$ to be the submodule of $\mathscr{D}_Y^{\mathbb{C}[s]} G^s \otimes_{\mathcal{O}_Y} \omega_Y$ spanned by $G^s \mu^*(dx)$ over $\mathscr{D}_Y^{\mathbb{C}[s]}$.

Lemma 3.3.1. *The right $\mathscr{D}_Y^{\mathbb{C}[s]}$-module $\mathscr{M}$ is relative holonomic.*

Proof. The left $\mathscr{D}_Y^{\mathbb{C}[s]}$-module $\mathscr{D}_Y^{\mathbb{C}[s]} G^s$ is relative holonomic following (Maisonobe, 2016, Result 1). Then the associated right-module $\mathscr{D}_Y^{\mathbb{C}[s]} G^s \otimes_{\mathcal{O}_Y} \omega_Y$ is also relative holonomic. Hence, Proposition 3.2.3 implies that the submodule $\mathscr{M}$ is also relative holonomic. □

$\mathscr{D}_X\langle s, t\rangle$-modules

Let $\mathscr{D}_X\langle s, t\rangle$ denote the sheaf of rings found from $\mathscr{D}_X^{\mathbb{C}[s]}$ found by adding a new variable t which commutes with sections of $\mathscr{D}_X$ and is subject to $s_j t = t(s_j + a_j)$ for every $j = 1, \dots, r$. The $\mathscr{D}_X^{\mathbb{C}[s]}$-module $\mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_Y} \omega_Y$ may be equipped with the structure of a right $\mathscr{D}_X\langle s, t\rangle$-module by the action

$$F^s dx P(x, \partial, s) \cdot t = F^{s+a} dx P(x, \partial, s + a).$$

In this formalism B_F^a is the Bernstein-Sato ideal of $(\mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_Y} \omega_Y)/(\mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_Y} \omega_Y)t$. An analogous $\mathscr{D}_Y\langle s, t\rangle$-module structure can be given to $\mathscr{M}$.

Lemma 3.3.2. *The Bernstein-Sato ideal $B_{\mathscr{M}/\mathscr{M}t}$ contains a polynomial of the form*

$$b(s) = \prod_E \prod_{j=1}^{N} (\mathrm{ord}_E(g_1)s_1 + \cdots + \mathrm{ord}_E(g_r)s_r + k_i + j)$$

where E ranges over the irreducible components of μ^*D and $N \in \mathbb{Z}_{\geq 0}$.

Proof. This follows from a local computation which is analogous to the corresponding local computation in of (Lichtin, 1989, Section 4). Suppose we are working near a point $y \in Y$ which is on the irreducible component E_i of μ^*D if and only if $i \in I$. Pick local coordinates $z_1, \dots, z_n$ where every E_i with $i \in I$ is determined by some z_{j_i}. After relabeling the E_i it may be assumed that $j_i = i$.

In these local coordinates

$$G^s = \prod_{j=1}^{r} u_j^{s_j} \prod_{i \in I} z_i^{\sum_{j=1}^r \mathrm{ord}_{E_i}(g_j)s_j}; \qquad \mu^*(dx) = v \prod_{i \in I} z_i^{k_i} dz$$

where u_j, v are local units. For any $i \in I$ set $\xi_i := -\partial_i - \sum_{j=1}^r s_j \partial_j(u_j)u_j^{-1}$ and let $P = v^{-1}(\prod_{j=1}^r u_j^{-a_j})(\prod_{i \in I} \xi_i^{\sum_{j=1}^r a_j \mathrm{ord}_{E_i}(g_j)})v$ then

$$G^{s+a}\mu^*(dx) \cdot P = q(s)G^s\mu^*(dx)$$

where

$$q(s) = \prod_{i \in I}((\sum_{j=1}^{r} \operatorname{ord}_{E_i}(g_j)s_j + a_j \operatorname{ord}_{E_i}(g_j)) + k_i) \cdots ((\sum_{j=1}^{r} \operatorname{ord}_{E_i}(g_j)s_j) + 1 + k_i).$$

□

Coherence as $\mathscr{D}_Y$-module

The goal of this section is to show that it may be assumed that $\mathscr{M}$ is coherent as a $\mathscr{D}_Y$-module.

Pick r independent linear polynomials $\sum_{j=1}^{r} d_{ij}s_j$ such that for any $i = n+1, \ldots, n+r$ there is no hyperplane parallel to $\sum_{j=1}^{r} d_{ij}s_j = 0$ in $Z(B^a_{F,x})$ and the d_{ij} are non-negative integers. Introduce new coordinates $z_{n+1}, \ldots, z_{n+r}$ and set $\widetilde{f}_j = f_j \prod_{i=n+1}^{n+r} z_i^{d_{ij}}$ on $X \times \mathbb{C}^r$. Note that the induced map $Y \times \mathbb{C}^r \to X \times \mathbb{C}^r$ is a resolution of singularities for $\prod \widetilde{f}_i$ and that $\widetilde{g}_j = g_j \prod_{i=n+1}^{n+i} z_i^{d_{ij}}$ is the pullback of $\widetilde{f}_i$.

For any $i = n+1, \ldots, n+r$ consider the action of ∂_i on the generator

$$\widetilde{G}^s \mu^*(dx) \cdot \partial_i = -\sum_{j=1}^{r} d_{ij}s_j z_i^{-1} \widetilde{G}^s \mu^*(dx).$$

Recall that the $\sum_{j=1}^{r} d_{ij}s_j$ are independent by assumption. It follows that an appropriate $\mathbb{C}$-linear combination of the vector fields $\partial_i z_i$ provides a vector field $\mathcal{S}_j$ acting as s_j on the generator. Hence, coherence as a $\mathscr{D}^{\mathbb{C}[s]}_{Y \times \mathbb{C}^r}$-module implies the coherence as a $\mathscr{D}_{Y \times \mathbb{C}^r}$-module.

Lemma 3.3.3. *For any $(x,p) \in X \times \mathbb{C}^r$ it holds that if $b \in B^a_{\widetilde{F},(x,p)}$ then $b \in B^a_{F,x}$.*

Proof. Take local coordinates $z_1, \ldots, z_{n+r}$ near (x,p) and let P be in the stalk of $\mathscr{D}^{\mathbb{C}[s]}_{X \times \mathbb{C}^r}$ at (x,p) such that $b\widetilde{F}^s = P\widetilde{F}^{s+a}$.

Similarly to the above there exists a $\mathbb{C}$-basis $\zeta_1, \ldots, \zeta_r$ for the span of $\partial_{n+1}, \ldots, \partial_{n+r}$ so that $\mathcal{S}'_j := z_{n+j}\zeta_j$ satisfies $\mathcal{S}_j \cdot \widetilde{F}^s = s_j\widetilde{F}^s$. Expand P as a polynomials in $\zeta_1, \ldots, \zeta_r$

$$P = \sum_{\alpha} P_\alpha \zeta_1^{\alpha_1} \cdots \zeta_r^{\alpha_r}$$

where the coefficients P_α live in the stalk of $\mathcal{O}_{X \times \mathbb{C}^r} \otimes_{\mathcal{O}_X} \mathscr{D}^{\mathbb{C}[s]}_X$ at (x,p).

Let N be greater than the maximal value of $|\alpha|$ then

$$(z_{n+1} \cdots z_{n+r})^N b\widetilde{F}^s = \left(\sum_{\alpha} \prod_{i=1}^{r} (s_i + a_i)^{\alpha_i} \sum_{\beta} Q_{\alpha\beta} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \right) \widetilde{F}^{s+a}$$

where the P_α were expanded as polynomials in $\partial_1, \ldots, \partial_n$ with coefficients $Q_{\alpha\beta}$ from the stalk of $\mathcal{O}_{X \times \mathbb{C}^r}$. Observe that $\partial_1, \ldots, \partial_n$ act on the formal symbol $\widetilde{F}^{s+a}$ the same as they act on the formal symbol F^{s+a}.

Now consider this functional equation on the analytification of $X \times \mathbb{C}^r$ and expand $(z_{n+1} \cdots z_{n+r})^N$ and the $Q_{\alpha\beta}$ as power series at p. Identifying powers of $(z_{n+1} - p_1) \cdots (z_{n+r} - p_r)$ on both sides a functional equation with analytical coefficients for F^s

follows. This establishes that b is in the analytic Bernstein-Sato ideal. It follows that $b \in B^a_{F,x}$ since analytic and algebraic Bernstein-Sato ideals agree Brianccon and Maisonobe (2002).

□

Replacing F by $\widetilde{F}$ leaves the estimate unchanged up to hyperplanes parallel to $\sum_{j=1}^r d_{ij}s_j = 0$. These are not in $Z(B^a_{F,x})$ by assumption so, by Lemma 3.3.3, it remains to prove the theorem for $\widetilde{F}$. For notational simplicity we write F instead of $\widetilde{F}$ and X instead of $X \times \mathbb{C}^r$ which has dimension $m = n + r$.

Reduction of the number of variables

Let $\ell_1, \ldots, \ell_{r-1} \in \mathbb{C}[s]$ be degree one polynomials which will be fixed later. For any $i = 0 \ldots, r-1$ let L_i be the ideal of $\mathbb{C}[s]$ generated by $\ell_1, \ldots, \ell_i$. Assume that the ℓ_i are chosen sufficiently generically so that $Z(L_{r-1})$ is a line.

Proposition 3.3.4. *The $\mathscr{D}_Y$-module $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is coherent and its (non-relative) characteristic variety satisfies $\mathrm{Ch}\, \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1} \subseteq \Lambda \cup W$ where Λ is isotropic and W is an irreducible variety of dimension $m+1$ which dominates Y.*

Proof. The coherence of $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is immediate from the coherence of $\mathscr{M}$ as $\mathscr{D}_Y$-module.

Let s_0 denote a new variable so that $\mathbb{C}[s]/L_{r-1} \cong \mathbb{C}[s_0]$. As in the proof of Lemma 3.3.2 we may pick local coordinates such that

$$G^s = \prod_{j=1}^r u_j^{s_j} \prod_{i \in I} z_i^{\sum_{j=1}^r \mathrm{ord}_{E_i}(g_j)s_j}; \qquad \mu^*(dx) = v \prod_{i \in I} z_i^{k_i} dz$$

where u_j, v are local units and we may assume that $\{n+1, \ldots, n+r\} \subseteq I$. For any $j = 1, \ldots, n+r$ consider the action of $v^{-1}\partial_j v z_j$ on $G^s\mu^*(dx)$ and employ the isomorphism $\mathbb{C}[s]/L_{r-1} \cong \mathbb{C}[s_0]$. This yields complex constants B_i, C_i, b_i, c_i such that

$$G^s\mu^*(dx) \cdot v^{-1}\partial_j v z_j = ((C_j s_0 + c_j) + z_j \sum_{i=1}^r B_i s_0 u_i^{-1} \partial_j(u_i)) G^s \mu^*(dx)$$

holds in $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$. Recall that the linear functions $\sum_{j=1}^r d_{ij}s_j$ with $i \in \{n+1, \ldots, n+r\}$ form a basis for the linear polynomials. Since these occur as the exponents in the final factors of $G^s\mu^*(dx)$ it may be assumed that there is at least one C_j with $j > n$ which is non-zero.

Set $h_j = C_j + z_j \sum_{i=1}^n B_i u_i^{-1}\partial_j(u_i)$ and $\xi_j = v^{-1}\partial_j v z_j - c_j$. Observe that $z_j \sum_{i=1}^n B_i u_i^{-1}\partial_j(u_i) = 0$ for $j > n$. Hence the h_j with $j > n$ are scalars and they are not all zero. By renumbering we may assume that h_1 is a non-zero scalar.

Now for any $j = 2, \ldots, n+r$ the vector field $\xi_j h_1 - \xi_1 h_j$ annihilates $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$. In particular the corresponding section of $\mathrm{gr}\, \mathscr{D}_Y$ annihilates $\mathrm{gr}\, \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$. This proves the desired bound on $\mathrm{Ch}\, \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$.

□

Proposition 3.3.5. *It holds that $\mathscr{M}$ is m-Cohen-Macaulay.*

Proof. Pick local coordinates $z_1, \dots, z_m$ on Y. For the sake of notational simplicity we apply the equivalence between left and right $\mathscr{D}_Y^{\mathbb{C}[s]}$-modules described in Section 3.2. It then suffices to show that there exists a free resolution of length $\leq m$ for any left $\mathscr{D}_Y^{\mathbb{C}[s]}$-module of the form $\mathscr{N} := \mathscr{D}_Y^{\mathbb{C}[s]} z_1^{p_1} \cdots z_n^{p_m}$ with $p_1, \dots, p_m \in \mathbb{C}[s]$.

Denote e_i for the i-th basis vector of $(\mathscr{D}_X^{\mathbb{C}[s]})^m$. For any k let $\wedge^k(\mathscr{D}_X^{\mathbb{C}[s]})^m$ be the left $\mathscr{D}_X^{\mathbb{C}[s]}$-module $\oplus_{1 \leq i_1 < \cdots < i_k < n} \mathscr{D}_X^{\mathbb{C}[s]}$ whose generators are denoted by $e_{i_1} \wedge \cdots \wedge e_{i_q}$ in the conventional fashion. We claim that the desired free resolution for $\mathscr{N}$ is given by the Koszul complex $K^\bullet := \wedge^\bullet(\mathscr{D}_X^{\mathbb{C}[s]})^m$ whose differentials are defined by

$$d_q : K^q \to K^{q-1} : \lambda e_{i_1} \wedge \cdots \wedge e_{i_q} \mapsto \lambda \sum_j (-1)^{j-1} (x_{i_j} \partial_{i_j} - p_{i_j}) e_{i_1} \wedge \cdots \hat{e}_{i_j} \cdots \wedge e_{i_q}.$$

First observe that the obvious surjection from $H^0(K^\bullet)$ onto $\mathscr{N}$ is injective. Indeed, suppose we are given some differential operator P which annihilates $z_1^{p_1} \cdots z_n^{p_m}$. In local coordinates we may write $P = \sum c_{\alpha\beta} z^\alpha (x\partial - p)^\beta$ where $(z\partial - p)^\beta = \prod_i (z_i \partial_i - p_i)^{\beta_i}$. That P annihilates $z_1^{p_1} \cdots z_n^{p_m}$ then means that $c_{\alpha 0} = 0$ for any α which proves the injectivity.

Observe that the elements $z_i \partial_i - p_i$ commute with each other. We will establish that right multiplication by $z_i \partial_i - p_i$ is injective on $\mathscr{D}_Y^{\mathbb{C}[s]} / (z_1 \partial_1 - p_1, \dots, z_{i-1} \partial_{i-1} - p_{i-1})$ then the proof of the acyclicity of the Koszul complex in (Weibel, 1995, Corollary 4.5.4) yields that $H^i(K^\bullet) = 0$ for $i > 0$ which is the desired result.

Observe that

$$\left(\sum_{\alpha,\beta} c_{\alpha\beta} z^\alpha \partial^\beta\right)(z_j \partial_j - p_j) = \sum_{\alpha,\beta} (c_{\alpha\beta}(\beta_j - p_j) + c_{(\alpha - e_j)(\beta - e_j)}) z^\alpha \partial^\beta$$

for any local section $\sum_{\alpha,\beta} c_{\alpha\beta} z^\alpha \partial^\beta$ of $\mathscr{D}_Y^{\mathbb{C}[s]}$. Now suppose by contraposition that right multiplication by $z_i \partial_i - p_i$ is not injective on $\mathscr{D}_Y^{\mathbb{C}[s]} / (z_1 \partial_1 - p_1, \dots, z_{i-1} \partial_{i-1} - p_{i-1})$. This means that we can find local sections $\lambda_1, \dots, \lambda_i$ of $\mathscr{D}_Y^{\mathbb{C}[s]}$ such that

$$\lambda_i (z_i \partial_i - p_i) = \sum_{j=1}^{i-1} \lambda_j (z_j \partial_j - p_j).$$

Expand every λ_j as a polynomial in $z^\alpha \partial^\beta$ with coefficients $c_{\alpha\beta}^{(j)}$. Take $k \geq 0$ to be maximal such that there exists some nonzero $c_{\alpha\beta}^{(i)}$ with $\alpha_i = k$. Then, for any α, β with $\alpha_i = k$ we have that

$$c_{\alpha\beta}^{(i)} = \sum_{j=1}^{i-1} c_{(\alpha + e_i)(\beta + e_i)}^{(j)} (\beta_j - p_j) + c_{(\alpha + e_i - e_j)(\beta + e_i - e_j)}^{(j)}.$$

This yields that

$$\sum_{\alpha_i = k} c_{\alpha\beta}^{(i)} z^\alpha \partial^\beta = \sum_{j=1}^{i-1} \left(\sum_{\alpha_i = k} c_{(\alpha + e_i)(\beta + e_i)}^{(j)} z^\alpha \partial^\beta\right)(z_j \partial_j - p_j).$$

Removing the left hand side from λ_i reduces k, proceed iteratively to find that $\lambda_i \in (z_1 \partial_1 - p_1, \dots, z_{i-1} \partial_{i-1} - p_{i-1})$.

$\square$

Corollary 3.3.6. *Any polynomial $b \in \mathbb{C}[s]$ which is not in L_i induces an injective endomorphisms on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$.*

Proof. Inductively applying Proposition 3.2.20 yields that $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ is m-Cohen-Macaulay over $\mathscr{D}_X^{\mathbb{C}[s]/L_i}$. In particular $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ is m-pure and the Bernstein-Sato ideal over $\mathbb{C}[s]/L_i$ is trivial by Remark 3.2.9. Now Proposition 3.2.12 yields the desired injectivity.

$\square$

Lemma 3.3.7. *One can pick $\ell_1, \dots, \ell_{r-1}$ such that $\mathcal{T}or_1^{\mathscr{D}_X^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}})$ is a relative holonomic right $\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}$-module of grade greater than or equal to $m+1$ for every $i = 1, \dots, r-1$. Here K_i denotes the kernel of ℓ_i on $\int^1(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1})$.*

Proof. Observe that the K_i are relative holonomic $\mathscr{D}_X^{\mathbb{C}[s]/L_i}$-modules due to Proposition 3.2.4 and Proposition 3.2.14. Using Proposition 3.2.21 one can inductively construct the ℓ_i and an affine open $\operatorname{Spec} R \subseteq \mathbb{C}^r$ such that for every i

(i) $Z(L_i) \cap \operatorname{Spec} R \neq 0$.

(ii) $\mathcal{E}xt^m_{\mathscr{D}_X^{R/L_i}}(K_i, \mathscr{D}_X^{R/L_i})$ is m-Cohen-Macaulay over $\mathscr{D}_X^{R/L_i}$ or zero.

(iii) $\mathcal{E}xt^{m+j}_{\mathscr{D}_X^{R/L_i}}(K_i, \mathscr{D}_X^{R/L_i}) = 0$ for every $j > 0$.

By consideration of a resolution of K_i by free right modules of finite rank one finds that

$$R\mathcal{H}om_{\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}}(K_i \otimes^L_{\mathscr{D}_X^{\mathbb{C}[s]/Li}} \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}, \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}})$$
$$\cong \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}} \otimes^L_{\mathscr{D}_X^{\mathbb{C}[s]/L_i}} R\mathcal{H}om_{\mathscr{D}_X^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{D}_X^{\mathbb{C}[s]/L_i})$$

where we note that $\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}$ is a $\mathscr{D}_X^{\mathbb{C}[s]/L_i}$-bimodule so that both tensor products produce complexes of left $\mathscr{D}_X^{R/b}$-modules. We compare the spectral sequences of both sides.

The spectral sequence on the left-hand-side has terms

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}}(\mathcal{T}or_{-q}^{\mathscr{D}_X^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}), \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}).$$

By Proposition 3.2.8 and Proposition 3.2.11 these $\mathcal{E}xt$-terms can only be non-zero for $p = m$ or $p = m+1$. In particular, the spectral sequence degenerates at E_2. By the same argument as in the proof of Proposition 3.2.17 it now suffices to show that the cohomology of degree $p + q = m - 1$ has grade $\geq m + 1$.

The spectral sequence on the right-hand-side has terms

$$E_2^{pq} = \mathcal{T}or_{-p}^{\mathscr{D}_X^{\mathbb{C}[s]/L_i}}(\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}, \mathcal{E}xt^q_{\mathscr{D}_X^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{D}_X^{\mathbb{C}[s]/L_i})).$$

We claim that that all terms with $p + q = m - 1$ vanish on $X \times (Z(L_{r-1}) \cap \operatorname{Spec} R)$. Then by Proposition 3.2.8 the terms have grade $\geq m + 1$ and it follows that the same must hold for the total cohomology.

Denote $\mathcal{E}_k := \mathcal{E}xt^m_{\mathscr{D}_X^{R/L_k}}(K_k, \mathscr{D}_X^{R/L_k})$ for any $k \geq 0$. Recall $\mathcal{E}_k$ is m-Cohen-Macaulay by construction. Hence, Proposition 3.2.20 shows that $R/L_i \otimes_{R/L_k} \mathcal{E}_k$ is m-Cohen-Macaulay or zero for any $k < i$. In particular the Bernstein-Sato ideal of $R/L_i \otimes_{R/L_k} \mathcal{E}_k$ is trivial by Remark 3.2.9 Now Proposition 3.2.12 yields that the polynomial ℓ_i induces an injection on $R/L_{i-1} \otimes_{R/L_k} \mathcal{E}_k$ for any $i > k$. This means that $\mathcal{T}or_p^{\mathscr{D}_X^{R/L_i}}(\mathscr{D}_X^{R/L_i}, \mathscr{D}_X^{R/L_{i-1}} \otimes_{\mathscr{D}_X^{R/L_k}} \mathcal{E}_k) = 0$ for all $p > 0$ and $i > k$. It follows that

$$\mathcal{T}or_1^{\mathscr{D}_X^{\mathbb{C}[s]}}(\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}, \mathcal{E}_k) = 0.$$

The left-hand-side is precisely the $E_2^{-1,m}$-term of the spectral sequence. Due to condition (iii) the $E_2^{p,q}$-terms with $p + q = m - 1$ and $p > m$ also vanish. The remaining term $E_2^{(m-1),0}$ is zero regardless since it involves $\mathcal{E}xt^{m-1}$ of a relative holonomic module. This concludes the proof.

□

Comparison $\mathscr{U}$ and $\mathscr{M}$

The $\mathscr{D}_X$-linear endomorphism t induces an endomorphism on $\int^0 \mathscr{M}$. The relation $s_i t = t(s_i + a_i)$ also holds on $\int^0 \mathscr{M}$ due to the functoriality of the direct image. Hence, $\int^0 \mathscr{M}$ is equipped with the structure of a $\mathscr{D}_X\langle s, t\rangle$-module.

The surjection of right $\mathscr{D}_Y^{\mathbb{C}[s]}$-modules $\mathscr{D}_Y^{\mathbb{C}[s]} \to \mathscr{M}$ defined by $1 \mapsto G^s \mu^*(dx)$ induces a morphism $\int^0 \mathscr{D}_Y^{\mathbb{C}[s]} \to \int^0 \mathscr{M}$. Observe that $\int^0 \mathscr{D}_Y^{\mathbb{C}[s]} = R^0\mu_*(\mathscr{D}_{Y\to X}^{\mathbb{C}[s]})$ contains global section corresponding to 1 in $\mathscr{D}_{Y\to X}^{\mathbb{C}[s]}$. We write u for the image of this section in $\int^0 \mathscr{M}$ and $\mathscr{U}$ for the right $\mathscr{D}_X\langle s, t\rangle$-module generated by u.

Lemma 3.3.8. *There is a surjective morphism of right $\mathscr{D}_X\langle s, t\rangle$-modules $\mathscr{U} \to \mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_X} \omega_X$ sending u to $F^s dx$.*

Proof. This is analogous to the corresponding absolute result (Bjork, 1979, Chapter 5, p246). It must be show that $(F^s dx)P = 0$ whenever $uP = 0$ for some differential operator P over an open $U \subseteq X$.

The resolution of singularities $Y \to X$ is an isomorphism on the complement of the divisor D determined by $\prod f_i$. Hence, an isomorphism $\mathscr{U} \cong \int^0 \mathscr{M} \cong \mathscr{D}_X F^s \otimes_{\mathcal{O}_X} \omega_X$ holds outside of D. It follows that the support of the coherent sheaf of $\mathcal{O}_U$-modules $\mathcal{O}_U(F^s dx)P$ lies in D. The Nullstellen Satz now yields that $(\prod f_i)^N (F^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that $\prod f_i$ is a non-zero divisor of $(\mathscr{D}_X F^s \otimes_{\mathcal{O}_X} \omega_X)(U)$. Therefore, $(f^s dx)P = 0$ on U as desired.

□

Lemma 3.3.9. *The relative holonomic $\mathscr{D}_X^{\mathbb{C}[s]}$-module $\int^0 \mathscr{M}/\mathscr{U}$ has grade $j(\int^0 \mathscr{M}/\mathscr{U}) \geq m + 1$.*

Proof. Let $\ell_1, \ldots, \ell_{r-1}$ be the degree one polynomials provided by Lemma 3.3.7. For any $i = 0, \ldots, r-1$ denote $\mathscr{M}_i = \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$.

Recall from Corollary 3.3.6 that ℓ_i is injective on $\mathscr{M}_{i-1}$. The injective endomorphism of ℓ_i on $\mathscr{M}_{i-1}$ induces a long exact sequence of $\mathscr{D}_X^{\mathbb{C}[s]/L_{i-1}}$-modules

$$0 \to \int^0 \mathscr{M}_{i-1} \xrightarrow{\ell_i} \int^0 \mathscr{M}_{i-1} \to \int^0 \mathscr{M}_i \to \cdots$$

whence $(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_i$ is a submodule of $\int^0 \mathscr{M}_i$. The quotient is isomorphic to the kernel K_i of ℓ_i on $\int^1 \mathscr{M}_{i-1}$.

Applying a tensor product with $\mathbb{C}[s]/L_{r-1}$ to the inclusion $(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_i \hookrightarrow \int^0 \mathscr{M}_i$ yields an exact sequence

$$\mathcal{T}or_1^{\mathbb{C}[s]/L_{i-1}}\left(K_i, \frac{\mathbb{C}[s]}{L_{r-1}}\right) \to \left(\int^0 \mathscr{M}_{i-1}\right) \otimes_{\mathbb{C}[s]/L_{i-1}} \frac{\mathbb{C}[s]}{L_{r-1}} \to \left(\int^0 \mathscr{M}_i\right) \otimes_{\mathbb{C}[s]/L_{i-1}} \frac{\mathbb{C}[s]}{L_{r-1}}.$$

By choice of the ℓ_i it holds that $\mathcal{T}or_1^{\mathscr{D}_X^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}})$ has grade $\geq m+1$. This implies the existence of a non-zero polynomial $b_i \in \mathbb{C}[s]/L_{r-1}$ which annihilates $\mathcal{T}or_1^{\mathbb{C}[s]/L_i}(K_i, \mathbb{C}[s]/L_{r-1})$.

Denote $B = \prod_{i=1}^{r-1} b_i$ and note that the kernels of the endomorphisms induced by B^N form an increasing sequence inside the coherent $\mathscr{D}_X^{\mathbb{C}[s]/L_{r-1}}$-module $(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{r-1}$. Such increasing sequences must stabilise for sufficiently large N. Then it follows that the intersection of $\operatorname{Im} \mathcal{T}or_1^{\mathbb{C}[s]/L_{i-1}}(K_i, \mathbb{C}[s]/L_{r-1})$ and $B^N(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{r-1}$ is trivial for any i. This means that there are injections

$$B^N\left(\int^0 \mathscr{M}_{i-1}\right) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{r-1} \hookrightarrow B^N\left(\int^0 \mathscr{M}_i\right) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{r-1}.$$

In particular $B^N(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is a submodule of $\int^0 \mathscr{M}_{r-1}$.

Theorem 4.2 of Kashiwara (1976) and Proposition 3.3.4 yield that $\int^0 \mathscr{M}_{r-1}$ is a coherent $\mathscr{D}_X$-module with characteristic variety $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda \cup W))$ with Λ isotropic and W irreducible of dimension $m+1$ dominating Y. Observe that $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is a subquotient of $\int^0 \mathscr{M}_{r-1}$ with support in the divisor D. Hence, $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is a coherent $\mathscr{D}_X$-module with

$$\operatorname{Ch}\left(B^N\left(\int^0 \mathscr{M}/\mathscr{U}\right) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\Lambda \cup W)\right) \cap \pi^{-1}(D)$$

where $\pi : T^*X \to X$ is the projection map.

By (Kashiwara, 1976, Proposition 4.9) $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda))$ remains isotropic. Moreover, $\widetilde{\mu}((T^*\mu)^{-1}(W))$ is irreducible of dimension $m+1$ and dominates X. Intersecting with $\pi^{-1}(D)$ yields a closed strict subset which necessarily has lower dimension. Hence, it follows that $\dim \operatorname{Ch} B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1} \leq m$. This means that $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is holonomic. By (Bjork, 1979, Proposition 3.11, Chapter 3) the Bernstein-Sato ideal of a holonomic module is non-zero. This implies that the Bernstein-Sato ideal of $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{r-1}$ is non-zero which means that it has grade $\geq m+1$ by Remark 3.2.9. Then also $\int^0 \mathscr{M}/\mathscr{U}$ has grade $\geq m+1$ by Proposition 3.2.17.

□

Theorem 3.3.10. *Every irreducible component of $Z(B^a_{F,x})$ of codimension 1 is a hyperplane of the form*

$$\operatorname{ord}_{E_i}(g_1)s_1 + \cdots + \operatorname{ord}_{E_i}(g_r)s_r + k_i + c = 0$$

with $c \in \mathbb{Z}_{>0}$.

Proof. The Bernstein-Sato ideals $B_{(\int^0 \mathscr{M}/\mathscr{U})t^n}$ form an increasing sequence of ideals in the Noetherian ring R. Hence there must exist some $N \geq 1$ such that $B_{(\int^0 \mathscr{M}/\mathscr{U})t^n} = B_{(\int^0 \mathscr{M}/\mathscr{U})t^{n+1}}$ for all $n \geq N$.

By Lemma 3.3.9 the $\mathscr{D}_X^{\mathbb{C}[s]}$-module $\int^0 \mathscr{M}/\mathscr{U}$ has grade greater than or equal to $m+1$ so Remark 3.2.9 provides some non-zero $q(s_1,\dots,s_r) \in B_{\mathscr{M}/\mathscr{U}}$. Then also $q \in B_{(\int^0 \mathscr{M}/\mathscr{U})t^N}$. Observe that one has the commutation relation

$$q(s_1,\dots,s_r)t = tq(s_1+a_1,\dots,s_r+a_r).$$

In particular it follows that $q(s+a) \in B_{(\int^0 \mathscr{M}/\mathscr{U})t^{N+1}}$. Due to the stabilisation $B_{(\int^0 \mathscr{M}/\mathscr{U})t^N} = B_{(\int^0 \mathscr{M}/\mathscr{U})t^{N+1}}$ it follows by iteration that $q(s+ja) \in B_{(\int^0 \mathscr{M}/\mathscr{U})t^N}$ for any integer $j \geq 0$. Due to the estimate for the slopes in Theorem 3.1.3 it follows that $Z(B_{(\int^0 \mathscr{M}/\mathscr{U})t^N})$ does not contain any codimension one component of $Z(B^a_{F,x})$. Hence, we can pick some polynomial $r(s)$ which annihilates $(\int^0 \mathscr{M}/\mathscr{U})t^N$ but does not vanish on any codimension one component of $Z(B^a_{F,x})$.

Let $b(s)$ be the Bernstein-Sato polynomial for $\mathscr{M}/\mathscr{M}t$ provided by Lemma 3.3.2. Set $B := r(s)\prod_{j=0}^{N+1} b(s+ja)$ then it follows that $B\mathscr{M} \otimes_{\mathbb{C}[s]} R \subseteq (\mathscr{U}t) \otimes_{\mathbb{C}[s]} R$. In particular, this means that $B \in B_{(\mathscr{U}/\mathscr{U}t)\otimes_{\mathbb{C}[s]} R}$.

Now recall from Lemma 3.3.8 that we have a surjection $\mathscr{U} \to \mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_X} \omega_X$. This remains a surjection after a tensor product with R so that B is in the Bernstein-Sato ideal of $((\mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_X} \omega_X)/(\mathscr{D}_X^{\mathbb{C}[s]} F^s \otimes_{\mathcal{O}_X} \omega_X)t) \otimes_{\mathbb{C}[s]} R$. This means that $Z(B^a_F) \subseteq Z(B)$ which yields the desired result.

□

Analytic case

The proof of Theorem 3.1.4 proceeds similarly in the local analytic case. If the smooth affine variety X is replaced with a germ of a complex manifold (X,x) then one should replace the sheaves $\mathscr{D}_X$ and $\mathscr{D}_X^R$ by the stalks of the corresponding analytic sheaves $\mathscr{D}_{X,x}$ and $\mathscr{D}_{X,x}^R$. Here R remains a regular commutative finitely generated $\mathbb{C}$-algebra integral domain.

Denote Y for the analytification of $\operatorname{Spec} R$. The role of the relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$ is replaced by $\mathrm{Ch}^{rel}\mathscr{M} \cap \pi^{-1}(\Omega_x) \times Y$ for some small ball Ω_x in X where $\pi : T^*X \to X$ is the projection map. Correspondingly, a $\mathscr{D}_X^R$-module $\mathscr{M}$ is said to be relative holonomic at x if $\mathrm{Ch}^{rel}\mathscr{M} \cap \pi^{-1}(\Omega_x) \times Y$ satisfies the analytic version of Definition 3.2.2.

Similarly one can considers grades, purity and j-Cohen-Macaulay at x which is to say that one considers these properties over $\mathscr{D}_{X,x}^R$. With these changes the homological results from Budur et al. (2021) given in section 3.2 generalise appropriately as is outlined outlined in section 3.6 of loc. cit.

The analytical direct image functor associated to a proper morphism of complex manifolds $\mu : Y \to X$ still preserves relative holonomicity provided one adds the assumption that the right $\mathscr{D}_Y^R$-module $\mathscr{M}$ is μ-good (Monteiro Fernandes and Sabbah, 2016, Theorem 1.17). An analytic $\mathscr{D}_Y^R$-module $\mathscr{M}$ is said to be μ-good if X has an open cover $\{V_j\}_j$ such that $\mathscr{M}$ admits a good filtration on $\mu^{-1}(V_j)$ for every j. This assumption is satisfied in our case because our modules are generated by a single symbol $G^s\mu^*(dx)$.

3.4 Lower bounds for the Bernstein-Sato zero locus

Trivial hyperplanes

The following result generalises the fact that one always has a trivial root -1 for the Bernstein-Sato polynomial when $r = 1$.

Proposition 3.4.1. *Let E be an irreducible component of D. Then $\sum_{j=1}^{r} \mathrm{ord}_E(f_j)s_j + c = 0$ determines an irreducible component of $Z(B_F^a)$ for $c = 1, \ldots, \sum_{j=1}^{r} \mathrm{ord}_E(f_j)a_j$.*

Proof. Let $x \in E$, pick some arbitrary $b(s) \in B_{F,x}^a$ and suppose that $P \in \mathscr{D}_{X,x}^{\mathbb{C}[s]}$ is a local differential operator which realises the Bernstein-Sato relation with $b(s)$ in a neighbourhood of x. Let $h \in \mathcal{O}_{X,x}$ be a local equation for the divisor E near x. Factor every f_j as $f_j = h^{\mathrm{ord}_E(f_j)} r_j$ for some $r_j \in \mathcal{O}_{X,x}$.

Pick some $\lambda \in \mathbb{Q}^r$ on the hyperplane $\sum_{j=1}^{r} \mathrm{ord}_E(f_j)s_j + c = 0$ and substitute in the Bernstein-Sato functional equation to find that

$$b(\lambda) h^{\sum_{j=1}^{r} \mathrm{ord}_E(f_j)\lambda_j} \prod_{j=1}^{r} r_j^{\lambda_j} = P h^{\sum_{j=1}^{r} \mathrm{ord}_E(f_j)(\lambda_j + a_j)} \prod_{j=1}^{r} r_j^{\lambda_j + a_j}.$$

By assumption, $\sum_{j=1}^{r} \mathrm{ord}_E(f_j)(\lambda_j + a_j)$ is a positive integer. It follows that the $\mathbb{Q}$-divisor determined by the right-hand-side has a positive coefficients on E. On the other hand, the $\mathbb{Q}$-divisor determined by $h^{\sum_{j=1}^{r} \mathrm{ord}_E(f_j)\lambda_j}$ has a negative coefficient on E. Hence, the equality is only possible if $b(\lambda) = 0$. Since λ was arbitrary this concludes the proof. $\square$

Jumping walls

In this section we establish Theorem 3.1.7 on the relation between the jumping walls and $Z(B_F^a)$.

By (Kollár, 1997, Corollary 3.12) one can rephrase the LCT-polytope and KLT_a-region more canonically as

$$\mathrm{LCT}(F) = \bigcap_E \{\lambda \in \mathbb{R}_{\geq 0}^r : (X, F^s) \text{ is log-canonical}\}$$
$$\mathrm{KLT}_a(F) = \bigcap_E \{\lambda \in \mathbb{R}_{\geq 0}^r : (X, F^{\lambda - a}) \text{ is Kawamata log-terminal}\}.$$

For our purposes an analytical reformulation of Kawatama log-canonicity (Kollár, 1997, Proposition 3.20) is most convenient

$$\mathrm{KLT}_a(F) = \{\lambda \in \mathbb{R}_{\geq 0}^r : \prod_{j=1}^{r} |f_j|^{-2(\lambda_j - a_j)} \text{ is integrable near any } x \in X\}.$$

Recall that the jumping walls were defined using the mixed multiplier ideal sheaf $\mathcal{J}(F^\lambda)$ of $\lambda \in \mathbb{R}_{\geq 0}^r$. Following (Libgober, 2002, Remark 2.6) this ideal sheaf may be reformulated in terms of ideals of quasiadjunction, this is to say that for any $x \in X$

$$\mathcal{J}(F^\lambda)_x = \{\phi \in \mathcal{O}_{X,x} : |\phi|^2 \prod_{j=1}^{r} |f_j|^{-2\lambda_j} \text{ is integrable near } x\}.$$

The proof of the following theorem is now analogous to a proof for $r = 1$ in (Ein et al., 2004, Theorem B) using the ideas from (Kollár, 1997, Theorem 10.6).

Theorem 3.4.2. *Suppose that* $\sum_{j=1}^{r} \operatorname{ord}_E(g_j)s_j = k_E + c$ *determines a facet of a jumping wall which intersects* $\mathrm{KLT}_a(F)$. *Then* $\sum_{j=1}^{r} \operatorname{ord}_E(g_j)s_j + k_E + c = 0$ *determines a component of* $Z(B_F^a)$.

Proof. Let λ be a point of the facet of $\sum_{j=1}^{r} \operatorname{ord}_E(g_j)s_j = k_E + c$ inside of $\mathrm{KLT}_a(F)$. Since λ is on a facet there must exist some $x \in D$ and $\phi \in \mathcal{O}_{X,x} \setminus \mathcal{J}(F^s)_x$ such that

$$\int |\phi|^2 \prod_{j=1}^{r} |f_j|^{-2(\lambda_j - \varepsilon_j)} \psi dx < \infty, \qquad \int \prod_{j=1}^{r} |f_j|^{-2(\lambda_j - a_j)} \psi dx < \infty.$$

for any $\varepsilon \in \mathbb{R}_{>0}^r$ and positive bump function ψ supported on a sufficiently small neighbourhood of x.

Pick some $b(s) \in B_F^a$ and take the support of ψ to be sufficiently small such that there exists some local differential operator P with $b(s)F^s = PF^{s+a}$. By conjugation it follows that $\overline{b}(s)\overline{F}^s = \overline{PF}^{s+a}$. Holomorphic and antiholomorphic differential operators commute so

$$|b(s)|^2 \prod_{j=1}^{r} |f_j|^{2s_j} = P\overline{P}|f_j|^{2(s_j + a_j)}.$$

Now assume that the real part of all $2(s_j + a_j)$ is strictly greater than the order of P. Then $|f_j|^{2(s_j+a_j))}$ has enough continuous derivatives to apply integration by parts. This yields that

$$|b(s)|^2 \int \prod_{j=1}^{r} |f_j|^{2s_j} \phi\psi dx = \int \prod_{j=1}^{r} |f_j|^{2(s_j+a_j))} P^* \overline{P^*} |\phi|^2 \psi dx$$

View this as an equality of meromorphic functions of s to conclude that the equality holds for arbitrary $s \in \mathbb{R}^r$ provided both integrals are finite.

Now take $s = -\lambda + \varepsilon$ and let ε tend to zero from above. Then, by dominated convergence, the integral on the right hand side converges to a finite number. On the other hand, since ϕ is not in $\mathcal{J}(F^s)_x$, the integral on the left hand side tends to infinity by the monotone convergence theorem. This means that the equality is only possible if $b(s)$ vanishes on $(-\lambda_1, \ldots, -\lambda_r)$. Since the point λ on the facet and $b(s) \in B_F^a$ were arbitrary we conclude that $\sum_j \operatorname{ord}_E(g_j)s_j + k_E + c = 0$ determines a component of $Z(B_F^a)$.

□

Corollary 3.4.3. *Suppose that* $\sum_{j=1}^{r} \operatorname{ord}_E(g_j)s_j = k_E + 1$ *determines a face of* $\mathrm{LCT}(F)$. *If* $a_j \neq 0$ *and* $\operatorname{ord}_E(g_j) \neq 0$ *for some* j, *then* $\sum_j \operatorname{ord}_E(g_j)s_j + k_E + 1 = 0$ *determines a component of* $Z(B_F^a)$.

Proof. Let λ be some λ is a relative interior point of the face determined by $\sum_{j=1}^{r} \operatorname{ord}_E(g_j)s_j = k_E + 1$ The condition that $a_j \neq 0$ and $\operatorname{ord}_E(g_j) \neq 0$ for some j ensures that

$$\sum_{j=1}^{r} \operatorname{ord}_E(g_j)(\lambda_j - a_j) < 0.$$

Since λ is an interior point of the face it holds that $\sum_{j=1}^{r} \operatorname{ord}_{E'}(g_j)(\lambda_j) < 0$ for all other divisors E' in $\mu^* D$. From this we conclude that $\lambda \in \mathrm{KLT}_a(F)$ which means that the facet intersects $\mathrm{KLT}_a(F)$. The desired result now follows from Theorem 3.1.7.

□

Real Jumping Walls

Finally, we establish the real analogues for the results in section 3.4. Let $X_{\mathbb{R}}$ be a real algebraic manifold and let $F = (f_1, \ldots, f_r)$ be a tuple of real algebraic functions on $X_{\mathbb{R}}$. Fix some $a \in \mathbb{Z}_{\geq 0}$ and set $f = \prod_{j=1}^r f_j$. The divisors corresponding to f and f_j are denoted D and D_j respectively. Denote $\mathcal{O}_{X_{\mathbb{R}}}$ for the sheaf of real analytic functions on $X_{\mathbb{R}}$.

The real Bernstein-Sato ideal $B^a_{\mathbb{R},F}$ consists of all polynomials $b(s) \in \mathbb{R}[s]$ such that

$$b(s)F^s \in \mathscr{D}_{X_{\mathbb{R}}}[s]F^{s+a}$$

where $\mathscr{D}_{X_{\mathbb{R}}}[s]$ denotes the sheaf of analytic differential operators on $X_{\mathbb{R}}$. Note that taking the the real part of the Bernstein-Sato functional equation on the complexification of $X_{\mathbb{R}}$ shows that $B^a_{\mathbb{R},F}$ is a non-empty subset of the Bernstein-Sato ideal of F on the complexification of $X_{\mathbb{R}}$. The converse is also true provided all singularities of F on the complexification lie inside of $X_{\mathbb{R}}$.

Let $\mu : Y_{\mathbb{R}} \to X_{\mathbb{R}}$ be a resolution of singularities for f by real algebraic manifolds. Denote $g_j = f_j \circ \mu$ and K_μ for the relative canonical divisor of μ. The real mixed multiplier ideal sheaf associated to $\lambda \in \mathbb{R}^r_{\geq 0}$ is given by $\mathcal{J}_{\mathbb{R}}(F^\lambda) := \mu_* \mathcal{O}_{Y_{\mathbb{R}}}(K_\mu - \lfloor \sum_{j=1}^r \lambda_j \mu^* D_j \rfloor)$. This sheaf may be restated analytically as in (Saito, 2007, Proposition 1). For any $x \in X$

$$\mathcal{J}_{\mathbb{R},x}(F^\lambda)_x = \{\phi \in \mathcal{O}_{X,x} : |\phi| \prod_{j=1}^r |f_j|^{-\lambda_j} \text{ is integrable near } x\}.$$

Associated to λ is the region $\mathcal{R}_{\mathbb{R},F}(\lambda) := \{\lambda' \in \mathbb{R}^r_{\geq 0} : \mathcal{J}_{\mathbb{R}}(F^\lambda) \subseteq \mathcal{J}_{\mathbb{R}}(F^{\lambda'})\}$. The real jumping wall associated to λ is the intersection of the boundary of $\mathcal{R}_{\mathbb{R},F}(\lambda)$ with $\mathbb{R}^r_{>0}$. The RLCT-polytope $\mathrm{RLCT}(F)$ is the closure of $\mathcal{R}_{\mathbb{R},F}(0)$. The facets of the jumping wall are cut out by hyperplanes of the form $\sum_{j=1}^r \mathrm{ord}_E(g_j)s_j = k_E + c$ with $c \in \mathbb{Z}_{>0}$ and E an irreducible component of $\mu^* D$. The RLCT-polytope is cut out by hyperplanes of the form $\sum_{j=1}^r \mathrm{ord}_E(g_j)s_j = k_E + 1$. The RKLT_a-region is given by

$$\mathrm{RKLT}_a(F) = \{\lambda \in \mathbb{R}^r_{\geq 0} : \prod_{j=1}^r |f_j|^{-(\lambda_j - a_j)} \text{ is integrable near any } x \in D\}.$$

The following theorem now follows similarly to Theorem 3.4.2.

Theorem 3.4.4. *Suppose that* $\sum_{j=1}^r \mathrm{ord}_E(g_j)s_j = k_E + c$ *determines a facet of a real jumping wall which intersects* $\mathrm{RKLT}_a(F)$. *Then* $\sum_{j=1}^r \mathrm{ord}_E(g_j)s_j + k_E + c = 0$ *determines a component of* $Z(B^a_{\mathbb{R},F})$.

Proof. Let λ be a point of the facet of $\sum_{j=1}^r \mathrm{ord}_E(g_j)s_j = k_E + c$ inside of $\mathrm{RKLT}_a(F)$. Since λ is on a facet there must exist some $x \in D$ and $\phi \in \mathcal{O}_{X,x} \setminus \mathcal{J}_{\mathbb{R}}(F^s)_x$ such that

$$\int |\phi| \prod_{j=1}^r |f_j|^{-(\lambda_j - \varepsilon_j)} \psi dx < \infty, \qquad \int \prod_{j=1}^r |f_j|^{-(\lambda_j - a_j)} \psi dx < \infty.$$

for any $\varepsilon \in \mathbb{R}^r_{>0}$ and positive bump function ψ supported on a sufficiently small neighbourhood of x.

Pick some $b(s) \in B^a_{\mathbb{R},F}$ and take the support of ψ to be sufficiently small such that there exists some local differential operator P with $b(s)F^s = PF^{s+a}$. Assume that $s_j + a_j$ has real part strictly greater than the order of P for all j. Then $b(s)\prod_{j=1}^r |f_j|^{s_j} = P\prod_{j=1}^r |f_j|^{s_j+a_j}$ and $|f_j|^{s_j+a_j}$ has enough continuous partial derivatives to apply integration by parts. This yields that

$$b(s)\int \prod_{j=1}^r |f_j|^{s_j}|\phi|\psi dx = \int \prod_{j=1}^r |f_j|^{s_j+a_j}P^*|\phi|\psi dx.$$

View this as an equality of meromorphic functions in s to deduce that the equality holds for arbitrary $s \in \mathbb{R}^r$ provided both integrals are finite.

Now take $s = -\lambda + \varepsilon$ and let ε tend to zero from above. Then, by dominated convergence, the integral on the right hand side stays finite as ε tends to zero. On the other hand, by monotone convergence, the integral on the left hand side tends to infinity since ϕ is not in $\mathcal{J}_{\mathbb{R}}(F^s)_x$. This means that $b(s)$ vanishes on $(-\lambda_1, \ldots, -\lambda_r)$. Since the point λ on the facet and $b(s) \in B^a_{\mathbb{R},F}$ were arbitrary we conclude that $\sum_j \mathrm{ord}_E(g_j)s_j + k_E + c = 0$ determines a component of $Z(B^a_{\mathbb{R},F})$.

$\square$

Precisely as with Corollary 3.4.3 one gets the following corollary.

Corollary 3.4.5. *Suppose that* $\sum_{j=1}^r \mathrm{ord}_E(g_j)s_j = k_E + 1$ *determines a face of* $\mathrm{RLCT}(F)$. *If* $a_j \neq 0$ *and* $\mathrm{ord}_E(g_j) \neq 0$ *for some* j, *then* $\sum_j \mathrm{ord}_E(g_j)s_j + k_E + 1 = 0$ *determines a component of* $Z(B^a_{\mathbb{R},F})$.

Chapter 4

Moments of Holonomic Distributions

In order to do mathematics one has to pick some class of objects which is sufficiently specific that one can make interesting observations but is also sufficiently general. For instance, many real-world functions are analytic but representing a general analytic function requires an infinite amount of information. Hence, the representation of general analytic functions is not possible inside of a computer. On the other hand, a polynomial only has a finite amount of information and can be dealt with in a computer but these are not sufficiently general for most real-world applications. It was observed by Zeilberger (1990) that the class of holonomic functions provides an acceptable compromise. Many functions are holonomic and one only requires a finite amount of information to encode the corresponding differential equations.

An introduction to the literature on holonomic functions is provided in section 4.1. A class of distributions of probabilistic interest is discussed in section 4.2. We demonstrate how the holonomic toolbox may be applied to such probabilistic problems in section 4.3.

4.1 Holonomic functions

Let $\mathbb{k}$ denote a field of characteristic zero. The Weil algebra $D_n(\mathbb{k})$ is the $\mathbb{k}$-algebra found from $\mathbb{k}[x_1, \dots, x_n]$ found by adjoining new variables $\partial_1, \dots, \partial_n$ subject to the usual commutation relations

$$\partial_i x_j = x_j \partial_i + \delta_{ij}; \qquad \partial_i \partial_j = \partial_j \partial_i$$

where δ_{ij} denotes the Kronecker delta. One has the corresponding notions of a order filtration, graded objects, characteristic varieties and holonomic $D_n(\mathbb{k})$-modules precisely as in chapter 2.

An object which gives rise to a $D_n(\mathbb{k})$-module is said to be holonomic over $\mathbb{k}$ precisely when the corresponding module is so. For instance, a holonomic function over $\mathbb{C}$ is an analytic function $f : U \to \mathbb{C}$ on some open $U \subseteq \mathbb{C}^n$ such that f satisfies a system of differential equations $P_1(x, \partial)f = 0, \dots, P_k(x, \partial)f = 0$ with $D_n(\mathbb{C})/(P_1, \dots, P_k)$ holonomic over $D_n(\mathbb{C})$. For a parameter-dependent function $f_s(x)$ one can speak of holonomicity over $\mathbb{C}(s_1, \dots, s_r)$. These notions also apply to distributions.

Example 4.1.1. *Let $f_s(x) = \exp(sx^2)$ on $\mathbb{C}$ then $\partial_x f_s - s f_s = 0$. It follows that the characteristic variety is the zero section of $T^*\mathbb{C}(s)$. Hence, the parameter-dependent function $f_s(x)$ is holonomic over $\mathbb{C}(s)$.*

More generally, for any polynomial p on $\mathbb{C}^n$ with undetermined coefficients $s_1, \dots, s_r$ it holds that $f_s = \exp(p(x))$ is holonomic over $\mathbb{C}(s_1, \dots, s_r)$.

Example 4.1.2. *Consider a Dirac delta δ_s at the point $(s_1, \dots, s_n)$ in $\mathbb{C}^n$. Then $\delta_s \cdot (x_i - s_i) = 0$ for all $i = 1, \dots, n$ whence it follows that δ_s is holonomic over $\mathbb{C}(s_1, \dots, s_n)$.*

In the foregoing chapter we had a notion of relative holonomicity over $\mathbb{C}[s]$ with $s = (s_1, \dots, s_r)$. One can go from relative holonomicity to holonomicity over $\mathbb{C}(s)$ by use of the following result.

Lemma 4.1.3. *For any $\mathscr{D}_{\mathbb{C}^n}[s]$-module $\mathscr{M}$ with $\dim \mathrm{Ch}^{rel}\mathscr{M} \leq n + r$ it holds that the global sections of $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$ form a holonomic $D_n(\mathbb{C}(s))$-module.*

Proof. Consider a good filtration on $\mathscr{M}$ and equip $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$ with the induced filtration. Since localisation is an exact functor it holds that $\mathrm{gr}^{rel}\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s) \cong \mathrm{gr}(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s))$.

Suppose that $\mathrm{Ch}(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s))$ has dimension strictly greater than n as a variety over $\mathbb{C}(s)$. Let $\mathfrak{m}$ be a maximal ideal of the coordinate ring of $\mathrm{Ch}(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s))$. The maximal ideal $\mathfrak{m}$ corresponds to a prime ideal $\mathfrak{p}$ of the coordinate ring of $\mathrm{Ch}^{rel}\mathscr{M}$ which does not intersect $\mathbb{C}[s]\setminus\{0\}$. Moreover it follows from the assumption that $\dim \mathrm{Ch}(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s)) > n$ that $\mathfrak{p}$ has height $> n$.

Now the subvariety $V = Z(\mathfrak{p})$ of $\mathrm{Ch}^{rel}\mathscr{M}$ has codimension $> n$ and is not contained in any set of the form $Z(b(s))$ for $b(s) \in \mathbb{C}[s] \setminus \{0\}$. Since $\dim \mathrm{Ch}^{rel}\mathscr{M} \leq n + s$ it follows that $\dim V < r$.

Denote $\pi : \mathbb{C}^{n+r} \to \mathbb{C}^r$ for the projection map and observe that $\dim \mathrm{cl}\, \pi(V) \leq \dim(V) < r$ where cl denotes the closure in the Zariski topology. This contradicts the assumption that $V \nsubseteq Z(b(s))$ for any $b(s) \in \mathbb{C}[s] \setminus \{0\}$ and we conclude that $\dim \mathrm{Ch}(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s)) \leq n$.

□

Closure properties

The class of holonomic functions is closed under the usual operations. These closure properties are typically effective. This means that there are algorithms to compute the relations over $D_n(\Bbbk)$ satisfied by the output of the operation. Such algorithms often rely on the theory of Gröbner bases and have been implemented in software packages such as SINGULAR or Mathematica.

Theorem 4.1.4. *(Zeilberger, 1990, Proposition 3.1) Let f, g be holonomic functions over $\Bbbk$. If $f + g$ is defined it is also holonomic over $\Bbbk$.*

Theorem 4.1.5. *(Zeilberger, 1990, Proposition 3.2) Let f, g be holonomic functions over $\Bbbk$ such that $f \cdot g$ is defined. If $f \cdot g$ is defined it is also holonomic over $\Bbbk$. When $\Bbbk = \mathbb{R}$, $\Bbbk = \mathbb{C}$ or $\Bbbk = \mathbb{C}(s)$ the statement also holds if g is replaced by a distribution.*

Theorem 4.1.6. *(Stanley, 1980, Theorem 2.7) Let f and a be monovariate holonomic functions and algebraic functions over $\Bbbk$ respectively. If $f \circ a$ is defined it is also holonomic over $\Bbbk$.*

4.2 Subgaussian random variables

4.3 Recursion on moments

Let μ be a fixed holonomic distribution on $\mathbb{R}^n$ and assume that the moments

$$m_\alpha = \int_{\mathbb{R}^n} x^\alpha d\mu$$

exist for all $\alpha \in \mathbb{Z}^n_{\geq 0}$.

From here on out we will assume that $n = 1$ which is the main case of interest in the probabilistic context.

Proposition 4.3.1. *Suppose that μ is annihilated by $P \in \mathscr{D}^n_{\mathbb{C}}$. If $P = \sum_{ij} c_{ij} x^i \partial^j$ then it holds that*

$$\sum_{ij} c_{ij}(n+i)(n+i-1)\cdots(n+i-j+1)m_{n+i-j} = 0$$

for any $n \geq 0$.

Bibliography

Berthelot, P., Ferrand, D., Grothendieck, A., Jouanolou, J.-P., Jussila, O., Illusie, L., Kleiman, S., Raynaud, M., and Serre, J.-P. (2006). *SGA 6: Theorie des Intersections et Theoreme de Riemann-Roch*, volume 225. Springer.

Bjork, J.-E. (1979). Rings of differential operators. *North-Holland Math. Library*, 21.

Blanco, G. (2019). Yano's conjecture. *arXiv preprint arXiv:1908.05917*.

Borel, A. (1987). *Algebraic D-modules*, volume 2. Academic Press.

Brianccon, J. and Maisonobe, P. (2002). Remarques sur l'idéal de Bernstein associé à des polynômes. prépublication Univ. *Nice-Sophia Antipolis*, (650).

Budur, N. (2015). Bernstein-Sato Polynomials . *Lecture Notes, UPC Barcelona*. Accessed in January 2021 at `https://perswww.kuleuven.be/~u0089821/Barcelona/BarcelonaNotes.pdf`.

Budur, N., van der Veer, R., Wu, L., and Zhou, P. (2020). Zero loci of Bernstein-Sato ideals - II. *arXiv preprint arXiv:2001.05728v1*.

Budur, N., van der Veer, R., Wu, L., and Zhou, P. (2021). Zero loci of bernstein–sato ideals. *Inventiones mathematicae*, pages 1–28.

Deligne, P. (1977). SGA 4 1/2: Cohomologie étale. *Lecture Notes in Mathematics*, 569.

Dimca, A. (2004). *Sheaves in topology.* Springer Science & Business Media.

Ein, L., Lazarsfeld, R., Smith, K. E., Varolin, D., et al. (2004). Jumping coefficients of multiplier ideals. *Duke Mathematical Journal*, 123(3):469–506.

Gyoja, A. (1993). Bernstein-Sato's polynomial for several analytic functions. *Journal of Mathematics of Kyoto University*, 33(2):399–411.

Hotta, R. and Tanisaki, T. (2007). *D-modules, perverse sheaves, and representation theory*, volume 236. Springer Science & Business Media.

Igusa, J.-i. (2007). *An introduction to the theory of local zeta functions*, volume 14. American Mathematical Soc.

Kashiwara, M. (1976). B-functions and holonomic systems. *Inventiones mathematicae*, 38(1):33–53.

Kashiwara, M. (2003). *D-modules and microlocal calculus*, volume 217. American Mathematical Soc.

Kollár, J. (1997). Singularities of pairs. In *Proceedings of Symposia in Pure Mathematics*, volume 62, pages 221–288. American Mathematical Society.

Libgober, A. (2002). Hodge decomposition of alexander invariants. *manuscripta mathematica*, 107(2):251–269.

Lichtin, B. (1989). Poles of $|f(z,w)|^{2s}$ and roots of the b-function. *Arkiv för Matematik*, 27(1-2):283–304.

Maisonobe, P. (2016). Filtration Relative, l'Idéal de Bernstein et ses pentes. *arXiv preprint arXiv:1610.03354*.

Malgrange, B. (1985). Sur les images directes de D-modules. *manuscripta mathematica*, 50(1):49–71.

Milnor, J. (1968). *Singular Points of Complex Hypersurfaces.* Number 61. Princeton University Press.

Monteiro Fernandes, T. and Sabbah, C. (2016). Riemann-Hilbert correspondence for mixed twistor D-Modules. *arXiv*, pages arXiv–1609.

Oaku, T. (1997). An algorithm of computing b-functions. *Duke Mathematical Journal*, 87(1):115–132.

Sabbah, C. (1987). Proximité évanescente. I. La structure polaire d'un D-module. *Compositio Mathematica*, 62(3):283–328.

Sabbah, C. (2011). Introduction to the theory of D-modules. *Lecture Notes, Nankai.* Accessed in October 2020 at `http://www.cmls.polytechnique.fr/perso/sabbah/livres/sabbah_nankai110705.pdf`.

Saito, M. (2007). On real log canonical thresholds. *arXiv preprint arXiv:0707.2308.*

Stanley, R. P. (1980). Differentiably finite power series. *European journal of combinatorics*, 1(2):175–188.

van der Veer, R. (2021). On the support of relative d-modules. *arXiv preprint arXiv:2102.09476.*

Weibel, C. A. (1995). *An introduction to homological algebra.* Number 38. Cambridge university press.

Yano, T. (1982). Exponents of singularities of plane irreducible curves. *Sci. Rep. Saitama Univ. Ser. A*, 10(2):21–28.

Zeilberger, D. (1990). A holonomic systems approach to special functions identities. *Journal of computational and applied mathematics*, 32(3):321–368.