# Chapter 1

# The Behaviour of $\mathscr{A}_X^R$-Modules

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The classical approximation of the roots of the $b$-polynomial due to Kashiwara (1976) relies on a quotient module $\int \mathscr{M}/\mathscr{D}_X u$ being holonomic. This is no longer true in the multivariate case but a refined assumption, called relative holonomicity, due to Maisonobe (2016) still holds. This refinement works with $\mathscr{D}_X \times \mathbb{C}[s]$-modules whence one gets characteristic varieties inside $T^*X \times \mathbb{C}^p$. p?

Maybe also mentic the example Robir put on the white-board? Possibly ir the main body?

## 1.1 Modules over $\mathscr{A}_X^R$

### Basic Definitions and Properties

Let $X$ be a smooth complex irreducible algebraic variety of dimension $n$ and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain $R$ we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. It will also be convenient to use the abbreviation $\mathcal{O}_X^R := \mathcal{O}_{X \times \operatorname{Spec} R}$.

The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration. The associated graded objects are denoted by $\mathrm{gr}^{rel}$.

Similarly to the case of $\mathscr{D}_X$ in the first chapter that it holds that $\mathscr{A}_X^R$ is the sheaf of rings generated by $\mathcal{O}_X^R$ and $\Theta_X$ inside of $\mathcal{E}nd_{\underline{\mathbb{C}}}(\mathcal{O}_X^R)$. Giving a left $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X^R$-module $\mathscr{M}$ with $\Theta_X$-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections $f$ of $\mathcal{O}_X^R$ and $\xi$ of $\Theta_X$. Similarly, giving a right $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with $\Theta_X$-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections $f$ of $\mathcal{O}_X^R$ and $\xi$ of $\Theta_X$. Cite when C1 is written

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2). Probably cite C1 instead

**Proposition 1.1.1.** *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\mathrm{gr}^{rel}\mathscr{M}$ is coherent over $\mathrm{gr}^{rel}\mathscr{A}_X^R$. Such a filtration is called a good filtration.*

**Proposition 1.1.2.** *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\mathrm{Ch}^{rel}\mathscr{M}$.*

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over $R$ if $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

## Basic Operations

For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ and left $\mathscr{D}_X$-module $\mathscr{N}$ the tensor product $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ comes equipped with a right $\mathscr{A}_X^R$-module structure defined by

$$f \cdot (m \otimes n) = mf \otimes n; \qquad \xi \cdot (m \otimes n) = m\xi \otimes n - m \otimes \xi n$$

for any sections $f$ of $\mathcal{O}_X^R$ and $\xi$ in $\Theta_X$. The same definition applies for a $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X^R} \mathscr{N}$ whenever $\mathscr{N}$ is a left $\mathscr{A}_X^R$-module.

Similarly, given a left $\mathscr{D}_X$-module $\mathscr{L}$ and a left $\mathscr{A}_X^R$-module $\mathscr{N}$ a left $\mathscr{A}_X^R$-module structure on $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{L}$ is defined by

$$f \cdot (\ell \otimes n) = \ell \otimes fn; \qquad \xi \cdot (\ell \otimes n) = \xi\ell \otimes n + \ell \otimes \xi n$$

for any sections $f$ of $\mathcal{O}_X^R$ and $\xi$ in $\Theta_X$.

**Lemma 1.1.3.** *Let $\mathscr{M}, \mathscr{N}$ be right and left $\mathscr{A}_X^R$-modules respectively and let $\mathscr{L}$ be a left $\mathscr{D}_X$-module. Then there is a isomorphism of left $\mathscr{A}_X^R$-modules*

$$(\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}) \otimes_{\mathcal{O}_X^R} \mathscr{N} \cong \mathscr{M} \otimes_{\mathcal{O}_X^R} (\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}).$$

*Proof.* This is immediate by checking that the obvious bijection conserves the $\mathscr{A}_X^R$-module structure. Note that the only nontrivial check is the action of a section $\xi$ from $\Theta_X$. □

**Lemma 1.1.4.** *Let $\mathscr{N}$ be a left $\mathscr{A}_X^R$-module which is locally free as a $\mathcal{O}_X^R$-module. Consider $\mathscr{A}_X^R$ as a right $\mathscr{A}_X^R$-module, then $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$ is locally free as a right $\mathscr{A}_X^R$-module.*

*Proof.* Consider local coordinates $x_1, \ldots, x_n$ on $X$ and a local $\mathcal{O}_X^R$-basis $\{n_\beta\}_\beta$ for $\mathscr{N}$. Then $\{1 \otimes n_\beta\}_\beta$ will be a local $\mathscr{A}_X^R$-basis for $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

To see that this generates the $\mathscr{A}_X^R$-module note that $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ is a $\mathcal{O}_X^R$-basis set when $\alpha$ runs over all multi-indices in $\mathbb{Z}_{\geq 0}^n$. These sections can be recovered using the $\mathscr{A}_X^R$-action on the proposed generating set by induction on $|\alpha|$. Indeed, $\xi^\alpha \cdot (1 \otimes n_\beta)$ equals $\xi^\alpha \otimes n_\beta$ up to a element in the $\mathcal{O}_X^R$-span of $\{\xi^\gamma \otimes n_\beta\}_{|\gamma|<|\alpha|}$.

For the freedom, suppose there is a local $\mathscr{A}_X^R$-relation $\sum_\beta P_\beta \cdot 1 \otimes n_\beta = 0$ with some $P_\beta$ nonzero. This is of the form $\sum_{\alpha,\beta} f_{\alpha,\beta}\xi^\alpha \cdot 1 \otimes n_\beta = 0$ with the $f_{\alpha,\beta}$ sections of $\mathcal{O}_X^R$ not all equal to zero. Pick some multi-index $\mu \in \mathbb{Z}_{\geq 0}^n$ and of maximal degree such that $f_{\mu,\beta}$ is non-zero for some $\beta$. Then, rewriting $\sum_{\alpha,\beta} f_\alpha \xi^\alpha \cdot 1 \otimes n_\beta = 0$ in terms of the $\mathcal{O}_X^R$-basis $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ one finds a non-zero coeffiencient at $\xi^\eta \otimes n_\beta$ for some $\beta$ which is a contradiction. □

## 1.2 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.

introduction

### Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties, by abuse of notation we will also denote $\mu$ for the induced map from $Y \times \operatorname{Spec} R$ to $X \times \operatorname{Spec} R$.

A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}\mathscr{A}_X^R$. Here, similar definitions to section 1.1 are used to define the left $\mathscr{A}_Y^R$-module structure and the right $\mu^{-1}\mathscr{A}_X^R$-module structure is just the action on the second component.

**Definition 1.2.1.** *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the $j$-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript $\mu$ will be surpressed whenever there is no ambiguity.*

To compute the direct image $\int^j \mathscr{M}$ a resolution for the transfer bimodule $\mathscr{A}_{Y\to X}$ is required.

**Definition 1.2.2.** *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\operatorname{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\operatorname{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map $\delta$ given by*

$$\begin{aligned} m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto & \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k \\ & - \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \end{aligned}$$

The following lemma and it's proof are a generalisation of exercise 1.20 in Sabbah (2011) to the relative case.

**Lemma 1.2.1.** *The relative Spencer complex $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_X^R$ as left $\mathscr{A}_X^R$-module.*

*Proof.* Define a filtration on $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ by the complexes $F_k \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ which have term $F_{k-\ell}\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot $\ell$. This filtration induces the complexes $\operatorname{gr}_k^{rel} \operatorname{Sp}_X^\bullet(\mathscr{A}_Y^R)$ with term $\operatorname{gr}_{k-\ell}^{rel} \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot $\ell$.

In local coordinates $x_1, \ldots, x_n$ one finds that $\operatorname{gr}^{rel}\operatorname{Sp}_Y^\bullet := \oplus_k \operatorname{gr}^{rel}{}_k \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is the Koszul complex of $\mathcal{O}_Y^R[\xi_1, \ldots, \xi_n] = \operatorname{gr}^{rel}\mathscr{A}_Y^R$ with respect to $\xi_1, \ldots, \xi_n$. Since $\xi_1, \ldots, \xi_n$ form a regular sequence a standard result on Koszul complexes yields that $\operatorname{gr}^{rel}\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_Y^R$ as $\operatorname{gr}^{rel}\mathscr{A}_Y^R$-module.

Should I explain what a Koszul complex is?

On the other hand, it is immediate that $F_0 \operatorname{Sp}^\bullet(\mathscr{A}_Y^R) = \operatorname{gr}_0^{rel} \operatorname{Sp}^\bullet(\mathscr{A}_Y^R)$ is $\mathcal{O}_Y \otimes \mathcal{O}_R$ viewed as a complex. Hence, there is no contribution to $\operatorname{gr}^{rel} \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ from the terms of $k > 0$. That is to say that $\operatorname{gr}_k^{rel} \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is quasi-isomorphic to the zero complex for

Give reference to some book

$k > 0$. Hence, $F_0 \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R) \hookrightarrow \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a quasi-isomorphism by the exactness of the direct limit. It follows that $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a resolution of $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_R$. That the terms of $\operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ are locally free follows from lemma 1.1.4 after some minor adjustments in the statement and proof. □

Would be nice to give a reference, proof may be foun on stackexchange

Define the transfer Spencer complex as the complex of $(\mathscr{A}_Y^R, f^{-1}\mathscr{A}_X)$-bimodules given by $\operatorname{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \operatorname{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X}^R$. The following lemma and it's proof are direct generalisation of exercise 3.4 in Sabbah (2011) to the relative case.

**Lemma 1.2.2.** *The transfer Spencer complex* $\operatorname{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R)$ *is a resolution of* $\mathscr{A}_{Y\to X}^R$ *as a bimodule by locally free left* $\mathscr{A}_Y^R$*-modules.*

*Proof.* To see that the terms of the complex are locally free recall from lemma 1.1.3 the following isomorphisms of left $\mathscr{A}_Y^R$-modules

$$(\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X} \cong \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y^R} (\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}).$$

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Note that $\mathscr{A}_{Y\to X}^R$ is a locally free $\mathcal{O}_Y^R$-module since it is the pullback of a locally free module on $X \times \operatorname{Spec} R$. Combined with the fact that $\wedge^\ell \Theta$ is a locally free $\mathcal{O}_Y$-module this yields that $\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}$ is a locally free $\mathcal{O}_Y^R$-module. Hence lemma 1.1.3 is applicable and yields that the terms of the transfer Spencer complex are locally free $\mathscr{A}_Y^R$-modules.

That the transfer Spencer complex is a resolution of $\mathscr{A}_{Y\to X}^R$ follows from lemma 1.2.1 by using that $\mathscr{A}_{Y\to X}^R$ is a locally free and hence flat over $\mathcal{O}_Y^R$. □

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

**Corollary 1.2.3.** *It holds that* $\int = \boldsymbol{R}\mu_*(- \otimes_{\mathscr{A}_Y^R} \operatorname{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$.

**Lemma 1.2.4.** *Construction of global section in* $\int^j \mathscr{M}$.

**Theorem 1.2.5.** *Long exact sequence*

## Functorial Properties of the Direct Image

**Theorem 1.2.6.** *Let* $\mu : Z \to Y$ *and* $\nu : Y \to X$ *be morphisms of smooth algebraic varieties. If* $\mu$ *is proper then* $\int_{\nu\circ\mu} = \int_\nu \int_\mu$.

*Proof.* See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` □

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $\boldsymbol{D}_{qc}^{b,r}(\mathscr{A}_Y^R)$ the full subcategory of $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ consisting of those complexes of right $\mathscr{A}_Y^R$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\operatorname{Spec} R}$. Similarly for $\boldsymbol{D}_{coh}^{b,r}(\mathscr{A}_Y^R)$ with the cohomology being coherent $\mathscr{A}_Y^R$-modules.

**Theorem 1.2.7.** *Let* $\mu : X \to Y$ *be a morphism of nonsingular algebraic varieties. Then the direct image* $\int$ *takes* $\boldsymbol{D}_{qc}^{b,r}(\mathscr{A}_Y^R)$ *into* $\boldsymbol{D}_{qc}^{b,r}(\mathscr{A}_X^R)$*. Moreover, when* $\mu$ *is proper the direct image takes* $\boldsymbol{D}_{coh}^{b,r}(\mathscr{A}_Y^R)$ *into* $\boldsymbol{D}_{coh}^{b,r}(\mathscr{A}_X^R)$.

*Proof.* See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture18.pdf` □

## Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j \mathscr{M}$ in terms of $\mathrm{Ch}^{rel}\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara.

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The idea of the following proof is due to Malgrange (1985) in a $K$-theoretic context. We follow the exposition of Sabbah (2011) and replace it with the corresponding relative notions.

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

**Theorem 1.2.8.** *Let $\mathscr{M}$ be a coherent $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

$$\mathrm{Ch}^{rel}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

Note that the statement is local so, after replacing $X$ by some affine open, we may assume that $X \times \operatorname{Spec} R$ and $Y \times \operatorname{Spec} R$ are compact. The first step is to note that a similar inclusion is easy for the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. The direct image functor on $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules $\mathcal{M}$ is defined by $\int^j \mathcal{M} := \mathscr{H}^j(\boldsymbol{R}\mu_*(\mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R)$. Looking at the supports the following result is immediate.

**Lemma 1.2.9.** *For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ it holds that*

$$\operatorname{supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1} \operatorname{supp} \mathcal{M}\right).$$

Applying this lemma to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to show that $\operatorname{supp} \mathrm{gr}^{rel}\int^j \mathscr{M} \subseteq \operatorname{supp} \int^j \mathrm{gr}^{rel}\mathscr{M}$. This is proved in proposition 1.2.15. The main technical ingredient in the proof is the Rees modules associated to a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$.

**Definition 1.2.3.** *Let $z$ be a new variable. The Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

Given a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it follows that $\mathcal{R}\mathscr{M}$ is a coherent $\mathcal{R}\mathscr{A}_Y^R$-module similarly to proposition 1.1.1. The following isomorphisms of filtered modules are essential. They mean that the Rees module can be viewed as a parametrisation of various relevant modules.

$$\frac{\mathcal{R}\mathscr{M}}{(z-1)\mathcal{R}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}\mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z^\ell\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}.$$

Here $\mathrm{gr}_{[\ell]}^{rel}$ takes a filtered object and returns $\oplus_k F_k/F_{k-\ell}$. The first formula may be be used to find a corresponding filtered $\mathscr{A}_Y^R$-module for any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

The $j$th direct image of a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is the sheaf of $\mathcal{R}\mathscr{A}_X^R$-modules on $X \times \operatorname{Spec} R$ defined by $\int^j \mathcal{M} = \mathscr{H}^j \boldsymbol{R}\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R)$. Here the filtration on $\mathscr{A}_{Y\to X}^R$ is defined by $F_i\mathscr{A}_{Y\to X}^R = \mathcal{O}_X^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}F_i\mathscr{A}_X^R$. The direct image may be restricted to the category of graded Rees modules in which case it returns a graded Rees module. Coherence is preserved similarly to theorem 1.2.7.

Recall that a $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules on $Y \times \operatorname{Spec} R$ could be be viewed as a sheaf on $T^*Y \times \operatorname{Spec} R$ and is already equipped with a direct image. The Rees module viewpoint agrees with the earlier definition by the following lemma.

**Lemma 1.2.10.** *Consider a filterd $\mathscr{A}_Y^R$-module $\mathscr{M}$. Then viewing $\int \mathcal{R}\mathscr{M}/z\mathcal{R}\mathscr{M}$ with it's $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module structure as a sheaf on $T^*X \times \operatorname{Spec} R$ recovers the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module direct image $\int \mathrm{gr}^{rel}\mathscr{M}$. Viewing $\int \mathcal{R}\mathscr{M}/(z-1)\mathscr{M}$ as a $\mathscr{A}_X^R$-module recovers $\int \mathscr{M}$.*

*Proof.* Will be easier to write when producing sheaves on $T^*X \times \operatorname{Spec} R$ is written formally. Comes down to noting that tensoring a free resolution of $\mathscr{A}_{Y\to X}$ yields free resolution of something else. $\square$

It turns out that one can directly compare $\mathrm{gr}_{[\ell]}^{rel}\int^j \mathscr{M}$ and $\int^j \mathrm{gr}_{[\ell]}^{rel}\mathscr{M}$ when $\ell$ is large. Some care is required since since $\int^j \mathcal{R}\mathscr{M}$ may have $\mathbb{C}[z]$-torsion.

**Lemma 1.2.11.** *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, for sufficiently large $\ell$, the kernel of $z^\ell$ in $\int^j \mathcal{R}\mathscr{M}$ stabilises. For such $\ell$ the quotient $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is the $\mathcal{R}\mathscr{A}_X^R$-coherent module associated to a good filtration on $\int^j \mathscr{M}$.*

*Proof.* By $\int \mathcal{R}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}_X^R$ one gets that $\ker z^\ell$ locally stabilises. This is sufficient since $X \times \operatorname{Spec} R$ is assumed to be compact.

Now consider the short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z-1} \mathcal{R}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z-1} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z-1} \cdots .$$

Since $\int^{j+1} \mathcal{R}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module it follows that $z-1$ is injective whence $\int^j \mathcal{R}\mathscr{M}/(z-1)\int^j \mathcal{R}\mathscr{M} \cong \int^j \mathscr{M}$. This yields the desired result using that $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is $\mathbb{C}[z]$-torsion free and the isomorphism

$$\frac{\int^j \mathcal{R}\mathscr{M}}{(z-1)\int^j \mathcal{R}\mathscr{M}} \cong \frac{\int^j \mathcal{R}\mathscr{M}/\ker z^\ell}{(z-1)(\int^j \mathcal{R}\mathscr{M}/\ker z^\ell)}.$$

$\square$

From now on we equip $\int^j \mathscr{M}$ with the good filtation inherited from the Rees module's direct image.

**Lemma 1.2.12.** *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, if $\ell$ is sufficiently large, $\mathrm{gr}_{[\ell]}^{rel}\int^j \mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}_{[\ell]}^{rel}\mathscr{M}$.*

*Proof.* The short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \mathcal{R}\mathscr{M} \to \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \cdots.$$

Hence, $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M})$ and it remains to show that $\mathcal{R}\int^j \mathscr{M}/z^\ell\mathcal{R}\int^j \mathscr{M}$ is a quotient of $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$.

Let $\ell$ be sufficiently large so that lemma 1.2.11 yields a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong \mathcal{R}\int^j \mathscr{M}$. The map $z^\ell$ induces a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong z^\ell \int^j \mathcal{R}\mathscr{M}$. Therefore $z^\ell \int^j \mathcal{R}\mathscr{M}/z^{2\ell}\int^j \mathcal{R}\mathscr{M} \cong \mathcal{R}\int^j \mathscr{M}/z^\ell\mathcal{R}\int^j \mathscr{M}$. The desired quotient follows by applying the map $m \mapsto z^\ell m$ on $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$. □

The main remaining task is to relate these results to the desired case of $\ell = 1$.

**Definition 1.2.4.** *For any $\ell \geq 1$ the $G$-filtration on a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is defined by the decreasing sequence of $\mathrm{gr}_{[\ell]}^{rel}\mathscr{A}_Y^R$-submodules $G_j\mathcal{M} := z^j\mathcal{M}$.*

**Lemma 1.2.13.** *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ there is the a isomorphism of $\mathrm{gr}\,\mathscr{A}_Y^R$-modules*

$$\mathrm{gr}^G \mathrm{gr}_{[\ell]}^{rel}\mathscr{M} \cong (\mathrm{gr}^{rel}\mathscr{M})^\ell.$$

*Proof.* This follows from directly from the fact that $G_j\, \mathrm{gr}_{[\ell]}^{rel}\mathscr{M} = \oplus_k F_{k-j}\mathscr{M}/F_{k-\ell}\mathscr{M}$. □

**Lemma 1.2.14.** *Consider a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$. Then one has a isomorphism $\mathrm{gr}^G \int \mathcal{M} \cong \int \mathrm{gr}^G \mathcal{M}$ in $\boldsymbol{D}^{b,r}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$.*

*Proof.* Writing out the direct image functors the desired result is a isomorphism

$$\mathrm{gr}^G \int \boldsymbol{R}\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R) \cong \boldsymbol{R}\mu_*(\mathrm{gr}^G \mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R).$$

The proof of the commutation proceeds in two steps corresponding to the two derived functors.

From lemma 1.2.2 one deduces that $\mathscr{F}^\bullet := \mathcal{R}\,\mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)$ is a locally free resolution for $\mathcal{R}\mathscr{A}_{Y\to X}^R$. There is a $G$-filtration on this complex given by $z^j(\mathcal{M}\otimes_{\mathcal{R}\mathscr{A}_Y^R}\mathscr{F}^\bullet) = (z^j\mathcal{M})\otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. By the flatness of locally free sheaves and the short exact sequence $0 \to \oplus_j z^j\mathcal{M} \to \oplus_j z^{j-1}\mathcal{M} \to \mathrm{gr}^G \mathcal{M} \to 0$ it follows that $\mathrm{gr}^G(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) \cong (\mathrm{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. Further, by the argument in the proof of lemma 1.2.10 the complex of $\mathrm{gr}^G \mathscr{A}_Y^R$-modules $(\mathrm{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ can be viewed as a representative of $(\mathrm{gr}^G \mathcal{M}) \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R$.

Explain in more detail and mention filtrations

Denote $\mathcal{G}(-)$ for the functor which takes a sheaf complex and returns its Godement resolution. Flabby resolutions are acyclic for $\mu_*$ so the Godement resolution may be used to compute $\boldsymbol{R}\mu_*$. Moreover, since the terms of a Godement resolution are essentially direct sums of formal products of stalks, it is immediate that $z^i\mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(z^i\mathcal{N}^\bullet)$ and that $\mathrm{gr}^G \mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(\mathrm{gr}^G \mathcal{N}^\bullet)$ for any complex of right $\mu^{-1}\mathcal{R}\mathscr{A}_X^R$-modules $\mathcal{N}^\bullet$. Applying $\mu_*$ to these equalities and setting $\mathcal{N}^\bullet = \mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ yields the desired result. □

**Proposition 1.2.15.** *For a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it holds that*

$$\mathrm{supp}\,\mathrm{gr}^{rel}\int^j \mathscr{M} \subseteq \mathrm{supp}\int^j \mathrm{gr}^{rel}\mathscr{M}.$$

d spectral se- e result on- add good refer-

? Seems to d on prefer- but should lly matter what for the entials.

*Proof.* Let $\ell \geq 0$ be sufficiently large so that lemma 1.2.12 holds, that is to say that $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$. By lemma 1.2.13 it holds that $\mathrm{gr}^G \mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M} \cong (\mathrm{gr}^{rel}\int \mathscr{M})^\ell$. Since $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ it remains to show that the support of $\mathrm{gr}^G \int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ is a subset of the support of $\int^j \mathrm{gr}\,\mathscr{M}$.

This can be established with the spectral sequence associated of the $G$-filtered complex $\int \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$. Since the $G$-filtration is finite on $\mathrm{gr}^{rel}_{[\ell]}\mathscr{A}_X^R$-modules the associated spectral sequence abbuts by general results. To be precise the associated spectral sequence with terms $E^2_{pq} = \mathscr{H}^{p+q}\,\mathrm{gr}^G \int \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ abuts to $\mathrm{gr}^G \int \mathscr{M}$. By lemma 1.2.14 and lemma 1.2.13 it holds that $E^2_{pq} \cong (\int^{p+q} \mathrm{gr}\,\mathscr{M})^\ell$. It follows that $\mathrm{supp}\,\mathrm{gr}^G \int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ is a subset of the support of $\int \mathrm{gr}\,\mathscr{M}$ which completes the proof. $\square$

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