KU LEUVEN

FACULTY OF SCIENCES

Relative Holonomicity and Estimation of Bernstein-Sato Zero Loci

Alexander VAN WERDE

Supervisor: Prof. N. Budur

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Chapter 1

Categorical Preliminaries

This chapter contains some categorical preliminaries on the topic of derived category theory and spectral sequences.

Derived category theory allows one to measure the lack of exactness in a functor $F : \mathcal{A} \to \mathcal{B}$ by encoding error-terms in derived functors $R^iF : \mathcal{A} \to \mathcal{B}$. For instance the non-exactness of the tensor product may be measured by $\mathcal{T}or$-functors.

Spectral sequences were historically developed by Leray to compute the cohomology of the pushforeward of a sheaf. There is some overlap between derived category theory and spectral sequences. In particular the Grothendieck spectral sequence allows one to compute the derived functor of some composition $F \circ G$ based on the derived functors of F and G individually. This theorem is a essential technical ingredient in the proofs of ??.

The discussion of derived category theory in this chapter summarises the relevant parts of chapters 1,2 and 5 of Dimca (2004). The section on spectral sequences is based on chapter 5 of Weibel (1995).

1.1 Spectral Sequences

Fix an abelian category $\mathcal{A}$. Denote $C(\mathcal{A})$ for the category with complexes of objects in $\mathcal{A}$

$$X^\bullet : \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots .$$

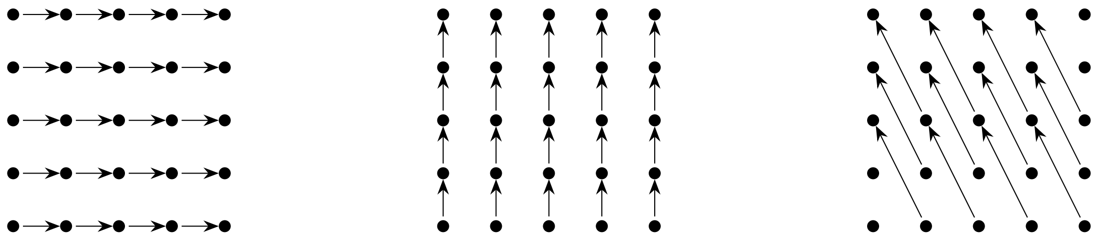
A double complex $E^{\bullet\bullet}$ gives rise to a total complex with terms $\mathrm{Tot}(E)^n = \oplus_{i+j=n} E^{ij}$. The motivating question behind spectral sequences is how the cohomology of the total complex may be computed.

First compute horizontal cohomology to get data E_1^{**}. By the commutativity of the double complex there are vertical differentials on E_1^{**} and one can compute the vertical cohomology to get E_2^{**}. Diagram chasing allows to construct higher-order differentials leading to the following notion.

Definition 1.1.1. *A cohomology spectral sequence starting at the a-th sheet consists of families of objects $\{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ for $r \geq a$ and maps $d_{pq}^r : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that*

(i) The maps d_{pq}^r are differentials in the sense that $d^r \circ d^r = 0$.

(ii) The $(r+1)$-st sheet is the cohomology of the r-th sheet $E_{pq}^{r+1} \cong \ker(d_{pq}^r)/\operatorname{Im}(d_{p-r,q+r-1}^r)$.



E_0^{**} E_1^{**} E_2^{**}

Definition 1.1.2. *A cohomology spectral sequence is said to be bounded if for each n there are finitely many non-zero terms E_a^{pq} with $p + q = n$.*

In a bounded complex there is, for each choice of p, q, a value r_0 such that $E_r^{pq} = E_{r+1}^{pq}$ for all $r \geq r_0$. This stable value is denoted E_∞^{pq}.

Definition 1.1.3. *A bounded spectral sequence is said to converge to a family of objects H^* if any H^n admits a finite filtration*

$$0 = F^s H^n \subseteq \cdots F^p H^n \subseteq F^{p+1} H^n \subseteq \cdots \subseteq F^t H^n = H^n$$

such that $E_\infty^{pq} \cong F^p H^{p+q} / F^{p-1} H^{p+q}$.

Observe that H^* is not necessarily uniquely identified by a convergent spectral sequence. The total complex in the motivating problem comes equipped with two filtrations. A vertical filtration $F^m \operatorname{Tot}(E)^n = \oplus_{p+q=n, p<m} E^{pq}$ and a similar horizontal filtration.

Definition 1.1.4. *A filtration of a complex $C_\bullet$ is a family of subcomplexes $\{F^m C^\bullet\}_{m \in \mathbb{Z}}$. The filtration is said to be exhaustive if $C^\bullet = \cup_m F^m C^\bullet$.*

Proposition 1.1.5. *(Weibel, 1995, Theorem 5.4.1.) A exhaustive filtration of a complex $C^\bullet$ determines a spectral sequence starting with $E_0^{pq} = F^p C^{p+q} / F^{p-1} C^{p+q}$ and $E_1^{pq} = H^{p+q} E_0^{p\bullet}$.*

Definition 1.1.6. *A filtration on a complex $C^\bullet$ is said to be bounded if, for each n, there are integers $s < t$ such that $F^s C^n = 0$ and $F^t C^n = C^n$.*

The following proposition may be used in the motivating problem to recover the cohomology of the total complex up to extension. This is to say that we know that the total cohomology $H^* = H^* \operatorname{Tot}(E)^\bullet$ admits a filtration such that $E_\infty^{pq} \cong F^p H^{p+q} / F^{p-1} H^{p+q}$.

Proposition 1.1.7. *(Weibel, 1995, Theorem 5.51.) Let $C^\bullet$ be a complex with a exhaustive bounded filtration. Then, the associated spectral sequence is bounded and converges to $H^*(C^\bullet)$.*

1.2 Derived Categories

The category $C(\mathcal{A})$ contains full subcategories $C^*(\mathcal{A})$ with $* \in \{+, -, b\}$ denoting that the complexes in $\mathcal{A}$ are bounded below, above or bounded on both sides respectively. For example $C^+(\mathcal{A})$ may contain complexes of the form $\cdots \to 0 \to \cdots \to X^{-1} \to X^0 \to \cdots$. For a complex $X^\bullet$ and $k \in \mathbb{Z}$ one has a shifted complex $X^\bullet[k]$ with $(X^\bullet[k])^s = X^{k+s}$. Further, denote $\operatorname{Hom}^k(X^\bullet, Y^\bullet) := \operatorname{Hom}(X^\bullet, Y^\bullet[k])$ for the chain maps that change the grading by k.

Definition 1.2.1. *Two complex morphisms $u, v : X^\bullet \to Y^\bullet$ are called homotopic if there exists $h \in \mathrm{Hom}^{-1}(X^\bullet, Y^\bullet)$ such that $u - v = d_Y h + h d_X$. This may be denoted $u \sim v$.*

Definition 1.2.2. *A morphism $u : X^\bullet \to Y^\bullet$ of complexes in $C^*(\mathcal{A})$ is called a quasi-isomorphism if the induced morphism in cohomology $H^k(u) : H^k(X^\bullet) \to H^k(Y^\bullet)$ is a isomorphism for all k.*

The idea behind the following definition is to retain the same objects as $C^*(\mathcal{A})$ but turn quasi-isomorphisms into isomorphisms. The technicalities may be found in chapter 8 of Deligne (1977).

Definition 1.2.3. *The derived category $D^*(\mathcal{A})$ is defined as the category obtained from $C^*(\mathcal{A})$ by localising with respect to the multiplicative system formed by the quasi-isomorphisms.*

This definition can be made more concrete provided the category has enough injectives.

Definition 1.2.4. *A abelian category $\mathcal{A}$ has enough injectives if for any object X in $\mathcal{A}$ there is an exact sequence $0 \to X \to I$ in $\mathcal{A}$ with I injective.*

Definition 1.2.5. *Let $\mathcal{A}$ be a abelian category. The homotopical category of complexes $K^*(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $C^*(\mathcal{A})$ and as morphisms*

$$\mathrm{Hom}_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet) := \mathrm{Hom}_{C^*(\mathcal{A})}(X^\bullet, Y^\bullet)/\sim .$$

Observe that two homotopic maps induce the same morphism in cohomology. It follows that there is a well-defined functor $p^*_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$.

Proposition 1.2.6. *(Dimca, 2004, Proposition 1.3.10.) Let $\mathcal{A}$ be a abelian category with enough injectives and denote $I(\mathcal{A})$ for the full subcategory of injective objects. Then the natural functor*

$$p^+_{\mathcal{A}} : K^+(I(\mathcal{A})) \to D^+(\mathcal{A})$$

is a equivalence of categories.

By passing to the opposite categories one gets a similar theorem in categories with enough projectives for $D^-(\mathcal{A})$.

1.3 Triangulated Categories

The categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ remain additive but may fail to be exact. In particular, the notion of short exact sequences no longer makes sense. Instead, $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ may be viewed as triangulated categories which is to say that they come equipped with a notion of exact triangles.

Definition 1.3.1. *Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. The mapping cone of u is the complex in $C^*(\mathcal{A})$ given by*

$$C^\bullet_u := Y^\bullet \oplus (X^\bullet[1])$$

with $d_u(y, x) = (dy + u(x), -dx)$.

The concept of a mapping cone originated in a construction from algebraic topology which explains the name. Observe that the mapping cone gives rise to a triangle

$$T_u : X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{q} C_u^\bullet \to X^\bullet[1]$$

which may be denoted more intuitively as

$$\begin{tikzcd} X^\bullet \arrow[rr, "u"] & & Y^\bullet \arrow[dl, "q"] \\ & C_u^\bullet \arrow[ul, "+1"] & \end{tikzcd}$$

The triangles T_u may be used to encode short exact sequences.

Proposition 1.3.2. *(Dimca, 2004, Proposition 1.1.23.) Given a short exact sequence in* $C^*(\mathcal{A})$

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

there exists a quasi-isomorphism $m : C_u^\bullet \to Z^\bullet$ *with* $m \circ q = v$.

This shows that a short exact sequence induces a triangle isomorphic to a standard triangle T_u in $D^*(\mathcal{A})$. Further evidence that the triangles T_u behave like short exact sequences is given by the following result.

Proposition 1.3.3. *(Dimca, 2004, Lemma 1.1.20, Proposition 1.1.21) Let* $u : X^\bullet \to Y^\bullet$ *be a morphism in* $C^*(\mathcal{A})$.

(i) The composition of any two consecutive maps in T_u *is homotopic to* 0.

(ii) The triangle T_u *induces a long exact sequence in cohomology*

$$\cdots \to H^k(X^\bullet) \xrightarrow{u} H^k(Y^\bullet) \to H^k(C_u^\bullet) \xrightarrow{\delta} H^{k+1}(X^\bullet) \to \cdots$$

where the connecting morphism δ *comes from the map* $C_u^\bullet \to X^\bullet[1]$.

Further investigation of the properties of T_u gives rise to the concept of a triangulated category. These definitions and properties are pleasant in their own right so we go into some detail.

The distinguished triangles $\mathcal{T}$ in $K^*(\mathcal{A})$ or $D^*(\mathcal{A})$ are the family of triangles which are isomorphic to a triangle of the form T_u. Observe that these categories have a shift functor T given by $TX^\bullet = X^\bullet[1]$.

Definition 1.3.4. *An additive category* $\mathcal{D}$ *equipped with a self-equivalence* T *and a family of distinguished triangles* $\mathcal{T}$ *is called a triangulated category if the following axioms are satisfied.*

(Tr1) Any triangle isomorphic to a distinguish triangle is distinguished. For any object X *the triangle* $X \xrightarrow{\mathrm{Id}} X \to 0 \to TX$ *is distinguished. Any morphism* $u : X \to Y$ *is part of some distinguished triangle* $X \xrightarrow{u} Y \to Z \to TX$.

(Tr2) A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ *is distinguished if and only if the triangle* $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{Tu} TY$ *is distinguished.*

(Tr3) A commutative diagram of the following form whose rows are distinguished triangles gives rise to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & TX \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & TA
\end{tikzcd}$$

(Tr4) For any triple of distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{x} A \longrightarrow TX$$

$$Y \xrightarrow{v} Z \longrightarrow B \xrightarrow{y} TY$$

$$X \xrightarrow{v \circ u} Z \longrightarrow C \longrightarrow TX$$

there is a distinguished triangle

$$A \xrightarrow{a} C \xrightarrow{b} B \xrightarrow{(Tx)\circ y} TA$$

such that (id_X, v, a) *and* (u, id_Z, b) *are morphisms of triangles.*

Proposition 1.3.5. *(Dimca, 2004, Proposition 1.2.4.) Let* $\mathcal{A}$ *be a abelian category. Then* $K^*(\mathcal{A})$ *and* $D^*(\mathcal{A})$ *are triangulated categories.*

A triangle $X \to Y \to Z \to TX$ will also be denoted $X \to Y \to Z \xrightarrow{+1} X$ and $T^m X$ may be denoted with $X[m]$. Now the data of the final axiom can be organised as follows. Correspondingly, $(Tr4)$ is also referred to as the octahedral axiom.

$$\begin{tikzcd}
& C \arrow[dr] \arrow[ddl] & \\
A \arrow[ur] \arrow[d] & & B \arrow[ll] \arrow[ddl] \\
X \arrow[rr, "v \circ u"] \arrow[dr, "u"'] & & Z \arrow[u] \arrow[uul] \\
& Y \arrow[ur, "v"'] \arrow[uul] &
\end{tikzcd}$$

Definition 1.3.6. *Let* $\mathcal{D}$ *be a triangulated category. A subcategory* $\mathcal{C}$ *of* $\mathcal{D}$ *is said to be stable under extensions if any distinguished triangle in* $\mathcal{D}$ *with two vertices in* $\mathcal{C}$ *also has its third vertex in* $\mathcal{D}$*.*

Definition 1.3.7. *Let* $\mathcal{C}$ *be a full additive subcategory of a triangulated category* $\mathcal{D}$*. One calls* $\mathcal{C}$ *is a triangulated subcategory if* $\mathcal{C}$ *is stable under extensions and* $T\mathcal{C} \subseteq \mathcal{C}$*.*

Definition 1.3.8. *Let* $\mathcal{D}$ *be a triangulated category and* $\mathcal{A}$ *a abelian category. An additive functor* $F : \mathcal{D} \to \mathcal{A}$ *is a cohomological functor if for any distinguished triangle in* $\mathcal{D}$

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{A}$. If F is a cohomological functor one sets $F^i = F \circ T^i$.

The family of functors F^i is conservative if for any distinguished triangle

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced long sequence

$$\cdots \to F^i(X) \to F^i(Y) \to F^i(Z) \to F^{i+1}(X) \to \cdots$$

is exact.

The key example for the above definition is given by the cohomological functor $H^0 : K^*(\mathcal{A}) \to \mathcal{A}$ and the conservative system of functors H^i.

Definition 1.3.9. *Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories. A functor $F : \mathcal{D} \to \mathcal{D}'$ is called a functor of triangulated categories if it is compatible with the shift functor and transforms distinguished triangles in $\mathcal{D}$ into distinguished triangles of $\mathcal{D}'$.*

1.4 Derived Functors

Given abelian categories $\mathcal{A}, \mathcal{B}$ and a functor of triangulated categories $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ one may wonder if there is a natural lift to the derived categories.

Definition 1.4.1. *Let F be as above. The right derived functor of F is a initial couple (R^*F, ξ_F) consisting of a functor of triangulated categories $R^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : p_B \circ F \to R^*F \circ p^*_{\mathcal{A}}$. By initial it is meant that for any other such couple (G, ζ) there is a unique natural transformation $\eta : R^*F \to G$ such that $\zeta = (\eta \circ p^*_{\mathcal{A}}) \circ \xi_F$.*

The dual notion is a left derived functor. This is a final couple (L^*F, ξ_F) consisting of a functor of triangulated categories $L^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : L^*F \circ p^*_{\mathcal{A}} \to p_B \circ F$. It is clear that, if a derived functor exists, it is unique up to unique isomorphism.

There are general theorems on the existence of derived functors which may be found in chapter 1 of Dimca (2004). The following will be sufficient for our applications.

Theorem 1.4.2. *(Dimca, 2004, Remark 1.3.15.) Consider a functor of triangulated categories $F : K^+(\mathcal{A}) \to K(\mathcal{B})$. If $\mathcal{A}$ has enough injectives and F is additive then the right derived functor R^+F exists.*

By dualising, a similar theorem applies to $F : K^-(\mathcal{A}) \to K(\mathcal{B})$ for the existence of L^-F in categories with enough projectives.

The main use of derived functors is to fix a lack of exactness in F. Recall from proposition 1.3.2 that a short-exact sequence in $C^+(\mathcal{A})$ induces a distinguished triangle in $D^+(\mathcal{A})$. Since R^+F is a functor of triangulated categories it will transform this distinguished triangle into a distinguished triangle of $D(\mathcal{B})$. Further, there is a associated long exact sequence whose higher-order terms measure to what degree the original functor failed to be exact.

Definition 1.4.3. *Let $F : K^*(\mathcal{A}) \to K(B)$ be a functor of triangulated categories such that R^*F exists. For any $n \in \mathbb{Z}$ one defines $R^nF : \mathcal{A} \to \mathcal{B}$ to be the composition*

$$\mathcal{A} \xrightarrow{\iota} D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

where ι sends a object to the chain complex with a single non-trivial term. Similarly, one defines $\mathbb{R}^nF : D^*(\mathcal{A}) \to \mathcal{B}$ as the composition

$$D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}.$$

Proposition 1.4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories. Suppose that the derived functor R^*F of the induced functor $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ exists. Then, for any short exact sequence in $\mathcal{A}$*

$$0 \to X \to Y \to Z \to 0$$

there is a long exact sequence in $\mathcal{B}$

$$\cdots \to R^iF(X) \to R^iF(Y) \to R^iF(Z) \to R^{i+1}F(X) \to \cdots .$$

Proof. This is immediate by R^*F being a functor of triangulated categories and the fact that the cohomology H^i forms a conservative system of functors.

□

In the situation of theorem 1.4.2 the derived functor can be computed explicitly. Pick some object $X^\bullet$ in $D^+(\mathcal{A})$. By proposition 1.2.6 there is a quasi-isomorphism $X^\bullet \to I^\bullet$ for some complex of injective objects $I^\bullet$. Then one has explicitly

$$R^+F(X^\bullet) \cong p_{\mathcal{B}} \circ F(I^\bullet).$$

Further, if F is exact one has that $F(I^\bullet)$ is quasi-isomorphic to $F(X^\bullet)$ whence $R^+F(X^\bullet)$ is isomorphic to $p_{\mathcal{B}} \circ F(X^\bullet)$.

In practice, it is often difficult to find a concrete injective resolution.

Definition 1.4.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A object X in $\mathcal{A}$ is F-acyclic if $R^iF(X) = 0$ for all $i \geq 1$.*

Computation derived functors can also be done using F-acyclic resolutions. One can show that injective objects are F-acyclic for any left-exact functor. Hence, this generalises the earlier computations.

Proposition 1.4.6. *(Dimca, 2004, Theorem 1.3.18.) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive functors between abelian categories with enough injective objects. Suppose that F is left-exact and that G transforms injective objects into F-acyclic objects, then there is an isomorphism*

$$R^+(F \circ G) \cong R^+F \circ R^+G.$$

The following theorem is known as the Grothendieck spectral sequence and will be used extensively in **??**.

Theorem 1.4.7. *(Dimca, 2004, Theorem 1.3.19.) Let F, G be as in the previous proposition. Then, for any object X of $\mathcal{A}$, there is a spectral sequence*

$$E_2^{pq} = R^p F(R^q G(X))$$

converging to $R^{p+q}(F \circ G)(X)$.

Proof. A modification of the proof is discussed in detail in ??.

The main idea is to consider the double complex $F(I^\bullet) \to J^{\bullet\bullet}$ provided by the dual of the following lemma. The total complex of $G(J^{\bullet\bullet})$ is equipped with two filtrations, a vertical and a horizontal filtration, comparing the spectral sequences from proposition 1.1.7 yields the desired result.

□

Proposition 1.4.8. *(Weibel, 1995, Lemma 5.7.2.) Suppose $\mathcal{A}$ admits enough projectives. Then, for any complex $X^\bullet$ there exists a lower half-plane double complex $P^{\bullet\bullet}$ of projective objects such that*

(i) There is a map $P^{0,\bullet} \to X^\bullet$ such that $P^{\bullet,q} \to X^q$ is a projective resolution for every p.

(ii) If $X^q = 0$ the corresponding column $P^{\bullet,q}$ is zero.

(iii) The horizontal cocycles, coboundaries and cohomology on $P^{\bullet,q}$ form projective resolutions for the q-th cocycles, coboundaries and cohomology of $X^\bullet$ respectively.

For later use on the structure of the Cartan-Eilenberg resolution we remark that the columns $P^{\bullet,q}$ are found as direct sums of projective resolutions for the boundaries and cohomology at level q and $q + 1$.

We conclude this section by considering a few important examples of derived functors which will be used later on.

Let X be a topological space equipped with a sheaf of rings $\mathcal{R}_X$ which need-not be commutative. The corresponding categories of complexes of left or right modules are denoted $C^{*,\ell}(\mathcal{R}_X)$ and $C^{*,r}(\mathcal{R}_X)$ respectively. Similarly, the category of complexes of bimodules is denoted $C^{*,\ell r}(\mathcal{R}_X)$. Using theorem 1.4.2 one can establish that the global sections functor $\Gamma(X, -)$ on $C^{+,*}(\mathcal{R}_X)$ has a derived functor $R^+\Gamma(X, -)$. The cohomology of a sheaf of modules is given by the functors $H^k(X, -) := R^k\Gamma(X, -)$ and the hypercohomology of a complex of modules is given by the functors $\mathbb{H}^k(X, -) := \mathbb{R}^k\Gamma(X, -)$.

Cohomology measures the failure of sections to be global. Correspondingly, acyclic objects are given by sheaves which have no such failure.

Definition 1.4.9. *A sheaf of $\mathcal{R}_X$-modules $\mathscr{F}$ is called flabby if for any open $U \subseteq X$ the restriction morphism $\rho_U^X : \mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.*

Proposition 1.4.10. *(Dimca, 2004, Proposition 2.1.8.) If $\mathscr{F}$ is a flabby sheaf of $\mathcal{R}_X$-modules then $\mathscr{F}$ is $\Gamma(X, -)$-acyclic.*

Let $f : Y \to X$ be a continuous map between topological spaces. The direct image of a sheaf $\mathcal{S}$ on Y is the sheaf $f_*\mathcal{S}$ on X defined by

$$(f_*\mathcal{S})(U) = \mathcal{S}(f^{-1}(U)).$$

Suppose that Y, X are equipped with sheaves of rings $\mathcal{R}_Y, \mathcal{R}_X$ respectively and that $f_*\mathcal{R}_Y$ is a $\mathcal{R}_X$-algebra. Then the direct image yields a functor from the category of $\mathcal{R}_Y$-modules to the category of $\mathcal{R}_X$-modules. This is a left-exact functor so the derived functor may be computed by injective resolutions. One can verify that flabby sheaves are f_*-acyclic so that flabby resolutions may also be used in the computations.

A classical example of a non-exact functor is given by the tensor product. This may be considered as a bifunctor

$$\otimes_{\mathcal{R}_X} : C^{-,\ell}(\mathcal{R}_X) \times C^{-,r}(\mathcal{R}_X) \to C^{-,\ell r}(\mathcal{R}_X)$$

where $(\mathscr{M}^\bullet \otimes_{\mathcal{R}_X} \mathscr{N}^\bullet)^n = \oplus_{i+j=n} \mathscr{M}^i \otimes_{\mathcal{R}_X} \mathscr{N}^j$ with differentials defined at $\mathscr{M}^i \otimes_{\mathcal{R}_X} \mathscr{N}^j$ by $d(m \otimes n) = dm \otimes n + (-1)^i m \otimes dn$. If it exists, the left-derived functor is denoted

$$\otimes^L_{\mathcal{R}_X} : D^{-,\ell}(\mathcal{R}_X) \times D^{-,r}(\mathcal{R}_X) \to D^{-,\ell r}(\mathcal{R}_X).$$

This yields $\mathcal{T}or$-sheaves $\mathcal{T}or_k^{\mathcal{R}_X}(X^\bullet, Y^\bullet) = H^{-k}(X^\bullet \otimes^L_{\mathcal{R}_X} Y^\bullet)$.

A similar procedure applies to the $\mathcal{H}om_{\mathcal{R}_X}$-bifunctor which is defined by $\mathcal{H}om^n_{\mathcal{R}_X}(\mathscr{M}^\bullet, \mathscr{N}^\bullet) = \prod_{j \in Z} \mathcal{H}om_{\mathcal{R}_X}(\mathscr{M}^j, \mathscr{N}^{n+j})$ with the differentials on $\mathcal{H}om^n_{\mathcal{R}_X}(M^\bullet, N^\bullet)$ given by $d\varphi = d_N \circ \varphi - (-1)^n \varphi \circ d_M$. If the right-derived functor exists it is denoted

$$R\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-) : D^{-,\ell}(\mathcal{R}_X)^{opp} \times D^{+,\ell r}(\mathcal{R}_X) \to D^r(\mathcal{R}_X).$$

This yields the $\mathcal{E}xt$-sheaves $\mathcal{E}xt^n_{\mathcal{R}_X}(M^\bullet, N^\bullet) = R^n\mathcal{H}om^\bullet_{\mathcal{R}_X}(M^\bullet, N^\bullet)$.

1.5 t-structures

A generalisation of positive and negatively supported complexes is given by the concept of a t-structure.

Definition 1.5.1. *A t-structure on a triangulated category $\mathcal{D}$ consists of two strictly full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ such that, setting $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, the following properties hold.*

(i) It holds that $\mathcal{D}^{\leq 0}$ is a subcategory of $\mathcal{D}^{\leq 1}$ and $D^{\geq 1}$ is a subcategory of $\mathcal{D}^{\geq 0}$.

(ii) For any objects X in $\mathcal{D}^{\leq 0}$ and Y of $\mathcal{D}^{\geq 1}$ it holds that $\mathrm{Hom}(X, Y) = 0$.

(iii) For any object X of $\mathcal{D}$ there is a distinguished triangle

$$A \to X \to B \xrightarrow{+1} A$$

with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

Definition 1.5.2. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure.*

In the motivating case of $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ the heart of the t-structure recovers the original abelian category $\mathcal{A}$.

Proposition 1.5.3. *(Dimca, 2004, Proposition 5.1.2.) The heart $\mathcal{C}$ of a t-structure is an abelian category which is stable by extensions.*

Observe that $D^*(\mathcal{A})$ comes equipped with truncation functors $\tau_{\leq m} : D^*(\mathcal{A}) \to D^-(\mathcal{A})$ which sends a complex $X^\bullet$ to

$$\tau_{\leq m} X^\bullet : \cdots \to X^{m-1} \to \ker d \to 0 \to 0 \to \cdots$$

and similarly truncation functors $\tau_{\geq m} : D^*(\mathscr{A}) \to D^+(\mathscr{A})$ are defined by

$$\tau_{\geq m} X^\bullet : \cdots \to 0 \to 0 \to \operatorname{coim} d \to X^{m+1} \to \cdots .$$

This generalises to t-structures.

Proposition 1.5.4. *(Dimca, 2004, Proposition 5.1.4.) Let $\mathcal{D}$ be a triangulated category with a t-structure. Then the inclusion of $\mathcal{D}^{\leq n}$ in $\mathcal{D}$ has a right adjoint functor $\tau_{\leq n}$. Similarly, the inclusion of $\mathcal{D}^{\geq n}$ in $\mathcal{D}$ has a left adjoint $\tau_{\geq n}$.*

Observe that in the example of $D^*(\mathcal{A})$ one has that $\tau_{\geq 0}\tau_{\leq 0}X^\bullet$ is the complex with a single entry $H^0(X^\bullet)$. This generalises to t-structures by viewing ${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0}$ as a functor from $\mathcal{D}$ to it's heart $\mathcal{C}$. Further, let ${}^tH^i := {}^tH^0 \circ T^i$.

Definition 1.5.5. *A t-structure is said to be non-degenerated if $\cap \mathcal{D}^{\leq n} = \cap \mathcal{D}^{\geq n} = \mathrm{Null}$ where Null denotes the family of objects which are isomorphic to the zero object in $\mathcal{D}$.*

Proposition 1.5.6. *(Dimca, 2004, Proposition 5.1.6.) Let $\mathcal{D}$ be a triangulated category with a t-structure. Then ${}^tH^0 : \mathcal{D} \to \mathcal{C}$ is a cohomological functor.*

Proposition 1.5.7. *(Dimca, 2004, Proposition 5.1.7.) Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then the system of functors ${}^tH^i$ is conservative.*

Proposition 1.5.8. *(Dimca, 2004, Proposition 5.1.7.) Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then $X \in \mathcal{D}^{\leq 0}$ if and only if ${}^tH^i(X) = 0$ for $i > 0$. Similarly $X \in \mathcal{D}^{\geq 0}$ if and only if ${}^tH^i(X) = 0$ for $i < 0$.*

Definition 1.5.9. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures. A functor of triangulated categories $F : \mathcal{D}_1 \to \mathcal{D}_2$ is called left or right t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ or $F(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$ respectively. The functor F is called t-exact if it is left and right t-exact.*

Definition 1.5.10. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. The perverse functor ${}^pF$ associated to F is the induced functor on the hearts ${}^pF ={}^t H^0 \circ F \circ j_1$ where j_1 denotes the inclusion functor $j_1 : \mathcal{C}_1 \to \mathcal{D}_1$.*

Proposition 1.5.11. *(Dimca, 2004, Proposition 5.1.9.) Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a t-exact functor of triangulated categories. Then F sends the heart $\mathcal{C}_1$ into the heart $\mathcal{C}_2$ and the induced functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is naturally isomorphic to ${}^pF$.*

Chapter 2

$\mathscr{D}_X$-modules

The goal of this chapter is to summarise some of the results and definitions which are common knowledge in the field of $\mathscr{D}_X$-modules.

The basic definitions of $\mathscr{D}_X$-module theory are given in section 2.1. The theory builds up to the Riemann-Hilbert correspondence in section 2.2 which states in general terms that a system of differential equations is equivalent to it's solutions. This result is powerful because it yields a connection between algebraic geometry and topology. A particular instantiation of this correspondence is the connection between Bernstein-Sato polynomials and monodromy discussed in section 2.3. Finally, we include the estimation of the roots of Bernstein-Sato polynomials due to Kashiwara and Lichtin.

Detailed treatments of the theory of $\mathscr{D}_X$-modules may be found in Bjork (1979), Kashiwara (2003), Hotta and Tanisaki (2007) or Borel (1987).

2.1 Sheaf of Differential Operators

Let X denote a smooth algebraic variety or a complex manifold.

Definition 2.1.1. *The sheaf of differential operators $\mathscr{D}_X$ is the subsheaf of rings in $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the vector fields Θ_X.*

Observe that $\mathscr{D}_X$ is a sheaf of non-commutative rings. Given local coordinates $x_1, \ldots, x_n$ on X it holds that

$$\partial_i x_j - x_j \partial_i = \delta_{ij}$$

where δ denotes the Kronecker delta.

Lemma 2.1.2. *(Hotta and Tanisaki, 2007, Proposition 1.4.6., Theorem 4.1.2) For any $x \in X$ the stalk $\mathscr{D}_{X,x}$ is left and right Noetherian. Moreover, in the algebraic case $\mathscr{D}_X$ is a left and right Noetherian sheaf of rings.*

Giving a left $\mathscr{D}_X$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\ m$ for any sections f of $\mathcal{O}_X$ and ξ of Θ_X. Similarly, giving a right $\mathscr{D}_X$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\ \xi(f)$ for any sections f of $\mathcal{O}_X$ and ξ of Θ_X.

Translation between left and right-modules is possible. Denote ω_X for the sheaf of top-level differential forms. Then ω_X comes equipped with the structure of a right $\mathscr{D}_X$-module where vector fields act by the Lie derivative.

For any left $\mathscr{D}_X$-module $\mathscr{M}$ a right $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X} \omega_X$ may be defined by

$$m \otimes \omega \cdot \xi = m \otimes \omega\xi - \xi m \otimes \omega.$$

For any right $\mathscr{D}_X$-module $\mathscr{M}$ a left $\mathscr{D}_X$-module structure on $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathscr{M})$ may be defined by

$$(\xi \cdot \varphi)(\omega) = \varphi(\omega \cdot \xi) - \varphi(\omega) \cdot \xi.$$

The following lemma follows by a direct computation.

Lemma 2.1.3. *The functor $- \otimes_{\mathcal{O}_X} \omega_X$ is a equivalence of categories with pseudoinverse $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, -)$.*

Example 2.1.4. *Consider the differential equation defining the square root function $f(z) = z^{-1/2}$*

$$zf'(z) - 1/2 f(z) = 0.$$

The use of f to describe the differential equation is somewhat arbitrary. For instance take $g(z) = \exp(z) f(z)$ which satisfies the differential equation

$$2zg'(z) - (2z+1)g(z) = 0.$$

The statement that non-trivial solutions of the differential equation for $f(z)$ can not be global on $\mathbb{C}^\times$ is equivalent to the same statement for the solutions to the differential equation for $g(z)$.

Thus one is led to the following question. Is it possible to present a differential equation without having to make some arbitrary choice of function f to describe it? This can indeed be accomplished by use of $\mathscr{D}_X$-modules.

In the current example one should consider the analytic left $\mathscr{D}_{\mathbb{C}}$-module $\mathscr{M}$ which occurs as cokernel of the map

$$P : \mathscr{D}_{\mathbb{C}} \to \mathscr{D}_{\mathbb{C}} : Q \mapsto Q(z\partial_x - 1/2).$$

The solutions are then encoded in the sheaf $\mathcal{H}om_{\mathscr{D}_{\mathbb{C}}}(\mathscr{M}, \mathcal{O}_{\mathbb{C}})$.

Remark 2.1.5. *More generally than the foregoing example for a system of differential equations*

$$\sum_{j=1}^{k} P_{ij}(x, \partial) f_j = 0; \qquad i = 1, \ldots, m$$

with unknown functions f_j on X and differential operators $P_{ij} \in \mathscr{D}_X(X)$ one can consider the cokernel $\mathscr{M}$ of the map

$$P : \mathscr{D}_X^k \to \mathscr{D}_X^m : (Q_1, \ldots, Q_k) \mapsto \left(\ldots, \sum_{j=1}^{k} Q_j P_{ij}, \ldots \right).$$

The solutions in a left $\mathscr{D}_X$-module $\mathscr{N}$ are encoded by $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$.

Example 2.1.6. *This example is essential in the study of Bernstein-Sato equations. Let $f \in \mathcal{O}_X(X)$ be some global section and introduce a new variable s. The free $\mathbb{C}[x, f^{-1}]$-module $\mathbb{C}[x, f^{-1}]f^s$ is equipped with the structure of left $\mathscr{D}_X$-module by setting*

$$\xi f^s = s\xi(f) f^{-1} f^s$$

for any section ξ of Θ_X. One denotes $\mathscr{D}_X f^s$ for the $\mathscr{D}_X$-submodule generated by f^s.

Filtrations

The non-commutativity of $\mathscr{D}_X$ exits the typical domain of algebraic geometry. This can be resolved by consideration of a graded structure. The essential observation is that differential operators commute up to an element of lower order.

Definition 2.1.7. *The order filtration on $\mathscr{D}_X$ is defined inductively to be given by the sheaves of $\mathcal{O}_X$-submodules $F_i\mathscr{D}_X$ such that $F_0\mathscr{D}_X = \mathcal{O}_X$ and $F_i\mathscr{D}_X$ is maximal with $[F_i\mathscr{D}_X, F_i\mathscr{D}_X] \subseteq F_{i-1}\mathscr{D}_X$.*

The term $F_i\mathscr{D}_X$ in the order filtration can be described as containing all differential operators of order less than or equal to i. Indeed, given local coordinates $x_1, \dots, x_n$ one can see that $F_i\mathscr{D}_X$ is the $\mathcal{O}_X$-module which is locally generated by $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where α is a multi-index with $|\alpha| \leq i$. The following observations are immediate.

Lemma 2.1.8. *The $F_i\mathscr{D}_X$ are coherent $\mathcal{O}_X$-modules and form a exhaustive filtration. This is to say that $\cup_{i\geq 0} F_i\mathscr{D}_X = \mathscr{D}_X$ and that for any $i, j \geq 0$ it holds that $F_i\mathscr{D}_X \cdot F_j\mathscr{D}_X \subseteq F_{i+j}\mathscr{D}_X$.*

One denotes $\operatorname{gr}\mathscr{D}_X = \oplus_{i\geq 0} F^i\mathscr{D}_X / F_{i-1}\mathscr{D}_X$ for the induced graded sheaf of rings. Observe that $\operatorname{gr}\mathscr{D}_X$ is commutative by definition of the order filtration.

Let $\pi : T^*X \to X$ be the projection map. It is known that $\operatorname{gr}\mathscr{D}_X \cong \pi_*\mathcal{O}_{T^*X}$ (Hotta and Tanisaki, 2007, Section 2.1). The symplectic structure of T^*X captures part of the non-commutativity. Indeed, consider two differential operators P, Q. Pick local coordinates $x_1, \dots, x_n$ and decompose $P = \sum_\alpha p_\alpha \partial^\alpha$ and $Q = \sum_\beta q_\beta \partial^\beta$. Let m_1, m_2 be the maximal values of $|\alpha|$ and $|\beta|$ with non-zero coefficients. Then the induced elements of P and Q in $\operatorname{gr}\mathscr{D}_X$ are of the form $p = \sum_{|\alpha|=m_1} p_\alpha \xi^\alpha$ and $q = \sum_{|\beta|=m_2} q_\beta \xi^\beta$ where ξ_i is the induced element of ∂_i. On the other hand, the induced element of $PQ - QP$ is $\sum_{i=1}^n \frac{\partial p}{\partial \xi_i}\frac{\partial q}{\partial x_i} - \frac{\partial q}{\partial \xi_i}\frac{\partial p}{\partial x_i}$. This is precisely $\{p, q\}$ where $\{-,-\}$ is the Poisson bracket.

There is a similar notion of filtrations on $\mathscr{D}_X$-modules $\mathscr{M}$. Assume that $\mathscr{M}$ is a left $\mathscr{D}_X$-module, the case for right modules is analogous. A filtration consists of a increasing sequence of quasi-coherent $\mathcal{O}_X$-submodules $F_i\mathscr{M}$ of $\mathscr{M}$ such that $\cup_i F_i\mathscr{M} = \mathscr{M}$ and $F_i\mathscr{D}_X \cdot F_j\mathscr{M} \subseteq F_{i+j}\mathscr{M}$ where i runs over $\mathbb{Z}_{\geq 0}$. The graded objects $\operatorname{gr}\mathscr{M} = \oplus_{i\geq 0} F_i\mathscr{M}/F_{i-1}\mathscr{M}$ are $\operatorname{gr}\mathscr{D}_X$-modules.

The $\operatorname{gr}\mathscr{D}_X$-module $\operatorname{gr}\mathscr{M}$ has a corresponding module on T^*X defined by $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\operatorname{gr}\mathscr{D}_X} \pi^{-1}\operatorname{gr}\mathscr{M}$. By abuse of notation this module is still denoted $\operatorname{gr}\mathscr{M}$ and it will always be implicitly assumed that $\operatorname{gr}\mathscr{D}_X$-modules live on T^*X unless it is explicitly mentioned otherwise.

Proposition 2.1.9. *(Hotta and Tanisaki, 2007, Theorem 2.1.3., Section 4.1) A $\mathscr{D}_X$-module $\mathscr{M}$ is coherent if and only if it locally admits a filtration such that $\operatorname{gr}\mathscr{M}$ is a coherent $\operatorname{gr}\mathscr{D}_X$-module. Such a filtration is called a good filtration. In the algebraic case the filtration can be taken globally.*

Proposition 2.1.10. *(Hotta and Tanisaki, 2007, Theorem 2.2.1.) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then the support of $\operatorname{gr}\mathscr{M}$ in T^*X is a independent of the chosen good filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\operatorname{Ch}\mathscr{M}$.*

Proposition 2.1.11. *(Hotta and Tanisaki, 2007, Theorem 2.3.1, 2) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then $\operatorname{Ch}\mathscr{M}$ is a homogeneous and involutive closed subset of T^*X.*

Remark 2.1.12. *The characteristic variety corresponds to the so-called method of characteristics in the classical study of partial differential equations. This method allows one to use the characteristic variety to determine qualitative properties such as the propagation of shock waves.*

Further relations between characteristic varieties and the properties of differential equations are made precise in the study of microlocal analysis.

Characteristic varieties behave well with respect to quotients and submodules.

Proposition 2.1.13. *Consider a short exact sequence of coherent $\mathscr{D}_X$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

then it holds that

$$\operatorname{Ch} \mathscr{M}_2 = \operatorname{Ch} \mathscr{M}_1 \cup \operatorname{Ch} \mathscr{M}_3.$$

Proof. A good filtration on $\mathscr{M}_2$ induces good filtrations on $\mathscr{M}_1$ and $\mathscr{M}_3$ and one has a short exact sequence

$$0 \to \operatorname{gr} \mathscr{M}_1 \to \operatorname{gr} \mathscr{M}_2 \to \operatorname{gr} \mathscr{M}_3 \to 0$$

whence the result follows.

□

The characteristic variety corresponding to the $\mathscr{D}_X$-module in Example 2.1.6 is understood and may provide some intuition for general characteristic varieties.

Proposition 2.1.14. *(Kashiwara, 1976, Theorem 5.3) The characteristic variety of the coherent $\mathscr{D}_X$-module $\mathscr{D}_X f^s$ is the closure of*

$$W_f = \{(x, sf^{-1}(x)df(x)); \qquad f(x) \neq 0, \quad s \in \mathbb{C}\}$$

in T^*X.

The following result follows from proposition 2.1.14 by establishing that the part of the closure of W_f above $f = 0$ is isotropic.

Proposition 2.1.15. *(Kashiwara, 1976, Proposition 5.6) One can write $\operatorname{Ch} \mathscr{D}_X f^s = \Lambda \cup W$ for some isotropic variety $\Lambda \subseteq T^*X$ and a irreducible $(n+1)$-dimensional variety W which dominates X.*

Direct Image

In this section we describe the direct image of $\mathscr{D}_Y$-modules. Let $\mu : Y \to X$ be a morphism of smooth algebraic varieties or complex manifolds.

A-priori, it is not even clear what $\mathscr{D}_X$-module should correspond to $\mathscr{D}_Y$. This issue may be resolved by use of the transfer $(\mathscr{D}_Y, \mu^{-1}\mathscr{D}_X)$-bimodule $\mathscr{D}_{Y\to X} := \mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\mathscr{D}_X$. Here, the right $\mu^{-1}\mathscr{D}_X$-module structure is just the action on the second component and the left $\mathscr{D}_Y$-module structure is defined by

$$f \cdot (g \otimes \mu^{-1}h) = fg \otimes \mu^{-1}h; \qquad \xi \cdot (g \otimes \mu^{-1}h) = \xi g \otimes \mu^{-1}h + g \otimes T\mu(\xi)\mu^{-1}h$$

for any sections f of $\mathcal{O}_Y$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Example 2.1.16. *If $\mu : \mathbb{C} \to \mathbb{C}^2 : y \mapsto (0, y)$ is the inclusion then sections of $\mathscr{D}_{\mathbb{C}\to\mathbb{C}^2}$ may be identified with finite sums of the form $\sum_{j=0}^{n} f_j(x, y)\partial_y^j$ where $f_j(x, y)$ are sections of $\mathcal{O}_{\mathbb{C}^2}$. The left $\mathscr{D}_{\mathbb{C}^2}$-module structure is such that*

$$\partial_x \cdot f\partial_y^j = \partial_x(f)\partial_y^j; \qquad \partial_y \cdot f\partial_y^j = \partial_y(f)\partial_y^j + f\partial_y^{j+1}.$$

Definition 2.1.17. *The direct image functor $\int_\mu : D^{b,r}(\mathscr{D}_Y) \to D^{b,r}(\mathscr{D}_X)$ is given by $R\mu_*(- \otimes^L_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$. For any $\mathscr{D}_Y$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{D}_X$-modules $\int_\mu^j \mathscr{M} = H^j \int_\mu \mathscr{M}$. The subscript μ will be suppressed whenever there is no ambiguity.*

Remark 2.1.18. *A explicit free resolution for the transfer module is known. This involves the Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{M})$ of a $\mathscr{D}_Y$-module $\mathscr{M}$ with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k\Theta_k$. The details may be found in Sabbah (2011).*

Remark 2.1.19. *A direct image functor for left $\mathscr{D}_Y$-modules is induced as*

$$\int \mathscr{M} := R\mathcal{H}om_{\mathcal{O}_X}\left(\omega_X, \int(\mathscr{M} \otimes_{\mathcal{O}_Y} \omega_Y)\right).$$

The definition for the direct image functor is somewhat subtle due to passing through derived categories but many nice properties follow. Most notably, it is immediate from the derived definition that one gets a long exact sequence.

Proposition 2.1.20. *For any short exact sequence of $\mathscr{D}_Y$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

there is a long exact sequence of $\mathscr{D}_X$-modules

$$0 \to \int^0 \mathscr{M}_1 \to \int^0 \mathscr{M}_2 \to \int^0 \mathscr{M}_3 \to \int^1 \mathscr{M}_1 \to \cdots.$$

Proposition 2.1.21. *(Borel, 1987, Chapter VI, Section 5) Let $\mu : Z \to Y$ and $\nu : Y \to X$ be a morphisms of smooth varieties. Then there is a isomorphism of functors $\int_{\nu\circ\mu} \cong \int_\nu \int_\mu$.*

A similar theorem applies to complex manifolds provided μ is proper (Sabbah, 2011, Theorem 3.3.6). Denote $D^{*,*}_{coh}(\mathscr{D}_X)$ for the full subcategory of $D^{*,*}(\mathscr{D}_X)$ consisting of those complexes of $\mathscr{D}_X$-modules with coherent cohomology. The coherence properties of the direct image in the analytic case require the following notion.

Definition 2.1.22. *A $\mathscr{D}_Y$-module $\mathscr{M}$ is said to be μ-good if there exists a open cover $\{V_j\}_{j\in J}$ of X such that $\mathscr{M}$ admits a good filtration on $\mu^{-1}(V_j)$ for any $j \in J$.*

Note that, by proposition 2.1.9, any coherent $\mathscr{D}_Y$-module on a algebraic variety is μ-good.

Theorem 2.1.23. *(Sabbah, 2011, Theorem 3.4.1.) Let $\mathscr{M}$ be a μ-good $\mathscr{D}_Y$-module and suppose that μ is proper on the support of $\mathscr{M}$. Then, $\int \mathscr{M}$ has coherent cohomology.*

Consider the following cotangent diagram.

$$\begin{tikzcd} & \mu^*T^*X \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y & & T^*X \end{tikzcd}$$

Proposition 2.1.24. *(Sabbah, 2011, Theorem 3.4.1.) Let $\mathscr{M}$ be a μ-good $\mathscr{D}_Y$-module and suppose that μ is proper on the support of $\mathscr{M}$. Then, for any $j \geq 0$*

$$\mathrm{Ch}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}\,\mathscr{M})\right).$$

2.2 Riemann-Hilbert Correspondence

This section concerns the Riemann-Hilbert correspondence which states that a system of differential equations is equivalent to it's system of solutions. The systems of differential equations are encoded in regular holonomic $\mathscr{D}_X$-modules. The solutions are given by perverse sheaves.

Holonomic Modules

A particularly nice class of $\mathscr{D}_X$-modules are given by maximally overdetermined systems of differential equations. This is to say that there are many relations for $\mathscr{M}$ or equivalently that $\mathrm{Ch}\,\mathscr{M}$ is small. Observe that the involutive part of proposition 2.1.11 implies that $\dim \mathrm{Ch}\,\mathscr{M} \geq n$ for any coherent $\mathscr{D}_X$-module $\mathscr{M}$.

Definition 2.2.1. *A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called holonomic if $\dim \mathrm{Ch}\,\mathscr{M} = n$.*

The full subcategory of $D^{*,*}(\mathscr{D}_X)$ consisting of complexes with holonomic cohomology is denoted $D_h^{*,*}(\mathscr{D}_X)$. For technical purposes it is mostly important that holonomic modules have finiteness properties.

Proposition 2.2.2. *(Kashiwara, 2003, Proposition 4.42) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then, for any $x \in X$, the stalk $\mathscr{M}_x$ is a $\mathscr{D}_{X,x}$-module of finite length.*

Proposition 2.2.3. *(Bjork, 1979, Chapter 5, Proposition 9.2) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ is $\mathbb{C}$-algebraic. This is to say that for any $\varphi \in \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ there exists some polynomial b with coefficients in $\mathbb{C}$ such that $b(\varphi) = 0$.*

Proposition 2.2.4. *(Kashiwara, 1976, Corollary 4.10) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module and suppose that $\mu : Y \to X$ is proper. Then $\int \mathscr{M}$ has holonomic cohomology.*

The above theorem can be established by combining proposition 2.1.24 and proposition 2.1.11 with the following facts.

Lemma 2.2.5. *(Sabbah, 2011, Theorem 4.3.4) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathscr{M}$ has a globally defined good filtration.*

Lemma 2.2.6. *(Kashiwara, 1976, Proposition 4.9.) Let $\mu : Y \to X$ be a proper morphism and $V \subseteq T^*Y$ an isotropic subvariety. Then $\widetilde{\mu}((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M}))$ is also isotropic.*

Regular singularities

Let $X = \mathbb{C}$ considered as a complex manifold and consider a ordinary differential operator $P(x, \partial) = \sum_{k=0}^m a_k(x)\partial^k$. Suppose that $a_m(x) \neq 0$ for any $x \neq 0$. Then $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$ is locally isomorphic to $\mathcal{O}_X^m$ as a $\mathscr{D}_X$-module near any point $x \neq 0$. In particular the solutions $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ form a locally constant sheaf of rank m outside of 0. The solutions near zero may be more subtle due to monodromy.

Observe that $\mathrm{Ch}\,\mathscr{M} \subseteq \{(x, \xi) : x\xi = 0\}$. Hence, for any filtration on $\mathscr{M}$ there exists some $N > 0$ such that $(x\xi)^N \operatorname{gr} \mathscr{M} = 0$.

Proposition 2.2.7. *(Kashiwara, 2003, Section 5.1) The following conditions are equivalent.*

1. *There exists a filtration on $\mathscr{M}$ such that $x\xi \operatorname{gr} \mathscr{M} = 0$.*

2. *The equation $P(x, \partial)u = 0$ has m linearly independent solutions of the form $x^\lambda \sum_{j=0}^{s} u_j \log(x)^j$ near 0 for some $s \geq 0$, $\lambda \in \mathbb{C}$ and holomorphic u_j.*

If these two equivalent conditions are satisfied one calls 0 a regular singularity of $\mathscr{M}$. This has the following generalisation to higher dimensions.

Definition 2.2.8. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a complex manifold X with characteristic variety determined by some ideal sheaf $\mathcal{I}$. Then $\mathscr{M}$ is called regular holonomic if it it admits a filtration such that $\mathcal{I} \operatorname{gr}(\mathscr{M}) = 0$.*

Denote $D^{**}_{rh}(\mathscr{D}_X)$ for the full subcategory of $D^{**}(\mathscr{D}_X)$ consisting of complexes with regular holonomic cohomology.

It appears that these definitions should generalise directly to the algebraic situation. However, this has unintended consequences for the Riemann-Hilbert correspondence. For a example, let $X = \mathbb{C}$ as before and consider the regular holonomic $\mathscr{D}_X$-modules $\mathcal{O}_X$ and $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X(\partial - 1)$. These are analytically isomorphic by the map which sends $f(x)$ to $f(x)\exp(x)$. In particular the Riemann-Hilbert correspondence shows that they have isomorphic systems of solutions. However, $\mathcal{O}_X$ and $\mathscr{M}$ are not algebraically isomorphic. This seems to suggest that the equivalence between differential equations and their systems of solutions would not hold in the algebraic case. The problem is that $\mathscr{M}$ is not regular at infinity.

There are a number of equivalent definitions for regularity in the algebraic case. The following definition expresses that the analytic definition may be used provided one adds the points at infinity. This uses the analytification functor which is provided by the GAGA principle and respects holonomicity.

Definition 2.2.9. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a smooth variety X. Denote $\iota : X \to \overline{X}$ for the smooth completion of X. Then $\mathscr{M}$ is called regular if $(\int_\iota \mathscr{M})^{an}$ is regular holonomic on the complex manifold $\overline{X}^{an}$.*

Perverse Sheaves

Classically, the solutions to a differential equation on a vector bundle produces a local system. One can not expect local systems in the case of general $\mathscr{D}_X$-modules since their support could be a proper subvariety.

Definition 2.2.10. *Let X be a complex manifold. A stratification of X consists of a locally finite partition $X = \sqcup_{j \in J} X_j$ into connected locally closed subsets, called strata, such that*

(i) For any $j \in J$ the fronteer $\partial X_j = \overline{X}_j \setminus X_j$ is a union of strata.

(ii) For any $j \in J$ the spaces $\overline{X}_j$ and ∂X_j are closed complex analytic subspaces.

The same definition applies on algebraic varieties by replacing the analytic subspaces by subvarieties.

Definition 2.2.11. *A $\mathbb{C}_X$-module $\mathcal{F}$ is called a constructible sheaf on X if there exists a stratification $X = \sqcup_{\alpha \in A} X_\alpha$ such that $\mathcal{F}|_{X_\alpha}$ is a local system of finite rank on X_α for any $\alpha \in A$.*

Denote $D^b_c(X)$ for the full subcategory of $D^b(\mathbb{C}_X)$ consisting of complexes with constructible cohomology. Such complexes are called constructible.

For a constructible complex $\mathcal{F}^\bullet$ in $D^b_c(X)$ the supports and cosupports are defined dually by

$$\operatorname{supp}^m \mathcal{F}^\bullet = \operatorname{supp} H^m\mathcal{F}^\bullet; \qquad \operatorname{cosupp}^m \mathcal{F}^\bullet = \operatorname{supp}^{-m} \mathbb{D}\mathcal{F}^\bullet$$

where $\mathbb{D}\mathcal{F}^\bullet := R\mathcal{H}om_{\mathbb{C}}(\mathcal{F}^\bullet, \mathbb{C}_X)$. The support $\operatorname{supp}\mathcal{F}^\bullet$ is the closure of the union of the $\operatorname{supp}^m \mathcal{F}^\bullet$.

Theorem 2.2.12. *(Dimca, 2004, Theorem 4.1.5.) Let $\mathcal{F}^\bullet$ be a constructible complex on Y and consider a morphism $\mu : Y \to X$ which is proper on $\operatorname{supp}\mathcal{F}^\bullet$. Then $Rf_*(\mathcal{F}^\bullet)$ is constructible on X.*

Theorem 2.2.13. *(Dimca, 2004, Theorem 4.1.16) Let $\mathcal{F}^\bullet$ be a complex of $D^b(\mathbb{C}_X)$. Then $\mathcal{F}^\bullet$ is constructible if and only if the dual $\mathbb{D}\mathcal{F}^\bullet$ is constructible.*

Let $D^{\leq 0}(X)$ denote the full subcategory of $D^b_c(X)$ consisting of complexes with $\dim\operatorname{supp}^{-m}\mathcal{F}^\bullet < m$ and $\dim\operatorname{supp}^m\mathcal{F}^\bullet = 0$ for all $m \geq 0$. Dually $D^{\geq 0}(X)$ consists of complexes with $\dim\operatorname{cosupp}^{-m}\mathcal{F}^\bullet < m$ and $\dim\operatorname{cosupp}^m\mathcal{F}^\bullet = 0$ for all $m \geq 0$.

Proposition 2.2.14. *(Dimca, 2004, Proposition 5.1.12) The pair $(D^{\leq 0}(X), D^{\geq 0}(X))$ is a non-degenerated t-structure on the triangulated category $D^b_c(X)$.*

Definition 2.2.15. *The heart of the t-structure on $D^b_c(X)$ are called the perverse sheaves $\operatorname{Perv}(X) = D^{\leq 0}(X) \cap D^{\geq 0}(X)$.*

Observe that the objects in $\operatorname{Perv}(X)$ are not sheaves but complexes. The reason for the terminology perverse sheaves is that the functor $U \mapsto \operatorname{Perv}(U)$ has the gluing property of sheaves. More precisely, it is a stack. Perverse sheaves still capture the local systems.

Theorem 2.2.16. *(Dimca, 2004, Theorem 5.1.20) Let X be a complex manifold of dimension n. Then $\mathcal{L}[n]$ is a perverse sheaf on X for any local system $\mathcal{L}$ on X.*

The following are immediate from proposition 1.5.3 and proposition 1.5.8.

Proposition 2.2.17. *A constructible complex $\mathcal{F}^\bullet$ is a perverse sheaf if and only if ${}^pH(\mathcal{F}^\bullet) = 0$ for all $k \neq 0$.*

Proposition 2.2.18. *For any distinguished triangle in $D^b_c(X)$*

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \xrightarrow{+1}$$

it holds that if two terms are perverse sheaves then so is the third.

Riemann-Hilbert Correspondence

Recall from example 2.1.4 and remark 2.1.5 that $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ encodes the solutions of a system of differential equations. More generally, the solutions complex is the functor $\operatorname{Sol}(-) := R\mathcal{H}om_{\mathscr{D}_X}(-, \mathcal{O}_X)[n]$ from $D^{b,\ell}(\mathscr{D}_X)^{opp}$ to $D^b(\mathbb{C}_X)$. This is a contravariant functor. The contravariance may be fixed using the duality functor

$$\mathbb{D} = R\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X) \otimes^L_{\mathcal{O}_X} \omega_X^{-1}[n]$$

from $D^{b,*}(\mathscr{D}_X)^{opp}$ to $D^{b,*}(\mathscr{D}_X)$. The de Rham complex of $\mathscr{M}^\bullet$ is defined by

$$\operatorname{DR}(\mathscr{M}^\bullet) := \Omega^\bullet_X \otimes_{\mathscr{D}_X} \mathscr{M}^\bullet[n].$$

Proposition 2.2.19. *(Dimca, 2004, Theorem 5.3.1.) There is a natural isomorphism* $\mathrm{Sol}(-) \cong \mathrm{DR}(\mathbb{D}-)$.

Proposition 2.2.20. *(Dimca, 2004, Theorem 5.3.1.) For any holonomic complex* $\mathscr{M}^\bullet$ *in* $D_h^{b,\ell}(\mathscr{D}_X)$ *the complexes* $\mathrm{Sol}(\mathscr{M}^\bullet)$ *and* $\mathrm{DR}(\mathscr{M}^\bullet)$ *are constructible.*

We are finally ready to state the Riemann-Hilbert correspondence on the equivalence between differential equations and their solutions.

Theorem 2.2.21 (Riemann-Hilbert Correspondence)**.** *The de Rham functor* $\mathrm{DR} : D_{rh}^{b,\ell}(\mathscr{D}_X) \to D_c^b(X)$ *is a t-exact equivalence of categories and commutes with direct images.*

Corollary 2.2.22. *The de Rham functor is a equivalence of categories between the category of regular holonomic* $\mathscr{D}_X$*-modules and* $\mathrm{Perv}(X)$.

Proof. Follows from the Riemann-Hilbert correspondence and proposition 1.5.11.

□

2.3 Interpretation and estimation of Bernstein-Sato polynomials

Philosophically, the Riemann-Hilbert correspondence states that there is a intimate connection between $\mathscr{D}_X$-modules and topology. The goal of this section is to investigate a particular instantiation of this connection, namely the connection between Bernstein-Sato polynomials and monodromy.

Further, we include Kashiwara and Lichtin's proof for the estimation of the roots of the Bernstein-Sato polynomial. This proof is a important framework for the generalisation in ??.

We focus on the local analytic case. The algebraic case will be discussed in detail in the next chapter. Consider $\mathbb{C}^n$ as a complex manifold and take a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with $f(x) = 0$.

Bernstein-Sato polynomials

Definition 2.3.1. *Let s be a new variable. The local Bernstein-Sato polynomial* $b_{f,0}(s) \in \mathbb{C}[s]$ *is the monic polynomial of minimal degree such that there exists some differential operator* $P(x, \partial, s)$ *in* $\mathscr{D}_{\mathbb{C}^n,0} \otimes_{\mathbb{C}} \mathbb{C}[s]$ *with*

$$P(x, \partial, s) f^{s+1} = b_{f,0}(s) f^s$$

in the stalk at x.

The fact that there always exists a Bernstein-Sato polynomial was proved by I.N. Bernstein, I.S. Gelfand and independently by Atiyah.

Remark 2.3.2. *Algorithms to compute the Bernstein-Sato polynomials are known due to Oaku (1997). These algorithms have been implemented in software packages such as SINGULAR. This package was used in the computation of the following examples.*

Example 2.3.3. *The monomial $f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ on $\mathbb{C}^n$ satisfies the Bernstein-Sato relation*

$$\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f^{s+1} = \prod_{i=1}^{n} (\alpha_i s + \alpha_i) \cdots (\alpha_i s + 1) f^s.$$

Example 2.3.4. *The hyperplane arrangement $f(x,y) = x(x+y)(x+2y)$ on $\mathbb{C}^2$ has local Bernstein-Sato polynomial*

$$b_{f,0} = (s+2/3)(s+1)^2(s+4/3).$$

Example 2.3.5. *The cusp singularity $f(x,y) = x^2 - y^2$ on $\mathbb{C}^2$ has local Bernstein-Sato polynomial*

$$b_{f,0}(s) = (s+5/6)(s+1)(s+7/6).$$

Example 2.3.6. *The cardoid $f(x,y) = (x^2+y^2+x)^2 - (x^2+y^2)$ on $\mathbb{C}^2$ has local Bernstein-Sato polynomial*

$$b_{f,0}(s) = (s+5/6)(s+1)(s+7/6).$$

Observe that this is the same local Bernstein-Sato polynomial as for the cusp in example 2.3.5. This is no coincidence, the analytic curve germs corresponding to the cusp and the cardoid at the are isomorphic and the Bernstein-Sato polynomial is an invariant of singularities.

Note that in all these examples the roots of the local Bernstein-Sato polynomial are negative rational numbers. This is a general fact due to Kashiwara (1976). The proof of this statement will be discussed in further on.

Let $Z(b_{f,0})$ denote the set of zeros of the Bernstein-Sato polynomial.

Proposition 2.3.7. *Whenever f is non-constant with $f(0) = 0$ it holds that $-1 \in Z(b_{f,0})$.*

Proof. Substitute $s = -1$ in the Bernstein-Sato equation

$$P(x, \partial, s) f^{s+1} = b_{f,0}(s) f^s$$

to get that $p = b_{f,0}(-1) f^{-1}$ for some analytic germ $p \in \mathcal{O}_{\mathbb{C}^n,0}$. In particular, p is a well-defined in 0 whereas f^{-1} has a pole in 0. This means that the equality is only possible if $b_{f,0}(-1) = 0$. □

The roots of the Bernstein-Sato polynomial provide an invariant of singularities. In particular, these are trivial whenever there are no singularities.

Proposition 2.3.8. *(Igusa, 2007, Section 4.2) If f is non-singular in 0 then $Z(b_{f,0}) = \{-1\}$.*

Remark 2.3.9. *There are a number of connections between Bernstein-Sato polynomials and other invariants of singularities. We will soon discuss how $Z(b_{f,0})$ is connected to the topological invariant of the eigenvalues of monodromy. A invariant called the jumping numbers will be encountered in* **??**.

A open problem, called the monodromy conjecture, concerns the relation between the roots of Bernstein-Sato polynomials and the poles of a certain meromorphic function called the Zeta function. Another connection to the world of topology is was given by a conjecture of Yano (1982). This conjecture uses the topological invariant of Puiseux characteristics in the case of a plane curve. The conjecture was proved in full generality by Blanco (2019).

Monodromy

Theorem 2.3.10. *(Milnor (1968)) Let $B \subseteq \mathbb{C}^n$ be a small ball and pick $t \in \mathbb{C}^\times$ close to zero. The diffeomorphism class of $F_{f,0} := f^{-1}(t) \cap B$ is independent of the choice of t. This diffeomorphism class is called the Milnor fiber.*

Going over a loop around the origin in $\mathbb{C}^\times$ induces a well-defined endomorphism M^* on the singular cohomology $H^j(F_{f,0}, \mathbb{C})$ for every $j \in \mathbb{Z}$. This is called the monodromy action and only depends on the local singularity $(f, 0)$. In particular, this means that the eigenvalues of M^* on $H^j(F_{f,0}, \mathbb{C})$ are invariants of the singularity. If $\lambda \in \mathbb{C}$ is a eigenvalue of M^* on some $H^j(F_{f,0}, \mathbb{C})$ it is called a eigenvalue of monodromy. The following theorem is due to Malgrange and Kashiwara.

Theorem 2.3.11. *The set of eigenvalues of monodromy is equal to the set* $\exp(2\pi i\, Z(b_{f,0}))$.

Monodromy is a topological notion whereas the Bernstein-Sato polynomial is defined in terms of $\mathscr{D}_X$-modules. This suggest that the Riemann-Hilbert correspondence is involved. Indeed, the monodromy of the Milnor fiber can be encoded in a constructible complex so that the Riemann-Hilbert correspondence is applicable.

Take a small open ball $B \subseteq \mathbb{C}^n$ such that f is defined on B. Let $\widetilde{\mathbb{C}}^\times$ denote the universal cover of $\mathbb{C}^\times$ and consider the projection $p : B \times \widetilde{\mathbb{C}}^\times \to B$. Denote $\iota : f^{-1}(0) \to B$ for the inclusion map.

Definition 2.3.12. *Deligne's nearby cycle functor from $D^b_c(B)$ to $D^b_c(f^{-1}(0))$ is given by $\psi_f := L\iota^* \circ Rp_* \circ Lp^*$.*

Denote $\iota_0 : \{0\} \to f^{-1}(0)$ for the inclusion map. The following theorem is due to Deligne.

Theorem 2.3.13. *There is a isomorphism*

$$H^i(F_{f,0}, \mathbb{C}) \cong \mathbb{H}^i(L\iota_0^*(\psi_f \mathbb{C}_B))$$

and the monodromy action on the cohomology of the Milnor fiber corresponds with the action of the covering transformations $\widetilde{\mathbb{C}}^\times \to \widetilde{\mathbb{C}}^\times$ on the nearby cycles.

To describe the $\mathscr{D}$-theoretic counterpart of this constructible complex requires the technical notion of V-filtrations. The interested reader may find these concepts in Budur (2015).

Estimation of $Z(b_{f,0})$

The main idea employed in the estimation of $Z(b_{f,0})$ by Kashiwara (1976) is that one can reduce to the monomial case of example 2.3.3 by a resolution of singularities. Hereafter one can use the $\mathscr{D}$-module direct image functor to relate the result on the resolution to the desired result on the original space. Fix a small ball B on which f may be defined.

Definition 2.3.14. *Let D be a divisor on B. A strong log-resolution of (B, D) consists of a projective morphism $\mu : Y \to B$ with Y smooth such that μ is a isomorphism over the complement of D and $\mu^* D$ a simple normal crossings divisor.*

Let D be the divisor determined by f. By Hironaka's resolution of singularities one can find a strong log-resolution $\mu : Y \to B$ for (B, D). Let $g = f \circ \mu$ denote the pullback of f to Y and let $\operatorname{mult}_E(g)$ denotes the order of vanishing of g on some irreducible component E of $\mu^* D$. Kashiwara was able to establish the following estimate by consideration of the direct image of the $\mathscr{D}_Y$-module $\mathscr{D}_Y g^s$.

Theorem 2.3.15. *(Kashiwara, 1976, Corollary 5.2) Every root of $b_{f,0}(s)$ is of the form $-c/\operatorname{mult}_E(g)$ with $c \in \mathbb{Z}_{>0}$. In particular $Z(b_{f,0}) \subseteq \mathbb{Q}_{<0}$.*

Combining this estimate with theorem 2.3.11 one gets the following theorem.

Theorem 2.3.16. *The eigenvalues of monodromy are roots of unity.*

Lichtin (1989) improved the estimate by similar computations for the right $\mathscr{D}_Y$-module $\mathscr{M}$ spanned by $g^s \mu^*(dx)$ inside $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. The advantage of this approach is that $\mu^*(dx)$ involves the local behaviour of μ. Denote k_E for the order of vanishing of the Jacobian $\operatorname{Jac} \mu$ on E, this is also the coefficients of the relative canonical divisor $K_{Y/B}$ on E.

Theorem 2.3.17. *Every root of $b_{f,0}(s)$ is of the form $-(k_E + c)/\operatorname{mult}_E(g)$ with $c \in \mathbb{Z}_{>0}$.*

We now provide the proof for this improved estimate following Lichtin and Kashiwara.

One can ensure that multiplication by s stays inside $\mathscr{D}_B f^s$ with the following trick. Introduce a new coordinate x_{n+1} and set $\widetilde{f} = x_{n+1} f$ on a small ball $\widetilde{B}$ of $\mathbb{C}^{n+1}$. Then $x_{n+1}\partial_{n+1}$ acts like s on $\widetilde{f}^s$. The induced map $\widetilde{Y} \to \widetilde{B}$ is a strong log resolution for the divisor determined by $\widetilde{f}$. Now suppose we can prove theorem 2.3.17 for $\widetilde{f}$. Then, the theorem also follows for f due to the following result.

Lemma 2.3.18. *The Bernstein-Sato polynomial $b_{f,0}(s)$ is a divisor of $b_{\widetilde{f},0}(s)$.*

Proof. Let P be in the stalk $\mathscr{D}_{\widetilde{B},0}$ such that $P\widetilde{f}^{s+1} = b_{\widetilde{f},0}(s)\widetilde{f}^s$. Expand $P = \sum_{j=1}^N P_j \partial_{n+1}^j$ with coefficients P_j in $\mathscr{D}_{B,0}$. Then

$$x_{n+1}^N b_{\widetilde{f},0}(s)\widetilde{f}^s = \left(\sum_{j=1}^{k} (s+1)^j \sum_{\alpha} Q_\alpha \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \right) \widetilde{f}^{s+1}$$

where the P_j were expanded as polynomials in $\partial_1, \ldots, \partial_n$ with coefficients Q_α in $\mathcal{O}_{B,0}$.

Observe that $\partial_1, \ldots, \partial_n$ act on the formal symbol $\widetilde{f}^{s+1}$ the same as they act on the formal symbol f^{s+1}. Expand the Q_α as power series in $x_1, \ldots, x_{n+1}$ and identify powers of x_{n+1} on both sides for the desired functional equation.

□

For notational simplicity we write f instead of $\widetilde{f}$ and B instead of $\widetilde{B}$ from here on. The dimension of B will be denoted by $m = n + 1$.

Let t be a new variable. The sheaf of rings $\mathscr{D}_B\langle s, t\rangle$ is found from $\mathscr{D}_B$ by adjoining s and t subject to $ts - st = 1$ where s, t commute with $\mathscr{D}_B$. One can view $\mathscr{D}_B f^s$ as a $\mathscr{D}_B\langle s, t\rangle$-module by the action $tP(s)f^s = P(s+1)f^{s+1}$ for any differential operator P in $\mathscr{D}_B[s]$. In this notation the functional equation for $b_{f,0}$ means that $b_{f,0} \in \operatorname{Ann}_{\mathbb{C}[s]}(\mathscr{D}_B f^s / t\mathscr{D}_B f^s)_x$.

There is a $\mathcal{O}_B$-linear isomorphism between any left $\mathscr{D}_B$-module $\mathscr{N}$ and it's right version $\mathscr{N} \otimes_{\mathcal{O}_B} \omega_B$. Concretely, any section u of $\mathscr{N}$ gives rise to the section $u^* := udx$. Further, for any operator P of $\mathscr{D}_B$ there is a adjoint P^* such that

$$(P \cdot u)^* = u^* \cdot P^*$$

for any section u of $\mathscr{N}$. For a vector field $\xi := \sum_i \xi_i \partial_i$ comparison of the definitions shows that $\xi^* := -\sum_i \partial_i \xi_i$ satisfies this equality and this extends to $\mathscr{D}_B$ by iterating. By this procedure the functional equation $Pf^{s+1} = b(s)f^s$ may equivalently be stated as the equation

$$f^{s+1}dx \cdot P^* = b(s)f^s dx$$

in $\mathscr{D}_{B,0}f^s \otimes_{\mathcal{O}_{B,0}} \omega_{B,0}$. The corresponding module $\mathscr{M}$ on Y will be the submodule of $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$ spanned by $g^s \mu^*(dx)$. Observe that $\mathscr{M}$ can be equipped with a $\mathscr{D}_Y\langle s,t\rangle$-module structure as before.

Lemma 2.3.19. *The polynomial $b(s) = \prod_i \prod_{j=1}^{\mathrm{mult}_{E_i}(g)} (\mathrm{mult}_{E_i}(g)s + k_i + j)$ annihilates $\mathscr{M}/t\mathscr{M}$ where E_i runs over the irreducible components of $\mu^* D$.*

Proof. This may be checked locally. If the chosen point is on none of divisors E_i of $\mu^* D$ then g is invertible so that $\mathscr{M}/t\mathscr{M}$ is trivial. Now suppose we are working near a point $y \in Y$ which is on E_i if and only if $i \in I$ with I non-empty. Then one can pick local coordinates y_i such that

$$g = \prod_{i\in I} y_i^{\mathrm{mult}_{E_i}(g)}; \qquad \mu^*(dx) = u \prod_{i\in I} y_i^{k_i} dy$$

where u is a local unit. Now set $P = u^{-1}(\prod_{i\in I} \partial_i^{\mathrm{mult}_{E_i}(g)})u$ to get

$$g^{s+1}\mu^*(dx) \cdot P^* = q(s) g^s \mu^*(dx)$$

where $q(s) = \prod_{i\in I} \prod_{j=1}^{\mathrm{mult}_{E_i}(g)} (\mathrm{mult}_{E_i}(g)s + k_i + j)$.

□

Observe that s, t can be viewed as $\mathscr{D}_Y$-linear injective endomorphisms on $\mathscr{M}$. The associated the long exact sequence of direct images yields a $\mathscr{D}_B\langle s,t\rangle$-module structure on the direct image $\int^0 \mathscr{M}$ where the functorial nature of the direct image is used to ensure that $ts - st = 1$. Similarly, the polynomial $b(s)$ provided by lemma 2.3.19 annihilates $\int^0 \mathscr{M}/t\int^0 \mathscr{M}$.

Consider the surjection $\mathscr{D}_Y \to \mathscr{M}$ induced by $1 \mapsto g^s \mu^*(dx)$. The associated long exact sequence includes a morphism $\int^0 \mathscr{D}_Y \to \int^0 \mathscr{M}$. Observe that $\int^0 \mathscr{D}_Y = R^0 \mu_*(\mathscr{D}_{Y\to B})$ contains a global section corresponding to the section 1 of $\mathscr{D}_{Y\to B}$. Let u be the image of this section in $\int^0 \mathscr{M}$ and denote $\mathscr{U}$ for the right $\mathscr{D}_B\langle s,t\rangle$-module generated by u.

Lemma 2.3.20. *There is a surjective morphism of right $\mathscr{D}_B\langle s,t\rangle$-modules $\mathscr{U} \to \mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B$ sending u to $f^s dx$.*

Proof. Pick some open set $V \subseteq B$. To show this yields a well-defined morphism of $\mathscr{D}_B$-modules it must be show that $(f^s dx)P = 0$ whenever $uP = 0$ in $\mathscr{U}(V)$.

The resolution of singularities $Y \to B$ is a isomorphism on the complement of the divisor D determined by f. Hence, $\mathscr{U}$, $\int^0 \mathscr{M}$ and $\mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B$ are isomorphic outside of D. It follows that the support of the coherent sheaf of $\mathcal{O}_V$-modules $\mathcal{O}_V(f^s dx)P$ lies in D. The

Nullstellen Satz now yields that $f^N(f^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that f is a non-zero divisor of $(\mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B)(V)$. Therefore, $(f^s dx)P = 0$ on V as desired.

Finally, observe that $tu = fu$ so that this morphism of $\mathscr{D}_B$-modules also commutes with the actions by t and s.

□

Due to lemma 2.3.19 there is a suitable b-polynomial for $\int^0 \mathscr{M}$. By lemma 2.3.20 it remains to compare $\int^0 \mathscr{M}$ and $\mathscr{U}$.

Lemma 2.3.21. *The quotient $\int^0 \mathscr{M}/\mathscr{U}$ is a holonomic $\mathscr{D}_B$-module.*

Proof. By proposition 2.1.13 the characteristic variety of $\mathscr{M}$ is a subset of the characteristic variety of $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. This has the same characteristic variety as $\mathscr{D}_Y g^s$ using the $\mathcal{O}_Y$-linear isomorphism between $\mathscr{D}_Y g^s$ and $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. By proposition 2.1.15 it follows that $\operatorname{Ch} \mathscr{M} \subseteq W \cup \Lambda$ for some isotropic $\Lambda \subseteq T^*Y$ and a irreducible $(m+1)$-dimensional variety W which dominates Y.

Observe that $\mathscr{M}$ is certainly μ-good since it admits a global good filtration $F_i\mathscr{M} := F_i\mathscr{D}_Y \cdot g^s \mu^*(dx)$. Hence, proposition 2.1.24 is applicable and yields that

$$\operatorname{Ch} \int^0 \mathscr{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\Lambda \cup W)\right).$$

By lemma 2.2.6 the set $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda))$ is still isotropic and will not form any obstruction to $\int^0 \mathscr{M}/\mathscr{U}$ being holonomic. Further, observe that $\widetilde{\mu}((T^*\mu)^{-1}(W))$ remains a irreducible $(m+1)$-dimensional variety which dominates B. On the other hand μ is a isomorphism outside of D so $\int^0 \mathscr{M}/\mathscr{U}$ is only supported on D. Intersecting $\widetilde{\mu}((T^*\mu)^{-1}(W))$ with D yields a m-dimensional variety whence the desired result follows.

□

Proposition 2.3.22. *For sufficiently large N it holds that $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = 0$.*

Proof. The sequence $t^n \int^0 \mathscr{M}/\mathscr{U}$ forms a decreasing sequence of holonomic $\mathscr{D}_B$-modules. By proposition 2.2.2 the induced sequence of $\mathscr{D}_{B,0}$ modules in the stalk at 0 must stabilise. Let N be sufficiently large such that $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0$ attains the stable value.

Applying proposition 2.2.3 to the $\mathscr{D}_{B,0}$-linear endomorphism s produces a non-zero polynomial $q(s) \in \mathbb{C}[s]$ which annihilates $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0$. Let $q(s)$ be of minimal degree with this property. Observe that $q(s+1)t = tq(s)$ so, using the stabilisation, it follows that

$$q(s+1)t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = tq(s)t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = 0.$$

This means that $q(s) - q(s+1)$ also annihilates $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0$. By the minimality of the degree of $q(s)$ it follows that $q(s) - q(s+1) = 0$ which is to say that $q(s)$ is a non-zero constant. This means that $t^N(\int^0 \mathscr{M})_0/\mathscr{U}_0 = 0$ as desired.

□

Putting all these facts together yields the proof of theorem 2.3.17.

Proof. Let N be as in proposition 2.3.22 and denote $b(s)$ for the polynomial provided by lemma 2.3.19. Set $\Pi(s) = b(s+N+1)b(s+N)\cdots b(s)$ and observe that $\Pi(s)\mathscr{M}_0 \subseteq t^{N+1}\mathscr{M}_0 \subseteq t\mathscr{U}_0$. In particular this means that $\Pi(s) \in \operatorname{Ann}_{\mathbb{C}[s]} \mathscr{U}_0/t\mathscr{U}_0$.

The $\mathscr{D}_B\langle s,t\rangle$-linear surjection $\mathscr{U} \to \mathscr{D}_B f^s \otimes_{\mathcal{O}_B} \omega_B$ from lemma 2.3.20 now implies that $b_{f,0}(s)$ divides $\Pi(s)$. This yields the desired estimate for $Z(b_{f,0})$.

□

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