Chapter 1

$\mathscr{D}_X$-modules and the Riemann-Hilbert Correspondence

Chapter 2

The Behaviour of $\mathscr{A}_X^R$-Modules

Mention BVWZ

The classical approximation of the roots of the b-polynomial due to Kashiwara (1976) relies on a quotient module $\int \mathscr{M}/\mathscr{D}_X u$ being holonomic. This is no longer true in the multivariate case but a refined assumption, called relative holonomicity, due to Maisonobe (2016) still holds. This refinement works with $\mathscr{D}_X \times \mathbb{C}[s]$-modules whence one gets characteristic varieties inside $T^*X \times \mathbb{C}^p$.

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Maybe also mention the example Robin put on the whiteboard? Possibly in the main body?

2.1 Modules over $\mathscr{A}_X^R$

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2).

Proposition 2.1.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\operatorname{gr}^{rel}\mathscr{M}$ is coherent over $\operatorname{gr}^{rel}\mathscr{A}_X^R$.*

Proposition 2.1.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\operatorname{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\operatorname{Ch}^{rel}\mathscr{M}$.*

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\operatorname{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.

This is all general knowledge, should anything be cited?

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties. A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := \mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\mathscr{A}_X^R$.

Definition 2.1.1. *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(\mathscr{M} \otimes_{\mathscr{D}_Y}^L -)$. In particular, for any $\mathscr{A}_Y^R$ module $\mathscr{M}$ one has $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$.*

To compute $\int_\mu^j \mathscr{M}$ a resolution for $\mathscr{A}_{Y\to X}$ is required.

Definition 2.1.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map δ given by*

$$m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k$$
$$- \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k$$

Lemma 2.1.3. *The complex of $(\mathscr{A}_X, f^{-1}\mathscr{A}_Y)$-bimodules $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}^R$ is a resolution of $\mathscr{A}_{Y\to X}^R$ as a bimodule by locally free left $\mathscr{A}_Y^R$-modules.*

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Proof. This is analogous to the case of $\mathscr{D}_Y$-modules in (Sabbah, 2011, p33). □

Since tensoring with locally free modules yields a exact functor it follows that $\int_\mu \mathscr{M} = \boldsymbol{R}\mu_*(\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$. This means that $\int_\mu^j \mathscr{M}$ is the sheaf of right $\mathscr{A}_X^R$-modules associated to the presheaf $V \mapsto H^j(\mu^{-1}(V), \mathscr{M} \otimes_{\mathscr{A}_Y} \mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$.

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Preservation of Relative Holonomicity

Kashiwara's Estimate for the Characteristic Variety

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Let $\mu : Y \to X$ be a proper morphism of COMPLEX MANIFOLDS. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j \mathscr{M}$ in terms of $\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara Kashiwara (1976). We note that the assumption that μ is proper can be relaxed but this version will suffice for our purposes.

The original proof by Kashiwara Kashiwara (1976) uses the theory of microlocal differential operators. The proof we consider here is adapted from a proof by Malgrange Malgrange (1985) and we follow the exposition by Sabbah (Sabbah, 2011, p36).

Consider the following cotangent diagram

$$\begin{tikzcd}
 & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\
T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R
\end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

Theorem 2.1.4. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

$$\mathrm{Ch}^{rel}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

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The first step is to note that a similar inclusion is easy for the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ define $\int^j \mathcal{M} := \mathscr{H}^j\left(\boldsymbol{R}\mu_*(\boldsymbol{L}(T^*\mu)^*\mathcal{M})\right)$. Note that $(T^*\mu)^*$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by the tensor product $- \otimes_{f^{-1}\mathcal{O}_X \times \mathcal{O}_{\operatorname{Spec} R}} \mathrm{gr}^{rel}\mathscr{A}_X^R$. Hence, looking at the supports, the following result is immediate.

Lemma 2.1.5. *For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ it holds that*

$$\operatorname{Supp}\int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}\operatorname{Supp}\mathcal{M}\right).$$

Applying this to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to understand the difference between $\mathrm{gr}^{rel}\int^j \mathscr{M}$ and $\int^j \mathrm{gr}^{rel}\mathscr{M}$. This may be done using relative Rees modules.

Definition 2.1.3. *Let z be a new variable. The relative Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}^{rel}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

The following obvious isomorphisms of filtered modules allow us to view the relative Rees module as a way to interpolate between $\mathscr{M}$ and $\mathrm{gr}^{rel}\mathscr{M}$

$$\frac{\mathcal{R}^{rel}\mathscr{M}}{(z-1)\mathcal{R}^{rel}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}^{rel}\mathscr{M}}{z\mathcal{R}^{rel}\mathscr{M}} = \mathrm{gr}^{rel}\mathscr{M}.$$

Conversely, the second formula may be used to produce a filtered $\mathscr{A}_Y^R$-module from any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

One can define $\int^j \mathcal{R}^{rel}\mathscr{M}$ similarly to the $\mathscr{D}$-module direct image and these are coherent $\mathscr{A}_X^R$-modules by the following adaptation of

theorem 3.4.1 in Sabbah

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