Chapter 1

Relative Holonomic Modules

1.1 Introduction

Fix a variety X and a morphism $F : X \to \mathbb{C}^p : x \mapsto (f_1(x), \ldots, f_p(x))$. Denote D for the divisor defined by $\prod f_i$ and let $\mu : Y \to X$ be a resolution of singularities for (X, D). This means that μ is a projective morphism which is a isomorphism over the complement of D and such that $\mu^* D = \sum_{i=1}^r \operatorname{mult}_{E_i} a_i E_i$ is in normal crossings form. Associated the behaviour of μ over D is measured by the relative canonical divisor $K_{Y/X} = \sum_{i=1}^r k_i E_i$ which is locally defined by the Jacobian of μ. Write $G : Y \to \mathbb{C}^p$ for the lift of F to Y. Introducing new variables $s_1, \ldots, s_p$ we abbreviate $F^s = f_1^{s_1} \cdots f_p^{s_p}$ and similarly for G^s.

The local Bernstein-Sato Ideal $B_{F,x}$ of the function germ of F at some point $x \in X$ consists of all polynomials $b(s_1, \ldots, s_n)$ such that there exists some local partial differential operator $P \in \mathscr{D}_{X,x} \otimes \mathbb{C}[s_1, \ldots, s_n]$ with the following equality in the stalk at x

$$b(s_1, \ldots, s_n) F^s = P \cdot F^{s+1}.$$

The global Bernstein-Sato Ideal B_F of F is the intersection of all local Bernstein-Sato Ideals.[1]

The goal of this chapter is to estimate the zero locust $Z(B_F) \subseteq \mathbb{C}^p$. This zero locust generalises the roots of the Bernstein-Sato polynomial in the monovariate case. The classical approximation of the roots of the b-polynomial is due to Kashiwara (1976) and this estimation was further refined by Lichtin (1989). The idea in both proofs is that it is easy to explicitly compute the Bernstein-Sato polynomial for monomials and that one can reduce to this case by use of the resolution of singularities. The main non-trivial step in these arguments is to translate the solution upstairs to a solution on X. This makes use of the direct image of $\mathscr{D}_X$-modules. The essential insight in the refined estimate due to Lichtin is that the direct image of $\mathscr{D}_X$-modules is more natural for right $\mathscr{D}_X$-modules than left $\mathscr{D}_X$-modules.

The estimate by Kashiwara has been generalised to the multivariate situation in Budur et al. (2020). The main challenge in such a multivariate generalisation is that the classical proof relies on modules of the type $\mathscr{D}_X f^s / \mathscr{D}_X f^{s+1}$ being holonomic. This is no longer the case for the multivariate generalisation $\mathscr{D}_X[s_1, \ldots, s_n] f^s / \mathscr{D}_X[s_1, \ldots, s_n] f^{s+1}$. Tthe notion of relative holonomicity, due to Maisonobe (2016), still holds.

[1]Note: Restate more generally with $+a$ when proof is done.

In this chapter we generalise the refined estimate by Lichtin (1989) to the multivariate situation. The main new ingredient is a induction argument which reduces the problem to the monovariate case where relative holonomicity becomes ordinary holonomicity. This induction is similar to the arguments in Budur et al. (2019).

Theorem 1.1.1. *With notation as above every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{\geq 0}$.

[2]

1.2 Relative Notions

Modules over $\mathscr{A}_X^R$

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. It will also be convenient to use the abbreviation $\mathcal{O}_X^R := \mathcal{O}_{X \times \operatorname{Spec} R}$.

The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration. The associated graded objects are denoted by gr^{rel}. Denote $\pi : T^*X \times \operatorname{Spec} R \to X \times \operatorname{Spec} R$ for the projection map. As in the case of $\mathscr{D}_X$-modules in chapter 1 [3] one can view $\pi^{-1}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$ as a subsheaf of $\mathcal{O}_{T^*X}^R$ and for any $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module $\mathcal{M}$ there is a corresponding module on $T^*X \times \operatorname{Spec} R$ defined by $\mathcal{O}_{T^*X}^R \otimes_{\pi^{-1}\mathrm{gr}^{rel}\mathscr{A}_X^R} \pi^{-1}\mathcal{M}$. By abuse of notation the corresponding module on $T^*X \times \operatorname{Spec} R$ is still denoted with $\mathcal{M}$ and we adopt the perspective that $\mathrm{gr}^{rel}\mathscr{A}_X^R$-modules always live on $T^*X \times \operatorname{Spec} R$ unless explicitly mentioned otherwise.

Similarly to the case of $\mathscr{D}_X$ in the first chapter that [4] it holds that $\mathscr{A}_X^R$ is the sheaf of rings generated by $\mathcal{O}_X^R$ and Θ_X inside of $\mathcal{E}nd_{\underline{\mathbb{C}}}(\mathcal{O}_X^R)$. Giving a left $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X^R$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X. Similarly, giving a right $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2). [5]

Proposition 1.2.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\mathrm{gr}^{rel}\mathscr{M}$ is coherent over $\mathrm{gr}^{rel}\mathscr{A}_X^R$. Such a filtration is called a good filtration.*

[2] Note: Should also give a overview of the results that are already known here.
[3] Note: cite
[4] Note: Cite when C1 is written
[5] Note: Probably cite C1 instead

Proposition 1.2.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\mathrm{Ch}^{rel}\mathscr{M}$.*

Basic Operations

For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ and left $\mathscr{D}_X$-module $\mathscr{N}$ the tensor product $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ comes equipped with a right $\mathscr{A}_X^R$-module structure defined by

$$f \cdot (m \otimes n) = mf \otimes n; \qquad \xi \cdot (m \otimes n) = m\xi \otimes n - m \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X. Putting multiplication by f on the other side of the tensor product this definition is also applicable for a right $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X^R} \mathscr{N}$ if $\mathscr{M}$ is a right $\mathscr{D}_X$-module and $\mathscr{N}$ is a left $\mathscr{A}_X^R$-module. If both are $\mathscr{A}_X^R$-modules there is a right $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

Similarly, given a left $\mathscr{D}_X$-module $\mathscr{L}$ and a left $\mathscr{A}_X^R$-module $\mathscr{N}$ a left $\mathscr{A}_X^R$-module structure on $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}$ is defined by

$$f \cdot (\ell \otimes n) = \ell \otimes fn; \qquad \xi \cdot (\ell \otimes n) = \xi\ell \otimes n + \ell \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

Lemma 1.2.3. *Let $\mathscr{M}, \mathscr{N}$ be right and left $\mathscr{A}_X^R$-modules respectively and let $\mathscr{L}$ be a left $\mathscr{D}_X$-module. Then there is a isomorphism of left $\mathscr{A}_X^R$-modules*

$$(\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}) \otimes_{\mathcal{O}_X^R} \mathscr{N} \cong \mathscr{M} \otimes_{\mathcal{O}_X^R} (\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}).$$

Proof. This is immediate by checking that the obvious bijection conserves the $\mathscr{A}_X^R$-module structure. Note that the only nontrivial check is the action of a section ξ from Θ_X. $\square$

Lemma 1.2.4. *Let $\mathscr{N}$ be a left $\mathscr{A}_X^R$-module which is locally free as a $\mathcal{O}_X^R$-module. Consider $\mathscr{A}_X^R$ as a right $\mathscr{A}_X^R$-module, then $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$ is locally free as a right $\mathscr{A}_X^R$-module.*

Proof. Consider local coordinates $x_1, \dots, x_n$ on X and a local $\mathcal{O}_X^R$-basis $\{n_\beta\}_\beta$ for $\mathscr{N}$. Then $\{1 \otimes n_\beta\}_\beta$ will be a local $\mathscr{A}_X^R$-basis for $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

To see that this generates the $\mathscr{A}_X^R$-module note that $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ is a $\mathcal{O}_X^R$-basis set when α runs over all multi-indices in $\mathbb{Z}_{\geq 0}^n$. These sections can be recovered using the $\mathscr{A}_X^R$-action on the proposed generating set by induction on $|\alpha|$. Indeed, $\xi^\alpha \cdot (1 \otimes n_\beta)$ equals $\xi^\alpha \otimes n_\beta$ up to a element in the $\mathcal{O}_X^R$-span of $\{\xi^\gamma \otimes n_\beta\}_{|\gamma|<|\alpha|}$.

For the freedom, suppose there is a local $\mathscr{A}_X^R$-relation $\sum_\beta P_\beta \cdot 1 \otimes n_\beta = 0$ with some P_β nonzero. This is of the form $\sum_{\alpha,\beta} f_{\alpha,\beta}\xi^\alpha \cdot 1 \otimes n_\beta = 0$ with the $f_{\alpha,\beta}$ sections of $\mathcal{O}_X^R$ not all equal to zero. Pick some multi-index $\mu \in \mathbb{Z}_{\geq 0}^n$ and of maximal degree such that $f_{\mu,\beta}$ is non-zero for some β. Then, rewriting $\sum_{\alpha,\beta} f_\alpha \xi^\alpha \cdot 1 \otimes n_\beta = 0$ in terms of the $\mathcal{O}_X^R$-basis $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ one finds a non-zero coeffiencient at $\xi^\eta \otimes n_\beta$ for some β which is a contradiction. $\square$

Lemma 1.2.5. *The functor $\Omega_X \otimes_{\mathcal{O}_X} -$ which takes a left $\mathscr{A}_X^R$-modules and returns a right $\mathscr{A}_X^R$-module is an equivalence of categories with pseudoinverse $\mathscr{H}om_{\mathcal{O}_X}(\Omega_X, -)$.*

Proof. For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ the left $\mathscr{A}_X^R$-module structure on $\mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathscr{M})$ is defined by

$$(f \cdot \varphi)(\omega) = \varphi(\omega) \cdot f; \qquad (\xi \cdot \varphi)(\omega) = \varphi(\omega \cdot \xi) - \varphi(\omega) \cdot \xi.$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

For any left $\mathscr{A}_X^R$-module $\mathscr{M}$ there is a natural isomorphism of $\mathcal{O}_X^R$-modules $\Omega_X \otimes_{\mathcal{O}_X} \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathscr{M}) \cong \mathscr{M}$ by sending $\omega \otimes \varphi$ to $\varphi(\omega)$. Similarly for any right $\mathscr{A}_X^R$-module $\mathscr{M}$ the isomorphism $\mathscr{M} \cong \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \Omega_X \otimes \mathscr{M})$ associates to a section m of $\mathscr{M}$ the morphism $\omega \mapsto \omega \otimes m$. A direct computation verifies these isomorphisms commute with the $\mathscr{A}_X^R$-module structure. □

Relative Holonomicity

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

Lemma 1.2.6. *The sheaf $\mathscr{M} := \mathscr{A}_Y G^s$ is relatively holonomic with relative characteristic variety*

$$\mathrm{Ch}^{rel}\mathscr{M} := \bigcup_{J \subseteq \{1,\dots,n\}} T^\perp Y_J \times \mathbb{C}^p$$

where $Y_J = \{y \in Y : g_j(y) = 0$ for all $j \in J\}$. [6]

Proof. Working on a affine open U we may assume that $G^s = x_1^{a_1 s_1} \cdots x_k^{a_k s_k} u_{k+1}^{s_{k+1}} \cdots , u_p^{s_p}$ for coordinate functions $x_1, \dots, x_p$, natural numbers $a_1, \dots, a_k > 0$ and invertible sections $u_{k+1}, \dots, u_p$ of $\mathcal{O}_Y$. We claim that $\mathscr{A}_U G^S \cong \mathscr{A}_U^R / \mathcal{I}$ where $\mathcal{I}$ is the left ideal sheaf generated by the $x_i \partial_i - a_i s_i$ and $\partial_j - s_j u_j^{-1}$.

Denoting $\varphi : \mathscr{A}_U \to \mathscr{A}_U G^s$ for the obvious surjection we certainly have that $\mathcal{I}$ is a subsheaf of $\ker \varphi$. It remains to show that $\ker\varphi/\mathcal{I} = 0$. Let $P = \sum c_{\alpha\beta} x^\alpha \partial^\beta$ represent some section in $\ker\varphi/\mathcal{I}$ where the non-zero $c_{\alpha\beta}$ do not vanish in 0. By the relations $\partial_j - s_j u_j^{-1}$ it can be assumed that the only nonzero components of the multi-indices β lie in $1, \dots, k$. By $\mathscr{A}_U$-linear combinations of $x_i \partial_i - a_i s_i$ it can further be enforced that the terms are either have $\alpha_i = 0$ or $\beta_i = 0$ for any $i = 1, \dots, k$. When acting on G^s with the remainder the coefficients all end on different monomial coefficients to G^s which means they have to be zero in order for P to be in the kernel. This shows $\ker\varphi = \mathcal{I}$ as desired.

It follows that $\mathrm{gr}^{rel}\mathcal{A}_U G^s \cong \mathrm{gr}^{rel}\mathscr{A}_U / \mathrm{gr}^{rel}\mathcal{I}$. It holds that $\mathrm{gr}^{rel}\mathcal{I}$ is generated by $x_i \xi_i$ and ξ_j whence the result follows. □

The following lemma and it's proof may be found in Maisonobe (2016).

Lemma 1.2.7. *Let $\mathscr{M}$ be a finitely generated $\mathscr{A}_Y^R$-module. Suppose that $\mathrm{Ch}^{rel}\mathscr{M} \subseteq \Lambda \times \operatorname{Spec} R$ for some, not necessarily irreducible, conic Lagrangian subvariety $\Lambda \subseteq T^*X$. Then $\mathscr{M}$ is relative holonomic over R.*

The Bernstein-Sato ideal may be defined more generally for any $\mathscr{A}_X^R$-module $\mathscr{M}$ as $B_{\mathscr{M}} := \operatorname{Ann}_R \mathscr{M}$. To see how this generalises B_F one considers $\mathscr{A}_X^R F^s$ as a $\mathscr{A}_X^R\langle t\rangle$-module.

[6] $T^\perp$ denotes covectors annihilating the tangent space.

Here t is a new variable which commutes with sections of $\mathscr{D}_X$ and satisfies $ts_i - s_i t = 1$ for any $i = 1, \ldots, n$. The $\mathscr{A}_X^R\langle t\rangle$-module structure on $\mathscr{A}_X^R F^s$ is then defined by extending $tF^s = F^{s+1}$. From this point of view $B_F = B_{\mathscr{A}_X^R F^s / t\mathscr{A}_X^R F^s}$.

Proposition 1.2.8. $Z(B_n) = p_2(\mathrm{Ch}^{rel}(\mathscr{M}))$

1.3 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.[7]

Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties, by abuse of notation we will also denote μ for the induced map from $Y \times \operatorname{Spec} R$ to $X \times \operatorname{Spec} R$.

A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}\mathscr{A}_X^R$. Here, the right $\mu^{-1}\mathscr{A}_X^R$-module structure is just the action on the second component and definitions like section 1.2 are used to define the left $\mathscr{A}_Y^R$-module structure. To be precise

$$f \cdot (g \otimes \mu^{-1}h_X) = fg \otimes \mu^{-1}h_X; \qquad \xi \cdot (g \otimes \mu^{-1}h_X) = \xi g \otimes \mu^{-1}h_X + g \otimes T\mu(\xi)\mu^{-1}h_X$$

for any sections f of $\mathcal{O}_Y^R$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Definition 1.3.1. *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be surpressed whenever there is no ambiguity.*

To compute the direct image $\int^j \mathscr{M}$ a resolution for the transfer bimodule $\mathscr{A}_{Y\to X}$ is required.

Definition 1.3.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map δ given by*

$$\begin{aligned} m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto & \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k \\ & - \sum_{i=1}^k (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \end{aligned}$$

The following lemma and it's proof are a generalisation of exercise 1.20 in Sabbah (2011) to the relative case.

Lemma 1.3.1. *The relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_X^R$ as left $\mathscr{A}_X^R$-module.*

[7] Note: more introduction

Proof. Define a filtration on $\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ by the complexes $F_k\,\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ which have term $F_{k-\ell}\mathscr{A}^R_Y \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot ℓ. This filtration induces the complexes $\mathrm{gr}^{rel}_k\,\mathrm{Sp}^\bullet_X(\mathscr{A}^R_Y)$ with term $\mathrm{gr}^{rel}_{k-\ell}\,\mathscr{A}^R_Y \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot ℓ.

In local coordinates $x_1, \dots, x_n$ one finds that $\mathrm{gr}^{rel}\mathrm{Sp}^\bullet_Y := \oplus_k\, \mathrm{gr}^{rel}{}_k\,\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ is the Koszul complex of $\mathcal{O}^R_Y[\xi_1, \dots, \xi_n] = \mathrm{gr}^{rel}\mathscr{A}^R_Y$ with respect to $\xi_1, \dots, \xi_n$.[8] Since $\xi_1, \dots, \xi_n$ form a regular sequence a standard result on Koszul complexes[9] yields that $\mathrm{gr}^{rel}\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ is a locally free resolution of $\mathcal{O}^R_Y$ as $\mathrm{gr}^{rel}\mathscr{A}^R_Y$-module.

On the other hand, it is immediate that $F_0\,\mathrm{Sp}^\bullet(\mathscr{A}^R_Y) = \mathrm{gr}^{rel}_0\,\mathrm{Sp}^\bullet(\mathscr{A}^R_Y)$ is $\mathcal{O}^R_Y$ viewed as a complex. Hence, there is no contribution to $\mathrm{gr}^{rel}\,\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ from the terms of $k > 0$. That is to say that $\mathrm{gr}^{rel}_k\,\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ is quasi-isomorphic to the zero complex for $k > 0$. Hence, $F_0\,\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y) \hookrightarrow \mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ is a quasi-isomorphism by the exactness of the direct limit. [10] It follows that $\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ is a resolution of $\mathcal{O}^R_X$. That the terms of $\mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y)$ are locally free follows from lemma 1.2.4 after some minor adjustments in the statement and proof. □

Define the transfer Spencer complex as the complex of $(\mathscr{A}^R_Y, f^{-1}\mathscr{A}_X)$-bimodules given by $\mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}^R_Y) := \mathrm{Sp}^\bullet_Y(\mathscr{A}^R_Y) \otimes_{\mathcal{O}^R_Y} \mathscr{A}^R_{Y\to X}$. The following lemma and it's proof are direct generalisation of exercise 3.4 in Sabbah (2011) to the relative case.

Lemma 1.3.2. *The transfer Spencer complex* $\mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}^R_Y)$ *is a resolution of* $\mathscr{A}^R_{Y\to X}$ *as a bimodule by locally free left* $\mathscr{A}^R_Y$*-modules.*

Proof. To see that the terms of the complex are locally free recall from lemma 1.2.3 the following isomorphisms of left $\mathscr{A}^R_Y$-modules

$$(\mathscr{A}^R_Y \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y) \otimes_{\mathcal{O}^R_Y} \mathscr{A}_{Y\to X} \cong \mathscr{A}^R_Y \otimes_{\mathcal{O}^R_Y} (\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}).$$

[11] Note that $\mathscr{A}^R_{Y\to X}$ is a locally free $\mathcal{O}^R_Y$-module since it is the pullback of a locally free module on $X \times \operatorname{Spec} R$. Combined with the fact that $\wedge^\ell \Theta$ is a locally free $\mathcal{O}_Y$-module this yields that $\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}$ is a locally free $\mathcal{O}^R_Y$-module. Hence lemma 1.2.3 is applicable and yields that the terms of the transfer Spencer complex are locally free $\mathscr{A}^R_Y$-modules.

That the transfer Spencer complex is a resolution of $\mathscr{A}^R_{Y\to X}$ follows from lemma 1.3.1 by using that $\mathscr{A}^R_{Y\to X}$ is a locally free and hence flat over $\mathcal{O}^R_Y$. □

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 1.3.3. *It holds that* $\int = \boldsymbol{R}\mu_*(- \otimes_{\mathscr{A}^R_Y} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}^R_Y))$.

A strategy one can employ in proving theorems on some space X is by first solving them on a nicer space Y equipped with a map $Y \to X$. This can then be related to the problem on X by use of the direct image. For this purpose it is useful that any global section of $\mathcal{M}$ induces a global section of the direct image. This is usually done in the language of left modules but for us it is more natural to work with right $\mathscr{A}^R_Y$-modules.

[8] Note: Should I explain what a Koszul complex is?

[9] Note: Give reference to some book

[10] Note: Would be nice to give a reference, proof may be found on stackexchange

[11] Note: May be possible to remove this step from the proof and removing need for minor adjustment of previous proof.

Lemma 1.3.4. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module. Then any global section $m \in \Gamma(Y, \mathscr{M})$ induces a global section of $\int^0 \mathscr{M})$.*

Proof. By the Leray spectral sequence there is a functorial isomorphism

$$\mathbb{H}^\bullet(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y \to X}(\mathscr{A}_Y^R)) \cong \mathbb{H}^\bullet(X, \boldsymbol{R}\mu_*(\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y \to X}(\mathscr{A}_Y^R))).$$

In particular it follows that $\mathbb{H}^0(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y \to X}(\mathscr{A}_Y^R)) \cong \Gamma(X, \int^0 \mathscr{M})$. The Čech spectral sequence now induces the desired global section in the direct image based on the section $m \otimes 1$ of $\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^0_{Y \to X}(\mathscr{A}_Y^R)$. □

Theorem 1.3.5. *Long exact sequence*

Functorial Properties of the Direct Image

Theorem 1.3.6. *Let $\mu : Z \to Y$ and $\nu : Y \to X$ be morphisms of smooth algebraic varieties. If μ is proper then $\int_{\nu \circ \mu} = \int_\nu \int_\mu$.*

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture17.pdf` □

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_Y^R)$ the full subcategory of $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ consisting of those complexes of right $\mathscr{A}_Y^R$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\mathrm{Spec}\, R}$. Similarly for $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_Y^R)$ with the cohomology being coherent $\mathscr{A}_Y^R$-modules.

Theorem 1.3.7. *Let $\mu : X \to Y$ be a morphism of nonsingular algebraic varieties. Then the direct image $\int$ takes $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_Y^R)$ into $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}_X^R)$. Moreover, when μ is proper the direct image takes $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_Y^R)$ into $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}_X^R)$.*

Proof. See `http://www.math.stonybrook.edu/~cschnell/mat615/lectures/lecture18.pdf` □

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ with relative characteristic variety $\mathrm{Ch}^{rel}\mathscr{M}$. We desire to estimate $\mathrm{Ch}^{rel}\int^j \mathscr{M}$ in terms of $\mathrm{Ch}^{rel}\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara.

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The idea of the following proof is due to Malgrange (1985) in a K-theoretic context. We follow the exposition of Sabbah (2011) and replace it with the corresponding relative notions.

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^* T^* X \times \mathrm{Spec}\, R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^* Y \times \mathrm{Spec}\, R & & T^* X \times \mathrm{Spec}\, R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

Theorem 1.3.8. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$, we have*

$$\mathrm{Ch}^{rel}\bigg(\int^j \mathscr{M}\bigg) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

Note that the statement is local so, after replacing X by some affine open, it may be assumed that $X \times \operatorname{Spec} R$ and $Y \times \operatorname{Spec} R$ are compact. The first step is to note that a similar inclusion is easy for the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules. The direct image functor on $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules $\mathcal{M}$ is defined by $\int^j \mathcal{M} := \boldsymbol{R}^j\widetilde{\mu}_*(\boldsymbol{L}(T^*\mu)^*\mathcal{M})$. Here, $(T^*\mu)^*(-)$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by $- \otimes_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R$. Looking at the supports the following result is immediate.

Lemma 1.3.9. *For any $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module $\mathcal{M}$ it holds that*

$$\operatorname{supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1} \operatorname{supp} \mathcal{M}\right).$$

Applying this lemma to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to show that $\operatorname{supp} \mathrm{gr}^{rel}\!\int^j\!\mathscr{M} \subseteq \operatorname{supp} \int^j \mathrm{gr}^{rel}\mathscr{M}$. This is proved in proposition 1.3.15. The main technical ingredient in the proof is the Rees modules associated to a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$.

Definition 1.3.3. *Let z be a new variable. The Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

Given a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it follows that $\mathcal{R}\mathscr{M}$ is a coherent $\mathcal{R}\mathscr{A}_Y^R$-module similarly to proposition 1.2.1. The following isomorphisms of filtered modules on $Y \times \operatorname{Spec} R$ are essential. They mean that the Rees module can be viewed as a parametrisation of various relevant modules.

$$\frac{\mathcal{R}\mathscr{M}}{(z-1)\mathcal{R}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}\mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z^\ell\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}.$$

Here $\mathrm{gr}^{rel}_{[\ell]}$ takes a filtered object and returns $\oplus_k F_k/F_{k-\ell}$. The first formula may be be used to find a corresponding filtered $\mathscr{A}_Y^R$-module for any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

The jth direct image of a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is the sheaf of $\mathcal{R}\mathscr{A}_X^R$-modules on $X \times \operatorname{Spec} R$ defined by $\int^j \mathcal{M} = \boldsymbol{R}^j\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R)$. Here the filtration on $\mathscr{A}_{Y\to X}^R$ is defined by $F_i\mathscr{A}_{Y\to X}^R = \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}F_i\mathscr{A}_X^R$. The direct image may be restricted to the category of graded Rees modules in which case it returns a graded Rees module. Coherence is preserved similarly to theorem 1.3.7.

Recall that a $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules on $Y \times \operatorname{Spec} R$ could be be viewed as a sheaf on $T^*Y \times \operatorname{Spec} R$ and is already equipped with a direct image. The Rees module viewpoint agrees with the earlier definition by the following lemma.

Lemma 1.3.10. *Consider a filterd $\mathscr{A}_Y^R$-module $\mathscr{M}$. Then viewing $\int^j \mathcal{R}\mathscr{M}/z\mathcal{R}\mathscr{M}$ with it's $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module structure as a sheaf on $T^*X \times \operatorname{Spec} R$ recovers the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module direct image $\int^j \mathrm{gr}^{rel}\mathscr{M}$. Viewing $\int^j \mathcal{R}\mathscr{M}/(z-1)\mathscr{M}$ as a $\mathscr{A}_X^R$-module recovers $\int^j\mathscr{M}$.*

Proof. We give the proof for $\int^j \mathrm{gr}^{rel}\mathscr{M}$, the proof for $\int^j \mathscr{M}$ is similar but easier.

Consider the following Cartesian square

$$\begin{tikzcd}
\mu^*T^*X \times \operatorname{Spec} R \arrow[r, "T^*\mu"] \arrow[d, "\widetilde{\mu}"] & T^*Y \times \operatorname{Spec} R \arrow[r, "\pi_Y"] & Y \times \operatorname{Spec} R \arrow[d, "\mu"] \\
T^*X \times \operatorname{Spec} R \arrow[rr, "\pi_X"] & & X \times \operatorname{Spec} R.
\end{tikzcd}$$

Since π_X is flat the derived version of the flat base change theorem yields that [12]

$$\boldsymbol{L}\pi_X^*\boldsymbol{R}\mu_*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}) = \boldsymbol{R}\widetilde{\mu}_*\boldsymbol{L}(T^*\mu\circ\pi_Y)^*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}).$$

Since π_X is flat it follows that $\mathscr{H}^j\boldsymbol{L}\pi_X^*(-) = \pi_X^*\mathscr{H}^j(-)$[13]. It now suffices to show that the right hand side is $\int \mathrm{gr}^{rel}\mathscr{M}$.

Since π_Y is flat it holds that $\boldsymbol{L}(T^*\mu\circ\pi_Y)^* = \boldsymbol{L}(T^*\mu)^*\circ\boldsymbol{L}\pi_Y^*$[14]. We show that $\boldsymbol{L}\pi_Y^*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}) \cong \mathrm{gr}^{rel}\mathscr{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \widetilde{\mu}^*\,\mathrm{gr}^{rel}\mathscr{A}_X^R$ from which the result follows immediately.

Let $\mathcal{F}^\bullet$ denote a bimodule resolution for $\mathcal{R}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathcal{R}\mathscr{A}^R_Y$-modules. Then $(\mathcal{R}\mathscr{A}_Y^R/z\mathcal{R}\mathscr{A}_Y^R)\otimes_{\mathcal{R}\mathscr{A}_Y^R}\mathcal{F}^\bullet$ is a bimodule resolution for $\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathrm{gr}^{rel}\mathscr{A}^R_Y$-modules. Now $\boldsymbol{L}\pi_Y^*$ just means applying $\pi^{-1}(-)\otimes\mathcal{O}_{T^*Y}$ to the terms of this free resolution. Due to flatness this yields a free resolution in $\pi^*\,\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules of $\pi^*\,\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X}$. Since $\mathrm{gr}^{rel}\mathscr{A}^R_{Y\to X} = \mathcal{O}_Y^R\otimes_{\mu^{-1}\mathcal{O}_X^R}\mu^{-1}\,\mathrm{gr}^{rel}\mathscr{A}_X^R$ and $\pi^*\mu^* = \widetilde{\mu}^*\pi^*$ the desired equality follows. [15] $\square$

It turns out that one can directly compare $\mathrm{gr}^{rel}_{[\ell]}\int^j\mathscr{M}$ and $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ when ℓ is large. Some care is required since since $\int^j \mathcal{R}\mathscr{M}$ may have $\mathbb{C}[z]$-torsion.

Lemma 1.3.11. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, for sufficiently large ℓ, the kernel of z^ℓ in $\int^j \mathcal{R}\mathscr{M}$ stabilises. For such ℓ the quotient $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is the $\mathcal{R}\mathscr{A}_X^R$-coherent module associated to a good filtration on $\int^j\mathscr{M}$.*

Proof. By $\int \mathcal{R}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}_X^R$ it follows that $\ker z^\ell$ locally stabilises. This is sufficient since $X\times\operatorname{Spec} R$ is assumed to be compact.

Now consider the short exact sequence $0\to\mathcal{R}\mathscr{M}\xrightarrow{z-1}\mathcal{R}\mathscr{M}\to\mathscr{M}\to 0$. This induces a long exact sequence

$$\cdots\to\int^j\mathcal{R}\mathscr{M}\xrightarrow{z-1}\int^j\mathcal{R}\mathscr{M}\to\int^j\mathscr{M}\to\int^{j+1}\mathcal{R}\mathscr{M}\xrightarrow{z-1}\cdots.$$

Since $\int^{j+1}\mathcal{R}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module it follows that $z-1$ is injective whence $\int^j\mathcal{R}\mathscr{M}/(z-1)\int^j\mathcal{R}\mathscr{M}\cong\int^j\mathscr{M}$. This yields the desired result using that $\int^j\mathcal{R}\mathscr{M}/\ker z^\ell$ is $\mathbb{C}[z]$-torsion free and the isomorphism

$$\frac{\int^j\mathcal{R}\mathscr{M}}{(z-1)\int^j\mathcal{R}\mathscr{M}}\cong\frac{\int^j\mathcal{R}\mathscr{M}/\ker z^\ell}{(z-1)(\int^j\mathcal{R}\mathscr{M}/\ker z^\ell)}.$$

$\square$

[12]Note: Check in detail that the theorem is applicable and has this conclusion due to flatness

[13]Note: $\mathscr{H}^j\boldsymbol{L}\pi_X^*(-) = \pi_X^*\mathscr{H}^j(-)$?

[14]Note: $\boldsymbol{L}(T^*\mu\circ\pi_Y)^* = \boldsymbol{L}(T^*\mu)^*\circ\boldsymbol{L}\pi_Y^*$?

[15]Note: Write out more

From now on we equip $\int^j\mathscr{M}$ with the good filtation inherited from the Rees module's direct image.

Lemma 1.3.12. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, if ℓ is sufficiently large, $\mathrm{gr}^{rel}_{[\ell]}\int^j\mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$.*

Proof. The short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \mathcal{R}\mathscr{M} \to \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \cdots.$$

Hence, $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M})$ and it remains to show that $\mathcal{R}\int^j\mathscr{M}/z^\ell\mathcal{R}\int^j\mathscr{M}$ is a quotient of $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$.

Let ℓ be sufficiently large so that lemma 1.3.11 yields a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong \mathcal{R}\int^j \mathscr{M}$. The map z^ℓ induces a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong z^\ell \int^j \mathcal{R}\mathscr{M}$. Therefore $z^\ell \int^j \mathcal{R}\mathscr{M}/z^{2\ell} \int^j \mathcal{R}\mathscr{M} \cong \mathcal{R}\int^j\mathscr{M}/z^\ell\mathcal{R}\int^j\mathscr{M}$. The desired quotient follows by applying the map $m \mapsto z^\ell m$ on $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$. $\square$

The main remaining task is to relate these results to the desired case of $\ell = 1$.

Definition 1.3.4. *For any $\ell \geq 1$ the G-filtration on a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is defined by the decreasing sequence of $\mathrm{gr}^{rel}_{[\ell]}\mathscr{A}_Y^R$-submodules $G_j\mathcal{M} := z^j\mathcal{M}$.*

Lemma 1.3.13. *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ there is the a isomorphism of $\mathrm{gr}\,\mathscr{A}_Y^R$-modules*

$$\mathrm{gr}^G\,\mathrm{gr}^{rel}_{[\ell]}\mathscr{M} \cong (\mathrm{gr}^{rel}\mathscr{M})^\ell.$$

Proof. This follows from directly from the fact that $G_j\,\mathrm{gr}^{rel}_{[\ell]}\mathscr{M} = \oplus_k F_{k-j}\mathscr{M}/F_{k-\ell}\mathscr{M}$. $\square$

Lemma 1.3.14. *Consider a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$. Then one has a isomorphism $\mathrm{gr}^G \int \mathcal{M} \cong \int \mathrm{gr}^G \mathcal{M}$ in $\boldsymbol{D}^{b,r}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$.*

Proof. Writing out the direct image functors the desired result is a isomorphism

$$\mathrm{gr}^G\,\boldsymbol{R}\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R) \cong \boldsymbol{R}\mu_*(\mathrm{gr}^G\,\mathcal{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R).$$

The proof of the commutation proceeds in two steps corresponding to the two derived functors.

Let $\mathscr{F}^\bullet$ be a bimodule resolution for $\mathcal{R}\mathscr{A}_{Y\to X}^R$ by locally free left $\mathcal{R}\mathscr{A}_Y^R$-modules. There is a G-filtration on this complex given by $z^j(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) = (z^j\mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. By the flatness of locally free sheaves and the short exact sequence $0 \to \oplus_j z^j\mathcal{M} \to \oplus_j z^{j-1}\mathcal{M} \to \mathrm{gr}^G\,\mathcal{M} \to 0$ it follows that $\mathrm{gr}^G(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) \cong (\mathrm{gr}^G\,\mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. Further, by the argument in the proof of lemma 1.3.10 the complex of $\mathrm{gr}^G\,\mathscr{A}_Y^R$-modules $(\mathrm{gr}^G\,\mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ can be viewed as a representative of $(\mathrm{gr}^G\,\mathcal{M}) \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R$.[16]

Denote $\mathcal{G}(-)$ for the functor which takes a sheaf complex and returns its Godement resolution. Flabby sheaves are acyclic for μ_* so the Godement resolution may be used to

[16]Note: Check after lemma is entirely proven

compute $\boldsymbol{R}\mu_*$. Moreover, since the terms of a Godement resolution are essentially direct sums of formal products of stalks, it is immediate that $z^i\mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(z^i\mathcal{N}^\bullet)$ and that $\operatorname{gr}^G \mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(\operatorname{gr}^G \mathcal{N}^\bullet)$ for any complex of right $\mu^{-1}\mathcal{R}\mathscr{A}_X^R$-modules $\mathcal{N}^\bullet$. Applying μ_* to these equalities and setting $\mathcal{N}^\bullet = \mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ yields the desired result. □

Proposition 1.3.15. *For a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it holds that*

$$\operatorname{supp} \operatorname{gr}^{rel} \int^j \mathscr{M} \subseteq \operatorname{supp} \int^j \operatorname{gr}^{rel} \mathscr{M}.$$

Proof. Let $\ell \geq 0$ be sufficiently large so that lemma 1.3.12 holds, that is to say that $\operatorname{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int^j \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$. By lemma 1.3.13 it holds that $\operatorname{gr}^G \operatorname{gr}^{rel}_{[\ell]}\int^j \mathscr{M} \cong (\operatorname{gr}^{rel}\int \mathscr{M})^\ell$. Since $\operatorname{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$ it remains to show that the support of $\operatorname{gr}^G \int^j \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$ is a subset of the support of $\int^j \operatorname{gr} \mathscr{M}$.

This can be established with the spectral sequence associated of the G-filtered complex $\int \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$. Since the G-filtration is finite on $\operatorname{gr}^{rel}_{[\ell]}\mathscr{A}_X^R$-modules the associated spectral sequence abbuts by general results[17]. To be precise the associated spectral sequence with terms $E^2_{pq} = \mathscr{H}^{p+q} \operatorname{gr}^G \int \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$ abuts to $\operatorname{gr}^G \int \mathscr{M}$. By lemma 1.3.14 and lemma 1.3.13 it holds that $E^2_{pq} \cong (\int^{p+q} \operatorname{gr} \mathscr{M})^\ell$. [18] It follows that $\operatorname{supp} \operatorname{gr}^G \int^j \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$ is a subset of the support of $\int \operatorname{gr} \mathscr{M}$ which completes the proof. □

1.4 Non-commutative Homological Notions

In this section we discuss homological notions associated to the Ext-functor over a ring A which may not be commutative. These notions are particularly well-behaved for relatively holonomic modules.

Auslander Regularity

Based on appendix Budur et al. (2019) which is itself based on Bjork (1979). X is assumed to be affine

Definition 1.4.1. *Homological dimension*

Definition 1.4.2. *Grade*

The following definition should be viewed in perspective of the Auslander-Buchsbaum-Serre theorem which asserts that a commutative local ring is regular if and only if it has finite global homological dimension. This connection to the commutative notion of regularity is further clarified by propositions ...

Definition 1.4.3. *Auslander regular ring*

Proposition 1.4.1. *Connection with commutative case* gr

[17]Note: Found spectral sequence result online, add good reference.

[18]Note: Or E^1? Seems to depend on preference but should actually matter somewhat for the differentials.

Proposition 1.4.2. *If A is regular commutative*

Proposition 1.4.3. *gl.dim in terms of $j(M)$*

Proposition 1.4.4. $j(-) = j(\mathrm{gr})$

Corollary 1.4.5. *For finitely generated $\mathscr{A}_X^R$-modules*

$$j(\mathscr{M}) + \dim \mathrm{Ch}^{rel}\mathscr{M} = 2n + \dim R$$

Pure Modules

Definition 1.4.4. *Pure*

Lemma 1.4.6. *Behaviour* Ext

Lemma 1.4.7. Ext *of relative holonomic remains so*

Lemma 1.4.8. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which is $(n+k)$-pure for some $0 \leq k \leq \dim R$. If $b \in R$ is not contained in any minimal prime ideal containing $B_{\mathscr{M}}$ then multiplication by b induces injective automorphisms on $\mathscr{N}$ and $\mathrm{Ext}^{n+k}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$.*

Cohen-Macauley

Definition 1.4.5. *Cohen-Macauley*

Lemma 1.4.9. *Relative holonomic with $j(\mathscr{M}) = n + k$ is $(n+k)$-CM over a open* $\mathrm{Spec}\, R$ *of codimension $> k$. Similarly $j(\mathscr{M}) > n + k$ yields zero.*

Proof. Like lemma 3.5.2 in Budur et al. (2019) □

Lemma 1.4.10. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which is $(n + k)$-Cohen-Macaulay. If $b \in R$ is not a element of any minimal prime ideal containing B_N then $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic and $(n + k)$-Cohen-Macaulay $\mathscr{A}_X^{R/(b)}$-module.*

Proof. Like prop 3.4.3 in Budur et al. (2019) □

The following lemma is a generalisations of a result by Kashiwara (1976) to the relative case. The proof and statement are more involved than the original result by Kashiwara but follow the same line of thought.

Lemma 1.4.11. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which comes equipped with the structure of a $\mathscr{A}_X^R\langle t\rangle$-module. Suppose that $\mathscr{M}$ has grade $j(\mathscr{M}) = n + k$ with $k \geq 1$. Then there exists a open* $\mathrm{Spec}\, R' \subseteq \mathrm{Spec}\, R$ *such that $\mathscr{M}' = \mathscr{M} \otimes_R R'$ is a relative holonomic $\mathscr{A}_X^{R'}$-module with $t^N\mathscr{M}' = 0$ for N sufficiently large. Moreover, it may be assumed that* $\mathrm{Spec}\, R \setminus \mathrm{Spec}\, R'$ *has codimension strictly greater than k.*

Proof. The proof is split in two main parts. The first part it is establises that $\mathrm{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ stabilises and that the $t^i\mathscr{M}'$ are $(n + k)$-Cohen-Macauley. It follows that $t^i\mathscr{M}'$ stabilises. The final part of the proof deduces that the stable value is zero.

Note that $\mathscr{M}/t\mathscr{M}$ is a coherent sheaves over the Noetherian sheaf of rings $\mathscr{A}_X^R$. Hence, the kernel of the morphisms $\mathscr{M}/t\mathscr{M} \to t^i\mathscr{M}/t^{i+1}\mathscr{M}$ stabilise. Let $N \geq 0$ be sufficiently large so that these kernels are constant for $i \geq N$.

By use of lemma 1.4.9 it may be assumed that $\operatorname{Spec} R'$ is such that $t^i\mathscr{M}'$, $\mathscr{M}'/t^i\mathscr{M}'$, $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ and the kernels K_i of the morphisms $\mathscr{M}'/t\mathscr{M}' \to t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ are zero or $(n+k)$-Cohen-Macauley for any $i = 0, \ldots, N$. Since localisation is an exact functor the stabilisation of kernels for $i \geq N$ is still valid over $\operatorname{Spec} R'$. The first few steps in this proof are to us these stabilisation results to estabish that these modules are actually $(n+k)$-Cohen-Macaulay for arbitrary $i \geq 0$.

The short exact sequence $0 \to K_i \to \mathscr{M}'/t\mathscr{M}' \to t^i\mathscr{M}'/t^{i+1}\mathscr{M}' \to 0$ induces a long exact sequence

$$0 \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}, \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(\frac{\mathscr{M}'}{t\mathscr{M}'}, \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(K_i, \mathscr{A}_X^{R'}) \to \cdots$$

In particular there is a isomorphism $\operatorname{Ext}^{n+k+1}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}') \cong \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(K_i)/\operatorname{Im}\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(\mathscr{M}'/t\mathscr{M}')$. For $i \leq N$ the left-hand-side of this isomorphism is known to vanish. Due to the stabilisation of K_i it follows that $\operatorname{Ext}^{n+k+1}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}') \cong 0$ for any $i \geq 0$. Using that the K_i and $\mathscr{M}'/t\mathscr{M}'$ are $(n+k)$-Cohen-Macaulay in the higher order terms of the long exact sequence yields $\operatorname{Ext}^{n+k+j}_{\mathscr{A}_X^{R'}}(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}, \mathscr{A}_X^{R'}) \cong 0$ for $j > 1$. This establishes that $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

The long exact sequence induced by $0 \to t^{i+1}\mathscr{M}' \to t^i\mathscr{M}' \to t^i\mathscr{M}'/t^{i+1}\mathscr{M}' \to 0$ yields exact sequences $\operatorname{Ext}^{n+k+j}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}', \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k+j}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k+j}_{\mathscr{A}_X^{R'}}(t^{i+1}\mathscr{M}', \mathscr{A}_X^{R'})$ for any $i \geq 0$. By induction on i it follows that $t^i\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay for any $i \geq 0$. Similarly the long exact sequence induced by $0 \to t^i\mathscr{M}'/t^{i+1}\mathscr{M}' \to \mathscr{M}/t^{i+1}\mathscr{M}' \to \mathscr{M}'/t^i\mathscr{M}'$ and induction on i yields that $\mathscr{M}/t^i\mathscr{M}$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

By $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ being $(n+k)$-Cohen-Macaulay it follows that the morphisms $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^{i+1}\mathscr{M}', \mathscr{A}_X^{R'})$ are surjective. Similarly, by $\mathscr{M}'/t^i\mathscr{M}'$ being $(n+k)$-Cohen-Macaulay or zero it follows that the morphisms $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(\mathscr{M}', \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ are surjective. Note that $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R}}(\mathscr{M}, \mathscr{A}_X^{R})$ is a coherent sheaf over the Noetherian sheaf of rings $\mathscr{A}_X^{R'}$. Hence the kernels of $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(\mathscr{M}', \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ stabilise, by increasing N it follows that the morphisms $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'}) \to \operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^{i+1}\mathscr{M}', \mathscr{A}_X^{R'})$ are injective for $i \geq N$. This establishes that $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^{i+1}\mathscr{M}', \mathscr{A}_X^{R'})$ is constant for $i \geq N$.

By $t^i\mathscr{M}'$ being $(n+k)$-Cohen-Macaulay it follows that $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'}), \mathscr{A}_X^{R'}) \cong t^i\mathscr{M}'$.[19] Conclude that $t^i\mathscr{M}'$ stabilises for $i \geq N$. It remains to show that this stable value is 0.

Recall that $t^N\mathscr{M}'$ is relative holonomic and if it is nonzero it is $(n+k)$-Cohen-Macaulay with $k > 1$. Let's assume that $t^N\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay and derive a contradiction. By corollary 1.4.5 and proposition 1.2.8 it follows that there exists some nonzero polynomial $b(s_1, \ldots, s_p) \in B_{t^N\mathscr{M}'}$. Note that one has the commutation relation

$$tb(s_1, \ldots, s_p) = b(s_1 + 1, \ldots, s_p + 1)t.$$

[19]Note: Provide reference, maybe include and prove earlier since this is easy from double complex

Since $t^{N+1}\mathscr{M}' = t^N\mathscr{M}'$ it follows by iteration that $b(s_1+n,\dots,s_p+n) \in B_{t^N\mathscr{M}'}$ for any $n \geq 0$. This implies that $Z(B_{t^N\mathscr{M}'}) = 0$ which means that $t^N\mathscr{M}' = 0$. $\square$

1.5 Estimation of the Bernstein-Sato Zero Locust

This section contains the main result of this chapter, namely a proof of the improved estimate for the Bernstein-Sato Zero Locust which was announced in theorem 1.1.1. The following proof is similar to the method employed by Lichtin (1989) and Kashiwara (1976).

This section employs the same notation as section 1.1. Recall that the global Bernstein-Sato Ideal is the intersection of all local ones. This means that the global Bernstein-Sato zero locust $Z(B_F)$ is the union of all local ones so it suffices to estimate $Z(B_{F,x})$. In particular, it may be assumed that X is affine and admits global coordinates $x_1,\dots,x_n$.

Due to these global coordinates there is a $\mathcal{O}_X^R$-linear isomorphism between any left $\mathscr{A}_X^R$-module $\mathscr{N}$ and it's right version $\Omega_X \otimes_{\mathcal{O}_X} \mathscr{N}$. Concretely, any section u of $\mathscr{N}$ gives rise to the section $u^* := udx$. Further, for any operator P of $\mathscr{A}_X^R$ there is a adjoint P^* such that

$$(P\cdot u)^* = u^* \cdot P^*$$

for any section u of $\mathscr{N}$. For a vector field $\xi := \sum_i \xi_i\partial_i$ comparison of the definitions shows that $\xi^* := \sum_i \partial_i\xi_i$ satisfies this equality and this extends to $\mathscr{A}_X^R$ by iterating.

By this procedure the functional equation $PF^{s+1} = bF^s$ may equivalently be stated as the equation

$$F^{s+1}dx \cdot P^* = bF^s dx$$

in $\mathscr{A}_X F^s \otimes_{\mathcal{O}_X} \Omega_X$. This viewpoint is convenient since the $\mathscr{A}_X$-module direct image is more natural for right modules. This makes it easier to transfer results along the resolution of singularities. Further, it explains how the relative canonical divisor becomes involved in the estimate of theorem 1.1.1. Define $\mathscr{M}$ to be the submodule of $\Omega_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_Y G^s$ spanned by $G^s\mu^*(dx)$. This may be viewed as a $\mathscr{A}_Y^R\langle t\rangle$-module by the action $tG^s\mu^*(dx) = G^{s+1}\mu^*(dx)$.

Lemma 1.5.1. *In the notation of section 1.1 a polynomial of the form $\prod_{i=1}^p\prod_{j=0}^N(\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + j)$ belongs to the Bernstein-Sato ideal $B_{\mathscr{M}/t\mathscr{M}}$ if $N \in \mathbb{Z}_{\geq 0}$ is sufficiently large.*

Proof. This may be checked locally. Take a open $U \subseteq Y$ sufficiently small so that it admits local coordinates $z_1,\dots,z_n$ where z_i determines E_i if $E_i \cap U \neq \emptyset$.

In these local coordinates $G^s = \prod_{i=1}^p z_i^{\sum_{j=1}^p M_{ij}s_j}$ and $\mu^*(dx) = u\prod_{i=1}^n z_i^{m_i}dz$ where $M_{ij} \leq \mathrm{mult}_{E_i}(g_j)$, $m_i \leq k_i$ and u is a local unit. For notational convenience define $M_{ij} = 0$ for $i > p$. Denote $N_i = \sum_j M_{ij}$ for any $i = 1,\dots,p$ and set $P = u^{-1}\partial_1^{N_1}\cdots\partial_p^{N_p}u$ then

$$u\prod_{i=1}^n z_i^{\sum_{j=1}^p M_{ij}(s_j+1)+m_i}dz \cdot P = bu\prod_{i=1}^n z_i^{\sum_{j=1}^p M_{ij}s_j+m_i}dz$$

where

$$b = \prod_{i=1}^p(\sum_{j=1}^p M_{ij}s_j + m_i + N_i)(\sum_{j=1}^p M_{ij}s_j + m_i + N_i - 1)\cdots(\sum_{j=1}^p M_{ij}s_j + m_i).$$

$\square$

By lemma 1.3.4 the global section $G^s\mu^*(dx)$ of $\mathscr{M}$ gives rise to a global section u of $\int^0\mathscr{M}$. Denote $\mathscr{U}$ for the right $\mathscr{A}_X^R$-module generated by u. The following lemma and it's proof are similar to the monovariate case which may be found in (Bjork, 1979, p246).

Lemma 1.5.2. *There is a surjective morphism right $\mathscr{A}_X^R$-modules $\mathscr{U} \to \mathscr{A}_X^R F^s \otimes_{\mathcal{O}_X} \Omega_X$ sending u to $F^s dx$.*

Proof. The resolution of singularities $Y \to X$ is a isomorphism on the complement of $\prod f_i = 0$. Hence, a isomorphism $\mathscr{U} = \int^0\mathscr{M} \cong \mathscr{A}_X^R F^s \otimes_{\mathcal{O}_X} \Omega_X$ holds outside of $\prod f_i = 0$.

Pick some open set $V \subseteq X$ we must show that whenever $uP = 0$ in $\mathscr{U}(V)$ it follows that $(F^s dx)P = 0$. Due to the isomorphism it is certainly the case that $(F^s dx)P = 0$ outside of $\prod f_i = 0$. Hence, the support of the coherent sheaf of $\mathcal{O}_V^R$-modules $\mathcal{O}_V^R(F^s dx)P$ lies in $\prod f_i = 0$. The Nullstellen Satz now yields that $(\prod f_i)^N(F^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that $\prod f_i$ is a non-zero divisor of $(F^s \otimes_{\mathcal{O}_X} \Omega_X)(V)$. Hence, it follows that $(F^s dx)P = 0$ on V as desired. □

Due to lemma 1.5.1 there is a b-polynomial of a desirable form in $B_{\int^0\mathscr{M}/t\int^0\mathscr{M}}$. The remaining difficulty in the proof is to go from there to a b-polynomial in $B_{\mathscr{U}/t\mathscr{U}}$.

1.6 Old Stuff

Sketch

1. It holds that $\mathscr{A}_X G^s$ is relative holonomic. Hence also $G^s \otimes \Omega_Y$ is relative holonomic. In particular the module $\mathscr{M}$ spanned by $G^s \otimes \mu^*(dx)$ is relative holonomic.

2. By Kashiwara's estimate $\int^0 \mathscr{M}$ is relative holonomic and this contains a global section u.

3. There is a surjection $u \to f^s$.

4. Goal: Given b which annihilates $\mathscr{M}/t\mathscr{M}$ show $B(s) = \prod b(s+k)$ annihilates u/tu.

   - Problem: the usual argument exploits that $\int^0 \mathscr{M}/u$ is finite length.
   - Induction step: If $B(\lambda)$ annihilates $(u/tu) \otimes \frac{\mathbb{C}}{(\ell)}$ for generic ℓ then $B(\lambda)$ annihilates u/tu. This is subtle due to a lack of Nakayama. Argument may be similar to Budur and Robin paper 1.
   - Will require Cohen-Macauley similarly to Nero and Robin paper 1.

5. In the final step of the induction we need to deduce that $B(s)$ annihilates $u/tu \otimes \frac{\mathbb{C}[s]}{L}$. The standard method comes down the following observations

   - $b \int^0 \mathscr{M} \subseteq t \int^0 \mathscr{M}$
   - $t^N \int^0 \mathscr{M}/u = 0$ for large N
   - Hence $Bu \subseteq B \int^0 \mathscr{M} \subseteq t^{N+1} \int^0 \mathscr{M} \subseteq tu$

   In the final step of the induction we get that $(\int^0 \mathscr{M} \otimes \frac{\mathbb{C}[s]}{L})/\widetilde{u}$ is holonomic from which we can deduce $t^N \int^0 \mathscr{M} \otimes \frac{\mathbb{C}[s]}{L}/\widetilde{u} = 0$. But we actually need $t^N(\int^0 \mathscr{M}/u) \otimes \frac{\mathbb{C}[s]}{L} = 0$

6. To get this note

   - The SES
   $$0 \to \ell\mathscr{M} \to \mathscr{M} \to \mathscr{M} \otimes \frac{\mathbb{C}[]}{(\ell)} \to 0$$
   yields
   $$0 \to \int^0 \ell\mathscr{M} \to \int^0 \mathscr{M} \to \int^0 \mathscr{M} \otimes \frac{\mathbb{C}[]}{(\ell)} \to \cdots$$
   - Provided ℓ does not contain irreducible parts of $Z(B)$ the map ℓ is injective upstairs by lemma 3.4.2 in paper 1 Nero and Robin so
   $$(\int^0 \mathscr{M}) \otimes \frac{\mathbb{C}[]}{\ell} \hookrightarrow \int^0 (\mathscr{M} \otimes \frac{\mathbb{C}[]}{(\ell)})$$

7. Now
$$0 \to u \to \int^0 \mathscr{M} \to \frac{\int^0 \mathscr{M}}{u} \to 0$$

induces

$$\begin{tikzcd}
Tor \arrow[r] & u \otimes \frac{\mathbb{C}[s]}{\ell} \arrow[r] \arrow[dr] & (\int^0 \mathscr{M}) \otimes \frac{\mathbb{C}[s]}{\ell} \arrow[r] \arrow[d, hook] & \frac{\int^0 \mathscr{M}}{u} \otimes \frac{\mathbb{C}[s]}{\ell} \arrow[r] & 0 \\
& & \int^0 \mathscr{M} \otimes \frac{\mathbb{C}[s]}{\ell} & &
\end{tikzcd}$$

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