Chapter 1

Relative Holonomic Modules

1.1 Introduction

Fix a smooth variety X and a morphism $F : X \to \mathbb{C}^p : x \mapsto (f_1(x), \dots, f_p(x))$. Denote D for the divisor defined by $\prod f_i$ and let $\mu : Y \to X$ be a resolution of singularities for (X, D). This means that μ is a projective morphism which is a isomorphism over the complement of D and such that $\mu^* D = \sum_{i=1}^r \operatorname{mult}_{E_i} a_i E_i$ is in normal crossings form. The behaviour of μ over D is measured by the relative canonical divisor $K_{Y/X} = \sum_{i=1}^r k_i E_i$ which is locally defined by the Jacobian of μ. Write $G : Y \to \mathbb{C}^p$ for the lift of F to Y. Introducing new variables $s_1, \dots, s_p$ we abbreviate $F^s = f_1^{s_1} \cdots f_p^{s_p}$ and similarly for G^s.

The local Bernstein-Sato Ideal $B_{F,x}$ of the function germ of F at some point $x \in X$ consists of all polynomials $b(s_1, \dots, s_n)$ such that there exists some local partial differential operator $P \in \mathscr{D}_{X,x} \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_n]$ with the following equality in the stalk at x

$$b(s_1, \dots, s_n) F^s = P \cdot F^{s+1}.$$

The global Bernstein-Sato Ideal B_F of F is the intersection of all local Bernstein-Sato Ideals.[1]

The goal of this chapter is to estimate the zero locust $Z(B_F) \subseteq \mathbb{C}^p$. This zero locust generalises the roots of the Bernstein-Sato polynomial in the monovariate case. The classical approximation of the roots of the b-polynomial is due to Kashiwara (1976) and this estimation was further refined by Lichtin (1989). The idea in both proofs is that it is easy to explicitly compute the Bernstein-Sato polynomial for monomials and that one can reduce to this case by use of the resolution of singularities. The main non-trivial step in these arguments is to translate the solution upstairs to a solution on X. This makes use of the direct image of $\mathscr{D}_X$-modules. The essential insight in the refined estimate due to Lichtin is that the direct image of $\mathscr{D}_X$-modules is more natural for right $\mathscr{D}_X$-modules than left $\mathscr{D}_X$-modules.

The estimate by Kashiwara has been generalised to the multivariate situation in Budur et al. (2020). The main challenge in such a multivariate generalisation is that the classical proof relies on modules of the type $\mathscr{D}_X f^s / \mathscr{D}_X f^{s+1}$ being holonomic. This is no longer the case for the multivariate generalisation $\mathscr{D}_X[s_1, \dots, s_n] f^s / \mathscr{D}_X[s_1, \dots, s_n] f^{s+1}$. Tthe notion of relative holonomicity, due to Maisonobe (2016), still holds.

[1]Note: Restate more generally with $+a$ when proof is done.

In this chapter we generalise the refined estimate by Lichtin (1989) to the multivariate situation. The main new ingredient is a induction argument which reduces the problem to the monovariate case where relative holonomicity becomes ordinary holonomicity. This induction is similar to the arguments in Budur et al. (2019).

Theorem 1.1.1. *With notation as above every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{\geq 0}$.

[2]

1.2 Relative Notions

Modules over $\mathscr{A}_X^R$

Let X be a smooth complex irreducible algebraic variety of dimension n and denote $\mathscr{D}_X$ for it's sheaf of rings of algebraic differential operators. For a regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R; \qquad \mathscr{A}_X = \mathscr{A}_X^{\mathbb{C}[s]}$$

where we abbreviated $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$. It will also be convenient to use the abbreviation $\mathcal{O}_X^R := \mathcal{O}_{X \times \operatorname{Spec} R}$.

The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\underline{\mathbb{C}}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration. The associated graded objects are denoted by gr^{rel}. Denote $\pi : T^*X \times \operatorname{Spec} R \to X \times \operatorname{Spec} R$ for the projection map. As in the case of $\mathscr{D}_X$-modules in chapter 1 [3] one can view $\pi^{-1}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$ as a subsheaf of $\mathcal{O}_{T^*X}^R$ and for any $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module $\mathcal{M}$ there is a corresponding module on $T^*X \times \operatorname{Spec} R$ defined by $\mathcal{O}_{T^*X}^R \otimes_{\pi^{-1}\mathrm{gr}^{rel}\mathscr{A}_X^R} \pi^{-1}\mathcal{M}$. By abuse of notation the corresponding module on $T^*X \times \operatorname{Spec} R$ is still denoted with $\mathcal{M}$ and we adopt the perspective that $\mathrm{gr}^{rel}\mathscr{A}_X^R$-modules always live on $T^*X \times \operatorname{Spec} R$ unless explicitly mentioned otherwise.

Similarly to the case of $\mathscr{D}_X$ in the first chapter that [4] it holds that $\mathscr{A}_X^R$ is the sheaf of rings generated by $\mathcal{O}_X^R$ and Θ_X inside of $\mathcal{E}nd_{\underline{\mathbb{C}}}(\mathcal{O}_X^R)$. Giving a left $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X^R$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X. Similarly, giving a right $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X.

The proof of the following results proceeds precisely like the case of $\mathscr{D}_X$-modules which may be found in (Hotta and Tanisaki, 2007, Chapter 2). [5]

Proposition 1.2.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\mathrm{gr}^{rel}\mathscr{M}$ is coherent over $\mathrm{gr}^{rel}\mathscr{A}_X^R$. Such a filtration is called a good filtration.*

[2] Note: Should also give a overview of the results that are already known here.

[3] Note: cite

[4] Note: Cite when C1 is written

[5] Note: Probably cite C1 instead

Proposition 1.2.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen good filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\mathrm{Ch}^{rel}\mathscr{M}$.*

Lemma 1.2.3. *Consider a short exact sequence of coherent $\mathscr{A}_X^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0.$$

Then it holds that

$$\mathrm{Ch}^{rel}\mathscr{M}_2 = \mathrm{Ch}^{rel}\mathscr{M}_1 \cup \mathrm{Ch}^{rel}\mathscr{M}_3.$$

Basic Operations

For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ and left $\mathscr{D}_X$-module $\mathscr{N}$ the tensor product $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ comes equipped with a right $\mathscr{A}_X^R$-module structure defined by

$$f \cdot (m \otimes n) = mf \otimes n; \qquad \xi \cdot (m \otimes n) = m\xi \otimes n - m \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X. Putting multiplication by f on the other side of the tensor product this definition is also applicable for a right $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N}$ if $\mathscr{M}$ is a right $\mathscr{D}_X$-module and $\mathscr{N}$ is a left $\mathscr{A}_X^R$-module. If both are $\mathscr{A}_X^R$-modules there is a right $\mathscr{A}_X^R$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

Similarly, given a left $\mathscr{D}_X$-module $\mathscr{L}$ and a left $\mathscr{A}_X^R$-module $\mathscr{N}$ a left $\mathscr{A}_X^R$-module structure on $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}$ is defined by

$$f \cdot (\ell \otimes n) = \ell \otimes fn; \qquad \xi \cdot (\ell \otimes n) = \xi\ell \otimes n + \ell \otimes \xi n$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

Lemma 1.2.4. *Let $\mathscr{M}, \mathscr{N}$ be right and left $\mathscr{A}_X^R$-modules respectively and let $\mathscr{L}$ be a left $\mathscr{D}_X$-module. Then there is a isomorphism of left $\mathscr{A}_X^R$-modules*

$$(\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}) \otimes_{\mathcal{O}_X^R} \mathscr{N} \cong \mathscr{M} \otimes_{\mathcal{O}_X^R} (\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{N}).$$

Proof. This is immediate by checking that the obvious bijection conserves the $\mathscr{A}_X^R$-module structure. Note that the only nontrivial check is the action of a section ξ from Θ_X. $\square$

Lemma 1.2.5. *Let $\mathscr{N}$ be a left $\mathscr{A}_X^R$-module which is locally free as a $\mathcal{O}_X^R$-module. Consider $\mathscr{A}_X^R$ as a right $\mathscr{A}_X^R$-module, then $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$ is locally free as a right $\mathscr{A}_X^R$-module.*

Proof. Consider local coordinates $x_1, \ldots, x_n$ on X and a local $\mathcal{O}_X^R$-basis $\{n_\beta\}_\beta$ for $\mathscr{N}$. Then $\{1 \otimes n_\beta\}_\beta$ will be a local $\mathscr{A}_X^R$-basis for $\mathscr{A}_X^R \otimes_{\mathcal{O}_X^R} \mathscr{N}$.

To see that this generates the $\mathscr{A}_X^R$-module note that $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ is a $\mathcal{O}_X^R$-basis set when α runs over all multi-indices in $\mathbb{Z}_{\geq 0}^n$. These sections can be recovered using the $\mathscr{A}_X^R$-action on the proposed generating set by induction on $|\alpha|$. Indeed, $\xi^\alpha \cdot (1 \otimes n_\beta)$ equals $\xi^\alpha \otimes n_\beta$ up to a element in the $\mathcal{O}_X^R$-span of $\{\xi^\gamma \otimes n_\beta\}_{|\gamma|<|\alpha|}$.

For the freedom, suppose there is a local $\mathscr{A}_X^R$-relation $\sum_\beta P_\beta \cdot 1 \otimes n_\beta = 0$ with some P_β nonzero. This is of the form $\sum_{\alpha,\beta} f_{\alpha,\beta}\xi^\alpha \cdot 1 \otimes n_\beta = 0$ with the $f_{\alpha,\beta}$ sections of $\mathcal{O}_X^R$ not all equal to zero. Pick some multi-index $\mu \in \mathbb{Z}_{\geq 0}^n$ and of maximal degree such that $f_{\mu,\beta}$ is non-zero for some β. Then, rewriting $\sum_{\alpha,\beta} f_\alpha \xi^\alpha \cdot 1 \otimes n_\beta = 0$ in terms of the $\mathcal{O}_X^R$-basis $\{\xi^\alpha \otimes n_\beta\}_{\alpha,\beta}$ one finds a non-zero coeffiencient at $\xi^\eta \otimes n_\beta$ for some β which is a contradiction. $\square$

Lemma 1.2.6. *The functor $\Omega_X \otimes_{\mathcal{O}_X} -$ which takes a left $\mathscr{A}_X^R$-modules and returns a right $\mathscr{A}_X^R$-module is an equivalence of categories with pseudoinverse $\mathscr{H}om_{\mathcal{O}_X}(\Omega_X, -)$.*

Proof. For any right $\mathscr{A}_X^R$-module $\mathscr{M}$ the left $\mathscr{A}_X^R$-module structure on $\mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathscr{M})$ is defined by

$$(f \cdot \varphi)(\omega) = \varphi(\omega) \cdot f; \qquad (\xi \cdot \varphi)(\omega) = \varphi(\omega \cdot \xi) - \varphi(\omega) \cdot \xi.$$

for any sections f of $\mathcal{O}_X^R$ and ξ in Θ_X.

For any left $\mathscr{A}_X^R$-module $\mathscr{M}$ there is a natural isomorphism of $\mathcal{O}_X^R$-modules $\Omega_X \otimes_{\mathcal{O}_X} \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathscr{M}) \cong \mathscr{M}$ by sending $\omega \otimes \varphi$ to $\varphi(\omega)$. Similarly for any right $\mathscr{A}_X^R$-module $\mathscr{M}$ the isomorphism $\mathscr{M} \cong \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \Omega_X \otimes \mathscr{M})$ associates to a section m of $\mathscr{M}$ the morphism $\omega \mapsto \omega \otimes m$. A direct computation verifies these isomorphisms commute with the $\mathscr{A}_X^R$-module structure. □

Relative Holonomicity

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

Lemma 1.2.7. *The sheaf of $\mathscr{A}_X^R$-modules $\mathscr{M} := \mathscr{A}_Y G^s$ is relatively holonomic with relative characteristic variety*

$$\mathrm{Ch}^{rel}\mathscr{M} := \bigcup_{J \subseteq \{1,\dots,n\}} T^{\perp}Y_J \times \mathbb{C}^p$$

where $Y_J = \{y \in Y : g_j(y) = 0$ for all $j \in J\}$. [6]

Proof. Working on a affine open U we may assume that $G^s = x_1^{a_1 s_1} \cdots x_k^{a_k s_k} u_{k+1}^{s_{k+1}} \cdots , u_p^{s_p}$ for coordinate functions $x_1, \dots, x_p$, natural numbers $a_1, \dots, a_k > 0$ and invertible sections $u_{k+1}, \dots, u_p$ of $\mathcal{O}_Y$. We claim that $\mathscr{A}_U G^S \cong \mathscr{A}_U^R / \mathcal{I}$ where $\mathcal{I}$ is the left ideal sheaf generated by the $x_i \partial_i - a_i s_i$ and $\partial_j - s_j u_j^{-1}$.

Denoting $\varphi : \mathscr{A}_U \to \mathscr{A}_U G^s$ for the obvious surjection we certainly have that $\mathcal{I}$ is a subsheaf of $\ker \varphi$. It remains to show that $\ker \varphi / \mathcal{I} = 0$. Let $P = \sum c_{\alpha\beta} x^\alpha \partial^\beta$ represent some section in $\ker \varphi / \mathcal{I}$ where the non-zero $c_{\alpha\beta}$ do not vanish in 0. By the relations $\partial_j - s_j u_j^{-1}$ it can be assumed that the only nonzero components of the multi-indices β lie in $1, \dots, k$. By $\mathscr{A}_U$-linear combinations of $x_i \partial_i - a_i s_i$ it can further be enforced that the terms are either have $\alpha_i = 0$ or $\beta_i = 0$ for any $i = 1, \dots, k$. When acting on G^s with the remainder the coefficients all end on different monomial coefficients to G^s which means they have to be zero in order for P to be in the kernel. This shows $\ker \varphi = \mathcal{I}$ as desired.

It follows that $\mathrm{gr}^{rel} \mathcal{A}_U G^s \cong \mathrm{gr}^{rel} \mathscr{A}_U / \mathrm{gr}^{rel} \mathcal{I}$. It holds that $\mathrm{gr}^{rel}\mathcal{I}$ is generated by $x_i \xi_i$ and ξ_j whence the result follows. □

The following lemma and it's proof may be found in Maisonobe (2016).

Lemma 1.2.8. *Let $\mathscr{M}$ be a finitely generated $\mathscr{A}_Y^R$-module. Suppose that $\mathrm{Ch}^{rel}\mathscr{M} \subseteq \Lambda \times \operatorname{Spec} R$ for some, not necessarily irreducible, conic Lagrangian subvariety $\Lambda \subseteq T^*X$. Then $\mathscr{M}$ is relative holonomic over R.*

[6] $T^{\perp}$ denotes covectors annihilating the tangent space.

The Bernstein-Sato ideal may be defined more generally for any $\mathscr{A}_X^R$-module $\mathscr{M}$ as $B_{\mathscr{M}} := \mathrm{Ann}_R \mathscr{M}$. To see how this generalises B_F one considers $\mathscr{A}_X^R F^s$ as a $\mathscr{A}_X^R\langle t\rangle$-module. Here t is a new variable which commutes with sections of $\mathscr{D}_X$ and satisfies $ts_i - s_i t = 1$ for any $i = 1, \dots, n$. The $\mathscr{A}_X^R\langle t\rangle$-module structure on $\mathscr{A}_X^R F^s$ is then defined by extending $tF^s = F^{s+1}$. From this point of view $B_F = B_{\mathscr{A}_X^R F^s / t\mathscr{A}_X^R F^s}$.

The Bernstein-Sato ideal may be recovered from the characteristic variety. The following result is due to Maisonobe (2016) in the analytical case and Budur et al. (2019) in the algebraic.

Proposition 1.2.9. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$ module. Then $Z(B_{\mathscr{M}}) = \pi_2(\mathrm{Ch}^{rel}(\mathscr{M}))$ where $\pi_2 : T^*X \times \operatorname{Spec} R \to \operatorname{Spec} R$ is the projection on the second coordinate.*

1.3 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.[7]

Transfer Modules and $\mathscr{A}_Y^R$-module Direct Image

Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties, by abuse of notation we will also denote μ for the induced map from $Y \times \operatorname{Spec} R$ to $X \times \operatorname{Spec} R$.

A-priori it is not even clear what $\mathscr{A}_X^R$-module should correspond to $\mathscr{A}_Y^R$ since there is no natural push forward of vector fields. This issue may be resolved by use of the transfer $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X^R)$-bimodule $\mathscr{A}_{Y\to X}^R := \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}\mathscr{A}_X^R$. Here, the right $\mu^{-1}\mathscr{A}_X^R$-module structure is just the action on the second component and definitions like section 1.2 are used to define the left $\mathscr{A}_Y^R$-module structure. To be precise

$$f \cdot (g \otimes \mu^{-1}h_X) = fg \otimes \mu^{-1}h_X; \qquad \xi \cdot (g \otimes \mu^{-1}h_X) = \xi g \otimes \mu^{-1}h_X + g \otimes T\mu(\xi)\mu^{-1}h_X$$

for any sections f of $\mathcal{O}_Y^R$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Definition 1.3.1. *The direct image functor $\int_\mu$ from $\boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ to $\boldsymbol{D}^{b,r}(\mathscr{A}_X^R)$ is defined to be $\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_\mu^j \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be surpressed whenever there is no ambiguity.*

To compute the direct image $\int^j \mathscr{M}$ a resolution for the transfer bimodule $\mathscr{A}_{Y\to X}$ is required.

Definition 1.3.2. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module, the relative Spencer complex $\mathrm{Sp}_Y^\bullet(\mathscr{M})$ is a complex of right $\mathscr{A}_Y^R$-modules, concentrated in negative degrees, with $\mathrm{Sp}_Y^{-k}(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_Y$ and as differential the right-$\mathscr{A}_Y^R$-linear map δ given by*

$$m \otimes \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \widehat{\xi_j} \wedge \cdots \wedge \xi_k$$
$$- \sum_{i=1}^{k} (-1)^i m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k$$

[7]Note: more introduction

The following lemma and it's proof are a generalisation of exercise 1.20 in Sabbah (2011) to the relative case.

Lemma 1.3.1. *The relative Spencer complex* $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ *is a locally free resolution of* $\mathcal{O}_X^R$ *as left* $\mathscr{A}_X^R$*-module.*

Proof. Define a filtration on $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ by the complexes $F_k \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ which have term $F_{k-\ell}\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot ℓ. This filtration induces the complexes $\mathrm{gr}_k^{rel} \mathrm{Sp}_X^\bullet(\mathscr{A}_Y^R)$ with term $\mathrm{gr}_{k-\ell}^{rel} \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y$ in spot ℓ.

In local coordinates $x_1, \ldots, x_n$ one finds that $\mathrm{gr}^{rel}\mathrm{Sp}_Y^\bullet := \oplus_k \mathrm{gr}^{rel}{}_k \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is the Koszul complex of $\mathcal{O}_Y^R[\xi_1, \ldots, \xi_n] = \mathrm{gr}^{rel}\mathscr{A}_Y^R$ with respect to $\xi_1, \ldots, \xi_n$.[8] Since $\xi_1, \ldots, \xi_n$ form a regular sequence a standard result on Koszul complexes[9] yields that $\mathrm{gr}^{rel}\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a locally free resolution of $\mathcal{O}_Y^R$ as $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module.

On the other hand, it is immediate that $F_0 \mathrm{Sp}^\bullet(\mathscr{A}_Y^R) = \mathrm{gr}_0^{rel} \mathrm{Sp}^\bullet(\mathscr{A}_Y^R)$ is $\mathcal{O}_Y^R$ viewed as a complex. Hence, there is no contribution to $\mathrm{gr}^{rel} \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ from the terms of $k > 0$. That is to say that $\mathrm{gr}_k^{rel} \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is quasi-isomorphic to the zero complex for $k > 0$. Hence, $F_0 \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \hookrightarrow \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a quasi-isomorphism by the exactness of the direct limit. [10] It follows that $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ is a resolution of $\mathcal{O}_X^R$. That the terms of $\mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R)$ are locally free follows from lemma 1.2.5 after some minor adjustments in the statement and proof. □

Define the transfer Spencer complex as the complex of $(\mathscr{A}_Y^R, \mu^{-1}\mathscr{A}_X)$-bimodules given by $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R) := \mathrm{Sp}_Y^\bullet(\mathscr{A}_Y^R) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X}^R$. The following lemma and it's proof are direct generalisation of exercise 3.4 in Sabbah (2011) to the relative case.

Lemma 1.3.2. *The transfer Spencer complex* $\mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R)$ *is a resolution of* $\mathscr{A}_{Y\to X}^R$ *as a bimodule by locally free left* $\mathscr{A}_Y^R$*-modules.*

Proof. To see that the terms of the complex are locally free recall from lemma 1.2.4 the following isomorphisms of left $\mathscr{A}_Y^R$-modules

$$(\mathscr{A}_Y^R \otimes_{\mathcal{O}_Y} \wedge^\ell \Theta_Y) \otimes_{\mathcal{O}_Y^R} \mathscr{A}_{Y\to X} \cong \mathscr{A}_Y^R \otimes_{\mathcal{O}_Y^R} (\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}).$$

[11] Note that $\mathscr{A}_{Y\to X}^R$ is a locally free $\mathcal{O}_Y^R$-module since it is the pullback of a locally free module on $X \times \operatorname{Spec} R$. Combined with the fact that $\wedge^\ell \Theta$ is a locally free $\mathcal{O}_Y$-module this yields that $\wedge^\ell \Theta_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_{Y\to X}$ is a locally free $\mathcal{O}_Y^R$-module. Hence lemma 1.2.4 is applicable and yields that the terms of the transfer Spencer complex are locally free $\mathscr{A}_Y^R$-modules.

That the transfer Spencer complex is a resolution of $\mathscr{A}_{Y\to X}^R$ follows from lemma 1.3.1 by using that $\mathscr{A}_{Y\to X}^R$ is a locally free and hence flat over $\mathcal{O}_Y^R$. □

Since tensoring with locally free modules yields a exact functor this simplifies the computation of the direct image as follows.

Corollary 1.3.3. *It holds that* $\int = \boldsymbol{R}\mu_*(- \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}_{Y\to X}^\bullet(\mathscr{A}_Y^R))$.

[8] Note: Should I explain what a Koszul complex is?

[9] Note: Give reference to some book

[10] Note: Would be nice to give a reference, proof may be found on stackexchange

[11] Note: May be possible to remove this step from the proof and removing need for minor adjustment of previous proof.

Proposition 1.3.4. *For any short exact sequence of $\mathscr{A}_X^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

there is a long exact sequence in direct images

$$0 \to \int^0 \mathscr{M}_1 \to \int^0 \mathscr{M}_2 \to \int^0 \mathscr{M}_3 \to \int^1 \mathscr{M}_1 \to \cdots .$$

[12] A strategy one can employ in proving theorems on some space X is by first solving them on a nicer space Y equipped with a map $Y \to X$. This can then be related to the problem on X by use of the direct image. For this purpose it is useful that any global section of $\mathscr{M}$ induces a global section of the direct image. This is usually done in the language of left modules but for us it is more natural to work with right $\mathscr{A}_Y^R$-modules.

Lemma 1.3.5. *Let $\mathscr{M}$ be a right $\mathscr{A}_Y^R$-module. Then any global section $m \in \Gamma(Y, \mathscr{M})$ induces a global section of $\int^0 \mathscr{M})$.*

Proof. By the Leray spectral sequence there is a functorial isomorphism

$$\mathbb{H}^\bullet(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)) \cong \mathbb{H}^\bullet(X, \boldsymbol{R}\mu_*(\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R))).$$

In particular it follows that $\mathbb{H}^0(Y, \mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^\bullet_{Y\to X}(\mathscr{A}_Y^R)) \cong \Gamma(X, \int^0 \mathscr{M})$. The Čech spectral sequence now induces the desired global section in the direct image based on the section $m \otimes 1$ of $\mathscr{M} \otimes_{\mathscr{A}_Y^R} \mathrm{Sp}^0_{Y\to X}(\mathscr{A}_Y^R)$. □

Functorial Properties of the Direct Image

The following properties are well-known for $\mathscr{D}_X$-modules. We only sketch the main steps of the proofs so that it is clear that no new problems for $\mathscr{A}_X^R$-modules and refer to chapters 6 and 9 in Borel (1987) for the details.

Theorem 1.3.6. *Let $\mu : Z \to Y$ and $\nu : Y \to X$ be morphisms of smooth algebraic varieties. There is a isomorphism of functors $\int_{\nu\circ\mu} = \int_\nu \int_\mu$.*

Proof. By definition of the direct image we must compare the following functors

$$\int_\nu \int_\mu = \boldsymbol{R}\nu_*(\boldsymbol{R}\mu_*(- \otimes^L_{\mathscr{A}_Z^R} \mathscr{A}^R_{Z\to Y}) \otimes^L_{\mathscr{A}_Y^R} \mathscr{A}^R_{Y\to X})$$

$$\int_{\nu\circ\mu} = \boldsymbol{R}(\nu\circ\mu)_*(- \otimes^L_{\mathscr{A}_Z^R} \mathscr{A}^R_{Z\to X})$$

Note that $\boldsymbol{R}(\nu\circ\mu)_* = \boldsymbol{R}\nu_* \circ \boldsymbol{R}\mu_*$. A calculation expanding the definitions shows that

$$\mathscr{A}^R_{Z\to X} \cong \mathscr{A}^R_{Z\to Y} \otimes^L_{\mu^{-1}\mathscr{A}_Y^R} f^{-1}\mathscr{A}^R_{Y\to X}.$$

The result now follows by applying the general observation that for any $\mathscr{F}^\bullet \in \boldsymbol{D}^{b,r}(\mu^{-1}\mathscr{A}_Y^R)$ and $\mathscr{G}^\bullet \in \boldsymbol{D}^{b,r}(\mathscr{A}_Y^R)$ there is a isomorphism

$$(\boldsymbol{R}\mu_*\mathscr{F}^\bullet) \otimes^L_{\mathscr{A}_Y^R} \mathscr{G}^\bullet \cong \boldsymbol{R}\mu_*(\mathscr{F}^\bullet \otimes^L_{\mu^{-1}\mathscr{A}_Y^R} \mu^{-1}\mathscr{G}^\bullet).$$

□

[12]Note: Provide reference

This theorem reduces the computation of direct images to closed embeddings and projections by writing $\mu = \pi \circ \iota$ for $\iota : Y \to Y \times X$ and $\pi : Y \times X \to X$.

Denote by $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}^R_Y)$ the full subcategory of $\boldsymbol{D}^{b,r}(\mathscr{A}^R_Y)$ consisting of those complexes of right $\mathscr{A}^R_Y$-modules whose cohomology sheaves are quasi-coherent over $\mathcal{O}_Y \times \mathcal{O}_{\operatorname{Spec} R}$. Similarly for $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}^R_Y)$ with the cohomology being coherent $\mathscr{A}^R_Y$-modules.

Theorem 1.3.7. *Let $\mu : Y \to X$ be a morphism of smooth algebraic varieties. Then the direct image $\int$ takes $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}^R_Y)$ into $\boldsymbol{D}^{b,r}_{qc}(\mathscr{A}^R_X)$. Moreover, when μ is proper the direct image takes $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}^R_Y)$ into $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}^R_X)$.*

Proof. This proof boils down to a reduction to a similar theorem for $\mathcal{O}^R_X$-modules.

One can show that every object in $\boldsymbol{D}^{b,r}_{coh}(\mathscr{A}^R_Y)$ can be represented by a complex with terms $\mathscr{F}^p \otimes_{\mathcal{O}^R_Y} \mathscr{A}^R_Y$ where $\mathscr{F}^p$ is a coherent $\mathcal{O}^R_Y$-modules. This ultimately reduces to computation of the direct image to a spectral sequence with

$$E_1^{p,q} = (R^q \mu_* \mathscr{F}^p) \otimes_{\mathcal{O}^R_X} \mathscr{A}^R_X.$$

The properness of μ yields that $R^q \mu_* \mathscr{F}^p$ is coherent as a $\mathcal{O}^R_X$-module. In particular the $E_1^{p,q}$-terms are coherent as $\mathscr{A}^R_X$ which implies that the same holds for the cohomology sheaves of the direct image. $\square$

Kashiwara's Estimate for the Characteristic Variety

Let $\mu : Y \to X$ be a proper morphism of smooth algebraic varieties. Given a coherent $\mathscr{A}^R_X$-module $\mathscr{M}$ with relative characteristic variety $\operatorname{Ch}^{rel}\mathscr{M}$. We desire to estimate $\operatorname{Ch}^{rel}\int^j\mathscr{M}$ in terms of $\operatorname{Ch}^{rel}\mathscr{M}$. Such a estimate in the non-relative case is known due to Kashiwara.

The original proof by Kashiwara (1976) uses the theory of microlocal differential operators. The idea of the following proof is due to Malgrange (1985) in a K-theoretic context. We follow the exposition of Sabbah (2011) and replace it with the corresponding relative notions.

Consider the following cotangent diagram

$$\begin{tikzcd} & \mu^*T^*X \times \operatorname{Spec} R \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y \times \operatorname{Spec} R & & T^*X \times \operatorname{Spec} R \end{tikzcd}$$

where the maps $T^*\mu$ and $\widetilde{\mu}$ act on the first component.

Theorem 1.3.8. *Let $\mathscr{M}$ be a coherent $\mathscr{A}^R_Y$-module. Then, for any $j \geq 0$, we have*

$$\operatorname{Ch}^{rel}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\operatorname{Ch}^{rel}\mathscr{M})\right).$$

The first step is to note that a similar inclusion is easy for the $\operatorname{gr}^{rel}\mathscr{A}^R_Y$-modules. The direct image functor on $\operatorname{gr}^{rel}\mathscr{A}^R_Y$-modules $\mathcal{M}$ is defined by $\int^j \mathcal{M} := \boldsymbol{R}^j\widetilde{\mu}_*(\boldsymbol{L}(T^*\mu)^*\mathcal{M})$. Here, $(T^*\mu)^*(-)$ produces a sheaf on $\mu^*T^*X \times \operatorname{Spec} R$ by $- \otimes_{\mu^{-1}\mathcal{O}^R_X} \operatorname{gr}^{rel}\mathscr{A}^R_X$. Looking at the supports the following result is immediate.

Lemma 1.3.9. *For any* $\mathrm{gr}^{rel}\mathscr{A}_Y^R$*-module* $\mathcal{M}$ *it holds that*

$$\operatorname{supp} \int^j \mathcal{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1} \operatorname{supp} \mathcal{M}\right).$$

Applying this lemma to $\mathrm{gr}^{rel}\mathscr{M}$ it remains to show that $\operatorname{supp} \mathrm{gr}^{rel} \int^j \mathscr{M} \subseteq \operatorname{supp} \int^j \mathrm{gr}^{rel}\mathscr{M}$. This is proved in proposition 1.3.15. The main technical ingredient in the proof is the Rees modules associated to a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$.

Definition 1.3.3. *Let z be a new variable. The Rees sheaf of rings $\mathcal{R}\mathscr{A}_Y^R$ is defined as the subsheaf $\oplus_p F_p\mathscr{A}_Y^R z^p$ of $\mathscr{A}_Y^R \otimes_{\mathbb{C}} \mathbb{C}[z]$. Similarly, any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ gives rise to a $\mathcal{R}\mathscr{A}_Y$-module $\mathcal{R}\mathscr{M} := \oplus_p F_p\mathscr{M} z^p$.*

Given a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it follows that $\mathcal{R}\mathscr{M}$ is a coherent $\mathcal{R}\mathscr{A}_Y^R$-module similarly to proposition 1.2.1. The following isomorphisms of filtered modules on $Y \times \operatorname{Spec} R$ are essential. They mean that the Rees module can be viewed as a parametrisation of various relevant modules.

$$\frac{\mathcal{R}\mathscr{M}}{(z-1)\mathcal{R}\mathscr{M}} \cong \mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}\mathscr{M}; \qquad \frac{\mathcal{R}\mathscr{M}}{z^\ell\mathcal{R}\mathscr{M}} \cong \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}.$$

Here $\mathrm{gr}^{rel}_{[\ell]}$ takes a filtered object and returns $\oplus_k F_k/F_{k-\ell}$. The first formula may be be used to find a corresponding filtered $\mathscr{A}_Y^R$-module for any graded $\mathcal{R}\mathscr{A}_Y^R$-module without $\mathbb{C}[z]$-torsion.

The jth direct image of a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is the sheaf of $\mathcal{R}\mathscr{A}_X^R$-modules on $X \times \operatorname{Spec} R$ defined by $\int^j \mathcal{M} = \boldsymbol{R}^j\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X})$. Here the filtration on $\mathscr{A}^R_{Y\to X}$ is defined by $F_i\mathscr{A}^R_{Y\to X} = \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1}F_i\mathscr{A}_X^R$. The direct image may be restricted to the category of graded Rees modules in which case it returns a graded Rees module. Coherence is preserved similarly to theorem 1.3.7.

Recall that a $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-modules on $Y \times \operatorname{Spec} R$ could be be viewed as a sheaf on $T^*Y \times \operatorname{Spec} R$ and is already equipped with a direct image. The Rees module viewpoint agrees with the earlier definition by the following lemma.

Lemma 1.3.10. *Consider a filterd $\mathscr{A}_Y^R$-module $\mathscr{M}$. Then viewing $\int^j \mathcal{R}\mathscr{M}/z\mathcal{R}\mathscr{M}$ with it's $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module structure as a sheaf on $T^*X \times \operatorname{Spec} R$ recovers the $\mathrm{gr}^{rel}\mathscr{A}_Y^R$-module direct image $\int^j \mathrm{gr}^{rel}\mathscr{M}$. Viewing $\int^j \mathcal{R}\mathscr{M}/(z-1)\mathscr{M}$ as a $\mathscr{A}_X^R$-module recovers $\int^j\mathscr{M}$.*

Proof. We give the proof for $\int^j \mathrm{gr}^{rel}\mathscr{M}$, the proof for $\int^j\mathscr{M}$ is similar but easier.

Consider the following Cartesian square

$$\begin{tikzcd}
\mu^*T^*X \times \operatorname{Spec} R \arrow[r, "T^*\mu"] \arrow[d, "\widetilde{\mu}"] & T^*Y \times \operatorname{Spec} R \arrow[r, "\pi_Y"] & Y \times \operatorname{Spec} R \arrow[d, "\mu"] \\
T^*X \times \operatorname{Spec} R \arrow[rr, "\pi_X"] & & X \times \operatorname{Spec} R.
\end{tikzcd}$$

Since π_X is flat the derived version of the flat base change theorem yields that [13]

$$\boldsymbol{L}\pi_X^*\boldsymbol{R}\mu_*\Big(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}\Big) = \boldsymbol{R}\widetilde{\mu}_*\boldsymbol{L}(T^*\mu \circ \pi_Y)^*\Big(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}\Big).$$

[13] Note: Check in detail that the theorem is applicable and has this conclusion due to flatness

Since π_X is flat it follows that $\mathscr{H}^j \boldsymbol{L}\pi_X^*(-) = \pi_X^* \mathscr{H}^j(-)$[14]. It now suffices to show that the right hand side is $\int \operatorname{gr}^{rel}\mathscr{M}$.

Since π_Y is flat it holds that $\boldsymbol{L}(T^*\mu \circ \pi_Y)^* = \boldsymbol{L}(T^*\mu)^* \circ \boldsymbol{L}\pi_Y^*$[15]. We show that $\boldsymbol{L}\pi_Y^*(\frac{\mathcal{R}\mathscr{M}}{z\mathcal{R}\mathscr{M}} \otimes^L_{\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}^R_{Y\to X}) \cong \operatorname{gr}^{rel}\mathscr{M} \otimes^L_{\mu^{-1}\mathcal{O}_X^R} \widetilde{\mu}^* \operatorname{gr}^{rel}\mathscr{A}_X^R$ from which the result follows immediately.

Let $\mathcal{F}^\bullet$ denote a bimodule resolution for $\mathcal{R}\mathscr{A}^R_{Y\to X}$ by locally free left $\mathcal{R}\mathscr{A}^R_Y$-modules. Then $(\mathcal{R}\mathscr{A}_Y^R / z\mathcal{R}\mathscr{A}_Y^R) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{F}^\bullet$ is a bimodule resolution for $\operatorname{gr}^{rel}\mathscr{A}^R_{Y\to X}$ by locally free left $\operatorname{gr}^{rel}\mathscr{A}_Y^R$-modules. Now $\boldsymbol{L}\pi_Y^*$ just means applying $\pi^{-1}(-) \otimes \mathcal{O}_{T^*Y}$ to the terms of this free resolution. Due to flatness this yields a free resolution in $\pi^* \operatorname{gr}^{rel}\mathscr{A}_Y^R$-modules of $\pi^* \operatorname{gr}^{rel}\mathscr{A}^R_{Y\to X}$. Since $\operatorname{gr}^{rel}\mathscr{A}^R_{Y\to X} = \mathcal{O}_Y^R \otimes_{\mu^{-1}\mathcal{O}_X^R} \mu^{-1} \operatorname{gr}^{rel}\mathscr{A}_X^R$ and $\pi^*\mu^* = \widetilde{\mu}^*\pi^*$ the desired equality follows. [16] ☐

It turns out that one can directly compare $\operatorname{gr}^{rel}_{[\ell]}\int^j\mathscr{M}$ and $\int^j \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$ when ℓ is large. Some care is required since since $\int^j \mathcal{R}\mathscr{M}$ may have $\mathbb{C}[z]$-torsion.

Lemma 1.3.11. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, for sufficiently large ℓ, the kernel of z^ℓ in $\int^j \mathcal{R}\mathscr{M}$ stabilises. For such ℓ the quotient $\int^j \mathcal{R}\mathscr{M} / \ker z^\ell$ is the $\mathcal{R}\mathscr{A}_X^R$-coherent module associated to a good filtration on $\int^j\mathscr{M}$.*

Proof. By $\int \mathcal{R}\mathscr{M}$ being coherent over the sheaf of Noetherian rings $\mathcal{R}\mathscr{A}_X^R$ it follows that $\ker z^\ell$ stabilises.

Now consider the short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z-1} \mathcal{R}\mathscr{M} \to \mathscr{M} \to 0$. This induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z-1} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z-1} \cdots.$$

Since $\int^{j+1} \mathcal{R}\mathscr{M}$ is a graded $\mathcal{R}\mathscr{A}_X^R$-module it follows that $z-1$ is injective whence $\int^j \mathcal{R}\mathscr{M}/(z-1)\int^j \mathcal{R}\mathscr{M} \cong \int^j\mathscr{M}$. This yields the desired result using that $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell$ is $\mathbb{C}[z]$-torsion free and the isomorphism

$$\frac{\int^j \mathcal{R}\mathscr{M}}{(z-1)\int^j \mathcal{R}\mathscr{M}} \cong \frac{\int^j \mathcal{R}\mathscr{M}/\ker z^\ell}{(z-1)(\int^j \mathcal{R}\mathscr{M}/\ker z^\ell)}.$$

☐

From now on we equip $\int^j\mathscr{M}$ with the good filtation inherited from the Rees module's direct image.

Lemma 1.3.12. *Consider a $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration. Then, if ℓ is sufficiently large, $\operatorname{gr}^{rel}_{[\ell]}\int^j\mathscr{M}$ is a subquotient of $\int^j \operatorname{gr}^{rel}_{[\ell]}\mathscr{M}$.*

Proof. The short exact sequence $0 \to \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \mathcal{R}\mathscr{M} \to \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to 0$ induces a long exact sequence

$$\cdots \to \int^j \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \int^j \mathcal{R}\mathscr{M} \to \int^j \mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M} \to \int^{j+1} \mathcal{R}\mathscr{M} \xrightarrow{z^\ell} \cdots.$$

[14] Note: $\mathscr{H}^j \boldsymbol{L}\pi_X^*(-) = \pi_X^* \mathscr{H}^j(-)$?
[15] Note: $\boldsymbol{L}(T^*\mu \circ \pi_Y)^* = \boldsymbol{L}(T^*\mu)^* \circ \boldsymbol{L}\pi_Y^*$?
[16] Note: Write out more

Hence, $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$ is a submodule of $\int^j(\mathcal{R}\mathscr{M}/z^\ell\mathcal{R}\mathscr{M})$ and it remains to show that $\mathcal{R}\int^j\mathscr{M}/z^\ell\mathcal{R}\int^j\mathscr{M}$ is a quotient of $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$.

Let ℓ be sufficiently large so that lemma 1.3.11 yields a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong \mathcal{R}\int^j \mathscr{M}$. The map z^ℓ induces a isomorphism $\int^j \mathcal{R}\mathscr{M}/\ker z^\ell \cong z^\ell \int^j \mathcal{R}\mathscr{M}$. Therefore $z^\ell \int^j \mathcal{R}\mathscr{M}/z^{2\ell} \int^j \mathcal{R}\mathscr{M} \cong \mathcal{R}\int^j\mathscr{M}/z^\ell\mathcal{R}\int^j\mathscr{M}$. The desired quotient follows by applying the map $m \mapsto z^\ell m$ on $\int^j \mathcal{R}\mathscr{M}/z^\ell \int^j \mathcal{R}\mathscr{M}$. □

The main remaining task is to relate these results to the desired case of $\ell = 1$.

Definition 1.3.4. *For any $\ell \geq 1$ the G-filtration on a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$ is defined by the decreasing sequence of $\mathrm{gr}_{[\ell]}^{rel}\mathscr{A}_Y^R$-submodules $G_j\mathcal{M} := z^j\mathcal{M}$.*

Lemma 1.3.13. *For any filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ and $\ell \geq 1$ there is the a isomorphism of $\mathrm{gr}\,\mathscr{A}_Y^R$-modules*

$$\mathrm{gr}^G \mathrm{gr}_{[\ell]}^{rel}\mathscr{M} \cong (\mathrm{gr}^{rel}\mathscr{M})^\ell.$$

Proof. This follows from directly from the fact that $G_j\,\mathrm{gr}_{[\ell]}^{rel}\mathscr{M} = \oplus_k F_{k-j}\mathscr{M}/F_{k-\ell}\mathscr{M}$. □

Lemma 1.3.14. *Consider a $\mathcal{R}\mathscr{A}_Y^R$-module $\mathcal{M}$. Then one has a isomorphism $\mathrm{gr}^G \int \mathcal{M} \cong \int \mathrm{gr}^G \mathcal{M}$ in $\boldsymbol{D}^{b,r}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$.*

Proof. Writing out the direct image functors the desired result is a isomorphism

$$\mathrm{gr}^G \boldsymbol{R}\mu_*(\mathcal{M} \otimes^L_{\mathcal{R}\mathscr{A}_Y^R} \mathcal{R}\mathscr{A}_{Y\to X}^R) \cong \boldsymbol{R}\mu_*(\mathrm{gr}^G \mathcal{M} \otimes^L_{\mu^{-1}\mathscr{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R).$$

The proof of the commutation proceeds in two steps corresponding to the two derived functors.

Let $\mathscr{F}^\bullet$ be a bimodule resolution for $\mathcal{R}\mathscr{A}_{Y\to X}^R$ by locally free left $\mathcal{R}\mathscr{A}_Y^R$-modules. There is a G-filtration on this complex given by $z^j(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) = (z^j\mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. By the flatness of locally free sheaves and the short exact sequence $0 \to \oplus_j z^j\mathcal{M} \to \oplus_j z^{j-1}\mathcal{M} \to \mathrm{gr}^G \mathcal{M} \to 0$ it follows that $\mathrm{gr}^G(\mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet) \cong (\mathrm{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$. Further, by the argument in the proof of lemma 1.3.10 the complex of $\mathrm{gr}^G \mathscr{A}_Y^R$-modules $(\mathrm{gr}^G \mathcal{M}) \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ can be viewed as a representative of $(\mathrm{gr}^G \mathcal{M}) \otimes^L_{\mu^{-1}\mathscr{O}_X^R} \mathrm{gr}^{rel}\mathscr{A}_X^R$.[17]

Denote $\mathcal{G}(-)$ for the functor which takes a sheaf complex and returns its Godement resolution. Flabby sheaves are acyclic for μ_* so the Godement resolution may be used to compute $\boldsymbol{R}\mu_*$. Moreover, since the terms of a Godement resolution are essentially direct sums of formal products of stalks, it is immediate that $z^i\mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(z^i\mathcal{N}^\bullet)$ and that $\mathrm{gr}^G \mathcal{G}(\mathcal{N}^\bullet) = \mathcal{G}(\mathrm{gr}^G \mathcal{N}^\bullet)$ for any complex of right $\mu^{-1}\mathcal{R}\mathscr{A}_X^R$-modules $\mathcal{N}^\bullet$. Applying μ_* to these equalities and setting $\mathcal{N}^\bullet = \mathcal{M} \otimes_{\mathcal{R}\mathscr{A}_Y^R} \mathscr{F}^\bullet$ yields the desired result. □

Proposition 1.3.15. *For a filtered $\mathscr{A}_Y^R$-module $\mathscr{M}$ with a good filtration it holds that*

$$\mathrm{supp}\,\mathrm{gr}^{rel}\int^j \mathscr{M} \subseteq \mathrm{supp}\int^j \mathrm{gr}^{rel}\mathscr{M}.$$

[17] Note: Check after lemma is entirely proven

Proof. Let $\ell \geq 0$ be sufficiently large so that lemma 1.3.12 holds, that is to say that $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$. By lemma 1.3.13 it holds that $\mathrm{gr}^G \mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M} \cong (\mathrm{gr}^{rel}\int \mathscr{M})^\ell$. Since $\mathrm{gr}^{rel}_{[\ell]}\int^j \mathscr{M}$ is a subquotient of $\int \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ it remains to show that the support of $\mathrm{gr}^G \int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ is a subset of the support of $\int^j \mathrm{gr}\, \mathscr{M}$.

This can be established with the spectral sequence associated of the G-filtered complex $\int \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$. Since the G-filtration is finite on $\mathrm{gr}^{rel}_{[\ell]}\mathscr{A}^R_X$-modules the associated spectral sequence abbuts by general results[18]. To be precise the associated spectral sequence with terms $E^2_{pq} = \mathscr{H}^{p+q} \mathrm{gr}^G \int \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ abuts to $\mathrm{gr}^G \int \mathscr{M}$. By lemma 1.3.14 and lemma 1.3.13 it holds that $E^2_{pq} \cong (\int^{p+q} \mathrm{gr}\, \mathscr{M})^\ell$. [19] It follows that $\mathrm{supp}\, \mathrm{gr}^G \int^j \mathrm{gr}^{rel}_{[\ell]}\mathscr{M}$ is a subset of the support of $\int \mathrm{gr}\, \mathscr{M}$ which completes the proof. □

1.4 Non-commutative Homological Notions

In this section we discuss homological notions associated to the Ext-functor the noncommutative sheaf of rings $\mathscr{A}^R_X$. These notions are particularly well-behaved for relatively holonomic modules. The results are sheaf-theoretic rewording of the similar results in Budur et al. (2019) which are themselves derived from the appendices of Björk (1993). Throughout X is assumed to be a smooth variety of dimension n.

Definition 1.4.1. *For a nonzero coherent sheaf of $\mathscr{A}^R_X$-modules $\mathscr{M}$ the smallest integer $k \geq 0$ such that $\mathrm{Ext}^k_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X) \neq 0$ is called the grade of $\mathscr{M}$ and is denoted $j(\mathscr{M})$.*

The following proposition gives geometrical meaning to grades.

Proposition 1.4.1. *For coherent $\mathscr{A}^R_X$-modules $\mathscr{M}$ it holds that*

$$j(\mathscr{M}) + \dim \mathrm{Ch}^{rel}\mathscr{M} = 2n + \dim R$$

where $\dim R$ *denotes the Krull dimension of the ring R.*

Proof. This is lemma 3.2.2 in Budur et al. (2019). □

Definition 1.4.2. *A nonzero coherent sheaf of $\mathscr{A}^R_X$-modules $\mathscr{M}$ is called j-pure if $j(\mathscr{N}) = j(\mathscr{M}) = j$ for every nonzero submodule $\mathscr{N}$.*

Lemma 1.4.2. *Let $\mathscr{M}$ be a nonzero coherent $\mathscr{A}_X$-module of grade j. Then $\mathrm{Ext}^k_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A})$ has grade greater than or equal to k for any $k \geq 0$ and $\mathrm{Ext}^j_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X)$ is a j-pure $\mathscr{A}^R_X$-module.*

Moreover $\mathscr{M}$ is j-pure if and only if $\mathrm{Ext}^j_{\mathscr{A}^R_X}(\mathrm{Ext}^k_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X), \mathscr{A}^R_X) = 0$ for every $k \neq j$.

Proof. This is lemma 4.3.5 in Budur et al. (2019). □

Lemma 1.4.3. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}^R_X$-module of grade j. Then $\mathrm{Ext}^j_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X)$ is a relative holonomic $\mathscr{A}^R_X$-module and*

$$\mathrm{Ch}^{rel}\, \mathrm{Ext}^j_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X) \subseteq \mathrm{Ch}^{rel}\mathscr{M}.$$

[18]Note: Found spectral sequence result online, add good reference.

[19]Note: Or E^1? Seems to depend on preference but should actually matter somewhat for the differentials.

Proof. This is lemma 3.2.4 in Budur et al. (2019). □

Lemma 1.4.4. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module and let $I \subseteq R$ be an ideal. Then, for any $k \geq 0$, $\operatorname{Tor}_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I})$ is a relative holonomic $\mathscr{A}_X^{R/I}$-module.*

Proof. Compute $\operatorname{Tor}_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I})$ with a locally free $\mathscr{A}_X^R$-resolution of $\mathscr{A}_X^{R/I}$. Then lemma 1.2.3 and lemma 1.2.8 show that it is a relative holonomic $\mathscr{A}_X^R$-module. This means that it admits a relative filtration over $\mathscr{A}_X^R$ such that the graded object has a support of the form

$$\operatorname{supp} \operatorname{gr}^{rel}_{\mathscr{A}_X^R} \operatorname{Tor}_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I}) = \bigcup \Lambda \times S_\Lambda$$

for Lagrangian subvarieties $\Lambda \subseteq T^*X \times \operatorname{Spec} R$ and algebraic varieties $S_\Lambda \subseteq \operatorname{Spec} R$.

On the other hand $\operatorname{Tor}_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I})$ is also a coherent $\mathscr{A}_X^{R/I}$-module. The earlier filtration descends to a filtration over $\mathscr{A}_X^{R/I}$ and it holds that

$$\operatorname{supp}^{rel}_{\mathscr{A}_X^{R/I}} \operatorname{Tor}_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I}) = (\operatorname{Id}_{T^*X} \times \Delta)^{-1}(\operatorname{supp} \operatorname{gr}^{rel}_{\mathscr{A}_X^R} \operatorname{Tor}_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/I}))$$

where $\Delta : \operatorname{Spec} R/I \to \operatorname{Spec} R$ is the closed embedding. This yields the desired result. □

Lemma 1.4.5. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which is $(n+k)$-pure for some $0 \leq k \leq \dim R$. If $b \in R$ is not contained in any minimal prime ideal containing $B_{\mathscr{M}}$ then multiplication by b induces injective automorphisms on $\mathscr{M}$ and $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$. Moreover, there exists a good filtration on $\mathscr{M}$ such that b induces a injection on $\operatorname{gr}^{rel}\mathscr{M}$.*

The proof of the following lemma is a slight modification on the proof of proposition 3.4.3 in Budur et al. (2019).

Lemma 1.4.6. *Let $\mathscr{M}$ be a non-zero relative holonomic $\mathscr{A}_X^R$-module of grade $j(\mathscr{M}) = n$ then, for any non-unit $b \in R$, it holds that $\mathscr{M} \otimes_R R/(b)$ is a non-zero relative holonomic $\mathscr{A}_X^{R/(\ell)}$-module of grade n.*

Proof. Applying lemma 1.4.4 with $k = 0$ yields that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic $\mathscr{A}_X^{R/(\ell)}$-module.

It remains to establish that $\mathscr{M} \otimes_R R/(b)$ is nonzero of grade n. By taking a free resolution of $\mathscr{M}$ one has that

$$\boldsymbol{R}\operatorname{Hom}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \otimes^L_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)} \cong \boldsymbol{R}\operatorname{Hom}_{\mathscr{A}_X^{R/(b)}}(\mathscr{M} \otimes^L_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}, \mathscr{A}_X^{R/(b)})$$

where we note that $\mathscr{A}_X^{R/(b)}$ is a $\mathscr{A}_X^R$-bimodule so that both tensor products are well-defined. We compare the Grothendieck spectral sequences of both sides of this isomorphism.

The spectral sequence associated with the right-hand-side has E^2-sheet

$$E^2_{pq} = \operatorname{Ext}^p_{\mathscr{A}_X^{R/(b)}}(\operatorname{Tor}_{-q}^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/(b)}), \mathscr{A}_X^{R/(b)}).$$

Recall from lemma 1.4.2 terms with $p > n$ have grade greater than n and due to **??** there are no nonzero terms with $p < n$. Hence, the only term with $p + q = n$ which could potentially have degree n is E^2_{n0}. If we can show that the total cohomology of degree n on the left-hand-side has grade n then it follows that $\operatorname{Ext}^n_{\mathscr{A}_X^{R/(b)}}(\mathscr{M} \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}), \mathscr{A}_X^{R/(b)}) \neq 0$.

The spectral sequence associated to the left-hand-side has E^2-sheet given by

$$E^2_{pq} = \operatorname{Tor}^{\mathscr{A}^R_X}_{-q}(\operatorname{Ext}^p_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X), \mathscr{A}^{R/(b)}_X).$$

Note that there are no nonzero differentials which map into E^j_{0n} or for $j \geq 2$. Further, the differentials out of the E^j_{0n} land in $E^j_{-j(n+j)}$ which is a subquotient of $\operatorname{Tor}^{\mathscr{A}^R_X}_j(\operatorname{Ext}^{n+j+1}_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X), \mathscr{A}^{R/(b)}_X)$. Since $\operatorname{Ext}^{n+j}_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X)$ is relative holonomic of degree $n+j$ proposition 1.4.1 and a argument similarly to that which established that $\mathscr{M} \otimes_R R/(b)$ is relative holonomic yield that $E_{j(n+j+1)^j}$ has grade greater than or equal to $n+j-1$. We will show that that E^2_{0n} has grade n. Then using proposition 1.4.1 and lemma 1.2.3 on the exact sequences

$$0 \to E^{j+1}_{0n} \to E^j_{0n} \to E_{j(n+j+1)^j}$$

it follows that E^j_{0n} has grade n for every $j \geq 2$. This shows that the total cohomology of degree n has grade n and concludes the proof.

Denote $\mathcal{E}^n := \operatorname{Ext}^n_{\mathscr{A}^R_X}(\mathscr{M}, \mathscr{A}^R_X)$, by lemma 1.4.2 it holds that $\mathcal{E}^n$ is a n-pure relative holonomic $\mathscr{A}^R_X$-module. By lemma 1.4.5 it follows that b induces injections on $\mathcal{E}^n$ and $\operatorname{gr}^{rel} \mathcal{E}^n$ for some appropriate filtration. In particular the vertical maps in the following diagram are injective

$$\begin{tikzcd}
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] \arrow[d, "b"] & F_i\mathcal{E}^n \arrow[r] \arrow[d, "b"] & \operatorname{gr}^{rel}_i \mathcal{E}^n \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] & F_i\mathcal{E}^n \arrow[r] & \operatorname{gr}^{rel}_i \mathcal{E}^n \arrow[r] & 0
\end{tikzcd}$$

so the snake lemma yields a short exact sequence

$$0 \longrightarrow F_{i-1}\mathcal{E}^n \otimes_R R/(b) \longrightarrow F_i\mathcal{E}^n \otimes_R R/(b) \longrightarrow \operatorname{gr}^{rel}_i \mathcal{E}^n \otimes_R R/(b) \longrightarrow 0.$$

The injectivity of b on $\operatorname{gr}^{rel} \mathcal{E}^n$ implies that b is also injective on $\mathcal{E}^n/F_i\mathcal{E}^n$. A similar appliccation of the snake lemma now yields a short exact sequence

$$0 \longrightarrow F_i\mathcal{E}^n \otimes_R R/(b) \longrightarrow \mathcal{E}^n \otimes_R R/(b) \longrightarrow (\mathcal{E}^n/F_i\mathcal{E}^n) \otimes_R R/(b) \longrightarrow 0.$$

A filtration on $\mathcal{E}^n \otimes_R R/(b)$ is induced by the image of $F_i\mathcal{E}^n$. By the injectivity of the short exact sequences one now has isomorphisms

$$F_i(\mathcal{E}^n \otimes_R R/(b)) \cong F_i\mathcal{E}^n/(F_i\mathcal{E}^n \cap b\mathcal{E}^n) \cong F_i\mathcal{E}^n/bF_i\mathcal{E}^n \cong (F_i\mathcal{E}^n) \otimes_R R/(b)$$

combined with the surjectivity of the first short exact sequence it follows that

$$\operatorname{gr}^{rel}(\mathcal{E}^n \otimes_R R/(b)) \cong \operatorname{gr}^{rel} \mathcal{E}^n \otimes_R R/(b).$$

It follows that

$$\operatorname{Ch}^{rel}(\mathcal{E}^n \otimes_{\mathscr{A}^R_X} \mathscr{A}^{R/(b)}_X) = (\operatorname{Id}_{T^*X} \times \Delta)^{-1}(\operatorname{Ch}^{rel}\mathscr{M})$$

with $\Delta : \operatorname{Spec} R/(b) \to \operatorname{Spec} R$ the closed embedding as before. Since $\mathscr{M}$ has grade n this equality and proposition 1.4.1 imply that $\operatorname{Ch}^{rel}(\mathcal{E}^n \otimes_{\mathscr{A}^R_X} \mathscr{A}^{R/(b)}_X)$ has dimension $n + \dim R - 1$. In particular it follows that $\mathcal{E}^n \otimes_{\mathscr{A}^R_X} \mathscr{A}^{R/(b)}_X$ is nonzero and has grade n. This concludes the proof. $\square$

By lemma 1.4.2 the following definition gives a class of j-pure modules.

Definition 1.4.3. *A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be j-Cohen-Macaulay for some $j \geq 0$ if $\operatorname{Ext}^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) = 0$ for any $k \neq j$.*

The property of being j-pure is not stable when restricting to a subscheme of $\operatorname{Spec} R$. For the subclass of j-Cohen-Macaulay modules the restriction is more well-behaved.

Lemma 1.4.7. *Let $\mathscr{M}$ be a relative holonomic and $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^R$-module. Let $b \in R$ be non-vanishing on every irreducible component of $Z(B_{\mathscr{M}})$. Then it holds that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^{R/(b)}$-module or zero.*

Proof. This is shown in the proof of proposition 3.4.3 in Budur et al. (2019). This proof is similar to the proof of lemma 1.4.6 which was based on Budur et al. (2019). □

Lemma 1.4.8. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade $n+k$. Then there exists a open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathscr{M} \otimes_R R'$ is a relative holonomic and $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^{R'}$ module. Moreover it may be assumed that the complement of $\operatorname{Spec} R'$ in $\operatorname{Spec} R$ has codimension $> k$.*

Proof. This is established in the proof of lemma 3.5.2 in Budur et al. (2019). □

The following lemma is a generalisations of a result by Kashiwara (1976) to the relative case. The proof and statement are more involved than the original result by Kashiwara but follow the same line of thought.

Lemma 1.4.9. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which comes equipped with the structure of a $\mathscr{A}_X^R\langle t\rangle$-module. Suppose that $\mathscr{M}$ has grade $j(\mathscr{M}) = n+k$ with $k \geq 1$. Then there exists a open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathscr{M}' = \mathscr{M} \otimes_R R'$ is a relative holonomic $\mathscr{A}_X^{R'}$-module with $t^N \mathscr{M}' = 0$ for N sufficiently large. Moreover, it may be assumed that $\operatorname{Spec} R \setminus \operatorname{Spec} R'$ has codimension strictly greater than k.*

Proof. The proof is split in two main parts. The first part it is establises that $\operatorname{Ext}^{n+k}_{\mathscr{A}_X^{R'}}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ stabilises and that the $t^i\mathscr{M}'$ are $(n+k)$-Cohen-Macauley. It follows that $t^i\mathscr{M}'$ stabilises and the final part of the proof deduces that the stable value is zero.

Note that $\mathscr{M}/t\mathscr{M}$ is a coherent sheaves over the Noetherian sheaf of rings $\mathscr{A}_X^R$. Hence, the kernel of the morphisms $\mathscr{M}/t\mathscr{M} \to t^i\mathscr{M}/t^{i+1}\mathscr{M}$ stabilise. Let $N \geq 0$ be sufficiently large so that these kernels are constant for $i \geq N$.

By use of lemma 1.4.8 it may be assumed that $\operatorname{Spec} R'$ is such that $t^i\mathscr{M}'$, $\mathscr{M}'/t^i\mathscr{M}'$, $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ and the kernels K_i of the morphisms $\mathscr{M}'/t\mathscr{M}' \to t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ are zero or $(n+k)$-Cohen-Macauley for any $i = 0, \ldots, N$. Since localisation is an exact functor the stabilisation of kernels for $i \geq N$ is still valid over $\operatorname{Spec} R'$. The first steps in this proof use the stabilisation to estabish that the modules are actually $(n+k)$-Cohen-Macaulay for arbitrary $i \geq 0$. For notational simplicity we abbreviate $\operatorname{Ext}^k(\mathscr{M}') := \operatorname{Ext}^k_{\mathscr{A}_X^{R'}}(\mathscr{M}', \mathscr{A}_X^{R'})$.

The surjection $\mathscr{M}'/t\mathscr{M}' \twoheadrightarrow t^i\mathscr{M}'/t^{i+1}$ induces a long exact sequence

$$0 \to \operatorname{Ext}^{n+k}\left(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}\right) \to \operatorname{Ext}^{n+k}\left(\frac{\mathscr{M}'}{t\mathscr{M}'}\right) \to \operatorname{Ext}^{n+k}(K_i) \to \operatorname{Ext}^{n+k+1}\left(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}\right) \to \cdots$$

In particular there is a isomorphism $\operatorname{Ext}^{n+k+1}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}') \cong \operatorname{Ext}^{n+k}(K_i)/\operatorname{Im}\operatorname{Ext}^{n+k}(\mathscr{M}'/t\mathscr{M}')$ whose left-hand-side is known to vanish when $i \leq N$. Since the righ-hand-side is constant for $i \geq N$ it follows that $\mathrm{Ext}^{n+k+1}(t^i\mathscr{M}'/t^{i+1}\mathscr{M}') \cong 0$ for any $i \geq 0$. The higher order terms of the long exact sequence yield $\mathrm{Ext}^{n+k+j}(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}) \cong 0$ for $j > 1$. This shows that $t^i\mathscr{M}'/t^{i+1}\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

The injection $t^{i+1}\mathscr{M}' \to t^i\mathscr{M}'$ induces exact sequences

$$\mathrm{Ext}^{n+k+j}\left(\frac{t^i\mathscr{M}'}{t^{i+1}\mathscr{M}'}\right) \to \mathrm{Ext}^{n+k+j}\left(t^i\mathscr{M}'\right) \to \mathrm{Ext}^{n+k+j}\left(t^{i+1}\mathscr{M}'\right).$$

By induction on i it follows that $t^i\mathscr{M}'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$. Similarly the long exact sequence induced by the surjection $\mathscr{M}/t^{i+1}\mathscr{M}' \twoheadrightarrow \mathscr{M}'/t^i\mathscr{M}'$ yields that $\mathscr{M}/t^i\mathscr{M}$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

By the Cohen-Macauley results which have been established it follows that the morphisms $\mathrm{Ext}^{n+k}(t^i\mathscr{M}') \to \mathrm{Ext}^{n+k}(t^{i+1}\mathscr{M}')$ and $\mathrm{Ext}^{n+k}(\mathscr{M}') \to \mathrm{Ext}^{n+k}(t^i\mathscr{M}', \mathscr{A}_X^{R'})$ are surjective. Note that $\mathrm{Ext}^{n+k}(\mathscr{M})$ is a coherent sheaf over the Noetherian sheaf of rings $\mathscr{A}_X^{R'}$. Hence the kernels of $\mathrm{Ext}^{n+k}(\mathscr{M}') \to \mathrm{Ext}^{n+k}(t^i\mathscr{M}')$ stabilise. After possibly increasing N it follows that the morphisms $\mathrm{Ext}^{n+k}(t^i\mathscr{M}') \to \mathrm{Ext}^{n+k}(t^{i+1}\mathscr{M}')$ are isomorphisms for $i \geq N$ which means that $\mathrm{Ext}^{n+k}(t^{i+1}\mathscr{M}')$ stabilises.

By $t^i\mathscr{M}'$ being $(n+k)$-Cohen-Macaulay it follows that $\mathrm{Ext}^{n+k}(\mathrm{Ext}^{n+k}(t^i\mathscr{M}')) \cong t^i\mathscr{M}'$.[20] It follows that $t^i\mathscr{M}'$ stabilises for $i \geq N$ and remains to show that this stable value is 0. If the stable value is nonzero then it is $(n+k)$-Cohen-Macaulay with $k > 1$. By proposition 1.4.1 and proposition 1.2.9 it follows that there exists some nonzero $b(s_1, \ldots, s_p) \in B_{t^N\mathscr{M}'}$. Note that one has the commutation relation

$$tb(s_1, \ldots, s_p) = b(s_1 + 1, \ldots, s_p + 1)t.$$

Since $t^{N+1}\mathscr{M}' = t^N\mathscr{M}'$ it follows by iteration that $b(s_1 + n, \ldots, s_p + n) \in B_{t^N\mathscr{M}'}$ for any $n \geq 0$. This implies that $Z(B_{t^N\mathscr{M}'}) = 0$ which means that $t^N\mathscr{M}' = 0$.[21] □

1.5 Estimation of the Bernstein-Sato Zero Locust

This section contains the main result of this chapter, namely a proof of the improved estimate for the Bernstein-Sato zero locust which was announced in theorem 1.1.1. We use the same notation as section 1.1. This proof is similar to the method employed by Lichtin (1989) and Kashiwara (1976) but a new induction argument is required in the proof of lemma 1.5.6.

Recall that the global Bernstein-Sato Ideal is the intersection of all local ones. This means that the global Bernstein-Sato zero locust $Z(B_F)$ is the union of all local ones so it suffices to estimate $Z(B_{F,x})$. In particular, it may be assumed that X is affine and admits global coordinates $x_1, \ldots, x_n$.

Due to these global coordinates there is a $\mathcal{O}_X^R$-linear isomorphism between any left $\mathscr{A}_X^R$-module $\mathscr{N}$ and it's right version $\Omega_X \otimes_{\mathcal{O}_X} \mathscr{N}$. Concretely, any section u of $\mathscr{N}$ gives rise to the section $u^* := udx$. Further, for any operator P of $\mathscr{A}_X^R$ there is a adjoint P^* such that

$$(P \cdot u)^* = u^* \cdot P^*$$

[20]Note: Provide reference, maybe include and prove earlier since this is easy from double complex

[21]Note: Geometrically obvious but may want to add formal argument

for any section u of $\mathscr{N}$. For a vector field $\xi := \sum_i \xi_i \partial_i$ comparison of the definitions shows that $\xi^* := \sum_i \partial_i \xi_i$ satisfies this equality and this extends to $\mathscr{A}_X^R$ by iterating.

By this procedure the functional equation $PF^{s+1} = bF^s$ may equivalently be stated as the equation

$$F^{s+1}dx \cdot P^* = bF^s dx$$

in $\mathscr{A}_X F^s \otimes_{\mathcal{O}_X} \Omega_X$. The corresponding module $\mathscr{M}$ on Y will be the submodule of $\Omega_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_Y G^s$ spanned by $G^s \mu^*(dx)$. This may be viewed as a $\mathscr{A}_Y\langle t\rangle$-module by the action $tG^s\mu^*(dx) = G^{s+1}\mu^*(dx)$. Replacing $\mathscr{A}_X F^s$ by it's right version $\mathscr{A}_X F^s \otimes_{\mathcal{O}_X} \Omega_X$ is convenient because the direct image functor is more natural for right modules. This will make it easier to transfer information along the resolution of singularities. Further, it explains why the relative canonical divisor occurs in the improved estimate of theorem 1.1.1. This is because $\mu^*(dx)$ gives a local equation for $K_{Y/X}$.

Lemma 1.5.1. *In the notation of section 1.1 a polynomial of the form* $\prod_{i=1}^p \prod_{j=0}^N (\mathrm{mult}_{E_i}(g_1) s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + j)$ *belongs to the Bernstein-Sato ideal* $B_{\mathscr{M}/t\mathscr{M}}$ *if* $N \geq 0$ *is sufficiently large.*

Proof. This may be checked locally. Take a open $U \subseteq Y$ which is sufficiently small to admit local coordinates $z_1, \ldots, z_n$ where z_i determines E_i if $E_i \cap U \neq \emptyset$.

In these local coordinates $G^s = \prod_{i=1}^p u_i^{s_i} \prod_{i=1}^n z_i^{\sum_{j=1}^p M_{ij}s_j}$ and $\mu^*(dx) = v\prod_{i=1}^n z_i^{m_i} dz$ where $M_{ij} \leq \mathrm{mult}_{E_i}(g_j)$, $m_i \leq k_i$ and u_i, v are local units. For notational convenience set $u_i = 1$ and $s_i = 0$ for $i > p$. Denote $N_i = \sum_j M_{ij}$ and $\xi_i = \partial_i - \sum_{j=1}^n s_j u_j \partial_i(u_j)$ for any $i = 1, \ldots, p$. Let $P = v^{-1}(\prod_{i=1}^p u_i^{-1})\xi_1^{N_1} \cdots \xi_p^{N_p} v$ then

$$v\prod_{i=1}^n u_i^{s_j+1} z_i^{\sum_{j=1}^p M_{ij}(s_j+1)+m_i} dz \cdot P = bv \prod_{i=1}^n u_i^{s_j} z_i^{\sum_{j=1}^p M_{ij}s_j + m_i} dz$$

where

$$b = \prod_{i=1}^p (\sum_{j=1}^p M_{ij}s_j + m_i + N_i)(\sum_{j=1}^p M_{ij}s_j + m_i + N_i - 1)\cdots(\sum_{j=1}^p M_{ij}s_j + m_i).$$

□

By lemma 1.3.5 the global section $G^s\mu^*(dx)$ of $\mathscr{M}$ gives rise to a global section u of $\int^0 \mathscr{M}$. Denote $\mathscr{U}$ for the right $\mathscr{A}_X$-module generated by u. From lemma 1.5.1 one gets a b-polynomial of a desirable form for $\int^0 \mathscr{M}/t\int^0 \mathscr{M}$. The main remaining difficulty in is to induce a b-polynomial for $\mathscr{U}/t\mathscr{U}$. This will exploit lemma 1.4.9 whence it is needed that $\int^0 \mathscr{M}/\mathscr{U}$ has grade at least $n+1$.

In what follows we want to consider the $\mathscr{A}_Y$-module $\mathscr{M}$ as a $\mathscr{D}_Y$-module. This could disturb coherence. To solve this one introduces new coordinates such that there are vector fields $\mathcal{S}_1, \ldots, \mathcal{S}_p$ which acts as $s_1, \ldots, s_p$ on the generator.

Note that there are finitely many codimension 1 components in $Z(B_F)$. Hence, there exist p independent linear polynomials $\sum_{i=1}^p d_{ij}s_i$ such that for any j there is no hyperplane parallel to $\sum_{i=1}^p d_{ij}s_i = 0$ in $Z(B_F)$. Moreover, it may be assumed that the d_{ij} are nonnegative integers. Introduce new coordinates $z_{n+1}, \ldots, z_{n+p}$ and set $\mathscr{X} := X \times \mathbb{C}^p$ and $\mathscr{Y} := Y \times \mathbb{C}^p$. For any $j = 1, \ldots, p$ set $\widetilde{f_j} = f_j \prod_{i=1}^p z_{n+i}^{d_{ij}}$. Note that the induced map

$\mathscr{Y} \to \mathscr{X}$ is a resolution of singularities for $\prod \widetilde{f}_i$ and that $\widetilde{g}_j = g_j \prod_{i=1}^p z_{n+i}^{d_{ij}}$ is the pullback of $\widetilde{f}_i$.

For any $i = 1, \dots, p$ it holds that

$$\widetilde{G}^s \mu^*(dx) \cdot \partial_{n+i} = \sum_{j=1}^p d_{ij} s_j x_j^{-1} \widetilde{G}^s \mu^*(dx).$$

Since the linear polynomials are independent a appropriate $\mathbb{C}$-linear combination provides a vector field ζ_j with $\widetilde{G}^s \mu^*(dx) \cdot \zeta_j = s_j z_j^{-1} \widetilde{G}^s \mu^*(dx)$. Set $\mathcal{S}_j = \zeta_j z_j$ so that $\widetilde{G}^s \mu^*(dx) \cdot \mathcal{S}_j = s_j \widetilde{G}^s \mu^*(dx)$. This solves the coherence isssue.

Lemma 1.5.2. *If $b \in B_{\widetilde{F},x}$ for any $x \in X \times \{0\}^p$ then $b \in B_F$.*

Proof. Take local coordinates $x_1, \dots, x_{n+p}$ near x and let P be in the stalk of $\mathscr{A}_{\mathscr{X}}$ at x such that $b\widetilde{F}^s = P\widetilde{F}^{s+1}$. Similarly to the above there is a $\mathbb{C}$-basis $\xi_1, \dots, \xi_p$ for the span of $\partial_{n+1}, \dots, \partial_{n+p}$ so that $\mathcal{S}_j := x_{n+j}\xi_j$ satisfies $\mathcal{S}_j \cdot \widetilde{F}^s = s_j \widetilde{F}^s$. Expand P as a polynomials in $\xi_1, \dots, \xi_p$

$$P = \sum_\alpha P_\alpha \xi_1^{\alpha_1} \cdots \xi_p^{\alpha_p}$$

where the coefficients P_α live in a stalk of $\mathcal{O}_{\mathscr{X}} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{D}_{\mathscr{X}}$.

Let N be greater than the maximal value of $|\alpha|$ then

$$(x_{n+1} \cdots x_{n+p})^N b\widetilde{F}^s = \left(\sum_\alpha \prod_{i=1}^p (s_i + 1)^{\alpha_i} \sum_\beta Q_{\alpha\beta} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \right) \widetilde{F}^{s+1}$$

where the P_α were expanded as polynomials in $\partial_1, \dots, \partial_n$ with coefficients $Q_{\alpha\beta}$ from $\mathcal{O}_{\mathscr{X}}$. Observe that $\partial_1, \dots, \partial_n$ act on the formal symbol $\widetilde{F}^{s+1}$ the same as they act on the formal symbol F^{s+1}.

Now consider this functional equation on the analytification of $\mathscr{X}$ and expand the $Q_{\alpha\beta}$ as power series at x. Identifying powers of $x_{n+1} \cdots x_{n+p}$ on both sides a functional equation with analytical coefficients for F^s follows. This establishes that $b \in B_{F,x}$ for any $x \in X$ provided analytical and algebraic Bernstein-Sato ideals are equal.[22] □

Note that replacing F by $\widetilde{F}$ leaves theorem 1.1.1 unchanged up to hyperplanes parallel to $\sum_{i=1}^p d_{ij} s_i = 0$. These are not in $Z(B_F)$ by assumption so, by lemma 1.5.2, it remains to prove the theorem for $\widetilde{F}$. For notational simplicity we simply write F instead of $\widetilde{F}$.

Let $\ell_1, \dots, \ell_{p-1} \in \mathbb{C}[s]$ be degree one polynomials which will be fixed later. For any $i = 0 \dots, p$ let L_i be the ideal of $\mathbb{C}[s]$ generated by $\ell_1, \dots, \ell_i$. Assume that the ℓ_i are chosen sufficiently generically so that $Z(L_{p-1})$ is a line.

Lemma 1.5.3. *The $\mathscr{D}_{\mathscr{Y}}$-module $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is coherent and it's characteristic variety satisfies $\operatorname{Ch} \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1} \subseteq V \cup W$ where V is isotropic and W is a irreducible variety of dimension $\dim \mathscr{Y} + 1$ which dominates $\mathscr{Y}$.*

[22] Note: Is this true?

Proof. Recall that we ensured that $\mathscr{M}$ is a coherent $\mathscr{D}_{\mathscr{Y}}$-module. Hence, also $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ will be a coherent $\mathscr{D}_{\mathscr{Y}}$-module.

Take local coordinates $z_1, \dots, z_n, z_{n+1}, \dots, z_{n+p}$ on $\mathscr{Y}$ as in the proof of lemma 1.5.1. This is to say that locally

$$G^s \mu^*(dx) = v \prod_{i=1}^{n} u_i^{s_j} z_i^{\sum_{j=1}^{p} M_{ij} s_j + m_i} \prod_{i=1}^{n} z_{n+i}^{\sum_{j=1}^{p} d_{ij} s_j} dz.$$

Let s_0 denote a new variable so that $\mathbb{C}[s]/L_{p-1} \cong \mathbb{C}[s_0]$. Then $\mathscr{M} \otimes_{\mathbb{C}[s_0]} R/L_{p-1}$ may be viewed as the $\mathscr{D}_{\mathscr{Y}}$-module which is locally generated by a formal symbol

$$[G^s \mu^*(dx)] = v \prod_{i=1}^{2n} u_i^{A_i s_0 + a_i} z_i^{B_i s_0 + b_i} dz$$

where A_i, B_i, a_i, b_i are complex numbers and we set $u_{n+i} = 1$. Moreover, since the linear functions $\sum_{j=1}^{p} d_{ij} s_j$ on the final terms in $G^s \mu^*(dx)$ formed a basis for the linear polynomials there will be at least one B_{i+n} which is nonzero.

Denote $w = v \prod_{i=1}^{n} u_i^{a_i}$ and consider for any $j = 1, \dots, n+p$ the operation of $w^{-1} \partial_j w z_j$ on the generator

$$[G^s \mu^*(dx)] \cdot w^{-1} \partial_j w = ((B_j s_0 + b_j) z_j^{-1} + \sum_{i=1}^{n} A_i s_0 u_i^{-1} \partial_j (u_i))[G^s \mu^*(dx)].$$

Recall that the $s_1, \dots, s_n$ could be produced by acting with a vector field. Since s_0 is found with affine relations it follows that there exists some differential operator $\mathcal{S}_0$ of degree 1 such that $s_0[G^s \mu^*(dx)] = [G^s \mu^*(dx)] \cdot \mathcal{S}_0$. Now we get a well-defined surjection $\mathscr{D}_{\mathscr{Y}}/I \twoheadrightarrow \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ where I denotes the right ideal generated by $w^{-1} \partial_j w z_j - b_j - \mathcal{S} h_i$ with $h_j = B_j + z_j \sum_{i=1}^{n} A_i u_i^{-1} \partial_j (u_i)$ for $j = 1, \dots, n+p$.

Note that $z_j \sum_{i=1}^{n} A_i u_i^{-1} \partial_j (u_i) = 0$ for $j > n$. Hence, the h_{n+j} are complex scalars and they are not all zero since there exists a nonzero B_{n+j}. After renumbering we now have that $h_1 \in \mathbb{C}^\times$. Denoting ζ_j, σ_0 for the elements of $\operatorname{gr} \mathscr{D}_{\mathscr{X}}$ which correspond to $\partial_j, \mathcal{S}_0$ respectively it holds that $\operatorname{gr} I$ contains $z_j \zeta_j - h_j \sigma_0$ for any $j = 1, \dots, n+p$. Then also $h_1 z_j \zeta_j - h_j z_1 \zeta_1$ is in $\operatorname{gr} I$ for any $j = 2, \dots, n+p$. This yields the desired bound for the characteristic variety. □

Lemma 1.5.4. *If the $\mathscr{A}_{\mathscr{X}}$-module $\int^0 \mathscr{M}/U$ has grade $\dim \mathscr{X}$ then the quotients $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ are relative holonomic $\mathscr{A}_{\mathscr{X}}^{R/L_i}$-modules of grade $\dim \mathscr{X}$.*

Proof. This follows by induction on $i = 0, \dots, p$ using lemma 1.4.6 which is applicable since grade $\dim \mathscr{X}$ means that the Bernstein-Sato ideals of $\int^0 \mathscr{M}/\mathscr{U} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$ is the zero ideal by proposition 1.4.1 and proposition 1.2.9. □

Lemma 1.5.5. *Any polynomial $b \in \mathbb{C}[s]$ which is not in L_i induces a injective automorphisms on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$.*

Proof. Similarly to the proof of **??** one can locally view $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ as being generated by some formal symbol $[G^s \mu^*(dx)]$. If b is not injective then it follows that $[G^s \mu^*(dx)] f b = 0$ for some nonzero sections f of $\mathcal{O}_X^{\mathbb{C}[s]/L_i}$. This is absurd, the formal symbol $[G^s \mu^*(dx)]$ is not annihilated by any section of $\mathcal{O}_X^{\mathbb{C}[s]/L_i}$. [23] □

[23]Note: More words

Lemma 1.5.6. *The relative holonomic $\mathscr{A}_{\mathscr{X}}$-module $\int^0 \mathscr{M}/\mathscr{U}$ has grade $j(\int^0 \mathscr{M}/\mathscr{U}) \geq \dim \mathscr{X} + 1$.*

Proof. Suppose that $\int^0 \mathscr{M}/\mathscr{U}$ has grade $\dim \mathscr{X}$. A contradiction will be derived by replacing $\int^0 \mathscr{M}/\mathscr{U}$ with a holonomic $\mathscr{D}_{\mathscr{X}}$-module. The first task is to understand how $\int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1})$ relates to $(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$.

Recall from lemma 1.5.5 that ℓ_i is injective on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$. This implies that $\ell_i \int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1} = \int^0 \ell_i \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$. The injective automorphisms of ℓ_i on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}$ induces a long exact sequence of $\mathscr{A}_X^{\mathbb{C}[s]/L_{i-1}}$-modules

$$0 \to \int^0 \left(\mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \xrightarrow{\ell_i} \int^0 \left(\mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \to \int^0 \left(\mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_i}\right) \to \cdots$$

whence $(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/(\ell_i)$ is a submodule of $\int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i)$. The quotient is isomorphic to the kernel K_i of ℓ_i on $\int^1(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1})$.

Denote R_{i+1} for the ideal $(\ell_{i+1}, \ldots, \ell_{p-1})$. Applying a tensor product with $\mathbb{C}[s]/R_{i+1}$ to the inclusion $(\int^0 \mathscr{M}/L_{i-1}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/(\ell_i) \hookrightarrow \int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i)$ yields a exact sequence

$$\cdots \to \operatorname{Tor}_1^{\mathbb{C}[s]}\left(K_i, \frac{\mathbb{C}[s]}{R_{i+1}}\right) \to \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{R_i} \to \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_i}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{R_{i+1}}.$$

We claim that the ℓ_i can be chosen so that $\operatorname{Tor}_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}$-module of grade greater than or equal to $\dim \mathscr{X} + 1$. Let's show how this suffices to finish the proof and prove this claim afterwards.

That the grade of $\operatorname{Tor}_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ is greater than or equal to $\dim \mathscr{X} + 1$ means that the Bernstein-Sato ideal is not trivial. Equivalently, for any i, there exists a non-zero polynomial $b_i \in \mathbb{C}[s]/L_{p-1}$ which annihilates $\operatorname{Tor}_1^{\mathbb{C}[s]/L_i}(K_i, \mathbb{C}[s]/R_{i+1}))$. The kernels of the automorphisms induced by b_i^m form a increasing sequence inside the coherent $\mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}$-module $(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/R_i$. Such a increasing sequence must stabilise which means that we can pick m_i sufficiently large so that

$$\operatorname{Im} \operatorname{Tor}_1^{\mathbb{C}[s]}\left(K_i, \frac{\mathbb{C}[s]}{R_{i+1}}\right) \bigcap b_i^{m_i} \left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{R_i} = 0.$$

Set $B = \prod_{i=0}^{p-2} b_i^{m_i}$ then we get injections

$$\cdots \hookrightarrow B\left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{i-1}}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{R_i} \hookrightarrow B\left(\int^0 \mathscr{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_i}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{R_{i+1}} \hookrightarrow \cdots$$

Since μ is proper the Kashiwara estimate is applicable and lemma 1.5.3 yields that $\int^0(\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1})$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module with characteristic variety $\widetilde{\mu}\left((T^*\mu)^{-1}(V \cup W)\right)$. It follows that $B(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module and

$$\operatorname{Ch} B\left(\left(\int^0 \mathscr{M}\right) \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{L_{p-1}}\right) \subseteq \mathcal{L} \cup \widetilde{\mu}\left((T^*\mu)^{-1}(V \cup W)\right).$$

Now observe that $B(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a quotient of $B(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ with support in the divisor D. Hence, $B(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module with

$$\operatorname{Ch}\left(B\left(\int^0 \mathscr{M}/\mathscr{U}\right) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}\right) \subseteq \mathcal{L} \cup \widetilde{\mu}\left((T^*\mu)^{-1}(V \cup W)\right) \cap (T^*\mathscr{X} \times_{\mathscr{X}} \operatorname{supp} D).$$

This means $B(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a holonomic $\mathscr{D}_{\mathscr{X}}$-module. Indeed, by ([24]) $\widetilde{\mu}((T^*\mu)^{-1}(V))$ remains isotropic and forms no obstruction to the characteristic variety being Lagrangian. Moreover, $\widetilde{\mu}((T^*\mu)^{-1}(W))$ is irreducible of dimension $\dim \mathscr{X} + 1$ and dominates $\mathscr{X}$. Intersecting with $T^*\mathscr{X} \times_{\mathscr{X}} \operatorname{supp} D$ then yields a closed strict subset which necessarily has lower dimension. Hence, it follows that $\dim \operatorname{Ch} B(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1} \leq \dim \mathscr{X}$. This means that $B(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is holonomic. By ([25]) the Bernstein-Sato ideal of holonomic modules is non-zero. But then also the Bernstein-Sato ideal of $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is non-zero. This contradicts lemma 1.5.4 and we conclude that the assumption the grade is $\dim \mathscr{X}$ must have been wrong.

It remains to show that the ℓ_j can be chosen so that $\operatorname{Tor}_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$-module of grade greater than or equal to $\dim \mathscr{X} + 1$. This means we must understand the Ext-functor of a Tor. Hence, we consider the interaction between the derived Hom-functor and the derived tensor product.

By a taking a $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}$-free resolution of K_i one finds that

$$\boldsymbol{R}\operatorname{Hom}_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}}(K_i \otimes^L_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}} \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}) \cong \boldsymbol{R}\operatorname{Hom}_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}) \otimes^L_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}} \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$$

where we note that $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$ is a $\mathscr{A}_X^{\mathbb{C}[s]/L_i}$-bimodule so that both tensor products are defined. We compare the Grothendieck spectral sequences of both sides.

The spectral sequence on the left-hand-side has terms

$$E^2_{rq} = \operatorname{Ext}^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}}(\operatorname{Tor}_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

Since $\operatorname{Tor}_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}$-module these terms are only nonzero for $r = \dim \mathscr{X}$ or $r = \dim \mathscr{X} + 1$. In particular, the spectral sequence degenerates at E^2. Note that the statement that $\operatorname{Tor}_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ has grade greater than or equal to $\dim \mathscr{X} + 1$ is equivalent to $E^2_{\dim \mathscr{X}, -1} = 0$.

The spectral sequence on the right-hand-side has terms

$$E^2_{rq} = \operatorname{Tor}_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\operatorname{Ext}^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

The claim follows if we can ensure that that all terms with $r - q = \dim \mathscr{X} - 1$ vanish on $\mathscr{X} \times \operatorname{Spec} R$ for some open subset $\operatorname{Spec} R \subseteq \mathbb{C}^p$. Indeed, then by proposition 1.2.9 and proposition 1.4.1 the terms have grade $\dim \mathscr{X} + 1$ and it follows that the same must hold for the terms of the spectral sequence on the left hand side. Since $\operatorname{Ext}^{\dim \mathscr{X}}$ of a relative holonomic module $\dim \mathscr{X}$-pure this means that the $E^2_{\dim \mathscr{X}, -1}$-term in the left-hand-side spectral sequence vanishes.

[24]Note: Put somewhere in Chapter 1 (Kashiwara, 1976, proposition 4.9).

[25]Note: Bjork (1979) holonomic means s is algebraic

The ℓ_i and the open $\operatorname{Spec} R$ are constructed by induction on i. For any i, j, k with $k \leq i$ denote $\mathscr{E}_{ik}^{n+j} := \operatorname{Ext}^{n+j}_{\mathscr{A}_{\mathscr{X}}^{R/L_k}}(K_k, \mathscr{A}_{\mathscr{X}}^{R/L_k}) \otimes_{\mathscr{A}_{\mathscr{X}}^{R/L_k}} \mathscr{A}_{\mathscr{X}}^{R/L_i}$. In every induction step it is ensured that

(i) $\mathscr{E}_{ii}^{n+j}$ is $(n+j)$-Cohen-Macaulay over $\mathscr{A}_X^{R/L_i}$ or zero for every $j \geq 0$.

(ii) $Z(L_i) \cap \operatorname{Spec} R \neq 0$.

(iii) ℓ_i induces a injection on $\mathscr{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$.

By abuse of notation L_i may also denote the ideal of R generated by $\ell_1, \ldots, \ell_i$.

Take some arbitrary ℓ_1 for the base-case and use lemma 1.4.8 to find a open $\operatorname{Spec} R \subseteq \mathbb{C}^p$ such that $\mathscr{E}_{11}^{n+j}$ is $(n+j)$-Cohen-Macaulay for every $j \geq 0$. This only requires removing a strict closed subset of $\operatorname{Spec} \mathbb{C}[s]/L_1$ so $Z(L_1) \cap \operatorname{Spec} R = \operatorname{Spec} R/L_1$ is non-empty. The final property is vacuous for $i = 1$.

Now assume that $i > 1$ and that $\ell_1, \ldots, \ell_{i-1}$ are already constructed. First let's ensure that ℓ_i induces a injection on $\mathscr{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$. By iterative application of lemma 1.4.7 it holds that $\mathscr{E}_{(i-1)k}^{n+j}$ is $(n+j)$-Cohen-Macaulay over $\mathscr{A}_{\mathscr{X}}^{L_{i-1}}$. Take ℓ_i so that the induduced element of R/L_{i-1} is non-constant and does not vanish on any irreducible component of the Bernstein-Sato zero locust of $\mathscr{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$. Then, by lemma 1.4.5 the desired injectivity follows. As before, lemma 1.4.8 can be used to to find a open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathscr{E}_{ii}^{n+j}$ is $(n+j)$-Cohen-Macaulay for every $j \geq 0$ and $Z(L_i) \cap \operatorname{Spec} R' = \operatorname{Spec} R'/L_i$ is non-empty. Note that replacing $\operatorname{Spec} R$ by $\operatorname{Spec} R'$ will conserve the induction hypothesis. This concludes the inductive construction of the ℓ_i.

Applying injectivity of ℓ_i on $\mathscr{E}_{(i-1)k}^{n+j}$ with the free resolution $\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}} \to \mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}$ for $\mathscr{A}_{\mathscr{X}}^{R/L_i}$ yields that $\operatorname{Tor}_m^{\mathscr{A}_{\mathscr{X}}^{R/L_i}}(\mathscr{E}_{(i-1)k}^{n+j}, \mathscr{A}_{\mathscr{X}}^{R/L_i}) = 0$ for all $m > 0$. By taking a $\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}$-free resolution of $\mathscr{E}_{(i-1)k}^{n+j}$ it follows that

$$\mathscr{E}_{(i-1)k}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}} \cong \mathscr{E}_{ik}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}.$$

Iterative application of the isomorphism yields $\mathscr{E}_{ii}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}} \cong \mathscr{E}_{(p-2)i}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}$. This means that

$$\operatorname{Tor}_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\operatorname{Ext}^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}) \cong \operatorname{Tor}_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathscr{E}^r_{(p-2)i}, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$$

The right right-hand-side of this isomorphism was already observed to vanish for any $-q > 0$ and the left-hand-side is precisely the E^2_{rq}-term of the spectral sequence. This establishes that the E^2_{rq}-terms with $r - q = \dim \mathscr{X} - 1$ vanish for $q > 0$. The remaining terms $E_{\dim \mathscr{X} -1, 0}$ is zero regardless since it involves $\operatorname{Ext}^{\dim X - 1}$ of a relative holonomic module. This shows that the $\operatorname{Tor}_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ are relative holonomic $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$-module of grade greater than or equal to $\dim \mathscr{X} + 1$ and concludes the proof. □

The following lemma and it's proof are similar to the monovariate case which may be found in (Bjork, 1979, p246).

Lemma 1.5.7. *There is a morphism right $\mathscr{A}_{\mathscr{X}}^R$-modules $\mathscr{U} \to \mathscr{A}_{\mathscr{X}}^R F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega_{\mathscr{X}}$ sending u to $F^s dx$.*

Proof. The resolution of singularities $\mathscr{Y} \to \mathscr{X}$ is a isomorphism on the complement of $\prod f_i = 0$. Hence, a isomorphism $\mathscr{U} = \int^0 \mathscr{M} \cong \mathscr{A}^R_{\mathscr{X}} F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega_{\mathscr{X}}$ holds outside of $\prod f_i = 0$.

Pick some open set $V \subseteq \mathscr{X}$ we must show that whenever $uP = 0$ in $\mathscr{U}(V)$ it follows that $(F^s dx)P = 0$. Due to the isomorphism it is certainly the case that $(F^s dx)P = 0$ outside of $\prod f_i = 0$. Hence, the support of the coherent sheaf of $\mathcal{O}^R_V$-modules $\mathcal{O}^R_V(F^s dx)P$ lies in $\prod f_i = 0$. The Nullstellen Satz now yields that $(\prod f_i)^N (F^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that $\prod f_i$ is a non-zero divisor of $(F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega_{\mathscr{X}})(V)$. Hence, it follows that $(F^s dx)P = 0$ on V as desired. □

Now all ingredients are in place for the proof of theorem 1.1.1.

Theorem 1.5.8. *With notation as in section 1.1 every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of the form*

$$\operatorname{mult}_{E_i}(g_1)s_1 + \cdots + \operatorname{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{\geq 0}$.

Proof. By lemma 1.5.6 the $\mathscr{A}_X$-module $\mathscr{M}/\mathscr{U}$ has grade greater than or equal to $\dim \mathcal{X} + 1$. Hence lemma 1.4.9 provides $N \geq 1$ such that $t^N \mathscr{M}/\mathscr{U} = 0$ on a open $\mathscr{X} \times \operatorname{Spec} R$ for some open $\operatorname{Spec} R \subseteq \mathbb{C}^p$ with complement of codimension strictly greater than 1.

Let $b(s_1, \ldots, s_p)$ denote the Bernstein-Sato polynomial for $\mathscr{M}/t\mathscr{M}$ provided by lemma 1.5.1. Set $B := \prod_{i=0}^{N+1} b(s_1 + i, \ldots, s_p + i)$ then it follows that $B\mathscr{M} \subseteq t\mathscr{U}$ on $\mathscr{X} \times \operatorname{Spec} R$. In particular this means that B is in the Bernstein-Sato ideal of $\mathscr{U}/t\mathscr{U}$ over $\operatorname{Spec} R$. By the surjection of lemma 1.5.7 this means that $B \in B_F$ over $\operatorname{Spec} R$. This proves which proves the theorem because the complement of $\operatorname{Spec} R$ has codimension strictly greater than 1. □

1.6 Varia

Relation Ch and Ch$^{\#}$

In the notation of Budur et al. (2020) $\mathrm{Ch}^{\#}$ yields a bound for the classical characteristic variety of cyclic $\mathscr{D}_X$-modules with a $\mathscr{A}_X^{\mathbb{C}[s]}$-structure where $s_1, \ldots, s_n$ act as a vector field on the generator. Any $P(x, \xi, s) \in \operatorname{Ann}_{\mathrm{gr}^{\#} \mathscr{A}_X^{\mathbb{C}[s]}} \mathrm{gr}^{\#} \mathscr{M}$ induces a element of $\operatorname{Ann}_{\mathrm{gr}\,\mathscr{D}_X} \mathrm{gr}\,\mathscr{M}$.

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