Introduction

The purpose of this thesis is provide a improved estimate for the zero locust of Bernstein-Sato ideals.

Historically, this problem arose when trying to extend a function to the entire complex plane. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be some fixed positive polynomial and $g \in \mathcal{C}_c^\infty$ be some test function. Gelfand asked if it is possible to find a meromorphic extension of the function

$$\Gamma_g(s) = \int_{\mathbb{R}^n} g f^s dx; \qquad \mathrm{Re}(s) > 0$$

to the entire complex plane. A proof by Bernstein relies on the existence of a differential operator $P(x, \partial, s)$ such that

$$P(x, \partial, s) f^{s+1} = b_f(s) f^s$$

for some polynomial $b_f(s) \in \mathbb{C}[s]$. The monic polynomial of minimal degree is called the Bernstein-Sato polynomial. In particular, this method shows that any pole of Γ_g is a root of $b_f(s)$ up a shift with a negative integer.

The relation between the roots of $b(s)$ and the poles of functions like Γ_g is the topic of a notable open problem called the monodromy conjecture. Further, the roots of $b_f(s)$ give insight into the nature of a singularity by the behaviour of the ill-defined function f^{-1}. Suppose that $f(0) = 0$ and consider f as a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let $t \in \mathbb{C}^\times$ be close to 0 and consider the intersection $X_t := f^{-1}(t) \cap B_\varepsilon$ where B_ε is some small ball centered at 0. Twisting t around the origin induces a diffeomorphism M of X_t which is called the geometric monodromy. In particular, one gets linear endomorphisms M^* on the vector spaces $H^j(X_t, \mathbb{C})$. The action M^* only depends on the singularity $(f^{-1}(0), 0)$. In particular, the eigenvalues of the action are invariants of the singularity. These eigenvalues are called the eigenvalues of monodromy. A classical theorem due to Malgrange and Kashiwara estabilshes that the eigenvalues of monodromy are precisely the set $\exp(2\pi i Z(b_f))$ where $Z(b_f)$ denotes the collection of roots of the Bernstein-Sato polynomial.

This shows that the roots of the Bernstein-Sato polynomial are a worthy topic of study. Estimation of the roots of the Bernstein-Sato polynomial has been done in terms of data from a resolution of singularities. The resolution reduces the problem to the case where f is a monomial in which case the Bernstein-Sato polynomial can be computed explicitly. The main difficulty is to connect the easier problem to the original problem. This relies on the sheaf-theoretic framework of $\mathscr{D}_X$-modules and their direct images. Here $\mathscr{D}_X$ denotes the sheaf of differential operators on a space X.

The original estimate due to Kashiwara establishes that the roots of $b(s)$ are negative rational numbers. A lower bound for the distance between the largest root and 0 was established by Lichtin. This refined estimate uses a similar methodology to Kashiwara but replaced f^s with the distribution $f^s dx$.

In this thesis a multivariate generalization of the problem is considered. Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a function germ with coordinate functions $f_1, \ldots, f_p$ and define variables $s_1, \ldots, s_p$. It is then still known that there exists a differential operator $P(x, \partial, s)$ such that

$$P(x, \partial, s) f_1^{s_1+1} \cdots f_p^{s_p+1} = b(s) f_1^{s_1} \cdots f_p^{s_p}$$

for some polynomial $b(s) \in \mathbb{C}[s_1, \ldots, s_p]$. The collection of all possible polynomial $b(s)$ form an ideal of $\mathbb{C}[s_1, \ldots, s_p]$ which is called the Bernstein-Sato ideal and is denoted B_F. The roots of Bernstein-Sato polynomials are generalised by the zero locust $Z(B_F)$.

There are generalisations of the monodromy conjecture and monodromy theorem to the multivariate situation involving $Z(B_F)$. Kashiwara's estimate for the roots of $b(s)$ has been generalised to a estimation of the Bernstein-Sato ideal by Budur et al. This thesis also generalises the refined estimate due to Lichtin.

The main new idea is a induction argument to reduce the number of s_i to one. Homological algebra is used to control error terms of the reduction process.

The first chapter in this thesis gives provides the categorical framework of derived categories. Such derived categories give a natural framework for dealing with homological algebra and are also necessary to define the direct image of $\mathscr{D}_X$-modules. The second chapter provides background on $\mathscr{D}_X$-modules building up to the Riemann-Hilbert correspondence which is a broad generalisation of the the equivalence between systems of differential equations and their solutions. Further, the significance of the roots of Bernstein-Sato polynomials and the estimates due to Kashiwara and Lichtin are discussed. Finally, the third chapter includes the new proof for the generalised estimate.

Contents

Chapter 1

Categorical Preliminaries

This chapter contains some categorical preliminaries on the topic of derived category theory and spectral sequences.

Derived category theory allows to measure the lack of exactness in a functor $F : \mathcal{A} \to \mathcal{B}$ by encoding error-terms in derived functors $R^i F : \mathcal{A} \to \mathcal{B}$. For instance the non-exactness of the tensor product may be measured by $\mathcal{T}or$-functors.

Spectral sequences were historically developed by Leray to compute the cohomology of the pushforward of a sheaf. There is some overlap between derived category theory and spectral sequences. In particular the Grothendieck spectral sequence allows one to compute the derived functor of some composition $F \circ G$ based on the derived functors of F and G individually. This theorem is a essential technical ingredient in the proofs of chapter 3.

The discussion of derived category theory in this chapter summarises the relevant parts of (Dimca, 2004, Chapters 1, 2 and 5). The section on spectral sequences is based on Weibel (1995). A

1.1 Spectral Sequences

Fix a abelian category $\mathcal{A}$. Denote $C(\mathcal{A})$ for the category with complexes of objects in $\mathcal{A}$

$$X^\bullet : \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots .$$

A double complex $E^{\bullet\bullet}$ gives rise to a total complex with terms $\mathrm{Tot}(E)^n = \oplus_{i+j=n} E^{ij}$. The motivating question behind spectral sequences is how the cohomology of the total complex may be computed.

We will not go into detail on this motivating question but the idea is that one can first compute horizontal cohomology to get data E_1^{**}. By the commutativity of the double complex there are vertical differentials on E_1^{**} and one can compute the vertical cohomology to get E_2^{**}. Diagram chasing allows to construct higher-order differentials leading to the following notion.

Definition 1.1.1. *A cohomology spectral sequence starting at the a-th sheet consists of families of objects $\{E_r^{p,q}\}_{p,q \in \mathbb{Z}}$ for $r \geq a$ and maps $d_{pq}^r : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that*

(i) The maps d_{pq}^r are differentials in the sense that $d^r \circ d^r = 0$.

(ii) The $(r+1)$-st sheet is the homology of the r-th sheet $E_{pq}^{r+1} \cong \ker(d_{pq}^r)/\operatorname{Im}(d_{p-r,q+r-1}^r)$.

Definition 1.1.2. *A cohomology spectral sequence is said to be bounded if for each n there are finitely many non-zero terms E_a^{pq} with $p+q=n$.*

In a bounded complex there is, for each choice of p, q, a value r_0 such that $E_r^{pq} = E_{r+1}^{pq}$ for all $r \geq r_0$. This stable value is denoted E_∞^{pq}.

Definition 1.1.3. *A bounded spectral sequence is said to converge to a family of objects H^* if any H^n admits a finite filtration*

$$0 = F^s H^n \subseteq \cdots F^p H^n \subseteq F^{p+1} H^n \subseteq \cdots \subseteq F^t H^n = H^n$$

such that $E_\infty^{pq} \cong F^p H^{p+q}/F^{p-1} H^{p+q}$.

Observe that H^* is not necessarily uniquely identified by a convergent spectral sequence. The total complex in the motivating problem comes equipped with a filtration $F^m \operatorname{Tot}(E)^n = \oplus_{p+q=n, p<m} E^{pq}$.

Definition 1.1.4. *A filtration of a complex $C_\bullet$ is a family of subcomplexes $\{F^m C^\bullet\}_{m \in \mathbb{Z}}$. The filtration is said to be exhaustive if $C^\bullet = \cup_m F^m C^\bullet$.*

Proposition 1.1.5. *A filtration of a complex $C^\bullet$ determines a spectral sequence starting with $E_0^{pq} = F^p C^{p+q}/F^{p-1} C^{p+q}$ and $E_1^{pq} = H^{p+q} E_0^{p\bullet}$.*

Proof. The construction for this spectral sequence may be found in (Weibel, 1995, Chapter 5) □

Definition 1.1.6. *A filtration on a complex $C^\bullet$ is said to be bounded if, for each n, there are integers $s < t$ such that $F^s C^n = 0$ and $F^t C^n = C^n$.*

The following proposition may be used in the motivating problem to recover some information on the cohomology of the total complex.

Proposition 1.1.7. *Let $C^\bullet$ be a complex with a bounded filtration. Then the associated spectral sequence is bounded and converges to $H^*(C^\bullet)$.*

Proof. This result may be found in (Weibel, 1995, Chapter 5). □

1.2 Derived Categories

The category $C(\mathcal{A})$ contains full subcategories $C^*(\mathcal{A})$ with $* \in \{+, -, b\}$ denoting that the complexes in $\mathcal{A}$ are bounded below, above or bounded on both sides respectively. For example $C^+(\mathcal{A})$ may contain complexes of the form $\cdots \to 0 \to \cdots X^{-1} \to X^0 \to \cdots$. For a complex $X^\bullet$ and $k \in \mathbb{Z}$ one has a shifted complex $X^\bullet[k]$ with $(X^\bullet[k])^s = X^{k+s}$. Further, denote $\operatorname{Hom}^k(X^\bullet, Y^\bullet) := \operatorname{Hom}(X^\bullet, Y^\bullet[k])$ which are the chain maps that change the grading by k.

Definition 1.2.1. *Two complex morphisms $u, v : X^\bullet \to Y^\bullet$ are called homotopic if there exists $h \in \operatorname{Hom}^{-1}(X^\bullet, Y^\bullet)$ such that $u - v = d_Y h + h d_X$.*

Definition 1.2.2. *A morphism $u : X^\bullet \to Y^\bullet$ of complexes in $C^*(\mathcal{A})$ is called a quasi-isomorphism if the induced morphism in cohomology $H^k(u) : H^k(X^\bullet) \to H^k(Y^\bullet)$ is a isomorphism for all k. This may be denoted $u \sim v$.*

The idea behind the following definition is to retain the same objects as $C^*(\mathcal{A})$ but turn quasi-isomorphisms into isomorphisms. The technicalities may be found in (Deligne, 1977, Chapter 8).

Definition 1.2.3. *The derived category $D^*(\mathcal{A})$ is defined as the category obtained from $C^*(\mathcal{A})$ by localising with respect to the multiplicative system formed by the quasi-isomorphisms.*

This definition can be made more concrete provided the category has enough injectives.

Definition 1.2.4. *A abelian category $\mathcal{A}$ has enough injectives if for any object X in $\mathcal{A}$ there is an exact sequence $0 \to X \to I$ in $\mathcal{A}$ with I injective.*

Definition 1.2.5. *Let $\mathcal{A}$ be a abelian category. The homotopical category of complexes $K^*(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $C^*(\mathcal{A})$ and as morphisms*

$$\mathrm{Hom}_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet) := \mathrm{Hom}_{C^*(\mathcal{A})}(X^\bullet, Y^\bullet)/\sim .$$

Observe that two homotopic maps induce the same morphism in cohomology. It follows that there is a well-defined functor $p^*_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$.

Proposition 1.2.6. *Let $\mathcal{A}$ be a abelian category with enough injectives and denote $I(\mathcal{A})$ for the full subcategory from the injective objects. Then the natural functor*

$$p^*_{\mathcal{A}} : K^+(I(\mathcal{A})) \to D^+(\mathcal{A})$$

is a equivalence of categories.

Proof. This result may be found in (Dimca, 2004, Chapter 1). □

By passing to the opposite categories one gets a similar theorem in categories with enough projectives for $D^-(\mathcal{A})$.

1.3 Triangulated Categories

The categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ remain additive but may fail to be exact. In particular, the notion of short exact sequences no longer makes sense. Instead, $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ may be viewed as triangulated categories which is to say that they come equipped with a notion of exact triangles.

Definition 1.3.1. *Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. The mapping cone of u is the complex in $C^*(\mathcal{A})$ given by*

$$C^\bullet_u := Y^\bullet \oplus (X^\bullet[1])$$

with $d_u(y, x) = (dy + u(x), -dx)$.

The concept of a mapping cone originated in a construction from algebraic topology which explains the name. Observe that the mapping cone gives rise to a triangle

$$T_u : X^\bullet \xrightarrow{u} Y^\bullet \to C_u^\bullet \to X^\bullet[1]$$

which may be denoted more intuitively as

$$\begin{tikzcd} X^\bullet \arrow[rr, "u"] & & Y^\bullet \arrow[dl, "q"] \\ & C_u^\bullet \arrow[ul, "+1"] & \end{tikzcd}$$

The triangles T_u may be used to encode short exact sequences.

Proposition 1.3.2. *Given a short exact sequence in $C^*(\mathcal{A})$*

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

there exists a quasi-isomorphism $m : C_u^\bullet \to Z^\bullet$ with $m \circ q = v$.

Proof. This result may be found in (Dimca, 2004, Chapter 1). □

This shows that a short exact sequence induces a triangle isomorphic to a standard triangle T_u in $D^*(\mathcal{A})$. Further evidence that the triangles T_u behave like short exact sequences is given by the following result.

Proposition 1.3.3. *Let $u : X^\bullet \to Y^\bullet$ be a morphism in $C^*(\mathcal{A})$.*

(i) The composition of any two consecutive maps in T_u is homotopic to 0.

(ii) The triangle T_u induces a long exact sequence in cohomology

$$\cdots \to H^k(X^\bullet) \xrightarrow{u} H^k(Y^\bullet) \to H^k(C_u^\bullet) \xrightarrow{\delta} H^{k+1}(X^\bullet) \to \cdots$$

where the connecting morphism δ comes from the map $C_u^\bullet \to X^\bullet[1]$.

Proof. This result may be found in (Dimca, 2004, Chapter 1). □

Further investigation of the properties of T_u gives rise to the concept of a triangulated category. These definitions and properties are pleasant in their own right so we go into some detail.

The distinguish triangles $\mathcal{T}$ in $K^*(\mathcal{A})$ or $D^*(\mathcal{A})$ are the family of triangles which are isomorphic to a triangle of the form T_u. Observe that these categories have a shift functor T given by $TX^\bullet = X^\bullet[1]$.

Definition 1.3.4. *An additive category $\mathcal{D}$ equipped with a self-equivalence T and family of distinguished triangles $\mathcal{T}$ is called a triangulated category if the following axioms are satisfied.*

(Tr1) Any triangle isomorphic to a distinguish triangle is distinguished. For any object X the triangle $X \to X \to 0 \to TX$ is distinguish where the first morphism is the identity. Any morphism $u : X \to Y$ is part of some distinguished triangle $X \xrightarrow{u} Y \to Z \to TX$.

(Tr2) A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is distinguished.

(Tr3) A commutative diagram of the following from whose rows are distinguished triangles gives rise to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & TX \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & TA
\end{tikzcd}$$

(Tr4) For any triple of distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{x} A \longrightarrow TX$$

$$Y \xrightarrow{v} Z \longrightarrow B \xrightarrow{y} TY$$

$$X \xrightarrow{vu} Z \longrightarrow C \longrightarrow TX$$

there is a distinguished triangle

$$A \xrightarrow{a} C \xrightarrow{b} B \xrightarrow{(Tx)y} TA$$

such that (id_X, v, a) and (u, id_Z, b) are morphisms of triangles.

Proposition 1.3.5. *Let $\mathcal{A}$ be a abelian category. Then $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ are triangulated categories.*

Proof. This result may be found in (Dimca, 2004, Chapter 1). □

A triangle $X \to Y \to Z \to TX$ will also be denoted $X \to Y \to Z \xrightarrow{+1} X$ and $T^m X$ may be denoted with $X[m]$. Now the data of the final axiom can be organised as follows. Correspondingly, $(Tr4)$ is also referred to as the octahedral axiom.

$$\begin{tikzcd}
 & C \arrow[dr] \arrow[ddl] & \\
A \arrow[ur] \arrow[d] & & B \arrow[ll] \arrow[ddl] \\
X \arrow[rr, "v \circ u"] \arrow[dr, "u"'] & & Z \arrow[u] \arrow[uul] \\
 & Y \arrow[uul] \arrow[ur, "v"'] &
\end{tikzcd}$$

Definition 1.3.6. *Let $\mathcal{D}$ be a triangulated category. A subcategory $\mathcal{C}$ of $\mathcal{D}$ is said to be stable under extensions if any distinguished triangle in $\mathcal{D}$ with two vertices in $\mathcal{C}$ also has it's third vertex in $\mathcal{D}$.*

Definition 1.3.7. *Let $\mathcal{C}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. One calls $\mathcal{C}$ is a triangulated subcategory if $\mathcal{C}$ is stable under extensions and $T\mathcal{C} \subseteq \mathcal{C}$.*

Definition 1.3.8. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ a abelian category. An additive functor $F : \mathcal{D} \to \mathcal{A}$ is a cohomological functor if for any distinguished triangle in $\mathcal{D}$*

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced sequence $F(X) \to F(Y) \to F(Z)$ is a exact in $\mathcal{A}$. If F is a cohomological functor one sets $F^i = F \circ T^i$.

The family of functors F^i is conservative if for any distinguished triangle

$$X \to Y \to Z \xrightarrow{+1} X$$

the induced long sequence

$$\cdots \to F^i(X) \to F^i(Y) \to F^i(Z) \to F^{i+1}(X) \to \cdots$$

is exact.

The key example for the above definition is given by the cohomological functor $H^0 : K^*(\mathcal{A}) \to \mathcal{A}$ and the conservative system of functors H^k.

Definition 1.3.9. *Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories. A functor $F : \mathcal{D} \to \mathcal{D}'$ is called a functor of triangulated categories if it is compatible with the shift functor and transforms distinguished triangles in $\mathcal{D}$ into distinguished triangles of $\mathcal{D}'$.*

1.4 Derived Functors

Given abelian categories $\mathcal{A}, \mathcal{B}$ and a functor of triangulated categories $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ one may wonder if there is a natural lift to the derived categories.

Definition 1.4.1. *Let F be as above. The right derived functor of F is a initial couple (R^*F, ξ_F) consisting of a functor of triangulated categories $R^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : p_B \circ F \to R^*F \circ p^*_{\mathcal{A}}$. By initial it is mean that for any other such couple (G, ζ) there is a unique natural transformation $\eta : R^*F \to G$ such that $\zeta = (\eta \circ p^*_{\mathcal{A}}) \circ \xi_F$.*

The dual notion is a left derived functor. This is a final couple (L^*F, ξ_F) consisting of a functor of triangulated categories $F^*F : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a natural transformation $\xi_F : L^*F \circ p^*_{\mathcal{A}} \to p_B \circ F$. It is clear that, if a derived functor exists, it is unique up to unique isomorphism.

There are general theorems on the existence of derived functors which may be found in (Dimca, 2004, Chapter 1). The following will be sufficient for our applications.

Theorem 1.4.2. *Consider a functor of triangulated categories $F : K^+(\mathcal{A}) \to K(\mathcal{B})$. If $\mathcal{A}$ has enough injectives and F is additive then the right derived functor R^+F exists.*

By dualising, a similar theorem applies to $F : K^-(\mathcal{A}) \to K(\mathcal{B})$ for the existence of L^-F in categories with enough projectives.

The main use of derived functors is to fix a lack of exactness in F. Recall from proposition 1.3.2 that a short-exact sequence in $C^+(\mathcal{A})$ induces a distinguished triangle in $D^+(\mathcal{A})$. Applying R^+F to the distinguished triangle returns a distinguished triangle by R^+F being a functor of triangulated categories. Further, there is a associated long exact sequence. The higher-order terms measure measures to what degree the original functor failed to be exact.

Definition 1.4.3. *Let $F : K^*(\mathcal{A}) \to K(B)$ be a functor of triangulated categories such that R^*F exists. For any $n \in \mathbb{Z}$ one defines $R^nF : \mathcal{A} \to \mathcal{B}$ to be the composition*

$$\mathcal{A} \xrightarrow{\iota} D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

where ι sends a object to the chain complex with a single non-trivial term. Similarly, one defines $\mathbb{R}^nF : D^*(\mathcal{A}) \to \mathcal{B}$ as the composition

$$D^*(\mathcal{A}) \xrightarrow{R^*F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}.$$

Proposition 1.4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of triangulated categories. Suppose that the derived functor R^*F of the induced functor $F : K^*(\mathcal{A}) \to K(\mathcal{B})$ exists. Then, for any short exact sequence in $\mathcal{A}$*

$$0 \to X \to Y \to Z \to 0$$

there is a long exact sequence in $\mathcal{B}$

$$\cdots \to R^iF(X) \to R^iF(Y) \to R^iF(Z) \to R^{i+1}F(X) \to \cdots .$$

Proof. This is immediate by R^*F being a functor of triangulated categories and the fact that the cohomologies H^k form a conservative system of functors. □

In the situation of theorem 1.4.2 the derived functor can be computed explicitly. Pick some object $X^\bullet$ in $D^+(\mathcal{A})$. By proposition 1.2.6 there is a quasi-isomorphism $X^\bullet \to I^\bullet$ for some complex of injective objects $I^\bullet$. Then one has explicitly

$$R^+F(X^\bullet) \cong p_{\mathcal{B}} \circ F(I^\bullet).$$

Further, if F is exact one has that $F(I^\bullet)$ is quasi-isomorphic to $F(X^\bullet)$ whence $R^+F(X^\bullet)$ is $p_{\mathcal{B}} \circ F(X^\bullet)$.

In practice, it is often difficult to find a concrete injective resolution.

Definition 1.4.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A object X in $\mathcal{A}$ is F-acyclic if $R^iF(X) = 0$ for all $i \geq 1$.*

Computation derived functors can also be done using F-acyclic resolutions. One can show that injective objects are F-acyclic for any left-exact functor. Hence, this generalises the earlier computations.

Proposition 1.4.6. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive functors between abelian categories with enough injective objects. Suppose that F is left-exact and that G transforms injective objects into F-acyclic objects, then there is an isomorphism*

$$R^+(F \circ G) = R^+F \circ R^+G.$$

Proof. This result may be found in (Dimca, 2004, Chapter 1). □

Theorem 1.4.7 (Grothendieck Spectral Sequence)**.** *Let F, G be as in the previous proposition. Then, for any object X of $\mathcal{A}$, there is a spectral sequence*

$$E_2^{pq} = R^pF(R^qG(X))$$

converging to $R^{p+q}(F \circ G)(X)$.

Proof. This result may be found in (Dimca, 2004, Chapter 1) or (Weibel, 1995, Chapter 5). The main ingredient is the spectral sequence associated to a double complex **?**. □

We conclude this section by considering a few important examples of derived functors which will be used later on.

Let X be a topological space equipped with a sheaf of rings $\mathcal{A}_X$ which need-not be commutative. The corresponding categories of complexes of left or right modules are denoted $C^{*,\ell}(\mathcal{A}_X)$ and $C^{*,r}(\mathcal{A}_X)$ respectively. Similarly, the category of complexes of bimodules is denoted $C^{*,\ell r}(\mathcal{A}_X)$. Using theorem 1.4.2 one can establish that the global sections functor $\Gamma(X,-)$ on $C^{*,*}(\mathcal{A}_X)$ has a derived functor $R^+\Gamma(X,-)$. The cohomology of a sheaf of modules is given by the functors $H^k(X,-) := R^k\Gamma(X,-)$ and the hypercohomology of a complex of modules is given by the functors $\mathbb{H}^k(X,-) := \mathbb{R}^k\Gamma(X,-)$. The cohomology sheaf of a complex $\mathscr{M}^\bullet$ is the sheaf associated to the presheaf $U \mapsto \mathbb{H}^k(U,\mathscr{M}^\bullet)$ and is denoted $\mathscr{H}^k(\mathscr{M}^\bullet)$. The cohomology sheaves of a module $\mathscr{M}$ are defined similarly and also denoted $\mathscr{H}^k(\mathscr{M})$.

Cohomology measures the failure of sections to be global. Correspondingly, acyclic objects are given by sheaves which have no such failure.

Definition 1.4.8. *A sheaf of $\mathcal{A}_X$-modules $\mathscr{F}$ is called flabby if for any open $U \subseteq X$ the restriction morphism $\rho_U^X : \mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.*

Proposition 1.4.9. *If $\mathscr{F}$ is flabby then $\mathscr{F}$ is $\Gamma(X,-)$-acyclic.*

Proof. This result may be found in (Dimca, 2004, Chapter 2). □

The hypercohomology of a sheaf complex can be computed using flabby resolutions. Concrete flabby resolutions may be found using the Godement resolution. A sheaf $\mathscr{M}$ gives rise to a flabby sheaf $\mathscr{F}$ by the formal product of stalks. The same argument applies to the cokernel of $\mathscr{M} \to \mathscr{F}$ and iterating the argument yields a flabby resolution for $\mathscr{M}$. For a sheaf complex $\mathscr{M}^\bullet$ the flabby resolutions of the $\mathscr{M}^j$ produce a double complex $\mathscr{F}^{\bullet,\bullet}$. The total complex of $\mathscr{F}^{\bullet,\bullet}$ yields a flabby resolution of $\mathscr{M}^\bullet$.

Let $f : Y \to X$ be a continuous map between topological spaces. The direct image of a sheaf $\mathcal{S}$ on Y is the sheaf $f_*\mathcal{S}$ on X defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

Suppose that Y, X are equipped with sheaves of rings $\mathcal{A}_Y, \mathcal{A}_X$ respectively and that $f_*\mathcal{A}_Y$ is a $\mathcal{A}_X$-algebra. Then the direct image yields a functor from the category of $\mathcal{A}_Y$-modules to the category of $\mathcal{A}_X$-modules. This is a left-exact functor so it may be computed by injective resolutions. One can verify that flabby sheaves are f_*-acyclic so that flabby resolutions may also be used in the computations.

A classical example of a non-exact functor is given by the tensor product. This may be considered as a bifunctor

$$\otimes_{\mathcal{A}_X} : C^{-,\ell}(\mathcal{A}_X) \times C^{-,r}(\mathcal{A}_X) \to C^{-,\ell r}(\mathcal{A}_X)$$

where $(\mathscr{M}^\bullet \otimes_{\mathcal{A}_X} \mathscr{N}^\bullet)^n = \oplus_{i+j=n}\mathscr{M}^i \otimes_{\mathcal{A}_X} \mathscr{N}^j$ with differentials defined at $\mathscr{M}^i \otimes_{\mathcal{A}_X} \mathscr{N}^j$ by $d(m \otimes n) = dm \otimes n + (-1)^i m \otimes dn$. The category of $\mathcal{A}_X$-modules admits locally free resolutions. In particular, it has enough projective objects. Essentially by the remark after theorem 1.4.2 it is then possible to construct a derived left-derived functor

$$\otimes^L_{\mathcal{A}_X} : D^{-,\ell}(\mathcal{A}_X) \times D^{-,r}(\mathcal{A}_X) \to D^{-,\ell r}(\mathcal{A}_X).$$

This yields $\mathcal{T}or$-sheaves $\mathcal{T}or_k^{\mathcal{A}_X}(X^\bullet, Y^\bullet) = H^{-k}(X^\bullet \otimes^L_{\mathcal{A}_X} Y^\bullet)$.

A similar procedure applies to the $\mathcal{H}om_{\mathcal{A}_X}$-bifunctor which is defined by $\mathcal{H}om^n_{\mathcal{A}_X}(\mathscr{M}^\bullet, \mathscr{N}^\bullet) = \prod_{j\in Z} \mathcal{H}om_{\mathcal{A}_X}(\mathscr{M}^j, \mathscr{N}^{n+j})$ with the differentials on $\mathcal{H}om^n_{\mathcal{A}_X}(M^\bullet, N^\bullet)$ given by $d\varphi = d_N \circ \varphi - (-1)^n \varphi \circ d_M$. There is a induced derived bifunctor

$$R\mathcal{H}om^\bullet_{\mathcal{A}_X}(-,-) :: D^{-,\ell}(\mathcal{A}_X)^{opp} \times D^{+,\ell r}(\mathcal{A}_X) \to D^r(\mathcal{A}_X).$$

This yields the $\mathcal{E}xt$-sheaves $\mathcal{E}xt^n_{\mathcal{A}_X}(M^\bullet, N^\bullet) = R^n\mathcal{H}om^\bullet_{\mathcal{A}_X}(M^\bullet, N^\bullet)$.

1.5 t-structures

A generalisation of positive and negatively supported complexes is given by the concept of a t-structure.

Definition 1.5.1. *A t-structure on a triangulated category $\mathcal{D}$ consists of two strictly full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ such that, setting $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, the following properties hold.*

(i) It holds that $D^{\leq 0}$ is a subcategory of $D^{\leq 1}$ and $D^{\geq 1}$ is a subcategory of $D^{\geq 0}$.

(ii) For any objects X in $\mathcal{D}^{\leq 0}$ and Y of $\mathcal{D}^{\geq 1}$ it holds that $\operatorname{Hom}(X, Y) = 0$.

(iii) For any object X of $\mathcal{D}$ there is a distinguished triangle

$$A \to X \to B \xrightarrow{+1} A$$

with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

Definition 1.5.2. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then $\mathcal{D} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure.*

In the motivating case of $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ the heart of the t-structure recovers the original abelian category $\mathcal{A}$.

Proposition 1.5.3. *The heart $\mathcal{D}$ of a t-structure is an abelian category which is stable by extensions.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Observe that $D^*(\mathcal{A})$ comes equipped with a truncation functors $\tau_{\leq m} : D^*(\mathcal{A}) \to D^-(\mathcal{A})$ which sends a complex $X^\bullet$ to

$$\tau_{\leq m} X^\bullet : \cdots \to X^{m-1} \to \ker d \to 0 \to 0 \to \cdots$$

and similarly a truncation functor $\tau_{\geq m}$ is defined by

$$\tau_{\geq m} X^\bullet : \cdots \to 0 \to 0 \to \operatorname{coim} d \to X^{m+1} \to \cdots .$$

This generalises to t-structures.

Proposition 1.5.4. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then the inclusion of $\mathcal{D}^{\leq n}$ in $\mathcal{D}$ has a right adjoint functor $\tau_{\leq n}$. Similarly, the inclusion of $\mathcal{D}^{\geq n}$ in $\mathcal{D}$ has a left adjoint $\tau_{\geq n}$.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Observe that in the example of $D^*(\mathcal{A})$ one has that $\tau_{\geq 0}\tau_{\leq 0}X^\bullet$ is the complex with a single entry $H^0(X^\bullet)$. This generalises to t-structures by viewing ${}^tH^0 := \tau_{\geq 0}\tau_{\leq 0}$ as a functor from $\mathcal{D}$ to it's heart $\mathcal{C}$. Further let ${}^tH^i :={}^t H^0 \circ T^i$.

Definition 1.5.5. *A t-structure is said to be non-degenerated if* $\cap \mathcal{D}^{\leq n} = \cap \mathcal{D}^{\geq n} = \mathrm{Null}$ *where* Null *denotes the family of objects which are isomorphic to the zero object in* $\mathcal{D}$.

Proposition 1.5.6. *Let $\mathcal{D}$ be a triangulated category with a t-structure. Then ${}^tH^0 : \mathcal{D} \to \mathcal{C}$ is a cohomological functor.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Proposition 1.5.7. *Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then the system of functors ${}^tH^i$ is conservative.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Proposition 1.5.8. *Let $\mathcal{D}$ be a triangulated category with a non-degenerated t-structure. Then $X \in \mathcal{D}^{\leq 0}$ if and only if ${}^tH^i(X) = 0$ for $i > 0$. Similarly $X \in \mathcal{D}^{\geq 0}$ if and only if ${}^tH^i(X) = 0$ for $i < 0$.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Definition 1.5.9. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures. A functor of triangulate categories $F : \mathcal{D}_1 \to \mathcal{D}_2$ is called left or right t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ or $F(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$ respectively. The functor F is called t-exact if it is left and right t-exact.*

Definition 1.5.10. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories equipped with t-structures and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. The perverse functor ${}^pF$ associated to F is the induced functor on the hearts ${}^pF ={}^t H^0 \circ F \circ j_1$ where j_1 denotes the inclusion functor $j_1 : \mathcal{C}_1 \to \mathcal{C}_2$.*

Proposition 1.5.11. *Let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a t-exact functor of triangulated categories. Then F sends the heart $\mathcal{C}_1$ into the heart $\mathcal{C}_2$ and the induced functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is naturally isomorphic to ${}^pF$.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Chapter 2

$\mathscr{D}_X$-modules and the Riemann-Hilbert Correspondence

The subject of this chapter are modules over rings of differential operators. Throughout X can be a smooth algebraic variety or a complex manifold of dimension n. The rings of differential operators $\mathscr{D}_X$ will be defined formally in the next section. For the purpose of this section it's sufficient to note that local sections of $\mathscr{D}_X$ are of the form $\sum c_\alpha \partial^\alpha$ with c_α local sections of $\mathcal{O}_X$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

A $\mathscr{D}_X$-modules gives a canonical description of systems of differential equations. Consider a system of differential equations

$$\sum_{j=1}^{k} P_{ij}(x, \partial) f_j = 0; \qquad i = 1, \ldots, m$$

with unknown functions f_j of $\mathcal{O}_X$ and differential operators P_{ij}. The functions f_j are somewhat arbitrary in the description of this system. For instance, take $g_j = \lambda_j f_j$ for certain non-zero functions λ_j. There is then a associated system of equations for g_j. A solution of the g_j-system corresponds uniquely to a solution of the f_j-system.

Consider the cokernel $\mathscr{M}$ of the map

$$P : \mathscr{D}_X^k \to \mathscr{D}_X^m : (Q_1, \ldots, Q_k) \mapsto \left(\ldots, \sum_{j=1}^{k} Q_j P_{ij}, \ldots \right).$$

This map is left $\mathscr{D}_X$-linear so $\mathscr{M}$ is a left $\mathscr{D}_X$-module. Note that it is necessary to distinguish between left and right modules because differential operators form a non-commutative ring. Direct verification shows that the solutions of the system of differential equations are encoded in $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$. More generally, for any $\mathscr{D}_X$-module $\mathscr{N}$ the solutions in $\mathscr{N}$ are encoded by $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$. This shows that $\mathscr{D}_X$-modules provide a canonical description of differential equations.

The goal of this section is to summarise some of the results and definitions which are common knowledge in the field of $\mathscr{D}_X$-modules. This builds up to the Riemann-Hilbert correspondence which states in very general terms that a system of differential equations is equivalent to it's solutions. This result is powerful because it yields a connection between algebraic geometry and topology. A particular instantiation of this correspondence is the connection between Bernstein-Sato polynomials and monodromy. Finally, we include the estimation of the roots of Bernstein-Sato polynomials due to Kashiwara and Lichtin.

Detailed treatments of the theory of $\mathscr{D}_X$-modules may be found in Bjork (1979), Kashiwara (2003), Hotta and Tanisaki (2007) or Borel (1987).

2.1 $\mathscr{D}_X$-modules

As stated in the introduction X may denote a smooth algebraic variety or a complex manifold.

Definition 2.1.1. *The sheaf of differential operators $\mathscr{D}_X$ is the subsheaf of rings in $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the vector fields Θ_X.*

Observe that $\mathscr{D}_X$ is a sheaf of non-commutative rings. Given local coordinates $x_1, \ldots, x_n$ on X it holds that

$$\partial_i x_j - x_j \partial_i = \delta_{ij}$$

where δ denotes the Kronecker delta.

Lemma 2.1.2. *For any $x \in X$ the stalk $\mathscr{D}_{X,x}$ is left and right Noetherian. Moreover, in the algebraic case $\mathscr{D}_X$ is a left and right Noetherian sheaf of rings.*

Proof. The algebraic and analytic cases may be found in chapters 1 and 4 of Hotta and Tanisaki (2007) respectively. □

Giving a left $\mathscr{D}_X$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X$ and ξ of Θ_X. Similarly, giving a right $\mathscr{D}_X$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X.

Translation between left and right-modules is possible. Denote ω_X for the top-level differential forms. Then ω_X comes equipped with the structure of a right $\mathscr{D}_X$-module where vector fields act by the Lie derivative.

For any left $\mathscr{D}_X$-module $\mathscr{M}$ a right $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathcal{O}_X} \omega_X$ may be defined by

$$m \otimes \omega \cdot \xi = m \otimes \omega\xi - \xi f \otimes \omega.$$

For any right $\mathscr{D}_X$-module $\mathscr{M}$ a left $\mathscr{D}_X$-module structure on $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathscr{M})$ may be defined by

$$(\xi \cdot \varphi)(\omega) = \varphi(\omega \cdot \xi) - \varphi(\omega) \cdot \xi.$$

The following lemma follows by a direct computation.

Lemma 2.1.3. *The functor $- \otimes_{\mathcal{O}_X} \omega_X$ is a equivalence of categories with pseudoinverse $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, -)$.*

Filtrations

This non-commutativity exits the typical domain of algebraic geometry. This can be resolved by consideration of a graded structure. The essential observation here is that differential operators commute up to a element of lower order.

Definition 2.1.4. *The order filtration on $\mathscr{D}_X$ is defined inductively to be given by the sheaves of $\mathcal{O}_X$-submodules $F_i\mathscr{D}_X$ such that $F_0\mathscr{D}_X = \mathcal{O}_X$ and $[F_i\mathscr{D}_X, F_i\mathscr{D}_X] \subseteq F_{i-1}\mathscr{D}_X$.*

The term $F_i\mathscr{D}_X$ in the order filtration can be described as containing all differential operators of order less than or equal to i. Indeed, given local coordinates $x_1, \dots, x_n$ one can show that $F_i\mathscr{D}_X$ is the $\mathcal{O}_X$-module locally generated by $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where α is a multi-index with $|\alpha| \leq n$. The following observations are immediate.

Lemma 2.1.5. *The $F_i\mathscr{D}_X$ are coherent $\mathcal{O}_X$-modules and form a exhaustive filtration. This is to say that $\cup_{i\geq 0} F_i\mathscr{D}_X = \mathscr{D}_X$ and that for any $i, j \geq 0$ it holds that $F_i\mathscr{D}_X \cdot F_j\mathscr{D}_X \subseteq F_{i+j}\mathscr{D}_X$.*

There is a similar notion of filtrations on $\mathscr{D}_X$-modules $\mathscr{M}$. Without any harm let's assume that $\mathscr{M}$ is a left $\mathscr{D}_X$-module, the case for right modules is analogous. A filtration consists of $\mathcal{O}_X$-submodules $F_i\mathscr{M}$ of $\mathscr{M}$ such that $\cup_i F_i\mathscr{M} = \mathscr{M}$ and $F_i\mathscr{D}_X \cdot F_j\mathscr{M} \subseteq F_{i+j}\mathscr{M}$. Stepping over to the graded object has the advantage that $\operatorname{gr}\mathscr{D}_X$ is commutative by definition of the order filtration whence the classical methods of algebraic geometry are applicable. One can view $\operatorname{gr}\mathscr{D}_X$ as a subsheaf of $\mathcal{O}_{T^*X}$. The symplectic structure of T^*X captures part of the non-commutativity. Indeed, given two differential operators P, Q. Pick local coordinates $x_1, \dots, x_n$ and decompose $P = \sum_\alpha p_\alpha \partial^\alpha$ and $Q = \sum_\beta q_\beta \partial^\beta$. Let m_1, m_2 be the maximal values of $|\alpha|$ and $|\beta|$ with non-zero coefficients. Then the induced elements of P and Q in $\operatorname{gr}\mathscr{D}_X$ are of the form $p = \sum_{|\alpha|=m_1} p_\alpha \xi^\alpha$ and $q = \sum_{|\beta|=m_2} q_\beta \xi^\beta$ where ξ_i is the induced element of ∂_i. On the other hand, the induced element of $PQ - QP$ is $\sum_{i=1}^n \frac{\partial p}{\partial \xi_i}\frac{\partial q}{\partial x_i} - \frac{\partial q}{\partial \xi_i}\frac{\partial p}{\partial x_i}$. This is precisely $\{p, q\}$ where $\{-,-\}$ is the Poisson bracket.

Denote $\pi : T^*X \to X$ for the projection map. Then any $\operatorname{gr}\mathscr{D}_X$-module $\mathcal{M}$ has a corresponding module on T^*X defined by $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\operatorname{gr}\mathscr{D}_X} \mathcal{M}$. By abuse of notation this module is still denoted $\mathcal{M}$ and it will always be implicitly assumed that $\operatorname{gr}\mathscr{D}_X$-modules live on T^*X unless it is explicitly mentioned otherwise. In particular, for a filtration of the $\mathscr{D}_X$-module $\mathscr{M}$ the graded object $\operatorname{gr}\mathscr{M} = \oplus_i F_i\mathscr{M}/F_{i-1}\mathscr{M}$ is a $\operatorname{gr}\mathscr{D}_X$-module.

Proposition 2.1.6. *A $\mathscr{D}_X$-module $\mathscr{M}$ is coherent if and only if it locally admits a filtration such that $\operatorname{gr}\mathscr{M}$ is a coherent $\operatorname{gr}\mathscr{D}_X$-module. Such a filtration is called a good filtration. Moreover, this filtration can be taken globally if X is a variety.*

Proof. A proof of this result may be found in (Hotta and Tanisaki, 2007, Chapter 2). □

Proposition 2.1.7. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then the support of $\operatorname{gr}^{rel}\mathscr{M}$ in T^*X is a independent of the chosen good filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\operatorname{Ch}\mathscr{M}$.*

Proof. A proof of this result may be found in (Hotta and Tanisaki, 2007, Chapter 2). □

Proposition 2.1.8. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, then $\operatorname{Ch}\mathscr{M}$ is a homogeneous and involutive closed subset of T^*X.*

Proof. These results may be found in (Kashiwara, 2003, Chapter 2). □

Characteristic varieties behave well with respect to quotients and submodules.

Proposition 2.1.9. *Consider a short exact sequence of coherent $\mathscr{D}_X$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

then it holds that

$$\operatorname{Ch}\mathscr{M}_2 = \operatorname{Ch}\mathscr{M}_1 \cup \operatorname{Ch}\mathscr{M}_3.$$

Proof. A good filtration on $\mathscr{M}_2$ induces good filtrations on $\mathscr{M}_1$ and $\mathscr{M}_3$ and one has a short exact sequence

$$0 \to \operatorname{gr} \mathscr{M}_1 \to \operatorname{gr} \mathscr{M}_2 \to \operatorname{gr} \mathscr{M}_3 \to 0$$

whence the result follows. □

Let f be a non-constant function of $\mathcal{O}_X$ and s a new variable. For the purposes of the study of Bernstein-Sato polynomials the coherent $\mathscr{D}_X$-module $\mathscr{D}_X f^s$ is essential. The corresponding characteristic variety is understood and may provide some intuition for general characteristic varieties.

Proposition 2.1.10. *The characteristic variety of the coherent $\mathscr{D}_X$-module $\mathscr{D}_X f^s$ is the closure of*

$$W_f = \{(x, sf^{-1}df(x)); \qquad f(x) \neq 0, s \in \mathbb{C}\}$$

in T^*X.

Proof. This proof of this result may be found in Kashiwara (1976). □

Proposition 2.1.11. *One can write $\operatorname{Ch} \mathscr{D}_X f^s = \Lambda \cup W$ for some isotropic variety $\Lambda \subseteq T^*X$ and a irreducible $(n+1)$-dimensional variety W which dominates X.*

Proof. The proof of this result may be found in Kashiwara (1976). It follows from proposition 2.1.10 by establishing that the part of the closure of W_f above $f = 0$ is isotropic. □

Direct Image

In this section we describe the direct image of $\mathscr{D}_Y$-modules. Let $\mu : Y \to X$ be some morphism of smooth algebraic varieties or complex manifolds.

A-priori, it is not even clear what $\mathscr{D}_X$-module should correspond to $\mathscr{D}_Y$ since there is no natural push forward of vector fields. For example consider the case where μ is the embedding of a curve in X. This issue may be resolved by use of the transfer $(\mathscr{D}_Y, \mu^{-1}\mathscr{D}_X)$-bimodule $\mathscr{D}_{Y\to X} := \mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\mathscr{D}_X$. Here, the right $\mu^{-1}\mathscr{D}_X$-module structure is just the action on the second component and the left $\mathscr{D}_Y$-module structure is defined by

$$f \cdot (g \otimes \mu^{-1}h_X) = fg \otimes \mu^{-1}h_X; \qquad \xi \cdot (g \otimes \mu^{-1}h_X) = \xi g \otimes \mu^{-1}h_X + g \otimes T\mu(\xi)\mu^{-1}h_X$$

for any sections f of $\mathcal{O}_Y$ and ξ of Θ_Y. Here $T\mu(\xi)$ is a local section of $\mathcal{O}_Y \otimes_{\mu^{-1}\mathcal{O}_X} \mu^{-1}\Theta_X$.

Definition 2.1.12. *The direct image functor $\int_\mu$ from $D^{b,r}(\mathscr{D}_Y)$ to $D^{b,r}(\mathscr{D}_X)$ is defined to be $R\mu_*(- \otimes^L_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$. For any $\mathscr{D}_Y$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{D}_X$-modules $\int^j_\mu \mathscr{M} = \mathscr{H}^j \int_\mu \mathscr{M}$. The subscript μ will be suppressed whenever there is no ambiguity.*

Let us remark that a explicit free resolution for the transfer module is known. This involves the Spencer complex $\operatorname{Sp}^\bullet_Y(\mathscr{M})$ of a $\mathscr{D}_Y$-module $\mathscr{M}$ with $\operatorname{Sp}^{-k}_Y(\mathscr{M}) = \mathscr{M} \otimes_{\mathcal{O}_Y} \wedge^k \Theta_k$. The details may be found in Sabbah (2011). A direct image functor for left $\mathscr{D}_Y$-modules is induced as

$$\int \mathscr{M} := R\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \int \mathscr{M} \otimes_{\mathcal{O}_Y} \omega_Y)).$$

The definition for the direct image functor is somewhat subtle due to passing through derived categories but many nice properties follow. Most notably, it is immediate from the derived definition that one gets a long exact sequence.

Proposition 2.1.13. *For any short exact sequence of $\mathscr{D}_Y$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

there is a long exact sequence in direct images

$$0 \to \int^0 \mathscr{M}_1 \to \int^0 \mathscr{M}_2 \to \int^0 \mathscr{M}_3 \to \int^1 \mathscr{M}_1 \to \cdots .$$

Proposition 2.1.14. *Let $\mu : Z \to Y$ and $\nu : Y \to X$ be a morphisms of smooth varieties. Then there is a isomorphism of functors $\int_{\nu\circ\mu} \cong \int_\nu \int_\mu$.*

Proof. A proof may be found in (Borel, 1987, Chapter 6). □

A similar theorem applies to complex manifolds provided μ is proper. Denote $D^{*,*}_{coh}(\mathscr{D}_X)$ for the full subcategory of $D^{*,*}(\mathscr{D}_X)$ consisting of those complexes of $\mathscr{D}_X$-modules with coherent cohomology. The coherence properties of the direct image in the analytic case require the following notion.

Definition 2.1.15. *A $\mathscr{D}_Y$-module $\mathscr{M}$ is said to be μ-good if there exists a open cover $\{V_j\}_{j\in J}$ of X such that $\mathscr{M}$ admits a good filtration on $\mu^{-1}(V_j)$ for any $j \in J$.*

Note that, by proposition 2.1.6, any coherent $\mathscr{D}_Y$-module on a algebraic variety is μ-good.

Theorem 2.1.16. *Let $\mathscr{M}$ be a μ-good $\mathscr{D}_Y$-module and suppose that μ is proper on the support of $\mathscr{M}$. Then, $\int \mathscr{M}$ has coherent cohomology.*

Proof. A proof may be found in (Sabbah, 2011, Chapter 3) □

Consider the following cotangent diagram.

$$\begin{tikzcd} & \mu^*T^*X \arrow[dl, "T^*\mu"'] \arrow[dr, "\widetilde{\mu}"] & \\ T^*Y & & T^*X \end{tikzcd}$$

Proposition 2.1.17. *Let $\mathscr{M}$ be a μ-good $\mathscr{D}_Y$-module and suppose that μ is proper on the support of $\mathscr{M}$. Then, for any $j \geq 0$*

$$\mathrm{Ch}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\mathrm{Ch}\,\mathscr{M})\right).$$

Proof. See remark 2.5.2 in (Hotta and Tanisaki, 2007, Chapter 2) for the algebraic case or (Sabbah, 2011, Chapter 3) for the analytic case. □

2.2 Riemann-Hilbert Correspondence

This section concerns the Riemann-Hilbert correspondence which states that a system of differential equations is equivalent to it's system of solutions. The systems of differential equations are encoded in regular holonomic $\mathscr{D}_X$-modules. The solutions are given by perverse sheaves.

Holonomic Modules

A particularly nice class of $\mathscr{D}_X$-modules are given by maximally overdetermined systems of differential equations. This is to say that there are many relations for $\mathscr{M}$ or equivalently that $\operatorname{Ch}\mathscr{M}$ is small. Observe that by the involutive part of proposition 2.1.8 it follows that $\dim \operatorname{Ch}\mathscr{M} \geq n$.

Definition 2.2.1. *A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called holonomic if* $\dim \operatorname{Ch}\mathscr{M} = n$.

The full subcategory of $D^{*,*}(\mathscr{D}_X)$ consisting of complexes with holonomic cohomology is denoted $D_h^{*,*}(\mathscr{D}_X)$. For technical purposes it is mostly important that holonomic modules have finiteness properties.

Proposition 2.2.2. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then, for any $x \in X$, the stalk $\mathscr{M}_x$ is a $(\mathscr{D}_X)_x$-module of finite length.*

Proof. This result may be found in (Kashiwara, 2003, Chapter 4). □

Proposition 2.2.3. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ is $\mathbb{C}$-algebraic. This is to say that for any $\varphi \in \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ there exists some polynomial b with coefficients in $\mathbb{C}$ such that $b(\varphi) = 0$.*

Proof. This result may be found in (Bjork, 1979, Chapter 5). □

Proposition 2.2.4. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module and suppose that $\mu : Y \to X$ is proper on the support of $\mathscr{M}$. Then $\int \mathscr{M}$ has holonomic cohomology. Moreover, the assumption that μ is proper may be removed if in the algebraic case.*

Proof. This result may be found in (Sabbah, 2011, Chapter 4) or (Hotta and Tanisaki, 2007, Chapter 3) in the algebraic case. □

When μ is proper this may be established by combining proposition 2.1.17 with the following results.

Lemma 2.2.5. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathscr{M}$ has a globally defined good filtration.*

Proof. This result may be found in (Sabbah, 2011, Chapter 4). □

Lemma 2.2.6. *Let $\mu : Y \to X$ be a proper morphism and $V \subseteq T^*Y$ an isotropic subvariety. Then $\widetilde{\mu}((T^*\mu)^{-1}(\operatorname{Ch}^{rel}\mathscr{M}))$ is also isotropic.*

Proof. This result may be found in Kashiwara (1976). □

Regular singularities

This section is based on (Kashiwara, 2003, Chapter 5) and (Hotta and Tanisaki, 2007, Chapter 6). Let $X = \mathbb{C}$ considered with it's analytical topology and consider a ordinary differential operator $P(x, \partial) = \sum_{k=0}^{m} a_k(x)\partial^k$. Suppose that $a_m(x) \neq 0$ for any $x \neq 0$. Then $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$ is locally of the form $\mathcal{O}_X^m$ as a $\mathscr{D}_X$-module near any point $x \neq 0$. In particular the solutions $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ form a locally constant sheaf of rank m outside of 0. The solutions near zero may be more subtle due to monodromy.

Observe that $\operatorname{Ch}\mathscr{M} \subseteq \{(x, \xi) : x\xi = 0\}$. Hence, for any filtration on $\mathscr{M}$ there exists some $N > 0$ such that $(x\xi)^N \operatorname{gr}\mathscr{M} = 0$.

Proposition 2.2.7. *The following conditions are equivalent.*

1. *There exists a filtration on $\mathscr{M}$ such that $x\xi \operatorname{gr} \mathscr{M} = 0$.*

2. *The equation $P(x, \partial)u$ has m linearly independent solutions of the form $x^\lambda \sum_{j=0}^{s} u_j \log(x)^j$ near 0 for some $s \geq 0$, $\lambda \in \mathbb{C}$ and holomorphic u_j if and only P has a regular singularity in 0.*

Proof. This result may be found in (Kashiwara, 2003, Chapter 5). $\square$

If these two equivalent conditions are satisfied one calls 0 a regular singularity of $\mathscr{M}$. This has the following generalisation to higher dimensions.

Definition 2.2.8. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a complex manifold X with characteristic variety determined by some ideal sheaf $\mathcal{I}$. Then $\mathscr{M}$ is called regular holonomic if it it admits a filtration such that $\mathcal{I} \operatorname{gr}(\mathscr{M}) = 0$.*

Denote $D^{**}_{rh}(\mathscr{D}_X)$ for the full subcategory of $D^{**}(\mathscr{D}_X)$ consisting of complexes with regular holonomic cohomology.

It appears that these definitions should generalise directly to the algebraic situation. However, this has unintended consequences for the Riemann-Hilbert correspondence. For a example, let $X = \mathbb{C}$ as before and consider the regular holonomic $\mathscr{D}_X$-modules $\mathcal{O}_X$ and $\mathscr{M} := \mathscr{D}_X / \mathscr{D}_X(\partial - 1)$. These are analytically isomorphic by the map which sends f to $f \exp(x)$. In particular the Riemann-Hilbert correspondence shows that they have isomorphic systems of solutions. However, $\mathcal{O}_X$ and $\mathscr{M}$ are not algebraically isomorphic. This seems to suggest that the equivalence between differential equations and their systems of solutions would not hold in the algebraic case. The problem is that $\mathscr{M}$ is not regular at infinity.

There are a number of equivalent definitions for regularity in the algebraic case. The following definition expresses that the analytic definition may be used provided one adds the points at infinity. This uses the analytification functor on coherent sheaves which is provided by the GAGA principle and respects holonomicity.

Definition 2.2.9. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module on a smooth variety X. Denote $\iota : X \to \overline{X}$ for the smooth completion of X. Then $\mathscr{M}$ is called regular if $(\int_\iota \mathscr{M})^{an}$ is regular holonomic on the complex manifold $\overline{X}^{an}$.*

Perverse Sheaves

Classically, the solutions to a differential equation on a vector bundle produce a local system. One can not expect local systems in the case of general $\mathscr{D}_X$-modules since their support could be a proper subvariety.

Definition 2.2.10. *Let X be a complex manifold. A stratification of X consists of a locally finite partition $X = \sqcup_{j \in J} X_j$ into connected locally closed subsets, called strata, such that*

(i) For any $j \in J$ the fronteer $\partial X_j = \overline{X}_j \setminus X_j$ is a union of strata.

(ii) For any $j \in J$ the spaces $\overline{X}_j$ and ∂X_j are closed complex analytic subspaces.

The same definition applies on algebraic varieties by replacing the analytic subspaces by subvarieties.

Definition 2.2.11. *A $\mathbb{C}_X$-module $\mathcal{F}$ is called a constructible sheaf on X if there exists a stratification $X = \sqcup_{\alpha \in A} X_\alpha$ such that $\mathcal{F}|_{X_\alpha}$ is a local system of finite rank on X_α for any $\alpha \in A$.*

Denote $D_c^b(X)$ for the full subcategory of $D^b(\mathbb{C}_X)$ consisting of complexes with constructible cohomology. Such complexes are called constructible.

For a constructible complex $\mathcal{F}^\bullet$ in $D_c^b(X)$ the supports and cosupports are defined dually by

$$\operatorname{supp}^m \mathcal{F}^\bullet = \operatorname{supp} \mathscr{H}^m \mathcal{F}^\bullet; \qquad \operatorname{cosupp}^m \mathcal{F}^\bullet = \operatorname{supp}^{-m} \mathbb{D}\mathcal{F}^\bullet.$$

The support $\operatorname{supp} \mathcal{F}^\bullet$ is the closure of the union of the $\operatorname{supp}^m \mathcal{F}^\bullet$.

Theorem 2.2.12. *Let $\mathcal{F}^\bullet$ be a constructible complex on Y and consider a morphism $\mu : Y \to X$ which is proper on* $\operatorname{supp} \mathcal{F}^\bullet$*. Then $Rf_*(\mathcal{F}^\bullet)$ is constructible on X.*

Proof. This result may be found in (Dimca, 2004, Chapter 4). □

Theorem 2.2.13. *Let $\mathcal{F}^\bullet$ be a complex of $D^b(\mathbb{C}_X)$. Then $\mathcal{F}^\bullet$ is constructible if and only if the dual $\mathbb{D}\mathcal{F}^\bullet := R\mathcal{H}om_{\mathbb{C}}(\mathcal{F}^\bullet, \mathbb{C}_X)$ is constructible.*

Proof. This result may be found in (Dimca, 2004, Chapter 4). □

Let $D^{\leq 0}(X)$ denote the full subcategory of $D_c^b(X)$ consisting of complexes with $\dim \operatorname{supp}^{-m} \mathcal{F}^\bullet < m$ and $\dim \operatorname{supp}^m \mathcal{F}^\bullet = 0$ for all $m \geq 0$. Dually $D^{\geq 0}(X)$ consists of complexes with $\dim \operatorname{cosupp}^{-m} \mathcal{F}^\bullet < m$ and $\dim \operatorname{cosupp}^m \mathcal{F}^\bullet = 0$ for all $m \geq 0$.

Proposition 2.2.14. *The pair $(D^{\leq 0}(X), D^{\geq 0}(X))$ is a non-degenerated t-structure on the triangulated category $D_c^b(X)$.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Definition 2.2.15. *The heart of the t-structure on $D_c^b(X)$ are called the perverse sheaves* $\operatorname{Perv}(X) = D^{\leq 0}(X) \cap D^{\geq 0}(X)$.

Observe that the objects in $\operatorname{Perv}(X)$ are not sheaves but complexes. The reason for the terminology perverse sheaves is that the functor $U \mapsto \operatorname{Perv}(U)$ has the gluing property of sheaves. More precisely, it is a stack. Perverse sheaves still capture the local systems.

Theorem 2.2.16. *Let X be a complex manifold of dimension n. Then $\mathcal{L}[n]$ is a perverse sheaf on X for any local system $\mathcal{L}$ on X.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

The following are immediate from proposition 1.5.3 and proposition 1.5.8.

Proposition 2.2.17. *A constructible complex $\mathcal{F}^\bullet$ is a perverse sheaf if and only if ${}^pH(\mathcal{F}^\bullet) = 0$ for all $k \neq 0$.*

Proposition 2.2.18. *For any distinguished triangle in $D_c^b(X)$*

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \xrightarrow{+1}$$

it holds that if two terms are perverse sheaves then so is the third.

Riemann-Hilbert Correspondence

Recall that $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ encodes the solutions of a system of differential equations. More generally, the solutions complex is the functor $\mathrm{Sol}(-) := R\mathcal{H}om_{\mathscr{D}_X}(-, \mathcal{O}_X)[n]$ from $D^{b,\ell}(\mathscr{D}_X)^{opp}$ to $D^b(\mathbb{C}_X)$. The contravariance may be fixed using the duality functor

$$\mathbb{D} = R\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X) \otimes^L_{\mathcal{O}_X} \omega_X^{-1}[n]$$

from $D^{b,*}(\mathscr{D}_X)^{opp}$ to $D^{b,*}(\mathscr{D}_X)$. The de Rham complex of $\mathscr{M}^\bullet$ is defined by

$$\mathrm{DR}(\mathscr{M}^\bullet) := \Omega^\bullet_X \otimes_{\mathscr{D}_X} \mathscr{M}^\bullet[n].$$

Proposition 2.2.19. *There is a natural isomorphism* $\mathrm{Sol}(-) \cong \mathrm{DR}(\mathbb{D}-)$.

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Proposition 2.2.20. *For any holonomic complex* $\mathscr{M}^\bullet$ *in* $D_h^{b,\ell}(\mathscr{D}_X)$ *the complexes* $\mathrm{Sol}(\mathscr{M}^\bullet)$ *and* $\mathrm{DR}(\mathscr{M}^\bullet)$ *are constructible.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

We are finally ready to state the Riemann-Hilbert correspondence on the equivalence between differential equations and their solutions.

Theorem 2.2.21 (Riemann-Hilbert Correspondence)**.** *The de Rham functor* $\mathrm{DR} : D^{b,\ell}_{rh}(\mathscr{D}_X) \to D^b_c(X)$ *is a t-exact equivalence of categories and commutes with direct images.*

Proof. This result may be found in (Dimca, 2004, Chapter 5). □

Corollary 2.2.22. *The de Rham functor is a equivalence of categories between the category of regular holonomic* $\mathscr{D}_X$*-modules and* $\mathrm{Perv}(X)$.

Proof. Follows from the Riemann-Hilbert correspondence and proposition 1.5.11. □

2.3 Monodromy and Bernstein-Sato Polynomials

The goal of this section is to motivate Bernstein-Sato polynomials by their connection to monodromy. Further, we include Kashiwara and Lichtin's proof for the estimation of the roots of the Bernstein-Sato polynomial. This proof is a important framework for the generalised proof in chapter 3.

We focus on the local analytic case. The algebraic case will be discussed in detail in the next chapter. Let X be a complex manifold and consider a holomorphic function $f : X \to \mathbb{C}$. Let $x \in X$ with $f(x) = 0$, we study the function germ $f : (X, x) \to (\mathbb{C}, 0)$.

Monodromy

Theorem 2.3.1. *Let* $B \subseteq X$ *be a small ball and pick* $t \in \mathbb{C}^\times$ *close to zero. The diffeomorphism class of* $F_{f,x} := f^{-1}(t) \cap B$ *is independent of the choice of* t.

Proof. The isolated singularity case is established in Milnor (1968) and the general case was established by Hamm-Le. □

Definition 2.3.2. *The diffeomorphism class of $F_{f,x}$ is called the Milnor fiber.*

Going over a loop around the origin in T induces a endomorphism M^* on the cohomology $H^j(F_{f,x}, \mathbb{C}_{F_{f,x}})$ for every $j \in \mathbb{Z}$. This is called the monodromy action and only depends on the local singularity $f^{-1}(0) \cap B$. In particular, this means that the eigenvalues of M^* on $H^j(F_{f,x}, \mathbb{C}_{F_{f,x}})$ are invariants of the singularity. If $\lambda \in \mathbb{C}$ is a eigenvalue of M^* on some $H^j(F_{f,x}, \mathbb{C}_{F_{f,x}})$ it is called a eigenvalue of monodromy.

Definition 2.3.3. *Let s be a new variable. The local Bernstein-Sato polynomial $b_{f,x}(s) \in \mathbb{C}[s]$ is the monic polynomial of minimal degree such that there exists some differential operator $P(x, \partial, s)$ in $\mathscr{D}_{X,x} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with*

$$P(x, \partial, s)f^{s+1} = b_{f,x}(s)f^s$$

in the stalk at x.

The existence of Bernstein-Sato polynomials was originally proved by I.N. Berntein. The following section gives a alternative method to prove the existence. In fact, it will even be established that all roots of $b_{f,x}(s)$ are negative rational numbers.

Let $Z(b_{f,x})$ denote the set of zeros of the Bernstein-Sato polynomial. These are connected to monodromy by the following classical theorem due to Malgrange and Kashiwara.

Theorem 2.3.4. *Denote* $\operatorname{Exp}(x) = \exp(2\pi i x)$ *then* $\operatorname{Exp}(Z(b_{f,x}))$ *is equal to the set of eigenvalues of monodromy.*

Proof. This theorem may be found in Budur (2015). □

Monodromy is a topological notion whereas the Bernstein-Sato polynomial is defined in terms of $\mathscr{D}_X$-modules. This suggest that the Riemann-Hilbert correspondence is involved. Indeed, monodromy of the Milnor fiber can be encoded in a constructible complex so that the Riemann-Hilbert correspondence is applicable.

Let $\widetilde{\mathbb{C}}^\times$ denote the universal cover of $\mathbb{C}^\times$ and consider the projection $p : X \times \widetilde{\mathbb{C}}^\times \to X$. Denote $\iota : f^{-1}(0) \to X$ for the inclusion map.

Definition 2.3.5. *Deligne's nearby cycle functor from $D^b_c(X)$ to $D^b_c(f^{-1}(0))$ is given by $\psi_f := L\iota^* \circ Rp_* \circ Lp^*$.*

Denote $\iota_x : \{x\} \to f^{-1}(0)$ for the inclusion map. The following theorem is due to Deligne.

Theorem 2.3.6. *There is a isomorphism*

$$H^i(F_{f,x}, \mathbb{C}_{F_{f,x}}) \cong \mathbb{H}^i(L\iota_x^*(\psi_f \mathbb{C}_X))$$

and the monodromy action on the cohomology of the Milnor fiber corresponds with the action of the covering transformations $\widetilde{\mathbb{C}}^\times \to \widetilde{\mathbb{C}}^\times$ on the nearby cycles.

Proof. This theorem may be found in Budur (2015). □

To describe the $\mathscr{D}_X$-theoretic counterpart of this constructible complex requires the technical notion of V-filtrations. The interested reader may find these concepts in Budur (2015).

Estimation of $Z(b_{f,0})$

Estimation of $Z(b_{f,x})$ is easy if f is a monomial. For instance, if $f = x_1^{r_1} \cdots x_n^{r_n}$ then set $P = \partial_1^{r_1} \cdots \partial_n^{r_n}$ to get

$$Pf^{s+1} = \prod_{i=1}^{n} (r_i s + r_i) \cdots (r_i s + 1) f^s.$$

The main idea employed in the proof by Kashiwara (1976) is that one can reduce to this situation by resolution of singularities and $\mathscr{D}_X$-module direct images.

Definition 2.3.7. *Let D be a divisor on X. A strong log resolution of (X, D) consists of a projective morphism $\mu : Y \to X$ which is a isomorphism over the complement of D such that Y is smooth and $\mu^* D$ is a simple normal crossings divisor.*

Let D be the divisor determined by f. By Hironaka's resolution of singularities one can find a resolution $\mu : Y \to X$ for (X, D). Let $g = f \circ \mu$ denote the pullback of f to Y, then $\mu^* D = \sum_{i=1}^{r} \operatorname{mult}_{E_i}(g) E_i$ where $\operatorname{mult}_{E_i}(g)$ denotes the order of vanishing of g on the irreducible hypersurface E_i. Using that g is locally a monomial Kashiwara was able to establish the following estimate by consideration of the direct image of the $\mathscr{D}_Y$-module $\mathscr{D}_Y g^s$.

Theorem 2.3.8. *Every root of $b_{f,x}(s)$ is of the form $-c_i / \operatorname{mult}_{E_i}(g)$ with $c_i \in \mathbb{Z}_{>0}$. In particular $Z(b_{f,x}) \subseteq \mathbb{Q}_{<0}$.*

Proof. This is proved in Kashiwara (1976). □

Combining this estimate with theorem 2.3.4 one gets the following theorem.

Theorem 2.3.9. *The eigenvalues of monodromy are roots of unity.*

Lichtin (1989) improved the estimate by similar computations for the right $\mathscr{D}_Y$-module $\mathscr{M}$ spanned by $g^s \mu^*(dx)$ inside $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. The advantage of this approach is that $\mu^*(dx)$ involves the local behaviour of μ which can detect redundant divisors in $\mu^* D$. The vanishing of $\mu^*(dx)$ is measured in the relative canonical divisor $K_{Y/X} = \sum k_i E_i$.

Theorem 2.3.10. *Every root of $b_{f,x}(s)$ is of the form $-(k_i + c_i) / \operatorname{mult}_{E_i}(g)$ with $c_i \in \mathbb{Z}_{>0}$.*

We will now reproduce the proof for this improved estimate following Lichtin and Kashiwara.

One can ensure that multiplication by s stays inside $\mathscr{D}_X f^s$ with the following trick. Introduce a new coordinate x_{n+1} and set $\widetilde{f} = x_{n+1} f$ on the space $X \times \mathbb{C}$. Then $x_{n+1}\partial_{n+1}$ acts like s on $\widetilde{f}^s$. The induced map $Y \times \mathbb{C} \to X \times \mathbb{C}$ is a strong log resolution for the divisor determined by $\widetilde{f}$. Now suppose we can prove theorem 2.3.10 for $\widetilde{f}$. Then, the theorem also follows for f due to the following result.

Lemma 2.3.11. *The Bernstein-Sato polynomial $b_{f,x}(s)$ is a divisor of $b_{\widetilde{f},x}(s)$.*

Proof. Take local coordinates $x_1, \ldots, x_{n+1}$ and let P be in the stalk $\mathscr{D}_{X \times \mathbb{C}, x}$ such that $b_{\widetilde{f},x} \widetilde{f}^s = P \widetilde{f}^{s+1}$. Expand $P = \sum_{j=1}^{N} P_j \partial_{n+1}^j$ with coefficients P_j in $\mathscr{D}_{X,x}$. Then

$$x_{n+1}^N b_{\widetilde{f},x}(s) \widetilde{f}^s = \left(\sum_{j=1}^{k} (s+1)^j \sum_{\alpha} Q_\alpha \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \right) \widetilde{f}^{s+1}$$

where the P_j were expanded as polynomials in $\partial_1, \dots, \partial_n$ with coefficients Q_α in $\mathcal{O}_{X,x}$.

Observe that $\partial_1, \dots, \partial_n$ act on the formal symbol $\widetilde{f}^{s+1}$ the same as they act on the formal symbol f^{s+1}. Expand the Q_α as power series in $x_1, \dots, x_{n+1}$ and identify powers of x_{n+1} on both sides for the desired functional equation. $\square$

For notational simplicity we will simply write f instead of $\widetilde{f}$ and X instead of $X \times \mathbb{C}$. The dimension of X will still be denoted with n.

Let t be a new variable. The sheaf of rings $\mathscr{D}_X\langle s,t\rangle$ is found from $\mathscr{D}_X$ by adjoining s and t subject to $ts - st = 1$ where s, t commute with $\mathscr{D}_X$. The notation with the angle brackets is to emphasise that s and t do not commute with each other. One can view $\mathscr{D}_X f^s$ as a $\mathscr{D}_X\langle s,t\rangle$-module by the action $tP(s)f^s = P(s+1)f^{s+1}$ for any differential operator P in $\mathscr{D}_X[s]$. In this notation the functional equation for $b_{f,x}$ means that $b_{f,x} \in \mathrm{Ann}_{\mathbb{C}[s]}(\mathscr{D}_X f^s / t\mathscr{D}_X f^s)_x$.

There is a $\mathcal{O}_X$-linear isomorphism between any left $\mathscr{D}_X$-module $\mathscr{N}$ and it's right version $\mathscr{N} \otimes_{\mathcal{O}_X} \omega_X$. Concretely, any section u of $\mathscr{N}$ gives rise to the section $u^* := u dx$. Further, for any operator P of $\mathscr{D}_X$ there is a adjoint P^* such that

$$(P \cdot u)^* = u^* \cdot P^*$$

for any section u of $\mathscr{N}$. For a vector field $\xi := \sum_i \xi_i \partial_i$ comparison of the definitions shows that $\xi^* := \sum_i \partial_i \xi_i$ satisfies this equality and this extends to $\mathscr{D}_X$ by iterating. By this procedure the functional equation $Pf^{s+1} = b(s)f^s$ may equivalently be stated as the equation

$$f^{s+1} dx \cdot P^* = b(s) f^s dx$$

in $\mathscr{D}_X f^s \otimes_{\mathcal{O}_X} \omega_X$. The corresponding module $\mathscr{M}$ on Y will be the submodule of $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$ spanned by $g^s \mu^*(dx)$. Observe that $\mathscr{M}$ may also be equipped with a $\mathscr{D}_X\langle s,t\rangle$-module structure.

Lemma 2.3.12. *The polynomial* $b(s) = \prod_{i=1}^r \prod_{j=1}^{\mathrm{mult}_{E_i}(g)} (\mathrm{mult}_{E_i}(g)s + k_i + j)$ *annihilates* $\mathscr{M}/t\mathscr{M}$.

Proof. This may be checked locally. If the chosen point is on none of divisors E_i then g is invertible so that $\mathscr{M}/t\mathscr{M}$ is trivial. Now suppose we are working near a point $y \in Y$ which is on E_i if and only if $i \in I$ with I non-empty. Then one can pick local coordinates y_i such that

$$g = \prod_{i \in I} y_i^{\mathrm{mult}_{E_i}(g)}; \qquad \mu^*(dx) = u \prod_{i \in I} y_i^{k_i}$$

where u is a local unit. Now set $P = u^{-1}(\prod_{i \in I} \partial_i^{\mathrm{mult}_{E_i}(g)})u$ to get

$$g^{s+1}\mu^*(dx) \cdot P = q(s) g^s \mu^*(dx)$$

where $q(s) = \prod_{i \in I} \prod_{j=1}^{\mathrm{mult}_{E_i}(g)} (\mathrm{mult}_{E_i}(g)s + k_i + j)$. $\square$

Observe that s, t can be viewed as $\mathscr{D}_X$-linear injective endomorphisms on $\mathscr{M}$. The associated the long exact sequence yields a $\mathscr{D}_X\langle s,t\rangle$-module structure on the direct image $\int^0 \mathscr{M}$ where the functorial nature of the direct image is used to ensure that $ts - st = 1$. Similarly, the polynomial $b(s)$ provided by lemma 2.3.12 annihilates $\int^0 \mathscr{M} / t \int^0 \mathscr{M}$.

Consider the surjection $\mathscr{D}_Y \to \mathscr{M}$ induced by $1 \mapsto g^s \mu^*(dx)$. The associated long exact sequence includes a morphism $\int^0 \mathscr{D}_Y \to \int^0 \mathscr{M}$. Observe that $\int^0 \mathscr{D}_Y = R^0\mu_*(\mathscr{D}_{Y\to X})$ contains a global section corresponding to the section 1 of $\mathscr{D}_{Y\to X}$. Let u be the image of this section in $\int^0 \mathscr{M}$ and denote $\mathscr{U}$ for the right $\mathscr{D}_X\langle s,t\rangle$-module generated by u.

Lemma 2.3.13. *There is a surjective morphism of right $\mathscr{D}_X\langle s,t\rangle$-modules $\mathscr{U} \to \mathscr{D}_X f^s \otimes_{\mathcal{O}_X} \omega_X$ sending u to $f^s dx$.*

Proof. The resolution of singularities $Y \to X$ is a isomorphism on the complement of the divisor D determined by f. Hence, a isomorphism $\mathscr{U} \cong \int^0 \mathscr{M} \cong \mathscr{D}_X f^s \otimes_{\mathcal{O}_X} \omega_X$ holds outside of D.

Pick some open set $V \subseteq X$. To show this yields a well-defined morphism of $\mathscr{D}_X$-modules it must be show that $(f^s dx)P = 0$ whenever $uP = 0$ in $\mathscr{U}(V)$. Due to the isomorphism it is certainly the case that $(f^s dx)P = 0$ outside of D. Hence, the support of the coherent sheaf of $\mathcal{O}_V$-modules $\mathcal{O}_V(f^s dx)P$ lies in D. The Nullstellen Satz now yields that $f^N(f^s dx)P = 0$ for some sufficiently large $N \geq 0$. Note that f is a non-zero divisor of $(\mathscr{D}_X f^s \otimes_{\mathcal{O}_X} \omega_X)(V)$. Hence, it follows that $(f^s dx)P = 0$ on V as desired.

Finally, observe that $tu = fu$ so that this morphism of $\mathscr{D}_X$-modules also commutes with the actions by t and s. □

Due to lemma 2.3.12 there is a suitable b-polynomial for $\int^0 \mathscr{M}$. By **??** it remains to compare $\int^0 \mathscr{M}$ and $\mathscr{U}$.

Lemma 2.3.14. *The quotient $\int^0 \mathscr{M}/\mathscr{U}$ is a holonomic $\mathscr{D}_X$-module.*

Proof. By proposition 2.1.9 the characteristic variety of $\mathscr{M}$ is a subset of the characteristic variety of $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. This has the same characteristic variety as $\mathscr{D}_Y g^s$ due to the behaviour of the $\mathcal{O}_Y$-linear isomorphism between $\mathscr{D}_Y g^s$ and $\mathscr{D}_Y g^s \otimes_{\mathcal{O}_Y} \omega_Y$. By proposition 2.1.11 it follows that $\operatorname{Ch} \mathscr{M} \subseteq W \cup \Lambda$ for some isotropic $\Lambda \subseteq T^*Y$ and a irreducible $(n+1)$-dimensional variety W which dominates Y.

Observe that $\mathscr{M}$ is certainly μ-good since it admits a global good filtration $F_i\mathscr{M} := F_i\mathscr{D}_X \cdot g^s \mu^*(dx)$. Hence, proposition 2.1.17 is applicable and yields that

$$\operatorname{Ch} \int^0 \mathscr{M} \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\Lambda \cup W)\right).$$

By lemma 2.2.6 the set $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda))$ is still isotropic and will not form any obstruction to $\int^0 \mathscr{M}/\mathscr{U}$ being holonomic. Further, observe that $\widetilde{\mu}((T^*\mu)^{-1}(W))$ remains a irreducible $(n+1)$-dimensional variety which dominates X. On the other hand μ is a isomorphism outside of $\{f = 0\}$ so $\int^0 \mathscr{M}/\mathscr{U}$ is only supported on $\{f = 0\}$. Intersecting $\widetilde{\mu}((T^*\mu)^{-1}(W))$ with $\{f = 0\}$ yields a n-dimensional variety whence the desired result follows. □

Proposition 2.3.15. *For sufficiently large N it holds that $t^N(\int^0 \mathscr{M})_x/\mathscr{U}_x = 0$.*

Proof. The sequence $t^n \int^0 \mathscr{M}/\mathscr{U}$ forms a decreasing sequence of holonomic $\mathscr{D}_X$-modules. By proposition 2.2.2 the induced sequence of $\mathscr{D}_{X,x}$ modules in the stalk at x must stabilise. Let N be sufficiently large such that $t^N(\int^0 \mathscr{M})_x/\mathscr{U}_x$ attains the stable value.

Applying proposition 2.2.3 to the $(\mathscr{D}_X)_x$-linear automorphism s produces a non-zero polynomial $q(s) \in \mathbb{C}[s]$ which annihilates $t^N(\int^0 \mathscr{M})_x/\mathscr{U}_x$. Let $q(s)$ be of minimal degree with this property. Observe that $q(s+1)t = tq(s)$ so, using the stabilisation, it follows that

$$q(s+1)t^N \left(\int^0 \mathscr{M}\right)_x / \mathscr{U}_x = tq(s)t^N \left(\int^0 \mathscr{M}\right)_x / \mathscr{U}_x = 0.$$

This means that $q(s) - q(s+1)$ also annihilates $t^N(\int^0 \mathscr{M})_x/\mathscr{U}_x$. By the minimality of the degree of $q(s)$ it follows that $q(s) - q(s+1) = 0$ which is to say that $q(s)$ is a non-zero constant. This means that $t^N(\int^0 \mathscr{M})_x/\mathscr{U}_x = 0$ as desired. □

Putting all these facts together yields the proof of theorem 2.3.10.

Proof. Let N be as in proposition 2.3.15 and denote $b(s)$ for the polynomial provided by lemma 2.3.12. Set $B(s) = b(s+N+1)b(s+N)\cdots b(s)$ and observe that $B(s)\mathscr{M}_x \subseteq t^{N+1}\mathscr{M}_x \subseteq t\mathscr{U}_x$. In particular this means that $B(s) \in \operatorname{Ann}_{\mathbb{C}[s]} \mathscr{U}_x/t\mathscr{U}_x$.

The $\mathscr{D}_X\langle s,t\rangle$-linear surjection $\mathscr{U} \to \mathscr{D}_X f^s \otimes_{\mathcal{O}_X} \omega_X$ from lemma 2.3.13 now implies that $b_{f,x}(s)$ divides $B(s)$. This yields the desired estimate for $Z(b_{f,x})$. □

Chapter 3

Relative Holonomic Modules

3.1 Introduction

Let X be a smooth algebraic variety and $F : X \to \mathbb{C}^p$ a morphism with coordinate functions $f_1, \ldots, f_p$. Introduce new variables $s_1, \ldots, s_p$ and abbreviate $F^s = f_1^{s_1} \cdots f_p^{s_p}$. The local Bernstein-Sato ideal $B_{F,x}$ of F at $x \in X$ is the collection of all $b(s_1, \ldots, s_p) \in \mathbb{C}[s]$ such that there exists some differential operator $P \in \mathscr{D}_{X,x} \otimes_{\mathbb{C}} \mathbb{C}[s]$ with

$$b(s_1, \ldots, s_p)F^s = P \cdot F^{s+1}$$

in the stalk at x. The local Bernstein-Sato ideal only depends on the local singularity at x, in particular there are only finitely many different $B_{F,x}$. The global Bernstein-Sato ideal B_F is the intersection of all local Bernstein-Sato ideals. The local Bernstein-Sato ideal is also a topic of interest when F is a analytic function germ.

In the case $p = 1$ the ring $\mathbb{C}[s]$ is a principal ideal domain and the monic generator of $B_{F,x}$ recovers the Bernstein-Sato polynomial. Recall from section 2.3 that the Bernstein-Sato polynomial encodes the eigenvalues of monodromy. A generalisation is possible for the Bernstein-Sato zero locust $Z(B_F)$.

Let D be the divisor determined by $\prod f_i$ and set $U = X \setminus D$. For any $\lambda \in (\mathbb{C}^*)^p$ let L_λ denote the line bundle on U found by the pullback over F of the line bundle on $(\mathbb{C}^*)^p$ with monodromy λ_i around the i-th missing coordinate hyperplane. The eigenvalues of monodromy are generalised by the set

$$S(F) = \{\lambda \in (\mathbb{C}^*)^p \mid \exists j \in \mathbb{Z}\ \exists x \in D : H^j(U_x, L_\lambda) \neq 0\}$$

where U_x is the intersection of U with a arbitrarily small ball centred at x. If F is a analytic function germ X should be replaced by a arbitrarily small neighbourhood of x and we denote the resulting set $S_x(F)$.

Theorem 3.1.1. *If F is a morphism of smooth affine algebraic varieties or then* $\mathrm{Exp}(Z(B_F)) = S(F)$. *If F is a holomorphic germ at x then* $\mathrm{Exp}(Z(B_{F,x})) = S_x(F)$.

Proof. This is proved in Budur et al. (2019). □

Theorem 3.1.2. *Every irreducible component of $Z(B_{F,x})$ of codimension > 1 can be translated by an element of $\mathbb{Z}^r$ into a component of codimension one.*

Proof. This result is due to Maisonobe (2016). □

This shows that $Z(B_F)$ is a natural topic of study. The rationality of the roots of the Bernstein-Sato polynomial admits the following generalisation of Kashiwara's estimate. Denote $\mu : Y \to X$ for the strong log resolution of (X, D). Then $\mu^* D = \sum a_i E_i$ is in normal crossings form and we let $G = F \circ \mu$ with coordinate functions $g_1, \dots, g_p$.

Theorem 3.1.3. *Every irreducible component of $Z(B_{F,x})$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + c_i$$

with $c_i \in \mathbb{Z}_{>0}$.

Proof. The rationality of the hyperplanes was established by Sabbah (1987) and Gyoja et al. (1993). The concrete estimate for the slopes was established by Budur et al. (2019). □

The goal of this chapter is to generalise Lichtin's estimate in theorem 2.3.10 using the canonical divisor $K_{Y/X} = \sum k_i E_i$.

Theorem 3.1.4. *Every irreducible component of $Z(B_{F,x})$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{>0}$.

The main new ingredient in our proof is a induction argument which reduces the problem to the monovariate case. This induction process involves a tensor product to reduce the number of variables. The non-exactness of the tensor product causes error-terms in the form of the $\mathcal{T}or$-functor. These error-terms may be controlled using homological results which are set up in section 3.4

In fact, it is known that analytical and algebraic local Bernstein-Sato ideals agree. Hence, the algebraic version follows from the analytical version. Nonetheless, for notational simplicity the proof will be done in the algebraic case and we discuss the necessary modifications for the local analytic case at the end.

Theorem 3.1.5. *The local Bernstein-Sato ideal of $B_{F,x}$ of a function germ F on X equals the local Bernstein-Sato ideal on the analytification X^{an}.*

Proof. The original proof is in the unpublished work Brianccon and Maisonobe (2002). We were unable to find a copy of this work. The proof may also be found in chapter 7 of Kruff et al. (2015). □

3.2 Relative Notions

For any regular commutative $\mathbb{C}$-algebra integral domain R we define a sheaf of rings on $X \times \operatorname{Spec} R$ by

$$\mathscr{A}_X^R = \mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_{\operatorname{Spec} R}.$$

It will also be convenient to use the abbreviations $\mathcal{O}_X^R := \mathcal{O}_{X \times \operatorname{Spec} R}$.

The order filtration $F_p\mathscr{D}_X$ extends to a filtration $F_p\mathscr{A}_X^R = F_p\mathscr{D}_X \otimes_{\mathbb{C}} \mathcal{O}_R$ on $\mathscr{A}_X^R$ which is called the relative filtration. The associated graded objects are denoted by gr^{rel}. Denote $\pi : T^*X \times \operatorname{Spec} R \to X \times \operatorname{Spec} R$ for the projection map. As in the case of $\mathscr{D}_X$-modules one can view $\pi^{-1}(\mathrm{gr}^{rel}\mathscr{A}_X^R)$ as a subsheaf of $\mathcal{O}_{T^*X}^R$ and for any $\mathrm{gr}^{rel}\mathscr{A}_X^R$-module $\mathcal{M}$ there is a corresponding module on $T^*X \times \operatorname{Spec} R$ defined by $\mathcal{O}_{T^*X}^R \otimes_{\pi^{-1}\mathrm{gr}^{rel}\mathscr{A}_X^R} \pi^{-1}\mathcal{M}$. By abuse of notation the corresponding module on $T^*X \times \operatorname{Spec} R$ is still denoted with $\mathcal{M}$ and we adopt the perspective that $\mathrm{gr}^{rel}\mathscr{A}_X^R$-modules always live on $T^*X \times \operatorname{Spec} R$ unless explicitly mentioned otherwise.

Similarly to the case of $\mathscr{D}_X$ it holds that $\mathscr{A}_X^R$ is the sheaf of rings generated by $\mathcal{O}_X^R$ and Θ_X inside of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X^R)$. Giving a left $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X^R$-module $\mathscr{M}$ with Θ_X-action such that $\xi \cdot (fm) = f(\xi \cdot m) + \xi(f)\, m$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X. Similarly, giving a right $\mathscr{A}_X^R$-module is equivalent to giving a $\mathcal{O}_X$-module $\mathscr{M}$ with Θ_X-action such that $(mf) \cdot \xi = (m \cdot \xi)f - m\, \xi(f)$ for any sections f of $\mathcal{O}_X^R$ and ξ of Θ_X.

The following results are analogous to those in chapter 2. They follow from general results in (Björk, 1993, Appendix III).

Proposition 3.2.1. *A quasi-coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is coherent if and only if it admits a filtration such that $\mathrm{gr}^{rel}\mathscr{M}$ is coherent over $\mathrm{gr}^{rel}\mathscr{A}_X^R$. Such a filtration is called a good filtration.*

Proposition 3.2.2. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^R$-module, then the support of $\mathrm{gr}^{rel}\mathscr{M}$ in $T^*X \times \operatorname{Spec} R$ is independent of the chosen good filtration. It is called the characteristic variety of $\mathscr{M}$ and denoted $\mathrm{Ch}^{rel}\mathscr{M}$.*

Lemma 3.2.3. *Consider a short exact sequence of coherent $\mathscr{A}_X^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

then it holds that

$$\mathrm{Ch}^{rel}\mathscr{M}_2 = \mathrm{Ch}^{rel}\mathscr{M}_1 \cup \mathrm{Ch}^{rel}\mathscr{M}_3.$$

A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be relative holonomic over R if $\mathrm{Ch}^{rel}\mathscr{M} = \cup_w \Lambda_w \times S_w$ for irreducible conic Lagrangian subvarieties $\Lambda_w \subseteq T^*X$ and irreducible closed subvarieties $S_w \subseteq \operatorname{Spec} R$.

Lemma 3.2.4. *The sheaf of $\mathscr{A}_X$-modules $\mathscr{A}_Y F^s$ is relative holonomic.*

Proof. This result may be found as proposition 13 in Maisonobe (2016). The proof applies in both the analytic and algebraic cases. □

Lemma 3.2.5. *Let $\mathscr{M}$ be a finitely generated $\mathscr{A}_Y^R$-module. Suppose that $\mathrm{Ch}^{rel}\mathscr{M} \subseteq \Lambda \times \operatorname{Spec} R$ for some, not necessarily irreducible, conic Lagrangian subvariety $\Lambda \subseteq T^*X$. Then $\mathscr{M}$ is relative holonomic.*

Proof. This result may be found in Maisonobe (2016) in the analytical case and Budur et al. (2019) in the algebraic case. □

The Bernstein-Sato ideal may be defined more generally for any $\mathscr{A}_X^R$-module $\mathscr{M}$ as $B_{\mathscr{M}} := \mathrm{Ann}_R\, \mathscr{M}$. To see how this generalises B_F one considers $\mathscr{A}_X^R F^s$ as a $\mathscr{A}_X^R\langle t\rangle$-module. Here t is a new variable which commutes with sections of $\mathscr{D}_X$ and satisfies $ts_i - s_i t = 1$ for any $i = 1, \ldots, p$. The $\mathscr{A}_X^R\langle t\rangle$-module structure on $\mathscr{A}_X^R F^s$ is then defined by $tP(s)F^s = P(s+1)F^{s+1}$ for any section P of $\mathscr{A}_X^R$. From this point of view $B_F = B_{\mathscr{A}_X^R F^s / t\mathscr{A}_X^R F^s}$.

The Bernstein-Sato ideal may be recovered from the characteristic variety.

Proposition 3.2.6. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$ module. Then $Z(B_{\mathscr{M}}) = \pi_2(\mathrm{Ch}^{rel}(\mathscr{M}))$ where $\pi_2 : T^*X \times \operatorname{Spec} R \to \operatorname{Spec} R$ is the projection on the second coordinate.*

Proof. This result may be found in Maisonobe (2016) in the analytical case and Budur et al. (2019) in the algebraic case. □

3.3 Direct Image Functor for $\mathscr{A}_X^R$-modules

In this section we state the natural generalisation of the direct image functor for $\mathscr{D}_X$-modules to the relative case of $\mathscr{A}_X^R$-modules. As with $\mathscr{D}$-modules this is the most natural for right-modules.

Consider some morphism $\mu : Y \to X$ and denote μ^R for the induced map from $Y \times \operatorname{Spec} R$ to $X \times \operatorname{Spec} R$. A transfer $(\mathscr{A}_Y^R, (\mu^R)^{-1}\mathscr{A}_X^R)$-bimodule is defined by $\mathscr{A}_{Y\to X}^R := \mathscr{D}_{Y\to X} \otimes_{\mathbb{C}} R$. Written out, this means that $\mathscr{A}_{Y\to X}^R = \mathcal{O}_Y^R \otimes_{(\mu^R)^{-1}\mathcal{O}_X^R} (\mu^R)^{-1}\mathscr{A}_X^R$.

Definition 3.3.1. *The direct image functor $\int_{\mu^R}$ from $D^{b,r}(\mathscr{A}_Y^R)$ to $D^{b,r}(\mathscr{A}_X^R)$ is defined to be $R\mu_*^R(- \otimes_{\mathscr{A}_Y^R}^L \mathscr{A}_{Y\to X}^R)$. For any $\mathscr{A}_Y^R$ module $\mathscr{M}$ the j-th direct image is the $\mathscr{A}_X^R$-modules $\int_{\mu^R}^j \mathscr{M} = H^j \int_{\mu^R} \mathscr{M}$. The subscript μ^R will be suppressed whenever there is no ambiguity.*

Observe that a free $\mathscr{A}_Y^R$-resolution for a complex $\mathscr{M}^\bullet$ is also a free $\mathscr{D}_Y$-resolution. Hence, the following isomorphism holds in $D^{b,r}(\mathscr{D}_X)$ on $Y \times \operatorname{Spec} R$

$$\mathscr{M}^\bullet \otimes_{\mathscr{A}_Y^R}^L \mathscr{A}_{Y\to X}^R \cong \mathscr{M}^\bullet \otimes_{\mathscr{D}_Y}^L \mathscr{D}_{Y\to X}.$$

Denote $p_X : X \times \operatorname{Spec} R \to X$ for the projection map. Due to proposition 1.4.6 one has a isomorphism

$$R(p_X)_* \circ R\mu_*^R \cong R(p_X \circ \mu^R)_* \cong R\mu^* \circ R(p_Y)_*.$$

Combining the right-hand-sides of these isomorphisms yields the $\mathscr{D}_X$-module direct image. This is to say that the $\mathscr{A}_Y^R$-module direct image computes the $\mathscr{D}_Y$-module direct image with additional structure.

By definition as a derived functor a long exact sequence is immediate.

Proposition 3.3.2. *For any short exact sequence of $\mathscr{A}_Y^R$-modules*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

there is a long exact sequence in direct images

$$0 \to \int^0 \mathscr{M}_1 \to \int^0 \mathscr{M}_2 \to \int^0 \mathscr{M}_3 \to \int^1 \mathscr{M}_1 \to \cdots.$$

Analogously to the case of $\mathscr{D}_X$-modules the direct image has the following conservation properties.

Definition 3.3.3. *A $\mathscr{A}_Y^R$-module $\mathscr{M}$ is said to be μ-good if X admits a open cover $\{V_j\}_{j\in J}$ such that $\mathscr{M}$ has a good filtration on $\mu^{-1}(V_j)$ for every $j \in J$.*

Theorem 3.3.4. *Suppose that μ is proper and let $\mathscr{M}$ be a μ-good $\mathscr{A}_Y^R$-module. Then $\int \mathscr{M}$ has coherent cohomology.*

Proof. The proof for $\mathscr{D}_X$-modules in chapter 3 of Sabbah (2011) is still applicable provided Grauert's coherence theorem is replaced with the algebraic version. This is theorem 3.2.1 of Grothendieck and Dieudonné (1961). $\square$

We want to establish that direct images also conserve holonomicity. A analytical version of this result is found in theorem 1.17 of Monteiro Fernandes and Sabbah (2016). The analytic analogue of $\mathscr{A}_Y^R$ is given by $\mathscr{D}_{Y\times U/U} := \mathscr{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y\times U}$ where U denotes the analytification of $\operatorname{Spec} R$. It is clear how notions such as the relative characteristic variety may be analytified.

Since analytification is a exact functor it is compatible with good filtrations and $(\mathrm{gr}^{rel}\mathscr{M})^{an} \cong \mathrm{gr}^{rel}\mathscr{M}^{an}$. In particular $\mathrm{Ch}^{rel}\mathscr{M}^{an}$ is the analytification of $\mathrm{Ch}^{rel}\mathscr{M}$. Using proposition 1.4.6 the exactness of analytification also implies that it commutes with the direct image.

Now consider the following cotangent diagram.

$$\begin{tikzcd} & \mu^* T^* X \times \operatorname{Spec} R \arrow[dl, "T^*\mu \times \mathrm{Id}_{\operatorname{Spec} R}"'] \arrow[dr, "\widetilde{\mu} \times \mathrm{Id}_{\operatorname{Spec} R}"] & \\ T^* Y \times \operatorname{Spec} R & & T^* X \times \operatorname{Spec} R \end{tikzcd}$$

Theorem 3.3.5. *Suppose that μ is proper and let $\mathscr{M}$ be a μ-good $\mathscr{A}_Y^R$-module. Then, for any $j \geq 0$*

$$\mathrm{Ch}^{rel}\left(\int^j \mathscr{M}\right) \subseteq \widetilde{\mu} \times \mathrm{Id}_{\operatorname{Spec} R}\left((T^*\mu \times \mathrm{Id}_{\operatorname{Spec} R})^{-1}(\mathrm{Ch}^{rel}\mathscr{M})\right).$$

Proof. The analytic version of this statement is provided by corollary 4.3 in Schapira and Schneiders (1994). This implies the algebraic version due to all involved notions commuting with analytification. □

Theorem 3.3.6. *Suppose that μ is proper and let $\mathscr{M}$ be a μ-good relative holonomic $\mathscr{A}_Y^R$-module. Then $\int \mathscr{M}$ has relative holonomic cohomology.*

Proof. Use lemma 2.2.6 on the upper bound of theorem 3.3.5. And apply lemma 3.2.5 for the desired result. □

3.4 Non-commutative Homological Notions

In this section we discuss homological notions associated to the $\mathcal{E}xt$-functor over the noncommutative sheaf of rings $\mathscr{A}_X^R$. These notions are particularly well-behaved for relatively holonomic modules. The results are sheaf-theoretic rewordings of the similar results in Budur et al. (2019) which are themselves derived from the appendices of Björk (1993).

Definition 3.4.1. *For a non-zero coherent sheaf of $\mathscr{A}_X^R$-modules $\mathscr{M}$ the smallest integer $k \geq 0$ such that $\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \neq 0$ is called the grade of $\mathscr{M}$ and is denoted $j(\mathscr{M})$.*

The following proposition gives geometrical meaning to grades.

Proposition 3.4.2. *For coherent $\mathscr{A}_X^R$-modules $\mathscr{M}$ it holds that*

$$j(\mathscr{M}) + \dim \mathrm{Ch}^{rel}\mathscr{M} = 2n + \dim R$$

where $\dim R$ *denotes the Krull dimension of the ring R.*

Proof. This is lemma 3.2.2 in Budur et al. (2019). □

Corollary 3.4.3. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module. Then $\mathscr{M}$ has grade strictly greater than n if and only if $B_{\mathscr{M}}$ is non-zero.*

Proof. This is immediate from proposition 3.2.6 and proposition 3.4.2. □

Definition 3.4.4. *A non-zero coherent sheaf of $\mathscr{A}_X^R$-modules $\mathscr{M}$ is called j-pure if $j(\mathscr{N}) = j(\mathscr{M}) = j$ for every non-zero submodule $\mathscr{N}$.*

Lemma 3.4.5. *Let $\mathscr{M}$ be a non-zero coherent $\mathscr{A}_X^R$-module of grade j. Then $\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A})$ has grade greater than or equal to k for any $k \geq 0$ and $\mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$ is a j-pure $\mathscr{A}_X^R$-module.*

Moreover $\mathscr{M}$ is j-pure if and only if $\mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^R) = 0$ for every $k \neq j$.

Proof. This is lemma 4.3.5 in Budur et al. (2019). □

Lemma 3.4.6. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade j. Then $\mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$ is a relative holonomic $\mathscr{A}_X^R$-module and*

$$\mathrm{Ch}^{rel}\, \mathcal{E}xt^j_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \subseteq \mathrm{Ch}^{rel} \mathscr{M}.$$

Proof. This is lemma 3.2.4 in Budur et al. (2019). □

Lemma 3.4.7. *Let $P \subseteq R$ be a prime ideal and let $\mathscr{M}$ be a coherent $\mathscr{A}_X^{R/P}$-module. If $\mathscr{M}$ is relative holonomic as a $\mathscr{A}_X^R$-module then it is also relative holonomic over $\mathscr{A}_X^{R/P}$.*

Proof. That $\mathscr{M}$ is relative holonomic over $\mathscr{A}_X^R$ means that it admits a good filtration such that

$$\operatorname{supp} \mathrm{gr}^{rel}_{\mathscr{A}_X^R} \mathscr{M} = \bigcup \Lambda \times S_\Lambda$$

for Lagrangian subvarieties $\Lambda \subseteq T^*X \times \operatorname{Spec} R$ and algebraic varieties $S_\Lambda \subseteq \operatorname{Spec} R$. This filtration descends to a good filtration over $\mathscr{A}_X^{R/P}$ and it holds that

$$\operatorname{supp} \mathrm{gr}^{rel}_{\mathscr{A}_X^{R/P}} \mathscr{M} = (\mathrm{Id}_{T^*X} \times \Delta)^{-1}(\operatorname{supp} \mathrm{gr}^{rel}_{\mathscr{A}_X^R} \mathscr{M})$$

where $\Delta : \operatorname{Spec} R/P \to \operatorname{Spec} R$ is the closed embedding. This yields the desired result. □

Lemma 3.4.8. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module and let $P \subseteq R$ be a prime ideal. Then, for any $k \geq 0$, $\mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/P})$ is a relative holonomic $\mathscr{A}_X^{R/P}$-module.*

Proof. Compute $\mathcal{T}or_k^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/P})$ with a locally free $\mathscr{A}_X^R$-resolution of $\mathscr{A}_X^{R/P}$. Then lemma 3.2.3 and lemma 3.2.5 show that it is a relative holonomic $\mathscr{A}_X^R$-module. The claim follows by the foregoing lemma. □

Lemma 3.4.9. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module which is $(n+k)$-pure for some $0 \leq k \leq \dim R$. If $b \in R$ is not contained in any minimal prime ideal containing $B_{\mathscr{M}}$ then multiplication by b induces injective automorphisms on $\mathscr{M}$ and $\mathcal{E}xt^{n+k}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$. Moreover, there exists a good filtration on $\mathscr{M}$ such that b induces a injection on $\mathrm{gr}^{rel} \mathscr{M}$.*

Proof. This is lemma 3.4.2 in Budur et al. (2019). □

The proof of the following lemma is a slight modification on the proof of proposition 3.4.3 in Budur et al. (2019).

Lemma 3.4.10. *Let $\mathscr{M}$ be a non-zero relative holonomic $\mathscr{A}_X^R$-module of grade $j(\mathscr{M}) = n$ then, for any irreducible $b \in R$, it holds that $\mathscr{M} \otimes_R R/(b)$ is a non-zero relative holonomic $\mathscr{A}_X^{R/(\ell)}$-module of grade n.*

Proof. Applying lemma 3.4.8 with $k = 0$ yields that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic $\mathscr{A}_X^{R/(\ell)}$-module.

It remains to establish that $\mathscr{M} \otimes_R R/(b)$ is non-zero of grade n. By taking a free resolution of $\mathscr{M}$ one has that

$$R\mathcal{H}om_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) \otimes^L_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)} \cong R\mathcal{H}om_{\mathscr{A}_X^{R/(b)}}(\mathscr{M} \otimes^L_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}, \mathscr{A}_X^{R/(b)})$$

where we note that $\mathscr{A}_X^{R/(b)}$ is a $\mathscr{A}_X^R$-bimodule so that both tensor products are well-defined. We compare the Grothendieck spectral sequences of both sides of this isomorphism.

The spectral sequence associated with the right-hand-side has E_2-sheet

$$E_2^{pq} = \mathcal{E}xt^p_{\mathscr{A}_X^{R/(b)}}(\mathcal{T}or_{-q}^{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^{R/(b)}), \mathscr{A}_X^{R/(b)}).$$

Recall from lemma 3.4.5 that terms with $p > n$ have grade greater than n and due to proposition 3.4.2 there are no non-zero terms with $p < n$. Hence, the only term with $p + q = n$ which could potentially have degree n is E_2^{n0}. If we can show that the total cohomology of degree n on the left-hand-side has grade n then it follows that $\mathcal{E}xt^n_{\mathscr{A}_X^{R/(b)}}(\mathscr{M} \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}, \mathscr{A}_X^{R/(b)}) \neq 0$ which is the desired result.

The spectral sequence associated to the left-hand-side has E_2-sheet given by

$$E_2^{pq} = \mathcal{T}or_{-p}^{\mathscr{A}_X^R}(\mathcal{E}xt^q_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^{R/(b)}).$$

Note that there are no non-zero differentials out of E_j^{0n} for $j \geq 2$. Further, the differentials into E_j^{0n} stem from $E_j^{-j(n+j-1)}$ which is a subquotient of $\mathcal{T}or_j^{\mathscr{A}_X^R}(\mathcal{E}xt^{n+j-1}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^{R/(b)})$. Observe that $\mathscr{A}_X^R \xrightarrow{b} \mathscr{A}_X^R$ yields a free resolution for $\mathscr{A}_X^{R/(b)}$. It follows that $E_j^{-j(n+j-1)} = 0$ for $j \geq 2$ whence $E_j^{0n} = E_2^{0n}$ for all $j \geq 2$. It remains to show that that E_2^{0n} has grade n, then the the total cohomology of degree n has grade n and this concludes the proof.

Denote $\mathcal{E}^n := \mathcal{E}xt^n_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R)$, by lemma 3.4.5 it holds that $\mathcal{E}^n$ is a n-pure relative holonomic $\mathscr{A}_X^R$-module. By lemma 3.4.9 it follows that b induces injections on $\mathcal{E}^n$ and $\mathrm{gr}^{rel}\,\mathcal{E}^n$ for some appropriate filtration. In particular the vertical maps in the following diagram are injective

$$\begin{tikzcd}
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] \arrow[d, "b"] & F_i\mathcal{E}^n \arrow[r] \arrow[d, "b"] & \mathrm{gr}_i^{rel}\,\mathcal{E}^n \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & F_{i-1}\mathcal{E}^n \arrow[r] & F_i\mathcal{E}^n \arrow[r] & \mathrm{gr}_i^{rel}\,\mathcal{E}^n \arrow[r] & 0
\end{tikzcd}$$

so the snake lemma yields a short exact sequence

$$0 \longrightarrow F_{i-1}\mathcal{E}^n \otimes_R R/(b) \longrightarrow F_i\mathcal{E}^n \otimes_R R/(b) \longrightarrow \mathrm{gr}_i^{rel}\,\mathcal{E}^n \otimes_R R/(b) \longrightarrow 0.$$

The injectivity of b on $\operatorname{gr}^{rel}\mathcal{E}^n$ implies that b is also injective on $\mathcal{E}^n/F_i\mathcal{E}^n$. A similar application of the snake lemma now yields a short exact sequence

$$0 \longrightarrow F_i\mathcal{E}^n \otimes_R R/(b) \longrightarrow \mathcal{E}^n \otimes_R R/(b) \longrightarrow (\mathcal{E}^n/F_i\mathcal{E}^n) \otimes_R R/(b) \longrightarrow 0.$$

A filtration on $\mathcal{E}^n \otimes_R R/(b)$ is induced by the image of $F_i\mathcal{E}^n$. By the injectivity of the short exact sequences one now has isomorphisms

$$F_i(\mathcal{E}^n \otimes_R R/(b)) \cong F_i\mathcal{E}^n/(F_i\mathcal{E}^n \cap b\mathcal{E}^n) \cong F_i\mathcal{E}^n/bF_i\mathcal{E}^n \cong (F_i\mathcal{E}^n) \otimes_R R/(b)$$

combined with the surjectivity of the first short exact sequence it follows that

$$\operatorname{gr}^{rel}(\mathcal{E}^n \otimes_R R/(b)) \cong \operatorname{gr}^{rel}\mathcal{E}^n \otimes_R R/(b).$$

It follows that

$$\operatorname{Ch}^{rel}(\mathcal{E}^n \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}) = (\operatorname{Id}_{T^*X} \times \Delta)^{-1}(\operatorname{Ch}^{rel}\mathscr{M})$$

with $\Delta : \operatorname{Spec} R/(b) \to \operatorname{Spec} R$ the closed embedding as before. Since $\mathscr{M}$ has grade n this equality and proposition 3.4.2 imply that $\operatorname{Ch}^{rel}(\mathcal{E}^n \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)})$ has dimension $n + \dim R - 1$. In particular it follows that $\mathcal{E}^n \otimes_{\mathscr{A}_X^R} \mathscr{A}_X^{R/(b)}$ is non-zero and has grade n. This concludes the proof. □

By lemma 3.4.5 the following definition gives a class of j-pure modules.

Definition 3.4.11. *A coherent $\mathscr{A}_X^R$-module $\mathscr{M}$ is said to be j-Cohen-Macaulay for some $j \geq 0$ if $\mathcal{E}xt^k_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R) = 0$ for any $k \neq j$.*

The property of being j-pure is not stable when restricting to a subscheme of $\operatorname{Spec} R$. For the subclass of j-Cohen-Macaulay modules the restriction is more well-behaved.

Lemma 3.4.12. *Let $\mathscr{M}$ be a relative holonomic and $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^R$-module. Let $b \in R$ be irreducible and non-vanishing on every irreducible component of $Z(B_{\mathscr{M}})$. Then it holds that $\mathscr{M} \otimes_R R/(b)$ is a relative holonomic $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^{R/(b)}$-module or zero.*

Proof. This is shown in the proof of proposition 3.4.3 in Budur et al. (2019). This proof is similar to the proof of lemma 3.4.10 which was based on Budur et al. (2019). □

Lemma 3.4.13. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade $n+k$. Then there exists a open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathscr{M} \otimes_R R'$ is a relative holonomic and $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^{R'}$ module. Moreover it may be assumed that the complement of $\operatorname{Spec} R'$ in $\operatorname{Spec} R$ has codimension $> k$.*

Proof. This is established in the proof of lemma 3.5.2 in Budur et al. (2019). □

Relative holonomic $\mathscr{A}_X^R$-modules are not necessarily of finite length. This seems to be a obstruction for generalising the proof of proposition 2.3.15. The following lemmas provide the necessary modifications and provide the same line of thought as the proof by Kashiwara (1976) for proposition 2.3.15.

Lemma 3.4.14. *For any $(n+k)$-Cohen-Macaulay $\mathscr{A}_X^R$-module $\mathscr{M}$ it holds that*

$$\mathcal{E}xt^{n+k}_{\mathscr{A}_X^R}(\mathcal{E}xt^{n+k}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^R) \cong \mathscr{M}.$$

Proof. By taking a free resolution one sees that

$$R\mathcal{H}om_{\mathscr{A}_X^R}(R\mathcal{H}om_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^R) \cong \mathscr{M}.$$

Due to the $(n+k)$-Cohen-Macaulay assumption the only non-zero term in the Grothendieck spectral sequence of the left-hand-side is $\mathcal{E}xt^{n+k}_{\mathscr{A}_X^R}(\mathcal{E}xt^{n+k}_{\mathscr{A}_X^R}(\mathscr{M}, \mathscr{A}_X^R), \mathscr{A}_X^R)$ whence the desired result follows. □

Lemma 3.4.15. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade $j(\mathscr{M}) = n + k$. Consider a chain of submodules $\cdots \to \mathscr{M}_2 \to \mathscr{M}_1 \to \mathscr{M}_0 = \mathscr{M}$ such that the quotients $\mathscr{M}_i/\mathscr{M}_{i+1}$ are isomorphic. Then there exists a open* $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ *and some sufficiently large N such that $\mathscr{M}_i' = \mathscr{M}_i \otimes_R R'$ stabilises for $i \geq N$. Moreover, it may be assumed that* $\operatorname{Spec} R \setminus \operatorname{Spec} R'$ *has codimension strictly greater than k.*

Proof. By use of lemma 3.4.13 it may be assumed that $\operatorname{Spec} R'$ is such that $\mathscr{M}_i'$, $\mathscr{M}'/\mathscr{M}_i'$ and $\mathscr{M}'/\mathscr{M}_1'$ are zero or $(n+k)$-Cohen-Macauley for any $i = 0, \ldots, N$. For notational simplicity we abbreviate $\mathcal{E}^k(\mathscr{M}') := \mathcal{E}xt^k_{\mathscr{A}_X^{R'}}(\mathscr{M}', \mathscr{A}_X^{R'})$.

Observe that $\mathscr{M}_i'/\mathscr{M}_{i+1}'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$. This may be used to establish that $\mathscr{M}'/\mathscr{M}_i$ and $\mathscr{M}_i$ are actually $(n+k)$-Cohen-Macaulay or zero for arbitrary $i \geq 0$.

The injection $\mathscr{M}_{i+1}' \to \mathscr{M}_i'$ induces exact sequences

$$\mathcal{E}^{n+k+j}\left(\frac{\mathscr{M}_i'}{\mathscr{M}_{i+1}'}\right) \to \mathcal{E}^{n+k+j}\left(\mathscr{M}_i'\right) \to \mathcal{E}^{n+k+j}\left(\mathscr{M}_{i+1}'\right) \to 0.$$

By induction on i it follows that $\mathscr{M}_i'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$. Similarly the long exact sequence induced by the surjection $\mathscr{M}'/\mathscr{M}_{i+1}' \twoheadrightarrow \mathscr{M}'/\mathscr{M}_i'$ yields that $\mathscr{M}'/\mathscr{M}_i'$ is $(n+k)$-Cohen-Macaulay or zero for any $i \geq 0$.

The fact that $\mathscr{M}_i'/\mathscr{M}_{i+1}'$ and $\mathscr{M}'/\mathscr{M}_i'$ are $(n+k)$-Cohen-Macauley implies that the morphisms $\mathcal{E}^{n+k}(\mathscr{M}_i') \to \mathcal{E}^{n+k}(\mathscr{M}_{i+1}')$ and $\mathcal{E}^{n+k}(\mathscr{M}') \to \mathcal{E}^{n+k}(\mathscr{M}_i')$ are surjective. Note that $\mathcal{E}^{n+k}(\mathscr{M})$ is a coherent sheaf over the Noetherian sheaf of rings $\mathscr{A}_X^{R'}$. Hence the kernels of $\mathcal{E}^{n+k}(\mathscr{M}') \to \mathcal{E}^{n+k}(\mathscr{M}_i')$ stabilise. This means that the maps $\mathcal{E}^{n+k}(\mathscr{M}_i') \to \mathcal{E}^{n+k}(\mathscr{M}_{i+1}')$ are isomorphisms for sufficiently large i. This establishes that $\mathcal{E}^{n+k}(\mathscr{M}_{i+1}')$ stabilises. By $\mathscr{M}_i'$ being $(n+k)$-Cohen-Macaulay this means that $\mathcal{E}^{n+k}(\mathcal{E}^{n+k}(\mathscr{M}_i')) \cong \mathscr{M}_i'$ stabilises. □

Lemma 3.4.16. *Let $\mathscr{M}$ be a relative holonomic $\mathscr{A}_X^R$-module of grade $j(\mathscr{M}) = n+k$ which comes equipped with the structure of a $\mathscr{A}_X^R\langle t\rangle$-module. Suppose that $k \geq 1$ and that the quotients $t^i\mathscr{M}/t^{i+1}\mathscr{M}$ are isomorphic. Then there exists a open* $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ *and some $N \geq 1$ such that $\mathscr{M}' = \mathscr{M} \otimes_R R'$ is a relative holonomic $\mathscr{A}_X^{R'}$-module with $t^N\mathscr{M}' = 0$. Moreover, it may be assumed that* $\operatorname{Spec} R \setminus \operatorname{Spec} R'$ *has codimension strictly greater than k.*

Proof. By lemma 3.4.15 a open $\operatorname{Spec} R'$ may be found such that $t^i\mathscr{M}'$ stabilises for $i \geq N$. By corollary 3.4.3 there exists some non-zero $b(s_1, \ldots, s_p) \in B_{\mathscr{M}'}$ which is necessarily also in $B_{t^N\mathscr{M}}$. Note that one has the commutation relation

$$tb(s_1, \ldots, s_p) = b(s_1 + 1, \ldots, s_p + 1)t.$$

Since $t^{N+1}\mathscr{M}' = t^N\mathscr{M}'$ it follows by iteration that $b(s_1 + n, \ldots, s_p + n) \in B_{t^N\mathscr{M}'}$ for any $n \geq 0$. This implies that $Z(B_{t^N\mathscr{M}'}) = \emptyset$ which means that $t^N\mathscr{M}' = 0$ due to proposition 3.2.6. □

3.5 Estimation of the Bernstein-Sato Zero Locust

This section contains the main result of this chapter, namely a proof of the improved estimate for the Bernstein-Sato zero locust which was announced in theorem 3.1.4. We use the same notation as section 3.1. This proof is similar to the method employed by Lichtin (1989) and Kashiwara (1976) but a new induction argument is required in the proof of lemma 3.5.8.

For the estimation of $Z(B_{F,x})$ it may be assumed that X is affine and admits global coordinates $x_1, \ldots, x_n$. Similarly to section 2.3 these global coordinates allow to restate the functional equation $PF^{s+1} = bF^s$ as the equation

$$F^{s+1}dx \cdot P^* = bF^s dx$$

in $\mathscr{A}_X F^s \otimes_{\mathcal{O}_X} \omega_X$. The corresponding module $\mathscr{M}$ on Y will be the submodule of $\omega_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_Y G^s$ spanned by $G^s \mu^*(dx)$. Observe that $\omega_Y \otimes_{\mathcal{O}_Y} \mathscr{A}_Y G^s$ is relative holonomic due to lemma 3.2.4 so the submodule $\mathscr{M}$ is certainly also relative holonomic. One can equip $\mathscr{M}$ with the structure of a $\mathscr{A}_Y\langle t\rangle$-module by the action $tG^s\mu^*(dx) = G^{s+1}\mu^*(dx)$. The following statement and proof are analogous to lemma 2.3.12.

Lemma 3.5.1. *In the notation of section 3.1 a polynomial of the form*

$$b(s) = \prod_{i=1}^{p}\prod_{j=1}^{N}(\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + j)$$

belongs to the Bernstein-Sato ideal $B_{\mathscr{M}/t\mathscr{M}}$ if $N \geq 0$ is sufficiently large.

Proof. This may be checked locally. Suppose we are working near a point $y \in Y$ which is on E_i if and only if $i \in I$. Pick local coordinates $z_1, \ldots, z_n$ where z_i determines E_i if $i \in I$.

In these local coordinates $G^s = \prod_{j=1}^{p} u_j^{s_j} \prod_{i\in I} z_i^{\sum_{j=1}^{p}\mathrm{mult}_{E_i}(g_j)s_j}$ and $\mu^*(dx) = v\prod_{i\in I} z_i^{k_i} dz$ For any $i \in I$ set $\xi_i := \partial_i - \sum_{j=1}^{p} s_i \partial_i(u_i)u_i^{-1}$ and let $P = v^{-1}(\prod_{j=1}^{p} u_j^{-1})(\prod_{i\in I}\xi_i^{\sum_{j=1}^{p}\mathrm{mult}_{E_i}(g_j)})v$ then

$$v\prod_{j=1}^{p} u_j^{s_j+1}\prod_{i\in I} z_i^{\sum_{j=1}^{p}\mathrm{mult}_{E_i}(g_j)(s_j+1)+k_i} dz \cdot P = q(s)v\prod_{j=1}^{p} u_j^{s_j}\prod_{i\in I} z_i^{\sum_{j=1}^{p}\mathrm{mult}_{E_i}(g_j)s_j+k_i}$$

where

$$q(s) = \prod_{i\in I}(\sum_{j=1}^{p}\mathrm{mult}_{E_i}(g_j)s_j + \mathrm{mult}_{E_i}(g_j) + k_i)\cdots(\sum_{j=1}^{p}\mathrm{mult}_{E_i}(g_j)s_j + 1 + k_i).$$

□

Recall that the $\mathscr{A}_Y^R$-module direct image computes the $\mathscr{D}_Y$-module direct image with additional structure. In particular the $\mathscr{D}_X$-linear endomorphism t induces a endomorphism on $\int^0 \mathscr{M}$. By the functoriality of the direct image $ts_i - s_i t = 1$ for any i so $\int^0 \mathscr{M}$ is equipped with the structure of a $\mathscr{A}_X^R\langle t\rangle$-module. Similarly, the functoriality and lemma 3.5.1 yields a b-polynomial of a desirable form for $\int^0 \mathscr{M}/t\int^0 \mathscr{M}$.

Consider the surjection $\mathscr{A}_Y^R \to \mathscr{M}$ induced by $1 \mapsto G^s\mu^*(dx)$. The associated long exact sequence includes a morphism $\int^0 \mathscr{A}_Y^R \to \int^0 \mathscr{M}$. Observe that $\int^0 \mathscr{A}_Y^R = R^0\mu_*(\mathscr{A}_{Y\to X}^R)$ contains global section corresponding to 1. Write u for the image of this section in $\int^0 \mathscr{M}$ and denote $\mathscr{U}$ for the right $\mathscr{A}_X^R\langle t\rangle$-module generated by u. Similarly to lemma 2.3.13 the followin lemma shows that the main difficulty is to tranfer the b-polynomial for $\int^0 \mathscr{M}/t\int^0 \mathscr{M}$ into a b-polynomial for $\mathscr{U}/t\mathscr{U}$. This will exploit lemma 3.4.16 whence it is needed that $\int^0 \mathscr{M}/\mathscr{U}$ has grade at least $n+1$.

Lemma 3.5.2. *There is a morphism right $\mathscr{A}_{\mathscr{X}}^R$-modules $\mathscr{U} \to \mathscr{A}_{\mathscr{X}}^R F^s \otimes_{\mathcal{O}_{\mathscr{X}}} \omega_{\mathscr{X}}$ sending u to $F^s dx$.*

Proof. The proof is identical to the proof of lemma 2.3.13 with f replaced by $\prod f_i$. □

In what follows we want to consider the $\mathscr{A}_Y$-module $\mathscr{M}$ as a $\mathscr{D}_Y$-module. This could disturb coherence. To solve this one introduces new coordinates such that there are vector fields $\mathcal{S}_1, \dots, \mathcal{S}_p$ which acts as $s_1, \dots, s_p$ on the generator.

Note that there are finitely many codimension 1 components in $Z(B_{F,x})$. Hence, there exist p independent linear polynomials $\sum_{i=1}^p d_{ij}s_i$ such that for any j there is no hyperplane parallel to $\sum_{i=1}^p d_{ij}s_i = 0$ in $Z(B_{F,x})$. Moreover, it may be assumed that the d_{ij} are non-negative integers. Introduce new coordinates $z_{n+1}, \dots, z_{n+p}$ and set $\widetilde{f}_j = f_j \prod_{i=1}^p z_{n+i}^{d_{ij}}$ on $\mathscr{X} := X \times \mathbb{C}^p$. Note that the induced map $\mathscr{Y} \to \mathscr{X}$ is a resolution of singularities for $\prod \widetilde{f}_i$ and that $\widetilde{g}_j = g_j \prod_{i=1}^p z_{n+i}^{d_{ij}}$ is the pullback of $\widetilde{f}_i$.

For any $i = 1, \dots, p$ it holds that

$$\widetilde{G}^s\mu^*(dx) \cdot \partial_{n+i} = \sum_{j=1}^p d_{ij}s_j x_j^{-1}\widetilde{G}^s\mu^*(dx).$$

Since the linear polynomials are independent a appropriate $\mathbb{C}$-linear combination provides a vector field ζ_j with $\widetilde{G}^s\mu^*(dx)\cdot\zeta_j = s_j z_j^{-1}\widetilde{G}^s\mu^*(dx)$. Set $\mathcal{S}_j = \zeta_j z_j$ so that $\widetilde{G}^s\mu^*(dx)\cdot\mathcal{S}_j = s_j\widetilde{G}^s\mu^*(dx)$. This solves the coherence issue. The following result and it's proof are analogous to lemma 2.3.11.

Lemma 3.5.3. *For any $x \in X \times \{0\}^p$ it holds that if $b \in B_{\widetilde{F},x}$ then $b \in B_{F,x}$.*

Proof. Take local coordinates $x_1, \dots, x_{n+p}$ near x and let P be in the stalk of $\mathscr{A}_{\mathscr{X}}$ at x such that $b\widetilde{F}^s = P\widetilde{F}^{s+1}$. Similarly to the above there is a $\mathbb{C}$-basis $\xi_1, \dots, \xi_p$ for the span of $\partial_{n+1}, \dots, \partial_{n+p}$ so that $\mathcal{S}_j := x_{n+j}\xi_j$ satisfies $\mathcal{S}_j \cdot \widetilde{F}^s = s_j\widetilde{F}^s$. Expand P as a polynomials in $\xi_1, \dots, \xi_p$

$$P = \sum_\alpha P_\alpha \xi_1^{\alpha_1} \cdots \xi_p^{\alpha_p}$$

where the coefficients P_α live in a stalk of $\mathcal{O}_{\mathscr{X}} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathscr{D}_{\mathscr{X}}$.

Let N be greater than the maximal value of $|\alpha|$ then

$$(x_{n+1}\cdots x_{n+p})^N b\widetilde{F}^s = \left(\sum_\alpha \prod_{i=1}^p (s_i+1)^{\alpha_i} \sum_\beta Q_{\alpha\beta}\partial_1^{\beta_1}\cdots\partial_n^{\beta_n}\right)\widetilde{F}^{s+1}$$

where the P_α were expanded as polynomials in $\partial_1, \dots, \partial_n$ with coefficients $Q_{\alpha\beta}$ from $\mathcal{O}_{\mathscr{X}}$. Observe that $\partial_1, \dots, \partial_n$ act on the formal symbol $\widetilde{F}^{s+1}$ the same as they act on the formal symbol F^{s+1}.

Now consider this functional equation on the analytification of $\mathscr{X}$ and expand the $Q_{\alpha\beta}$ as power series at x. Identifying powers of $x_{n+1} \cdots x_{n+p}$ on both sides a functional equation with analytical coefficients for F^s follows. This establishes that b is in the analytic Bernstein-Sato ideal. It follows that $b \in B_{F,x}$ since analytic and algebraic Bernstein-Sato ideals agree by theorem 3.1.5. $\square$

Note that replacing F by $\widetilde{F}$ leaves theorem 3.1.4 unchanged up to hyperplanes parallel to $\sum_{i=1}^p d_{ij} s_i = 0$. These are not in $Z(B_{F,x})$ by assumption so, by lemma 3.5.3, it remains to prove the theorem for $\widetilde{F}$. For notational simplicity we simply write F instead of $\widetilde{F}$. We write $m = n + p$ for the dimension of $\mathscr{X}$ and $\mathscr{Y}$.

Let $\ell_1, \dots, \ell_{p-1} \in \mathbb{C}[s]$ be degree one polynomials which will be fixed later. For any $i = 0 \dots, p$ let L_i be the ideal of $\mathbb{C}[s]$ generated by $\ell_1, \dots, \ell_i$. Assume that the ℓ_i are chosen sufficiently generically so that $Z(L_{p-1})$ is a line.

Lemma 3.5.4. *The $\mathscr{D}_{\mathscr{Y}}$-module $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is coherent and it's characteristic variety satisfies $\operatorname{Ch} \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1} \subseteq \Lambda \cup W$ where Λ is isotropic and W is a irreducible variety of dimension $m+1$ which dominates $\mathscr{Y}$.*

Proof. Recall that we ensured that $\mathscr{M}$ is a coherent $\mathscr{D}_{\mathscr{Y}}$-module. Hence, also $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ will be a coherent $\mathscr{D}_{\mathscr{Y}}$-module.

Take local coordinates $z_1, \dots, z_n, z_{n+1}, \dots, z_{n+p}$ on $\mathscr{Y}$ as in the proof of lemma 3.5.1. This is to say that locally

$$G^s \mu^*(dx) = v \prod_{i=1}^n u_i^{s_j} z_i^{\sum_{j=1}^p M_{ij} s_j + m_i} \prod_{i=1}^n z_{n+i}^{\sum_{j=1}^p d_{ij} s_j} dz.$$

Let s_0 denote a new variable so that $\mathbb{C}[s]/L_{p-1} \cong \mathbb{C}[s_0]$. Then $\mathscr{M} \otimes_{\mathbb{C}[s_0]} R/L_{p-1}$ may be viewed as the $\mathscr{D}_{\mathscr{Y}}$-module which is locally generated by a formal symbol

$$[G^s \mu^*(dx)] = v \prod_{i=1}^{2n} u_i^{A_i s_0 + a_i} z_i^{B_i s_0 + b_i} dz$$

where A_i, B_i, a_i, b_i are complex numbers and we set $u_{n+i} = 1$. Moreover, since the linear functions $\sum_{j=1}^p d_{ij} s_j$ on the final terms in $G^s \mu^*(dx)$ formed a basis for the linear polynomials there will be at least one B_{i+n} which is non-zero.

Denote $w = v \prod_{i=1}^n u_i^{a_i}$ and consider for any $j = 1, \dots, n+p$ the operation of $w^{-1} \partial_j w z_j$ on the generator

$$[G^s \mu^*(dx)] \cdot w^{-1} \partial_j w = ((B_j s_0 + b_j) z_j^{-1} + \sum_{i=1}^n A_i s_0 u_i^{-1} \partial_j(u_i))[G^s \mu^*(dx)].$$

Recall that the $s_1, \dots, s_n$ could be produced by acting with a vector field. Since s_0 is found with affine relations it follows that there exists some differential operator $\mathcal{S}_0$ of degree 1 such that $s_0[G^s \mu^*(dx)] = [G^s \mu^*(dx)] \cdot \mathcal{S}_0$. Now we get a well-defined surjection $\mathscr{D}_{\mathscr{Y}}/I \twoheadrightarrow \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ where I denotes the right ideal generated by $w^{-1} \partial_j w z_j - b_j - \mathcal{S} h_i$ with $h_j = B_j + z_j \sum_{i=1}^n A_i u_i^{-1} \partial_j(u_i)$ for $j = 1, \dots, n+p$.

Note that $z_j \sum_{i=1}^n A_i u_i^{-1} \partial_j(u_i) = 0$ for $j > n$. Hence, the h_{n+j} are complex scalars and they are not all zero since there exists a non-zero B_{n+j}. After renumbering we now have that $h_1 \in \mathbb{C}^\times$. Denoting ζ_j, σ_0 for the elements of $\operatorname{gr} \mathscr{D}_{\mathscr{X}}$ which correspond to $\partial_j, \mathcal{S}_0$ respectively it holds that $\operatorname{gr} I$ contains $z_j \zeta_j - h_j \sigma_0$ for any $j = 1, \dots, n+p$. Then also $h_1 z_j \zeta_j - h_j z_1 \zeta_1$ is in $\operatorname{gr} I$ for any $j = 2, \dots, n+p$. This yields the desired bound for the characteristic variety. □

Lemma 3.5.5. *If the $\mathscr{A}_{\mathscr{X}}$-module $\int^0 \mathscr{M}/U$ has grade m then the quotients $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ are relative holonomic $\mathscr{A}_{\mathscr{X}}^{R/L_i}$-modules of grade m.*

Proof. This follows by induction on $i = 0, \dots, p$ using lemma 3.4.10 which is applicable by corollary 3.4.3. □

Lemma 3.5.6. *Any polynomial $b \in \mathbb{C}[s]$ which is not in L_i induces a injective automorphisms on $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$.*

Proof. Observe that $\mathscr{M}$ has a trivial Bernstein-Sato ideal so that it has degree m by corollary 3.4.3. By inductively applying lemma 3.4.10 it holds that $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ has degree m. In particular, it has a trivial Bernstein-Sato ideal.

Similarly to the proof of lemma 3.5.4 one can pick local coordinates z_i such that $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ is generated by some formal symbol $[G^s \mu^*(dx)]$. Further pick some isomorphism $\mathbb{C}[s]/L_i \cong \mathbb{C}[\widetilde{s}]$. By definition of the formal symbol ∂_i acts on $[G^s \mu^*(dx)]$ as a polynomial in $\widetilde{s}$ with rational functions of the z_i as coefficients.

If b is not injective it follows by clearing denominators that there is some non-zero polynomial $f = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \widetilde{s}^\beta$ with $[G^s \mu^*(dx)] f = 0$. Further, it can be assumed that the degree of f in z is zero. Indeed, if z_i occurs in f then one can find a non-zero polynomial g of lesser degree such that

$$[G^s \mu^*(dx)] f \partial_1 = [G^s \mu^*(dx)] \partial_1 f + [G^s \mu^*(dx)] g.$$

The left-hand-side of this equality vanishes and the term $[G^s \mu^*(dx)] \partial_1 f$ must also vanish since ∂_i acts as a rational function. This means that $[G^s \mu^*(dx)] g = 0$. Repeating this procedure it may be assumed that f is a non-zero polynomial in $\mathbb{C}[\widetilde{s}]$.

Since $\mathbb{C}[\widetilde{s}]$ commutes with $\mathscr{A}_Y^{\mathbb{C}[\widetilde{s}]}$ it follows that f is a non-zero Bernstein-Sato polynomial of $\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$ which is a contradiction. □

The proof that $\int^0 \mathscr{M}/\mathscr{U}$ has a grade $\geq m+1$ involves a tensor product to reduce the number of variables. The following lemma allows us to choose the ℓ_i appropriately such that the $\mathcal{T}or$-terms associated to the tensor product can be controlled.

Lemma 3.5.7. *One can pick $\ell_1, \dots, \ell_{p-1}$ such that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}$-module of grade greater than or equal to $m+1$ for every $i = 1, \dots, p-1$. Here K_i denotes the kernel of ℓ_i on $\int^1 (\mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{i-1})$.*

Proof. By a taking a $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}$-free resolution of K_i one finds that

$$R\mathcal{H}om_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}}(K_i \otimes^L_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}} \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}) \cong R\mathcal{H}om_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}) \otimes^L_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}} \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$$

where we note that $\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}$ is a $\mathscr{A}_X^{\mathbb{C}[s]/L_i}$-bimodule so that both tensor products are defined. We compare the Grothendieck spectral sequences of both sides.

The spectral sequence on the left-hand-side has terms

$$E_2^{rq} = \mathcal{E}xt^r_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}}(\mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

Since $\mathcal{T}or_{-q}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}})$ is a relative holonomic $\mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}$-module these terms are only non-zero for $r = m$ or $r = m + 1$. In particular, the spectral sequence degenerates at E_2. By lemma 3.4.5 the statement that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ has grade $\geq m + 1$ is equivalent to the total cohomology of degree $r + q = m - 1$ having grade $\geq m + 1$.

The spectral sequence on the right-hand-side has terms

$$E_2^{rq} = \mathcal{T}or_{-r}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathcal{E}xt^q_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

The claim follows if we can ensure that that all terms with $r + q = m - 1$ vanish on $\mathscr{X} \times \operatorname{Spec} R$ for some open subset $\operatorname{Spec} R \subseteq \mathbb{C}^p$. Indeed, then by corollary 3.4.3 the terms have grade $m + 1$ and it follows that the same must hold for the total cohomology.

The ℓ_i and the open $\operatorname{Spec} R$ are constructed by induction on i. For any i, j, k with $k \leq i$ denote $\mathcal{E}_{ik}^{n+j} := \mathcal{E}xt^{n+j}_{\mathscr{A}_{\mathscr{X}}^{R/L_k}}(K_k, \mathscr{A}_{\mathscr{X}}^{R/L_k}) \otimes_{\mathscr{A}_{\mathscr{X}}^{R/L_k}} \mathscr{A}_{\mathscr{X}}^{R/L_i}$. In every induction step it is ensured that

(i) $\mathcal{E}_{ii}^{n+j}$ is $(n + j)$-Cohen-Macaulay over $\mathscr{A}_X^{R/L_i}$ or zero for every $j \geq 0$.

(ii) $Z(L_i) \cap \operatorname{Spec} R \neq 0$.

(iii) ℓ_i induces a injection on $\mathcal{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$.

By abuse of notation L_i may also denote the ideal of R generated by $\ell_1, \ldots, \ell_i$.

Take some arbitrary ℓ_1 for the base-case and use lemma 3.4.13 to find a open $\operatorname{Spec} R \subseteq \mathbb{C}^p$ such that $\mathcal{E}_{11}^{n+j}$ is $(n+j)$-Cohen-Macaulay for every $j \geq 0$. This only requires removing a strict closed subset of $\operatorname{Spec} \mathbb{C}[s]/L_1$ so $Z(L_1) \cap \operatorname{Spec} R = \operatorname{Spec} R/L_1$ is non-empty. The final property is vacuous for $i = 1$.

Now assume that $i > 1$ and that $\ell_1, \ldots, \ell_{i-1}$ are already constructed. First let's ensure that ℓ_i induces a injection on $\mathcal{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$. By iterative application of lemma 3.4.12 it holds that $\mathcal{E}_{(i-1)k}^{n+j}$ is $(n + j)$-Cohen-Macaulay over $\mathscr{A}_{\mathscr{X}}^{L_{i-1}}$. Take ℓ_i so that the induced element of R/L_{i-1} is non-constant and does not vanish on any irreducible component of the Bernstein-Sato zero locust of $\mathcal{E}_{(i-1)k}^{n+j}$ for every $j \geq 0$ and $k < i$. Then, by lemma 3.4.9 the desired injectivity follows. As before, lemma 3.4.13 can be used to to find a open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ such that $\mathcal{E}_{ii}^{n+j}$ is $(n + j)$-Cohen-Macaulay for every $j \geq 0$ and $Z(L_i) \cap \operatorname{Spec} R' = \operatorname{Spec} R'/L_i$ is non-empty. Note that replacing $\operatorname{Spec} R$ by $\operatorname{Spec} R'$ will conserve the induction hypothesis. This concludes the inductive construction of the ℓ_i.

Applying injectivity of ℓ_i on $\mathcal{E}_{(i-1)k}^{n+j}$ with the free resolution $\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}} \xrightarrow{\ell_i} \mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}$ for $\mathscr{A}_{\mathscr{X}}^{R/L_i}$ yields that $\mathcal{T}or_m^{\mathscr{A}_{\mathscr{X}}^{R/L_i}}(\mathcal{E}_{(i-1)k}^{n+j}, \mathscr{A}_{\mathscr{X}}^{R/L_i}) = 0$ for all $m > 0$. By taking a $\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}$-free resolution of $\mathcal{E}_{(i-1)k}^{n+j}$ it follows that

$$\mathcal{E}_{(i-1)k}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_{i-1}}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}} \cong \mathcal{E}_{ik}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}.$$

Iterative application of the isomorphism yields $\mathcal{E}_{ii}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_i}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}} \cong \mathcal{E}_{(p-2)i}^{n+j} \otimes^L_{\mathscr{A}_{\mathscr{X}}^{R/L_{p-2}}} \mathscr{A}_{\mathscr{X}}^{R/L_{p-1}}$. This means that

$$\mathcal{T}or_{-r}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathcal{E}xt^q_{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}), \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}) \cong \mathcal{T}or_{-r}^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(\mathcal{E}^q_{(p-2)i}, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}}).$$

The right right-hand-side of this isomorphism was already observed to vanish for any $r < 0$ and the left-hand-side is precisely the E_2^{rq}-term of the spectral sequence. This establishes that the E_2^{rq}-terms with $r + q = m - 1$ vanish for $q > m - 1$. The remaining term $E_2^{m-1,0}$ is zero regardless since it involves $\mathcal{E}xt^{m-1}$ of a relative holonomic module. This concludes the proof. □

Lemma 3.5.8. *The relative holonomic $\mathscr{A}_{\mathscr{X}}$-module $\int^0 \mathscr{M}/\mathscr{U}$ has grade $j(\int^0 \mathscr{M}/\mathscr{U}) \geq m + 1$.*

Proof. Let $\ell_1, \ldots, \ell_{p-1}$ be the degree one polynomials provided by lemma 3.5.7. For any $i = 0, \ldots, p-1$ denote $\mathscr{M}_i = \mathscr{M} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_i$. The first task is to relate $(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ with $\int^0 \mathscr{M}_{p-1}$.

Recall from lemma 3.5.6 that ℓ_i is injective on $\mathscr{M}_{i-1}$. This implies that $\ell_i \int^0 \mathscr{M}_{i-1} = \int^0 \ell_i \mathscr{M}_{i-1}$. The injective automorphisms of ℓ_i on $\mathscr{M}_{i-1}$ induces a long exact sequence of $\mathscr{A}_X^{\mathbb{C}[s]/L_{i-1}}$-modules

$$0 \to \int^0 \mathscr{M}_{i-1} \xrightarrow{\ell_i} \int^0 \mathscr{M}_{i-1} \to \int^0 \mathscr{M}_i \to \cdots$$

whence $(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_i$ is a submodule of $\int^0 \mathscr{M}_i$. The quotient is isomorphic to the kernel K_i of ℓ_i on $\int^1 \mathscr{M}_{i-1}$.

Applying a tensor product with $\mathbb{C}[s]/L_{p-1}$ to the inclusion $\int^0 \mathscr{M}_{i-1} \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_i \hookrightarrow \int^0 \mathscr{M}_i$ yields a exact sequence

$$\mathcal{T}or_1^{\mathbb{C}[s]/L_{i-1}}\left(K_i, \frac{\mathbb{C}[s]}{L_{p-1}}\right) \to \left(\int^0 \mathscr{M}_{i-1}\right) \otimes_{\mathbb{C}[s]/L_{i-1}} \frac{\mathbb{C}[s]}{L_{p-1}} \to \left(\int^0 \mathscr{M}_i\right) \otimes_{\mathbb{C}[s]/L_{i-1}} \frac{\mathbb{C}[s]}{L_{p-1}}.$$

By choice of the ℓ_i it holds that $\mathcal{T}or_1^{\mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_i}}(K_i, \mathscr{A}_{\mathscr{X}}^{\mathbb{C}[s]/L_{p-1}})$ has grade $\geq m + 1$. This implies the existence of a non-zero polynomial $b_i \in \mathbb{C}[s]/L_{p-1}$ which annihilates $\mathcal{T}or_1^{\mathbb{C}[s]/L_i}(K_i, \mathbb{C}[s]/L_{p-1}))$.

Denote $B = \prod_{i=1}^{p-1} b_i$ and note that the kernels of the automorphisms induced by B^N form a increasing sequence inside the coherent $\mathscr{A}_X^{\mathbb{C}[s]/L_{p-1}}$-module $(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{p-1}$. Such a increasing sequence must stabilise for sufficiently large N. Then it follows that the intersection of $\operatorname{Im} \mathcal{T}or_1^{\mathbb{C}[s]/L_{i-1}}(K_i, \mathbb{C}[s]/L_{p-1})$ and $B^N((\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{p-1})$ is trivial. This means that there are injections

$$B^N(\int^0 \mathscr{M}_{i-1}) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{p-1} \hookrightarrow B^N(\int^0 \mathscr{M}_i) \otimes_{\mathbb{C}[s]/L_{i-1}} \mathbb{C}[s]/L_{p-1}.$$

In particular $B^N(\int^0 \mathscr{M}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a submodule of $\int^0 \mathscr{M}_{p-1}$.

Since μ is proper proposition 2.1.17 yields that $\int^0 \mathscr{M}_{p-1}$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module with characteristic variety $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda \cup W))$ with Λ isotropic and W irreducible of dimension $m + 1$ dominating $\mathscr{Y}$. Observe that $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a subquotient of

$\int^0 \mathscr{M}_{p-1}$ with support in the divisor D. Hence, $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L$ is a coherent $\mathscr{D}_{\mathscr{X}}$-module with

$$\mathrm{Ch}\left(B^N\left(\int^0 \mathscr{M}/\mathscr{U}\right) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1})\right) \subseteq \widetilde{\mu}\left((T^*\mu)^{-1}(\Lambda \cup W)\right) \cap \pi^{-1}(D)$$

where $\pi : T^*X \to X$ is the projection map.

This means $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is a holonomic $\mathscr{D}_{\mathscr{X}}$-module. Indeed, by lemma 2.2.6 $\widetilde{\mu}((T^*\mu)^{-1}(\Lambda))$ remains isotropic and forms no obstruction to the characteristic variety being Lagrangian. Moreover, $\widetilde{\mu}((T^*\mu)^{-1}(W))$ is irreducible of dimension $m+1$ and dominates $\mathscr{X}$. Intersecting with $\pi^{-1}(D)$ yields a closed strict subset which necessarily has lower dimension. Hence, it follows that $\dim \mathrm{Ch}\, B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1} \leq m$. This means that $B^N(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is holonomic. By proposition 2.2.3 the Bernstein-Sato ideal of holonomic module is non-zero. This implies that the Bernstein-Sato ideal of $(\int^0 \mathscr{M}/\mathscr{U}) \otimes_{\mathbb{C}[s]} \mathbb{C}[s]/L_{p-1}$ is non-zero which means that it has grade $\geq m+1$ by corollary 3.4.3. Then also $\int^0 \mathscr{M}/\mathscr{U}$ has grade $\geq m+1$ by lemma 3.4.10. ☐

Now all ingredients are in place for the proof of theorem 3.1.4.

Theorem 3.5.9. *With notation as in section 3.1 every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of the form*

$$\mathrm{mult}_{E_i}(g_1)s_1 + \cdots + \mathrm{mult}_{E_i}(g_r)s_r + k_i + c_i = 0$$

with $c_i \in \mathbb{Z}_{\geq 0}$.

Proof. By lemma 3.5.8 the $\mathscr{A}_X$-module $\mathscr{M}/\mathscr{U}$ has grade greater than or equal to $m+1$. Hence lemma 3.4.16 provides $N \geq 1$ such that $t^N \mathscr{M}/\mathscr{U} = 0$ on a open $\mathscr{X} \times \operatorname{Spec} R$ for some open $\operatorname{Spec} R \subseteq \mathbb{C}^p$ with complement of codimension strictly greater than 1.

Let $b(s_1, \dots, s_p)$ denote the Bernstein-Sato polynomial for $\mathscr{M}/t\mathscr{M}$ provided by lemma 3.5.1. Set $B := \prod_{i=0}^{N+1} b(s_1+i, \dots, s_p+i)$ then it follows that $B\mathscr{M} \subseteq t\mathscr{U}$ on $\mathscr{X} \times \operatorname{Spec} R$. In particular this means that B is in the Bernstein-Sato ideal of $\mathscr{U}/t\mathscr{U}$ over $\operatorname{Spec} R$. By the surjection of lemma 3.5.2 this means that $B \in B_F$ over $\operatorname{Spec} R$. This proves which proves the theorem because the complement of $\operatorname{Spec} R$ has codimension strictly greater than 1. ☐

Bibliography

Bjork, J.-E. (1979). Rings of differential operators. *North-Holland Math. Library*, 21.

Björk, J.-E. (1993). *Analytic D-modules and applications*. Kluwer Academic Publishers Group, Dordrecht.

Borel, A. (1987). *Algebraic D-modules*, volume 2. Academic Pr.

Brianccon, J. and Maisonobe, P. (2002). Remarques sur l'idéal de bernstein associé à des polynômes. prépublication univ. *Nice-Sophia Antipolis*, (650).

Budur, N. (2015). Bernstein-sato polynomials (lecture notes, upc barcelona 2015).

Budur, N., van der Veer, R., Wu, L., and Zhou, P. (2019). Zero loci of bernstein-sato ideals. *arXiv preprint arXiv:1907.04010*.

Deligne, P. (1977). Sga 4 1/2–cohomologie étale. *Lecture Notes in Mathematics*, 569.

Dimca, A. (2004). *Sheaves in topology*. Springer Science & Business Media.

Grothendieck, A. and Dieudonné, J. (1961). Éléments de géométrie algébrique iii.

Gyoja, A. et al. (1993). Bernstein-sato's polynomial for several analytic functions. *Journal of Mathematics of Kyoto University*, 33(2):399–411.

Hotta, R. and Tanisaki, T. (2007). *D-modules, perverse sheaves, and representation theory*, volume 236. Springer Science & Business Media.

Kashiwara, M. (1976). B-functions and holonomic systems. *Inventiones mathematicae*, 38(1):33–53.

Kashiwara, M. (2003). D-modules and microlocal calculus, translations of mathematical monographs, 217 american math.

Kruff, N. et al. (2015). Ore localisation with applications in d-module theory, master's thesis. *RWTH Aachen University*.

Lichtin, B. (1989). Poles of $|f(z, w)|^{2s}$ and roots of the b-function. *Arkiv för Matematik*, 27(1-2):283–304.

Maisonobe, P. (2016). Filtration relative, l'idéal de bernstein et ses pentes. *arXiv preprint arXiv:1610.03354*.

Milnor, J. (1968). Singular points of complex hypersurfaces (am-61).

Monteiro Fernandes, T. and Sabbah, C. (2016). Riemann-hilbert correspondence for mixed twistor d-modules. *arXiv*, pages arXiv–1609.

Sabbah, C. (1987). Proximité évanescente. i. la structure polaire d'un D-module. *Compositio Mathematica*, 62(3):283–328.

Sabbah, C. (2011). Introduction to the theory of D-modules. *Lecture Notes, Nankai.* Accessed in October 2020 at `http://www.cmls.polytechnique.fr/perso/sabbah/livres/sabbah_nankai110705.pdf`.

Schapira, P. and Schneiders, J.-P. (1994). Index theorem for elliptic pairs. *Astérisque*, (224):1–113.

Weibel, C. A. (1995). *An introduction to homological algebra.* Number 38. Cambridge university press.