Lecture Notes on General Relativity (GR)

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(https://github.com/lazierthanthou/Lecture_Notes_GR)

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Contents

Abstract

These are lecture notes on General Relativity.

They are based on the Central Lecture Course by **Dr. Frederic P. Schuller** (**A thorough introduction to the theory of general relativity**) introducing the mathematical and physical foundations of the theory in 24 self-contained lectures at the International Winter School on Gravity and Light in Linz/Austriafor the WE Heraeus International Winter School of Gravity and Light, 2015 in Linz as part of the world-wide celebrations of the 100th anniversary of Einstein's theory of general relativity and the International Year of Light 2015.

These lectures develop the theory from first principles and aim at an audience ranging from ambitious undergraduate students to beginning PhD students in mathematics and physics. Satellite Lectures (see other videos on this channel) by Bernard F Schutz (Gravitational Waves), Domenico Giulini (Canonical Formulation of Gravity), Marcus C Werner (Gravitational Lensing) and Valeria Pettorino (Cosmic Microwave Background) expand on the topics of this central lecture course and take students to the research frontier.

Spacetime is the physical key object, we shall be concerned about.

Spacetime is a **4-dimensional topological manifold** with a **smooth atlas** carrying a **torsion-free connection** compatible with a **Lorentzian metric** and a **time orientation** satisfying the **Einstein equations**.

1 Topology

Motivation: At the coarsest level, spacetime is a set. But, a set is not enough to talk about continuity of maps, which is required for classical physics notions such as trajectory of a particle. We do not want jumps such as a particle disappearing at some point on its trajectory and appearing somewhere. So we require continuity of maps. There could be many structures that allow us to talk about continuity, e.g., distance measure. But we need to be very minimal and very economic in order not to introduce undue assumptions. So we are interested in the weakest structure that can be established on a set which allows a good definition of continuity of maps. Mathematicians know that the weakest such structure is topology. This is the reason for studying topological spaces.

1.1 Topological Spaces

Definition 1. Let M be a set and $\mathcal{P}(M)$ be the power set of M, i.e., the set of all subsets of M.
A set $\mathcal{O} \subseteq \mathcal{P}(M)$ is called a **topology**, if it satisfies the following:

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$

(ii) $U \in \mathcal{O}, \quad V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$

(iii) $U_\alpha \in \mathcal{O}, \quad \alpha \in \mathcal{A}$ ($\mathcal{A}$ is an index set $\implies \left(\bigcup_{\alpha\in\mathcal{A}} U_\alpha\right) \in \mathcal{O}$

Terminology:

1. the tuple $(M, \mathcal{O})$ is a **topological space**.
2. $\mathcal{U} \in M$ is an **open set** if $\mathcal{U} \in \mathcal{O}$.
3. $\mathcal{U} \in M$ is a **closed set** if $M \setminus \mathcal{U} \in \mathcal{O}$.

Definition 2. $(M, \mathcal{O})$, where $\mathcal{O} = \{\emptyset, M\}$ is called the **chaotic topology**.
Definition 3. $(M, \mathcal{O})$, where $\mathcal{O} = \mathcal{P}(M)$ is called the **discrete topology**.
Definition 4. A **soft ball** at the point p in $\mathbb{R}^d$ is the set

$$\mathcal{B}_r(p) := \left\{ (q_1, q_2, ..., q_d) \quad | \quad \sum_{i=1}^{d} (q_i - p_i)^2 < r^2 \right\} \text{ where } r \in \mathbb{R}^+ \tag{1.1}$$

Definition 5. $(\mathbb{R}^d, \mathcal{O}_{std})$ is the **standard topology**, provided that $U \in \mathcal{O}_{std}$ iff
$\forall p \in U, \exists r \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq U$

Proof. $\emptyset \in \mathcal{O}_{std}$ since $\forall\, p \in \emptyset$, $\exists r \in \mathbb{R}^+$: $\mathcal{B}_r(p) \subseteq \emptyset$ (i.e. satisfied "vacuously")
$\mathbb{R}^d \in \mathcal{O}_{std}$ since $\forall p \in \mathbb{R}^d$, $\exists r = 1 \in \mathbb{R}^+$: $\mathcal{B}_r(p) \subseteq \mathbb{R}^d$

Suppose $U, V \in \mathcal{O}_{std}$. Let $p \in U \cap V \implies \exists\, r_1, r_2 \in \mathbb{R}^+$ s.t. $\quad \mathcal{B}_{r_1}(p) \subseteq U, \quad \mathcal{B}_{r_2}(p) \subseteq V$.
Let $r = \min\{r_1, r_2\} \implies \mathcal{B}_r(p) \subseteq U$ and $\mathcal{B}_r(p) \subseteq V \implies \mathcal{B}_r(p) \subseteq U \cap V \implies U \cap V \in \mathcal{O}_{std}$.

Suppose, $U_\alpha \in \mathcal{O}_{std}, \forall\, \alpha \in \mathcal{A}$. Let $p \in \bigcup_{\alpha\in\mathcal{A}} U_\alpha \implies \exists \alpha \in \mathcal{A} : p \in U_\alpha$
$\implies \exists\, r \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\mathcal{A}} U_\alpha \implies \bigcup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{O}_{std}$. □

1.2 Continuous maps

A map f, $f : M \longrightarrow N$, connects each element of a set M (domain set) to an element of a set N (target set).

Terminology:

1. If f maps $m \in M$ to $n \in N$, then we may say $f(m) = n$, or m maps to n, or $m \mapsto f(m)$ or $m \mapsto n$.
2. If $V \subseteq N, \text{preim}_f(V) := \{m \in M | f(m) \in V\}$
3. If $\forall n \in N, \exists m \in M : n = f(m)$, then f is **surjective**. Or, $f : M \twoheadrightarrow N$.
4. If $m_1, m_2 \in M, m_1 \neq m_2 \implies f(m_1) \neq f(m_2)$, then f is **injective**. Or, $f : M \hookrightarrow N$.

Definition 6. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. A map $f : M \longrightarrow N$ is called **continuous** w.r.t. $\mathcal{O}_M$ and $\mathcal{O}_N$ if $V \in \mathcal{O}_N \implies (\text{preim}_f(V)) \in \mathcal{O}_M$.

Mnemonic*: A map is continuous iff the preimages of all open sets are open sets.*

1.3 Composition of continuous maps

Definition 7. If $f : M \longrightarrow N$ and $g : N \longrightarrow P$, then

$$g \circ f : M \longrightarrow P \text{ such that } m \mapsto (g \circ f)(m) := g(f(m))$$

Theorem 1. *If $f : M \longrightarrow N$ is continuous w.r.t. $\mathcal{O}_M$ and $\mathcal{O}_N$ and $g : N \longrightarrow P$ is continuous w.r.t. $\mathcal{O}_N$ and $\mathcal{O}_P$, then $g \circ f : M \longrightarrow P$ is continuous w.r.t. $\mathcal{O}_M$ and $\mathcal{O}_P$.*

Proof. Let $W \in \mathcal{O}_P$.

$$\begin{aligned}
\text{preim}_{g \circ f}(W) &= \{m \in M | g(f(m)) \in W\} && \because (g \circ f)(m) = g(f(m)) \\
&= \{m \in M | f(m) \in \text{preim}_g(W)\} && \text{preim}_g(W) \in \mathcal{O}_N \because g \text{ is continuous} \\
&= \text{preim}_f(\text{preim}_g(W)) && \in \mathcal{O}_M \because f \text{ is continuous} \\
&\implies g \circ f \text{ is continuous}
\end{aligned}$$

□

1.4 Inheriting a topology

Given a topological space $(M, \mathcal{O}_M)$, one way of inheriting a topology from it is the subspace topology.

Theorem 2. *If $(M, \mathcal{O}_M)$ is a topological space and $S \subseteq M$, then the set $\mathcal{O}|_S \subseteq \mathcal{P}(S)$ such that $\mathcal{O}|_S := \{S \cap U | U \in \mathcal{O}_M\}$ is a topology. $\mathcal{O}|_S$ is called the* ***subspace topology*** *inherited from $\mathcal{O}_M$.*

Proof.

1. $\emptyset, S \in \mathcal{O}|_S \because \emptyset = S \cap \emptyset, S = S \cap M$.
2. $S_1, S_2 \in \mathcal{O}|_S \implies \exists U_1, U_2 \in \mathcal{O}_M : S_1 = S \cap U_1, S_2 = S \cap U_2 \implies U_1 \cap U_2 \in \mathcal{O}_M$
$\implies S \cap (U_1 \cap U_2) \in \mathcal{O}|_S \implies (S \cap U_1) \cap (S \cap U_2) \in \mathcal{O}|_S \implies S_1 \cap S_2 \in \mathcal{O}|_S$.
3. Let $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an index set. Then $S_\alpha \in \mathcal{O}|_S \implies \exists U_\alpha \in \mathcal{O}_M : S_\alpha = S \cap U_\alpha$.
Further, let $\mathcal{U} = \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right)$. Therefore, $\mathcal{U} \in \mathcal{O}_M$.
Now, $\left(\bigcup_{\alpha \in \mathcal{A}} S_\alpha\right) = \left(\bigcup_{\alpha \in \mathcal{A}} (S \cap U_\alpha)\right) = S \cap \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right) = S \cap \mathcal{U} \implies \left(\bigcup_{\alpha \in \mathcal{A}} S_\alpha\right) \in \mathcal{O}|_S$.

□

Theorem 3. *If $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ are topological spaces, and $f : M \longrightarrow N$ is continuous w.r.t $\mathcal{O}_M$ and $\mathcal{O}_N$, then the restriction of f to $S \subseteq M, f|_S : S \longrightarrow N$ s.t. $f|_S(s \in S) = f(s)$, is continuous w.r.t $\mathcal{O}|_S$ and $\mathcal{O}_N$.*

Proof. Let $V \in \mathcal{O}_N$. Then, $\text{preim}_f(V) \in \mathcal{O}_M$.
Now $\text{preim}_{f|_S}(V) = S \cap \text{preim}_f(V) \implies \text{preim}_{f|_S}(V) \in \mathcal{O}|_S \implies f|_S$ is continuous. □

2 Manifolds

Motivation: There exist so many topological spaces that mathematicians cannot even classify them. For spacetime physics, we may focus on topological spaces $(M, \mathcal{O})$ that can be charted, analogously to how the surface of the earth is charted in an atlas.

2.1 Topological manifolds

Definition 8. A topological space $(M, \mathcal{O})$ is called a **d-dimensional topological manifold** if $\forall p \in M : \exists U \in \mathcal{O} : p \in U, \exists x : U \longrightarrow x(U) \subseteq \mathbb{R}^d$ satisfying the following:

(i) x **is invertible**: $x^{-1} : x(U) \longrightarrow U$

(ii) x **is continuous** w.r.t. $(M, \mathcal{O})$ and $(\mathbb{R}^d, \mathcal{O}_{std})$

(iii) x^{-1} **is continuous**

2.2 Terminology

1. The tuple (U, x) is a **chart** of $(M, \mathcal{O})$,
2. An **atlas** of $(M, \mathcal{O})$ is a set $\mathcal{A} = \{(U_\alpha, x_\alpha) | \alpha \in A, \text{an index set}\} : \bigcup_{\alpha \in A} U_\alpha = M$.
3. The map $x : U \longrightarrow x(U) \subseteq \mathbb{R}^d$ is called the **chart map**.
4. The chart map x maps a point $p \in U$ to a d-tuple of real numbers $x(p) = (x^1(p), x^2(p), \ldots, x^d(p))$. This is equivalent to d-many maps $x^i(p) : U \longrightarrow \mathbb{R}$, which are called the **coordinate maps**.
5. If $p \in U$, then $x^i(p)$ is the **ith coordinate of** p w.r.t. the chart (U, x).

2.3 Chart transition maps

Imagine 2 charts (U, x) and (V, y) with overlapping regions, i.e., $U \cap V \neq \emptyset$.

$$\begin{tikzcd}
& U \cap V \arrow[dl, "x"'] \arrow[dr, "y"] & \\
\mathbb{R}^d \supseteq x(U \cap V) \arrow[ur, bend left, "x^{-1}"] \arrow[rr, "y \circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^d
\end{tikzcd}$$

The map $y \circ x^{-1}$ is called the **chart transition map**, which maps an open set of $\mathbb{R}^d$ to another open set of $\mathbb{R}^d$. This map is continuous because it is composition of two continuous maps, Informally, these chart transition maps contain instructions on how to glue together the charts of an atlas,

2.4 Manifold philosophy

Often it is desirable (or indeed the only way) to define properties (e.g., 'continuity') of real-world object (e.g., the curve $\gamma : \mathbb{R} \longrightarrow M$) by judging suitable coordinates not on the 'real-world' object itself, but on a chart-representation of that real world object.

For example, in the picture below, we can use the map $x \circ \gamma$ to infer the continuity of the curve γ in $U \subseteq M$.

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & U \subseteq M \arrow[d, "x"] \\
& x(U) \subseteq \mathbb{R}^d
\end{tikzcd}
$$

However, we need to ensure that the defined property does not change if we change our chosen chart. For example, in the picture below, continuity in $x \circ \gamma$ should imply $y \circ \gamma$. This is true, since $y \circ \gamma = y \circ (x^{-1} \circ x) \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma)$ is continuous because it is a composition of two continuous functions, thanks to the continuity of the chart transition map $y \circ x^{-1}$.

$$
\begin{tikzcd}
& y(U) \subseteq \mathbb{R}^d \\
\mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & U \subseteq M \arrow[u, "y"'] \arrow[d, "x"] \\
& x(U) \subseteq \mathbb{R}^d \arrow[u, bend left, "x^{-1}"] \arrow[uu, bend right, "y \circ x^{-1}"']
\end{tikzcd}
$$

What about differentiability? Does differentiability of $x \circ \gamma$ guarantee differentiability of $y \circ \gamma$? No. Since composition of a differentiable map and a continuous map might only be continuous, The solution is to restrict the atlas by removing those charts which are not differentiable. Thus, we have got rid of our problem. However, we must remember that with the present structure, we cannot define differentiability at manifold level since we do not know how to subtract or divide in $U \subseteq M$. Therefore, differentiability of $\gamma : \mathbb{R} \longrightarrow M$ makes no sense yet.

3 Multilinear Algebra

Motivation: The essential object of study of linear algebra is vector space. However, a word of warning here. We will not equip space(time) with vector space structure. This is evident since, unlike in vector space, expressions such as $5 \cdot$ Paris and Paris $+$ Vienna do not make any sense. If multilinear algebra does not further our aim of studying spacetime, then why do we study it? The tangent spaces T_pM (defined in Lecture 5) at a point p of a smooth manifold M (defined in Lecture 4) carries a vector space structure in a natural way even though the underlying position space(time) does not have a vector space structure. Once we have a notion of tangent space, we have a derived notion of a tensor. Tensors are very important in differential geometry.
It is beneficial to study vector spaces (and all that comes with it) abstractly for two reasons: (i) for construction of T_pM, one needs an intermediate vector space $C^\infty(M)$, and (ii) tensor techniques are most easily understood in an abstract setting.

3.1 Vector Spaces

Definition 9. A $\mathbb{R}$-**vector space** is a triple $(V, +, \cdot)$, where

i) V is a set,

ii) $+ : V \times V \longrightarrow V$ (addition), and

iii) $. : \mathbb{R} \times V \longrightarrow V$ (S-multiplication)

satisfying the following:

a) $\forall u, v \in V : u + v = v + u$ (commutativity of $+$)

b) $\forall u, v, w \in V : (u + v) + w = u + (v + w)$ (associativity of $+$)

c) $\exists O \in V : \forall v \in V : O + v = v$ (neutral element in $+$)

d) $\forall v \in V : \exists(-v) \in V : v + (-v) = 0$ (inverse of element in $+$)

e) $\forall \lambda, \mu \in \mathbb{R}, \forall v \in V : \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$ (associativity in $\cdot$)

f) $\forall \lambda, \mu \in \mathbb{R}, \forall v \in V : (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ (distributivity of $\cdot$)

g) $\forall \lambda \in \mathbb{R}, \forall u, v \in V : \lambda \cdot u + \lambda \cdot v = \lambda \cdot (u + v)$ (distributivity of $\cdot$)

h) $\exists 1 \in \mathbb{R} : \forall v \in V : 1 \cdot v = v$ (unit element in $\cdot$)

Terminology: If $(V, +, \cdot)$ is a vector space, an element of V is often referred to, informally, as a **vector**. But, we should remember that it makes no sense to call an element of V a vector unless the vector space itself is specified.

Example: Consider a set of polynomials of fixed degree,

$$P := \left\{ p : (-1, +1) \longrightarrow \mathbb{R} \;\middle|\; p(x) = \sum_{n=0}^{N} p_n \cdot x^n, \text{ where } p_n \in \mathbb{R} \right\}$$

with $\oplus : P \times P \longrightarrow P$ with $(p, q) \mapsto p \oplus q : (p \oplus q)(x) = p(x) + q(x)$ and
$\odot : \mathbb{R} \times P \longrightarrow P$ with $(\lambda, p) \mapsto \lambda \odot p : (\lambda \odot p)(x) = \lambda \cdot p(x)$. $(P, \oplus, \odot)$ is a vector space.

Caution: *We are considering real vector spaces, that is S-multiplication with the elements of $\mathbb{R}$. We shall often use same symbols '$+$' and '$\cdot$' for different vector spaces, but the context should make things clear. When $\mathbb{R}, \mathbb{R}^2$, etc. are used as vector spaces, the obvious (natural) operations shall be understood to be used.*

3.2 Linear Maps

These are the structure-respecting maps between vector spaces.

Definition 10. If $(V, +_v, \cdot_v)$ and $(W, +_w, \cdot_w)$ are vector spaces, then $\phi : V \longrightarrow W$ is called a **linear map** if

i) $\forall v, \tilde{v} \in V : \phi(v +_v \tilde{v}) = \phi(v) +_w \phi(\tilde{v})$, and

ii) $\forall \lambda \in \mathbb{R}, v \in V : \phi(\lambda \cdot_v v) = \lambda \cdot_w \phi(v)$.

Notation: $\phi : V \longrightarrow W$ is a linear map $\iff \phi : V \xrightarrow{\sim} W$

Example: Consider the vector space $(P, \oplus, \odot)$ from the above example,
Then, $\delta : P \longrightarrow P$ with $p \mapsto \delta(p) := p'$ is a linear map, because
$\forall p, q \in P : \delta(p \oplus q) = (p \oplus q)' = p' \oplus q' = \delta(p) \oplus \delta(q)$ and
$\forall \lambda \in \mathbb{R}, p \in P : \delta(\lambda \odot p) = (\lambda \odot p)' = \lambda \odot p'$.

Theorem 4. *If $\phi : U \xrightarrow{\sim} V$ and $\psi : V \xrightarrow{\sim} W$, then $\psi \circ \phi : U \xrightarrow{\sim} W$.*

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[rr, bend right, "\psi \circ \phi"'] & V \arrow[r, "\psi"] & W \end{tikzcd}$$

Proof. $\forall u, \tilde{u} \in U, (\psi \circ \phi)(u +_u \tilde{u}) = \psi(\phi(u +_u \tilde{u})) = \psi(\phi(u) +_v \phi(\tilde{u})) = \psi(\phi(u)) +_w \psi(\phi(\tilde{u})) = (\psi \circ \phi)(u) +_w (\psi \circ \phi)(\tilde{u})$.

$\forall \lambda \in \mathbb{R}, u \in U, (\psi \circ \phi)(\lambda \cdot_u u) = \psi(\phi(\lambda \cdot_u u)) = \psi(\lambda \cdot_v \phi(u)) = \lambda \cdot_w \psi(\phi(u)) = \lambda \cdot_w (\psi \circ \phi)(u)$ □

Example: Consider the vector space $(P, \oplus, \odot)$ and the differential $\delta : P \longrightarrow P$ with $p \mapsto \delta(p) := p'$ from previous example. Then, p'', the second differential is also linear since it is composition of two linear maps, i.e., $\delta \circ \delta : P \xrightarrow{\sim} P$.

3.3 Vector Space of Homomorphisms

Definition 11. If $(V, +, \cdot)$ and $(W, +, \cdot)$ are vector spaces, then $Hom(V, W) := \left\{ \phi : V \xrightarrow{\sim} W \right\}$.

Theorem 5. *$(Hom(V, W), +, \cdot)$ is a vector space with*
$+ : Hom(V, W) \times Hom(V, W) \longrightarrow Hom(V, W)$ *with* $(\phi, \psi) \mapsto \phi + \psi : (\phi + \psi)(v) = \phi(v) + \psi(v)$ *and*
$\cdot : \mathbb{R} \times Hom(V, W) \longrightarrow Hom(V, W)$ *with* $(\lambda, \phi) \mapsto \lambda \cdot \phi : (\lambda \cdot \phi)(v) = \lambda \cdot \phi(v)$.

Example: $(Hom(P, P), +, \cdot)$ is a vector space. $\delta \in Hom(P, P)$, $\delta \circ \delta \in Hom(P, P)$, $\delta \circ \delta \circ \delta \in Hom(P, P)$, etc. Therefore, maps such as $5 \cdot \delta + \delta \circ \delta \in Hom(P, P)$. Thus, mixed order derivatives are in $Hom(P, P)$, and hence linear.

3.4 Dual Vector Spaces

Definition 12. If $(V, +, \cdot)$ is a vector space, and $V^* := \left\{ \phi : V \xrightarrow{\sim} \mathbb{R} \right\} = Hom(V, \mathbb{R})$ then
$(V^*, +, \cdot)$ is called the **dual vector space to V**.

Terminology: $\omega \in V^*$ is called, informally, a **covector**.

Example: Consider $I : P \xrightarrow{\sim} \mathbb{R}$, i.e., $I \in P^*$. We define $I(p) := \int_0^1 p(x)\,\mathrm{d}x$, which can be easily checked to be linear with $I(p + q) = I(p) + I(q)$ and $I(\lambda \cdot p) = \lambda \cdot I(p)$. Thus I is a covector, which is the integration operator $\int_0^1 (\quad)\,\mathrm{d}x$ which eats a function.

Remarks: We shall also see later that the gradient is a covector. In fact, lots of things in physicist's life, which are covectors, have been called vectors not to bother you with details. But covectors are neither esoteric nor unnatural.

3.5 Tensors

We can think of tensors as multilinear maps.
Definition 13. Let $(V, +, \cdot)$ be a vector space. An **(r,s) -tensor** T over V is a multilinear map

$$T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{\text{r times}} \times \underbrace{V \times V \times \cdots \times V}_{\text{s times}} \xrightarrow{\sim} \mathbb{R}$$

Example: If T is a (1,1)-tensor, then
$T(\omega_1 + \omega_2, v) = T(\omega_1, v) + T(\omega_2, v)$,
$T(\omega, v_1 + v_2) = T(\omega, v_1) + T(\omega, v_2)$,
$T(\lambda \cdot \omega, v) = \lambda \cdot T(\omega, v)$, and
$T(\omega, \lambda \cdot v) = \lambda \cdot T(\omega, v)$.
Thus, $T(\omega_1 + \omega_2, v_1 + v_2) = T(\omega_1, v_1) + T(\omega_1, v_2) + T(\omega_2, v_1) + T(\omega_2, v_2)$.

Remarks: Sometimes it is said that a (1,1)-tensor is something that eats a vector and outputs a vector. Here is why. For $T : V^* \times V \xrightarrow{\sim} \mathbb{R}$, define $\phi_T : V \xrightarrow{\sim} (V^*)^*$ with $v \mapsto T((\cdot), v)$. But, clearly $T((\cdot), v) : V^* \xrightarrow{\sim} \mathbb{R}$, which eats a covector and spits a number. In other words, $T((\cdot), v) \in (V^*)^*$. Although we are yet to define dimension, let us just trust, for the time being, that for finite-dimensional vector spaces, $(V^*)^* = V$. So, $\phi_T : V \xrightarrow{\sim} V$.

Example: Let $g : P \times P \xrightarrow{\sim} \mathbb{R}$ with $(p, q) \mapsto \int_{-1}^{1} p(x) \cdot q(x)\, \mathrm{d}x$. Then, g is a (0,2)-tensor over P.

3.6 Vectors and Covectors as Tensors

Theorem 6. *If $(V, +, \cdot)$ is a vector space, $\omega \in V^*$ is a (0,1)-tensor.*

Proof. $\omega \in V^*$ and, by definition, $V^* := \left\{\phi : V \xrightarrow{\sim} \mathbb{R}\right\}$, which is a collection of (0,1)-tensors. □

Theorem 7. *If $(V, +, \cdot)$ is a vector space, $v \in V$ is a (1,0)-tensor.*

Proof. We have already stated, without proof and without defining dimensions, that $V = (V^*)^*$ for finite-dimensional vector spaces. Therefore, $v \in V \implies v \in (V^*)^* \implies v \in \left\{\phi : V^* \xrightarrow{\sim} \mathbb{R}\right\} \implies v$ is a (1,0)-tensor. □

3.7 Bases

Definition 14. Let $(V, +, \cdot)$ is a vector space. A subset $B \subseteq V$ is called a **basis** if
$\forall v \in V, \exists! e_1, e_2, \ldots, e_n \in B, \exists! v_1, v_2, \ldots, v_n \in \mathbb{R} : v = \sum_{i=1}^{n} v_i \cdot e_i$.
Definition 15. A vector space $(V, +, \cdot)$ with a basis B is said to be d**-dimensional** if B has d elements. In other words, $dimV := d$.

Remarks: The above definition is well-defined only if every basis of a vector space has the same number of elements.

Remarks: Let $(V, +, \cdot)$ is a vector space. Having chosen a basis $e_1, e_2, \ldots, e_n$, we may uniquely associate $v \mapsto (v_1, v_2, dotsc, v_n)$, these numbers being the components of v w.r.t. chosen basis where $v = \sum_{i=1}^{n} v_i \cdot e_i$.

3.8 Basis for the Dual Space

Let $(V, +, \cdot)$ is a vector space. Having chosen a basis $e_1, e_2, \ldots, e_n$ for V, we can choose a basis $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ for V^* entirely independent of basis of V. However, it is more economical to require that

$$\epsilon^a(e_b) = \delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

This uniquely determines $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ from choice of $e_1, e_2, \ldots, e_n$.

Remarks: The reason for using indices as superscripts or subscripts is to be able to use the Einstein summation convention, which will be helpful in dropping cumbersome $\sum$ symbols in several equations.

Definition 16. For a basis $e_1, e_2, \ldots, e_n$ of vector space $(V, +, \cdot)$, $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$ is called the **dual basis** of the dual space, if $\epsilon^a(e_b) = \delta^a_b$.

Example: Consider polynomials P of degree 3. Choose $e_0, e_1, e_2, e_3 \in P$ such that $e_0(x) = 1, e_1(x) = x, e_2(x) = x^2 and e_3(x) = x^3$. Then, it can be easily verified that the dual basis is $\epsilon^a = \frac{1}{a!}\partial^a\Big|_{x=0}$.

3.9 Components of Tensors

Definition 17. Let T be a (r, s)-tensor over a d-dimensional (finite) vector space $(V, +, \cdot)$. Then, with respect to some basis $\{e_1, \ldots, e_r\}$ and the dual basis $\{\epsilon^1, \ldots, \epsilon^s\}$, define $(r + s)^d$ real numbers

$$T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} := T(\epsilon^{i_1}, \ldots, \epsilon^{i_r}, e_{j_1}, \ldots, e_{j_s})$$

such that the indices $i_1, \ldots, i_r, j_1, \ldots, j_s$ take all possible values in the set $\{1, \ldots, d\}$. These numbers $T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ are called the **components of the tensor** T w.r.t. the chosen basis.

This is useful because knowing components (and the basis w.r.t which these components have been chosen), one can reconstruct the entire tensor.

Example: If T is a $(1, 1)$-tensor, then $T^i{}_j := T(\epsilon^i, e_j)$. Then

$$T(\omega, v) = T\left(\sum_{i=1}^{d} \omega_i \cdot \epsilon^i, \sum_{j=1}^{d} v^j \cdot e_j\right) = \sum_{i=1}^{d}\sum_{j=1}^{d} \omega_i v^j T(\epsilon^i, e_j) = \sum_{i=1}^{d}\sum_{j=1}^{d} \omega_i v^j T^i{}_j =: \omega_i v^j T^i{}_j$$

4 Differential Manifolds

Motivation: So far we have dealt with topological manifolds which allow us to talk about continuity. But to talk about smoothness of curves on manifolds, or velocities along these curves, we need something like differentiability. Does the structure of topological manifold allow us to talk about differentiability? The answer is a resounding no.

So this lecture is about figuring out what structure we need to add on a topological manifold M to start talking about differentiability of curves ($\mathbb{R} \longrightarrow M$) on a manifold, or differentiability of functions ($M \longrightarrow \mathbb{R}$) on a manifold, or differentiability of maps ($M \longrightarrow N$) from one manifold M to another manifold N.

$$\begin{tikzcd}
\gamma : \mathbb{R} \arrow[r] \arrow[dr, "x \circ \gamma"'] & U \arrow[d, "x"] \\
 & x(U) \subseteq \mathbb{R}^d
\end{tikzcd}$$

<u>idea</u>. try to "lift" the undergraduate notion of differentiability of a curve on $\mathbb{R}^d$ to a notion of differentiability of a curve on M

<u>Problem</u> Can this be well-defined under change of chart?

$$\begin{tikzcd}
 & y(U \cap V) \subseteq \mathbb{R}^d \\
\gamma : \mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r] \arrow[dr, "x \circ \gamma"'] & U \cap V \neq \emptyset \arrow[u, "y"'] \arrow[d, "x"] \\
 & x(U \cap V) \subseteq \mathbb{R}^d \arrow[uu, bend right=60, "y \circ x^{-1}"']
\end{tikzcd}$$

$x \circ \gamma$ undergraduate differentiable ("as a map $\mathbb{R} \longrightarrow \mathbb{R}^d$")

$$\underbrace{y \circ \gamma}_{\text{maybe only continuous, but not undergraduate differentiable}} = \underbrace{\overbrace{(y \circ x^{-1})}^{\mathbb{R}^d \longrightarrow \mathbb{R}^d}}_{\text{continuous}} \circ \underbrace{\overbrace{(x \circ \gamma)}^{\mathbb{R} \longrightarrow \mathbb{R}^d}}_{\text{undergrad differentiable}} = y \circ (x^{-1} \circ x) \circ \gamma$$

At first sight, strategy does not work out.

4.1 Compatible charts

In section 1, we used any imaginable charts on the top. mfd. $(M, \mathcal{O})$.

To emphasize this, we may say that we took U and V from the *maximal atlas* $\mathcal{A}$ of $(M, \mathcal{O})$.
Definition 18. Two charts (U, x) and (V, y) of a top. mfd. are called ❀-compatible if either

(a) $U \cap V = \emptyset$ or

(b) $U \cap V \neq \emptyset$

chart transition maps have undergraduate ❀ property.

EY : 20151109 e.g. since $\mathbb{R}^d \longrightarrow \mathbb{R}^d$, can use undergraduate ❀ property such as continuity or differentiability.

$$y \circ x^{-1} : x(U \cap V) \subseteq \mathbb{R}^d \longrightarrow y(U \cap V) \subseteq \mathbb{R}^d$$
$$x \circ y^{-1} : y(U \cap V) \subseteq \mathbb{R}^d \longrightarrow x(U \cap V) \subseteq \mathbb{R}^d$$

Philosophy:

Definition 19. An atlas $\mathcal{A}_{❀}$ is a ❀-compatible atlas if any two charts in $\mathcal{A}_{❀}$ are ❀-compatible.

Definition 20. A **❀-manifold** is a triple $(\underbrace{M, \mathcal{O}}_{\text{top. mfd.}}, \mathcal{A}_{❀}) \qquad \mathcal{A}_{❀} \subseteq \mathcal{A}_{\text{maximal}}$

❀	undergraduate ❀	
C^0	$C^0(\mathbb{R}^d \longrightarrow \mathbb{R}^d) =$	continuous maps w.r.t. $\mathcal{O}$
C^1	$C^1(\mathbb{R}^d \longrightarrow \mathbb{R}^d) =$	differentiable (once) and is continuous
C^k		k-times continuously differentiable
D^k		k-times differentiable
$\vdots$		
C^∞	$C^\infty(\mathbb{R}^d \longrightarrow \mathbb{R}^d)$	
$\cup\!\mid$		
C^ω	$\exists$ multi-dim. Taylor exp.	
$\mathbb{C}^\infty$	satisfy Cauchy-Riemann equations, pair-wise	

EY : 20151109 Schuller says: C^k is easy to work with because you can judge k-times cont. differentiability from existence of all partial derivatives **and** their continuity. There are examples of maps that partial derivatives exist but are not D^k, k-times differentiable.

Theorem 8 (Whitney)**.** *Any $C^{k\geq 1}$-atlas, $\mathcal{A}_{C^{k\geq 1}}$ of a topological manifold* contains *a C^∞-atlas.*

Thus we may w.l.o.g. always consider C^∞-manifolds, "smooth manifolds", unless we wish to define Taylor expandibility/complex differentiability . . .

EY : 20151109 Hassler Whitney [1]

Definition 21. A smooth manifold $(\underbrace{M, \mathcal{O}}_{\text{top. mfd.}}, \underbrace{\mathcal{A}}_{C^\infty-\text{atlas}})$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & M \arrow[d, "x"] \\
& \mathbb{R}^d
\end{tikzcd}$$

EY: 20151109 Schuller was explaining that the trajectory is real in M; the coordinate maps to obtain coordinates is $x \circ \gamma$

4.2 Diffeomorphisms

$M \xrightarrow{\phi} N$

If M, N are naked sets, the structure preserving maps are the bijections (invertible maps).

e.g. $\{1, 2, 3\} \longrightarrow \{a, b\}$

Definition 22. $M \cong_{\text{set}} N$ (set-theoretically) isomorphic if $\exists$ bijection $\phi : M \longrightarrow N$

Examples. $\mathbb{N} \cong_{\text{set}} \mathbb{Z}$

$\mathbb{N} \cong_{\text{set}} \mathbb{Q}$ (EY: 20151109 Schuller says from diagonal counting)

[1] http://mathoverflow.net/questions/8789/can-every-manifold-be-given-an-analytic-structure

$\mathbb{N} \ncong_{\text{set}} \mathbb{R}$

Now $(M, \mathcal{O}_M) \cong_{\text{top}} (N, \mathcal{O}_N)$ (topl.) isomorphic = "homeomorphic" $\exists$ bijection $\phi : M \longrightarrow N$
ϕ, ϕ^{-1} are continuous.

$(V, +, \cdot) \cong_{\text{vec}} (W, +_w, \cdot_w)$ (EY: 20151109 vector space isomorphism) if
$\exists$ bijection $\phi : V \longrightarrow W$ linearly

finally

Definition 23. Two C^∞-manifolds
$(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ are said to be **diffeomorphic** if $\exists$ bijection $\phi : M \longrightarrow N$ s.t.

$$\begin{aligned} \phi &: M \longrightarrow N \\ \phi^{-1} &: N \longrightarrow M \end{aligned}$$

are both C^∞-maps

$$\begin{tikzcd}
 & \mathbb{R}^d \arrow[rr, "\widetilde{y}\circ\phi\circ\widetilde{x}^{-1}"] & & \mathbb{R}^e \\
C^\infty & M \supseteq U \arrow[u, "\widetilde{x}"] \arrow[d, "x"] \arrow[rr, "\phi"] & & V \subseteq N \arrow[u, "\widetilde{y}"'] \arrow[d, "y"] \\
 & \mathbb{R}^d \arrow[rr, "\substack{y\circ\phi\circ x^{-1} \\ \text{undergraduate } C^\infty}"] \arrow[uu, bend left] & & \mathbb{R}^e \arrow[uu, bend right]
\end{tikzcd}$$

Theorem 9. *$\#$ = number of C^∞-manifolds one can make out of a given C^0-manifolds (if any) - up to diffeomorphisms.*

dimM	$\#$	
1	*1*	*Morse-Radon theorems*
2	*1*	*Morse-Radon theorems*
3	*1*	*Morse-Radon theorems*
4	*uncountably infinitely many*	
5	*finite*	*surgery theory*
6	*finite*	*surgery theory*
$\vdots$	*finite*	*surgery theory*

EY : 20151109 cf. `http://math.stackexchange.com/questions/833766/closed-4-manifolds-with-uncountably-many-`
The wild world of 4-manifolds

5 Tangent Spaces

Lead question: "What is the velocity of a curve $\gamma : \mathbb{R} \longrightarrow M$ at the point p of the curve in M?"

5.1 Velocities

Definition 24. Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold. Let there be a curve $\gamma : \mathbb{R} \longrightarrow M$, which is at least C^1. Suppose $\gamma(\lambda_0) = p$. The **velocity** of γ at the point p of the curve γ is the linear map

$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R} \text{ with } f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0) \tag{5.1}$$

where $C^\infty(M) := \{f : M \longrightarrow \mathbb{R} \,|\, f \text{ is a smooth function }\}$ equipped with
$(f \oplus g)(p) := f(p) + g(p)$ and $(\lambda \otimes g)(p) := \lambda \cdot g(p)$ is a vector space.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Figure 1: $f \circ \gamma$. Intuition: If the first $\mathbb{R}$ is thought of as time, and f as temperature, then $f \circ \gamma$ relates time and temperature and $(f \circ \gamma)'$ is the rate of change of temperature as you run around the curve.

past: " $\underbrace{v^i}_{\text{vector in past}} (\partial_i f) = (\underbrace{v^i \partial_i}_{\text{vector as map}})f$

In an imprecise way, we could say that we want vectors to survive as the directional derivatives they induce. This is a very slight shift of perspective which is extremely powerful and leads to idea of tangent space in differential geometry.

Terminology: If X is a vector seen as a map, then X acting on a function f, i.e. Xf is called the **directional derivative of f in the X direction**.

5.2 Tangent vector space

Definition 25. For each point $p \in M$, the **tangent space** to M at the point p is the set

$$T_pM := \{v_{\gamma,p} \,|\, \text{ for all smooth curves } \gamma \text{ through } p\} \tag{5.2}$$

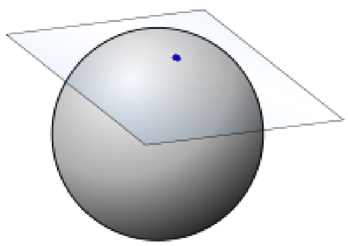

Figure 2: A pictorial representation of the tangent space T_xM of a single point, x, on a manifold. A vector in this T_xM can represent a possible velocity at x. After moving in that direction to a nearby point, one's velocity would then be given by a vector in the tangent space of that nearby point — a different tangent space, not shown. *By Alexwright at English Wikipedia - Transferred from en.wikipedia to Commons by Ylebru., Public Domain https://commons.wikimedia.org/w/index.php?curid=3941393*

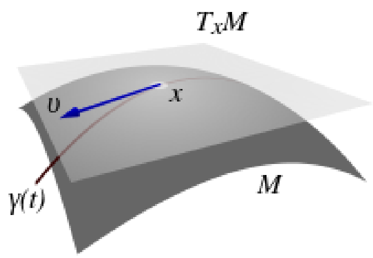

Figure 3: The tangent space T_xM and a tangent vector $v \in T_xM$, along a curve travelling through $x \in M$. *By derivative work: McSush (talk)Tangentialvektor.png: TNThe original uploader was TN at German Wikipedia - Tangentialvektor.png, Public Domain, https://commons.wikimedia.org/w/index.php?curid=4821938*

***Caution**: Although the Fig. 2 and 3 refer to an ambient space in which M is embedded, the tangent space has been defined intrinsically. There is a velocity corresponding to each curve along a different path in M passing* *through p. Velocity along two different curves could be same, or curves along same paths but having different parameter speeds would yield different velocities.*

Theorem 10. $(T_pM, \oplus, \otimes)$ *is a vector space with*

$$\begin{aligned}\oplus :& T_pM \times T_pM \longrightarrow Hom(C^\infty(M), \mathbb{R})\\ &(v_{\gamma,p} \oplus v_{\delta,p})(\underbrace{f}_{\in C^\infty(M)}) := v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f)\\ \odot :& \mathbb{R} \times T_pM \longrightarrow Hom(C^\infty(M), \mathbb{R})\\ &(\alpha \odot v_{\gamma,p})(f) := \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f)\end{aligned}$$

Proof. Various conditions that must be satisfied by a vector space, are trivially satisfied. It remains to be shown that

i) For product, $\exists\, \tau$ curve : $\alpha \odot v_{\gamma,p} = v_{\tau,p}$

ii) For sum, $\exists\, \sigma$ curve : $v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$

Product: Let $\tau : \mathbb{R} \longrightarrow M$ with $\lambda \mapsto \tau(\lambda) := \gamma(\alpha\lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda)$ where $\mu_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ with $r \mapsto \alpha \cdot r + \lambda_0$. Then $\tau(0) = \gamma(\lambda_0) = p$, and

$$v_{\tau,p} = (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) = \mu'_\alpha(0) \cdot (f \circ \gamma)'(\mu_\alpha(0)) = \alpha \cdot (f \circ \gamma)'(\lambda_0) = \alpha \cdot v_{\gamma,p}$$

Sum: Choose a chart (U, x) and $p \in U$. *(If the proof will depend on the choice of a chart, alarm bells should ring. But we shall see that the result is finally independent of the chart.)*
Let $p = \gamma(\lambda_0) = \delta(\lambda_1)$.
Now define $\sigma : \mathbb{R} \longrightarrow M$ with $\lambda \mapsto \sigma(\lambda) := x^{-1}(\underbrace{(x \circ \gamma)(\lambda_0 + \lambda)}_{\mathbb{R} \longrightarrow \mathbb{R}^d} + (x \circ \delta)(\lambda_1 + \lambda) - (x \circ \gamma)(\lambda_0))$.

Then, $\sigma_x(0) = x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_1) - (x \circ \gamma)(\lambda_0)) = \delta(\lambda_1) = p$.
Now

$$\begin{aligned}v_{\sigma_x,p}(f) &:= (f \circ \sigma_x)'(0)\\ &= (\underbrace{(f \circ x^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}} \circ \underbrace{(x \circ \sigma_x)}_{\mathbb{R} \longrightarrow \mathbb{R}^d})'(0)\\ &= \underbrace{(x \circ \sigma_x)'(0)}_{(x\circ\gamma)'(\lambda_0)+(x\circ\delta)'(\lambda_1)} \cdot \big(\partial_i(f \circ x^{-1})\big)\,(x(\underbrace{\sigma(0)}_{p}))\\ &= (x \circ \gamma)'(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) + (x \circ \delta)(\lambda_1)(\partial_i(f \circ x^{-1}))(x(p))\\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_1)\\ &= v_{\gamma,p}(f) + v_{\delta,p}(f) \qquad\qquad \forall\, f \in C^\infty(M)\end{aligned}$$

□

picture: (cf. https://youtu.be/pepU_7NJSGM?t=39m5s)

If we push γ and δ to one chart, and add them there, then bring the sum back to M, we would get a curve which would be different from the curve we would get if we used another chart. But it turns out, irrespective of the charts selected, we get the same tangent/velocity. Conclusion: Adding trajectories is chart dependent; hence, bad. Adding velocities is good because, whatever the charts, they yield the same derivative at the point of intersection. Of course, you cannot add two curves $(\gamma \oplus \delta)(\lambda) := \gamma(\lambda) +_M \delta(\lambda)$ because there is no addition $+_M$ in M. Defining $+$ through charts results in chart-dependent results, which is, therefore, not real.

5.3 Components of a vector w.r.t. a chart

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & M \arrow[r, "f"] \arrow[d, "x"] & \mathbb{R} \\
 & \mathbb{R}^d \arrow[ur, "f\circ x^{-1}"'] &
\end{tikzcd}
$$

Let $(U, x) \in \mathcal{A}_{\text{smooth}}$, $\gamma : \mathbb{R} \longrightarrow U$ and $\gamma(0) = p$. Then

$$
\begin{aligned}
v_{\gamma,p}(f) &:= (f \circ \gamma)'(0) \\
&= (\underbrace{(f \circ x^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}} \circ \underbrace{(x \circ \gamma)}_{\mathbb{R} \longrightarrow \mathbb{R}^d})'(0) \\
&= \left((x \circ \gamma)'\right)^i (0) \cdot \left(f \circ x^{-1}\right)'_i (x(p)) \\
&= \underbrace{\left((x \circ \gamma)'\right)^i (0)}_{=: \dot{\gamma}^i_x(0)} \cdot \underbrace{(\partial_i (f \circ x^{-1}))(x(p))}_{=: \left(\frac{\partial f}{\partial x^i}\right)_p} \\
&= \dot{\gamma}^i_x(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p f && \forall\, f \in C^\infty(M), f : M \longrightarrow \mathbb{R}
\end{aligned}
$$

Definition 26. For velocity $v_{\gamma,p}$, as a map under use of a chart (U, x),

$$
\boxed{v_{\gamma,p} = \dot{\gamma}^i_x(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p} \tag{5.3}
$$

where

$$
\dot{\gamma}^i_x = \left((x \circ \gamma)'\right)^i \tag{5.4}
$$

are the **components of the velocity** $v_{\gamma,p}$ and

$$
\left(\frac{\partial}{\partial x^i}\right) = \partial_i \left(\,\cdot \circ x^{-1}\right) = \left(\left(\,\cdot \circ x^{-1}\right)'\right)^i \tag{5.5}
$$

which eat a function, form a basis of T_pM w.r.t. which the components of the velocity need to be understood.

Note: The components of a vector are always w.r.t. a chart. In M, there is just the vector, no components.
Picture: https://youtu.be/pepU_7NJSGM?t=1h16s
Theorem 11. *For a chart (U, x),*

$$
\boxed{\frac{\partial x^i}{\partial x^j} = \delta^i_j} \tag{5.6}
$$

Proof.

$$
\begin{aligned}
\frac{\partial x^i}{\partial x^j} &= \partial_j (x^i \circ x^{-1})(x(p)) \\
&= \delta^i_j && \text{since } x^i \circ x^{-1} : \mathbb{R}^d \longrightarrow \mathbb{R} \text{ s.t. } (\alpha^1, \ldots, \alpha^d) \mapsto \alpha^i
\end{aligned}
$$

□

5.4 Chart-induced basis

Definition 27. If $(U, x) \in \mathcal{A}_{\text{smooth}}$, then $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_pU \subseteq T_pM$ constitute a **chart-induced basis** of T_pU.

Proof. We have already shown that any vector in T_pU can be expressed in terms of $\left(\frac{\partial}{\partial x^i}\right)_p$. It remains to be shown that they are linearly independent. That is, we require $\lambda^i \left(\frac{\partial}{\partial x^i}\right)_p = 0 \implies \lambda^i = 0$ for all $i = 1, \dots, d$. Or,

$$
\begin{aligned}
0 &= \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^j) && x^j : U \longrightarrow \mathbb{R} \text{ is differentiable} \\
&= \lambda^i \partial_i (x^j \circ x^{-1})(x(p)) && \text{by Eq. 5.5} \\
&= \lambda^i \delta_i^j && \text{by Theorem 11} \\
&= \lambda^j && \text{for all } j = 1, \dots, d
\end{aligned}
$$

□

Corollary 1. *$\dim T_pM = d = \dim M$.*
This follows from the fact d vectors are needed to express any vector in T_pM, and these d vectors arise from the d coordinates of chart which shows that M has d dimensions.

<u>Terminology:</u> $X \in T_pM \implies \exists \gamma : \mathbb{R} \longrightarrow M : X = v_{\gamma,p}$ and $\exists \underbrace{X^1, \dots, X^d}_{\in \mathbb{R}} : X = X^i \left(\frac{\partial}{\partial x^i}\right)_p$. X^i are called **components of the vector X w.r.t chart-induced basis**.

5.5 Change of vector components under a change of chart

✘ A vector does not change under change of chart. It is the vector components that transform under a change of chart.

Let (U, x) and (V, y) be overlapping charts and $p \in U \cap V$. Let $X \in T_pM$. Then, X can be expanded in terms of chart-induced basis of the two charts as follows:

$$
X^i_{(y)} \cdot \left(\frac{\partial}{\partial y^i}\right)_p \underbrace{=}_{(V,y)} X \underbrace{=}_{(U,x)} X^i_{(x)} \cdot \left(\frac{\partial}{\partial x^i}\right)_p \tag{5.7}
$$

Now,

$$
\begin{aligned}
\left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i (f \circ x^{-1})(x(p)) \\
&= \partial_i (\underbrace{(f \circ y^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}} \circ \underbrace{(y \circ x^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}^d})(x(p)) \\
&= (\partial_i (y \circ x^{-1})^j)(x(p)) \cdot (\partial_j (f \circ y^{-1}))(y(p)) \\
&= (\partial_i (y^j \circ x^{-1}))(x(p)) \cdot (\partial_j (f \circ y^{-1}))(y(p)) \\
&= \boxed{\left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial f}{\partial y^j}\right)_p}
\end{aligned}
$$

$$
\therefore \boxed{\left(\frac{\partial}{\partial x^i}\right)_p = \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p} \tag{5.8}
$$

Using Eq. 5.7 and Eq. 5.8,

$$X^i_{(x)} \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_p = X^j_{(y)} \left(\frac{\partial}{\partial y^j}\right)_p$$

$$\therefore \boxed{X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)}} \tag{5.9}$$

5.6 Cotangent spaces

Since T_pM is a vector space, therefore it is trivial to define cotangent space as follows.
Definition 28. For the tangent space T_pM at $p \in M$, **cotangent space** is defined as

$$(T_pM)^* := \{\varphi : T_pM \xrightarrow{\sim} \mathbb{R}\} \tag{5.10}$$

Definition 29. If $f \in C^\infty(M)$, then the **gradient of f at the point** $p \in M$ is defined as

$$\begin{aligned} (df)_p :& T_pM \xrightarrow{\sim} \mathbb{R} \\ & X \mapsto (df)_p(X) := Xf \end{aligned} \tag{5.11}$$

i.e. $\boxed{(df)_p \in T_pM^*}$

$(df)_p$ is a $(0,1)$-tensor over the underlying vector space T_pM. We define the components of the gradient the same way as we define the components of a tensor (refer section 3.9).
Definition 30. Components of gradient w.r.t. chart-induced basis of (U, x) are defined as

$$((df)_p)_j := (df)_p \left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \left(\frac{\partial f}{\partial x^j}\right)_p = \partial_j (f \circ x^{-1})(x(p)) \tag{5.12}$$

Theorem 12. *A chart $(U, x) \implies x^i : U \longrightarrow \mathbb{R}$ are smooth functions. Then, $(dx^1)_p, (dx^2)_p, \ldots, (dx^d)_p$ form a basis of T^*_pM.*

Proof. In fact, $(dx^i)_p$ form a dual basis since

$$(dx^a)_p \left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\frac{\partial x^a}{\partial x^b}\right)_p = \delta^a_b \text{ (using Theorem 11)} \tag{5.13}$$

□

5.7 Change of components of a covector under a change of chart

✘ A covector does not change under change of chart. It is the covector components that transform under a change of chart.

Let (U, x) and (V, y) be overlapping charts and $p \in U \cap V$. Let $\omega \in T^*_pM$. Then, ω can be expanded in terms of chart-induced basis of the two charts as follows:

$$\omega_{(y)j}(dy^j)_p \underbrace{=}_{(V,y)} \omega \underbrace{=}_{(U,x)} \omega_{(x)i}(dx^i)_p \tag{5.14}$$

Now,

$$
\begin{aligned}
& & \omega_{(y)j}(dy^j)_p &= \omega_{(x)i}(dx^i)_p && \text{by Eq. 5.14}\\
&\implies & \omega_{(y)j}(dy^j)_p \left(\frac{\partial}{\partial y^k}\right)_p &= \omega_{(x)i}(dx^i)_p \left(\frac{\partial}{\partial y^k}\right)_p &&\\
&\implies & \omega_{(y)j}(dy^j)_p \left(\frac{\partial}{\partial y^k}\right)_p &= \omega_{(x)i}(dx^i)_p \left(\frac{\partial x^q}{\partial y^k}\right)_p \cdot \left(\frac{\partial}{\partial x^q}\right)_p && \text{by Eq. 5.8}\\
&\implies & \omega_{(y)j} \left(\frac{\partial y^j}{\partial y^k}\right)_p &= \omega_{(x)i} \left(\frac{\partial x^q}{\partial y^k}\right)_p \cdot \left(\frac{\partial x^i}{\partial x^q}\right)_p && \text{by Eq. 5.11}\\
&\implies & \omega_{(y)j}\delta^j_k &= \omega_{(x)i} \left(\frac{\partial x^q}{\partial y^k}\right)_p \cdot \delta^i_q && \text{by Theorem 11}\\
&\implies & \omega_{(y)k} &= \omega_{(x)i} \left(\frac{\partial x^i}{\partial y^k}\right)_p &&
\end{aligned}
$$

Or, with a change of indices,

$$
\boxed{\omega_{(y)i} = \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j}} \tag{5.15}
$$

$$
\begin{aligned}
& & \omega_{(y)i} &= \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j}\\
&\implies & \omega_{(y)i}(dy^i)_p &= \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j}(dy^i)_p\\
&\implies & \omega &= \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j}(dy^i)_p\\
&\implies & \omega(dx^j)_p &= \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j}(dy^i)_p(dx^j)_p\\
&\implies & \omega(dx^j)_p &= \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega(dy^i)_p\\
&\implies & (dx^j)_p &= \left(\frac{\partial x^j}{\partial y^i}\right)_p (dy^i)_p
\end{aligned}
$$

$$
\therefore \boxed{(dx^j)_p = \left(\frac{\partial x^j}{\partial y^i}\right)_p (dy^i)_p} \tag{5.16}
$$

6 Fields

So far, we have focussed technically on a single tangent space and a vector/ covector in it, a basis if we chose a chart. As physicists, we are interested in things such as vector fields such that at any point of a manifold, there is a vector. The proper way to deal with it technically is *theory of bundles*.

6.1 Bundles

Definition 31. A **bundle** is a triple $\boxed{E \xrightarrow{\pi} M}$, where
E is a smooth manifold, called the **total space**,
M is a smooth manifold, called the **base space**, and
π is a smooth map (surjective), called the **projection map**.

Definition 32. Let $E \xrightarrow{\pi} M$ be a bundle and $p \in M$. Then, **fibre over** $p := \text{preim}_\pi(\{p\})$.

Definition 33. A **section** σ of a bundle $E \xrightarrow{\pi} M$ is the map $\sigma : M \longrightarrow E$ such that $\pi \circ \sigma = id_M$.

$$\begin{tikzcd} E \arrow[rr, "\pi"] & & M \arrow[ll, bend left, "\sigma"] \end{tikzcd}$$

<u>Example</u>: E is a cylinder, M a circle and π maps vertical lines on the cylinder to the point of intersection of this line with the circle.

<u>Example</u>: If the fibre of $p \in M$ is a tangent space, the section would pick one vector from the tangent space.

<u>Aside</u>: In quantum mechanics, $\psi : M \longrightarrow \mathbb{C}$ is called a wavefunction, but it is actually a section which selects one value from $\mathbb{C}$ for each $p \in M$.

6.2 Tangent bundle of smooth manifold

For this entire subsection, let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold and let $d := dim\, M$.

Define the set,

$$\boxed{TM := \dot{\bigcup}_{p\in M} T_pM} \tag{6.1}$$

Now define a surjective map π as follows:

$$\boxed{\begin{aligned} \pi :&TM \longrightarrow M \\ &X \mapsto \pi(X) := p \in M \text{ such that } X \in T_pM \end{aligned}} \tag{6.2}$$

<u>Situation</u>: $\underbrace{TM}_{\text{set}} \quad \underbrace{\xrightarrow{\pi}}_{\text{surjective map}} \quad \underbrace{M}_{\text{smooth manifold}}$

For a bundle, TM should be a smooth manifold and π a smooth map. Let us construct a topology on TM that is the coarsest topology such that π is just continuous. (**initial topology** with respect to π). Define

$$\boxed{\mathcal{O}_{TM} := \{\text{preim}_\pi(U) | U \in \mathcal{O}\}} \tag{6.3}$$

It can be shown that $(TM, \mathcal{O}_{TM})$ is a topological space. But we meed a smooth atlas.

Construction of a C^∞-atlas on TM from the C^∞-atlas $\mathcal{A}$ on M

Define

$$\boxed{\begin{aligned} \mathcal{A}_{TM} :=& \{(TU, \xi_x) \,|\, (U,x) \in \mathcal{A}\} \text{ where} \\ \xi_x :& TU \longrightarrow \mathbb{R}^{2d} \\ & X \mapsto \left(\underbrace{(x^1 \circ \pi)(X), \dots, (x^d \circ \pi)(X)}_{(U,x)-\text{ coords of } \pi(X)\,(d\text{-many})}, \underbrace{(dx^1)_{\pi(X)}(X), \dots, (dx^d)_{\pi(X)}(X)}_{\text{components of } X \text{ w.r.t } (U,x)\,(d\text{-many})} \right) \end{aligned}} \tag{6.4}$$

In the above, $(x^1 \circ \pi)(X) = x^1(\pi(X)) = x^1(p) = x^1$-coordinate, and
$X \in T_{\pi(X)}M \implies X = X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)} \implies (dx^j)_{\pi(X)}(X) = (dx^j)_{\pi(X)} \left(X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)} \right) = X^i_{(x)} \delta^j_i = X^j_{(x)}$.
Thus ξ_x maps X to the coordinates of its base point $\pi(X)$ under the chart (U,x) and the components of the vector X w.r.t the basis induced by this chart.

We can write ξ_x^{-1} as follows:

$$\boxed{\begin{aligned} \xi_x^{-1} :& \underbrace{\xi_x(TU)}_{\subseteq \mathbb{R}^{2d}} \longrightarrow TU \\ & (\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) := \beta^i \left(\frac{\partial}{\partial x^i}\right)_{\underbrace{x^{-1}(\alpha^1, \dots, \alpha^d)}_{\pi(X)}} \end{aligned}} \tag{6.5}$$

Now we check, whether the atlas $\mathcal{A}_{TM}$ smooth. That is, are the transitions between its charts smooth?

Theorem 13. *$\mathcal{A}_{TM}$ is a smooth atlas.*

Proof. Let $(U, \xi_x) \in \mathcal{A}_{TM}$, $\quad (V, \xi_y) \in \mathcal{A}_{TM} \quad$ and $\quad U \cap V \neq \emptyset$. Calculate the chart transition

$$(\xi_y \circ \xi_x^{-1})(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) = \xi_y \left(\beta^i \left(\frac{\partial}{\partial x^i}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \right) \qquad \text{by Eq. 6.5}$$

$$= \left(\dots, (y^i \circ \pi) \left(\beta^m \cdot \left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \right), \dots, \dots, (dy^i)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \left(\beta^m \left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \right), \dots \right) \qquad \text{by Eq. 6.4}$$

$$= \left(\dots, y^i \left(\underbrace{\pi \left(\beta^m \cdot \left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \right)}_{\text{the base point}, \, x^{-1}(\alpha^1,\dots,\alpha^d)} \right), \dots, \dots, (\beta^m \underbrace{(dy^i)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \left(\left(\frac{\partial}{\partial x^m}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \right)}_{\left(\frac{\partial y^i}{\partial x^m}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)}}, \dots \right)$$

$$= \left(\dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \dots, \beta^m \left(\left(\frac{\partial y^i}{\partial x^m}\right)_{x^{-1}(\alpha^1,\dots,\alpha^d)} \right), \dots \right)$$

$$= \left(\dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \dots, \beta^m \left(\partial_m (y^i \circ x^{-1})(x(x^{-1}(\alpha^1, \dots, \alpha^d))) \right), \dots \right)$$

$$= \left(\dots, \underbrace{(y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d)}_{\text{smooth } \because (A) \text{ is smooth atlas}}, \dots, \dots, \underbrace{\beta^m \left(\partial_m (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d) \right)}_{\text{smooth } \because \text{ chart transition map is } C^\infty \text{ smooth}}, \dots \right)$$

$\implies (\xi_y \circ \xi_x^{-1})$ is smooth $\implies \mathcal{A}_{TM}$ is smooth

$\square$

Further, the surjective map π is a smooth map because, in the chart representation, π takes the $2d$ components of $X \in TM$ to the d-coordinates of the base point in M, which can be seen to happen smoothly by seeing how the components are mapped. Therefore, we have the following definition.

Definition 34. Then, using the smooth manifold $(M, \mathcal{O}, \mathcal{A})$ as the base space and the smooth manifold $(TM, \mathcal{O}_{TM}, \mathcal{A}_{TM})$ as the total space, the **tangent bundle** is the triple

$$\boxed{TM \xrightarrow{\pi} M} \tag{6.6}$$

6.3 Vector fields

Why did we put so much effort in making a smooth atlas on TM and defining a tangent bundle? The answer is in the following definition of *smooth* vector field, not just any vector field.

Definition 35. For a tangent bundle $TM \xrightarrow{\pi} M$, a **smooth vector field** χ is a smooth map such that $\pi \circ \chi = id_M$.

$$\begin{tikzcd} TM \arrow[r, "\pi"] & M \arrow[l, bend left, "\chi"] \end{tikzcd}$$

Remarks: χ is a section, which couldn't have been a smooth map unless we had both M and TM as smooth manifolds.

6.4 The $C^\infty(M)$-module $\Gamma(TM)$

We already know that $C^\infty(M)$, the collection of all smooth functions is a vector space with S-multiplication with $\mathbb{R}$. But we may also consider the structure $(C^\infty(M), +, \cdot)$ with point-wise addition between elements of $C^\infty(M)$ and point-wise multiplication between elements of $C^\infty(M)$. This structure satisfies all the requirements of a field (commutativity, associativity, neutral element, inverse element under both operations, and distributivity) except that there is no inverse for all non-zero elements under multiplication. This is so because a function that is not zero everywhere, may be zero at some points and then point-wise multiplication with no function would result in the value 1 everywhere. Such a structure is called a ring.

A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring.

Let us consider the module made from the set of all smooth vector fields over the ring $C^\infty(M)$. Define

$$\Gamma(TM) = \{\chi : M \longrightarrow TM \,|\, \chi \text{ is a smooth section}\} \tag{6.7}$$

Definition 36. $(\Gamma(TM), \oplus, \odot)$ is a $C^\infty(M)$-module over the ring of $C^\infty(M)$ functions with $\chi, \widetilde{\chi} \in \Gamma(TM)$ and $g \in C^\infty(M)$, such that

$(\chi \oplus \widetilde{\chi})(f) := (\chi f) \underbrace{+}_{C^\infty(M)} (\widetilde{\chi} f)$

$(g \odot \chi)(f) := g \underbrace{\cdot}_{C^\infty(M)} (\chi f)$

<u>Facts</u>: Besides other differences, there are following 2 important facts:

(1) Proving that *every vector space has a basis* depends upon the choice of set theory; in particular, on the Axiom of Choice in ZFC theory.

(2) No such result exists for modules.

This is a shame, because otherwise, we could have chosen (for any manifold) vector fields, $\chi_{(1)}, \ldots, \chi_{(d)} \in \Gamma(TM)$ and would be able to write every vector field χ in terms of component functions f^i as $\chi = f^i \cdot \chi_{(i)}$.

Simple counterexample: Take a sphere. Can we find a smooth vector field over the entire sphere. Can you comb the sphere? No. For the field to be smooth, there is a problem. Morse Theory tells us that every smooth vector field on a sphere must vanish at 2 points $\implies$ basis cannot be chosen. We cannot choose a global basis. Therefore, if required, we only expand a vector field in terms of a basis on a domain where it is possible.

Remarks: Although we cannot have a global basis for $\Gamma(TM)$*, it is possible to do so locally. Thus, for the chart* (U, x) *we can take the **chart-induced basis of the vector field** in the chart domain* U *as the map*

$$\begin{aligned} \frac{\partial}{\partial x^i} : U &\xrightarrow{smooth} TU \\ p &\mapsto \left(\frac{\partial}{\partial x^i}\right)_p \end{aligned} \tag{6.8}$$

6.5 Tensor fields

So far we have constructed the sections over the tangent bundle. That is, $\Gamma(TM) =$"set of smooth vector fields" as a $C^\infty(M)$-module.

Exactly along the same lines we can construct the **cotangent bundle** $\Gamma(T^*M) =$ "set of covector fields" as a $C^\infty(M)$-module, by mapping a covector to the coordinates of its base point and components of the covector. $\Gamma(TM)$ and $\Gamma(T^*M)$ are the basic building blocks for every tensor field.
Definition 37. An (r, s)**-tensor field** T is a multilinear map

$$T : \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{r} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{s} \xrightarrow{\sim} C^\infty(M) \tag{6.9}$$

Remarks: the multilinearity is in $C^\infty(M)$*, in terms of addition in the modules and S-multiplication with functions in* $C^\infty(M)$*.*

Example: Let $f \in C^\infty(M)$. Then, define a $(0, 1)$-tensor field df as

$$\begin{aligned} df : &\Gamma(TM) \xrightarrow{\sim} C^\infty(M) \\ &\chi \mapsto df(\chi) := \chi f \qquad \text{such that } (\chi f)(\underbrace{p}_{\in M}) := \underbrace{\chi(p)}_{\in T_pM} f \end{aligned}$$

It can be checked that df is $C^\infty-$linear.

7 Lecture 7: Connections

Motivation: So far, all we have dealt with (e.g., sets, topological manifolds, smooth manifolds, fields, bundles, etc.) are structures that we have to provide by hand before we can start doing physics as we know it. Why? Because we don't have equations which determine what we have done so far. These are assumptions you need to submit before you can do physics.

In this lecture we introduce yet another structure called connections which are determined by Einstein's equations. Everything from now on will be objects that are the subject of Einstein's equations depending on the matter in the Universe. Connections are also called covariant derivatives. Even though these are different, for our purposes we shall not distinguish the two and use the more general connections.

So far, we saw that a vector field X can be used to provide a directional derivative of a function $f \in C^\infty(M)$ in the direction X

$$\nabla_X f := Xf$$

Isn't this a notational overkill? We already know

$$\nabla_X f = Xf = (df)X$$

Actually, they are not quite the same because

$$\begin{aligned} X : C^\infty(M) &\longrightarrow C^\infty(M) \\ df : \Gamma(TM) &\longrightarrow C^\infty(M) \\ \nabla_X : C^\infty(M) &\longrightarrow C^\infty(M) \end{aligned}$$

where ∇_X can be generalized to eat an arbitrary (p,q)-tensor field and yield a (p,q)-tensor field whereas X can only eat functions.

$$\begin{tikzcd} \nabla_X : C^\infty(M) \arrow[r] \arrow[d, squiggly, dotted] & C^\infty(M) \arrow[d, squiggly, dotted] \\ \nabla_X : (p,q)\text{-tensor field} \arrow[r] & (p,q)\text{-tensor field} \end{tikzcd}$$

We need ∇_X to provide the new structure to allow us to talk about directional derivatives of tensor fields and vector fields. Of course, only in cases where ∇_X acts on function f which is a $(0,0)$-tensor, it is exactly the same as Xf.

7.1 Directional derivatives of tensor fields

We formulate a wish list of properties which ∇_X acting on a tensor field should have. We put this in form of a definition. There may be many structures that satify this wish list. Any remaining freedom in choosing such a ∇ will need to be provided as additional structure beyond the structure we already have. And we assume all this takes place on a smooth manifold.

Definition 38. A **connection** ∇ on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a map that takes a pair consisting of a vector (field) X and a (p,q)-tensor field T and sends them to a (p,q)-tensor (field) $\nabla_X T$ satisfying

i) $\nabla_X f = Xf \quad \forall f \in C^\infty M$

ii) $\nabla_X(T+S) = \nabla_X T + \nabla_X S \quad$ where T, S are (p,q)-tensors

iii) **Leibnitz rule:** $\nabla_X T(\omega_1, \ldots, \omega_p, Y_1, \ldots, Y_q) = (\nabla_X T)(\omega_1, \ldots, \omega_p, Y_1, \ldots, Y_q)$
$+ T(\nabla_X \omega_1, \ldots, \omega_p, Y_1, \ldots, Y_q) + \cdots + T(\omega_1, \ldots, \nabla_X \omega_p, Y_1, \ldots, Y_q)$
$+ T(\omega_1, \ldots, \omega_p, \nabla_X Y_1, \ldots, Y_q) + \cdots + T(\omega_1, \ldots, \omega_p, Y_1, \ldots, \nabla_X Y_q)$ where T is a (p, q)-tensor

> Note that for a (p, q)-tensor T and a (r, s)-tensor S, since:
> $(T \otimes S)(\omega_{(1)}, \ldots, \omega_{(p+r)}, Y_{(1)}, \ldots, Y_{(q+s)}) =$
> $T(\omega_{(1)}, \ldots, \omega_{(p)}, Y_{(1)}, \ldots, Y_{(q)}) \cdot S(\omega_{(p+1)}, \ldots, \omega_{(p+r)}, Y_{(q+1)}, \ldots, Y_{(q+s)})$,
> Leibnitz rule implies $\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S)$.

iv) C^∞**-linearity:** $\forall f \in C^\infty(M), \nabla_{fX+Z} T = f \nabla_X T + \nabla_Z T$

> C^∞-linearity means that no matter how the function f scales the vectors at different points of the manifold, the effect of the scaling at any point is independent of scaling in the neighbourhood and depends only on how the scaling happens at that point.

A **manifold with a connection** ∇ is a quadruple $(M, \mathcal{O}, \mathcal{A}, \nabla)$, where M is a set, $\mathcal{O}$ is a topology and $\mathcal{A}$ is a smooth atlas.

Remark: If $\nabla_X(\cdot)$ can be seen as an extension of X,
then $\nabla_{(\cdot)}(\cdot)$ can be seen as an extension of d.

7.2 New structure on $(M, \mathcal{O}, \mathcal{A})$ required to fix ∇

How much freedom do we have in choosing such a structure?

Consider vector fields X, Y and chart $(U, x) \in \mathcal{A}$. Then

$$\begin{aligned}
\nabla_X Y &= \nabla_{(X^i \frac{\partial}{\partial x^i})} \left(Y^m \frac{\partial}{\partial x^m} \right) && \text{by expanding in chart-induced basis} \\
&= X^i \cdot \nabla_{(\frac{\partial}{\partial x^i})} \left(Y^m \frac{\partial}{\partial x^m} \right) && \text{by } C^\infty\text{-linearity} \\
&= X^i \underbrace{\left(\nabla_{(\frac{\partial}{\partial x^i})} Y^m \right)}_{= \frac{\partial}{\partial x^i} Y^m} \frac{\partial}{\partial x^m} + X^i \cdot Y^m \cdot \underbrace{\left(\nabla_{(\frac{\partial}{\partial x^i})} \frac{\partial}{\partial x^m} \right)}_{\text{a vector field, by defn.}} && \text{using Leibnitz rule} \\
&= X^i \left(\frac{\partial}{\partial x^i} Y^m \right) \frac{\partial}{\partial x^m} + X^i \cdot Y^m \cdot \left(\Gamma^q{}_{mi} \frac{\partial}{\partial x^q} \right)
\end{aligned}$$

Thus, by change of indices,

$$\boxed{(\nabla_X Y)^i = X^m \left(\frac{\partial}{\partial x^m} Y^i \right) + X^m \cdot Y^n \cdot \Gamma^i{}_{nm}} \tag{7.1}$$

So we need $(dim\, M)^3$-many functions to define directional derivative of a vector field.
Definition 39. Given $(M, \mathcal{O}, \mathcal{A}, \nabla)$ and $(U, x) \in \mathcal{A}$, then the **connection coefficient functions** (Γs) on M of ∇ w.r.t (U, x) are $(dim\, M)^3$-many functions given by

$$\begin{aligned}
\Gamma^i{}_{jk} : \quad & U \longrightarrow \mathbb{R} \\
& p \mapsto \Gamma^i{}_{jk}(p) := \left(dx^i \left(\nabla_{(\frac{\partial}{\partial x^k})} \frac{\partial}{\partial x^j} \right) \right)(p)
\end{aligned} \tag{7.2}$$

Note: $\frac{\partial}{\partial x^j}$ is a vector field; $\therefore$ $\nabla_{(\frac{\partial}{\partial x^k})} \frac{\partial}{\partial x^j}$ is a vector field, and dx^i is a covector which will result in a function after acting on a vector field.

On a chart domain U, choice of the $(dim\, M)^3$-many functions $\Gamma^i{}_{jk}$ suffices to fix the action of ∇ on a vector field. What about the directional derivative of a covector field, or a tensor field? Will we have to provide more and more coefficients? Fortunately, the same $(dim\, M)^3$-many functions fix the action of ∇ on any tensor field.

We know that, for a covector, $\nabla_{\frac{\partial}{\partial x^m}} \left(dx^i\right) = \Sigma^i{}_{jm} dx^j$, since dx^i form a dual basis. Are these Σs independent of Γs? Consider the following.

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial x^m}} \left(dx^i \left(\frac{\partial}{\partial x^j} \right) \right) = \nabla_{\frac{\partial}{\partial x^m}} \delta^i_j = \frac{\partial}{\partial x^m} (\delta^i_j) = 0 \\
&\implies \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left(\frac{\partial}{\partial x^j} \right) + dx^i \underbrace{\left(\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j} \right)}_{\Gamma^q{}_{jm} \frac{\partial}{\partial x^q}} = 0 \\
&\implies \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left(\frac{\partial}{\partial x^j} \right) + dx^i \Gamma^q{}_{jm} \frac{\partial}{\partial x^q} = 0 \\
&\implies \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left(\frac{\partial}{\partial x^j} \right) = -dx^i \Gamma^q{}_{jm} \frac{\partial}{\partial x^q} = -\Gamma^q{}_{jm} dx^i \frac{\partial}{\partial x^q} = -\Gamma^q{}_{jm} \delta^i_q = -\Gamma^i{}_{jm} \\
&\implies \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \underbrace{\left(\frac{\partial}{\partial x^j} \right) dx^j}_{=\delta^j_j=1} = -\Gamma^i{}_{jm} dx^j \\
&\implies \boxed{\nabla_{\frac{\partial}{\partial x^m}} dx^i = -\Gamma^i{}_{jm} dx^j} \\
&\implies \boxed{\left(\nabla_{\frac{\partial}{\partial x^m}} dx^i \right)_j = -\Gamma^i{}_{jm}}
\end{aligned}
$$

In summary,

$$
(\nabla_X Y)^i = X(Y^i) + \Gamma^i{}_{jm} Y^j X^m \tag{7.3}
$$

$$
(\nabla_X \omega)_i = X(\omega_i) - \Gamma^j{}_{im} \omega_j X^m \tag{7.4}
$$

Note that for the immediately above expression for $(\nabla_X Y)^i$, in the second term on the right hand side, $\Gamma^i{}_{jm}$ has the last entry at the bottom, m going in the direction of X, so that it matches up with X^m. This is a good mnemonic to memorize the index positions of Γ.

Similarly, as an example, by further application of Leibnitz rule, for a $(1,2)$-tensor field T,

$$
(\nabla_X T)^i{}_{jk} = X\left(T^i{}_{jk}\right) + \Gamma^i{}_{sm} T^s{}_{jk} X^m - \Gamma^s{}_{jm} T^i{}_{sk} X^m - \Gamma^s{}_{km} T^i{}_{js} X^m
$$

7.3 Change of Γ's under change of chart

Let (U, x), $(V, y) \in \mathcal{A}$ and $U \cap V \neq \emptyset$.

$$
\begin{aligned}
\left(\Gamma_{(y)}\right)^{i}{}_{jk} &:= dy^i \left(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}\right) \\
&= \frac{\partial y^i}{\partial x^q} dx^q \left(\nabla_{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j}\frac{\partial}{\partial x^s}\right) \\
&= \frac{\partial y^i}{\partial x^q} dx^q \left(\frac{\partial x^p}{\partial y^k}\left[\left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j}\left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^s}\right)\right]\right) && \because \nabla \text{ is } C^\infty - linear \\
&= \frac{\partial y^i}{\partial x^q} \underbrace{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}}_{\frac{\partial}{\partial y^k}} \frac{\partial x^s}{\partial y^j}\delta^q_s + \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}\left(\Gamma_{(x)}\right)^{q}{}_{sp}
\end{aligned}
$$

$$
\left(\Gamma_{(y)}\right)^{i}{}_{jk} = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^k \partial y^j} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\left(\Gamma_{(x)}\right)^{q}{}_{sp} \tag{7.5}
$$

Eq. (7.5) is the change of connection coefficient function under the change of chart $(U \cap V, x) \longrightarrow (U \cap V, y)$. Γ is not a tensor due to the first term on left hand side in Eq. (7.5). However, for linear transformation between coordinates in two charts, the term $\frac{\partial^2 x^q}{\partial y^k \partial y^j}$ always vanishes and then, if Γs are zero in one chart, they will be zero in the other chart too. However, there is no reason not to select a coordinate which is not a linear transformation of another one.

7.4 Normal Coordinates

Can we find a coordinate system that makes the Γs vanish?

Theorem 14. *Let $p \in M$ of $(M, \mathcal{O}, \mathcal{A}, \nabla)$. Having chosen a point p, one can construct a chart (U, x) with $p \in U$ such that the symmetric part of Γs vanish at the point p (not necessarily in any neighbourhood). That is, $\forall\, p \in M, \exists\, (U, x) \in \mathcal{A} : p \in U$ and $\left(\Gamma_{(x)}\right)^{i}{}_{(jk)}(p) = 0$.*
Such (U, x) is called a ***normal coordinate chart*** *of ∇ at $p \in M$.*

Proof. Let $(V, y) \in \mathcal{A}$ and $p \in V$. Then consider a new chart (U, x) to which one transits using the map $(x \circ y^{-1})$ whose i^{th} component is given by

$$
\begin{aligned}
&\left(x \circ y^{-1}\right)^i \left(\alpha^1, \ldots, \alpha^d\right) := \alpha^i - \left(\Gamma_{(y)}\right)^{i}{}_{(jk)} \alpha^j \alpha^k && \text{where the } \Gamma\text{s are taken at the point } p \\
&\implies \frac{\partial x^i}{\partial y^j} = \partial_j \left(x^i \circ y^{-1}\right) = \delta^i_j - \left(\Gamma_{(y)}\right)^{i}{}_{(jm)} \alpha^m \\
&\qquad\qquad \implies \frac{\partial^2 x^i}{\partial y^k \partial y^j} = -\left(\Gamma_{(y)}\right)^{i}{}_{(jk)}
\end{aligned}
$$

TODO: Not understood by me. ☐

8 Parallel Transport & Curvature

8.1 Parallelity of vector fields

Definition 40. Let $(M, \mathcal{O}, \mathcal{A}, \nabla)$ be a smooth manifold with connection ∇.

(1) A vector field X on M is said to be **parallely transported** along a smooth curve $\gamma : \mathbb{R} \longrightarrow M$ if

$$\boxed{\nabla_{v_\gamma} X = 0} \tag{8.1}$$

(2) A slightly weaker condition is "**parallel**" if, for $\mu : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\left(\nabla_{v_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)} \tag{8.2}$$

Note: Even though **parallely transported** sounds like an action, it is a property.

8.2 Autoparallely transported curves

Definition 41. A curve $\gamma : \mathbb{R} \longrightarrow M$ is called **autoparallely transported** if

$$\nabla_{v_\gamma} v_\gamma = 0 \tag{8.3}$$

Note: Sometimes, this curve is called an autoparallel curve. But we wish to call a curve autoparallel if $\nabla_{v_\gamma} v_\gamma = \mu v_\gamma$.

8.3 Autoparallel equation

Express $\nabla_{v_\gamma} v_\gamma = 0$ in terms of chart representation.

$$\begin{aligned}
0 &= \left(\nabla_{v_\gamma} v_\gamma\right) \\
&= \left(\nabla_{\left(\dot\gamma^m_{(x)} \frac{\partial}{\partial x^m}\right)} \dot\gamma^n_{(x)} \frac{\partial}{\partial x^n}\right) && \text{remember that } \gamma^m_{(x)} := x^m \circ \gamma \\
&= \dot\gamma^m \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)} \dot\gamma^n\right) \frac{\partial}{\partial x^n} + \dot\gamma^m \dot\gamma^n \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)} \frac{\partial}{\partial x^n}\right) && \text{x index is understood, hence suppressed} \\
&= \dot\gamma^m \left(\frac{\partial}{\partial x^m} \dot\gamma^n\right) \frac{\partial}{\partial x^n} + \dot\gamma^m \dot\gamma^n \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)} \frac{\partial}{\partial x^n}\right) \\
&= \dot\gamma^m \left(\frac{\partial}{\partial x^m} \dot\gamma^q\right) \frac{\partial}{\partial x^q} + \dot\gamma^m \dot\gamma^n \left(\Gamma^q{}_{nm} \frac{\partial}{\partial x^q}\right) && \text{change of index in 1st term} \\
&= \left(\dot\gamma^m \frac{\partial}{\partial x^m} \dot\gamma^q + \dot\gamma^m \dot\gamma^n \Gamma^q{}_{nm}\right) \frac{\partial}{\partial x^q} \\
&= \left(\ddot\gamma^q + \dot\gamma^m \dot\gamma^n \Gamma^q{}_{nm}\right) \frac{\partial}{\partial x^q} && \text{1st term is 2nd derivative by Fig. ??}
\end{aligned}$$

In summary:

$$\boxed{\ddot\gamma^q_{(x)}(\lambda) + (\Gamma_{(x)})^q{}_{mn}(\gamma(\lambda)) \dot\gamma^m_{(x)}(\lambda) \dot\gamma^n_{(x)}(\lambda) = 0} \tag{8.4}$$

Eq. (8.4) is the chart expression of the condition that γ be autoparallely transported.

8.4 Torsion

Definition 42. torsion of a connection ∇ is the $(1,2)$-**tensor field**

$$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \tag{8.5}$$

(Inside a cloud)

$[X, Y]$ vector field defined by

$$[X, Y]f := X(Yf) - Y(Xf)$$

Proof. check T is C^∞-linear in each entry

$$T(\omega, fX, Y) = \omega(\nabla_{fX} Y - \nabla_Y (fX) - [fX, Y])$$

□

Definition 43. A $(M, \mathcal{O}, \mathcal{A}, \nabla)$ is called torsion-free if $T = 0$

In a chart

$$\begin{aligned} T^i{}_{ab} := T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= dx^i(\dots) \\ &= \Gamma^i{}_{ab} - \Gamma^i{}_{ba} = 2\Gamma^i{}_{[ab]} \end{aligned}$$

From now on, in these lectures, we only use torsion-free connections.

8.5 Curvature

Definition 44. Riemann curvature of a connection ∇ is the $(1,3)$-tensor field

$$\mathrm{Riem}(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{8.6}$$

Proof. do it: C^∞-linear in each slot. □

<u>Tutorials</u> $\mathrm{Riem}^i{}_{jab} = \dots$

9 Lecture 9: Newtonian spacetime is curved!

Axiom 1 (Newton I:). *A body on which* no *force acts moves uniformly along a straight line*
Axiom 2 (Newton II:). *Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.*

Remark:

(1) 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space ...

(2) Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

9.1 Laplace's questions

Laplace $*1749$
$\dagger 1827$

Q: "Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to more along straight lines in this curved space?"

Answer: No!

Proof. gravity is a force point of view

$$m\ddot{x}^\alpha(t) = F^\alpha(x(t))$$
$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t))$$

$-\partial_\alpha f^\alpha = 4\pi G\rho$ (Poisson)
ρ mass density of matter

(EY : 20150330) You know this, $F = Gm_1m_2/r^2$

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

Laplace asks: Is this ($\ddot{x}(t)$) of the form

$$\ddot{x}^\alpha(t) + \Gamma^\alpha_{\beta\gamma}(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

Conclusion: One cannot find Γ s such that Newton's equation takes the form of an autoparallel.

$\square$

9.2 The full wisdom of Newton I

use also the information from Newton's first law that particles (no force) move uniformly

introduce the appropriate setting to talk about the difference easily insight: in spacetime $\boxed{\text{uniform \& straight motion}}$ is simply straight motion

So let's try in spacetime:

let $x : \mathbb{R} \longrightarrow \mathbb{R}^3$

be a particle's trajectory in space $\longleftrightarrow$ worldline (history) of the particle $\begin{array}{l} X : \mathbb{R} \longrightarrow \mathbb{R}^4 \\ t \mapsto (t, x^1(t), x^2(t), x^3(t)) := \\ := (X^0(t), X^1(t), X^2(t), X^3(t)) \end{array}$

That's all it takes:

Trivial rewritings:

$$\dot{X}^0 = 1$$

$$\Longrightarrow \boxed{\begin{array}{ll} \ddot{X}^0 & = 0 \\ \ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 & = 0 \end{array}} \quad (\alpha = 1, 2, 3) \Longrightarrow \begin{array}{c} a = 0, 1, 2, 3 \\ \boxed{\ddot{X}^a + \Gamma^a_{\ bc} \dot{X}^b \dot{X}^c = 0} \\ \text{antoparallel eqn in spacetime} \end{array}$$

Yes, choosing $\Gamma^0_{\ ab} = 0$

$\Gamma^\alpha_{\ \beta\gamma} = 0 = Gamma^\alpha_{\ 0\beta} = \Gamma^\alpha_{\ \beta 0}$

only: $\boxed{\Gamma^\alpha_{\ 00} \stackrel{!}{=} -f^\alpha}$

Question: Is this a coordinate-choice artifact?

No, since $R^\alpha_{\ 0\beta 0} = -\frac{\partial}{\partial x^\beta} f^\alpha$ (only non-vanishing components) (tidal force tensor, – the Hessian of the force component)

Ricci tensor $\Longrightarrow R_{00} = R^m_{\ 0m0} = -\partial_\alpha f^\alpha = 4\pi G\rho$

Poisson: $-\partial_\alpha f^\alpha = 4\pi G \cdot \rho$

writing: $T_{00} = \frac{1}{2} s$

$$\Longrightarrow \boxed{R_{00} = 8\pi G T_{00}}$$

Einstein in 1912 $\boxed{\cancel{R_{ab} = 8\pi G T_{ab}}}$

Conclusion: Laplace's idea works in spacetime

Remark

$$\Gamma^\alpha_{\ 00} = -f^\alpha$$
$$R^\alpha_{\ \beta\gamma\delta} = 0 \qquad \alpha, \beta, \gamma, \delta = 1, 2, 3$$
$$\boxed{R_{00} = 4\pi G\rho}$$

Q: What about transformation behavior of LHS of

$$\underbrace{\underbrace{\ddot{x}^a + \Gamma^a_{\ bc} \dot{X}^b \dot{X}^c}_{(\nabla_{v_X} v_X)^a}}_{:= a^a \text{ “acceleration vector”}} = 0$$

9.3 The foundations of the geometric formulation of Newton's axiom

new start

Definition 45. A **Newtonian spacetime** is a quintuple

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where $(M, \mathcal{O}, \mathcal{A})$ 4-dim. smooth manifold

$$t : M \longrightarrow \mathbb{R} \text{ smooth function}$$

(i) "There is an absolute space"

$$(dt)_p \neq 0 \qquad \forall\, p \in M$$

(ii) "absolute time flows uniformly"

$$\nabla dt \qquad \underbrace{=}_{\text{space of } (0,2)\text{-tensor fields}} \qquad 0 \qquad \text{everywhere}$$

∇dt is a $(0,2)$-tensor field

(iii) add to axioms of Newtonian spacetime $\nabla = 0$ torsion free

Definition 46. absolute space at time τ

$$S_\tau := \{p \in M | t(p) = \tau\}$$

$$\xrightarrow{dt \neq 0} M = \coprod S_\tau$$

Definition 47. A vector $X \in T_pM$ is called

(a) future-directed if

$$dt(X) > 0$$

(b) spatial if

$$dt(X) = 0$$

(c) past-directed if

$$dt(X) < 0$$

<u>picture</u>

<u>Newton I</u>: The worldline of a particle under the influence of no force (gravity isn't one, anyway) is a <u>future-directed autoparalle</u>
i.e.

$$\nabla_{v_X} v_X = 0$$
$$dt(v_X) > 0$$

<u>Newton II</u>:

$$\nabla_{v_X} v_X = \frac{F}{m} \Longleftrightarrow m \cdot a = F$$

where F is a spatial vector field:

$$dt(F) = 0$$

Convention: restrict attention to atlases $\mathcal{A}_{\text{stratefied}}$ whose charts $(\mathcal{U}, x)$ have the property

$$\begin{aligned} &x^0 : \mathcal{U} \longrightarrow \mathbb{R} \\ &x^1 : \mathcal{U} \longrightarrow \mathbb{R} \\ &\vdots \quad \vdots \\ &x^3 \end{aligned} \qquad x^0 = t|_{\mathcal{U}} \qquad \Longrightarrow \qquad \begin{aligned} 0 &\overset{\text{"absolute time flows uniformly"}}{=} \nabla dt \\ 0 &= \nabla_{\frac{\partial}{\partial x^a}} dx^0 = -\Gamma^0{}_{ba} \qquad a = 0,1,2,3 \end{aligned}$$

Let's evaluate in a chart $(\mathcal{U}, x)$ of a stratified atlas $\mathcal{A}_{\text{sheet}}$: Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m}$$

in a chart.

$$\begin{aligned} (X^0)'' + \cancel{\Gamma^0{}_{cd} (X^a)' (X^b)'}^{\text{stratified atlas}} &= 0 \\ (X^\alpha)'' + \Gamma^\alpha_{\gamma\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha_{00} X^{0'} X^{0'} + 2\Gamma^\alpha_{\gamma 0} X^{\gamma'} X^{0'} &= \frac{F^\alpha}{m} \qquad \alpha = 1,2,3 \end{aligned}$$

$$\Longrightarrow (X^0)''(\lambda) = 0 \Longrightarrow X^0(\lambda) = a\lambda + b \qquad \text{constants } a, b \ \text{ with}$$
$$X^0(\lambda) = (x^0 \circ X)(\lambda) \overset{\text{stratified}}{=} (t \circ X)(\lambda)$$

<u>convention</u> parametrize worldline by absolute time

$$\frac{d}{d\lambda} = a \frac{d}{dt}$$

$$\begin{aligned} a^2 \ddot{X}^\alpha + a^2 \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + a^2 \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0 &= \frac{F^\alpha}{m} \\ \Longrightarrow \underbrace{\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0}_{a^\alpha} &= \frac{1}{a^2} \frac{F^\alpha}{m} \end{aligned}$$

10 Lecture 10: Metric Manifolds

cf. Lecture 10: Metric Manifolds (International Winter School on Gravity and Light 2015)

We establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space).

From this structure, one can then define a notion of length of a curve.
Then we can look at shortest curves.

Requiring then that the shortest curves coincide with the straightest curves (wrt ∇) will result in ∇ being determined by the metric structure.

$$g \quad \overset{\text{straight=short}}{\overset{T=0}{\rightsquigarrow}} \quad \nabla \rightsquigarrow \text{Riem}$$

10.1 Metrics

Definition 48. A metric g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying

(i) symmetry $g(X,Y) = g(Y,X) \quad \forall\, X, Y$ vector fields

(ii) non-degeneracy: the musical map

$$\begin{aligned} \text{"flat"}\ \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\mapsto \flat(X) \end{aligned}$$

where $\quad \flat(X)(Y) := g(X,Y)$

$\qquad\qquad \flat(X) \in \Gamma(T^*M)$

In thought bubble: $\flat(X) = g(X, \cdot)$

... is a C^∞-isomorphism in other words, it is invertible.

Remark: $(\flat(X))_a \quad$ or
X_a
$(\flat(X))_a := g_{am}X^m$

Thought bubble: $\flat^{-1} = \sharp$

$\flat^{-1}(\omega)^a := g^{am}\omega_m$
$\flat^{-1}(\omega)^a := (g^{\text{“}-1''})^{am}\omega_m \implies$ not needed. (all of this is not needed)

Definition 49. The $(2,0)$-tensor field $g^{\text{“}-1''}$ with respect to a metric g is the symmetric

$$\begin{aligned} g^{\text{“}-1''} : \Gamma(T^*M) \times \Gamma(T^*M) &\to C^\infty(M) \\ (\omega, \sigma) &\mapsto \omega(\flat^{-1}(\sigma)) \qquad \flat^{-1}(\sigma) \in \Gamma(TM)) \end{aligned}$$

chart: $g_{ab} = g_{ba}$

$\qquad (g^{-1})^{am} g_{mb} = \delta^a_b$

Example: $(S^2, \mathcal{O}, \mathcal{A})$

$\qquad$ chart $(\mathcal{U}, x)$

$\varphi \in (0, 2\pi)$
$\theta \in (0, \pi)$

define the metric

$$g_{ij}(x^{-1}(\theta, \varphi)) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{bmatrix}_{ij}$$

$R \in \mathbb{R}^+$

"the metric of the round sphere of radius R"

10.2 Signature

$$A^a{}_m v^m = \lambda v^a \qquad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Linear algebra:

$$g_{am} v^m = \lambda \cdot v^a ? \rightsquigarrow \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

$(1, 1)$ tensor has eigenvalues
$(0, 2)$ has signature (p, q) (well-defined)

$$\left.\begin{matrix} (+++) \\ (++-) \\ (+--) \\ (---) \end{matrix}\right\} d+1 \text{ if } p+q = \text{dim} V$$

Definition 50. A metric is called
Riemannian if its signature is $(++\cdots+)$
Lorentzian if $(+-\cdots-)$

10.3 Length of a curve

Let γ be a smooth curve.
Then we know its veloctiy $v_{\gamma,\gamma(\lambda)}$ at each $\gamma(\lambda) \in M$.
Definition 51. On a Riemannian metric manifold $M, \mathcal{O}, \mathcal{A}, g)$, the **speed** of a curve at $\gamma(\lambda)$ is the number

$$(\sqrt{g(v_\gamma, v_\gamma)})_{\gamma(\lambda)} = s(\lambda)$$

F. Schuller: "I feel the need for speed." -Top Gun.

(I feel the need for speed, then I feel the need for a metric)

Aside: $[v^a] = \frac{1}{T}$
$[g_{ab}] = L^2$
$[\sqrt{g_{ab}v^a v^b}] = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$

Definition 52. Let $\gamma : (0,1) \longrightarrow M$ a smooth curve.
Then the **length of** γ is the number

$$\mathbb{R} \ni L[\gamma] := \int_0^1 d\lambda s(\lambda) = \int_0^1 d\lambda \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}$$

F. Schuller: "velocity is more fundamental than speed, speed is more fundamental than length"

Example: reconsider the round sphere of radius R

Consider its equator:
$\theta(\lambda) := (x^1 \circ \gamma)(\lambda) = \dfrac{\pi}{2}$
$\varphi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3$

$\theta'(\lambda) = 0$
$\varphi'(\lambda) = 6\pi\lambda^2$

on the same chart $g_{ij} = \begin{bmatrix} R^2 & \\ & R^2 \sin^2\theta \end{bmatrix}$

F.Schuller: do everything in this chart

$$L[\gamma] = \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} = \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda)) 36\pi^2\lambda^4} =$$
$$= 6\pi R \int_0^1 d\lambda \lambda^2 = 6\pi R[\frac{1}{3}\lambda^3]_0^1 = 2\pi R$$

Theorem 15. $\gamma : (0,1) \longrightarrow M$ *and*
$\sigma : (0,1) \longrightarrow (0,1)$ *smooth bijective and increasing "reparametrization"*

$$L[\gamma] = L[\gamma \circ \sigma]$$

Proof. $\Longrightarrow$ Tutorials □

10.4 Geodesics

Definition 53. A curve $\gamma : (0,1) \longrightarrow M$ is called a **geodesic** on a Riemannian manifold $(M, \mathcal{O}, \mathcal{A}, g)$ if its a *stationary* curve with respect to a length functional L.

Thought bubble: in classical mechanics, deform the curve a little, ϵ times this deformation, to first order, it agrees with $L[\gamma]$

Theorem 16. γ *geodesic iff it satisfies the Euler-Lagrange equations for the Lagrangian*

$$\begin{aligned} \mathcal{L} :& TM \longrightarrow \mathbb{R} \\ & X \mapsto \sqrt{g(X, X)} \end{aligned}$$

In a chart, the Euler Lagrange equations take the form:

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m}\right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial x^m} = 0$$

F.Schuller: this is a chart dependent formulation

here:

$$\mathcal{L}(\gamma^i, \dot\gamma^i) = \sqrt{g_{ij}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)}$$

Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot\gamma^m} &= \frac{1}{\sqrt{\cdots}} g_{mj}(\gamma(\lambda))\dot\gamma^j(\lambda) \\
\left(\frac{\partial \mathcal{L}}{\partial \dot\gamma^m}\right)^{\cdot} &= \left(\frac{1}{\sqrt{\cdots}}\right)^{\cdot} g_{mj}(\gamma(\lambda))\cdot\dot\gamma^j(\lambda) + \frac{1}{\sqrt{\cdots}}\left(g_{mj}(\gamma(\lambda))\ddot\gamma^j(\lambda) + \dot\gamma^s(\partial_s g_{mj})\dot\gamma^j(\lambda)\right)
\end{aligned}$$

Thought bubble: reparametrize $g(\dot\gamma, \dot\gamma) = 1$ (it's a condition on my reparametrization)

By a clever choice of reparametrization $(\frac{1}{\sqrt{\cdots}})^{\cdot} = 0$

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{\cdots}} \partial_m g_{ij}(\gamma(\lambda))\dot\gamma^i(\lambda)\dot\gamma^j(\lambda)$$

putting this together as Euler-Lagrange equations:

$$g_{mj}\ddot\gamma^j + \partial_s g_{mj}\dot\gamma^s\dot\gamma^j - \frac{1}{2}\partial_m g_{ij}\dot\gamma^i\dot\gamma^j = 0$$

Multiply on both sides $(g^{-1})^{qm}$

$$\ddot\gamma^q + (g^{-1})^{qm}(\partial_i g_{mj} - \frac{1}{2}\partial_m g_{ij})\dot\gamma^i\dot\gamma^j = 0$$

$$\boxed{\ddot\gamma^q + (g^{-1})^{qm}\frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})\dot\gamma^i\dot\gamma^j = 0}$$

geodesic equation for γ in a chart.

$$\boxed{(g^{-1})^{qm}\frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) =: \Gamma^q{}_{ij}(\gamma(\lambda))}$$

Thought bubble: $\left(\frac{\partial \mathcal{L}}{\partial \xi_x^{a+\dim M}}\right)^{\cdot}_{\sigma(x)} - \left(\frac{\partial \mathcal{L}}{\partial xi_x^a}\right)_{\sigma(x)} = 0$

Definition 54. "Christoffel symbol" ${}^{\text{L.C.}}\Gamma$ are the connection coefficient functions of the so-called Levi-Civita connection ${}^{\text{L.C.}}\nabla$

We usually make this choice of ∇ if g is given.

$(M, \mathcal{O}, \mathcal{A}, g) \longrightarrow (M, \mathcal{O}, \mathcal{A}, g, {}^{\text{L.C.}}\nabla)$

abstract way: $\nabla g = 0$ and $T = 0$ (torsion)
$\Longrightarrow \nabla = {}^{\text{L.C.}}\nabla$

Definition 55. (a) The Riemann-Christoffel curvature is defined by

$$R_{abcd} := g_{am}R^m{}_{bcd}$$

(b) Ricci: $R_{ab} = R^m{}_{amb}$
Thought bubble: with a metric, ${}^{\text{L.C.}}\nabla$

(c) (Ricci) scalar curvature:

$$R = g^{ab}R_{ab}$$

Thought bubble: ${}^{\text{L.C.}}\nabla$

Definition 56. Einstein curvature $(M, \mathcal{O}, \mathcal{A}, g)$

$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$$

Convention: $g^{ab} := (g^{``-1''})^{ab}$

F. Schuller: these indices are not being pulled up, because what would you pull them up with

(student) Question: Does the Einstein curvature yield new information?

Answer:

$$g^{ab} G_{ab} = R_{ab} g^{ab} - \frac{1}{2} g_{ab} g^{ab} R = R - \delta^a_a R = R - \frac{1}{2} \mathrm{dim} M \, R = (1 - \frac{d}{2}) R$$

11 Lecture 11: Symmetry

EY : 20150321 This lecture tremendously and lucidly clarified, for me at least, what a symmetry of the Lie algebra is, and in comparing structures $(M, \mathcal{O}, \mathcal{A})$ vs. $(M, \mathcal{O}, \mathcal{A}, \nabla)$, clarified differences, and asking about differences is a good way to learn, the difference between $\mathcal{L}$ and ∇, respectively.

Feeling that the round sphere

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$$

has rotational symmetry, while
the potato

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$$

does not.

11.1

11.2

Important

11.3 Flow of a complete vector field

Let $(M, \mathcal{O}, \mathcal{A})$ smooth X vector field on M
Definition 57. A curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is called an integral curve of X
if

$$v_{\gamma,\gamma(\lambda)} = X_{\gamma(\lambda)}$$

Definition 58. A vector filed X is **complete** if all integral curves have $I = \mathbb{R}$ EY: 20150321 (i.e. domain is all of $\mathbb{R}$)

Ex. minute 48:30 EY : reall good explanation by F.P.Schuller; take a pt. out for an incomplete vector field.
Theorem 17. *compactly supported smooth vector field is complete.*
Definition 59. The flow of a complete vector field X is a 1-parameter family

$$h^X = \mathbb{R} \times M \longrightarrow M$$

where $\gamma_p : \mathbb{R} \longrightarrow M$ is the integral curve of X with

$\gamma(0) = p$

Then for fixed $\lambda \in \mathbb{R}$

$$h^X_\lambda : M \longrightarrow M \text{ smooth}$$

picture $h^X_{\underline{\lambda}}(S) \neq S($ if $X \neq 0)$

11.4 Lie subalgebras of the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of vector fields

(a) $\Gamma(TM) = \{$ set of all vector fields $\}$ $\quad C^\infty(M)$-module = $\mathbb{R}$-vector space

$$\Longrightarrow [X, Y] \in \Gamma(TM) \qquad [X, Y]f := X(Yf) - Y(Xf)$$

(i) $[X, Y] = -[Y, X]$

(ii) $[\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]$

(iii) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

$(\Gamma(TM), [\cdot, \cdot])$ <u>Lie algebra</u>

(b) Let $X_1 \dots X_s$ for s (many) vector fields on M, <u>such that</u>

12 Integration

12.1

12.2

12.3 Volume forms

Definition 60. On a smooth manifold $(M, \mathcal{O}, \mathcal{A})$
a $(0, \text{dim}M)$-tensor field Ω is called a volume form if

(a) Ω vanishes nowhere (i.e. $\Omega \neq 0 \; \forall \, p \in M$)

(b) totally antisymmetric

$$\Omega(\dots, \underbrace{X}_{i\text{th}}, \dots, \underbrace{Y}_{j\text{th}} \dots) = -\Omega(\dots, \underbrace{Y}_{i\text{th}}, \dots, \underbrace{X}_{j\text{th}} \dots)$$

In a chart:

$$\Omega_{i_1 \dots i_d} = \Omega_{[i_1 \dots i_d]}$$

Example $(M, \mathcal{O}, \mathcal{A}, g)$ metric manifold

construct volume form Ω from g

In any chart: (U, x)

$$\Omega_{i_1 \dots i_d} := \sqrt{\det(g_{ij}(x))} \epsilon_{i_1 \dots i_d}$$

where **Levi-Civita symbol** $\epsilon_{i_1 \dots i_d}$ is defined as $\epsilon_{123 \dots d} = +1$

$$\epsilon_{1 \dots d} = \epsilon_{[i_1 \dots i_d]}$$

Proof. (well-defined) Check: What happens under a change of charts

$$\begin{aligned}
\Omega(y)_{i_1 \dots i_d} &= \sqrt{\det(g(y)_{ij})} \epsilon_{i_1 \dots i_d} = \\
&= \sqrt{\det(g_{mn}(x) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j})} \frac{\partial y^{m_1}}{\partial x^{i_1}} \dots \frac{\partial y^{m_d}}{\partial x^{i_d}} \epsilon_{[m_1 \dots m_d]} = \\
&= \sqrt{|\det g_{ij}(x)|} \left| \det \left(\frac{\partial x}{\partial y} \right) \right| \det \left(\frac{\partial y}{\partial x} \right) \epsilon_{i_1 \dots i_d} = \sqrt{\det g_{ij}(x)} \epsilon_{i_1 \dots i_d} \text{sgn} \left(\det \left(\frac{\partial x}{\partial y} \right) \right)
\end{aligned}$$

$\square$

EY : 20150323

Consider the following:

$$\begin{aligned}
\Omega(y)(Y_{(1)}\dots Y_{(d)}) &= \Omega(y)_{i_1\dots i_d}Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} = \\
&= \sqrt{\det(g_{ij}(y))}\epsilon_{i_1\dots i_d}Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} = \\
&= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\epsilon_{i_1\dots i_d}\frac{\partial y^{i_1}}{\partial x^{m_1}}\dots\frac{\partial y^{i_d}}{\partial x^{m_d}}X^{m_1}\dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d}X^{m_1}\dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))}\left|\det\left(\frac{\partial x}{\partial y}\right)\right|\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d}X^{m_1}\dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))}\epsilon_{m_1\dots m_d}\mathrm{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right)X^{m_1}\dots X^{m_d} = \mathrm{sgn}(\det\left(\frac{\partial x}{\partial y}\right))\Omega_{m_1\dots m_d}(x)X^{m_1}\dots X^{m_d}
\end{aligned}$$

If $\det\left(\frac{\partial y}{\partial x}\right) > 0$,

$$\Omega(y)(Y_{(1)}\dots Y_{(d)}) = \Omega(x)(X_{(1)}\dots X_{(d)})$$

This works also if Levi-Civita symbol $\epsilon_{i_1\dots i_d}$ doesn't change at all under a change of charts. (around 42:43 https://youtu.be/2XpnbvPy-Zg)

Alright, let's require,

restrict the smooth atlas $\mathcal{A}$

to a subatlas ($\mathcal{A}^{\uparrow}$ still an atlas)

$$\mathcal{A}^{\uparrow} \subseteq \mathcal{A}$$

s.t. $\forall (U,x),(V,y)$ have chart transition maps $y\circ x^{-1}$

$x\circ y^{-1}$

s.t. $\det\left(\frac{\partial y}{\partial x}\right) > 0$

such $\mathcal{A}^{\uparrow}$ called an **oriented** atlas

$$(M,\mathcal{O},\mathcal{A},g) \Longrightarrow (M,\mathcal{O},\mathcal{A}^{\uparrow},g)$$

Note: associated bundles.

Note also: $\det\left(\frac{\partial y^b}{\partial x^a}\right) = \det(\partial_a(y^b x^{-1}))$ $\frac{\partial y^b}{\partial x^a}$ is an endomorphism on vector space V. $\varphi : V \longrightarrow V$

$\det\varphi$ independent of choice of b

g is a $(0,2)$ tensor field, not endomorphism (not independent of choice of basis) $\sqrt{|\det(g_{ij}(y))|}$

Definition 61. Ω be a volume form on $(M,\mathcal{O},\mathcal{A}^{\uparrow})$ and consider chart (U,x)

Definition 62. $\omega_{(X)} := \Omega_{i_1\dots i_d}\epsilon^{i_1\dots i_d}$ same way $\epsilon^{12\dots d} = +1$

$\epsilon^{[\dots]}$

one can show

$$\boxed{\omega_{(y)} = \det\left(\frac{\partial x}{\partial y}\right)\omega_{(x)}} \qquad \text{scalar density}$$

12.4 Integration on <u>one</u> chart domain U

Definition 63.

$$\boxed{\int_U f :\stackrel{(U,y)}{=} \int_{y(U)} d^d\beta\,\omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta)} \tag{12.1}$$

Proof. : Check that it's (well-defined), how it changes under change of charts

$$\int_U f :\stackrel{(U,y)}{=} \int_{y(U)} d^d\beta\,\omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta) = \underset{(U,y)}{=} \int_{x(U)} \int d^d\alpha \left|\det\left(\frac{\partial y}{\partial x}\right)\right| f_{(x)}(\alpha)\omega_{(x)}(x^{-1}(\alpha)\det\left(\frac{\partial x}{\partial y}\right) =$$

$$= \int_{x(U)} d^d\alpha\,\omega_{(x)}(x^{-1}(x)) f_{(x)}(\alpha)$$

□

On an oriented metric manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$

$$\int_U f := \int_{x(U)} d^d\alpha \underbrace{\sqrt{\det(g_{ij}(x))(x^{-1}(\alpha))}}_{\sqrt{g}} f_{(x)}(\alpha)$$

12.5 Integration on the entire manifold

13 Lecture 13: Relativistic spacetime

Recall, from Lecture 9, the definition of Newtonian spacetime

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t) \qquad \begin{array}{l} \nabla \text{ torsion free} \\ t \in C^{\infty}(M) \\ dt \neq 0 \\ \nabla dt = 0 \quad \text{(uniform time)} \end{array}$$

and the definition of relativistic spacetime (before Lecture)

$$(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T) \qquad \begin{array}{l} \nabla \text{ torsion-free} \\ g \text{ Lorentzian metric}(+ - --) \\ T \text{ time-orientation} \end{array}$$

13.1 Time orientation

Definition 64. $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$ a Lorentzian manifold. Then a time-orientation is given by a vector field T that

(i) does **not** vanish anywhere

(ii) $g(T,T) > 0$

Newtonian vs. relativistic
Newtonian
X was called future-directed if

$$dt(X) > 0$$

$\forall\, p \in M$, take half plane, half space of T_pM
also stratified atlas so make planes of constant t straight
relativistic
half cone $\forall\, p, q \in M$, half-cone $\subseteq T_pM$

This definition of <u>spacetime</u>

Question
I see how the cone structure arises from the new metric. I don't understand however, how the T, the time orientation, comes in

Answer
$(M, \mathcal{O}, \mathcal{A}, g)$ $g \overset{(}{\leftarrow} + - --)$

requiring $g(X,X) > 0$, select cones
T chooses which cone

This definition of <u>spacetime</u> has been made to enable the following physical postulates:

(P1) The worldline γ of a <u>massive</u> particle satisfies

   (i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$

   (ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$

(P2) Worldlines of massless particles satisfy

(i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0$

(ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$

picture: spacetime:

Answer (to a question) T is a smooth vector field, T determines future vs. past, "general relativity: we have such a time orientation; smoothness makes it less arbitrary than it seems" -FSchuller,

Claim: 9/10 of a metric are determined by the cone

spacetime determined by distribution, only one-tenth error

13.2 Observers

$(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$
Definition 65. An observer is a worldline γ with

$$g(v_\gamma, v_\gamma) > 0$$
$$g(T, v_\gamma) > 0$$

together with a choice of basis

$$v_{\gamma,\gamma(\lambda)} \equiv e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$

of each $T_{\gamma(\lambda)}M$ where the observer worldline passes, if $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{ab}$

precise: observer = smooth curve in the frame bundle LM over M

13.2.1 Two physical postulates

(P3) A **clock** carried by a specific observer (γ, e) will measure a **time**

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

between the two "events"

$$\gamma(\lambda_0) \qquad \text{“start the clock”}$$

and

$$\gamma(\lambda_1) \qquad \text{“stop the clock”}$$

Compare with Newtonian spacetime:

$$t(p) = 7$$

Thought bubble: proper time/eigentime τ

$$\underline{\text{Application/Example.}} \quad \begin{aligned} &M = \mathbb{R}^4 \\ &\mathcal{O} = \mathcal{O}_{\text{st}} \\ &\mathcal{A} \ni (\mathbb{R}^4, \text{id}_{\mathbb{R}^4}) \\ &g : g_{(x)ij} = \eta_{ij} \quad ; \qquad T^i_{(x)} = (1,0,0,0)^i \end{aligned}$$

$$\Longrightarrow \Gamma^i_{(x)\ jk} = 0 \text{ everywhere}$$

$\Longrightarrow (M, \mathcal{O}, \mathcal{A}^{\uparrow}, g, T, \nabla) \qquad \text{Riemm} = 0$
$\Longrightarrow$ spacetime is flat

This situation is called special relativity.

Consider two observers:

$$\begin{aligned} &\gamma : (0,1) \longrightarrow M \\ &\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i \\ &\qquad\qquad \delta : (0,1) \longrightarrow M \\ &\alpha \in (0,1) : \delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases} \end{aligned}$$

let's calculate:

$$\begin{aligned} \tau_\gamma &:= \int_0^1 \sqrt{g_{(x)ij} \dot{\gamma}^i_{(x)} \dot{\gamma}^j_{(x)}} = \int_0^1 d\lambda 1 = 1 \\ \tau_\delta &:= \int_0^{1/2} d\lambda \sqrt{1-\alpha^2} + \int_{1/2}^1 \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1-\alpha^2} = \sqrt{1-\alpha^2} \end{aligned}$$

Note: piecewise integration

Taking the clock postulate (P3) seriously, one better come up with a realistic clock design that supports the postulate. <u>idea</u>.

2 little mirrors

(P4) <u>Postulate</u>

Let (γ, e) be an observer, and
δ be a *massive* particle worldline that is parametrized s.t. $g(v_\gamma, v_\gamma) = 1$ (for parametrization/normalization convenience)

Suppose the observer and the particle *meet* somewhere (in spacetime)

$$\delta(\tau_2) = p = \gamma(\tau_1)$$

This observer measures the 3-velocity (spatial velocity) of this particle as

$$v_\delta : \epsilon^\alpha(v_{\delta, \delta(\tau_2)}) e_\alpha \qquad \alpha = 1,2,3 \tag{13.1}$$

where $\epsilon^0, \boxed{\epsilon^1, \epsilon^2, \epsilon^3}$ is the unique dual basis of $e_0, \boxed{e_1, e_2, e_3}$

EY:20150407

There might be a major correction to Eq. (13.1) from the Tutorial 14 : Relativistic spacetime, matter, and Gravitation, see the second exercise, Exercise 2, third question:

$$v := \frac{\epsilon^\alpha(v_\delta)}{\epsilon^0(v_\delta)} e_\alpha \tag{13.2}$$

Consequence: An observer (γ, e) will extract quantities measurable in his laboratory from objective spacetime quantities always like that.

Ex: F Faraday $(0,2)$-tensor of electromagnetism:

$$F(e_a, e_b) = F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

observer frame e_a, e_b

$E_\alpha := F(e_0, e_\alpha)$
$B^\gamma := F(e_\alpha, e_\rho)\epsilon^{\alpha\beta\gamma}$ where $\epsilon^{123} = +1$ totally antisymmetric

13.3 Role of the Lorentz transformations

Lorentz transformations emerge as follows:
Let (γ, e) and $(\widetilde{\gamma}, \widetilde{e})$ be observers with $\gamma(\tau_1) = \widetilde{\gamma}(\tau_2)$

(for simplicity $\gamma(0) = \widetilde{\gamma}(0)$

Now

$$\begin{aligned} e_0, \ldots, e_1 \qquad & \text{at } \tau = 0 \\ \text{and } \widetilde{e}_0, \ldots, \widetilde{e}_1 \qquad & \text{at } \tau = 0 \end{aligned}$$

both bases for the same $T_{\gamma(0)}M$

Thus: $\widetilde{e}_a = \Lambda^b{}_a e_b \qquad \Lambda \in GL(4)$

Now:

$$\begin{aligned} \eta_{ab} = g(\widetilde{e}_a, \widetilde{e}_b) = g(\Lambda^m{}_a e_m, \Lambda^n{}_b e_n) = \\ = \Lambda^m{}_a \Lambda^n{}_b \underbrace{g(e_m, e_n)}_{\eta_{mn}} \end{aligned}$$

i.e. $\Lambda \in O(1,3)$

Result: Lorentz transformations relate the *frames* of *any two observers* at the same point.

"$\widetilde{x}^\mu - \Lambda^\mu{}_\nu x^\nu$" is utter nonsense

Tutorial

I didn't see a tutorial video for this lecture, but I saw that the Tutorial sheet number 14 had the relevant topics. Go there.

14 Lecture 14: Matter

two types of matter

point matter

field matter

point matter

massive point particle

more of a phenomenological importance

field matter

electromagnetic field

more fundamental from the GR point of view

both classical matter types

14.1 Point matter

Our postulates (P1) and (P2) already constrain the possible particle worldlines.

But what is their precise law of motion, possibly in the presence of "forces",

(a) without external forces

$$S_{\text{massive}}[\gamma] := m \int d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

with:

$$g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$$

dynamical law Euler-Lagrange equation

similarly

$$S_{\text{massless}}[\gamma, \mu] = \int d\lambda \mu g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})$$

$$\begin{array}{ll} \delta_\mu & g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0 \\ \delta_\gamma & \text{e.o.m.} \end{array}$$

Reason for describing equations of motion by actions is that composite systems have an action that is the sum of the actions of the parts of that system, possibly including "interaction terms."

Example.

$$S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$$

(b) presence of external forces
or rather presence of fields to which a particle "couples"

Example

$$S[\gamma; A] = \int d\lambda m \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})} + qA(v_{\gamma,\gamma(\lambda)})$$

where A is a **covector field** on M. A fixed (e.g. the electromagnetic potential)

Consider Euler-Lagrange eqns. $L_{\text{int}} = qA_{(x)}\dot{\gamma}^m_{(x)}$

$$m(\nabla_{v_\gamma} v_\gamma)_a + \underbrace{\left(\frac{\partial \dot{L}_{\text{int}}}{\partial \cdot^{\,m}_{(x)}}\right)\gamma - \frac{\partial L_{\text{int}}}{\partial \gamma^m_{(x)}}}_{*} = 0 \Longrightarrow \boxed{m(\nabla_{v_\gamma} v_\gamma)^a = \underbrace{-qF^a_{\ m}\dot{\gamma}^m}_{\text{Lorentz force on a charged particle in an electromagnetic field}}}$$

$$\frac{\partial L}{\partial \dot{\gamma}^a} = qA_{(x)a}, \qquad \left(\frac{\partial \dot{L}}{\partial \cdot^{\,m}}\right)\gamma = q \cdot \frac{\partial}{\partial x^m}(A_{(x)m}) \cdot \dot{\gamma}^m_{(x)}$$

$$\frac{\partial L}{\partial \gamma^a} = q \cdot \frac{\partial}{\partial x^a}(A_{(x)m})\dot{\gamma}^m$$

$$* = q\left(\frac{\partial A_a}{\partial x^m} - \frac{\partial A_m}{\partial x^a}\right)\dot{\gamma}^m_{(x)} \quad = q \cdot F_{(x)am}\dot{\gamma}^m_{(x)}$$

$F \leftarrow$ Faraday

$$S[\gamma] = \int (m\sqrt{g(v_\gamma, v_\gamma)} + qA(v_\gamma))d\lambda$$

14.2 Field matter

Definition 66. Classical (non-quantum) field matter is any tensor field on spacetime where equations of motion derive from an action.

Example:

$$S_{\text{Maxwell}}[A] = \frac{1}{4}\int_M d^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd}$$

A $(0,1)$-tensor field
= thought cloud: for simplicity one chart covers all of M
$-$ for $\sqrt{-g}$ $(+---)$

$F_{ab} := 2\partial_{[a}A_{b]} = 2(\nabla_{[a}A)_{b]}$

Euler-Lagrange equations for fields

$$0 = \frac{\partial \mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s}\left(\frac{\partial \mathcal{L}}{\partial \partial_s A_m}\right) + \frac{\partial}{\partial x^s}\frac{\partial}{\partial x^t}\frac{\partial^2 \mathcal{L}}{\partial \partial_t \partial_s A_m}$$

Example ...

$$(\nabla_{\frac{\partial}{\partial x^m}}F)^{ma} = j^a$$

inhomogeneous Maxwell

thought bubble $j = qv_\gamma$

$$\partial_{[a}F_{b]} - ()$$

homogeneous Maxwell

Other example well-liked by textbooks

$$S_{\text{Klein-Gordon}}[\phi] := \int_M d^4x\sqrt{-g}[g^{ab}(\partial_a\phi)(\partial_b\phi) - m^2\phi^2]$$

ϕ $(0,0)$-tensor field

14.3 Energy-momentum tensor of matter fields

At some point, we want to write down an action for the metric tensor field itself.

But then, this action $S_{\text{grav}}[g]$ will be added to any $S_{\text{matter}}[A, \phi, \dots]$ in order to describe the total system.

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]$$

$$\begin{aligned} \delta A \quad & :\Longrightarrow \text{ Maxwell's equations} \\ \delta g_{ab} \quad & : \boxed{\frac{1}{16\pi G} G^{ab}} + (-2T^{ab}) = 0 \end{aligned}$$

G Newton's constant

$$G^{ab} = 8\pi G_N T^{ab}$$

Definition 67. $S_{\text{matter}}[\Phi, g]$ is a matter action, the **so-called energy-momentum tensor** is

$$T^{ab} := \frac{-2}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{ab}} - \partial_s \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_s g_{ab}} + \dots \right)$$

$-$ of $\frac{-2}{\sqrt{g}}$ is Schrödinger minus (EY : 20150408 F.Schuller's joke? but wise)

choose all sign conventions s.t.

$$T(\epsilon^0, \epsilon^0) > 0$$

Example: For S_{Maxwell}:

$$T_{ab} = F_{am} F_{bn} g^{mn} - \frac{1}{4} F_{mn} F^{mn} g_{ab}$$

$T_{ab} \equiv T_{\text{Maxwell}ab}$

$$T(e_0, e_0) = \underline{E}^2 + \underline{B}^2$$

$$T(e_0, e_\alpha) = (E \times B)_\alpha$$

Fact: One often does not specify the fundamental action for some matter, but one is rather satisfied to assume certain properties / forms of

$$T_{ab}$$

Example Cosmology: (homogeneous & isotropic)

perfect fluid

of pressure p and density ρ modelled by

$$T^{ab} = (\rho + p) u^a u^b - p g^{ab}$$

radiative fluid

What is a fluid of photons:

$$T^{ab}_{\text{Maxwell}} g_{ab} = 0$$

observe: $T^{ab}_{\text{p.f.}} g_{ab} \stackrel{!}{=} 0$
$$= (\rho + p) u^a u^b g_{ab} - p \underbrace{g^{ab} g_{ab}}_{4}$$

$$\leftrightarrow \rho_p 04p = 0$$
$$\rho = 3p$$

$p = \frac{1}{3}\rho$

Reconvene at 3 pm? (EY : 20150409 I sent a Facebook (FB) message to the International Winter School on Gravity and Light: there was no missing video; it continues on Lecture 15 immediately)

Tutorial 14: Relativistic Spacetime, Matter and Gravitation

Exercise 2: Lorentz force law.

Question electromagnetic potential.

15 Lecture 15: Einstein gravity

Recall that in Newtonian spacetime, we were able to reformulate the Poisson law $\Delta\phi = 4\pi G_N \rho$ in terms of the Newtonian spacetime curvature as

$$R_{00} = 4\pi G_N \rho$$

R_{00} with respect to ∇_{Newton}

G_N = Newtonian gravitational constant

This prompted Einstein to postulate < 1915 that the relativistic field equations for the Lorentzian metric g of (relativistic) spacetime

$$R_{ab} = 8\pi G_N T_{ab}$$

However, this equation suffers from a problem

LHS: $(\nabla_a R)^{ab} \neq 0$
generically

RHS:

$$(\nabla_a T)^{ab} = 0$$

thought bubble: = formulated from an action

Einstein tried to argue this problem away.

Nevertheless, the equations cannot be upheld.

15.1 Hilbert

Hilbert was a specialist for variational principles.

To find the appropriate left hand side of the gravitational field equations, Hibert suggested to start from an action

$$S_{\text{Hilbert}}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}$$

thought bubble = "simplest action"

<u>aim</u>: varying this w.r.t. metric g_{ab} will result in some tensor

$$G^{ab} = 0$$

15.2 Variation of S_{Hilbert}

$$0 \overset{!}{=} \underbrace{\delta}_{g_i} S_{\text{Hilbert}}[g] = \int_M [\underbrace{\delta\sqrt{-g} g^{ab} R_{ab}}_{1} + \underbrace{\sqrt{-g}\delta g^{ab} R_{ab}}_{2} + \underbrace{\sqrt{-g} g^{ab} \delta R_{ab}}_{3}]$$

$$\text{and } 1: \delta\sqrt{-g} = \frac{-(\det g) g^{mn} \delta g_{mn}}{2\sqrt{-g}} = \frac{1}{2}\sqrt{-g} g^{mn} \delta g_{mn}$$

thought bubble

$$\delta \det(g) = \det(g) g^{mn} \delta g_{mn}$$

e.g. from

$$\det(g) = \exp \operatorname{tr} \ln g$$

ad 2: $g^{ab}g_{bc} = \delta^a_c$

$$\begin{aligned} &\Longrightarrow (\delta g^{ab})g_{bc} + g^{ab}(\delta g_{bc}) = 0 \\ &\Longrightarrow \delta g^{ab} = -g^{am}g^{bn}\delta g_{mn} \end{aligned}$$

ad 3:

$$\begin{aligned} \Delta R_{ab} \underbrace{=}_{\text{normal coords at point}} & \delta\partial_b \Gamma^m_{\ \ am} - \delta\partial_m \Gamma^m_{\ \ ab} + \Gamma\Gamma - \Gamma\Gamma = \\ & = \partial_b \delta\Gamma^m_{\ \ am} - \partial_m \delta\Gamma^m_{\ \ ab} = \\ & = \nabla_b(\delta\Gamma)^m_{\ \ am} - \nabla_m(\delta\Gamma)^m_{\ \ ab} \\ & \Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab} = \sqrt{-g} \end{aligned}$$

"if you formulate the variation properly, you'll see the variation δ commute with ∂_b" EY : 20150408 I think one uses the integration at the bounds, integration by parts trick

$\Gamma^i_{(x)\ jk} - \widetilde{\Gamma}^i_{(x)\ jk}$ are the components of a $(1,2)$-tensor.

Notation: $(\nabla_b A)^i_{\ \ g} =: A^i_{\ \ j;b}$

$$\begin{aligned} & \Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab} \\ \underbrace{=}_{\nabla g = 0} & \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ \ am})_{;b} - \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ \ ab})_{;m} = \sqrt{-g}A^b_{\ \ ;b} - \sqrt{-g}B^m_{\ \ ,m} \end{aligned}$$

Question: Why is the difference of coefficients a tensor?

Answer:

$$\Gamma^i_{(y)\ jk} = \frac{\partial y^i}{\partial x^m}\frac{\partial x^m}{\partial y^j}\frac{\partial x^q}{\partial y^k}\Gamma^m_{(x)\ \ ,nq} + \frac{\partial y^i}{\partial x^m}\frac{\partial^2 x^m}{\partial y^j \partial y^k}$$

Collecting terms, one obtains

$$\begin{aligned} 0 \overset{!}{=} \delta S_{\text{Hilbert}} = \int_M [\frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}g^{ab}R_{ab} - \sqrt{-g}g^{am}g^{bn}\delta g_{mn}R_{ab} + \underbrace{(\sqrt{-g}A^a)_{,a}}_{\text{surface}} - \underbrace{(\sqrt{-g}B^b)_{,b}}_{\text{surface term}}] \\ = \int_M \sqrt{-g}\delta \underbrace{g_{mn}}_{\text{arbitrary variation}} [\frac{1}{2}g^{mn}R - R^{mn}] \Longrightarrow G^{mn} = R^{mn} - \frac{1}{2}g^{mn}R \end{aligned}$$

Hence Hilbert, from this "mathematical" argument, concluded that one may take

$$\boxed{R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab}}$$

Einstein equations

$$S_{E-H}[g] = \int_M \sqrt{-g}R$$

15.3 Solution of the $\nabla_a T^{ab} = 0$ issue

One can show ($\longrightarrow$ Tutorials) that the Einstein curvature

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$

satisfy the so-called contracted differential Bianchi identity

$$(\nabla_a G)^{ab} = 0$$

15.4 Variants of the field equations

(a) a simple rewriting:

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_N T_{ab} = T_{ab}$$

$G_N = \frac{1}{8\pi}$

Contract on both sides g^{ab}

$$\begin{aligned} R_{ab} - \frac{1}{2} g_{ab} R &= T_{ab} || g^{ab} \\ R - 2R = T &:= T_{ab} g^{ab} \\ \implies R &= -T \\ \implies R_{ab} + \frac{1}{2} g_{ab} T &= T_{ab} \\ \iff R_{ab} = (T_{ab} - \frac{1}{2} T g_{ab}) &=: \widehat{T}_{ab} \end{aligned}$$

$$\boxed{R_{ab} = \widehat{T}_{ab}}$$

(b)

$$S_{E-H}[g] := \int_M \sqrt{-g}(R + 2\Lambda)$$

thought bubble: Λ cosmological constant

History:

1915: $\Lambda < 0$ (Einstein) in order to get a non-expanding universe

>1915: $\Lambda = 0$ Hubble

today $\Lambda > 0$ to account for an accelerated expansion

$\Lambda \neq 0$ can be interpreted as a contribution

$-\frac{1}{2}\Lambda g$ to the energy-momentum

"dark energy"

Question: surface terms scalar?

Answer: for a careful treatment of the surface terms which we discarded, see, e.g. E. Poisson, "A relativist's toolkit" C.U.P. "excellent book"

Question: What is a constant on a manifold?

Answer: $\int \sqrt{-g}\Lambda = \Lambda \int \sqrt{-g} 1$

[back to dark energy]

[Weinberg, QCD, calculated]
idea: 1 could arise as the vacuum energy of the standard model fields

$\Lambda_{\text{calculated}} = 10^{120} \times \Lambda_{\text{obs}}$

"worst prediction of physics"

Tutorials: check that

- Schwarzscheld metric (1916)

- FRW metric
- pp-wave metric
- Reisner-Nordstrom

$\Longrightarrow$ are solutions to Einstein's equations

in high school

$m\ddot{x} + m\omega^2 x^2 = 0$

$x(t) = \cos(\omega t)$

<u>ET</u>: [elementary tutorials]

study motion of particles & observers in Schwarzscheld S.T.

<u>Satellite</u>: Marcus C. Werner

Gravitational lensing

odd number of pictures Morse theory (EY:20150408 Morse Theory !!!)

<u>Domenico Giulini</u>

Hamiltonian form Canonical Formulations

Key to Quantum Gravity

16 Lecture 22: Black Holes

Only depends on Lectures 1-15, so does lecture on "Wednesday"

Schwarzschild solution also vacuum solution (from tutorial EY : oh no, must do tutorial)

Study the Schwarzschild as a vacuum solution of the Einstein equation:

$m = G_N M$ where M is the "mass"

$$g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \frac{1}{1 - \frac{2m}{r}} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi)$$

in the so-called Schwarzschild coordinates

t	r	θ	φ
$(-\infty, \infty)$	$(0, \infty)$	$(0, \pi)$	$(0, 2\pi)$

What staring at this metric for a while, two questions naturally pose themselves:

(i) What exactly happens $r = 2m$?

t	r	θ	φ
$(-\infty, \infty)$	$(0, 2m) \cup (2m, \infty)$	$(0, \pi)$	$(0, 2\pi)$

(ii) Is there anything (in the real world) beyond $t \longrightarrow -\infty$?
$t \longrightarrow +\infty$

idea: Map of Linz, blown up

Insight into these two issues is afforded by stopping to stare.

Look at *geodesic* of g, instead.

16.1 Radial null geodesics

null - $g(v_\gamma, v_\gamma) = 0$

Consider null geodesic in "Schd"

$$S[\gamma] = \int d\lambda \left[\left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2) \right]$$

with $[\dots] = 0$

and one has, in particular, the t-eqn. of motion:

$$\left(\left(1 - \frac{2m}{r}\right)\dot{t}\right)^{\cdot} = 0$$

$\Longrightarrow$

$$\boxed{\left(1 - \frac{2m}{r}\right)\dot{t} = k} = \text{ const.}$$

Consider radial null geodesics
$\theta \overset{!}{=}$ const. $\qquad \varphi =$ const.

From □ and □

$$\Longrightarrow \dot{r}^2 = k^2 \leftrightarrow \dot{r} = \pm k$$

$$\Longrightarrow r(\lambda) = \pm k \cdot \lambda$$

Hence, we may consider

$$\widetilde{t}(r) := t(\pm k\lambda)$$

Case A: $\oplus$

$$\frac{d\widetilde{t}}{dr} = \frac{\dot{\widetilde{t}}}{\dot{r}} = \frac{k}{\left(1 - \frac{2m}{r}\right)k} = \frac{r}{r-2m}$$

$$\Longrightarrow \widetilde{t}_+(r) = r + 2m \ln|r-2m|$$

(**outgoing** null geodesics)

Case b. $\pm$ (Circle around $-$, consider $-$):

$$\widetilde{t}_-(r) = -r - 2m \ln|r-2m|$$

(**ingoing** null geodesics)

Picture

16.2 Eddington-Finkelstein

Brilliantly simple idea:

change (on the domain of the Schwarzschild coordinates) to different coordinates, s.t. in those new coordinates,
ingoing null geodesics appear as straight lines, of slope -1

This is achieved by

$$\bar{t}(t, r, \theta, \varphi) := t + 2m \ln|r-2m|$$

Recall: ingoing null geodesic has

$$\widetilde{t}(r) = -(r + 2m \ln|r-2m|) \qquad (Schdcoords)$$

$$\Longleftrightarrow \bar{t} - 2m \ln|r-2m| = -r - 2m \ln|r-2m| + \text{ const.}$$

$$\therefore \bar{t} = -r + \text{ const.}$$

(Picture)

outgoing null geodesics

$$\bar{t} = r + 4m \ln|r-2m| + \text{ const.}$$

Consider the new chart (V, g) while (U, x) was the Schd chart.

$$\underbrace{U}_{\text{Schd}} \bigcup \{ \text{ horizon } \} = V$$

"chart image of the horizon"

Now calculate the *Schd metric* g w.r.t. Eddington-Finkelstein coords.

$$\begin{aligned}
\bar{t}(t, r, \theta, \varphi) &= t + 2m \ln |r - 2m| \\
\bar{r}(t, r, \theta, \varphi) &= r \\
\bar{\theta}(t, r, \theta, \varphi) &= \theta \\
\bar{\varphi}(t, r, \theta, \varphi) &= \varphi
\end{aligned}$$

EY : 20150422 I would suggest that after seeing this, one would calculate the metric by your favorite CAS. I like the Sage Manifolds package for Sage Math.

Schwarzschild_BH.sage on github

Schwarzschild_BH.sage on Patreon

Schwarzschild_BH.sage on Google Drive

```
sage: load(``Schwarzschild_BH.sage'')
4-dimensional manifold 'M'
  expr = expr.simplify_radical()
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
Launched png viewer for Graphics object consisting of 4 graphics primitives
```

Then calculate the Schwarzschild metric g but in Eddington-Finkelstein coordinates. Keep in mind to calculate the set of coordinates that uses $\bar{t}$, not $\widetilde{t}$:

```
sage: gI.display()
gI = (2*m - r)/r dt*dt - r/(2*m - r) dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: gI.display( X_EF_I_null.frame())
gI = (2*m - r)/r dtbar*dtbar + 2*m/r dtbar*dr + 2*m/r dr*dtbar + (2*m + r)/r dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
```

References

[1] Eric Poisson, **A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics**, Cambridge University Press, 2004. ISBN 0 521 83091 5