



# Lecture Notes on General Relativity (GR)

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### Abstract

These are lecture notes on General Relativity.

They are based on the [Central Lecture Course](#) by **Dr. Frederic P. Schuller** (**A thorough introduction to the theory of general relativity**) introducing the mathematical and physical foundations of the theory in 24 self-contained lectures at the International Winter School on Gravity and Light in Linz/Austria for the WE Heraeus International Winter School of Gravity and Light, 2015 in Linz as part of the world-wide celebrations of the 100th anniversary of Einstein's theory of general relativity and the International Year of Light 2015.

These lectures develop the theory from first principles and aim at an audience ranging from ambitious undergraduate students to beginning PhD students in mathematics and physics. Satellite Lectures (see other videos on this channel) by Bernard F Schutz (Gravitational Waves), Domenico Giulini (Canonical Formulation of Gravity), Marcus C Werner (Gravitational Lensing) and Valeria Pettorino (Cosmic Microwave Background) expand on the topics of this central lecture course and take students to the research frontier.

Spacetime is the physical key object, we shall be concerned about.

**Spacetime** is a **4-dimensional topological manifold** with a **smooth atlas** carrying a **torsion-free connection** compatible with a **Lorentzian metric** and a **time orientation** satisfying the **Einstein equations**.

# 1 Lecture 1: Topology

**Motivation:** At the coarsest level, spacetime is a set. But, a set is not enough to talk about continuity of maps, which is required for classical physics notions such as trajectory of a particle. We do not want jumps such as a particle disappearing at some point on its trajectory and appearing somewhere. So we require continuity of maps. There could be many structures that allow us to talk about continuity, e.g., distance measure. But we need to be very minimal and very economic in order not to introduce undue assumptions. So we are interested in the weakest structure that can be established on a set which allows a good definition of continuity of maps. Mathematicians know that the weakest such structure is topology. This is the reason for studying topological spaces.

## 1.1 Topological Spaces

**Definition 1.** Let  $M$  be a set and  $\mathcal{P}(M)$  be the power set of  $M$ , i.e., the set of all subsets of  $M$ . A set  $\mathcal{O} \subseteq \mathcal{P}(M)$  is called a **topology**, if it satisfies the following:

- (i)  $\emptyset \in \mathcal{O}, M \in \mathcal{O}$
- (ii)  $U \in \mathcal{O}, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
- (iii)  $U_\alpha \in \mathcal{O}, \alpha \in \mathcal{A} \text{ (}\mathcal{A} \text{ is an index set)} \implies \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha\right) \in \mathcal{O}$

**Terminology:**

- 1. the tuple  $(M, \mathcal{O})$  is a **topological space**,
- 2.  $U \in M$  is an **open set** if  $U \in \mathcal{O}$
- 3.  $U \in M$  is a **closed set** if  $M \setminus U \in \mathcal{O}$

**Definition 2.**  $(M, \mathcal{O})$ , where  $\mathcal{O} = \{\emptyset, M\}$  is called the **chaotic topology**.

**Definition 3.**  $(M, \mathcal{O})$ , where  $\mathcal{O} = \mathcal{P}(M)$  is called the **discrete topology**.

**Definition 4.** A **soft ball** at the point  $p$  in  $\mathbb{R}^d$  is the set

$$\mathcal{B}_r(p) := \left\{ (q_1, q_2, \dots, q_d) \mid \sum_{i=1}^d (q_i - p_i)^2 < r^2 \right\} \text{ where } r \in \mathbb{R}^+$$

**Definition 5.**  $(\mathbb{R}^d, \mathcal{O}_{std})$  is the **standard topology**, provided that  $U \in \mathcal{O}_{std}$  iff  $\forall p \in U, \exists r \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq U$

*Proof.*  $\emptyset \in \mathcal{O}_{std}$  since  $\forall p \in \emptyset, \exists r \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq \emptyset$  (i.e. satisfied “vacuously”)  
 $\mathbb{R}^d \in \mathcal{O}_{std}$  since  $\forall p \in \mathbb{R}^d, \exists r = 1 \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq \mathbb{R}^d$

Suppose  $U, V \in \mathcal{O}_{std}$ . Let  $p \in U \cap V \implies \exists r_1, r_2 \in \mathbb{R}^+$  s.t.  $\mathcal{B}_{r_1}(p) \subseteq U, \mathcal{B}_{r_2}(p) \subseteq V$ .

Let  $r = \min\{r_1, r_2\} \implies \mathcal{B}_r(p) \subseteq U$  and  $\mathcal{B}_r(p) \subseteq V \implies \mathcal{B}_r(p) \subseteq U \cap V \implies U \cap V \in \mathcal{O}_{std}$ .

Suppose,  $U_\alpha \in \mathcal{O}_{std}, \forall \alpha \in \mathcal{A}$ . Let  $p \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha \implies \exists \alpha \in \mathcal{A} : p \in U_\alpha$   
 $\implies \exists r \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha \implies \bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{O}_{std}$ . □

## 1.2 Continuous maps

A map  $f, f : M \longrightarrow N$ , connects each element of a set  $M$  (domain set) to an element of a set  $N$  (target set).

**Terminology:**

1. If  $f$  maps  $m \in M$  to  $n \in N$ , then we may say  $f(m) = n$ , or  $m$  maps to  $n$ , or  $m \mapsto f(m)$  or  $m \mapsto n$ .
2. If  $V \subseteq N$ ,  $\text{preim}_f(V) := \{m \in M \mid f(m) \in V\}$
3. If  $\forall n \in N, \exists m \in M : n = f(m)$ , then  $f$  is **surjective**. Or,  $f : M \twoheadrightarrow N$ .
4. If  $m_1, m_2 \in M, m_1 \neq m_2 \implies f(m_1) \neq f(m_2)$ , then  $f$  is **injective**. Or,  $f : M \hookrightarrow N$ .

**Definition 6.** Let  $(M, \mathcal{O}_M)$  and  $(N, \mathcal{O}_N)$  be topological spaces. A map  $f : M \longrightarrow N$  is called **continuous** w.r.t.  $\mathcal{O}_M$  and  $\mathcal{O}_N$  if

$$V \in \mathcal{O}_N \implies (\text{preim}_f(V)) \in \mathcal{O}_M.$$

**Mnemonic:** A map is continuous iff the preimages of all open sets are open sets.

### 1.3 Composition of continuous maps

**Definition 7.** If  $f : M \longrightarrow N$  and  $g : N \longrightarrow P$ , then

$$g \circ f : M \longrightarrow P \text{ such that } m \mapsto (g \circ f)(m) := g(f(m))$$

**Theorem 1.** If  $f : M \longrightarrow N$  is continuous w.r.t.  $\mathcal{O}_M$  and  $\mathcal{O}_N$  and  $g : N \longrightarrow P$  is continuous w.r.t.  $\mathcal{O}_N$  and  $\mathcal{O}_P$ , then  $g \circ f : M \longrightarrow P$  is continuous w.r.t.  $\mathcal{O}_M$  and  $\mathcal{O}_P$ .

*Proof.* Let  $W \in \mathcal{O}_P$ .

$$\begin{aligned} \text{preim}_{g \circ f}(W) &= \{m \in M \mid g(f(m)) \in W\} && \because (g \circ f)(m) = g(f(m)) \\ &= \{m \in M \mid f(m) \in \text{preim}_g(W)\} && \text{preim}_g(W) \in \mathcal{O}_N \because g \text{ is continuous} \\ &= \text{preim}_f(\text{preim}_g(W)) && \in \mathcal{O}_M \because f \text{ is continuous} \\ &\implies g \circ f \text{ is continuous} \end{aligned}$$

□

### 1.4 Inheriting a topology

Given a topological space  $(M, \mathcal{O}_M)$ , one way of inheriting a topology from it is the subspace topology.

**Theorem 2.** If  $(M, \mathcal{O}_M)$  is a topological space and  $S \subseteq M$ , then the set  $\mathcal{O}|_S \subseteq \mathcal{P}(S)$  such that  $\mathcal{O}|_S := \{S \cap U \mid U \in \mathcal{O}_M\}$  is a topology.  $\mathcal{O}|_S$  is called the **subspace topology** inherited from  $\mathcal{O}_M$ .

*Proof.* 1.  $\emptyset \in \mathcal{O}_M \implies S \cap \emptyset \in \mathcal{O}|_S \implies \emptyset \in \mathcal{O}|_S$

2.  $M \in \mathcal{O}_M \implies S \cap M \in \mathcal{O}|_S \implies S \in \mathcal{O}|_S$

3.

$$\begin{aligned} S_1, S_2 \in \mathcal{O}|_S &\implies \exists U_1, U_2 \in \mathcal{O}_M : S_1 = S \cap U_1, S_2 = S \cap U_2 \\ &\implies U_1 \cap U_2 \in \mathcal{O}_M \\ &\implies S \cap (U_1 \cap U_2) \in \mathcal{O}|_S \\ &\implies (S \cap U_1) \cap (S \cap U_2) \in \mathcal{O}|_S \\ &\implies S_1 \cap S_2 \in \mathcal{O}|_S \end{aligned}$$

4. Let  $\alpha \in \mathcal{A}$ , where  $\mathcal{A}$  is an index set. Then  $S_\alpha \in \mathcal{O}|_S \implies \exists U_\alpha \in \mathcal{O}_M : S_\alpha = S \cap U_\alpha$ .

Further, let  $\mathcal{U} = (\bigcup_{\alpha \in \mathcal{A}} U_\alpha)$ . Therefore,  $\mathcal{U} \in \mathcal{O}_M$ .

Now,  $(\bigcup_{\alpha \in \mathcal{A}} S_\alpha) = (\bigcup_{\alpha \in \mathcal{A}} (S \cap U_\alpha)) = S \cap (\bigcup_{\alpha \in \mathcal{A}} U_\alpha) = S \cap \mathcal{U} \implies (\bigcup_{\alpha \in \mathcal{A}} S_\alpha) \in \mathcal{O}|_S$ .

□

**Theorem 3.** *If  $(M, \mathcal{O}_M)$  and  $(N, \mathcal{O}_N)$  are topological spaces, and  $f : M \longrightarrow N$  is continuous w.r.t  $\mathcal{O}_M$  and  $\mathcal{O}_N$ , then the restriction of  $f$  to  $S \subseteq M$ ,  $f|_S : S \longrightarrow N$  s.t.  $f|_S(s \in S) = f(s)$ , is continuous w.r.t  $\mathcal{O}|_S$  and  $\mathcal{O}_N$ .*

*Proof.* Let  $V \in \mathcal{O}_N$ . Then,  $\text{preim}_f(V) \in \mathcal{O}_M$ .

Now  $\text{preim}_{f|_S}(V) = S \cap \text{preim}_f(V) \implies \text{preim}_{f|_S}(V) \in \mathcal{O}|_S \implies f|_S$  is continuous.

□

## 2 Lecture 2: Manifolds

**Motivation:** There exist so many topological spaces that mathematicians cannot even classify them. For spacetime physics, we may focus on topological spaces  $(M, \mathcal{O})$  that can be charted, analogously to how the surface of the earth is charted in an atlas.

### 2.1 Topological manifolds

**Definition 8.** A topological space  $(M, \mathcal{O})$  is called a **d-dimensional topological manifold** if  $\forall p \in M : \exists U \in \mathcal{O} : p \in U, \exists x : U \rightarrow x(U) \subseteq \mathbb{R}^d$  satisfying the following:

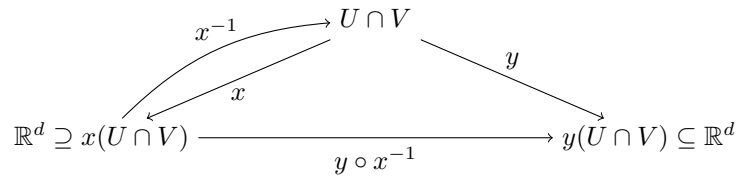
- (i)  $x$  is invertible:  $x^{-1} : x(U) \rightarrow U$
- (ii)  $x$  is continuous w.r.t.  $(M, \mathcal{O})$  and  $(\mathbb{R}^d, \mathcal{O}_{std})$
- (iii)  $x^{-1}$  is continuous

### 2.2 Terminology

1. The tuple  $(U, x)$  is a **chart** of  $(M, \mathcal{O})$ ,
2. An **atlas** of  $(M, \mathcal{O})$  is a set  $\mathcal{A} = \{(U_\alpha, x_\alpha) | \alpha \in A, \text{an index set}\} : \bigcup_{\alpha \in A} U_\alpha = M$ .
3. The map  $x : U \rightarrow x(U) \subseteq \mathbb{R}^d$  is called the **chart map**.
4. The chart map  $x$  maps a point  $p \in U$  to a d-tuple of real numbers  $x(p) = (x^1(p), x^2(p), \dots, x^d(p))$ . This is equivalent to d-many maps  $x^i(p) : U \rightarrow \mathbb{R}$ , which are called the **coordinate maps**.
5. If  $p \in U$ , then  $x^i(p)$  is the **ith coordinate of  $p$**  w.r.t. the chart  $(U, x)$ .

### 2.3 Chart transition maps

Imagine 2 charts  $(U, x)$  and  $(V, y)$  with overlapping regions, i.e.,  $U \cap V \neq \emptyset$ .



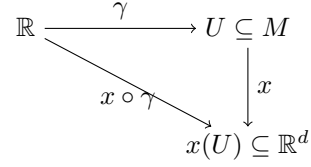
The map  $y \circ x^{-1}$  is called the **chart transition map**, which maps an open set of  $\mathbb{R}^d$  to another open set of  $\mathbb{R}^d$ . This map is continuous because it is composition of two continuous maps. Informally, these chart transition maps contain instructions on how to glue together the charts of an atlas,

### 2.4 Manifold philosophy

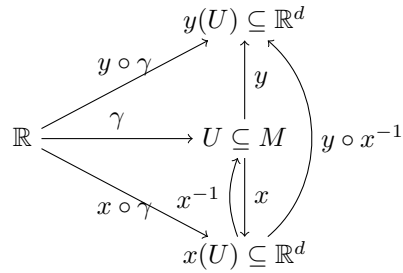
Often it is desirable (or indeed the only way) to define properties (e.g., ‘continuity’) of real-world object (e.g., the curve  $\gamma : \mathbb{R} \rightarrow M$ ) by judging suitable coordinates not on the ‘real-world’ object itself, but on a chart-representation of that real world object.



For example, in the picture below, we can use the map  $x \circ \gamma$  to infer the continuity of the curve  $\gamma$  in  $U \subseteq M$ .



However, we need to ensure that the defined property does not change if we change our chosen chart. For example, in the picture below, continuity in  $x \circ \gamma$  should imply  $y \circ \gamma$ . This is true, since  $y \circ \gamma = y \circ (x^{-1} \circ x) \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma)$  is continuous because it is a composition of two continuous functions, thanks to the continuity of the chart transition map  $y \circ x^{-1}$ .



What about differentiability? Does differentiability of  $x \circ \gamma$  guarantee differentiability of  $y \circ \gamma$ ? No. Since composition of a differentiable map and a continuous map might only be continuous, The solution is to restrict the atlas by removing those charts which are not differentiable. Thus, we have got rid of our problem. However, we must remember that with the present structure, we cannot define differentiability at manifold level since we do not know how to subtract or divide in  $U \subseteq M$ . Therefore, differentiability of  $\gamma : \mathbb{R} \rightarrow M$  makes no sense yet.

### 3 Multilinear Algebra

**Motivation:** The essential object of study of linear algebra is vector space. However, a word of warning here. We will not equip space(time) with vector space structure. This is evident since, unlike in vector space, expressions such as  $5 \cdot \text{Paris}$  and  $\text{Paris} + \text{Vienna}$  do not make any sense. If multilinear algebra does not further our aim of studying spacetime, then why do we study it? The tangent spaces  $T_p M$  (defined in Lecture 5) at a point  $p$  of a smooth manifold  $M$  (defined in Lecture 4) carries a vector space structure in a natural way even though the underlying position space(time) does not have a vector space structure. Once we have a notion of tangent space, we have a derived notion of a tensor. Tensors are very important in differential geometry.

It is beneficial to study vector spaces (and all that comes with it) abstractly for two reasons: (i) for construction of  $T_p M$ , one needs an intermediate vector space  $C^\infty(M)$ , and (ii) tensor techniques are most easily understood in an abstract setting.

#### 3.1 Vector Spaces

**Definition 9.** A  $\mathbb{R}$ -vector space is a triple  $(V, +, \cdot)$ , where

- i)  $V$  is a set,
- ii)  $+: V \times V \longrightarrow V$  (addition), and
- iii)  $\cdot: \mathbb{R} \times V \longrightarrow V$  ( $S$ -multiplication)

satisfying the following:

- a)  $\forall u, v \in V : u + v = v + u$  (commutativity of  $+$ )
- b)  $\forall u, v, w \in V : (u + v) + w = u + (v + w)$  (associativity of  $+$ )
- c)  $\exists O \in V : \forall v \in V : O + v = v$  (neutral element in  $+$ )
- d)  $\forall v \in V : \exists (-v) \in V : v + (-v) = O$  (inverse of element in  $+$ )
- e)  $\forall \lambda, \mu \in \mathbb{R}, \forall v \in V : \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$  (associativity in  $\cdot$ )
- f)  $\forall \lambda, \mu \in \mathbb{R}, \forall v \in V : (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$  (distributivity of  $\cdot$ )
- g)  $\forall \lambda \in \mathbb{R}, \forall u, v \in V : \lambda \cdot u + \lambda \cdot v = \lambda \cdot (u + v)$  (distributivity of  $\cdot$ )
- h)  $\exists 1 \in \mathbb{R} : \forall v \in V : 1 \cdot v = v$  (unit element in  $\cdot$ )

**Terminology:** If  $(V, +, \cdot)$  is a vector space, an element of  $V$  is often referred to, informally, as a **vector**. But, we should remember that it makes no sense to call an element of  $V$  a vector unless the vector space itself is specified.

**Example:** Consider a set of polynomials of fixed degree,

$$P := \left\{ p : (-1, +1) \longrightarrow \mathbb{R} \mid p(x) = \sum_{n=0}^N p_n \cdot x^n, \text{ where } p_n \in \mathbb{R} \right\}$$

with  $\oplus : P \times P \longrightarrow P$  with  $(p, q) \mapsto p \oplus q : (p \oplus q)(x) = p(x) + q(x)$  and

$\odot : \mathbb{R} \times P \longrightarrow P$  with  $(\lambda, p) \mapsto \lambda \odot p : (\lambda \odot p)(x) = \lambda \cdot p(x)$ .  $(P, \oplus, \odot)$  is a vector space.

**Caution:** We are considering real vector spaces, that is  $S$ -multiplication with the elements of  $\mathbb{R}$ . We shall often use same symbols ‘ $+$ ’ and ‘ $\cdot$ ’ for different vector spaces, but the context should make things clear. When  $\mathbb{R}, \mathbb{R}^2$ , etc. are used as vector spaces, the obvious (natural) operations shall be understood to be used.

### 3.2 Linear Maps

These are the structure-respecting maps between vector spaces.

**Definition 10.** If  $(V, +_v, \cdot_v)$  and  $(W, +_w, \cdot_w)$  are vector spaces, then  $\phi : V \longrightarrow W$  is called a **linear map** if

- i)  $\forall v, \tilde{v} \in V : \phi(v +_v \tilde{v}) = \phi(v) +_w \phi(\tilde{v})$ , and
- ii)  $\forall \lambda \in \mathbb{R}, v \in V : \phi(\lambda \cdot_v v) = \lambda \cdot_w \phi(v)$ .

**Notation:**  $\phi : V \longrightarrow W$  is a linear map  $\iff \phi : V \xrightarrow{\sim} W$

**Example:** Consider the vector space  $(P, \oplus, \odot)$  from the above example,

Then,  $\delta : P \longrightarrow P$  with  $p \mapsto \delta(p) := p'$  is a linear map, because

$\forall p, q \in P : \delta(p \oplus q) = (p \oplus q)' = p' \oplus q' = \delta(p) \oplus \delta(q)$  and

$\forall \lambda \in \mathbb{R}, p \in P : \delta(\lambda \odot p) = (\lambda \odot p)' = \lambda \odot p'.$

**Theorem 4.** If  $\phi : U \xrightarrow{\sim} V$  and  $\psi : V \xrightarrow{\sim} W$ , then  $\psi \circ \phi : U \xrightarrow{\sim} W$ .

$$\begin{array}{ccccc} U & \xrightarrow{\phi} & V & \xrightarrow{\psi} & W \\ & \searrow & & \nearrow & \\ & & \psi \circ \phi & & \end{array}$$

*Proof.*  $\forall u, \tilde{u} \in U, (\psi \circ \phi)(u +_u \tilde{u}) = \psi(\phi(u +_u \tilde{u})) = \psi(\phi(u) +_v \phi(\tilde{u})) = \psi(\phi(u)) +_w \psi(\phi(\tilde{u})) = (\psi \circ \phi)(u) +_w (\psi \circ \phi)(\tilde{u}).$

$\forall \lambda \in \mathbb{R}, u \in U, (\psi \circ \phi)(\lambda \cdot_u u) = \psi(\phi(\lambda \cdot_u u)) = \psi(\lambda \cdot_v \phi(u)) = \lambda \cdot_w \psi(\phi(u)) = \lambda \cdot_w (\psi \circ \phi)(u)$  □

**Example:** Consider the vector space  $(P, \oplus, \odot)$  and the differential  $\delta : P \longrightarrow P$  with  $p \mapsto \delta(p) := p'$  from previous example. Then,  $p''$ , the second differential is also linear since it is composition of two linear maps, i.e.,  $\delta \circ \delta : P \xrightarrow{\sim} P$ .

### 3.3 Vector Space of Homomorphisms

**Definition 11.** If  $(V, +, \cdot)$  and  $(W, +, \cdot)$  are vector spaces, then  $Hom(V, W) := \{ \phi : V \xrightarrow{\sim} W \}$ .

**Theorem 5.**  $(Hom(V, W), +, \cdot)$  is a vector space with

$+: Hom(V, W) \times Hom(V, W) \longrightarrow Hom(V, W)$  with  $(\phi, \psi) \mapsto \phi + \psi : (\phi + \psi)(v) = \phi(v) + \psi(v)$  and

$\cdot : \mathbb{R} \times Hom(V, W) \longrightarrow Hom(V, W)$  with  $(\lambda, \phi) \mapsto \lambda \cdot \phi : (\lambda \cdot \phi)(v) = \lambda \cdot \phi(v).$

**Example:**  $(Hom(P, P), +, \cdot)$  is a vector space.  $\delta \in Hom(P, P)$ ,  $\delta \circ \delta \in Hom(P, P)$ ,  $\delta \circ \delta \circ \delta \in Hom(P, P)$ , etc. Therefore, maps such as  $5 \cdot \delta + \delta \circ \delta \in Hom(P, P)$ . Thus, mixed order derivatives are in  $Hom(P, P)$ , and hence linear.

### 3.4 Dual Vector Spaces

**Definition 12.** If  $(V, +, \cdot)$  is a vector space, and  $V^* := \{ \phi : V \xrightarrow{\sim} \mathbb{R} \} = Hom(V, \mathbb{R})$  then  $(V^*, +, \cdot)$  is called the **dual vector space to V**.

**Terminology:**  $\omega \in V^*$  is called, informally, a **covector**.

**Example:** Consider  $I : P \xrightarrow{\sim} \mathbb{R}$ , i.e.,  $I \in P^*$ . We define  $I(p) := \int_0^1 p(x) dx$ , which can be easily checked to be linear with  $I(p + q) = I(p) + I(q)$  and  $I(\lambda \cdot p) = \lambda \cdot I(p)$ . Thus  $I$  is a covector, which is the integration operator  $\int_0^1 ( ) dx$  which eats a function.

*Remarks:* We shall also see later that the gradient is a covector. In fact, lots of things in physicist's life, which are covectors, have been called vectors not to bother you with details. But covectors are neither esoteric nor unnatural.

### 3.5 Tensors

We can think of tensors as multilinear maps.

**Definition 13.** Let  $(V, +, \cdot)$  be a vector space. An  $(\mathbf{r}, \mathbf{s})$ -**tensor**  $T$  over  $V$  is a multilinear map

$$T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{\mathbf{r} \text{ times}} \times \underbrace{V \times V \times \cdots \times V}_{\mathbf{s} \text{ times}} \xrightarrow{\sim} \mathbb{R}$$

**Example:** If  $T$  is a  $(1,1)$ -tensor, then

$$T(\omega_1 + \omega_2, v) = T(\omega_1, v) + T(\omega_2, v),$$

$$T(\omega, v_1 + v_2) = T(\omega, v_1) + T(\omega, v_2),$$

$$T(\lambda \cdot \omega, v) = \lambda \cdot T(\omega, v), \text{ and}$$

$$T(\omega, \lambda \cdot v) = \lambda \cdot T(\omega, v).$$

$$\text{Thus, } T(\omega_1 + \omega_2, v_1 + v_2) = T(\omega_1, v_1) + T(\omega_1, v_2) + T(\omega_2, v_1) + T(\omega_2, v_2).$$

*Remarks:* Sometimes it is said that a  $(1,1)$ -tensor is something that eats a vector and outputs a vector. Here is why. For  $T : V^* \times V \xrightarrow{\sim} \mathbb{R}$ , define  $\phi_T : V \xrightarrow{\sim} (V^*)^*$  with  $v \mapsto T((\cdot), v)$ . But, clearly  $T((\cdot), v) : V^* \xrightarrow{\sim} \mathbb{R}$ , which eats a covector and spits a number. In other words,  $T((\cdot), v) \in (V^*)^*$ . Although we are yet to define dimension, let us just trust, for the time being, that for finite-dimensional vector spaces,  $(V^*)^* = V$ . So,  $\phi_T : V \xrightarrow{\sim} V$ .

**Example:** Let  $g : P \times P \xrightarrow{\sim} \mathbb{R}$  with  $(p, q) \mapsto \int_{-1}^1 p(x) \cdot q(x) dx$ . Then,  $g$  is a  $(0,2)$ -tensor over  $P$ .

### 3.6 Vectors and Covectors as Tensors

**Theorem 6.** If  $(V, +, \cdot)$  is a vector space,  $\omega \in V^*$  is a  $(0,1)$ -tensor.

*Proof.*  $\omega \in V^*$  and, by definition,  $V^* := \left\{ \phi : V \xrightarrow{\sim} \mathbb{R} \right\}$ , which is a collection of  $(0,1)$ -tensors. □

**Theorem 7.** If  $(V, +, \cdot)$  is a vector space,  $v \in V$  is a  $(1,0)$ -tensor.

*Proof.* We have already stated, without proof and without defining dimensions, that  $V = (V^*)^*$  for finite-dimensional vector spaces. Therefore,  $v \in V \implies v \in (V^*)^* \implies v \in \left\{ \phi : V^* \xrightarrow{\sim} \mathbb{R} \right\} \implies v$  is a  $(1,0)$ -tensor. □

### 3.7 Bases

**Definition 14.** Let  $(V, +, \cdot)$  is a vector space. A subset  $B \subseteq V$  is called a **basis** if

$$\forall v \in V, \exists! e_1, e_2, \dots, e_n \in B, \exists! v_1, v_2, \dots, v_n \in \mathbb{R} : v = \sum_{i=1}^n v_i \cdot e_i.$$

**Definition 15.** A vector space  $(V, +, \cdot)$  with a basis  $B$  is said to be  $d$ -**dimensional** if  $B$  has  $d$  elements. In other words,  $\dim V := d$ .

*Remarks:* The above definition is well-defined only if every basis of a vector space has the same number of elements.

**Remarks:** Let  $(V, +, \cdot)$  is a vector space. Having chosen a basis  $e_1, e_2, \dots, e_n$ , we may uniquely associate  $v \mapsto (v_1, v_2, \text{dotsc}, v_n)$ , these numbers being the components of  $v$  w.r.t. chosen basis where  $v = \sum_{i=1}^n v_i \cdot e_i$ .

### 3.8 Basis for the Dual Space

Let  $(V, +, \cdot)$  is a vector space. Having chosen a basis  $e_1, e_2, \dots, e_n$  for  $V$ , we can choose a basis  $\epsilon^1, \epsilon^2, \dots, \epsilon^n$  for  $V^*$  entirely independent of basis of  $V$ . However, it is more economical to require that

$$\epsilon^a(e_b) = \delta_b^a = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

This uniquely determines  $\epsilon^1, \epsilon^2, \dots, \epsilon^n$  from choice of  $e_1, e_2, \dots, e_n$ .

*Remarks: The reason for using indices as superscripts or subscripts is to be able to use the Einstein summation convention, which will be helpful in dropping cumbersome  $\sum$  symbols in several equations.*

**Definition 16.** For a basis  $e_1, e_2, \dots, e_n$  of vector space  $(V, +, \cdot)$ ,  $\epsilon^1, \epsilon^2, \dots, \epsilon^n$  is called the **dual basis** of the dual space, if  $\epsilon^a(e_b) = \delta_b^a$ .

**Example:** Consider polynomials  $P$  of degree 3. Choose  $e_0, e_1, e_2, e_3 \in P$  such that  $e_0(x) = 1, e_1(x) = x, e_2(x) = x^2$  and  $e_3(x) = x^3$ . Then, it can be easily verified that the dual basis is  $\epsilon^a = \frac{1}{a!} \partial^a \Big|_{x=0}$ .

### 3.9 Components of Tensors

**Definition 17.** Let  $T$  be a  $(r, s)$ -tensor over a  $d$ -dimensional (finite) vector space  $(V, +, \cdot)$ . Then, with respect to some basis  $\{e_1, \dots, e_r\}$  and the dual basis  $\{\epsilon^1, \dots, \epsilon^s\}$ , define  $(r + s)^d$  real numbers

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} := T(\epsilon^{i_1}, \dots, \epsilon^{i_r}, e_{j_1}, \dots, e_{j_s})$$

such that the indices  $i_1, \dots, i_r, j_1, \dots, j_s$  take all possible values in the set  $\{1, \dots, d\}$ . These numbers  $T^{i_1 \dots i_r}_{j_1 \dots j_s}$  are called the **components of the tensor**  $T$  w.r.t. the chosen basis.

This is useful because knowing components (and the basis w.r.t which these components have been chosen), one can reconstruct the entire tensor.

**Example:** If  $T$  is a  $(1, 1)$ -tensor, then  $T^i_j := T(\epsilon^i, e_j)$ . Then

$$T(\omega, v) = T\left(\sum_{i=1}^d \omega_i \cdot \epsilon^i, \sum_{j=1}^d v^j \cdot e_j\right) = \sum_{i=1}^d \sum_{j=1}^d \omega_i v^j T(\epsilon^i, e_j) = \sum_{i=1}^d \sum_{j=1}^d \omega_i v^j T^i_j =: \omega_i v^j T^i_j$$

## 4 Differential Manifolds

$$\begin{array}{ccc} \gamma : \mathbb{R} & \xrightarrow{\quad} & U \\ & \searrow x \circ \gamma & \downarrow x \\ & & x(U) \subseteq \mathbb{R}^d \end{array}$$

idea. try to “lift” the undergraduate notion of differentiability of a curve on  $\mathbb{R}^d$  to a notion of differentiability of a curve on  $M$

Problem Can this be well-defined under change of chart?

$$\begin{array}{ccccc} & & y(U \cap V) \subseteq \mathbb{R}^d & & \\ & \nearrow y \circ \gamma & \uparrow y & \nwarrow y \circ x^{-1} & \\ \gamma : \mathbb{R} & \xrightarrow{\quad} & U \cap V \neq \emptyset & & \\ & \searrow x \circ \gamma & \downarrow x & & \\ & & x(U \cap V) \subseteq \mathbb{R}^d & & \end{array}$$

$x \circ \gamma$  undergraduate differentiable (“as a map  $\mathbb{R} \rightarrow \mathbb{R}^d$ ”)

$$\underbrace{y \circ \gamma}_{\text{maybe only continuous, but not undergraduate differentiable}} = \underbrace{(y \circ x^{-1})}_{\text{continuous}} \circ \underbrace{(x \circ \gamma)}_{\text{undergrad differentiable}} = y \circ (x^{-1} \circ x) \circ \gamma$$

At first sight, strategy does not work out.

### 4.1 Compatible charts

In section 1, we used any imaginable charts on the top. mfd.  $(M, \mathcal{O})$ .

To emphasize this, we may say that we took  $U$  and  $V$  from the *maximal atlas*  $\mathcal{A}$  of  $(M, \mathcal{O})$ .

**Definition 18.** Two charts  $(U, x)$  and  $(V, y)$  of a top. mfd. are called  $\ast$ -compatible if either

- (a)  $U \cap V = \emptyset$  or
- (b)  $U \cap V \neq \emptyset$

chart transition maps have undergraduate  $\ast$  property.

EY : 20151109 e.g. since  $\mathbb{R}^d \rightarrow \mathbb{R}^d$ , can use undergraduate  $\ast$  property such as continuity or differentiability.

$$\begin{aligned} y \circ x^{-1} : x(U \cap V) \subseteq \mathbb{R}^d &\longrightarrow y(U \cap V) \subseteq \mathbb{R}^d \\ x \circ y^{-1} : y(U \cap V) \subseteq \mathbb{R}^d &\longrightarrow x(U \cap V) \subseteq \mathbb{R}^d \end{aligned}$$

Philosophy:

**Definition 19.** An atlas  $\mathcal{A}_\ast$  is a  $\ast$ -compatible atlas if any two charts in  $\mathcal{A}_\ast$  are  $\ast$ -compatible.

**Definition 20.** A  $\ast$ -manifold is a triple  $(\underbrace{M, \mathcal{O}}_{\text{top. mfd.}}, \mathcal{A}_\ast)$   $\mathcal{A}_\ast \subseteq \mathcal{A}_{\text{maximal}}$

$\clubsuit$	undergraduate $\clubsuit$	
$C^0$	$C^0(\mathbb{R}^d \rightarrow \mathbb{R}^d) =$	continuous maps w.r.t. $\mathcal{O}$
$C^1$	$C^1(\mathbb{R}^d \rightarrow \mathbb{R}^d) =$	differentiable (once) and is continuous
$C^k$		$k$ -times continuously differentiable
$D^k$		$k$ -times differentiable
$\vdots$		
$C^\infty$	$C^\infty(\mathbb{R}^d \rightarrow \mathbb{R}^d)$	
$\bigcup$		
$C^\omega$	$\exists$ multi-dim. Taylor exp.	
$\mathbb{C}^\infty$	satisfy Cauchy-Riemann equations, pair-wise	

EY : 20151109 Schuller says:  $C^k$  is easy to work with because you can judge  $k$ -times cont. differentiability from existence of all partial derivatives **and** their continuity. There are examples of maps that partial derivatives exist but are not  $D^k$ ,  $k$ -times differentiable.

**Theorem 8** (Whitney). Any  $C^{k \geq 1}$ -atlas,  $\mathcal{A}_{C^{k \geq 1}}$  of a topological manifold contains a  $C^\infty$ -atlas.

Thus we may w.l.o.g. always consider  $C^\infty$ -manifolds, “smooth manifolds”, unless we wish to define Taylor expandibility/complex differentiability ...

EY : 20151109 Hassler Whitney <sup>1</sup>

**Definition 21.** A smooth manifold  $(\underbrace{M, \mathcal{O}}_{\text{top. mfd.}}, \underbrace{\mathcal{A}}_{C^\infty\text{-atlas}})$

$$\begin{array}{ccc}
 \mathbb{R} & \xrightarrow{\gamma} & M \\
 & \searrow x \circ \gamma & \downarrow x \\
 & & \mathbb{R}^d
 \end{array}$$

EY: 20151109 Schuller was explaining that the trajectory is real in  $M$ ; the coordinate maps to obtain coordinates is  $x \circ \gamma$

## 4.2 Diffeomorphisms

$$M \xrightarrow{\phi} N$$

If  $M, N$  are naked sets, the structure preserving maps are the bijections (invertible maps).

e.g.  $\{1, 2, 3\} \rightarrow \{a, b\}$

**Definition 22.**  $M \cong_{\text{set}} N$  (set-theoretically) isomorphic if  $\exists$  bijection  $\phi : M \rightarrow N$

Examples.  $\mathbb{N} \cong_{\text{set}} \mathbb{Z}$

$\mathbb{N} \cong_{\text{set}} \mathbb{Q}$  (EY: 20151109 Schuller says from diagonal counting)

$\mathbb{N} \cong_{\text{set}} \mathbb{R}$

Now  $(M, \mathcal{O}_M) \cong_{\text{top}} (N, \mathcal{O}_N)$  (topl.) isomorphic = “homeomorphic”  $\exists$  bijection  $\phi : M \rightarrow N$   
 $\phi, \phi^{-1}$  are continuous.

$(V, +, \cdot) \cong_{\text{vec}} (W, +_w, \cdot_w)$  (EY: 20151109 vector space isomorphism) if

$\exists$  bijection  $\phi : V \rightarrow W$  linearly

finally

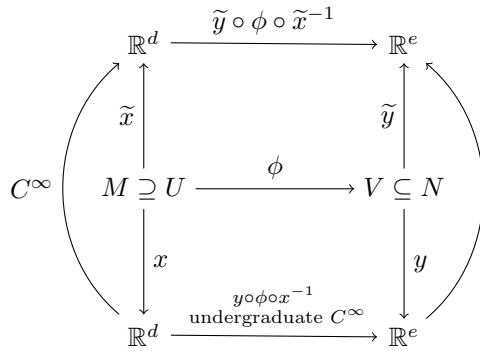
<sup>1</sup><http://mathoverflow.net/questions/8789/can-every-manifold-be-given-an-analytic-structure>

**Definition 23.** Two  $C^\infty$ -manifolds

$(M, \mathcal{O}_M, \mathcal{A}_M)$  and  $(N, \mathcal{O}_N, \mathcal{A}_N)$  are said to be **diffeomorphic** if  $\exists$  bijection  $\phi : M \longrightarrow N$  s.t.

$$\begin{aligned}\phi &: M \longrightarrow N \\ \phi^{-1} &: N \longrightarrow M\end{aligned}$$

are both  $C^\infty$ -maps



**Theorem 9.**  $\#$  = number of  $C^\infty$ -manifolds one can make out of a given  $C^0$ -manifolds (if any) - up to diffeomorphisms.

$\dim M$	$\#$	
1	1	Morse-Radon theorems
2	1	Morse-Radon theorems
3	1	Morse-Radon theorems
4	uncountably infinitely many	
5	finite	surgery theory
6	finite	surgery theory
$\vdots$	finite	surgery theory

EY : 20151109 cf. <http://math.stackexchange.com/questions/833766/closed-4-manifolds-with-uncountably-many-diffeomorphisms>  
The wild world of 4-manifolds



## 5 Lecture 5: Tangent Spaces

lead question: “what is the velocity of a curve  $\gamma$  point  $p$ ?”

### 5.1 Velocities

**Definition 24.**  $(M, \mathcal{O}, \mathcal{A})$  smooth mfd.

curve  $\gamma : \mathbb{R} \rightarrow M$  at least  $C^1$ .

Suppose  $\gamma(\lambda_0) = p$

The **velocity** of  $\gamma$  at  $p$  is the linear map

$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$$

$C^\infty(M) := \{f : M \rightarrow \mathbb{R} \mid f \text{ smooth function} \}$  equipped with  $(f \oplus g)(p) := f(p) + g(p)$

$$(\lambda \otimes g)(p) := \lambda \cdot g(p)$$

$\sim$  denotes linear map on top of  $\rightarrow$ .

$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$$

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\gamma} & M & \xrightarrow{f} & \mathbb{R} \\ & \searrow & & \nearrow & \\ & & f \circ \gamma & & \end{array}$$

intuition

Schuller says: children run around the world. Temperature function as temperature contour lines. You feel the temperature. You observe the rate of change of temperature as you run around.  $f$  is temperature.

past: “ $\underbrace{v^i}_{\text{vector}}(\partial_i f) = (\underbrace{v^i \partial_i}_{\text{vector}})f$ ”

### 5.2 Tangent vector space

**Definition 25.** For each point  $p \in M$

def the **set** “tangent space  $\neq_0 M$  at  $p$ ”

$$T_p M := \{v_{\gamma,p} \mid \gamma \text{ smooth curves} \}$$

picture:

rather  $M$  than (embedded)  $p \in T_p M$  EY : 20151109 see [https://youtu.be/pepU\\_7NJSGM?t=12m38s](https://youtu.be/pepU_7NJSGM?t=12m38s) for the picture

Observation:  $T_p M$  can be made into a vector space.

$$\begin{aligned} \oplus : T_p M \times T_p M &\longrightarrow \\ (v_{\gamma,p} \oplus v_{\delta,p})(\underbrace{f}_{\in C^\infty(M)}) &:= v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f) \\ \odot : \mathbb{R} \times T_p M &\longrightarrow \text{Hom}(C^\infty(\mathbb{R}), \mathbb{R}) \\ (\alpha \odot v_{\gamma,p})(f) &:= \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f) \end{aligned}$$

Remains to be shown that

$$(i) \exists \sigma \text{ curve} : v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$$

$$(ii) \exists \tau \text{ curve} : \alpha \odot v_{\gamma,p} = v_{\tau,p}$$

Claim:  $\tau : \mathbb{R} \longrightarrow M$

where  $\mu_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ , does the trick.

$$\mapsto \tau(\lambda) := \gamma(\alpha\lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \quad r \mapsto \alpha \cdot r + \lambda_0$$

$$\tau(0) = \gamma(\lambda_0) = p$$

$$\begin{aligned} v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) \\ &= (f \circ \gamma)'(\lambda_0) \cdot \alpha = \\ &= \alpha \cdot v_{\gamma,p} \end{aligned}$$

Now for the sum:

$$v_{\gamma,p} \oplus v_{\delta,p} \stackrel{?}{=} v_{\sigma,p}$$

make a choice of chart  $(\underbrace{U}_{\ni p}, x)$  In cloud: ill definition alarm bells.

and define:

Claim:

$$\begin{aligned} \sigma : \mathbb{R} &\longrightarrow M \\ \sigma(\lambda) &:= x^{-1}(\underbrace{(x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \delta)(\lambda_1 + \lambda)}_{\mathbb{R} \longrightarrow \mathbb{R}^d} - (x \circ \gamma)(\lambda_0)) \end{aligned}$$

does the trick.

*Proof.* Since:

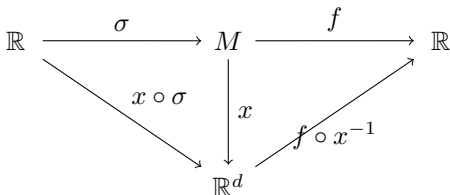
$$\begin{aligned} \sigma_x(0) &= x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_1) - (x \circ \gamma)(\lambda_0)) \\ &= \delta(\lambda_1) = p \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma_x,p}(f) &:= (f \circ \sigma_x)'(0) = \\ &= \underbrace{((f \circ x^{-1}) \circ (x \circ \sigma_x))'(\gamma)}_{\mathbb{R}^d \longrightarrow \mathbb{R}} = \underbrace{(x \circ \sigma_x)'(0)}_{(x \circ \gamma)'(\lambda_0) + (x \circ \delta)'(\lambda_1)} \cdot (\partial_i(f \circ x^{-1}))(x(\underbrace{\sigma(0)}_p)) = \\ &= (x \circ \gamma)'(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) + (x \circ \delta)'(\lambda_1)(\partial_i(f \circ x^{-1}))(x(p)) \\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_1) = \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall f \in C^\infty(M) \end{aligned}$$

$$\boxed{v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}}$$

□



picture: (cf. [https://youtu.be/pepU\\_7NJSGM?t=39m5s](https://youtu.be/pepU_7NJSGM?t=39m5s))

$$\begin{aligned}\gamma : \mathbb{R} &\longrightarrow M \\ \delta : \mathbb{R} &\longrightarrow M\end{aligned}$$

$$(\gamma \oplus)(\lambda) := \gamma(\lambda) + \delta(\lambda)$$

EY : 20151109 Schuller says adding trajectories is chart dependent, bad. Adding velocities is good.

### 5.3 Components of a vector wrt a chart

**Definition 26.** Let  $(U, x) \in \mathcal{A}_{\text{smooth}}$ .

$$\text{Let } \begin{aligned} \gamma : \mathbb{R} &\longrightarrow U \\ \gamma(0) &= p \end{aligned}.$$

Calculate

$$\begin{aligned} v_{\gamma,p}(f) &:= (f \circ \gamma)'(0) = \underbrace{(f \circ x^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}} \circ \underbrace{(x \circ \gamma)}_{\mathbb{R} \longrightarrow \mathbb{R}^d}'(0) \\ &= \underbrace{(x \circ \gamma)^{i'}(0)}_{\dot{\gamma}_x^i(0)} \cdot \underbrace{(\partial_i(f \circ x^{-1}))(x(p))}_{=:(\frac{\partial f}{\partial x^i})_p} \end{aligned}$$

think cloud  $f : M \longrightarrow \mathbb{R}$

$$= \boxed{\dot{\gamma}_x^i(0) \cdot \left( \frac{\partial}{\partial x^i} \right)_p} f \quad \forall f \in C^\infty(M)$$

$\therefore$  as a map.

$$v_{\gamma,p} \underbrace{=}_{\text{use of chart}} \underbrace{\dot{\gamma}_x^i(0)}_{\text{"components of the velocity } v_{\gamma,p}"} \underbrace{\left( \frac{\partial}{\partial x^i} \right)}_{\text{basis elements of the } T_p M \text{ wrt which the components need to be understood. "chart induced basis of } T_p M \text{"}}$$

Picture: [https://youtu.be/pepU\\_7NJSGM?t=1h16s](https://youtu.be/pepU_7NJSGM?t=1h16s)

### 5.4 Chart-induced basis

**Definition 27.**  $(U, x) \in \mathcal{A}_{\text{smooth}}$

the  $\left( \frac{\partial}{\partial x^1} \right)_p, \dots, \left( \frac{\partial}{\partial x^d} \right)_p \in T_p U \subseteq T_p M$

constitute a **basis** of  $T_p U$

*Proof.* remains: linearly independent

$$\begin{aligned} \lambda^i \left( \frac{\partial}{\partial x^i} \right)_p &\stackrel{!}{=} 0 \\ \implies \lambda^i \left( \frac{\partial}{\partial x^i} \right)_p (x^j) &= \lambda^i \partial_i (\underbrace{x^j \circ x^{-1}})(x(p)) = \begin{aligned} &x^j \circ x^{-1} : \mathbb{R}^d \longrightarrow \mathbb{R} \\ &(\alpha^1, \dots, \alpha^d) \mapsto \alpha^j \end{aligned} \\ &= \lambda^i \delta_i^j = \lambda^j \quad j = 1, \dots, d \end{aligned}$$

in cloud:  $x^j : U \longrightarrow \mathbb{R}$  differentiable

□

**Corollary 1.**  $\dim T_p M = d = \dim M$

Terminology:  $X \in T_p M \longrightarrow \exists \gamma : \mathbb{R} \longrightarrow M : X = v_{\gamma,p}$  and  
 $\underbrace{\exists X_1^1, \dots, X^d : X = X^i \left( \frac{\partial}{\partial x^i} \right)_p}_{\in \mathbb{R}}$

## 5.5 Change of vector components under a change of chart

✖ vector does **not** change under change of chart.

Let  $(U, x)$  and  $(V, y)$  be overlapping charts and  $p \in U \cap V$ .

Let  $X \in T_p M$

$$X_{(y)}^i \cdot \left( \frac{\partial}{\partial y^i} \right)_p \underbrace{=}_p X \underbrace{=}_{(U,x)} X_x^i \left( \frac{\partial}{\partial x^i} \right)_p$$

to study change of components formula:

$$\begin{aligned} \left( \frac{\partial}{\partial x^i} \right)_p f &= \partial_i (f \circ x^{-1})(x(p)) = \\ &= \partial_i \underbrace{(f \circ y^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}} \circ \underbrace{(y \circ x^{-1})}_{\mathbb{R}^d \longrightarrow \mathbb{R}^d} (x(p)) \\ &= (\partial_i (y^i \circ x^{-1}))(x(p)) \cdot (\partial_j (f \circ y^{-1}))(y(p)) = \\ &= \boxed{\left( \frac{\partial y^p}{\partial x^i} \right)_p \cdot \left( \frac{\partial f}{\partial y^j} \right)_p} f \\ \implies X_{(x)}^i \left( \frac{\partial y^j}{\partial x^i} \right)_p \left( \frac{\partial}{\partial y^j} \right)_p &= X_{(y)}^j \left( \frac{\partial}{\partial y^j} \right)_p \\ \implies \boxed{X_{(y)}^j} &= \left( \frac{\partial y^j}{\partial x^i} \right)_p X_{(x)}^i \end{aligned}$$

## 5.6 Cotangent spaces

$$T_p M = V$$

$$\text{trivial } (T_p M)^* := \{\varphi : T_p M \xrightarrow{\sim} \mathbb{R}\}$$

Example:  $f \in C^\infty(M)$

$$\begin{aligned} (df)_p : T_p M &\xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (df)_p(X) \end{aligned}$$

$$\text{i.e. } \boxed{(df)_p \in T_p M^*}$$

$(df)_p$  called the gradient of  $f$   $p \in M$ .

Calculate components of gradient w.r.t. chart-induced basis  $(U, x)$

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left( \left( \frac{\partial}{\partial x^j} \right)_p \right) \\ &= \left( \frac{\partial f}{\partial x^j} \right)_p = \partial_j (f \circ x^{-1})(x(p)) \end{aligned}$$

**Theorem 10.** Consider chart  $(U, x) \implies x^i : U \longrightarrow \mathbb{R}$

Claim:  $(dx^1)_p, (dx^2)_p, \dots, (dx^d)_p$  basis of  $T_p^*M$

$\implies$  In fact: dual basis:

$$(dx^a)_p \left( \left( \frac{\partial}{\partial x^b} \right)_p \right) = \left( \frac{\partial x^a}{\partial x^b} \right)_p = \dots = \delta_b^a$$

**5.7** Change of components of a covector under a change of chart:

$$\underbrace{T_p^*M}_{\ni \omega} \text{ with } \omega_{(y)}(dy^j)_p = \omega = \omega_{(x)i}(dx^i)_p$$

$$\implies \boxed{\omega_{(y)i} = \frac{\partial x^j}{\partial y^i} \omega_{(x)j}}$$

## 6 Lecture 6: Fields

So far:

$$\begin{array}{c} T_p M \\ \vdots \downarrow \\ T_p^* M \\ \vdots \downarrow \\ \vdots \end{array},$$

now

in Thought Cloud: theory of bundles

### 6.1 Bundles

**Definition 28.** A **bundle** is a triple

$$E \xrightarrow{\pi} M$$

$E$  smooth manifold “**total space**”

$\pi$  smooth map (surjective) “**projection map**”

$M$  smooth manifold “**base space**”

Example  $E = \text{cylinder}$   $M = \text{circle}$

**Definition 29.** define **fibre over**  $p$

$$:= \text{preim}_{\pi}(\{p\})$$

**Definition 30.** A **section**  $\sigma$  of a bundle

$$\begin{array}{c} E \\ \left( \begin{array}{c} \downarrow \\ \sigma \end{array} \right) \\ M \end{array}$$

require  $\pi \circ \sigma = \text{id}_M$

Schuller says: in quantum mechanics, Aside:  $\psi : M \rightarrow \mathbb{C}$

### 6.2 Tangent bundle of smooth manifold

$(M, \mathcal{O}, \mathcal{A})$  smooth manifold

(a) as a **set**  $TM := \dot{\bigcup}_{p \in M} T_p M$

(b) surjective  $\pi : TM \rightarrow M$  the *unique* point  $p \in M$ ,  $X \in T_p M$

$$X \mapsto p$$

situation:  $\underbrace{TM}_{\text{set}} \xrightarrow{\underbrace{\pi}_{\text{surjective map}}} \underbrace{M}_{\text{smooth manifold}}$

(c) Construct topology on  $TM$  that is the coarsest topology such that  $\pi$  (just) continuous. (“initial topology with respect to  $\pi$ ”).

$$\mathcal{O}_{TM} := \{\text{preim}_\pi(U) | U \in \mathcal{O}\}$$

Show: Tutorial  $\mathcal{O}_{TM}$  Schuller says this is shown in the tutorial

$(TM, \mathcal{O}_{TM})$

Construction of a  $C^\infty$ -atlas on  $TM$  from the  $C^\infty$ -atlas  $\mathcal{A}$  on  $M$ .

$$\mathcal{A}_{TM} := \{(TU, \xi_x) | (U, x) \in \mathcal{A}\}$$

where

$$\begin{aligned} \xi_x : TU &\longrightarrow \mathbb{R}^{2 \cdot \dim M} \\ X &\mapsto \underbrace{((x^1 \circ \pi)(X), \dots, (x^d \circ \pi)(X), (dx^1)_{\pi(X)}(X), \dots, (dx^d)_{\pi(X)}(X))}_{(U, x) \text{-- coords of } \pi(X) \text{ (d many)}} \end{aligned}$$

where  $X \in T_{\pi(X)}M$

$$X = X_{(x)}^i \left( \frac{\partial}{\partial x^i} \right)_{\pi(X)}$$

$$\begin{aligned} (dx^j)_{\pi(X)}(X) &= (dx^j)_{\pi(X)} \left( X_{(x)}^i \left( \frac{\partial}{\partial x^i} \right)_{\pi(X)} \right) = \\ &= X_{(x)}^i \delta_i^j = X_{(x)}^j \end{aligned}$$

Write  $\xi_x^{-1} : \xi_x(TU) \subseteq \mathbb{R}^{2 \dim M} \longrightarrow TU$

$$(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) := \beta^i \left( \frac{\partial}{\partial x^i} \right) \underbrace{x^{-1}(\alpha^1, \dots, \alpha^d)}_{\pi(X)}$$

Check:

$$\begin{aligned} (\xi_y \circ \xi_x^{-1})(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) &= \\ &= \xi_y \left( \beta^i \left( \frac{\partial}{\partial x^i} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \right) \\ &= \left( \dots, (y^i \circ \pi)(\beta^m \cdot \left( \frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}), \dots, \dots, (dy^i)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \left( \beta^m \left( \frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \right), \dots \right) = \\ &= \left( \dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \dots, \underbrace{\beta^m (dy^i)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \left( \left( \frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \right)}_{\beta^m \left( \frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)}} \right) \end{aligned}$$

$$\left( \frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} = \partial_m (y^i \circ x^{-1})(x \circ (x^{-1}(\alpha^1, \dots, \alpha^d))) = \partial_m (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d) \text{ smooth.}$$

upshot

$$\underbrace{TM}_{\text{smooth manifold}} \xrightarrow{\pi} \underbrace{M}_{\text{smooth manifold}}$$

bundle, called the tangent bundle.

### 6.3 Vector fields

**Definition 31.** A smooth vector field  $\chi$  is a *smooth* map

### 6.4 The $C^\infty(M)$ -module $\Gamma(TM)$

set  $\Gamma(TM) = \{\chi : M \rightarrow TM \mid \text{smooth section}\}$

$$(\chi \oplus \tilde{\chi})(f) := (\chi f) + \underbrace{\tilde{\chi}}_{C^\infty(M)}(f)$$

$$(g \odot \xi)(f) := g \cdot \underbrace{\chi}_{C^\infty(M)}(f)$$

upshot: set of all smooth vector fields can be made into a  $C^\infty(M)$ -module.

Fact:

- (1) ZFC  $\implies$  every vector space has a basis.
- (2) no such result exists for modules.

This is a shame, because otherwise, we could have chosen (for any manifolds) vector fields,

$$\Xi_{(1)}, \dots, \Xi_{(d)} \in \Gamma(TM)$$

and would be able to write every vector field  $\Xi$

$$\Xi = \underbrace{f^i}_{\text{component functions}} \cdot \Xi_{(i)}$$

### Simple counterexample

Schuller says: Take a sphere, Morse Theorem, every smooth vector field must vanish at 2 pts. “mustn’t choose a global basis”

However:  $\frac{\partial}{\partial x^i} : U \xrightarrow{\text{smooth}} TU$

$$p \mapsto \left( \frac{\partial}{\partial x^i} \right)_p$$

### 6.5 Tensor fields

so far

$\Gamma(M)$  = “set of vector fields”  $C^\infty(M)$ -module

$\Gamma(T^*M)$  = “covector fields”  $C^\infty(M)$ -module

**Definition 32.** An  $(r, s)$ -tensor field  $T$  is a multi-linear map

$$T : \underbrace{\Gamma(T^*M) \times \dots \times \Gamma(T^*M)}_r \times \Gamma(TM) \times \dots \times \Gamma(TM) \xrightarrow{\sim} C^\infty(M)$$



Example:  $f \in C^\infty(M)$

$$df : \Gamma(TM) \xrightarrow{\sim} C^\infty(M)$$

$$\Xi \mapsto df(\Xi) := \Xi[f]$$

$df$   $(0,1)$ -T.F. (tensor field)

where  $(\Xi f)(\underbrace{p}_{\in M}) := \underbrace{\Xi(p)}_{\in T_p M} f$

can check:  $df$  is  $C^\infty$ -linear

## 7 Lecture 7: Connections

$$\nabla_X f = Xf = (df)(X) \text{ but (not quite)}$$

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$

$$df : \Gamma(TM) \longrightarrow C^\infty(M)$$

$$\nabla_X : C^\infty(M) \longrightarrow C^\infty(M)$$

$$\begin{array}{ccc} \nabla_X : C^\infty(M) & \longrightarrow & C^\infty(M) \\ \vdots \downarrow & & \vdots \downarrow \end{array}$$

$$\nabla_X : \begin{array}{c} TM^p \otimes T^*M^q \text{ i.e.} \\ \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \text{ tensor field} \end{array} \longrightarrow \begin{array}{c} TM^p \otimes T^*M^q \text{ i.e.} \\ \left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \text{ tensor field} \end{array}$$

### 7.1 Directional derivatives of tensor fields

manifold with connection is quadruple  $(M, \mathcal{O}, \mathcal{A}, \nabla)$

topology  $\mathcal{O}$

atlas  $\mathcal{A}$

Consider chart  $(U, x) \in \mathcal{A}$

**Definition 33.**  $\forall$  pair  $(X, (p, q) - \text{tensor field}) \equiv (X, (p, q) - TF)$ ,

**connection**  $\nabla$  on smooth manifold  $(M, \mathcal{O}, \mathcal{A})$

$\nabla : (X, (p, q) - TF) \longrightarrow (p, q) - TF$  s.t.

$$(i) \quad \nabla_X f = Xf$$

$$(ii) \quad \nabla_X (T + S) = \nabla_X T + \nabla_X S$$

(iii)

$$\nabla_X (T(\omega, Y)) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$$

“Leibnitz” rule.

As

$$T \otimes S(\omega_{(1)} \dots \omega_{(p+r)}, Y_{(1)} \dots Y_{(q+s)}) = T(\omega_{(1)} \dots \omega_{(p)}, Y_{(1)} \dots Y_{(q)}) \cdot S(\omega_{(p+1)} \dots \omega_{(p+r)}, Y_{(q+1)} \dots Y_{(q+s)})$$

so

$$\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes \nabla_X S$$

$$(iv) \quad \nabla_{fX+Z} T = f \nabla_X T + \nabla_Z T \text{ } C^\infty\text{-linear}$$

## 7.2 New structure on $(M, \mathcal{O}, \mathcal{A})$ required to fix $\nabla$

There are  $(\dim M)^3$  many  $\Gamma_{jk}^i$

$$\begin{aligned}\Gamma_{jk}^i &: U \longrightarrow \mathbb{R} \\ p &\mapsto \left( dx^i \left( \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x^j} \right) \right) (p)\end{aligned}$$

Now  $\nabla_{\frac{\partial}{\partial x^m}} (dx^i) = ?$

$$\begin{aligned}\underbrace{\nabla_{\frac{\partial}{\partial x^m}} \left( dx^i \left( \underbrace{\frac{\partial}{\partial x^j}}_{\delta_j^i} \right) \right)}_{\parallel \quad (iii)} &= \frac{\partial}{\partial x^m} (\delta_j^i) = 0 \\ &= \left( \nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left( \frac{\partial}{\partial x^j} \right) + dx^i \underbrace{\left( \nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j} \right)}_{\Gamma_{jm}^q \frac{\partial}{\partial x^q}} = 0 \\ \implies \left( \nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left( \frac{\partial}{\partial x^j} \right) &= -\Gamma_{jm}^i \\ \nabla_{\frac{\partial}{\partial x^m}} dx^i &= -\Gamma_{jm}^i dx^j\end{aligned}$$

Hence

$$\begin{aligned}(\nabla_X Y)^i &= X(Y^i) + \Gamma_j^i \underbrace{\quad}_m Y^j X^m \\ &\quad \text{last entry goes in direction of } X \\ (\nabla_X \omega)_i &= X(\omega_i) - \Gamma_{im}^j \omega_j X^m\end{aligned}$$

Note that for the immediately above expression for  $(\nabla_X Y)^i$ , in the second term on the right hand side,  $\Gamma_{jm}^i$  has the last entry at the bottom,  $m$  going in the direction of  $X$ , so that it matches up with  $X^m$ . This is a good mnemonic to memorize the index positions of  $\Gamma$ .

summary so far:

$$\begin{aligned}(\nabla_X Y)^i &= X(Y^i) + \Gamma_{jm}^i Y^j X^m \\ (\nabla_X \omega)_i &= X(\omega_i) - \Gamma_{im}^j \omega_j X^m\end{aligned}$$

similarly, by further application of Leibnitz

$T$  a  $(1, 2)$ -TF (tensor field)

$$(\nabla_X T)_{jk}^i = X(T_{jk}^i) + \Gamma_{sm}^i T_{jk}^s X^m - \Gamma_{jm}^s T_{sk}^i X^m - \Gamma_{km}^s T_{js}^i X^m$$

What is a Euclidean space:

$(M = \mathbb{R}^n, \mathcal{O}_{\text{st}}, \mathcal{A})$  smooth manifold.

Assume  $(\mathbb{R}^n, \text{id}_{\mathbb{R}^n}) \in \mathcal{A}$  and

$$(\Gamma_{(x)}^i)_{jk} = dx^i \left( (\nabla_{\underline{E}})_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) \stackrel{!}{=} 0$$

## 7.3 Change of $\Gamma$ 's under change of chart

$(U, x), (V, y) \in \mathcal{A}$  and  $U \cap V \neq \emptyset$

$$\Gamma_{jk}^i(y) := dy^i \left( \nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} \right) = \frac{\partial y^i}{\partial x^q} dx^q \left( \nabla_{\frac{\partial x^p}{\partial y^k} \frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j} \frac{\partial}{\partial x^s} \right)$$

Note  $\nabla_{fX}$  is  $C^\infty$ -linear for  $fX$

covector  $dy^i$  is  $C^\infty$ -linear in its argument

$$\begin{aligned} \Rightarrow \Gamma_{jk}^i(y) &= \frac{\partial y^i}{\partial x^q} dx^q \left( \frac{\partial x^p}{\partial y^k} \left[ \left( \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j} \right) \frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j} \left( \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^s} \right) \right] \right) = \\ &= \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial}{\partial x^p} \frac{\partial x^s}{\partial y^j} \delta_s^q + \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial x^s}{\partial y^j} \Gamma_{sp}^q(x) \end{aligned}$$

$$\Gamma_{jk}^i(y) = \frac{\partial y^i}{\partial x^q} \frac{\partial^2 x^q}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial x^q} \frac{\partial x^s}{\partial y^j} \frac{\partial x^p}{\partial y^k} \Gamma_{sp}^q(x) \quad (7.1)$$

Eq. (7.1) is the change of connection coefficient function under the change of chart  $(U \cap V, x) \longrightarrow (U \cap V, y)$

## 7.4 Normal Coordinates

## 8 Lecture 8: Parallel Transport & Curvature

### 8.1 Parallelity of vector fields

**Definition 34.** (1) **parallelly transported** along smooth curve  $\gamma : \mathbb{R} \longrightarrow M$

if

$$\boxed{\nabla_{v_\gamma} X = 0} \quad (8.1)$$

(2) A slightly weaker condition

is “parallel”

$$(\nabla_{v_\gamma, \gamma(\lambda)} X)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}$$

### 8.2 Autoparallely transported curves

**Definition 35.** curve  $\gamma : \mathbb{R} \longrightarrow M$  is called **autoparallely transported** if

$$\nabla_{v_\gamma} v_\gamma \stackrel{!}{=} 0 \quad (8.2)$$

### 8.3 Autoparallel equation

$$\nabla_{v_\gamma} v_\gamma = 0$$

in summary:

$$\ddot{\gamma}_{(x)}^m(\lambda) + (\Gamma_{(x)}^m)_{ab}(\gamma(\lambda)) \dot{\gamma}_{(x)}^a(\lambda) \dot{\gamma}_{(x)}^b(\lambda) = 0 \quad (8.3)$$

### 8.4 Torsion

**Definition 36.** **torsion** of a connection  $\nabla$  is the  $(1,2)$ -**tensor field**

$$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \quad (8.4)$$

(Inside a cloud)

$[X, Y]$  vector field defined by

$$[X, Y]f := X(Yf) - Y(Xf)$$

*Proof.* check  $T$  is  $C^\infty$ -linear in each entry

$$T(\omega, fX, Y) = \omega(\nabla_{fX} Y - \nabla_Y(fX) - [fX, Y])$$

□

**Definition 37.** A  $(M, \mathcal{O}, \mathcal{A}, \nabla)$  is called **torsion-free** if  $T = 0$

In a chart

$$\begin{aligned} T_{ab}^i &:= T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = dx^i(\dots) \\ &= \Gamma_{ab}^i - \Gamma_{ba}^i = 2\Gamma_{[ab]}^i \end{aligned}$$

From now on, in these lectures, we only use torsion-free connections.

## 8.5 Curvature

**Definition 38. Riemann curvature** of a connection  $\nabla$  is the  $(1,3)$ -tensor field

$$\text{Riem}(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z) \quad (8.5)$$

*Proof.* do it:  $C^\infty$ -linear in each slot. □

Tutorials  $\text{Riem}^i_{jab} = \dots$

## 9 Lecture 9: Newtonian spacetime is curved!

**Axiom 1** (Newton I:). *A body on which no force acts moves uniformly along a straight line*

**Axiom 2** (Newton II:). *Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.*

Remark:

- (1) 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space  
...
- (2) Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

### 9.1 Laplace's questions

Laplace \* 1749

†1827

Q: "Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to move along straight lines in this curved space?"

Answer: No!

*Proof.* gravity is a force point of view

$$m\ddot{x}^\alpha(t) = F^\alpha(x(t))$$

$$m\ddot{x}^\alpha(t) = \underbrace{m f^\alpha}_{F^\alpha}(x(t))$$

$$-\partial_\alpha f^\alpha = 4\pi G\rho \text{ (Poisson)}$$

$\rho$  mass density of matter

(EY : 20150330) You know this,  $F = Gm_1m_2/r^2$

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

Laplace asks: Is this  $(\ddot{x}(t))$  of the form

$$\ddot{x}^\alpha(t) + \Gamma_{\beta\gamma}^\alpha(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

Conclusion: One cannot find  $\Gamma$  s such that Newton's equation takes the form of an autoparallel.

□

### 9.2 The full wisdom of Newton I

use also the information from Newton's first law that particles (no force) move uniformly

introduce the appropriate setting to talk about the difference easily

insight: in spacetime uniform & straight motion is simply straight motion

So let's try in spacetime:

let  $x : \mathbb{R} \longrightarrow \mathbb{R}^3$

be a particle's trajectory in space  $\longleftrightarrow$  worldline (history) of the particle  $X : \mathbb{R} \longrightarrow \mathbb{R}^4$   
 $t \mapsto (t, x^1(t), x^2(t), x^3(t)) :=$   
 $:= (X^0(t), X^1(t), X^2(t), X^3(t))$

That's all it takes:

Trivial rewritings:

$$\dot{X}^0 = 1$$

$$\Rightarrow \begin{array}{l} \ddot{X}^0 = 0 \\ \ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 = 0 \end{array} \quad (\alpha = 1, 2, 3) \Rightarrow \begin{array}{l} a = 0, 1, 2, 3 \\ \ddot{X}^a + \Gamma^a_{bc} \dot{X}^b \dot{X}^c = 0 \end{array}$$

antoparallel eqn in spacetime

Yes, choosing  $\Gamma^0_{ab} = 0$

$$\Gamma^\alpha_{\beta\gamma} = 0 = \textit{Gamma}^\alpha_{0\beta} = \Gamma^\alpha_{\beta 0}$$

only:  $\Gamma^\alpha_{00} \stackrel{!}{=} -f^\alpha$

Question: Is this a coordinate-choice artifact?

No, since  $R^\alpha_{0\beta 0} = -\frac{\partial}{\partial x^\beta} f^\alpha$  (only non-vanishing components) (tidal force tensor, – the Hessian of the force component)

$$\text{Ricci tensor} \Rightarrow R_{00} = R^m_{0m0} = -\partial_\alpha f^\alpha = 4\pi G\rho$$

$$\text{Poisson: } -\partial_\alpha f^\alpha = 4\pi G \cdot \rho$$

$$\text{writing: } T_{00} = \frac{1}{2}s$$

$$\Rightarrow \boxed{R_{00} = 8\pi G T_{00}}$$

$$\text{Einstein in 1912 } \boxed{\cancel{R_{ab}} \Rightarrow \cancel{8\pi G T_{ab}}}$$

Conclusion: Laplace's idea works in spacetime

Remark

$$\begin{array}{l} \Gamma^\alpha_{00} = -f^\alpha \\ R^\alpha_{\beta\gamma\delta} = 0 \quad \alpha, \beta, \gamma, \delta = 1, 2, 3 \\ \boxed{R_{00} = 4\pi G\rho} \end{array}$$

Q: What about transformation behavior of LHS of

$$\underbrace{\ddot{x}^a + \Gamma^a_{bc} \dot{X}^b \dot{X}^c}_{\underbrace{(\nabla_{v_X} v_X)^a}_{:= a^a \text{ "acceleration vector"}}} = 0$$



### 9.3 The foundations of the geometric formulation of Newton's axiom

new start

**Definition 39.** A **Newtonian spacetime** is a quintuple

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where  $(M, \mathcal{O}, \mathcal{A})$  4-dim. smooth manifold

$t : M \rightarrow \mathbb{R}$  smooth function

(i) “There is an absolute space”

$$(dt)_p \neq 0 \quad \forall p \in M$$

(ii) “absolute time flows uniformly”

$$\underbrace{\nabla dt}_{\text{space of } (0,2)\text{-tensor fields}} = 0 \quad \text{everywhere}$$

$\nabla dt$  is a  $(0,2)$ -tensor field

(iii) add to axioms of Newtonian spacetime  $\nabla = 0$  torsion free

**Definition 40.** absolute space at time  $\tau$

$$S_\tau := \{p \in M | t(p) = \tau\}$$

$$\xrightarrow{dt \neq 0} M = \coprod S_\tau$$

**Definition 41.** A vector  $X \in T_p M$  is called

(a) future-directed if

$$dt(X) > 0$$

(b) spatial if

$$dt(X) = 0$$

(c) past-directed if

$$dt(X) < 0$$

picture

Newton I: The worldline of a particle under the influence of no force (gravity isn't one, anyway) is a future-directed autoparalle  
i.e.

$$\begin{aligned} \nabla_{v_X} v_X &= 0 \\ dt(v_X) &> 0 \end{aligned}$$

Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m} \iff m \cdot a = F$$

where  $F$  is a spatial vector field:

$$dt(F) = 0$$

**Convention:** restrict attention to atlases  $\mathcal{A}_{\text{stratified}}$  whose charts  $(\mathcal{U}, x)$  have the property

$$\begin{array}{llll} x^0 : \mathcal{U} \longrightarrow \mathbb{R} \\ x^1 : \mathcal{U} \longrightarrow \mathbb{R} \\ \vdots \quad \vdots \\ x^3 \end{array} \quad x^0 = t|_{\mathcal{U}} \quad \Longrightarrow \quad \begin{array}{l} 0 \stackrel{\text{"absolute time flows uniformly"}}{=} \nabla dt \\ 0 = \nabla_{\frac{\partial}{\partial x^a}} dx^0 = -\Gamma_{ba}^0 \quad a = 0, 1, 2, 3 \end{array}$$

Let's evaluate in a chart  $(\mathcal{U}, x)$  of a stratified atlas  $\mathcal{A}_{\text{sheet}}$ : Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m}$$

in a chart.

$$\begin{aligned} (X^0)'' + \Gamma_{cd}^0 (X^a)' (X^b)' \stackrel{\text{stratified atlas}}{=} 0 \\ (X^\alpha)'' + \Gamma_{\gamma\delta}^\alpha X^{\gamma'} X^{\delta'} + \Gamma_{00}^\alpha X^{0'} X^{0'} + 2\Gamma_{\gamma 0}^\alpha X^{\gamma'} X^{0'} = \frac{F^\alpha}{m} \quad \alpha = 1, 2, 3 \end{aligned}$$

$$\Longrightarrow (X^0)''(\lambda) = 0 \Longrightarrow X^0(\lambda) = a\lambda + b \quad \text{constants } a, b \text{ with}$$

$$X^0(\lambda) = (x^0 \circ X)(\lambda) \stackrel{\text{stratified}}{=} (t \circ X)(\lambda)$$

convention parametrize worldline by absolute time

$$\frac{d}{d\lambda} = a \frac{d}{dt}$$

$$\begin{aligned} a^2 \ddot{X}^\alpha + a^2 \Gamma_{\gamma\delta}^\alpha \dot{X}^\gamma \dot{X}^\delta + a^2 \Gamma_{00}^\alpha \dot{X}^0 \dot{X}^0 + 2\Gamma_{\gamma 0}^\alpha \dot{X}^\gamma \dot{X}^0 &= \frac{F^\alpha}{m} \\ \Longrightarrow \underbrace{\ddot{X}^\alpha + \Gamma_{\gamma\delta}^\alpha \dot{X}^\gamma \dot{X}^\delta + \Gamma_{00}^\alpha \dot{X}^0 \dot{X}^0 + 2\Gamma_{\gamma 0}^\alpha \dot{X}^\gamma \dot{X}^0}_{a^\alpha} &= \frac{1}{a^2} \frac{F^\alpha}{m} \end{aligned}$$

# 10 Lecture 10: Metric Manifolds

cf. [Lecture 10: Metric Manifolds \(International Winter School on Gravity and Light 2015\)](#)

We establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space).

From this structure, one can then define a notion of length of a curve.

Then we can look at shortest curves.

Requiring then that the shortest curves coincide with the straightest curves (wrt  $\nabla$ ) will result in  $\nabla$  being determined by the metric structure.

$$g \xrightarrow[\sim]{\substack{\text{straight=short} \\ T=0}} \nabla \rightsquigarrow \text{Riem}$$

## 10.1 Metrics

**Definition 42.** A metric  $g$  on a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$  is a  $(0, 2)$ -tensor field satisfying

- (i) symmetry  $g(X, Y) = g(Y, X) \quad \forall X, Y$  vector fields
- (ii) non-degeneracy: the musical map

$$\begin{aligned} \text{“flat” } \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\mapsto \flat(X) \end{aligned}$$

$$\text{where} \quad \flat(X)(Y) := g(X, Y)$$

$$\flat(X) \in \Gamma(T^*M)$$

$$\text{In thought bubble: } \flat(X) = g(X, \cdot)$$

... is a  $C^\infty$ -isomorphism in other words, it is invertible.

Remark:  $(\flat(X))_a$  or

$$X_a$$

$$(\flat(X))_a := g_{am} X^m$$

Thought bubble:  $\flat^{-1} = \sharp$

$$\flat^{-1}(\omega)^a := g^{am} \omega_m$$

$$\flat^{-1}(\omega)^a := (g^{-1})^{am} \omega_m \implies \text{not needed. (all of this is not needed)}$$

**Definition 43.** The  $(2, 0)$ -tensor field  $g^{-1}$  with respect to a metric  $g$  is the symmetric

$$\begin{aligned} g^{-1} : \Gamma(T^*M) \times \Gamma(T^*M) &\rightarrow C^\infty(M) \\ (\omega, \sigma) &\mapsto \omega(\flat^{-1}(\sigma)) \quad \flat^{-1}(\sigma) \in \Gamma(TM) \end{aligned}$$

chart:  $g_{ab} = g_{ba}$

$$(g^{-1})^{am} g_{mb} = \delta_b^a$$

Example:  $(S^2, \mathcal{O}, \mathcal{A})$

chart  $(\mathcal{U}, x)$

$$\varphi \in (0, 2\pi)$$

$$\theta \in (0, \pi)$$

define the metric

$$g_{ij}(x^{-1}(\theta, \varphi)) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{bmatrix}_{ij}$$

$$R \in \mathbb{R}^+$$

“the metric of the round sphere of radius  $R$ ”

## 10.2 Signature

$$A^a_m v^m = \lambda v^a$$

Linear algebra:

$$g_{am} v^m = \lambda \cdot v^a? \rightsquigarrow$$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$$

(1, 1) tensor has eigenvalues

(0, 2) has signature  $(p, q)$  (well-defined)

$$\left. \begin{matrix} (+ + +) \\ (+ + -) \\ (+ - -) \\ (- - -) \end{matrix} \right\} d + 1 \text{ if } p + q = \dim V$$

**Definition 44.** A metric is called

**Riemannian** if its signature is  $(+ + \cdots +)$

**Lorentzian** if  $(+ - \cdots -)$

## 10.3 Length of a curve

Let  $\gamma$  be a smooth curve.

Then we know its velocity  $v_{\gamma, \gamma(\lambda)}$  at each  $\gamma(\lambda) \in M$ .

**Definition 45.** On a Riemannian metric manifold  $M, \mathcal{O}, \mathcal{A}, g$ , the **speed** of a curve at  $\gamma(\lambda)$  is the number

$$(\sqrt{g(v_\gamma, v_\gamma)})_{\gamma(\lambda)} = s(\lambda)$$

F. Schuller: “I feel the need for speed.” -Top Gun.

(I feel the need for speed, then I feel the need for a metric)

Aside:  $[v^a] = \frac{1}{T}$   
 $[g_{ab}] = L^2$   
 $[\sqrt{g_{ab}v^av^b}] = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$

**Definition 46.** Let  $\gamma : (0, 1) \rightarrow M$  a smooth curve.

Then the **length of  $\gamma$**  is the number

$$\mathbb{R} \ni L[\gamma] := \int_0^1 d\lambda s(\lambda) = \int_0^1 d\lambda \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}$$

F. Schuller: “velocity is more fundamental than speed, speed is more fundamental than length”

Example: reconsider the round sphere of radius  $R$

Consider its equator:

$$\theta(\lambda) := (x^1 \circ \gamma)(\lambda) = \frac{\pi}{2}$$

$$\varphi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3$$

$$\theta'(\lambda) = 0$$

$$\varphi'(\lambda) = 6\pi\lambda^2$$

on the same chart  $g_{ij} = \begin{bmatrix} R^2 & \\ & R^2 \sin^2 \theta \end{bmatrix}$

F.Schuller: do everything in this chart

$$\begin{aligned} L[\gamma] &= \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} = \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda)) 36\pi^2 \lambda^4} = \\ &= 6\pi R \int_0^1 d\lambda \lambda^2 = 6\pi R [\frac{1}{3}\lambda^3]_0^1 = 2\pi R \end{aligned}$$

**Theorem 11.**  $\gamma : (0, 1) \rightarrow M$  and

$\sigma : (0, 1) \rightarrow (0, 1)$  smooth bijective and increasing “reparametrization”

$$L[\gamma] = L[\gamma \circ \sigma]$$

*Proof.*  $\Rightarrow$  Tutorials

□

## 10.4 Geodesics

**Definition 47.** A curve  $\gamma : (0, 1) \rightarrow M$  is called a **geodesic** on a Riemannian manifold  $(M, \mathcal{O}, \mathcal{A}, g)$  if its a *stationary* curve with respect to a length functional  $L$ .

Thought bubble: in classical mechanics, deform the curve a little,  $\epsilon$  times this deformation, to first order, it agrees with  $L[\gamma]$

**Theorem 12.**  $\gamma$  geodesic iff it satisfies the Euler-Lagrange equations for the Lagrangian

$$\begin{aligned} \mathcal{L} : TM &\rightarrow \mathbb{R} \\ X &\mapsto \sqrt{g(X, X)} \end{aligned}$$

In a chart, the Euler Lagrange equations take the form:

$$\left( \frac{\partial \mathcal{L}}{\partial \dot{x}^m} \right)' - \frac{\partial \mathcal{L}}{\partial x^m} = 0$$

F.Schuller: this is a chart dependent formulation

here:

$$\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}$$

Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} &= \frac{1}{\sqrt{\dots}} g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \\ \left( \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right)' &= \left( \frac{1}{\sqrt{\dots}} \right)' g_{mj}(\gamma(\lambda)) \cdot \dot{\gamma}^j(\lambda) + \frac{1}{\sqrt{\dots}} (g_{mj}(\gamma(\lambda)) \ddot{\gamma}^j(\lambda) + \dot{\gamma}^s (\partial_s g_{mj}) \dot{\gamma}^j(\lambda)) \end{aligned}$$

Thought bubble: reparametrize  $g(\dot{\gamma}, \dot{\gamma}) = 1$  (it's a condition on my reparametrization)

By a clever choice of reparametrization  $(\frac{1}{\sqrt{\dots}})' = 0$

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{\dots}} \partial_m g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)$$

putting this together as Euler-Lagrange equations:

$$g_{mj} \ddot{\gamma}^j + \partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j - \frac{1}{2} \partial_m g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

Multiply on both sides  $(g^{-1})^{qm}$

$$\ddot{\gamma}^q + (g^{-1})^{qm} (\partial_i g_{mj} - \frac{1}{2} \partial_m g_{ij}) \dot{\gamma}^i \dot{\gamma}^j = 0$$

$$\boxed{\ddot{\gamma}^q + (g^{-1})^{qm} \frac{1}{2} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) \dot{\gamma}^i \dot{\gamma}^j = 0}$$

geodesic equation for  $\gamma$  in a chart.

$$\boxed{(g^{-1})^{qm} \frac{1}{2} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) =: \Gamma_{ij}^q(\gamma(\lambda))}$$

Thought bubble:  $\left( \frac{\partial \mathcal{L}}{\partial \xi_x^{a+\dim M}} \right)_{\sigma(x)} - \left( \frac{\partial \mathcal{L}}{\partial x_i^a} \right)_{\sigma(x)} = 0$

**Definition 48.** “Christoffel symbol”  ${}^{\text{L.C.}}\Gamma$  are the connection coefficient functions of the so-called Levi-Civita connection  ${}^{\text{L.C.}}\nabla$

We usually make this choice of  $\nabla$  if  $g$  is given.

$$(M, \mathcal{O}, \mathcal{A}, g) \longrightarrow (M, \mathcal{O}, \mathcal{A}, g, {}^{\text{L.C.}}\nabla)$$

abstract way:  $\nabla g = 0$  and  $T = 0$  (torsion)

$$\implies \nabla = {}^{\text{L.C.}}\nabla$$

**Definition 49.** (a) The Riemann-Christoffel curvature is defined by

$$R_{abcd} := g_{am} R_{bcd}^m$$

(b) Ricci:  $R_{ab} = R_{amb}^m$

Thought bubble: with a metric,  ${}^{\text{L.C.}}\nabla$

(c) (Ricci) scalar curvature:

$$R = g^{ab} R_{ab}$$

Thought bubble:  $\text{L.C.}\nabla$

**Definition 50.** Einstein curvature  $(M, \mathcal{O}, \mathcal{A}, g)$

$$G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R$$

Convention:  $g^{ab} := (g^{-1})^{ab}$

F. Schuller: these indices are not being pulled up, because what would you pull them up with

(student) Question: Does the Einstein curvature yield new information?

Answer:

$$g^{ab}G_{ab} = R_{ab}g^{ab} - \frac{1}{2}g_{ab}g^{ab}R = R - \delta_a^a R = R - \frac{1}{2}\dim M R = (1 - \frac{d}{2})R$$

## 11 Lecture 11: Symmetry

EY : 20150321 This lecture tremendously and lucidly clarified, for me at least, what a symmetry of the Lie algebra is, and in comparing structures  $(M, \mathcal{O}, \mathcal{A})$  vs.  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ , clarified differences, and asking about differences is a good way to learn, the difference between  $\mathcal{L}$  and  $\nabla$ , respectively.

Feeling that the round sphere

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$$

has rotational symmetry, while  
the potato

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$$

does not.

### 11.1

### 11.2

Important

### 11.3 Flow of a complete vector field

Let  $(M, \mathcal{O}, \mathcal{A})$  smooth  $X$  vector field on  $M$

**Definition 51.** A curve  $\gamma : I \subseteq \mathbb{R} \rightarrow M$  is called an integral curve of  $X$  if

$$v_{\gamma, \gamma(\lambda)} = X_{\gamma(\lambda)}$$

**Definition 52.** A vector field  $X$  is **complete** if all integral curves have  $I = \mathbb{R}$  EY: 20150321 (i.e. domain is all of  $\mathbb{R}$ )

Ex. minute 48:30 EY : reall good explanation by F.P.Schuller; take a pt. out for an incomplete vector field.

**Theorem 13.** *compactly supported smooth vector field is complete.*

**Definition 53.** The flow of a complete vector field  $X$  is a 1-parameter family

$$h^X = \mathbb{R} \times M \rightarrow M$$

where  $\gamma_p : \mathbb{R} \rightarrow M$  is the integral curve of  $X$  with

$$\gamma(0) = p$$

Then for fixed  $\lambda \in \mathbb{R}$

$$h_\lambda^X : M \rightarrow M \text{ smooth}$$

picture  $h_\lambda^X(S) \neq S$  ( if  $X \neq 0$  )

### 11.4 Lie subalgebras of the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of vector fields

(a)  $\Gamma(TM) = \{ \text{set of all vector fields} \}$   $C^\infty(M)$ -module =  $\mathbb{R}$ -vector space

$$\implies [X, Y] \in \Gamma(TM) \quad [X, Y]f := X(Yf) - Y(Xf)$$



$$(i) \quad [X, Y] = -[Y, X]$$

$$(ii) \quad [\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]$$

$$(iii) \quad [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

$$(\Gamma(TM), [\cdot, \cdot]) \text{ Lie algebra}$$

(b) Let  $X_1 \dots X_s$  for  $s$  (many) vector fields on  $M$ , such that

## 12 Integration

### 12.1

### 12.2

### 12.3 Volume forms

**Definition 54.** On a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$   
a  $(0, \dim M)$ -tensor field  $\Omega$  is called a volume form if

- (a)  $\Omega$  vanishes nowhere (i.e.  $\Omega \neq 0 \forall p \in M$ )
- (b) totally antisymmetric

$$\Omega(\dots, \underbrace{X}_{i\text{th}}, \dots, \underbrace{Y}_{j\text{th}}, \dots) = -\Omega(\dots, \underbrace{Y}_{i\text{th}}, \dots, \underbrace{X}_{j\text{th}}, \dots)$$

In a chart:

$$\Omega_{i_1 \dots i_d} = \Omega_{[i_1 \dots i_d]}$$

Example  $(M, \mathcal{O}, \mathcal{A}, g)$  metric manifold

construct volume form  $\Omega$  from  $g$

In any chart:  $(U, x)$

$$\Omega_{i_1 \dots i_d} := \sqrt{\det(g_{ij}(x))} \epsilon_{i_1 \dots i_d}$$

where **Levi-Civita symbol**  $\epsilon_{i_1 \dots i_d}$  is defined as  $\epsilon_{123 \dots d} = +1$

$$\epsilon_{1 \dots d} = \epsilon_{[i_1 \dots i_d]}$$

*Proof.* (well-defined) Check: What happens under a change of charts

$$\begin{aligned} \Omega(y)_{i_1 \dots i_d} &= \sqrt{\det(g(y)_{ij})} \epsilon_{i_1 \dots i_d} = \\ &= \sqrt{\det(g_{mn}(x) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j})} \frac{\partial y^{m_1}}{\partial x^{i_1}} \dots \frac{\partial y^{m_d}}{\partial x^{i_d}} \epsilon_{[m_1 \dots m_d]} = \\ &= \sqrt{|\det g_{ij}(x)|} \left| \det \left( \frac{\partial x}{\partial y} \right) \right| \det \left( \frac{\partial y}{\partial x} \right) \epsilon_{i_1 \dots i_d} = \sqrt{\det g_{ij}(x)} \epsilon_{i_1 \dots i_d} \operatorname{sgn} \left( \det \left( \frac{\partial x}{\partial y} \right) \right) \end{aligned}$$

□

EY : 20150323

Consider the following:

$$\begin{aligned}
\Omega(y)(Y_{(1)} \dots Y_{(d)}) &= \Omega(y)_{i_1 \dots i_d} Y_{(1)}^{i_1} \dots Y_{(d)}^{i_d} = \\
&= \sqrt{\det(g_{ij}(y))} \epsilon_{i_1 \dots i_d} Y_{(1)}^{i_1} \dots Y_{(d)}^{i_d} = \\
&= \sqrt{\det(g_{mn}(x)) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} \epsilon_{i_1 \dots i_d} \frac{\partial y^{i_1}}{\partial x^{m_1}} \dots \frac{\partial y^{i_d}}{\partial x^{m_d}} X^{m_1} \dots X^{m_d}} = \\
&= \sqrt{\det(g_{mn}(x)) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}} \det\left(\frac{\partial y}{\partial x}\right) \epsilon_{m_1 \dots m_d} X^{m_1} \dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))} \left| \det\left(\frac{\partial x}{\partial y}\right) \right| \det\left(\frac{\partial y}{\partial x}\right) \epsilon_{m_1 \dots m_d} X^{m_1} \dots X^{m_d} = \\
&= \sqrt{\det(g_{mn}(x))} \epsilon_{m_1 \dots m_d} \operatorname{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right) X^{m_1} \dots X^{m_d} = \operatorname{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right) \Omega_{m_1 \dots m_d}(x) X^{m_1} \dots X^{m_d}
\end{aligned}$$

If  $\det\left(\frac{\partial y}{\partial x}\right) > 0$ ,

$$\Omega(y)(Y_{(1)} \dots Y_{(d)}) = \Omega(x)(X_{(1)} \dots X_{(d)})$$

This works also if Levi-Civita symbol  $\epsilon_{i_1 \dots i_d}$  doesn't change at all under a change of charts. (around 42:43 <https://youtu.be/2XpnbvPy-Zg>)

Alright, let's require,

restrict the smooth atlas  $\mathcal{A}$   
to a subatlas ( $\mathcal{A}^\uparrow$  still an atlas)

$$\mathcal{A}^\uparrow \subseteq \mathcal{A}$$

s.t.  $\forall (U, x), (V, y)$  have chart transition maps  $y \circ x^{-1}$   
 $x \circ y^{-1}$

s.t.  $\det\left(\frac{\partial y}{\partial x}\right) > 0$   
such  $\mathcal{A}^\uparrow$  called an **oriented** atlas

$$(M, \mathcal{O}, \mathcal{A}, g) \implies (M, \mathcal{O}, \mathcal{A}^\uparrow, g)$$

Note: associated bundles.

Note also:  $\det\left(\frac{\partial y^b}{\partial x^a}\right) = \det(\partial_a(y^b x^{-1}))$   $\frac{\partial y^b}{\partial x^a}$  is an endomorphism on vector space  $V$ .  $\varphi : V \longrightarrow V$   
 $\det \varphi$  independent of choice of basis

$g$  is a  $(0, 2)$  tensor field, not endomorphism (not independent of choice of basis)  $\sqrt{|\det(g_{ij}(y))|}$

**Definition 55.**  $\Omega$  be a volume form on  $(M, \mathcal{O}, \mathcal{A}^\uparrow)$  and consider chart  $(U, x)$

**Definition 56.**  $\omega_{(X)} := \Omega_{i_1 \dots i_d} \epsilon^{i_1 \dots i_d}$  same way  $\epsilon^{12 \dots d} = +1$   
 $\epsilon^{[\dots]}$

one can show

$$\boxed{\omega_{(y)} = \det\left(\frac{\partial x}{\partial y}\right) \omega_{(x)}} \quad \text{scalar density}$$

## 12.4 Integration on one chart domain $U$

**Definition 57.**

$$\boxed{\int_U f \stackrel{(U,y)}{=} \int_{y(U)} d^d \beta \omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta)} \quad (12.1)$$

*Proof.* : Check that it's (well-defined), how it changes under change of charts

$$\begin{aligned} \int_U f \stackrel{(U,y)}{=} \int_{y(U)} d^d \beta \omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta) &= \int_{(U,y)} \int_{x(U)} d^d \alpha \left| \det \left( \frac{\partial y}{\partial x} \right) \right| f_{(x)}(\alpha) \omega_{(x)}(x^{-1}(\alpha)) \det \left( \frac{\partial x}{\partial y} \right) = \\ &= \int_{x(U)} d^d \alpha \omega_{(x)}(x^{-1}(\alpha)) f_{(x)}(\alpha) \end{aligned}$$

□

On an oriented metric manifold  $(M, \mathcal{O}, \mathcal{A}^\uparrow, g)$

$$\int_U f := \int_{x(U)} d^d \alpha \underbrace{\sqrt{\det(g_{ij}(x))}}_{\sqrt{g}}(x^{-1}(\alpha)) f_{(x)}(\alpha)$$

## 12.5 Integration on the entire manifold

# 13 Lecture 13: Relativistic spacetime

Recall, from Lecture 9, the definition of Newtonian spacetime

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t) \quad \begin{array}{l} \nabla \text{ torsion free} \\ t \in C^\infty(M) \\ dt \neq 0 \\ \nabla dt = 0 \quad (\text{uniform time}) \end{array}$$

and the definition of relativistic spacetime (before Lecture )

$$(M, \mathcal{O}, \mathcal{A}^\dagger, \nabla, g, T) \quad \begin{array}{l} \nabla \text{ torsion-free} \\ g \text{ Lorentzian metric}(+ - - -) \\ T \text{ time-orientation} \end{array}$$

## 13.1 Time orientation

**Definition 58.**  $(M, \mathcal{O}, \mathcal{A}^\dagger, g)$  a Lorentzian manifold. Then a time-orientation is given by a vector field  $T$  that

- (i) does **not** vanish anywhere
- (ii)  $g(T, T) > 0$

Newtonian vs. relativistic

Newtonian

$X$  was called future-directed if

$$dt(X) > 0$$

$\forall p \in M$ , take half plane, half space of  $T_p M$

also stratified atlas so make planes of constant  $t$  straight

relativistic

half cone  $\forall p, q \in M$ , half-cone  $\subseteq T_p M$

This definition of spacetime

Question

I see how the cone structure arises from the new metric. I don't understand however, how the  $T$ , the time orientation, comes in

Answer

$$(M, \mathcal{O}, \mathcal{A}, g) \quad g \leftarrow (+ - - -)$$

requiring  $g(X, X) > 0$ , select cones

$T$  chooses which cone

This definition of spacetime has been made to enable the following physical postulates:

(P1) The worldline  $\gamma$  of a massive particle satisfies

$$(i) \quad g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) > 0$$

$$(ii) \quad g_{\gamma(\lambda)}(T, v_{\gamma, \gamma(\lambda)}) > 0$$

(P2) Worldlines of massless particles satisfy

$$(i) \quad g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) = 0$$

$$(ii) \quad g_{\gamma(\lambda)}(T, v_{\gamma, \gamma(\lambda)}) > 0$$

picture: spacetime:

Answer (to a question)  $T$  is a smooth vector field,  $T$  determines future vs. past, “general relativity: we have such a time orientation; smoothness makes it less arbitrary than it seems” -FSchuller,

Claim: 9/10 of a metric are determined by the cone

spacetime determined by distribution, only one-tenth error

## 13.2 Observers

$(M, \mathcal{O}, \mathcal{A}^\dagger, \nabla, g, T)$

**Definition 59.** An observer is a worldline  $\gamma$  with

$$g(v_\gamma, v_\gamma) > 0$$

$$g(T, v_\gamma) > 0$$

together with a choice of basis

$$v_{\gamma, \gamma(\lambda)} \equiv e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$

of each  $T_{\gamma(\lambda)}M$  where the observer worldline passes, if  $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{ab}$

precise: observer = smooth curve in the frame bundle  $LM$  over  $M$

### 13.2.1 Two physical postulates

(P3) A **clock** carried by a specific observer  $(\gamma, e)$  will measure a **time**

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})}$$

between the two “events”

$$\gamma(\lambda_0) \quad \text{“start the clock”}$$

and

$$\gamma(\lambda_1) \quad \text{“stop the clock”}$$

Compare with Newtonian spacetime:

$$t(p) = 7$$

Thought bubble: proper time/eigentime  $\tau$

$$\begin{aligned}
M &= \mathbb{R}^4 \\
\mathcal{O} &= \mathcal{O}_{\text{st}} \\
\text{Application/Example. } \mathcal{A} &\ni (\mathbb{R}^4, \text{id}_{\mathbb{R}^4}) \\
g : g_{(x)ij} &= \eta_{ij} \quad ; \quad T_{(x)}^i = (1, 0, 0, 0)^i \\
&\implies \Gamma_{(x)jk}^i = 0 \text{ everywhere}
\end{aligned}$$

$$\begin{aligned}
&\implies (M, \mathcal{O}, \mathcal{A}^\dagger, g, T, \nabla) \quad \text{Riem} = 0 \\
&\implies \text{spacetime is flat}
\end{aligned}$$

This situation is called special relativity.

Consider two observers:

$$\begin{aligned}
\gamma &: (0, 1) \longrightarrow M \\
\gamma_{(x)}^i &= (\lambda, 0, 0, 0)^i \\
\delta &: (0, 1) \longrightarrow M \\
\alpha \in (0, 1) : \delta_{(x)}^i &= \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases}
\end{aligned}$$

let's calculate:

$$\begin{aligned}
\tau_\gamma &:= \int_0^1 \sqrt{g_{(x)ij} \dot{\gamma}_{(x)}^i \dot{\gamma}_{(x)}^j} = \int_0^1 d\lambda = 1 \\
\tau_\delta &:= \int_0^{1/2} d\lambda \sqrt{1 - \alpha^2} + \int_{1/2}^1 \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1 - \alpha^2} = \sqrt{1 - \alpha^2}
\end{aligned}$$

Note: piecewise integration

Taking the clock postulate (P3) seriously, one better come up with a realistic clock design that supports the postulate. idea.

2 little mirrors

(P4) Postulate

Let  $(\gamma, e)$  be an observer, and

$\delta$  be a *massive* particle worldline that is parametrized s.t.  $g(v_\gamma, v_\gamma) = 1$  (for parametrization/normalization convenience)

Suppose the observer and the particle *meet* somewhere (in spacetime)

$$\delta(\tau_2) = p = \gamma(\tau_1)$$

*This* observer measures the 3-velocity (spatial velocity) of this particle as

$$v_\delta : \epsilon^\alpha(v_{\delta, \delta(\tau_2)}) e_\alpha \quad \alpha = 1, 2, 3 \quad (13.1)$$

where  $\epsilon^0, \boxed{\epsilon^1, \epsilon^2, \epsilon^3}$  is the unique dual basis of  $e_0, \boxed{e_1, e_2, e_3}$

EY:20150407

There might be a major correction to Eq. (13.1) from the Tutorial 14 : Relativistic spacetime, matter, and Gravitation, see the second exercise, Exercise 2, third question:

$$v := \frac{\epsilon^\alpha(v_\delta)}{\epsilon^0(v_\delta)} e_\alpha \quad (13.2)$$

Consequence: An observer  $(\gamma, e)$  will extract quantities measurable in his laboratory from objective spacetime quantities always like that.

Ex:  $F$  Faraday  $(0, 2)$ -tensor of electromagnetism:

$$F(e_a, e_b) = F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

observer frame  $e_a, e_b$

$$E_\alpha := F(e_0, e_\alpha)$$

$$B^\gamma := F(e_\alpha, e_\rho) \epsilon^{\alpha\beta\gamma} \text{ where } \epsilon^{123} = +1 \text{ totally antisymmetric}$$

### 13.3 Role of the Lorentz transformations

Lorentz transformations emerge as follows:

Let  $(\gamma, e)$  and  $(\tilde{\gamma}, \tilde{e})$  be observers with  $\gamma(\tau_1) = \tilde{\gamma}(\tau_2)$

(for simplicity  $\gamma(0) = \tilde{\gamma}(0)$ )

Now

$$\begin{array}{ll} e_0, \dots, e_1 & \text{at } \tau = 0 \\ \text{and } \tilde{e}_0, \dots, \tilde{e}_1 & \text{at } \tau = 0 \end{array}$$

both bases for the same  $T_{\gamma(0)}M$

$$\text{Thus: } \tilde{e}_a = \Lambda^b_a e_b \quad \Lambda \in GL(4)$$

Now:

$$\begin{aligned} \eta_{ab} &= g(\tilde{e}_a, \tilde{e}_b) = g(\Lambda^m_a e_m, \Lambda^n_b e_n) = \\ &= \Lambda^m_a \Lambda^n_b \underbrace{g(e_m, e_n)}_{\eta_{mn}} \end{aligned}$$

i.e.  $\Lambda \in O(1, 3)$

Result: Lorentz transformations relate the *frames* of *any two observers* at the same point.

“ $\tilde{x}^\mu - \Lambda^\mu_\nu x^\nu$ ” is utter nonsense

## Tutorial

I didn't see a tutorial video for this lecture, but I saw that the Tutorial sheet number 14 had the relevant topics. Go there.



## 14 Lecture 14: Matter

two types of matter

point matter

field matter

point matter

massive point particle

more of a phenomenological importance

field matter

electromagnetic field

more fundamental from the GR point of view

both classical matter types

### 14.1 Point matter

Our postulates (P1) and (P2) already constrain the possible particle worldlines.

But what is their precise law of motion, possibly in the presence of “forces”,

(a) without external forces

$$S_{\text{massive}}[\gamma] := m \int d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})}$$

with:

$$g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) > 0$$

dynamical law Euler-Lagrange equation

similarly

$$S_{\text{massless}}[\gamma, \mu] = \int d\lambda \mu g(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})$$

$$\begin{array}{ll} \delta_\mu & g(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) = 0 \\ \delta_\gamma & \text{e.o.m.} \end{array}$$

Reason for describing equations of motion by actions is that composite systems have an action that is the sum of the actions of the parts of that system, possibly including “interaction terms.”

Example.

$$S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$$

(b) presence of external forces

or rather presence of fields to which a particle “couples”

Example

$$S[\gamma; A] = \int d\lambda m \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})} + qA(v_{\gamma, \gamma(\lambda)})$$

where  $A$  is a **covector field** on  $M$ .  $A$  fixed (e.g. the electromagnetic potential)

Consider Euler-Lagrange eqns.  $L_{\text{int}} = qA_{(x)}\dot{\gamma}_{(x)}^m$

$$\underbrace{m(\nabla_{v_\gamma} v_\gamma)_a + \left( \frac{\partial \dot{L}_{\text{int}}}{\partial \dot{\gamma}_{(x)}^m} \right)}_* - \frac{\partial L_{\text{int}}}{\partial \gamma_{(x)}^m} = 0 \Rightarrow \boxed{m(\nabla_{v_\gamma} v_\gamma)^a = \underbrace{-qF_m^a \dot{\gamma}^m}_{\text{Lorentz force on a charged particle in an electromagnetic field}}}$$

$$\frac{\partial L}{\partial \dot{\gamma}^a} = qA_{(x)a}, \quad \left( \frac{\partial \dot{L}}{\partial \dot{\gamma}^m} \right) = q \cdot \frac{\partial}{\partial x^m} (A_{(x)m}) \cdot \dot{\gamma}_{(x)}^m$$

$$\frac{\partial L}{\partial \gamma^a} = q \cdot \frac{\partial}{\partial x^a} (A_{(x)m}) \dot{\gamma}^m$$

$$* = q \left( \frac{\partial A_a}{\partial x^m} - \frac{\partial A_m}{\partial x^a} \right) \dot{\gamma}_{(x)}^m = q \cdot F_{(x)am} \dot{\gamma}_{(x)}^m$$

$F \leftarrow$  Faraday

$$S[\gamma] = \int (m\sqrt{g(v_\gamma, v_\gamma)} + qA(v_\gamma)) d\lambda$$

## 14.2 Field matter

**Definition 60.** Classical (non-quantum) field matter is any tensor field on spacetime where equations of motion derive from an action.

Example:

$$S_{\text{Maxwell}}[A] = \frac{1}{4} \int_M d^4x \sqrt{-g} F_{ab} F_{cd} g^{ac} g^{bd}$$

$A$   $(0,1)$ -tensor field

= thought cloud: for simplicity one chart covers all of  $M$

– for  $\sqrt{-g}$  (+ – – –)

$$F_{ab} := 2\partial_{[a} A_{b]} = 2(\nabla_{[a} A_{b]})$$

Euler-Lagrange equations for fields

$$0 = \frac{\partial \mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s} \left( \frac{\partial \mathcal{L}}{\partial \partial_s A_m} \right) + \frac{\partial}{\partial x^s} \frac{\partial}{\partial x^t} \frac{\partial^2 \mathcal{L}}{\partial \partial_t \partial_s A_m}$$

Example ...

$$(\nabla_{\frac{\partial}{\partial x^m}} F)^{ma} = j^a$$

**inhomogeneous** Maxwell

thought bubble  $j = qv_\gamma$

$$\partial_{[a} F_{b]} - ()$$

homogeneous Maxwell

Other example well-liked by textbooks

$$S_{\text{Klein-Gordon}}[\phi] := \int_M d^4x \sqrt{-g} [g^{ab} (\partial_a \phi) (\partial_b \phi) - m^2 \phi^2]$$

$\phi$   $(0,0)$ -tensor field

### 14.3 Energy-momentum tensor of matter fields

At some point, we want to write down an action for the metric tensor field itself.

But then, this action  $S_{\text{grav}}[g]$  will be added to any  $S_{\text{matter}}[A, \phi, \dots]$  in order to describe the total system.

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]$$

$$\delta A \quad \Longrightarrow \quad \text{Maxwell's equations}$$

$$\delta g_{ab} \quad : \quad \boxed{\frac{1}{16\pi G} G^{ab}} + (-2T^{ab}) = 0$$

$G$  Newton's constant

$$G^{ab} = 8\pi G_N T^{ab}$$

**Definition 61.**  $S_{\text{matter}}[\Phi, g]$  is a matter action, the **so-called energy-momentum tensor** is

$$T^{ab} := \frac{-2}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{ab}} - \partial_s \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_s g_{ab}} + \dots \right)$$

– of  $\frac{-2}{\sqrt{g}}$  is Schrödinger minus (EY : 20150408 F.Schuller's joke? but wise)

choose all sign conventions s.t.

$$T(\epsilon^0, \epsilon^0) > 0$$

Example: For  $S_{\text{Maxwell}}$ :

$$T_{ab} = F_{am} F_{bn} g^{mn} - \frac{1}{4} F_{mn} F^{mn} g_{ab}$$

$$T_{ab} \equiv T_{\text{Maxwell}ab}$$

$$T(e_0, e_0) = \underline{E}^2 + \underline{B}^2$$

$$T(e_0, e_\alpha) = (E \times B)_\alpha$$

Fact: One often does not specify the fundamental action for some matter, but one is rather satisfied to assume certain properties / forms of

$$T_{ab}$$

Example Cosmology: (homogeneous & isotropic)

perfect fluid

of pressure  $p$  and density  $\rho$  modelled by

$$T^{ab} = (\rho + p)u^a u^b - p g^{ab}$$

radiative fluid

What is a fluid of photons:

$$T_{\text{Maxwell}}^{ab} g_{ab} = 0$$

$$\begin{aligned} \text{observe: } T_{\text{p.f.}}^{ab} g_{ab} &\stackrel{!}{=} 0 \\ &= (\rho + p) u^a u^b g_{ab} - p \underbrace{g^{ab} g_{ab}}_4 \end{aligned}$$

$$\leftrightarrow \rho_p 04p = 0$$

$$\rho = 3p$$

$$p = \frac{1}{3}\rho$$

Reconvene at 3 pm? (EY : 20150409 I sent a Facebook (FB) message to the International Winter School on Gravity and Light: there was no missing video; it continues on Lecture 15 immediately)

## Tutorial 14: Relativistic Spacetime, Matter and Gravitation

**Exercise 2: Lorentz force law.**

**Question electromagnetic potential.**

## 15 Lecture 15: Einstein gravity

Recall that in Newtonian spacetime, we were able to reformulate the Poisson law  $\Delta\phi = 4\pi G_N \rho$  in terms of the Newtonian spacetime curvature as

$$R_{00} = 4\pi G_N \rho$$

$R_{00}$  with respect to  $\nabla_{\text{Newton}}$

$G_N$  = Newtonian gravitational constant

This prompted Einstein to postulate < 1915 that the relativistic field equations for the Lorentzian metric  $g$  of (relativistic) spacetime

$$R_{ab} = 8\pi G_N T_{ab}$$

However, this equation suffers from a problem

LHS:  $(\nabla_a R)^{ab} \neq 0$

generically

RHS:

$$(\nabla_a T)^{ab} = 0$$

thought bubble: = formulated from an action

Einstein tried to argue this problem away.

Nevertheless, the equations cannot be upheld.

### 15.1 Hilbert

Hilbert was a specialist for variational principles.

To find the appropriate left hand side of the gravitational field equations, Hilbert suggested to start from an action

$$S_{\text{Hilbert}}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}$$

thought bubble = “simplest action”

aim: varying this w.r.t. metric  $g_{ab}$  will result in some tensor

$$G^{ab} = 0$$

### 15.2 Variation of $S_{\text{Hilbert}}$

$$0 \stackrel{!}{=} \underbrace{\delta}_{g_i} S_{\text{Hilbert}}[g] = \int_M \left[ \underbrace{\delta\sqrt{-g} g^{ab} R_{ab}}_1 + \underbrace{\sqrt{-g} \delta g^{ab} R_{ab}}_2 + \underbrace{\sqrt{-g} g^{ab} \delta R_{ab}}_3 \right]$$

and 1 :  $\delta\sqrt{-g} = \frac{-(\det g) g^{mn} \delta g_{mn}}{2\sqrt{-g}} = \frac{1}{2} \sqrt{-g} g^{mn} \delta g_{mn}$

thought bubble

$$\delta \det(g) = \det(g) g^{mn} \delta g_{mn}$$

e.g. from

$$\det(g) = \exp \text{tr} \ln g$$

ad 2:  $g^{ab}g_{bc} = \delta_c^a$

$$\begin{aligned} \implies (\delta g^{ab})g_{bc} + g^{ab}(\delta g_{bc}) &= 0 \\ \implies \delta g^{ab} &= -g^{am}g^{bn}\delta g_{mn} \end{aligned}$$

ad 3:

$$\begin{aligned} \Delta R_{ab} &\stackrel{\text{normal coords at point}}{=} \delta \partial_b \Gamma_{am}^m - \delta \partial_m \Gamma_{ab}^m + \Gamma \Gamma - \Gamma \Gamma = \\ &= \partial_b \delta \Gamma_{am}^m - \partial_m \delta \Gamma_{ab}^m = \\ &= \nabla_b (\delta \Gamma)_{am}^m - \nabla_m (\delta \Gamma)_{ab}^m \\ &\implies \sqrt{-g} g^{ab} \delta R_{ab} = \sqrt{-g} \end{aligned}$$

“if you formulate the variation properly, you’ll see the variation  $\delta$  commute with  $\partial_b$ ” EY : 20150408 I think one uses the integration at the bounds, integration by parts trick

$\Gamma_{(x)jk}^i - \tilde{\Gamma}_{(x)jk}^i$  are the components of a  $(1,2)$ -tensor.

Notation:  $(\nabla_b A)^i_g =: A^i_{j;b}$

$$\begin{aligned} &\implies \sqrt{-g} g^{ab} \delta R_{ab} \\ &\stackrel{\nabla g=0}{=} \sqrt{-g} (g^{ab} \delta \Gamma_{am}^m)_{;b} - \sqrt{-g} (g^{ab} \delta \Gamma_{ab}^m)_{;m} = \sqrt{-g} A^b_{;b} - \sqrt{-g} B^m_{;m} \end{aligned}$$

Question: Why is the difference of coefficients a tensor?

Answer:

$$\Gamma_{(y)jk}^i = \frac{\partial y^i}{\partial x^m} \frac{\partial x^m}{\partial y^j} \frac{\partial x^q}{\partial y^k} \Gamma_{(x),nq}^m + \frac{\partial y^i}{\partial x^m} \frac{\partial^2 x^m}{\partial y^j \partial y^k}$$

Collecting terms, one obtains

$$\begin{aligned} 0 &\stackrel{!}{=} \delta S_{\text{Hilbert}} = \int_M \left[ \frac{1}{2} \sqrt{-g} g^{mn} \delta g_{mn} g^{ab} R_{ab} - \sqrt{-g} g^{am} g^{bn} \delta g_{mn} R_{ab} + \underbrace{(\sqrt{-g} A^a)_{,a}}_{\text{surface}} - \underbrace{(\sqrt{-g} B^b)_{,b}}_{\text{surface term}} \right] \\ &= \int_M \sqrt{-g} \delta \underbrace{g_{mn}}_{\text{arbitrary variation}} \left[ \frac{1}{2} g^{mn} R - R^{mn} \right] \implies G^{mn} = R^{mn} - \frac{1}{2} g^{mn} R \end{aligned}$$

Hence Hilbert, from this “mathematical” argument, concluded that one may take

$$\boxed{R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_N T_{ab}}$$

Einstein equations

$$S_{E-H}[g] = \int_M \sqrt{-g} R$$

### 15.3 Solution of the $\nabla_a T^{ab} = 0$ issue

One can show ( $\longrightarrow$  Tutorials) that the Einstein curvature

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

satisfy the so-called contracted differential Bianchi identity

$$(\nabla_a G)^{ab} = 0$$

## 15.4 Variants of the field equations

(a) a simple rewriting:

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab} = T_{ab}$$

$$G_N = \frac{1}{8\pi}$$

Contract on both sides  $g^{ab}$

$$R_{ab} - \frac{1}{2}g_{ab}R = T_{ab} || g^{ab}$$

$$R - 2R = T := T_{ab}g^{ab}$$

$$\implies R = -T$$

$$\implies R_{ab} + \frac{1}{2}g_{ab}T = T_{ab}$$

$$\iff R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}) =: \hat{T}_{ab}$$

$$\boxed{R_{ab} = \hat{T}_{ab}}$$

(b)

$$S_{E-H}[g] := \int_M \sqrt{-g}(R + 2\Lambda)$$

thought bubble:  $\Lambda$  cosmological constant

History:

1915:  $\Lambda < 0$  (Einstein) in order to get a non-expanding universe

>1915:  $\Lambda = 0$  Hubble

today  $\Lambda > 0$  to account for an accelerated expansion

$\Lambda \neq 0$  can be interpreted as a contribution

$-\frac{1}{2}\Lambda g$  to the energy-momentum

“dark energy”

Question: surface terms scalar?

Answer: for a careful treatment of the surface terms which we discarded, see, e.g. E. Poisson, “A relativist’s toolkit” C.U.P. “excellent book”

Question: What is a constant on a manifold?

Answer:  $\int \sqrt{-g}\Lambda = \Lambda \int \sqrt{-g}1$

[back to dark energy]

[Weinberg, QCD, calculated]

idea: 1 could arise as the vacuum energy of the standard model fields

$$\Lambda_{\text{calculated}} = 10^{120} \times \Lambda_{\text{obs}}$$

“worst prediction of physics”

Tutorials: check that

- Schwarzschild metric (1916)

- FRW metric
- pp-wave metric
- Reissner-Nordström

$\implies$  are solutions to Einstein's equations

in high school

$$m\ddot{x} + m\omega^2 x = 0$$

$$x(t) = \cos(\omega t)$$

ET: [elementary tutorials]

study motion of particles & observers in Schwarzschild S.T.

Satellite: Marcus C. Werner

Gravitational lensing

odd number of pictures Morse theory (EY:20150408 Morse Theory !!!)

Domenico Giulini

Hamiltonian form Canonical Formulations

Key to Quantum Gravity



## 16 Lecture 22: Black Holes

Only depends on Lectures 1-15, so does lecture on “Wednesday”

Schwarzschild solution also vacuum solution (from tutorial EY : oh no, must do tutorial)

Study the Schwarzschild as a vacuum solution of the Einstein equation:

$m = G_N M$  where  $M$  is the “mass”

$$g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \frac{1}{1 - \frac{2m}{r}} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi)$$

in the so-called Schwarzschild coordinates

$$\begin{array}{cc} t & r \\ (-\infty, \infty) & (0, \infty) \end{array} \quad \begin{array}{cc} \theta & \varphi \\ (0, \pi) & (0, 2\pi) \end{array}$$

What staring at this metric for a while, two questions naturally pose themselves:

- (i) What exactly happens  $r = 2m$ ?

$$\begin{array}{cc} t & r \\ (-\infty, \infty) & (0, 2m) \cup (2m, \infty) \end{array} \quad \begin{array}{cc} \theta & \varphi \\ (0, \pi) & (0, 2\pi) \end{array}$$

- (ii) Is there anything (in the real world) beyond  $t \rightarrow -\infty$ ?

$$t \rightarrow +\infty$$

idea: Map of Linz, blown up

Insight into these two issues is afforded by stopping to stare.

Look at *geodesic* of  $g$ , instead.

### 16.1 Radial null geodesics

$$\text{null} - g(v_\gamma, v_\gamma) = 0$$

Consider null geodesic in “Schd”

$$S[\gamma] = \int d\lambda \left[ \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) \right]$$

with  $[\dots] = 0$

and one has, in particular, the  $t$ -eqn. of motion:

$$\left( \left(1 - \frac{2m}{r}\right) \dot{t} \right)' = 0$$

$\implies$

$$\boxed{\left(1 - \frac{2m}{r}\right) \dot{t} = k} = \text{const.}$$

Consider radial null geodesics

$$\theta \stackrel{!}{=} \text{const.} \quad \varphi = \text{const.}$$

From  $\square$  and  $\square$

$$\implies \dot{r}^2 = k^2 \leftrightarrow \dot{r} = \pm k$$

$$\implies r(\lambda) = \pm k \cdot \lambda$$

Hence, we may consider

$$\tilde{t}(r) := t(\pm k\lambda)$$

Case A:  $\oplus$

$$\frac{d\tilde{t}}{dr} = \frac{\dot{\tilde{t}}}{\dot{r}} = \frac{k}{\left(1 - \frac{2m}{r}\right)k} = \frac{r}{r - 2m}$$

$$\implies \tilde{t}_+(r) = r + 2m \ln |r - 2m|$$

(**outgoing** null geodesics)

Case b.  $\pm$  (Circle around  $-$ , consider  $-$ ):

$$\tilde{t}_-(r) = -r - 2m \ln |r - 2m|$$

(**ingoing** null geodesics)

Picture

## 16.2 Eddington-Finkelstein

Brilliantly simple idea:

change (on the domain of the Schwarzschild coordinates) to different coordinates, s.t.

in those new coordinates,

*ingoing* null geodesics appear as straight lines, of slope  $-1$

This is achieved by

$$\bar{t}(t, r, \theta, \varphi) := t + 2m \ln |r - 2m|$$

Recall: ingoing null geodesic has

$$\tilde{t}(r) = -(r + 2m \ln |r - 2m|) \quad (\text{Schdcoords})$$

$$\iff \bar{t} - 2m \ln |r - 2m| = -r - 2m \ln |r - 2m| + \text{const.}$$

$$\therefore \bar{t} = -r + \text{const.}$$

(Picture)

*outgoing* null geodesics

$$\bar{t} = r + 4m \ln |r - 2m| + \text{const.}$$

Consider the new chart  $(V, g)$  while  $(U, x)$  was the Schd chart.

$$\bigcup_{\text{Schd}} \{ \text{horizon} \} = V$$

“chart image of the horizon”

Now calculate the *Schd metric*  $g$  w.r.t. Eddington-Finkelstein coords.

$$\bar{t}(t, r, \theta, \varphi) = t + 2m \ln |r - 2m|$$

$$\bar{r}(t, r, \theta, \varphi) = r$$

$$\bar{\theta}(t, r, \theta, \varphi) = \theta$$

$$\bar{\varphi}(t, r, \theta, \varphi) = \varphi$$

EY : 20150422 I would suggest that after seeing this, one would calculate the metric by your favorite CAS. I like the Sage Manifolds package for Sage Math.

[Schwarzschild\\_BH.sage on github](#)

[Schwarzschild\\_BH.sage on Patreon](#)

[Schwarzschild\\_BH.sage on Google Drive](#)

```
sage: load('Schwarzschild_BH.sage')
4-dimensional manifold 'M'
  expr = expr.simplify_radical()
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
Launched png viewer for Graphics object consisting of 4 graphics primitives
```

Then calculate the Schwarzschild metric  $g$  but in Eddington-Finkelstein coordinates. Keep in mind to calculate the set of coordinates that uses  $\bar{t}$ , not  $\tilde{t}$ :

```
sage: gI.display()
gI = (2*m - r)/r dt*dt - r/(2*m - r) dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: gI.display( X_EF_I_null.frame())
gI = (2*m - r)/r dtbar*dtbar + 2*m/r dtbar*dr + 2*m/r dr*dtbar + (2*m + r)/r dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
```

## References

- [1] Eric Poisson, **A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics**, Cambridge University Press, 2004. ISBN 0 521 83091 5