Curve Construction

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Reading List

- John Hull, Options, Futures and Other Derivatives, Prentice Hall 2006.
- Steve Shreve, Stochastic Calculus for Finance II, Springer 2004.
- Martin Baxter and Andrew Rennie, Financial Calculus: An Introduction to Derivative Pricing, (CUP 1996).
- Mark Joshi, C++ Design Patterns and Derivative Pricing, (Cambridge University Press, 2004).
- Leif B.G. Andersen, Vladimir Piterbarg, Interest Rate Modeling, Vols. I, II, III, Atlantic Financial Press 2010.

1. Part I : Interest Rates Basics

2. Part II: Yield Curve Bootstrapping

3. Part III: Merton Model

Part I

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Zero-Coupon Bonds

- Zero-coupon bond is a contract that guarantees the holder $GBP1$,- at T. Its price at time t is denoted by $B(t,T)$. By definition

$$B(T,T) = 1 \quad , \tag{1}$$

and usually

$$0 < B(t,T) < 1 \quad . \tag{2}$$

To avoid arbitrage the bond function should be monotonically decreasing, i.e. $B(t,T_1) > B(t,T_2)$ for any $T_1 < T_2$, as otherwise you can sell the later bond and make more money by buying the shorter one. At T_1 receive £1.- and pay that later at T_2. Even without earning interest in the period $T_1 \to T_2$, there will be a profit of $B(t,T_2) - B(t,T_1)$ from this simple strategy.

The dependence of $B(t,T)$ on T is known as **Term-Structure** of discount factors, or **Zero-Coupon Curve** at time t. The curve is a decreasing function of maturity.

Simply Compounded Interest Rates

If one uses bonds of different maturities as investment instruments, in trying to figure out the yield of each bond investment, it would be helpful to have an way to strip out the maturity off of each bond, so as one can compare apples with apples; simply looking at the prices will not be useful as they correspond to different time lengths. However, simply dividing the bond price by the maturity years will not serve this purpose. One need to focus on the return of the investment and for the length of time the return is gained. The capital of GBP1 invested at $t = 0$ grows at time T_1 to $1/B(0, T_1)$, therefore the return would be $1/B(0, T_1) - 1$. Dividing this for the time period will give the quantity called yield which is now invariant of the maturity of each bond

$$1 + T_1 F(0, T_1) = \frac{1}{B(0, T_1)} \quad . \tag{3}$$

And from each bond $B(0, T_i)$ we extract a rate $F_i = F(0, T_i)$. The is the so-called yield per annum earned on bonds of maturity T_i, respectively.

The **simply compounded spot rate** at time t for maturity T_1, defined as annualized rate of return from holding the bond from time t to time T_1,

- $F(t, T_1)$

$$B(t, T_1) = \frac{1}{1 + (T_1 - t)F(t, T_1)} \quad , \tag{4}$$

resulting in

$$F(t, T_1) = \frac{1}{(T_1 - t)} \left(\frac{1}{B(t, T_1)} - 1 \right) \quad . \tag{5}$$

Interest from t to T_2, $F(t, T_2)$, is usually different from $F(t, T_1)$ reflecting the fact that interest earned on the two sub-periods $[t, T_1]$, $[T_1, T_2]$, are different, e.g. if interest rates are expected to rise or fall in the future.

Simply Compounded Interest Rates

But knowledge of any Spot term-structure leads to knowledge of Forward term-structure. Examples here are the forward rates and forward Bonds. This concept is similar to the inference of travel times between points B and C when travel times between A and B and A and C are known. If for example it takes 2 hours to travel between A and B and 3 hours to travel between A and C, obviously you can infer the time of travel between B and C. Similarly if mortgage rates for two years are 2%, and for 4 years are 4%, that would imply an interest in the vicinity of 6% between years 2 and 4.

The same obvious concept applies to the forward interest rate cases. If time value of money between now and year 10, and between now and year 15, are known, that would imply the current market consensus for the time value of money between year 10 and year 15 and that would be what the forward bonds are. In the same context we will use the concepts of forward rates, both for simple rates as well as for continuous rates.

Simply Compounded Interest Rates

If one decomposes time into small interest rate periods $[T_i, T_{i+1}]$

$$\begin{aligned} 1 + F(t, T_k)\,(T_k - t) \quad &= \quad \Big(1 + L(t; T_0, T_1)(T_1 - T_0)\Big) \cdot \\ &\qquad \cdots \\ &\qquad \cdot \Big(1 + L(t; T_{k-1}, T_k)(T_k - T_{k-1})\Big) \quad , \end{aligned} \tag{6}$$

here $t \leq T_0$. If we know all $F(t, T_i)$, $i = 1, \ldots k$, then we can calculate

$$F(t; T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1 + F(t, T_i)(T_i - t)}{1 + F(t, T_{i-1})(T_{i-1} - t)} - 1 \right) \quad . \tag{7}$$

$L(t; T_i, T_{i+1})$ is the interest earned over the future time period $[T_i, T_{i+1}]$ as given by a contract fixed at t. It is rational to represent each such rate by its own stochastic process.
Usually $F(t, T_i)$, $i = 1, \ldots k$, are known from the yield curve at t (see Eq.(5)).

- Example of simply compounded rate is the **London Interbank Offered Rate (LIBOR)**.

Forward Interest Rates

- What would cost to enter into an agreement to buy a T_2-bond at time T_1, where $t < T_1 < T_2$?

  At time T_1 we are going to pay A to buy the T_2-Bond. The time t value of the cashflows will be

  $$-AB(t, T_1) + B(t, T_2) = 0 \quad . \tag{8}$$

  This will be the value at time t of the forward discount bond

  $$A = B(t; T_1, T_2) = \frac{B(t, T_2)}{B(t, T_1)} \quad . \tag{9}$$

  The time t value of the simply compounded annualized forward rate from time T_1 to time T_2,

  $$B(t; T_1, T_2) = \frac{B(t, T_2)}{B(t, T_1)} = \frac{1}{1 + (T_2 - T_1)F(t; T_1, T_2)} \quad . \tag{10}$$

Forward Bond Replication

The forward bond can be replicated at time t, by taking a short position on a T_1-bond, to generate just enough funding to buy one T_2-bond $B(t, T_2)$. Therefore we need to take a short position at the cost of $B(t, T_2)$ on $B(t, T_1)$ bonds, meaning we sell $B(t, T_2)/B(t, T_1)$ amount of $B(t, T_1)$ bonds. The overall cost is zero, as shown below

$$-\left(A \equiv \frac{B(t, T_2)}{B(t, T_1}\right) B(t, T_1) + B(t, T_2) = 0 \quad . \tag{11}$$

At time T_1 we have to pay the investment on the T_1 bond which equals $B(t, T_2)/B(t, T_1)$. And at T_2 we will receive one unit of currency from the long position in the T_2-bond.
Therefore at t we have locked the payment at T_1 to receive £1 at T_2. The amount paid at T_1 will be $B(t, T_2)/B(t, T_1)$.
The above strategy, simply put, is holding a long position on $B(0, T_2)$ and a short position on $B(0, T_1)$.
(There are two rates related with the ratio of these two bonds, a flat-continuous-compounded $R(t; T_1, T_2)$, and a simple compounded $F(t; T_1, T_2)$ - which one is bigger?).

Forward Bond Replication

In the above, the notional of $B(t, T_1)/B(t, T_2)$ paid at T_1 grows $(T_2 - T_1)$ years later to £1

$$\frac{B(t, T_2)}{B(t, T_1)}\Big(1 + (T_2 - T_1)F\Big) = 1 \quad , \tag{12}$$

resulting in a rate of (notice that there is no payment made at t)

$$F(t; T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{B(t, T_1)}{B(t, T_2)} - 1\right) \quad . \tag{13}$$

This is the first example of replication in finance and concept of price. If an instrument can be replicated with simpler instruments, the price of the replicating portfolio is the price of the more exotic instrument. Both the product and the replicating portfolio are stochastic but they replicate each other perfectly in this case.

For instance the amount that will need to be paid at T_1 for interest accruing to T_2 is not known now, and it is in general a stochastic quantity, the so-called Libor rate. But it can be replicated now thus we can talk of a price for it. In fact this is an example of static replication, where we set the replicating portfolio at the very beginning of the contract and then wait till the expiry. Our portfolio will generate that exact payoff, and thus we don't need to do anything else any more.

Concept of Replication - Hedging Libor

Notice that from Eq.(10), the Forward Rate and Forward Bond can be used interchangeably. If the Forward Bond is known then also the forward rate is known and vice-versa. Therefore the above replication of the Forward Bond, is an implicit replication of the Forward rate for the period $[T_1, T_2]$.

Lets assume we hold a long position on $B(0, T_1)$ and a short position in $B(0, T_2)$, i.e. we possess a portfolio of two bonds of value $B(0, T_1) - B(0, T_2)$. As we approach T_1 our portfolio will have a value of

$$B(T_1, T_1) - B(T_1, T_2) = \tau_1 B(T_1, T_2) \left[\frac{1}{\tau_1} \left(\frac{B(T_1, T_1)}{B(T_1, T_2)} - 1 \right) \right] \quad , \tag{14}$$

and this is the same as the Libor Rate paid at T_2 but discounted to T_1. The price of this payment at $t = 0$ is $B(0, T_1) - B(0, T_2)$, which further can be written as

$$B(0, T_2) L(0; T_1, T_2) = E\Big[D(0, T_{i-1}) B(T_1, T_2) L(T_1; T_1, T_2) \Big] \quad . \tag{15}$$

This gives the expected value (price) of the future stochastic Libor payment. The calculation has been done in the risk neutral-measure where the payoff $B(T_{i-1}, T_{i-1}) - B(T_{i-1}, T_i)$ when discounted is martingale.

Concept of Replication - Hedging Libor

Its value at T_2 will be $\tau_1 L(T_1; T_1, T_2)$, indeed the payment expected from the Libor leg of a swap. Therefore the portfolio of bonds as follows

$$(B(t, T_0) - B(t, T_1)) + \ldots (B(t, T_{n-1}) - B(t, T_n)) = B(t, T_0) - B(t, T_n) \quad (16)$$

will replicate perfectly the float leg payments of a swap, no matter of the fact that the Libor is stochastic and it will change. The reason being that our portfolio of bonds is also stochastic and matches the changes of the Libors perfectly. Therefore the price of the floating leg of the swap is equal to

$$\sum_{i=1}^{n} \tau_i E\Big[B(T_{i-1}, T_i) L(T_{i-1}; T_{i-1}, T_i) | \mathcal{F}_0\Big] = B(0, T_0) - B(0, T_n) \quad , \qquad (17)$$

no matter how the Libor rates will change over time. This is the price that the investors are willing to pay for a Libor leg payments now. Any other price, higher or lower, of the portfolio that can replicate the payoff perfectly, if it existed, it will create arbitrage opportunities in the market (which presumably the market will move quickly to take advantage and thus remove it).
$B(T_{i-1}, T_i)$ and $L(T_{i-1}; T_{i-1}, T_i)$ are both stochastic in the entire interval $[0, T_{i-1}]$. Libor $L(T_{i-1}; T_{i-1}, T_i)$ resets at T_{i-1}, whereas $B(T_{i-1}, T_i)$ will continue to be stochastic till T_i.

Concept of Replication - Hedging Libor

Before we move on to other concepts, let us think of what would happen if we change the payment time of the Libor. We assume the Libor that fixes at T_1 is paid at this same time, T_1, instead of at T_2 which was the case in the standard Swaps.
At T_1 we will have a payoff of

$$\tau_1 B(T_1, T_1) \frac{1}{\tau_1} \left(\frac{B(T_1, T_1)}{B(T_1, T_2)} - 1 \right) \quad . \tag{18}$$

This is a portfolio of bonds at T_1

$$\left(\frac{B(T_1, T_1)}{B(T_1, T_2)} \right) B(T_1, T_1) - B(T_1, T_1) \quad , \tag{19}$$

whose value at $t = 0$ cannot be written any more in terms of the $B(0, T_1)$ and $B(0, T_2)$. However, we can make an approximations and assume that the ratio is not very stochastic and freeze it to its initial value,

$$\frac{B(T_1, T_1)}{B(T_1, T_2)} \approx \frac{B(0, T_1)}{B(0, T_2)} \quad . \tag{20}$$

Concept of Replication - Hedging Libor

The best portfolio that comes close to hedging this product is holding long and short position of same bonds at T_1, as follows

$$\left(\frac{B(0,T_1)}{B(0,T_2)}\right) B(T_1,T_1) - B(T_1,T_1) \quad , \tag{21}$$

but this means that T_1 we will only replicate the following payoff

$$\tau_1 B(T_1,T_1)\frac{1}{\tau_1}\left(\frac{B(0,T_1)}{B(0,T_2)} - 1\right) = \tau_1 B(T_1,T_1) L(0;T_1,T_2) \quad . \tag{22}$$

What remains now to be hedged is $L(T_1;T_1,T_2) - L(0;T_1,T_2)$ or equivalently the difference between the inverse of the forward bonds

$$\frac{1}{B(T_1;T_1,T_2)} - \frac{1}{B(0;T_1,T_2)} = \frac{CVX}{B(0;T_1,T_2)} - \frac{1}{B(0;T_1,T_2)} \quad , \tag{23}$$

which are martingales in the T_2-forward measure, which is different from the one of the payment date T_1. The calculation of the convexity, CVX for brief notation, will be given later in the short-rate modeling.

Continuously compounded rates

Rates quoted in the market are always simply compounded. But it can be mathematically more convenient to work with continuously compounded rates. Denoted by $R(t, T)$ and expressed in terms of T-Bond as

$$B(t, T) = e^{-R(t,T)(T-t)} \quad , \quad \text{(time is annualized)} \quad . \tag{24}$$

$$R(t, T) = -\frac{\ln B(t, T)}{(T - t)} \quad . \tag{25}$$

The continuously compounded spot rate can be thought of as a measure of the implied interest rate offered by the bond for the period $[t, T]$ and is referred to as the **yield to maturity**.
The graph of $R(t, T)$ versus maturity T is known as the **yield curve**.

Continuously Compounded Forward Rate

In addition to simply compounded forward rates $F(t; S, T)$ we can define also the continuously compounded forward rates

$$B(t; S, T) = e^{-R(t;S,T)(T-S)} \quad , \tag{26}$$

which can be expressed as

$$R(t; S, T) = -\frac{\ln B(t; S, T)}{(T-S)} = -\frac{\ln B(t, T) - \ln B(t, S)}{(T-S)} \quad . \tag{27}$$

Using Eq. (25)

$$R(t; S, T) = \frac{R(t, T)(T-t) - R(t, S)(S-t)}{(T-S)} \quad . \tag{28}$$

This is a flat rate observed now at t which compounds continuously from S to T. This rate is a measure of the ratio of two bonds of different maturities $B(t, T)/B(t, S)$. The non-arbitrage condition of first page, expressed in terms of $R(t; S, T)$ means that this rate should be always positive!

Continuously Compounded Forward Rate

Notice that the use of the exponentials simplifies the expressions, as can be seen by comparing the simple and the continuous compounding rates

$$R(t; S, T) = \frac{1}{T-S}\Big[R(t,T)(T-t) - R(t,S)(S-t)\Big] \quad , \tag{29}$$

$$F(t; S, T) = \frac{1}{T-S}\left[\frac{F(t,T)(T-t) - F(t,S)(S-t)}{1+F(t,T)(T-t)}\right] \quad . \tag{30}$$

Instantaneous Rates

- Instantaneous forward rate $f(t,T)$.

  At time t for maturity T, the instantaneous fwd rate $f(t,T)$, can be thought of as the rate of return over an infinitesimally small time interval $[T, T+\delta T]$

$$f(t,T) = \lim_{\delta T \to 0} R(t; T, T+\delta T) = -\frac{\ln B(t,T+\delta T) - \ln B(t,T)}{\delta T}$$

$$= -\frac{\partial \ln B(t,T)}{\partial T} \quad . \tag{31}$$

  The dependence of $f(t,T)$ on T is called term structure of forward rates at time t.

- Short Rate $r(t)$ - the instantaneous short rate or the **risk free rate** is the rate of return over an infinitesimally small time interval $[t, t+\delta t]$

$$r(t) = f(t,t) \quad . \tag{32}$$

Relationship between yield and inst fwd rates

What did the forward bond agreement tell us? It told us that \$1 at time T, will have at time S a price different from \$1. In fact the market now thinks that the fair price for that contract will be $A = B(t,T)/B(t,S)$. When we reach the time S the price for \$1 at T will most likely change, but A discussed above, is the fair price today.

By comparing the price for \$1 received at T with its market value at t for payment at S, we can infer what the todays value for discounting between the time points S and T is, namely $F(t;S,T)$ or $L(t;S,T)$

$$L(t;S,T) = \frac{1}{(T-S)}\left(\frac{B(t,S)}{B(t,T)} - 1\right) = \frac{1}{(T-S)}\left(\frac{1}{B(t;S,T)} - 1\right) . \quad (33)$$

Taking the small distance limit of $T \to S$, or expressed differently $F(t;T,T+\delta T) \equiv f(t,T)$

$$f(t;T) = \frac{1}{\delta T}\left(\frac{B(t,T)}{B(t,T+\delta T)} - 1\right) \Leftrightarrow B(t;T,T+\delta T) = e^{-f(t,T)\delta T} . \quad (34)$$

This is the consensus now for the rate of discounting between two future time intervals.

Relationship between yield and inst fwd rates

Putting together time intervals between t and T we have

$$\begin{aligned} B(t,T) &= B(t;t,t+\delta t)B(t;t+\delta t,t+2\delta t)\cdots B(t;T-\delta t,T) \quad , \\ &= \frac{B(t,t+\delta t)}{B(t,t)}\cdot\frac{B(t,t+2\delta t)}{B(t,t+\delta t)}\cdots\frac{B(t,T)}{B(t,T-\delta t)} \quad , \qquad (35) \\ &= \exp\Big\{-f(t,t)\delta t\Big\}\cdot\exp\Big\{-f(t,t+\delta t)\delta t\Big\}\cdots\exp\Big\{-f(t,T)\delta t\Big\} \, , \\ &= \exp\Big\{-\int_t^T f(t,u)\mathrm{d}u\Big\} \quad . \qquad (36) \end{aligned}$$

If we were to enter into a contract to receive Libor at T for a fixed payment K at T the logic would go similarly. For the forward bond the market sets today the fair strike to be A, which in terms of Libor rate that resets at S would be [\$1 at S, will get \$ $(1+(T-S)F(t;S,T))$ or A at S, \$1 at T]. To remain self-consistent, for A we must have $A = 1/[1+(T-S)F(t;S,T)]$. For the Libor rate the fair strike should be $K = F(t;S,T) = (1/B(t;S,T)-1)/(T-S)$. This is the FRA, thus equivalent to the forward bond discussion. Conditions on forward bonds can translate to conditions on forward rates.

Forward bonds and forward rates

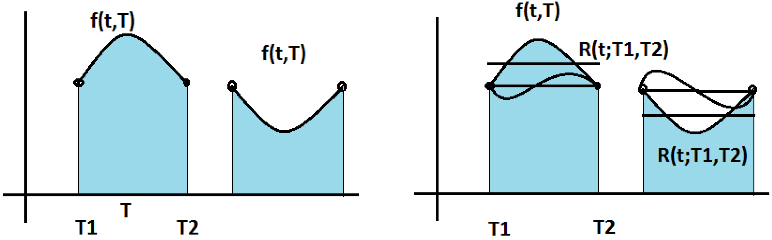


Figure: The forward bond price is dependent on the area under the forward rate $\int_{T_1}^{T_2} f(t,T)\mathrm{d}T$. The discrete forward rate $R(t;T_1,T_2)$ does not capture intra-interval $[T_1,T_2]$ fluctuations of f, Eq. (37). Small f first and larger later is equivalent to larger f first and smaller later so long as area under f is invariant.

What is important in the forward bond equation is the area under the instantaneous forward bonds, not just the start and end points,

$$B(t,T_1,T_2) = \exp\left\{-\int_{T_1}^{T_2} f(t,T)\mathrm{d}T\right\} \quad . \tag{37}$$

Forward bonds and forward rates

The area under the forward rate $f(t, T)$ is a measure of the future discounting between T_1 and T_2. A large area signifies strong discounting, a small area signifies small future discounting.
The discrete forward rate is flat and does not capture any shapes changes of the forward rate unless they affect the whole area under the curve of the rate f.
We will see in the short-rate modeling later that even fluctuations that leave the area under f invariant will still affect expected values of future discounting through its non-zero variance.

Relationship between yield and inst fwd rates

Going back to Eq. (31)

$$
\begin{aligned}
f(t,T) &= -\frac{\partial \ln B(t,T)}{\partial T} = \frac{\partial\Big(R(t,T)(T-t)\Big)}{\partial T} \quad , &(38)\\
&= R(t,T) + R'(t,T)(T-t) \quad . &(39)
\end{aligned}
$$

The forward rate is higher than the yield curve when the yield curve is normal, and below when it is inverted.
Integrating Eq. (38),

$$
\int_{T_i}^{T_{i+1}} f(t,T)\mathrm{d}T = \int_{T_i}^{T_{i+1}} \mathrm{d}\Big(R(t,T)(T-t)\Big) \quad , \tag{40}
$$

$$
R(t,T_{i+1})(T_{i+1}-t) - R(t,T_i)(T_i-t) = \int_{T_i}^{T_{i+1}} f(t,T)\mathrm{d}T \quad . \tag{41}
$$

Notice that due to curvature of the R-curve the surfaces under $R_i T_i$ and $R_{i-1} T_{i-1}$ don't cancel exactly below T_{i-1}, therefore to adjust the F curve has to be a bit displaced upward/downward depending on the sign of R'_{i-1}, unless the yield curve is flat, in which case the yield curve and f-curve coincide.

Relationship between yield and fwd rates (cont.)

Dividing both sides of Eq. (41) we get

$$\frac{R(t,T_{i+1})(T_{i+1}-t)-R(t,T_i)(T_i-t)}{T_{i+1}-T_i}=\frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}}f(t,T)\mathrm{d}T\ . \qquad (42)$$

From Eqs. (27)-(28) [here we note $R_i = R(t, T_i)$]

$$R(t;T_{i-1},T_i)=\frac{R_i(T_i-t)-R_{i-1}(T_{i-1}-t)}{T_i-T_{i-1}}=\frac{1}{T_i-T_{i-1}}\int_{T_{i-1}}^{T_i}f(t,u)\mathrm{d}u\ . \qquad (43)$$

The rate $R(t;T_{i-1},T_i)$ is a flat rate in (T_{i-1},T_i) which compounds continuosly. The f rate is also continuously compunded but is flat only in the smaller interval $(u,u+\delta u)\in(T_{i-1},T_i)$. The average of the instantaneous fwd rates in the integral is equal to the discrete fwd rate $R(t;T_{i-1},T_i)$ of the interval $[T_{i-1},T_i]$. Another useful relationship between the rates

$$R(t,T)(T-t)=R(t,T_{i-1})(T_{i-1}-t)+\int_{T_{i-1}}^{T}f(t,u)\mathrm{d}u\ ,\ T\in[T_{i-1},T_i]\ . \qquad (44)$$

If f is constant, then "$R(T)T$" grows linearly from $R_{i-1}T_{i-1}$.

Money Market Account (Continuous)

The money market account is security where interest accrues continuously at the risk free rate $r(t)$. Money market account at t is denoted by $B(t)$, and is defined by the differential equation

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t \quad , \quad B(0) = 1 \quad . \tag{45}$$

Solving (45) we find

$$B(t) = \exp\left(\int_0^t r(u)\mathrm{d}u\right) \quad . \tag{46}$$

Money invested at t earns $r(t)\delta t$ over the time period $[t, t+\delta t]$ and it is re-invested continuously.
With the money market account we usually build the bank-account numeraire.
Notice: The area under $\int_0^t r(u)\mathrm{d}u$ is what counts in the value of the above. We will see that fluctuations that leave the area unchanged will affect the calculations in the short rate modeling later. In stochastic calculus variances are also important in addition to mean values.

Money Market Account (Discrete)

If 1 unit of currency is invested at $T_0 = 0$, at T_1 we receive $1 + L(T_0; T_0, T_1)(T_1 - T_0)$. If we reinvest this over the next period and so on, over $[T_j, T_{j+1}]$, for $j = 1, 2, \dots$ then at T_N the one unit of ccy invested at T_0 has become

$$B(T_N) = \prod_{j=0}^{N-1} \Big(1 + L(T_j; T_j, T_{j+1})(T_{j+1} - T_j)\Big) \quad . \tag{47}$$

In terms of forward rates

$$B(T_N) = \prod_{j=0}^{N-1} \exp\left(\int_{T_j}^{T_{j+1}} f(T_j, u)\mathrm{d}u\right) \quad . \tag{48}$$

If consider small intervals $[T_j, T_j + \delta]$, the above becomes

$$B(T_N) = \prod_{j=0}^{N-1} \exp\left(\int_{T_j}^{T_{j+1}} f(T_j, T_j)\mathrm{d}T_j\right) = \exp\left(\int_{T_0}^{T_N} r(u)\mathrm{d}u\right) \quad , \tag{49}$$

same as in previous page.

Simple Market Instruments: Coupon Bonds, Swaps, Caps, Floors

Bonds

In most Bond markets the most traded instruments are the coupon Bonds. Usually there are very few zero-coupon Bonds. Coupon Bonds issued by the US Treasury are usually of semi-annual coupon payments, whereas in European markets bonds issues are usually annual.
Bond issued by the US treasury are divided into three categories:

- Bills: zero-coupon bonds with maturity less than one year.
- Notes: coupon bonds (semi-annual) with time to maturity between 2 and 10 years.
- Bonds: coupon bonds with time to maturity between 10 and 30 years.

Coupon-bearing Bonds

A **fixed-coupon bond** pays the holder deterministic amounts $c_1, c_2, \ldots c_n$ which are called coupon payments at times $T_1, T_2, \ldots T_n$ where $T_0 < T_1 < \ldots T_n$. At maturity time T_n the holder receives the coupon c_n but also the full notional or face value N. The value of the coupon-bond at t is

$$B_{\text{fixed}}(t) = \sum_{i=1}^{n} c_i B(t, T_i) + NB(t, T_n) \quad . \tag{50}$$

A **floating-coupon bond** pays at T_i the floating Libor rate $L(T_{i-1}, T_i)$ multiplied by $\tau_i = T_{i-1} - T_i$ and N-notional

$$c_i = \tau_i NL(T_{i-1}, T_i) = N \left(\frac{B(T_{i-1}, T_{i-1})}{B(T_{i-1}, T_i)} - 1 \right) \quad . \tag{51}$$

Let us look at the coupon payments one by one. At T_i, we receive

$$N\left(\frac{B(T_{i-1},T_{i-1})-B(T_{i-1},T_i)}{B(T_{i-1},T_i)}\right) \quad . \tag{52}$$

If we discount this payment to T_{i-1}, the value of the floating coupon at T_{i-1}

$$N\left(B(T_{i-1},T_{i-1})-B(T_{i-1},T_i)\right) \quad , \tag{53}$$

which is a portfolio of bonds. The time t value of these portfolio of bonds is

$$N\Big(B(t,T_{i-1})-B(t,T_i)\Big) \quad . \tag{54}$$

The above is just the today's value of floating coupon received at T_i. Putting them all together, we have

$$\begin{aligned} B_{\text{floating}} &= N\sum_{i=1}^{n}\left(B(t,T_{i-1}-B(t,T_i)\right)+NB(t,T_n) \quad , && (55)\\ &= NB(t,T_0) \quad . && (56) \end{aligned}$$

Yield and Duration

For zero-coupon bonds the yield of the bond corresponds to the zero-rate $R(t, T)$

$$B(t, T) = e^{-R(t,T)(T-t)} \quad . \tag{57}$$

For coupon bonds with fixed payments $c_1, c_2, \ldots c_n$ at times $T_1, T_2, \ldots T_n$ where $T_0 < T_1 < \ldots T_n$, the yield-to-maturity is given by matching the price of the bond with the one given by a continuously compounded flat rate $y(t)$ on all payment dates [are these fixed coupons say 5%, multpilied by τ_i??? XXX]

$$B(t) = \sum_{i=1}^{n} c_i B(t, T_i) \quad , \tag{58}$$

and equate this with

$$B(t) = e^{-y(t) D_{\mathrm{Mac}}} \quad . \tag{59}$$

$y(t)$ is the yield taken to be the same for all payment dates such that it matches the bond price

$$\sum_{i=1}^{n} c_i e^{-y(t)(T_i - t)} = \sum_{i=1}^{n} c_i e^{-R(t,T_i)(T_i - t)} \quad . \tag{60}$$

Yield and Duration

The quantity above is called the Macaulay duration and can be taken by Eq.(59) by taking the ratio of the y-derivative of the bond with the bond price,

$$D_{\mathrm{Mac}} = -\frac{\partial_y B(t)}{B(t)} \quad , \tag{61}$$

take here $t = 0$, to simplify notation, and we have

$$D_{\mathrm{Mac}} = \sum_{i=1}^{n} T_i \left[\frac{c_i e^{-yT_i}}{B(0)} \right] = \sum_{i=1}^{n} T_i w_i \quad , \quad \sum_{i=1}^{n} w_i = 1 \quad . \tag{62}$$

The weights w_i signify the portion of the bond paid at each time T_i.
The Macaulay duration can be thought of as an average time to coupon payment

$$D_{\mathrm{Mac}} = \sum_{i=1}^{n} T_i w_i = \bar{T} \quad . \tag{63}$$

The Macaulay duration measure the sensitivity of the bond versus changes of the yield-to-maturity of the bond.

The Macaulay duration measures the time in point where all the payments can be reproduced by a zero-coupon bond, (the zero-coupon bond has the same Macaulay duration as its maturity). This quantity measures the first-order sensitivity of the coupon-bond to its yield to maturity.
However, in the above no term-structure of the yield has been assumed. Instead all payments have been discounted with the same, yield-to-maturity rate $y(t)$. A better quantity is the Duration of the coupon-Bond, where all future coupon payments are discounted by the actual zero-rate $y(T_i)$

$$D = \sum_{i=1}^{n} T_i \left[\frac{c_i e^{-y_i T_i}}{B(0)} \right] = \sum_{i=1}^{n} T_i \left[\frac{c_i B_i(0)}{B(0)} \right] \quad , \tag{64}$$

where $y_i = R(0, T_i)$. This quantity measures the sensitivity of the price of the bond versus parallel shifts of the yield curve

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{i=1}^{n} c_i e^{-(y_i+s) T_i} \right) \bigg|_{s=0} = -DB(0) \quad . \tag{65}$$

The Duration therefore is like the Delta sensitivity for the Bonds against parallel shifts of the yield curve, similar to the stock sensitivity of the option on stocks. The second order sensitivity against parallel shifts of the yield curve is called Convexity and it is similar to Gamma of the stock options

$$C = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\sum_{i=1}^{n} c_i e^{-(y_i+s)T_i}\right)\Bigg|_{s=0} = \sum_{i=1}^{n} T_i^2 c_i e^{-y_i T_i} \quad . \tag{66}$$

These two sensitivities give the change in price of the bond versus parallel shifts of the yield curve

$$\Delta B \approx -D\,B\,\Delta y + \frac{1}{2}C(\Delta y)^2 \quad , \tag{67}$$

where $\Delta y = s$ is the parallel shift, the spread s in the previous equation.

Duration and Convexity Clean Price, Dirty Price, and Accrued Interest

Figure: The time horizon for a swap is split over $T_0, T_1, \dots T_n$. The floating rate accrues over the intervals $[T_{i-1}, T_i]$ for $i = 1, \dots, n$. For each period $[T_{i-1}, T_i]$ the Libor rate $L(t_{i-1}; T_{i-1}, T_i)$ fixes at t_{i-1}, usually two business deays before T_{i-1}. In the standard swaps the Libor rate is paid at T_i.

The Libor rates fulfill

$$L(t_{i-1}; T_{i-1}, T_i) = \frac{1}{\tau_i}\left(\frac{B(t_{i-1}, T_{i-1})}{B(t_{i-1}, T_i)} - 1\right) \quad , \tag{68}$$

where τ_i is the length of the interval $[T_{i-1}, T_i]$ in the appropriate day count convention.

Swaps

Allowing here for different notionals per period, i.e. amortizing Swap with deterministic notional, the floating side of the swap is written as

$$V_{float}(t) = \sum_{i=1}^{n} N_i B(t, T_i) \tau_i L(t_{i-1}; T_{i-1}, T_i) \quad . \tag{69}$$

Based on the martingale fact of Libor rate being ratio of two consecutive bonds

$$E^{Q_{T_i}} \left[L(t_{i-1}; T_{i-1}, T_i) \right] = L(t; T_{i-1}, T_i) \quad , \tag{70}$$

giving for the floating side the following

$$\begin{aligned} V_{float}(t) &= \sum_{i=1}^{n} N_i B(t, T_i) \tau_i L(t; T_{i-1}, T_i) \quad , && (71) \\ &= \sum_{i=1}^{n} N_i \left(B(t, T_{i-1}) - B(t, T_i) \right) \quad . && (72) \end{aligned}$$

For non-amortizing swap, $N_1 = \ldots = N_n$, the sum simplifies to

$$V_{float}(t) = N \left(B(t, T_0) - B(t, T_n) \right) \quad . \tag{73}$$

Swaps

A **payer swap** is a swap that receives a floating payment and pays a fixed rate

$$
\begin{aligned}
PS(t) &= \sum_{i=1}^{n}(B(t, T_{i-1}) - B(t, T_i)) - K \sum_{i=1}^{n} \tau_i B(t, T_i) \ , && (74) \\
&= B(t, T_0) - B(t, T_n) - K \sum_{i=1}^{n} \tau_i B(t, T_i) \quad , && (75) \\
&= \sum_{i=0}^{n} c_i B(t, T_i) \quad , && (76)
\end{aligned}
$$

where

$$
c_0 = 1 \ , \ c_i = -\tau_i K \ , \ c_n = -(1 + \tau_n K) \quad . \tag{77}
$$

Swap rate, the rates that makes the swap's value equal to zero

$$
S_{0,n}(t) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \quad . \tag{78}
$$

Annuity is the sum of dicounted accrual periods. This measure the contraction of the time away from $T_n - T_0$ that is caused by discounting.

Swaps as weighted sums of forward rates

The Swap Rate can be written as an weighted sum of Libor rates

$$S_{0,n}(t) = \sum_{j=0}^{n-1} w_j L_j \ , \quad \text{with } w_j \geq 0 \ , \quad \sum_{j=0}^{n-1} w_j = 1 \ . \tag{79}$$

The weights are given by

$$w_j(t) = \frac{B(t, T_{j+1})\,(T_{j+1} - T_j)}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \quad , \tag{80}$$

since

$$\begin{aligned} w_j L_j \quad &= \quad \frac{B(t, T_{j+1})\,(T_{j+1} - T_j)}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \frac{1}{(T_{j+1} - T_j)} \left(\frac{B(t, T_j) - B(t, T_{j+1})}{B(t, T_{j+1})} \right) \quad , \\ &= \quad \frac{B(t, T_j) - B(t, T_{j+1})}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \quad . \end{aligned} \tag{81}$$

Obviously, the sum over all $j = 0, \cdots n-1$, gives for the denominator $B(t, T_0) - B(t, T_n)$. Notice that weights $w_j(t)$ are stochastic and correlated with L_j's.

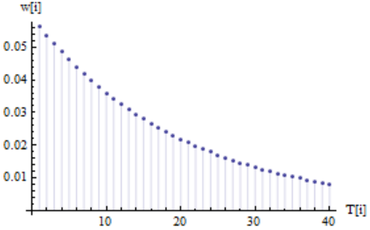


Figure: Swap weights w_i of Eq. (80) for annual 40y Swap. The rate here is taken to be flat at $r = 5\%$. The biggest contribution in the Swap Rate comes from the short end Libors on the curve. Here $w_1/w_{40} \approx 10$.

Swaps as weighted sums of forward rates

We will see later, once the Hull-White model has been introduced, that the stochastic process for the weights

$$w_i(t) = \frac{\tau_i B(t, T_i)}{A(t)} \quad . \tag{82}$$

are martingales in the annuity measure. They have the form

$$\frac{\tau_i B(t, T_i)}{A(t)} = \frac{\tau_i B(0, T_i)}{A(0)} \exp\left\{ \sigma \sqrt{\frac{e^{2kt} - 1}{2k}} \sum_{j=1}^{n} w_j(0) \left(\frac{e^{-kT_i} - e^{-kT_j}}{k} \right) X - \frac{1}{2}\alpha_i^2(t) \right\}$$

where the variances of the weight processes are

$$\alpha_i^2(t) = \sigma^2 \left(\frac{e^{2kt} - 1}{2k} \right) \left[\sum_{j=1}^{n} w_j(0) \left(\frac{e^{-kT_i} - e^{-kT_j}}{k} \right) \right]^2 \quad . \tag{84}$$

Swaps with Libor in Arrears

What if i make a very tiny modification to the contractual terms of the standard Swap, namely, if I ask for the Libors that reset at T_i to be also paid at T_i instead of paid in the following date of T_{i+1} as is the case for the standard Swaps? It will turn out that this tiny modification is not as innocent as it looks. We will notice that convexity corrections need to be added to the pricing and it will be discussed in detail later, but for now, the floating side payoff will look like

$$V_{float}^{Arrears}(t) = \sum_{i=1}^{n} N_i \underline{\underline{B(T_{i-1}, T_{i-1})}} \tau_i L(T_{i-1}; T_{i-1}, T_i) \quad , \tag{85}$$

as compared with

$$V_{float}^{Standard}(t) = \sum_{i=1}^{n} N_i \underline{\underline{B(T_{i-1}, T_i)}} \tau_i L(T_{i-1}; T_{i-1}, T_i) \quad . \tag{86}$$

What if I change the notional of the in Arrears Swap and re-scale each of them to $N_i = NB(t, T_i)/B(t, T_{i-1})$, will the two swaps be the same then?

The payments of the in Arrears Swap with rescaled notionals versus the standard Swap will look like

$$V_{float}^{Arrears}(t) = \sum_{i=1}^{n} \left(N \frac{B(t, T_i)}{B(t, T_{i-1}} \right) \underline{\underline{B(T_{i-1}, T_{i-1})}} \tau_i L(T_{i-1}; T_{i-1}, T_i) \quad , \tag{87}$$

as compared with

$$V_{float}^{Standard}(t) = \sum_{i=1}^{n} N \underline{\underline{B(T_{i-1}, T_i)}} \tau_i L(T_{i-1}; T_{i-1}, T_i) \quad . \tag{88}$$

They seem to be exactly the same. That is not correct! The proper calculation needs to take into account the fact that the two quantities in the calculation of the expected vale are stochastic and there is a covariance between them. This covariance is called convexity and will discuss about it in detail later

$$\begin{aligned} E\left[B(T_{i-1}, T_{i-1}) L(T_{i-1}; T_{i-1}, T_i)\right] &= B(0, T_i) E^{Q_{T_i}} \left[\frac{B(T_{i-1}, T_{i-1})}{B(T_{i-1}, T_i)} L_i(T_{i-1}) \right] , \\ &\neq B(0, T_{i-1}) L(0; T_{i-1}, T_i) \quad . \end{aligned} \tag{89}$$

If we push the calculations a bit further we can get some useful hints. Let us use the following equation

$$E[XY] = E[X]E[Y] + \mathrm{Cov}[XY]\,,\ \mathrm{Cov}[XY] = E[(X-\bar{X})(Y-\bar{Y})] \quad . \tag{90}$$

Therefore

$$\begin{aligned} E^{Q_{T_i}}\left[\frac{B(t,T_{i-1})}{B(t,T_i)}L(t;T_{i-1},T_i)\right] &= E^{Q_{T_i}}\left[\frac{B(t,T_{i-1})}{B(t,T_i)}\right]E^{Q_{T_i}}\Big[L(t;T_{i-1},T_i)\Big] \\ &+ \mathrm{Cov}\left[\frac{B(t,T_{i-1})}{B(t,T_i)},L(t;T_{i-1},T_i)\right] \end{aligned} \tag{91}$$

Both quantities are martingales in the T_i-forward measure and therefore

$$\begin{aligned} = \frac{B(0,T_{i-1})}{B(0,T_i)}L(0;T_{i-1},T_i) &+ E^{Q_{T_i}}\Big[\big(1+\tau_i L(t;T_{i-1},T_i)\big)L(t;T_{i-1},T_i)\Big] \\ &-\big(1+\tau_i L(0;T_{i-1,T_i})\big)L(0;T_{i-1},T_i) \end{aligned} \tag{92}$$

Therefore the difference between the re-scaled notional of the in Arrears Swap and the standard Swap is

$$\frac{B^2(0,T_i)}{B(0,T_{i-1})}\left(E^{Q_{T_i}}\left[\big(1+\tau_i L_i(T_{i-1})\big)L_i(T_{i-1})\right]-\big(1+\tau_i L_i(0)\big)L_i(0)\right)$$

$$= \frac{B^2(0,T_i)}{B(0,T_{i-1})}E^{Q_{T_i}}\left[\tau_i L_i^2(T_{i-1})) - \tau_i L_i^2(0)\right]$$

$$= \frac{B(0,T_i)\tau_i L^2(0;T_{i-1,T_i})}{1+\tau_i L(0;T_{i-1},T_i)}\left(e^{\sigma^2 T_{i-1}}-1\right) \quad , \tag{93}$$

$$\approx \frac{B(0,T_i)\tau_i L^2(0;T_{i-1,T_i})}{1+\tau_i L(0;T_{i-1},T_i)}\ \sigma^2 T_{i-1} \quad . \tag{94}$$

Notice that if there is no volatility then there is no difference between the rescaled in Arrears Swap and the standard Swap. The difference is proportional $\sigma^2 T_{i-1}$. If we adjust for the strike on the in Arrears Swap then it is worth drawing the PV difference as function of rates and volatility.

Summary of the Section: understanding questions

- I. Can you derive the relationship between instantaneous forward rates $f(0, T)$ and the zero-Rates $R(T)$

$$f(T) = R(T) + R'(T)T \quad . \tag{95}$$

: Going back to Eq. (31)

$$\begin{aligned} f(t, T) &= -\frac{\partial \ln B(t, T)}{\partial T} = \frac{\partial\Big(R(t, T)(T - t)\Big)}{\partial T} \quad , & (96) \\ &= R(t, T) + R'(t, T)(T - t) \quad . & (97) \end{aligned}$$

The forward rate is higher than the yield curve when the yield curve is normal, and below when it is inverted.

Summary of the Section: understanding questions

- Derive the fact that the flat continuous forward rate is the average of the instantaneous forward rates

$$R(t; T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T_i} f(t, u) \mathrm{d}u \quad . \tag{98}$$

Also note that the zero-Rates are flat-continuous forward rates but with term from $T_{i-1} = 0$ to $T_i = T$, same as above

$$R(0; T) = \frac{1}{T} \int_0^T f(0, u) \mathrm{d}u \quad . \tag{99}$$

- If $f(t) = \sqrt{T}$, what is $R(T)$? What are the quarterly flat-continuous forward rates $R(0; T_{i-1}, T_i)$, and what are the quarterly $L(0; T_{i-1}, T_i)$? Why are Libor rates higher than the flat continuous forward rates? Remember the following relationship

$$L(t; T_{i-1}, T_i) \approx R(t; T_{i-1}, T_i) + \frac{1}{2} R^2(t; T_{i-1}, T_i)(T_i - T_{i-1}) + \dots . \tag{100}$$

Summary of the Section: understanding questions

- Why the price of floating leg of a swap is $B(0, T_0) - B(0, T_n)$ - i.e. not stochastic at all even though the Libor rate is stochastic and it will change over time? Another example is the case of European Option on a stock with infinite volatility with very high upside on payoff, but with price no more than S_0.

  : Because the future Libor payments can be replicated by a portfolio of bonds which you can buy at $T = 0$. These will be also stochastic but replicate the Libors movements perfectly.

  In the case of the stock with high volatility we can forget about about the strike as $S(T)$

Table: Spot and Forward Rates in IR Modeling

Rates	Notation	Definition
Spot - Simple	$F(t,T)$	$\frac{1}{T-t}\left(\frac{1}{B(t,T)}-1\right)$
Spot - Continuous	$R(t,T)$	$-\frac{1}{T-t}\ln B(t,T)$
Forward - Simple	$F(t;T_1,T_2)$	$\frac{1}{T_2-T_1}\left(\frac{B(t,T_1)}{B(t,T_2)}-1\right)$
Forward - Continuous	$R(t;T_1,T_2)$	$-\frac{1}{T_2-T_1}\ln\left(\frac{B(t,T_2)}{B(t,T_1)}\right)$
Spot - Instantaneous	$r(t)$	$f(t,t)$
Forward - Instantaneous	$f(t,T)$	$-\frac{\partial}{\partial T}\ln B(t,T)$

Table: Accrual of Notional invested at t for Spot and at T_1 for Forwards

Rates	Notation	Notional Accrual
Spot - Simple	$F(t, T)$	$1 \longrightarrow 1 + (T - t)F(t, T)$
Spot - Continuous	$R(t, T)$	$1 \longrightarrow e^{R(t,T)(T-t)}$
Forward - Simple	$F(t; T_1, T_2)$	$1 \longrightarrow 1 + (T_2 - T_1)F(t; T_1, T_2)$
Forward - Continuous	$R(t; T_1, T_2)$	$1 \longrightarrow e^{R(t;T_1,T_2)(T_2-T_1)}$
Spot - Instantaneous	$r(t)$	$1 \longrightarrow e^{\int_t^T r(u)\mathrm{d}u}$
Forward - Instantaneous	$f(t, T)$	$e^{\int_{T_1}^{T_2} f(t,T)\mathrm{d}T}$

Caps, Floors, Swaptions

Caps, Floors

A Cap is an interest rate contract intended to prevent losses to an investor that pays Libor rate. The contract is extended over a time period $T_0, \cdots, T_n$, say of quarterly spacing for example (as is the case in USD). The investor worried about Libor rate exceeding a fixed rate K during this period enters into the Cap whereby the investor would receive the difference $(L_i - K)^+$ at the end of each period T_i, for $i = 1, \cdots, n$, in which the interest rate exceeds the agreed strike. Each one of these constituent options are called Caplets.
Of course, the investor with Libor payments, can enter into a Swap whereby he pays fixed and receives floating which he then transfers to his counteraparty. The Fixed-Float Swap will protect the invetor against a rise in interest rates. However, if the Libor rates fall below K, he will not be able to take advantage this way of the low rates environments. Caps instead allow him to take advantage of low rates while offering protection against higher rates environments.

A Floor acts in the opposite way, it offers protection to the investor receiving floating rates against low-rates environments while taking advantage of high rate environments. The payoff is $(K - L_i)^+$ at the end of each period T_i, for $i = 1, \cdots, n$.

A Cap is a strip of Caplets and has the following mathematical form

$$\sum_{i=1}^{n} B(t, T_i) N \tau_i \Big(L(t_{i-1}; T_{i-1}, T_i) - K \Big)^{+} \quad , \tag{101}$$

where the fixing date t_{i-1} is usually two business days before T_{i-1}. To exercise a Cap, its purchaser generally does not have to notify the seller, because the Cap will be exercised automatically if the interest rate exceeds the strike rate. Note that this automatic exercise feature is different from most other types of options. Each Caplet is settled in cash at the end of the period to which it relates to.
It is customary in the market to price Caps with the Black-Scholes formula

$$\mathrm{Cap}(K, t; T_0, \cdots, T_n) = \sum_{i=1}^{n} \mathrm{Bl}\Big(L(t; T_{i-1}, T_i), \sigma(t)\sqrt{T_{i-1} - t}; K \Big) \tag{102}$$

where $\sigma(t)$ is the Cap implied volatility and this is the same for all Caplets belonging to a Cap.

Black's formula for the ith Caplet is

$$\mathrm{Bl}\Big(L_i(t), \sigma(t)\sqrt{T_{i-1}-t}; K\Big) = N\tau_i B(t, T_i)\Big(L_i(t)N(d_{i1}) - KN(d_{i2})\Big) \quad , \tag{103}$$

where

$$d_{i1,2} = \frac{\log\left(\frac{L_i(t)}{K}\right) \pm \frac{1}{2}\sigma^2(T_{i-1}-t)}{\sigma\sqrt{T_{i-1}-t}} \quad . \tag{104}$$

A similar formula applies for the case of Floorlets.
Caps/Floors are quoted in the market in terms of their implied volatilities. The typical case is of $t=0$, $T_0=\tau_0$, $\tau_i = T_i - T_{i-1}$ equal to 3 months for the US market and 6 months for the EUR market.
A table of US, EUR, and GBP implied ATM vols for all these markets and a table for ATM implied vols for different maturities.

Swaptions

There are two kinds of Swaptions, 1) with Cash settlement, and 2) with Swap settlement. The underlying of the Swaption with cash settlement is the stochastic swap rate. These are typical for the European markets, usually the EUR and GBP Swaptions. The payoff at expiry time T_e is:

$$V(T_e) = \Big(SR(T_e) - K\Big)^+ \sum_{i=1}^{n} N\tau_i B(T_e, T_i) \quad , \tag{105}$$

$$= \Big(SR(T_e) - K\Big)^+ \sum_{i=1}^{n} \frac{N\tau_i}{(1+\tau_i SR(T_e))^{T_i - T_e}} \quad , \tag{106}$$

where the standard discounting has been replaced by

$$B(T_e, T_i) = \frac{1}{\left(1 + \frac{SR(T_e)}{m}\right)^{\frac{T_i - T_e}{m}}} \quad . \tag{107}$$

Here m is the swap frequency, $m = 4$ for quarterly swaps. Using standard pricing theory the value of the Swaption at $t = 0$ can be written as

$$V(0) = B(0, T_e) E^{Q_{T_e}} \left[V(T_e)\right] \quad . \tag{108}$$

Swaptions

The Swaptions with Swap settlements is a product whereby the buyer has the right to enter into a Swap where the payer pays fixed coupons and receives variable payments. In a receiver Swap the payment types are reversed.
The payoff of a Swaption with Swap settlement is given by

$$
\begin{aligned}
V(T_e) &= \Big(V_{float}(T_e) - V_{fix}(T_e)\Big)^+ \quad , &(109)\\
&= \Big(S(T_e)A(T_e) - V_{fix}(T_e)\Big)^+ \quad , &(110)
\end{aligned}
$$

where the Annuity $A(\cdot)$ at expiry T_e is given by

$$
A(T_e) = \sum_{i=1}^{n} \tau_i B(T_e, T_i) \quad . \tag{111}
$$

It is not unusual for Swaptions to contain also barriers like *down and in barrier* and *up and in barrier*

$$
V(T_e) = \Big(S(T_e)A(T_e) - V_{fix}(T_e)\Big)^+ 1_{\{S(T_e)>U\}} 1_{\{S(T_e)<D\}} \quad . \tag{112}
$$

Swaptions

A Swaption with expiry in x years and underlying swap of tenor y years is called a $x \times y$ Swaption. Liquid Swaptions are 5y5y Swaptions, 10y10y, 10y15y, etc.

Swaptions can be used to synthetically create callable bonds. Assume a company has issued a 10y bond to fund itself, paying in return an annual fixed coupon of a fixed rate $K = 5\%$ plus the notional at the end. The company might want to call the bond early, say in 5y and prepay, in case its own funding becomes cheaper at that point. In such a case the company buys a receiver Swaption, with same fixed rate of 5%. If in 5y the floating rates are cheaper than the fixed rate, then the Swaption will be in the money and thus exercised. The fixed payments will cancel the fixed coupons of the bonds. The floating payments on the other hand plus the notional payment at year 10, is equivalent to prepaying the notional at year 5, as desired.

CMS Rates, CMS Caps/Floors and CMS Swaptions

Exercise 1. Consider a Swap with reset and cashflow dates

$$0 < T_0 < T_1 < \cdots < T_n \tag{113}$$

T_0 is the first reset date. Take the accrual periods to be equal to $\tau_i \equiv \tau = 0.25$y, with fixed rate payment $\tau K N$, where N is the notional.
(a) Show that at $t = 0$ the value of a payer Swap equals

$$\Pi_p(t) = N\tau(R_{swap}(t) - K)\sum_{i=1}^{n} B(t, T_i) \quad . \tag{114}$$

(b) Now consider a numerical example: today $t = 0$, first reset date is $T_0 = 0.25$y. Cash flow dates are $T_i = T_{i-1} + 0.25$, maturity is $T_7 = 2$y. The forward curve is given by

$$\begin{bmatrix} F(0;0,0.25) \\ \vdots \\ F(0;1.75,2) \end{bmatrix} = \begin{bmatrix} 6\% \\ 9\% \\ 10\% \\ 10\% \\ 10\% \\ 9\% \\ 9\% \\ 9\% \end{bmatrix}$$

Find the term structure of the bonds

$$B(0,T_0),\ldots,B(0,T_7) \quad .$$

(c) Find the corresponding Swap Rate

$$R_{swap}(0) \quad .$$

Exercise 2. Consider a Swap Cap, Floor determined by the reset and cashflow dates

$$0 < T_0 < T_1 < \cdots < T_n \tag{115}$$

T_0 is the first reset date. Take the accrual periods to be equal to $\tau_i \equiv \tau = 0.25$y, with fixed rate payment $\tau K N$, where N is the notional.

(a) Show that the cash flow of the ith caplet

$$\tau\Big(L(T_{i-1}, T_i) - K\Big)^+$$

at time T_i is equivalent to the cash flow at $T_{i-1}$of a put option on a T_i bond

$$(1 + \tau K)\left(\frac{1}{1 + \tau K)} - B(T_{i-1}, T_i)\right)$$

with modified strike and notional.

Exercises

(b) Show that a payer Swaption price is always dominated by the corresponding Cap price.

(c) Prove the parity relationship

$$Cp(t) - Fl(t) = \Pi_p(t) \quad , \quad Swpt_p(t) - Swpt_p(t) = \Pi_p(t) \quad .$$

Exercise 3. Now assume we have a coupon bond of semi-annual coupons (American coupon bonds are semi-annual, and European Coupon bonds are annual) with $c_i = 5\%$ and nominal $N = 100$

(3.a) What is the price of the coupon bond with 2 year maturity?

(3.b) What is its continuously compounded yield-to-maturity y?

(3.c) Compute the yield curve $y_i = R(0, T_i)$, $i = 1, \dots 8$.

(3.d) Compute the Macaulay duration D_{Mac}, the duration D, and the convexity C of the bond.

(3.e) Consider a parallel shift of the yield curve

$$y_i \to \tilde{y}_i + s \quad , \quad i = 1 \dots , 8 \quad ,$$

by $s = 0.0001$ (one basis point) and $s = 0.01$. How does the bond price $b(t)$ change?

(2.f) Consider the first- and second-order approximations

$$b(t) - D_{Mac}bs \quad , \quad b - Dbs \quad , \quad b - Dbs + \frac{1}{2}Cs^2 \quad .$$

1. Part I : Interest Rates Basics
2. Part II: Yield Curve Bootstrapping
3. Part III: Merton Model

Part II: Yield Curve Bootstrapping

Yield Curve Construction

The yield curve represents the cost of borrowing for banks across various maturities and it is closely related with the Libor rates. The Libor rates are collected daily by the FCA (Bank of England and averaged across the various banks). Usually it represents the funding costs of the banks on an average sense. Depending on the perceived strength of each bank, the costs of borrowing can be at various spreads from the Libor, for some these spreads will be negative and for some positive and varying over time.
The yield curve is usually bootstrapped from liquid market instruments. The liquidity varies for different maturities in the market. In the first three months usually Cash Deposits are used, for one day, one week, one month, two months and three months. Then from three months to two years the Futures are the most liquidly traded instruments. Beyond that point the Interest Rate Swaps are liquid to long maturities usually up to 30 years or more depending on the currency.

Yield Curve Construction

After the financial crisis of 2008 and with the collapse of Lehman Brothers, counterparties have started to ask each other for collateral, as a from of protection against default. This has changed the standard practice of single curves and introduced an additional curve, representing the cost of the collateral and it is thus used for discounting. Usually this is obtained from the Overnight Index Swaps (OIS). More recently this will change to Secured Overnight Funding Rates (SOFR) Swap rates.
We can start here with a simplified version of the current market situation and leave the complexities for later chapters.

There are several ways of building a yield curve from quoted swap rates in the market. We are given a number of spot start swap rates of various maturities T_{n_i}, usually $T_{n_i} = 1, \ldots 5, 7, 10, 12, 15, 20, 25, 30$. For quarterly paying swaps, with 4 Libors per year, $n_i = 4T_i = 4, 8, \ldots 120$. As can be observed some swap rates for non-liquid points are missing. Filling the gap points, like the 6y, 8y, 9y etc, will be discussed as part of the interpolation-bootstrapping and there are several ways of proceeding. This point will be discussed later.
For a start we assume that we know all the Swap Rates starting from zero up to the end of the time horizon, say up to $t_n = 30$y. The time horizon is split in $t_0, t_1, \ldots, t_n$, which have distance of 1y (simplified to annual for now). Thus we assume that we know all the spot starting Swap Rates SR_j, $j = 1, 2 \ldots, 30$. We show next that from this knowledge we can extract all the bonds $B(0, t_j) \equiv B_j$.

Yield Curve Construction (cont. 2)

From the Swap Rates relationship we have

$$1 - B_1 = SR_1 \left(\tau_1 B_1\right) \Longrightarrow B_1 = \frac{1}{1 + \tau_1 SR_1} \quad . \tag{116}$$

Looking into the second spot starting Swap, where below $A_j = \sum_{i=1}^{j} \tau_i B_i$

$$1 - B_2 = SR_2 \sum_{i=1}^{2} \tau_i B_i \Longrightarrow B_2 = \frac{1 - SR_2 \left(\tau_1 B_1\right)}{1 + \tau_2 SR_2} = \frac{1 - SR_2 A_1}{1 + \tau_2 SR_2} \tag{117}$$

$$B_2 = \frac{1 - \tau_1 \left(SR_2 - SR_1\right)}{\left(1 + \tau_1 SR_1\right)\left(1 + \tau_2 SR_2\right)} \quad , \tag{118}$$

where there is a correction to B_2 below, of the amount $\tau_1(SR_2 - SR_1)$ discounted with B_2^0, (but no correction if the curve is flat $SR_1 = SR_2$)

$$B_2^0 = \frac{1}{\left(1 + \tau_1 SR_1\right)\left(1 + \tau_2 SR_2\right)} \quad , \tag{119}$$

$$B_2 = B_2^0 - B_2^0 \tau_1 (SR_2 - SR_1) \quad . \tag{120}$$

Yield Curve Construction (cont. 2)

Before taking a real example, we can extract the yield curve by the knowledge of the Swap Rates, at first semi-analytically. From the previous page, we have

$$R_1 = -\frac{\ln B_1}{\tau_1} = \frac{1}{\tau_1}\ln(1+\tau_1 SR_1) \approx \frac{1}{\tau_1}\tau_1 SR_1 = SR_1 \quad . \tag{121}$$

For the second bond we have

$$\begin{aligned} R_2 &= -\frac{1}{\tau_1+\tau_2}\ln B_2^0(1-\tau_1(SR_2-SR_1)) \quad , & (122)\\ &= \frac{1}{\tau_1+\tau_2}\left[\ln(1+\tau_1 SR_1)(1+\tau_2 SR_2) - \ln(1-\tau_1(SR_2-SR_1))\right] \quad , & \\ &\approx \frac{1}{\tau_1+\tau_2}(\tau_1 SR_1 + \tau_2 SR_2 + \tau_1 SR_2 - \tau_1 SR_1) \quad , & (123)\\ R_2 &= SR_2 \quad . & (124) \end{aligned}$$

This means that the yield curve looks very similar to the Swap-Rate curve, with minor differences due to Taylor approximation above. Now use a real example to compare the yield curve with the Swap-Rate curve!

Yield Curve Construction (cont. 3)

From SR_1 we calculated therefore B_1, then from this we calculate A_1. From knowledge of SR_2 and A_1, we calculate B_2, and thus A_2, and so on, from A_{j-1} and SR_j we can extract B_j.

$$1 - B_j = SR_j \sum_{i=1}^{j} \tau_i B_i = SR_j(A_{j-1} + \tau_j B_j) \quad . \tag{125}$$

Grouping the last bonds together, we get

$$B_j(1 + \tau_j SR_j) = 1 - SR_j A_{j-1} \tag{126}$$

B_j in terms of SR_j and A_{j-1} is given as follows

$$B_j = \frac{1 - SR_j A_{j-1}}{1 + \tau_j SR_j} \quad , \tag{127}$$

meaning that the knowledge of $SR_1, SR_2, \ldots SR_n$, we can calculate $B_1, B_2, \ldots, B_n$. In fact based on co-initial or co-terminal Swap Rates, Gallucio and Hunter have introduced the Swap Rates Model as an alternative to the Libor Market Model.

Yield Curve Construction (cont. 3)

The calculations here simplify considerably if we consider the yield curve to be flat. This will give for the Annuities

$$A_j = \frac{\tau}{1+\tau S} + \frac{\tau}{(1+\tau S)(1+\tau S)} + \cdots \frac{\tau}{(1+\tau S)^j} \quad , \tag{128}$$

$$= \frac{\tau}{1+\tau S} \frac{\left(1 - \frac{1}{(1+\tau S)^j}\right)}{1 - \frac{1}{1+\tau S}} = \frac{1}{S}\left(1 - \frac{1}{(1+\tau S)^j}\right) \quad . \tag{129}$$

Using the previous pages equation

$$B_j = \frac{1 - SR_j A_{j-1}}{1 + \tau_j SR_j} \quad , \tag{130}$$

we get

$$B_j = \frac{1 - SA_{j-1}}{1+\tau S} = \frac{1 - S\frac{1}{S}\left(1 - \frac{1}{(1+\tau S)^{j-1}}\right)}{1+\tau S} = \frac{1}{(1+\tau S)^j} \quad , \tag{131}$$

which is of course self-consistent with the Eq (128).

Similar calculations can be used to extract bond prices from the co-terminal Swap Rates. Let us denote with A_j the annuity of the Swap starting at T_j and ending at the last payment date T_n. Then we can express the ratios of the bonds B_j/B_n, starting from the shortest co-terminal Swap, we can express $B_n/B_n = 1$, B_{n-1}/B_n and our way backwards to B_1/B_n, as follows

$$SR_{n-1} = \frac{B_{n-1} - B_n}{A_{n-1}} \Longrightarrow \frac{B_{n-1}}{B_n} = 1 + SR_{n-1}\frac{A_{n-1}}{B_n} \quad . \tag{132}$$

Similarly assuming that for j the ratios B_k/B_n for $k > j$ have been found, we can calculate

$$\frac{B_j}{B_n} = 1 + SR_j\frac{A_j}{B_n} = 1 + SR_j\sum_{k>j}\tau_k\frac{B_k}{B_n} \quad . \tag{133}$$

This way we can calculate the ratio B_1/B_n, in which B_1 can be calculated by $B_1 = (1 + \tau_1 SR_1)^{-1}$, giving therefore from the bond ratios B_j/B_n for $j = 1, \ldots, n$, all the bond prices B_j.

Yield Curve Construction (cont. 5)

The most usual way to build the yield curve is by assuming that the instantaneous forward rates are piece-wise flat between the two swap maturity dates T_{n_i} and $T_{n_{i+1}}$, where $n_1 < n_2 \cdots$

$$f(t,T) = \begin{cases} f_1 \,, & \text{for } t \leq T \leq T_{n_1} \,, \\ f_{i+1} \,, & \text{for } T_{n_i} \leq T \leq T_{n_{i+1}} \,. \end{cases} \tag{134}$$

For $i = 1$ and $j = 0, \ldots, n_1$

$$B(t,T_j) = \exp\left(- \int_t^{T_j} f(t,u)\mathrm{d}u \right) = \exp\left(-f_1(T_j - t) \right) \quad . \tag{135}$$

Yield Curve Construction (cont. 6)

Bond $B(t, T_{n_1})$:

$$\exp(-f_1(T_0 - t)) - \exp(-f_1(T_{n_1} - t)) = S_1 \sum_{j=1}^{n_1} \tau_j \exp(-f_1(T_j - t)) \,. \tag{136}$$

Solve this for f_1 by Newton-Raphson method.
For the ith forward rate proceed iteratively. We know all the forward rates up to maturity T_{n_i}. We assume the next forward rate is constant f_{i+1} out to maturity $T_{n_{i+1}}$. We now solve the following for f_{i+1}

$$\begin{aligned} B(t, T_0) - B(t, T_{n_{i+1}}) &= S_{i+1} \sum_{j=1}^{n_{i+1}} \tau_j B(t, T_j) \quad , \qquad (137) \\ &= S_{i+1} \left(\sum_{j=1}^{n_i} \tau_j B(t, T_j) + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t, T_j) \right) \quad . \end{aligned}$$

$$B(t, T_{n_{i+1}}) + S_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t, T_j) = B(t, T_0) - S_{i+1} \sum_{j=1}^{n_i} \tau_j B(t, T_j) \,. \tag{138}$$

For $j = n_i + 1, \dots, n_{i+1}$ we have

$$B(t, T_j) = \exp\left(-\int_t^{T_j} f(t,u)\mathrm{d}u\right) = B(t, T_{n_i})\exp\left(-\int_{T_{n_i}}^{T_j} f(t,u)\mathrm{d}u\right) ,$$

$$= B(t, T_{n_i})\exp\left(-f_{i+1}(T_j - T_{n_i})\right) \quad . \tag{139}$$

We now solve (138) for f_{i+1}

$$\exp\left(-f_{i+1}(T_{n_{i+1}} - T_{n_i})\right) + S_{i+1}\sum_{j=n_i+1}^{n_{i+1}} \tau_j \exp\left(-f_{i+1}(T_j - T_{n_i})\right) \tag{140}$$

$$= B^{-1}(t, T_{n_i})\left(B(t, T_0) - S_{i+1}\sum_{j=1}^{n_i}\tau_j B(t, T_j)\right) .$$

In the simple case of all τ_i being all equal to 0.25, the above equations can be solved semi-analytically. For the first year

$$1-\exp\{-f_1\}=\frac{1}{n}S_1\sum_{i=1}^{T_{n_1}}\exp\left\{-f_1\frac{i}{n}\right\}=\frac{1}{n}S_1\exp\left\{-\frac{f_1}{n}\right\}\frac{1-\exp\{-f_1\}}{1-\exp\left\{-\frac{f_1}{n}\right\}} \tag{141}$$

where $n=4$ and $T_{n_1}=1$. Canceling $(1-\exp\{-f_1\})$ on both sides, we get

$$1=\frac{S_1}{n}\frac{\exp\left\{-\frac{f_1}{n}\right\}}{1-\exp\left\{-\frac{f_1}{n}\right\}} \quad , \tag{142}$$

resulting in

$$f_1=n\ln\left(1+\frac{S_1}{n}\right)\approx S_1 \quad . \tag{143}$$

Yield Curve Construction (cont. 9)

We can now continue iteratively, as follows

$$B(t,T_0) - B(t,T_{n_{i+1}}) = S_{i+1} \sum_{j=1}^{n_{i+1}} \tau_j B(t,T_j) \quad , \tag{144}$$

$$B(t,T_0) - B(t,T_{n_{i+1}}) = S_{i+1} \left(\sum_{j=1}^{n_i} \tau_j B(t,T_j) + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t,T_j) \right) \quad , \tag{145}$$

$$B(t,T_0) - B(t,T_{n_{i+1}}) = S_{i+1} \left(\frac{B(t,T_0) - B(t,T_{n_i})}{S_i} + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t,T_j) \right) \tag{146}$$

We can split using $Z_{i+1} = \exp\{-f_{i+1}/n\}$, where usually $n = 4$,

$$\begin{aligned} B(t,T_{n_i+1}) &= B(t,T_{n_i}) Z_{i+1} \quad , & (147) \\ B(t,T_{n_i+2}) &= B(t,T_{n_i}) Z_{i+1}^2 \quad , & (148) \\ B(t,T_{n_{i+1}}) &= B(t,T_{n_i}) Z_{i+1}^n \quad . & (149) \end{aligned}$$

Yield Curve Construction (cont. 10)

Substituting in Eq. (146)

$$B(t,T_0) - B(t,T_{n_i}) Z_{i+1}^n = S_{i+1} \left[\frac{B(t,T_0) - B(t,T_{n_i})}{S_i} + \frac{B(t,T_{n_i})}{n} \sum_{j=1}^{n} Z_{i+1}^j \right]$$

Equivalently

$$\begin{aligned} B(t,T_0) - \left(B(t,T_0) - B(t,T_{n_i})\right) \frac{S_{i+1}}{S_i} &= B(t,T_{n_i}) Z_{i+1}^n \\ &+ B(t,T_{n_i}) \frac{S_{i+1}}{n} \frac{Z_{i+1}(1 - Z_{i+1}^n)}{1 - Z_{i+1}} \quad . \end{aligned} \tag{150}$$

This leads to an $(n+1)$-order equation

$$Z_{i+1}^n + \frac{S_{i+1}}{n} \frac{Z_{i+1}(1 - Z_{i+1}^n)}{1 - Z_{i+1}} = \frac{B(t,T_0)}{B(t,T_{n_i})} - \left(\frac{B(t,T_0)}{B(t,T_{n_i})} - 1 \right) \frac{S_{i+1}}{S_i} \quad . \tag{151}$$

For $n = 1$, the equation reduces to first order

$$Z_{i+1} = \frac{1}{1 + S_{1+1}} \left[\frac{B(t,T_0)}{B(t,T_{n_i})} - \left(\frac{B(t,T_0)}{B(t,T_{n_i})} - 1 \right) \frac{S_{i+1}}{S_i} \right] \quad . \tag{152}$$

Yield Curve Construction (cont. 11)

Think in terms of filling missing values of the swaps at empty spaces in the tenor structure. Filling out the missing info is usually done through an interpolation. There are several kinds. The one described above is linear in the $R(t,T)T$. But there are others, like linear interpolation in the yield $R(t,T)$, etc.
To make use of the interpolation schemes given below we can think as follows.
Write Eq. (78)

$$
\begin{aligned}
B(t,T_0) - B(t,T_n) &= S_n \sum_{j=1}^{n} \tau_j B(t,T_j) \quad , \qquad (153)\\
B(t,T_0) - B(t,T_n) &= S_n \sum_{j=1}^{n-1} \tau_j B(t,T_j) + S_n \tau_n B(t,T_n) \, ,\\
B(t,T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t,T_j) &= B(t,T_n) + S_n \tau_n B(t,T_n) \quad , \qquad (154)
\end{aligned}
$$

$$
B(t,T_n)(1 + S_n \tau_n) = B(t,T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t,T_j) \quad . \qquad (155)
$$

In case of missing nodes, the following is a useful iteration formula!
We can write

$$B(t, T_n) = \frac{B(t, T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t, T_j)}{1 + S_n \tau_n} \quad . \tag{156}$$

From here we write

$$R(t, T_n) = -\frac{1}{\tau_n} \ln \left[\frac{B(t, T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t, T_j)}{1 + S_n \tau_n} \right] \quad . \tag{157}$$

Some of the discount factors inside the sum on the right of Eq. (157) are unknown and we need to interpolate with one of the schemes below. Those interpolated points will depend on the unknown value of R_n and the known values from the previous tenors, like $R_{n_i - 1}$ (using previous notation of T_{n_i}, $i = 1, 2, \ldots, n$).

Yield Curve Construction (cont. 13)

In building a proper yield curve we need to keep in mind several considerations:

1. The yield curve needs to reprice the market instruments that were used as inputs.
2. The yield curve should not allow arbitrage, the forward rates flat-discrete (1m, 3m, or 6m ones) or instantaneous forward rates should remain positive.
3. The interpolation method needs to be local. Small changes of market input should not change far away points in the curve. Changes should affect only the neighboring points which are most correlated with the change.
4. Are the forwards continuous and stable? Stability can be defined by bps changes in forwards for 1bp change in one of of the input points.

One of the first issues we need to resolve while building a yield curve is the issue of filling the missing points. Usually we will have knowledge of rates SR_1, SR_2, $\ldots$, SR_n at the points τ_1, τ_2, $\ldots$, τ_n and need to determine $SR(\tau)$ at τ at points different from any of the given τ_i's above and then proceed to extract the discount factors and the zero rates R_i.

One way to fill the missing data is to interpolate market input to the missing points. Say if the 1y, 2y, ..., 6y and 10y are given, to linearly interpolate between 6y and 10y to obtain swap rates at the points 7y, 8y, 9y.
While this method is simple, it has a drawback in that it decouples the bootstrap method from the interpolation. The best way that we will discuss below is to express the unknown discount factors in terms of the last known bond B_6 and the last as of yet unknown bond B_n^* and then through a zero root-finding we find all missing bonds such that we reprice the 10y Swap instrument. In our example here express $B_7 = f(B_{10}^*)$, $B_8 = f(B_{10}^*)$, $B_9 = f(B_{10}^*)$ and substitute these in Eq. (156), then solve equation for the SR_{10} to find B_{10}^*. This is the only unknown in (156).

For the interpolation above, use one of the following schemes:

- **(1).** Piece-wise flat-constant forward rates, or equivalently Linear in $R(t,T)(T-t)$ - the one explained above. Linear in "R(T)T" means constant in $f(t,T) = (R(t,T)T)'$.

  To simplify notation, set $t = 0$, and $R(0, T_{i-1}) = R_{i-1}$, then Eq. (44),

$$R(T)T = R_{i-1}T_{i-1} + \int_{T_{i-1}}^{T} f(u)\mathrm{d}u \quad , \tag{158}$$

$$= R_{i-1}T_{i-1} + f \cdot (T - T_{i-1}) \quad . \tag{159}$$

  This means that knowledge at one discrete point T_{i-1}, and the fact that $f(T)$ between T_{i-1} and T_i is constant, will lead to knowledge of $R(T)$ in all points between T_{i-1} and T_i. First we find the constant f between the two points

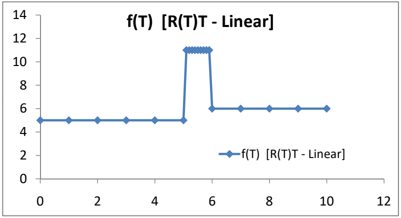
$$f(0,T) = \frac{R_i T_i - R_{i-1}T_{i-1}}{T_i - T_{i-1}} \ , \ T \in [T_{i-1}, T_i] \quad . \tag{160}$$

Yield Curve Construction (cont. 15-2)

Then the zero-rates between these two points are

$$
\begin{aligned}
R(0,T)\,T &= R_{i-1}\,T_{i-1} + (T - T_{i-1})\frac{R_i\,T_i - R_{i-1}\,T_{i-1}}{T_i - T_{i-1}}\;, && (161)\\
&= \frac{T - T_{i-1}}{T_i - T_{i-1}} R_i\,T_i + \frac{T_i - T}{T_i - T_{i-1}} R_{i-1}\,T_{i-1}\;, && (162)
\end{aligned}
$$

thus the “linear R(T) T” naming.



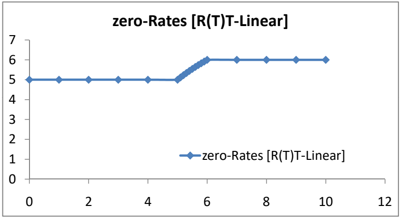


Figure: We take the case of known bonds $B(T_i)$ at the discrete points $T_i = 1y, \dots 5y$, or equivalently yields $R(T_i) = 5\%$ at $T_i = 1y, \dots 5y$ and $R(T_i) = 6\%$ at $T_i = 6y, \dots 10y$. We interpolate the yield in all intermediate points with here the scheme of flat constant forward rate or equivalently zero-rates interpolated with the $R(T)T$-linear interpolations method.

Yield Curve Construction (cont. 17)

Summarize the concepts of rates introduced so far

$$R(t;T_{i-1},T_i) = -\frac{\ln B(t;T_{i-1},T_i)}{T_i - T_{i-1}} = -\frac{\ln B(t,T_i) - \ln B(t,T_{i-1})}{T_i - T_{i-1}} \quad ,$$
$$= \frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} \quad .$$

This is related to the forward rate by

$$f(t,T) = -\frac{\mathrm{d}\ln B(t,T)}{\mathrm{d}T} = \frac{\mathrm{d}(R(T)T)}{\mathrm{d}T} \quad ,$$
$$\int_{T_{i-1}}^{T_i} f(t,u)\mathrm{d}u = \int_{T_{i-1}}^{T_i} \mathrm{d}(R(T)T) = R_i T_i - R_{i-1} T_{i-1} \quad ,$$
$$R(t;T_{i-1},T_i) = \frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} = \frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T_i} f(t,u)\mathrm{d}u \quad .$$

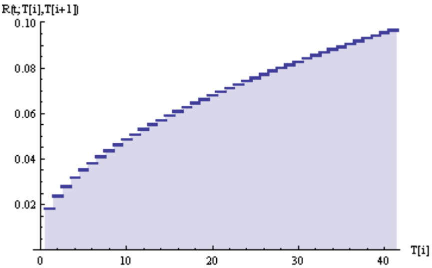
The discrete continuous compounding forward rate is equal to the average of the instantaneous forward rate for that same interval $[T_{i-1}, T_i]$.

Yield Curve Construction (cont. 18)

Let us see the relationship between the Libor rate and the discrete continuously compounded forward rate. From Eq. (33) we have

$$
\begin{aligned}
(t; T_{i-1}, T_i) &= \frac{1}{T_i - T_{i-1}} \left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1 \right) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{B(t; T_{i-1}, T_i)} - 1 \right), \; , \\
&= \frac{1}{T_i - T_{i-1}} \left(e^{R(t; T_{i-1}, T_i)(T_i - T_{i-1})} - 1 \right) \quad , \\
&= \frac{1}{T_i - T_{i-1}} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(R(t; T_{i-1}, T_i)(T_i - T_{i-1}) \right)^n - 1 \right) \quad , \\
&\approx R(t; T_{i-1}, T_i) + \frac{1}{2} R^2(t; T_{i-1}, T_i)(T_i - T_{i-1}) \quad .
\end{aligned}
$$

The Libor rate is marginally higher than the discrete forward rate, which in turn is the average of the instantaneous forward rate in the interval $[T_{i-1}, T_i]$. For $R = 3\%$, we have $L = 3\% + (3\%)^2(0.25/2) = 3.01125\%$.



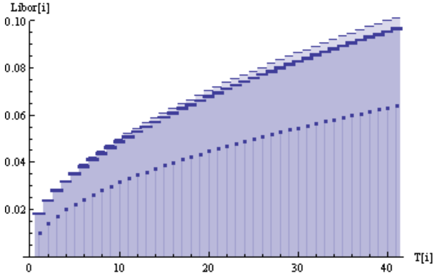


Figure: The discrete continuous compound rates $R(0; T_{i-1}, T_i)$ (thick piece-wise flat line), Libor Rates $L(0; T_{i-1}, T_i)$ (thin piece-wise flat line on top), and the yield, are shown. I have taken here for illustration the zero-Rate $R(0, T) \sim \sqrt{T}$ which gives for $f(T) \sim 3/2\sqrt{T}$ (50% higher than $R(0, T)$). The Libor rates are somewhat higher than $R(0; T_{i-1}, T_i)$ as shown on previous page.

Yield Curve Construction (cont. 20)

- **(2).** Linear in R(0,T)

$$R(0,T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} R_i + \frac{T_i - T}{T_i - T_{i-1}} R_{i-1} \quad , \tag{163}$$

resulting in

$$f(0,T) \;\; = \;\; \frac{2T - T_{i-1}}{T_i - T_{i-1}} R_i + \frac{T_i - 2T}{T_i - T_{i-1}} R_{i-1} \quad . \tag{164}$$

Notice some important points. In Eq. (163) as $T \to T_i$, $R(T_i^-) \to R_i$ and the rate loses memory of R_{i-1}. This is not the case for $f(0,T)$. This rate will hold memory of R_{i-1} as $T \to T_i^-$. And it will have information also from R_{i+1} as $T \to T_i^+$. Therefore the linear interpolation in $R(T)$ produces a linear interpolation in $f(0,T)$ but with jumps at the pillar points T_i. Notice also that $T_i - 2T$ becomes negative as $T \to T_i$. If the rate R_{i-1} is high and R_i is low, the second negative term will overtake the first positive term at some T, leading to negative forward rates (increasing discount - arbitrage!).

Yield Curve Construction (cont. 21)

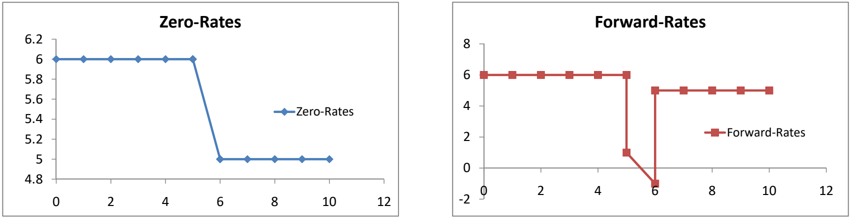


Figure: Forward rates and zero-rates interpolated with the $R(T)$-linear interpolations method. In this arrangement the forward rates can become negative when the yield curve is inverted, thus allowing arbitrage. This usually happens in intervals where $R_{i-1} > R_i$.

Yield Curve Construction (cont. 15)

- **(3).** Linear on log of rates

$$\ln R(0,T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} \ln R_i + \frac{T_i - T}{T_i - T_{i-1}} \ln R_{i-1} \quad , \tag{165}$$

  resulting in (does not allow for negative rates)

$$R(0,T) = R_i^{\frac{T - T_{i-1}}{T_i - T_{i-1}}} R_{i-1}^{\frac{T_i - T}{T_i - T_{i-1}}} \quad . \tag{166}$$

This method has the drawback that it cannot work at negative rates.

- **(4).** Linear on discount factors

For $T_{i-1} < T < T_i$,

$$B(0,T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} B(0,T_i) + \frac{T_i - T}{T_i - T_{i-1}} B(0,T_{i-1}) \quad , \qquad (167)$$

which for the rates results in

$$R(0,T) = -\frac{1}{T} \ln\left[\frac{T - T_{i-1}}{T_i - T_{i-1}} e^{-R_i T_i} + \frac{T_i - T}{T_i - T_{i-1}} e^{-R_{i-1} T_{i-1}}\right] \quad . \qquad (168)$$

Yield Curve Construction (cont. 23)

Let us consider another case of interpolation, the continuous & piece-wise linear in instantaneous forward rates [P. Hagan & G. West 2008]. After all this looks like a simple improvement to the piece-wise flat method (1) discussed earlier. This example illustrates the relationship between the discrete/continuous forwards. Let us assume that we know the zero-rate $R_i = 5\%$, for $i = 0, 1, \dots 5$, and $R_i = 6\%$, for $i = 6, 7, \dots 10$. Since the zero-rate is flat at 5% below 5y , the forward rate should be flat at 5% as well in that period (to ensure continuity as $f(0,0) = R_0 = 5\%$). Consider the interval $[5y, 6y]$. The flat forward rate for that period $R(t; 5, 6)$ will be 11%, since $R_6 = 6\%$, $R_5 = 5\%$

$$R(t; T_{i-1}, T_i) \quad = \quad \frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} = \frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T_i} f(t, u) \mathrm{d}u \quad .$$

On the other hand $R(t; 5, 6)$ is the average of the forward rate. To reach to this average we should have $f(t, 6) = 17\%$ since $f(t, 5) = 5\%$. The discrete forward rate in $[6, 7]$ should be 6%. This in turn results in $f(7) = -5\%$. We will have an alternation between 17% and -5%. This yield curve is oscillatory and negative in parts allowing arbitrage.

Yield Curve Construction (cont. 24)

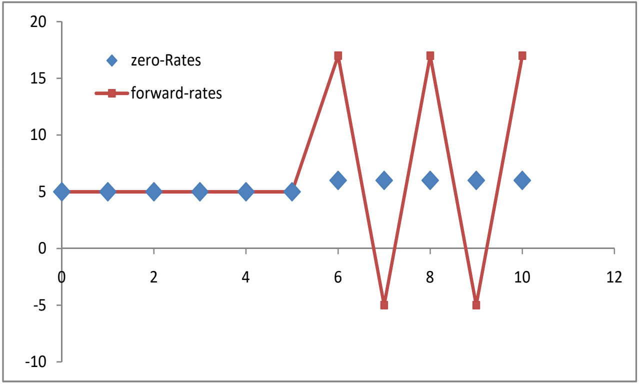


Figure: The forward-rates (red-line) produced by piece-wise linear interpolation in forward rates. The zero-rates (blue-line) are taken to be 5% from [0,5] and 6% in [6,10]. The forward rates are continuous but oscillatory.

Yield Curve Construction (cont. 25)

Use the following to interpolate the zero-rates

$$R(t;T)T = R_{i-1}T_{i-1} + \frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T} f(t,u)\mathrm{d}u \quad .$$

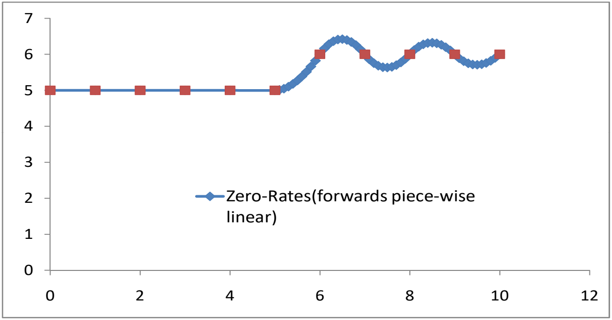
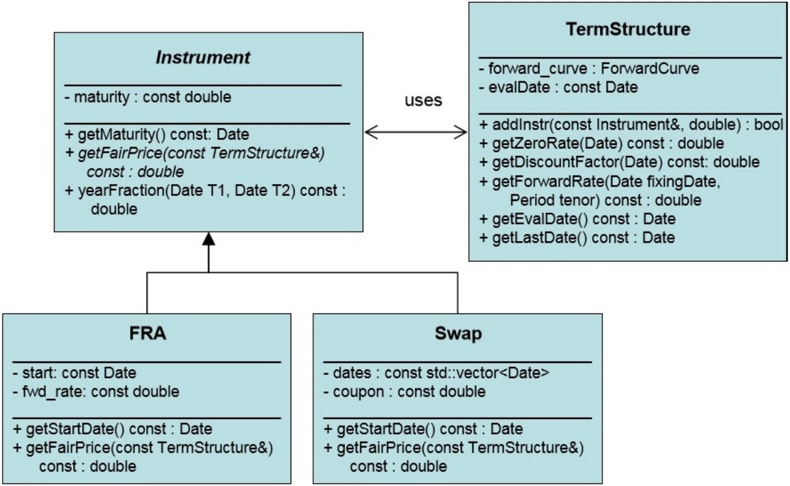


Figure: The zero-rates can be obtained by the formula above. Also zero-rates show an oscillatory behavior.

Pricing Framework



TimeFunction Base Class

Define a TimeFunction interface to allow for function evaluation and interpolation.

```
class TimeFunction {

public:

        TimeFunction( const Date & start) : m_start( start ) {}
        virtual double operator() ( Date T ) const = 0 ;
        virtual double integrate( Date T1, Date T2 ) = 0 ;
        Date getStart( ) { return m_start ; }
        virtual Date getEnd( ) = 0;
private:

        const Date m_start;
}
```

Class ForwardCurve

```
class ForwardCurve : public TimeFunction {

public:

        ForwardCurve( Date start, const std::vector <Date>& dates,
                    const std::vector <double>forward_rates );
        ForwardCurve(const ForwardCurve & rhs);
        virtual double operator() (Date T) const ; // linear interp
        virtual double integrate(Date T1, Date T2) ; // trapezoidal
        virtual Date getEnd() ;
        void removeDate (Date T ) ;
        void setForwardRate (Date T, double rate ) ;
private:

        std::map<Date, double >m_forwardRates;
}
```

Instrument Base Class

```
class TermStructure;
typedef double Date;
class Instrument {

public:

        Instrument(Date maturity) : m_maturity(maturity) { }
        Instrument(const Instrument& rhs):
                m_maturity(rhs.m_maturity(rhs.m_maturity) {}
        Date getMaturity() const { return m_maturity ; }
        virtual double getFairPrice(const TermStructure& ts) const=0;
        // calculate #years from T1 to T2
        static double yearFraction(Date T1, Date T2);
private:

        const Date m_maturity;
}
```

Swap Rates

Swaps

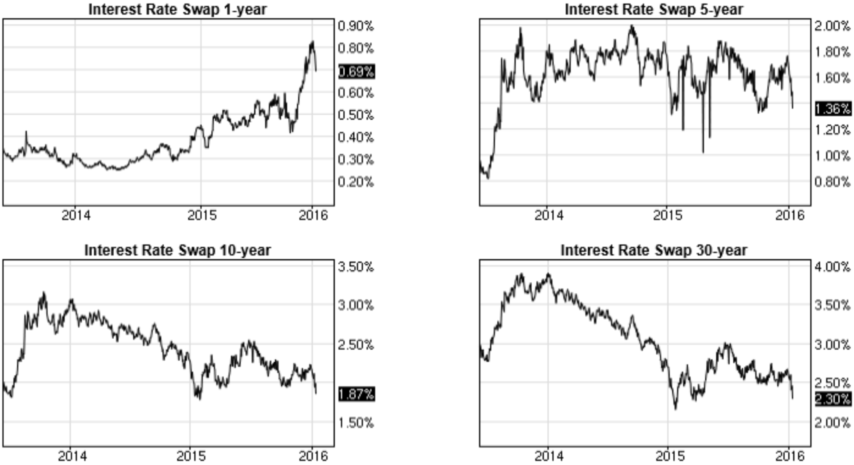
FlipCharts

Name	Current	1M ago	3M ago	6M ago	1Y ago	Date	Links
Interest Rate Swap 1 Year	0.69%	0.82%	0.55%	0.48%	0.35%	01/15/16	
Interest Rate Swap 2 Years	0.94%	1.13%	0.81%	0.91%	0.64%	01/19/16	
Interest Rate Swap 3 Years	1.11%	1.35%	1.08%	1.25%	0.92%	01/19/16	
Interest Rate Swap 4 Years	1.30%	1.62%	1.32%	1.54%	1.26%	01/14/16	
Interest Rate Swap 5 Years	1.40%	1.72%	1.49%	1.78%	1.36%	01/19/16	
Interest Rate Swap 7 Years	1.63%	1.96%	1.79%	2.13%	1.58%	01/19/16	
Interest Rate Swap 10 Years	1.89%	2.19%	2.03%	2.43%	1.89%	01/19/16	
Interest Rate Swap 15 Years	2.05%	2.35%	2.20%	2.62%	2.05%	01/14/16	
Interest Rate Swap 30 Years	2.32%	2.65%	2.52%	2.88%	2.35%	01/19/16	

In an interest rate swap agreement, one party undertakes payments linked to a floating interest rate index and receives a stream of fixed interest payments. The second party undertakes the reverse arrangement. The interest rate swap rate represents the fixed rate paid on a rate swap to receive payments based on a floating rate. Our Dollar Interest Rate Swaps page shows 1-, 5-, 10-, and 30-year rate swap charts, as well as historical rate swap data tables.

http://www.interestrateswapstoday.com/swap-rates.html
http://www.barchart.com/economy/swaps.php
http://www.thefinancials.com/free/EX_Interest_Swaps.html

Swap Rates (cont. 2)



Swap Rates (cont. 3)

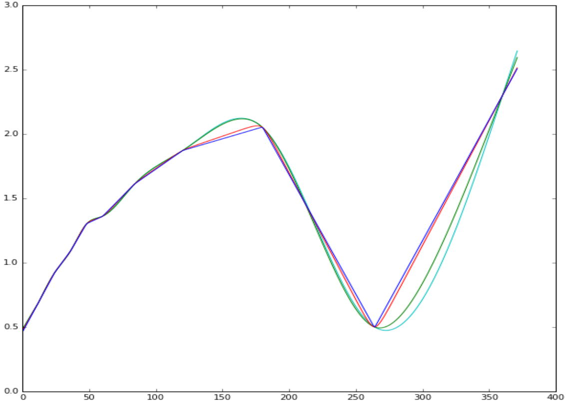


Figure: Swap Rates interpolated with tension Splines of increasing σ. $\sigma = 0.001$, $\sigma = 0.01$, $\sigma = 0.1$. The limit $\sigma \to 0$ approaches the cubic spline. Strong σ approaches the linear interpolation.

The cubic spline $f(x)$ interpolating a set of given points (x_i, f_i), $i = 1, \ldots, N$. Such spline is piecewise linear in its second derivative

$$f''(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} f_i'' + \frac{x - x_i}{x_{i+1} - x_i} f_{i+1}'' \quad , \tag{169}$$

where the second derivative is continuous at knot points, $\lim_{x \to x_i^-} f''(x) = \lim_{x \to x_i^+} f''(x)$. We supply these equations with $f''(x_1) = f''(x_N) = 0$. This type of interpolation is called *natural cubic spline*. An improvement of the cubic spline is by applying tensile force at the end points, measured by σ. Formally this is accomplished by replacing Eq. (169) by

$$f''(x) - \sigma^2 f(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} (f_i'' - \sigma^2 f_i) + \frac{x - x_i}{x_{i+1} - x_i} (f_{i+1}'' - -\sigma^2 f_{i+1}) \quad . \tag{170}$$

See figure of previous strengths of σ. At strong σ the interpolation approaches a linear Swap Rate interpolation. Small σ limit approaches the natural cubic spline.

1 Part I : Interest Rates Basics

2 Part II: Yield Curve Bootstrapping

3 Part III: Merton Model

Part III: Short Rate Modeling

Merton and Vasicek Models

The short rate modeling consists on calculating the prices and the respective probabilities (i.e. the distributions) of the bonds between two future time points $B(t, T)$. The initial shape of the yield-curve is considered as known, meaning that we know the bond prices $B(0, T)$ for all T. The reason for setting such a task can become clear, for instance, in evaluating a European Swaption we need to calculate the Bond price distribution for all bonds $B(T_0, T_i)$, for $i = 1, \ldots, n$. The Swaption payoff at Expiry T_0, as an example, is

$$
\begin{aligned}
V_{\text{Swaption}}(T_0) &= (\mathrm{Swap}(\mathrm{T}_0))^+ \quad , \\
&= \left(1 - B(T_0, T_n) - K \sum_{i=1}^{n} \tau_i B(T_0, T_i)\right)^+ , \ (\text{payer}), \\
&= \left(K \sum_{i=1}^{n} \tau_i B(T_0, T_i) - 1 + B(T_0, T_n)\right)^+ , \ (\text{receiver}).
\end{aligned}
\tag{171}
$$

All time points T_0 and T_i are in the future. As it can be seen in the above equation, the price of the Swaption can be obtained by taking an average of the basket of bonds as expressed in Eq. (171). The question is how do we go about modeling the prices of future bonds knowing the prices of all current bonds.

The answer to the modeling question of the distribution of the future bonds in the family of Short-Rate models is taken by modeling the very first point of the yield-curve, the short-rate, $r(t)$. The rest of the curve is taken as initial data for the calibration through the input of all $B(0,T)$ in the Bond reconstruction formula that will be obtained in the next few sections. Usually one is given a diffusion equation for the first point of the curve, and the distribution of the bonds for all the above future points will be found, for some of the models below this calculation can be carried out even analytically.

In a **one-factor short-rate model** we assume that $r(t)$ satisfies the following

$$\mathrm{d}r(t) = \mu(t, r(t))\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W(t) \quad . \tag{172}$$

Table: Short Rate Models

Model	$\mu(t, r(t))$	$\sigma(t, r(t))$
Merton - •	θ	σ
Vasicek	$\theta - \kappa r(t)$	σ
Cox–Ingersoll–Ross	$\kappa(\beta - r(t))$	$\sigma\sqrt{r(t)}$
Dothan	$\kappa r(t)$	$\sigma r(t)$
Black–Derman–Toy	$\theta(t) r(t)$	$\sigma r(t)$
Ho–Lee - •	$\theta(t)$	$\sigma(t)$
Hull–White (extended Vasicek)	$\theta(t) - \kappa r(t)$	$\sigma(t)$
Black–Karasinski	$r(t)(\theta(t) - \kappa \ln r(t))$	$\sigma r(t)$

Bond Price

Starting point, $\frac{B(t,T)}{B(t)}$ is a martingale under the Q risk-neutral measure. $B(t) = \exp\left(\int_0^t r(s)\mathrm{d}s\right)$ and

$$\frac{B(t,T)}{B(t)} = E^Q\left[\left.\frac{B(T,T)}{B(T)}\right|\mathcal{F}_t\right] = E^Q\left[\left.\frac{1}{\exp\left(\int_0^T r(s)\mathrm{d}s\right)}\right|\mathcal{F}_t\right] \ . \tag{173}$$

$$\begin{aligned} B(t,T) &= B(t)E^Q\left[\left.e^{-\int_0^T r(s)\mathrm{d}s}\right|\mathcal{F}_t\right] \quad , &(174)\\ &= E^Q\left[\left.e^{-\int_t^T r(s)\mathrm{d}s}\right|\mathcal{F}_t\right] \quad . &(175)\end{aligned}$$

Compare this with the expected values for the stocks case

$$E_0^Q\left[(S(T)-K)^+\right] \ , \text{ and } E_0^Q\left[\left(E_S^Q\left[\exp\left(-\int_S^T S(u)\mathrm{d}u\right)\right]-K\right)^+\right] \ . \tag{176}$$

The equation to the right demonstrates the complexity of interest rate modeling compared with the equity modeling.

Merton and Vasicek Models

In these lectures we will focus on these two models:
1.

$$
\begin{aligned}
\mathrm{d}r(t) &= \theta \mathrm{d}t + \sigma \mathrm{d}W(t) \quad (\text{Merton}) \quad , && (177)\\
\mathrm{d}r(t) &= \theta(t) \mathrm{d}t + \sigma(t) \mathrm{d}W(t) \quad (\text{Ho} - \text{Lee}) \quad , && (178)
\end{aligned}
$$

and
2.

$$
\begin{aligned}
\mathrm{d}r(t) &= \big(\theta - \kappa r(t)\big) \mathrm{d}t + \sigma \mathrm{d}W(t) \quad (\text{Vasicek}) \quad , && (179)\\
\mathrm{d}r(t) &= \big(\theta(t) - \kappa r(t)\big) \mathrm{d}t + \sigma \mathrm{d}W(t) \quad (\text{Hull} - \text{White}) \quad , && (180)
\end{aligned}
$$

the second line of each equation is a more generalized version of the first.

At the very beginning of these notes I would like to focus the attention on two points:

1. What role does the drift θ or more general $\theta(t)$ plays. Do the option price results depend on θ, if not why not. Follow the process of its disappearance in the calculations.
2. What role does the speed of mean reversion κ or $\kappa(t)$ plays. This parameter is absent in Merton/Ho-Lee but present in Vasicek/Hull-White.

Another question I would like to pay attention is:

1. In the final answer on the bond-reconstruction formula which term is dominant, the θ term of the drift or the diffusion part term $\sigma \mathrm{d}W(t)$.