Yield Curve Construction Redux

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Abstract

In this paper we consider several methods for building the yield curve from swap rates. In particular given the complete list of swap rates, we obtain an analytic formula for the zero rate. A continuous limit of this equation is discussed and used in order to extract the continuous yield curve. In real world applications, a complete list of swap rates is rarely present, thus one needs "bootstrapping" techniques. We review and compare several methods bootstrapping techniques and add some new bootstrapping techniques. This is done by using the continuous limit equation discussed in the first part of the paper and several variational models. In our experiments, we found that the new methods are better suited to reconstruct the yield curve.

1 Introduction

The yield curve is a quantity of a fundamental interest in Finance. Indeed, most of the interest rate models require the yield curve as initial datum. In this paper we review the current status of yield curve construction methods and add some new results.

The construction and composition of a yield curve has changed over the years. The financial crisis of 2008 with the Lehman Brothers collapse made it necessary to introduce a change since the funding rate and the OIS spreads were observed to widen to a significant value to be ignored in pricing trades. This resulted in the construction of yield curves with different rates for Libor forecast and discounting. In the years that followed, the practice of CSA agreements and the counterparty collateral posting become commonly used. Later on, the option to select among several currencies for collateral posting, namely the currency that offers the highest rate of return on the collateral, was also offered by the banks which then lead to extensive work on multi-currency collateral yield curve construction. While originally the Libor-OIS spreads were assumed to remain deterministic, later on the effect of stochasticity of Libor-OIS spreads has been investigated. The stochastic spreads are correlated with the Libor and thus also such modifications in pricing of even linear products like FRAs and Swaps have been investigated which leads to Libor/OIS spreads Volatility dependence in the derivatives pricing.

In this paper we look into the single curve case and examine various aspects of building the curve. The latest developments on curve construction with CSA and multiple currencies will be subject of future publications.

The yield curve cannot be determined uniquely from market data. Indeed, one typically tries to build the interest rate curve from quoted swap rates, which are liquid only at certain maturities $T_i = 1, \dots 5, 7, 10, 12, 15, 20, 25, 30$. Thus one needs to find a curve $y(t)$ whose implied swap rates coincide with the market data. Notice that the problem is under-determined, namely there is not a unique curve whose implied swap rates coincide with market data. For this reason one requires a number of additional qualitative features, which will act as a selection principle in order to select an appropriate yield curve. To this end a number of methods have been proposed.

The most common method of bootstrapping and interpolating a yield curve is a parametric one: typically one assumes that the yield curve is defined in a piece-wise fashion[1], and in any interval the yield curve depends

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[1] The domain is partitioned in intervals and in each interval the yield curve is defined in terms of simple functions

on one or more parameters. Afterwards a calibration is made so that the curve fits market data at the pillar points. Often polynomials are used in the parametric method.

Although it might happen in the real world, a good yield curve should not have negative forwards as this would signify presence of arbitrage in the market as later maturity bonds would be more expensive than the earlier ones. Thus yield curves that allow negative forwards should be avoided if possible.

It is well-known in the literature that using polynomials might lead to having negative forwards. Several methods have been proposed in order to avoid this issue. In [4], tension splines are used, and the tension parameter is increased manually until positive forwards are achieved. In [8] the authors propose a monotone preserving cubic spline.

Another way of bootstrapping and interpolating is to postulate that the yield curve is the minima of a functional. Such approach is rather broad and appears naturally in many modeling situations ranging from mathematical physics to statistical learning. As we will see many of the parametric methods can be viewed in this variational framework. For more details on how to choose such a functional to be minimized see Section 5.

In the finance literature the variational approach appears in several papers. In [5, 7], the authors minimize the integrated derivative of the forward squared. Instead, in [1, 11] the authors use the integrated squared second order derivative of the forward curve, while [6] the authors consider a similar model and use the Moore-Penrose pseudo-inverse to find the minimizers. In [12, 2, 10] the authors consider combinations of these two measures and show that this results in so called hyperbolic or tension splines.

The paper is organized as follows: In Section 2, setting the stage for the later parts of the paper, we look into the simpler case of construction of the curve in the case of availability in the market of complete set of points for annual Swap Rates. In Section 2.1, annual Bonds prices are shown to be obtained analytically in such a case, given by Eq. (21). A continuous limit of this equation is discussed further down on the same section, and investigated further in Section 2.2. In Section 2.3 we investigate the same ideas but for a complete set of co-terminal annual Swap Rates. In Section 2.4 we investigate approximations to the more complicated analytic bond price of Eq. (21). In Section 3 we discuss the bootstrapping and interpolation of the yield curve in presence of incomplete Swap Rate data. Different types of popular interpolations are discussed and whenever possible analytic formulas are provided (e.g. Section 3.1). In Section 4 the continuous limit derived in Section 2.2 in order to bootstrap. In Section 5, we discuss a variational approach to bootstrapping the yield curve.

2 Annual Swap Rates - complete set of annual points

For a start we discuss the case of construction of the yield curve from a complete set of Annual Swap Rates. We assume that we know all of them starting from zero up to the end of the time horizon. The spot starting Swap Rates are denoted by SR_j, $j = 1, 2 \ldots, n$. From this knowledge we show how to extract all the bonds $B(0, t_j) \equiv B_j$, for $j = 1, 2 \ldots, n$.

The procedure of calculating the bonds from the Swap Rates is an iterative one and is explained in this section. In the next one we show how to solve analytically the iterative procedure presented here.

From SR_1 we calculate B_1, and $A_1 = \tau_1 B_1$. From knowledge of SR_2 and A_1, we calculate B_2, and thus $A_2 = A_1 + \tau_2 B_2$, and so on, from SR_j and A_{j-1} we can extract B_j

$$1 - B_j = SR_j \sum_{i=1}^{j} \tau_i B_i = SR_j (A_{j-1} + \tau_j B_j) \quad , \quad j = 1, \ldots, n \quad , \tag{1}$$

where $A_0 = 0$ in the above. Grouping the last bond together, we get

$$B_j (1 + \tau_j SR_j) = 1 - SR_j A_{j-1} \quad , \quad j = 1, \ldots, n \quad . \tag{2}$$

B_j in terms of SR_j and A_{j-1} is given as follows

$$B_j = \frac{1 - SR_j A_{j-1}}{1 + \tau_j SR_j} \quad , \quad j = 1, \ldots, n \quad , \tag{3}$$

meaning that the knowledge of $SR_1, SR_2, \ldots SR_n$, leads to a unique set $B_1, B_2, \ldots, B_n$.
A similar process can be followed from knowledge of the full set of co-terminal annual Swap Rates. Let us denote with $A_{j,n}$ the annuity of the Swap starting at T_j and ending at the last payment date T_n. We can express the ratios of the bonds B_j/B_n, $j = n, \ldots, 1$, in terms of the given Swap Rates. We express the bond ratios $B_n/B_n = 1$, B_{n-1}/B_n and work our way backwards to B_1/B_n, as follows

$$SR_{n-1,n} = \frac{B_{n-1} - B_n}{A_{n,n}} \quad , \quad \frac{B_{n-1}}{B_n} = 1 + SR_{n-1,n}\frac{A_{n,n}}{B_n} \quad . \tag{4}$$

Similarly assuming that the ratios B_k/B_n for $k > j$ have been found, we can calculate

$$\frac{B_j}{B_n} = 1 + SR_{j,n}\frac{A_{j+1,n}}{B_n} = 1 + SR_{j,n}\sum_{k>j}^{n}\tau_k\frac{B_k}{B_n} \quad . \tag{5}$$

B_n itself can be calculated from the boundary $B_0/B_n = 1/B_n$ and knowledge of $S_{0,n}$ and all other $S_{j,n}$. This way we calculate all the bond ratios B_j/B_n for $j = 1, \ldots, n$, and thus the bond prices B_j.

2.1 Analytic calculation of the Bonds from Complete Set of Annual Swap Rates

The calculation described above for the spot starting Swap Rates case is carried out analytically in this section. Starting from (3), multiplying both sides by τ_j, we obtain the basic Swap Rate relationship

$$\tau_j B_j = \tau_j\frac{1 - SR_j A_{j-1}}{1 + \tau_j SR_j} \quad , \quad j = 1, \ldots, n \quad , \tag{6}$$

where $A_j := \sum_{i=1}^{j}\tau_i B_i$. Equivalently,

$$\tau_j B_j + \frac{\tau_j SR_j}{1 + \tau_j SR_j}A_{j-1} = \frac{\tau_j}{1 + \tau_j SR_j} \quad , \quad j = 1, \ldots, n \quad . \tag{7}$$

The first term in the above equation is nothing but the increment of the Annuity A_{j-1} to A_j, $\tau_j B_j = A_j - A_{j-1} = \Delta A_{j-1}$. Replacing this in Eq (7), we obtain

$$\Delta A_{j-1} + \tilde{S}_j A_{j-1} = \tilde{\tau}_j \quad , \quad j = 1, \ldots, n \quad , \tag{8}$$

where for notation brevity we introduced two new symbols,

$$\tilde{\tau}_j = \frac{\tau_j}{1 + \tau_j SR_j} \quad , \quad \text{and} \quad \tilde{S}_j = \tilde{\tau}_j SR_j = \frac{\tau_j SR_j}{1 + \tau_j SR_j} \quad . \tag{9}$$

If we multiply both sides of Eq. (8) with a function of the Swap Rates, γ_j, still to be found, but defined such that $\Delta\gamma_{j-1} = \tilde{S}_j\gamma_j$ [2], then the left hand side of Eq. (8) turns into a full discrete differential which can then be solved analytically

$$\gamma_j\Delta A_{j-1} + \tilde{S}_j\gamma_j A_{j-1} = \tilde{\tau}_j\gamma_j \quad , \tag{11}$$

$$\gamma_j\Delta A_{j-1} + (\Delta\gamma_j)\, A_{j-1} = \tilde{\tau}_j\gamma_j \quad , \quad (j = 1, \ldots, n) \quad , \tag{12}$$

$$\Delta(\gamma_{j-1}A_{j-1}) = \tilde{\tau}_j\gamma_j \quad . \tag{13}$$

[2] Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences and let us denote by

$$\zeta'_n = \zeta_{n+1} - \zeta_n.$$

With the above equation one has that

$$\begin{aligned}(\alpha_n\beta_n)' &= \alpha_{n+1}\beta_{n+1} - \alpha_n\beta_n \\ &= \alpha_{n+1}\beta_{n+1} - \alpha_{n+1}\beta_n + \alpha_{n+1}\beta_n - \alpha_n\beta_n \\ &= \alpha_{n+1}\beta'_n + \alpha'_n\beta_n\end{aligned} \tag{10}$$

The asymmetry in (10) is due to the fact that we are working in a discrete setting.

Now we can integrate both sides of this equation from $j = 1$ to n,

$$\gamma_n A_n - \gamma_0 A_0 = \sum_{j=1}^{n} \tilde{\tau}_j \gamma_j \quad . \tag{14}$$

In the above $A_0 = 0$, and $\gamma_0 = 1$, as we see below.
Similarly, we integrate the equation for γ_j, for $j = 1$ to n, to obtain

$$\Delta\gamma_{j-1} = \tilde{S}_j \gamma_j \quad , \quad \gamma_j - \gamma_{j-1} = \tilde{S}_j \gamma_j \quad , \tag{15}$$

$$\gamma_j = \frac{\gamma_{j-1}}{(1 - \tilde{S}_j)} \quad , \quad \gamma_1 = \frac{\gamma_0}{(1 - \tilde{S}_1)} \quad , \tag{16}$$

thus

$$\gamma_n = \prod_{i=1}^{n} \frac{\gamma_0}{1 - \tilde{S}_i} \quad . \tag{17}$$

Given that $1 - \tilde{S}_j = (1 + \tau_j SR_j)^{-1}$, from (17), for γ_n we get

$$\gamma_n = \prod_{i=1}^{n} \frac{1}{1 - \tilde{S}_i} = \prod_{i=1}^{n} (1 + \tau_i SR_i) \quad . \tag{18}$$

To keep the notation simple, we eliminate $\tilde{\tau}_i$ from the Annuity equation Eq. (14). From the structure of $\tilde{\tau}_i$ we have $\tilde{\tau}_i \gamma_i = \tau_i \gamma_{i-1}$. By using (18), for the Annuity we find

$$A_n = \frac{1}{\gamma_n} \sum_{i=1}^{n} \tau_i \gamma_{i-1} \quad , \tag{19}$$

where in the above the boundary value $\gamma_0 = 1$ as mentioned previously. Given that $A_n - A_{n-1} = \tau_n B_n$ one can immediately obtain also the Bond prices B_n

$$B_n = \frac{1}{\tau_n} \left(\frac{1}{\gamma_n} \sum_{i=1}^{n} \tau_i \gamma_{i-1} - \frac{1}{\gamma_{n-1}} \sum_{i=1}^{n-1} \tau_i \gamma_{i-1} \right) \quad . \tag{20}$$

If we put things together, we obtain the most important result of this section, an analytic formula for the bond prices from the Swap Rates

$$B_n = \prod_{j=1}^{n} \frac{1}{1 + \tau_j SR_j} \sum_{i=1}^{n} \frac{\tau_i}{\tau_n} \prod_{k=1}^{i-1} (1 + \tau_k SR_k) - \prod_{j=1}^{n-1} \frac{1}{1 + \tau_j SR_j} \sum_{i=1}^{n-1} \frac{\tau_i}{\tau_n} \prod_{k=1}^{i-1} (1 + \tau_k SR_k) \quad , \tag{21}$$

and the Annuity, respectively

$$A_n = \prod_{j=1}^{n} \frac{1}{1 + \tau_j SR_j} \sum_{i=1}^{n} \tau_i \prod_{k=1}^{i-1} (1 + \tau_k SR_k) \quad . \tag{22}$$

To our knowledge, the above formula which derives the bond prices from the market values of Swap Rates is not present in the literature. Since the above expression looks slightly complicated, we introduce a number of simplifications in section 2.4.
We also define the bond price to the zeroth order as it will be further discussed in Section 2.3,

$$B_n^{(0)} = \frac{1}{\gamma_n} = \prod_{i=1}^{n} \frac{1}{1 + \tau_i SR_i} \quad . \tag{23}$$

Let us now discuss the continuous analogs of the above results. Consider (1) where $\tau_i \downarrow 0$. For simplicity let us assume that all τ_j are equal to τ. The parameter τ can be seen as a discretization step. Thus writing analogously as in (3)

$$1 - B_{n\tau} = SR^{\tau}(n\tau) \sum_{j=1}^{n} \tau B_{k\tau} \quad ,$$

and passing to the limit as $\tau \downarrow 0$, we obtain

$$1 - B(t) = SR^{0}(t) \int_0^t B(s)\mathrm{d}s \quad . \tag{24}$$

The integrated term in the above can be seen as the continuous annuity,

$$A^{\mathrm{cont}}(t) = \int_0^t B(s)\mathrm{d}s \quad .$$

Notice that in taking the continuous limit, the Bond function remains unchanged at the pillar points for which Eq. (21) applies. The continuous Annuity on the other hand is different from the discrete Annuity even at the pillar points. Indeed, the discrete sum and the continuous integral differ due to the fact that the bonds are convex function of time.

From (24), one has that

$$1 - \frac{\mathrm{d}}{\mathrm{d}t} A^{\mathrm{cont}}(t) = SR^{0}(t) A^{\mathrm{cont}}(t) \quad .$$

Which can be integrated, namely resulting in

$$\mathrm{d}\left(e^{\int_0^t SR^0(u)\mathrm{d}u} A^{\mathrm{cont}}(t) \right) = e^{\int_0^t SR^0(u)\mathrm{d}u} \mathrm{d}t \quad , \tag{25}$$

with solution

$$A^{\mathrm{cont}}(T) = \frac{1}{\gamma(T)} \int_0^T \gamma(t)\mathrm{d}t = \int_0^T e^{-\int_t^T SR^0(u)\mathrm{d}u} \mathrm{d}t \quad , \tag{26}$$

where

$$\gamma(t) = \exp\left\{ \int_0^t SR^0(u)\mathrm{d}u \right\} \quad . \tag{27}$$

Notice that $\gamma(t)$ defined in (27) is similar to the discrete γ_n of Eq.(18), where in the continuous case SR_i is replaced by $SR^0(t)$. Here $\gamma(t)$ resembles a bank account where accrual occurs continuously with rate equal to the Swap Rate $SR^0(t)$.

At this point we reached the stated aim of this section. Namely, we are able to express the Bonds and the Annuity solely as functions of the Swap Rates

$$A_n = \frac{1}{\gamma_n} \sum_{i=1}^{n} \tau_i \gamma_{i-1} \quad , \quad B_n = \frac{1}{\tau_n} \left(\frac{1}{\gamma_n} \sum_{i=1}^{n} \tau_i \gamma_{i-1} - \frac{1}{\gamma_{n-1}} \sum_{i=1}^{n-1} \tau_i \gamma_{i-1} \right) \quad , \tag{28}$$

for the discrete case, with

$$\gamma_n = \prod_{i=1}^{n} (1 + \tau_i SR_i) \quad , \tag{29}$$

and

$$A^{\mathrm{cont}}(T) = \int_0^T e^{-\int_t^T SR^0(u)\mathrm{d}u} \mathrm{d}t \quad , \quad B(T) = 1 - SR^0(T) \int_0^T e^{-\int_t^T SR^0(u)\mathrm{d}u} \mathrm{d}t \quad , \tag{30}$$

for the continuous case.
In order to be able to apply the continuous case equations, like Eq. (24) and (30), we need to obtain an expression for the continuous swap rate SR^0 as function of the market known of a given frequency Swap Rates SR_i's. Given that from the market data, one can obtain only the discrete swap rate, the continuous swap rate needs to be estimated. This is done in the next section.
Similarly to (23), one can define the price of the Bonds to the zeroth order, for the continuous case as

$$B^{(0)}(T) = \exp\left\{-\int_0^T SR^0(u)\mathrm{d}u\right\} \quad . \tag{31}$$

The differences between B_n and $B_n^{(0)}$ lie in the difference between the $SR^0(T)$ and the instantaneous forward rate $f(0,T)$, as seen from the below

$$B(T) = \exp\left\{-\int_0^T f(0,u)\mathrm{d}u\right\} \quad . \tag{32}$$

Differentiating with respect to time on both sides of (24) and using that $\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{cont}}(t) = B(t)$, one has that

$$-f(0,t)B(t) = -\frac{\mathrm{d}}{\mathrm{d}t}SR^0(t)A^{\mathrm{cont}}(t) - SR^0(t)\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{cont}}(t) \quad ,$$

resulting in

$$f(0,t) = SR^0(t) + SR^{0\prime}(t)\frac{A^{\mathrm{cont}}(t)}{B(t)} \quad . \tag{33}$$

This in particular implies that whenever the continuous swap rate is increasing (namely in expanding economical periods) one has that the swap rate is lower than the instantaneous forward rate whereas in economic contraction the opposite holds.

2.2 Continuous Approximations

In order to be able to apply Eq. (24) or analogously Eq. (30), we need to obtain an expression for the continuous swap rate $SR^0(t)$. Given that from the market data one can obtain only the discrete swap rates, the continuous swap rate needs to be estimated.
For simplicity, we will assume that market data in our possession are the annual swap rates $SR^1(i)$, namely the swap rates with maturity i for i in $1,\ldots,n$ and frequency $1y$.
Denote by $F(t)$ the cubic interpolation of the discrete swap rates $SR^1(i)$, with $F(t_i) = SR^1(i)$. If we solve (24) just with $F(t)$ instead of $SR^0(t)$ we obtain a continuous rate, $R^C(t)$, that is expected to a rough degree to be close to the exact analytic discrete zero-rate R_i^D at the pillar points, where R_i^D is defined. R_i^D for i in $1,\ldots,n$ is the exact zero-rate obtained analytically from Eq. (21).
$R^C(t)$ obtained as above is shown with the blue line on Fig.1, left. Compare this with with R_i^D, green line of Fig.1, left.
Starting from the discrete swap rates $SR^1(i)$, we will find an expression for $SR^0(i)$ in such a way that the corresponding zero rate R^C almost coincides with R_i^D at the pillar points, namely $R^C(t_i) \approx R^D(t_i)$ with high precision. This obtained by substituting $SR^0(t_i) = f(SR^1(i))$ or more in general $SR^0(t) = f(F(t))$. The function $f(SR^1(i))$ is what we are looking for in this section.
Such a solution, to second order approximation, is obtained here and shown with the red line in Fig.1, left. The differences between the two R^C's, $R_F^C(t_i) - R_i^D$ is of order of 4bps, whereas $R_{SR^0}^C(t_i) - R_i^D$ is two orders of magnitude smaller.
From the formula

$$1 - B(n) = SR^\tau(n)\left(\sum_{k\tau\leq n}\tau B(k\tau)\right) \quad , \tag{34}$$

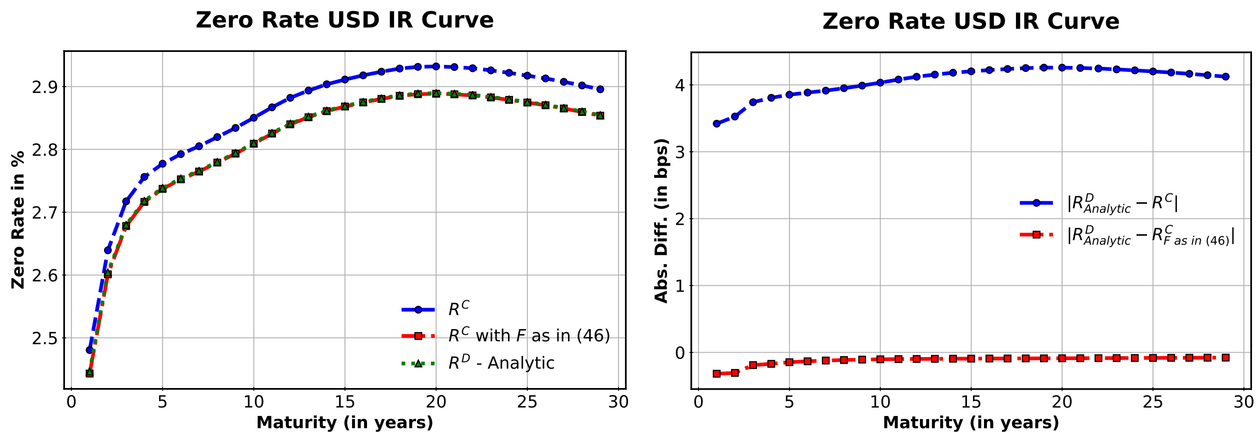

Figure 1: In the above graph, R_i^D, the green line, is the "discrete" zero rate calculated analytically from Eq. (21). R^C, the blue line, is the continuous zero rate, obtained using the cubic interpolation $F(t)$ of swap rates instead of $SR^0(t)$ in (30). A better approximation holds by using for $SR^0(t)$ the approximation $SR^0(t) = F(t)\exp\{-(F(t)/2)\}/(1 + F^2(t)/24)$ of Eq. (46). This is illustrated with the red line in the graph. The results obtained with this latter approximation, shown also in Figs. 4 - 5 and Tables 1-2, demonstrate a much better approximation to the exact values of the zero-rates.

where τ is the frequency of the swaps in consideration, one has that $SR^\tau(n)$ has to depend on the frequency τ. The continuous swap rate $SR^0(n)$ corresponds to the swap rate with maturity n and frequency $\tau \to 0$. Thus we have that

$$SR^\tau(n)\left(\sum_{k\tau \le n} \tau B(k\tau)\right) = SR^1(n)\left(\sum_{k=1}^{n} B(k\tau)\right) \quad . \tag{35}$$

As we sent the frequency $\tau \to 0$, we have that

$$SR^0(n)\left(\sum_{k=0}^{n-1} \int_k^{k+1} B(t)\mathrm{d}t\right) = SR^1(n)\left(\sum_{k=1}^{n} B(k)\right) \quad . \tag{36}$$

In estimating the bond integrals on the left of Eq. (36), notice that the actual bond function is given by $B(t) = \exp\Big\{-tR^C(t)\Big\}$, where R^C is the solution of Eq. (24) with $SR^0(t)$. As discussed this function is reasonably close to $F(t)$ thus reasonably smooth function.

Furthermore, for the purposes of this calculation, as intermediary step, we modify the bond function so that it almost agrees with the exact Bond values at the pillar points, and extend the Bond function as

$$B(t) \approx B(t_i)\exp\Big\{-m(t - t_i)\Big\} \quad , \quad t \in [t_i, t_{i+1}) \quad , \quad i = 0 \ldots, n-1 \quad , \tag{37}$$

where m is equal to the average of the markets annual swap rates $m = \bar{S}^1 = \sum_{i=1}^{n} SR^1(i)/n$. The Bond function is thus thought as right piece-wise continuous, approximating the actual $B(t)$ function in the whole interval, leaving the area underneath the function approximately invariant.

By making Taylor approximation on $t = 1/2$, say at the first time interval, we have that [3] [4]

$$\int_0^1 B(t)\mathrm{d}t \approx B(1)\left(1+\frac{m^2}{24}\right)e^{\frac{m}{2}} \quad . \tag{42}$$

By applying the above approximation each of the $\int_k^{k+1} B(t)\mathrm{d}t$ in (44), one has

$$\int_k^{k+1} B(t)\mathrm{d}t \approx B(k+1)\left(1+\frac{m^2}{24}\right)e^{\frac{m}{2}} \quad . \tag{43}$$

Thus, we have

$$\begin{aligned} SR^1(n)\left(\sum_{k=1}^{n} B(k)\right) &= SR^0(n)\left(\sum_{k=0}^{n-1}\int_k^{k+1} B(t)\mathrm{d}t\right) \quad , \\ &\approx \left[SR^0(n)\left(1+\frac{m^2}{24}\right)e^{\frac{m}{2}}\right]\left(\sum_{k=1}^{n} B(k)\right) \quad . \end{aligned} \tag{44}$$

At the pillar points we leave the bond function unchanged, thus the sum of the bonds on the left and right are equal. Hence, from the above second order approximation, the relation between $SR^1(n)$ and $SR^0(n)$ is

$$SR^0(n) = SR^1(n)\frac{e^{-\frac{1}{2}\left(\sum_{j=1}^{n}\frac{SR^1(j)}{n}\right)}}{1+\frac{1}{24}\left(\sum_{j=1}^{n}\frac{SR^1(j)}{n}\right)^2} \quad . \tag{45}$$

This approximation can be extended to the whole time interval by taking

$$SR^0(t) := F(t)\frac{e^{-\frac{1}{2}F(t)}}{\left(1+\frac{F^2(t)}{24}\right)} \quad . \tag{46}$$

Thus let us define by $B^{\mathrm{cont}}(t)$ the solution to Eq. (24), with SR^0 from Eq. (46), namely

$$B^{\mathrm{cont}}(t) = 1 - SR^0(t)\int_0^t \exp\left\{-\int_u^t SR^0(w)\mathrm{d}w\right\}\mathrm{d}u \quad , \tag{47}$$

and plot the Annuity $A(T) = \sum_{i=1}^{T} B^{\mathrm{cont}}(T_i)$. We can make a judgment of the accuracy of the approximation in Eq. (46) by comparing it with the analytic Annuity (22). See Fig. 2.

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[3] Notice that the variations in the rates are much smaller than the rates themselves. In particular, given that the variations of R^C and F are in general of a lower order of magnitude than R^C and F, namely $\frac{\mathrm{d}}{\mathrm{d}t}R^C \ll R^C$ and $\frac{\mathrm{d}}{\mathrm{d}t}F(t) \ll F(t)$, we have also that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}B(t) &\approx -F(t)B(t) \approx -m\exp(-tm) \approx -mB(t) \\ \frac{\mathrm{d}^2}{\mathrm{d}t2}B(t) &\approx -F(t)^2B(t) \approx -m^2\exp(-tm) \approx -m^2B(t) \quad . \end{aligned} \tag{38}$$

[4]

$$\int_0^1 B(t)\mathrm{d}t \approx B\left(\frac{1}{2}\right) + \int_0^1 B'\left(\frac{1}{2}\right)\left(t-\frac{1}{2}\right)\mathrm{d}t + \frac{1}{2}\int_0^1 B''\left(\frac{1}{2}\right)\left(t-\frac{1}{2}\right)^2\mathrm{d}t \quad , \tag{39}$$

$$= B\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^1 B''\left(\frac{1}{2}\right)\left(t-\frac{1}{2}\right)^2\mathrm{d}t = B\left(\frac{1}{2}\right) + \frac{1}{2}m^2B\left(\frac{1}{2}\right)\int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t \quad , \tag{40}$$

$$= B\left(\frac{1}{2}\right) + \frac{m^2}{24}B\left(\frac{1}{2}\right) = B(1)\left(1+\frac{m^2}{24}\right)/B(1/2) = B(1)\left(1+\frac{m^2}{24}\right)e^{\frac{m}{2}} \quad . \tag{41}$$

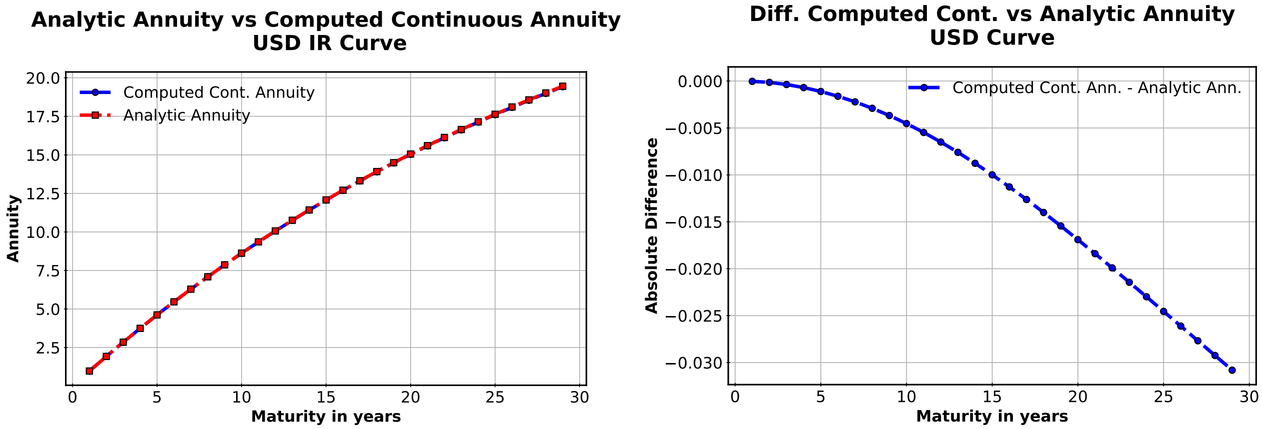

Figure 2: In the above graph, on the left, discrete Annuities are shown, whereas on the right, the corresponding differences. The blue line (discrete Annuity) here is calculated by taking $A(T) = \sum_{i=1}^{T} B^{\text{cont}}(T_i)$. The red line is the exact analytic Annuity obtained in (22). On the blue line we use (46) to define the continuous swap rate and afterwards we use (47) to calculate the bond price for the computation of the discrete Annuity. Small differences arise between the two computations due to the second order approximation of SR^0 of Eq. (46).

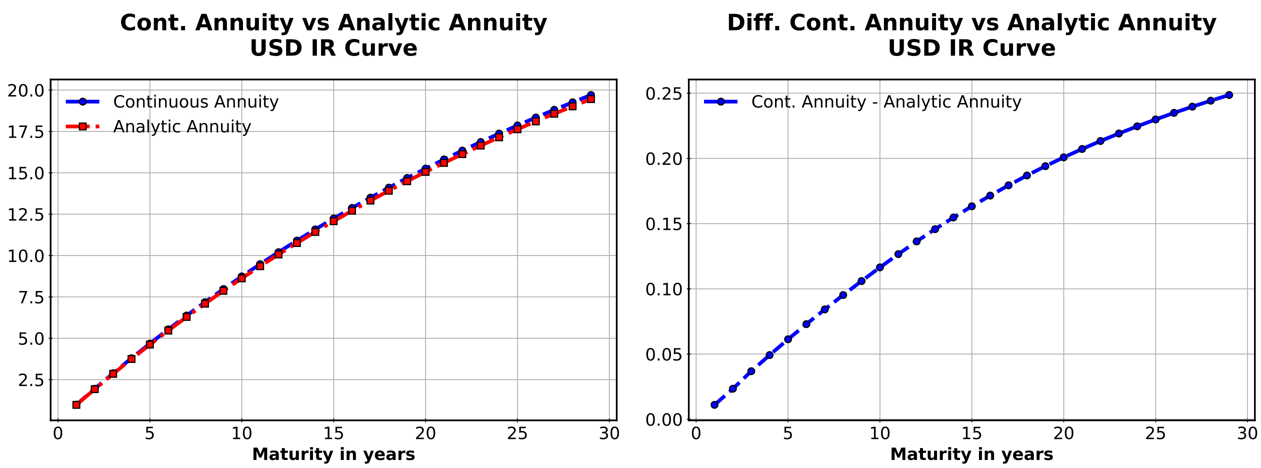

Figure 3: Here the discrete Annuity $A(T)$ of Eq. (22), blue line, and the continuous Annuity $A^{\text{cont}}(T)$ of Eq. (26), red line, are shown. In the continuous Annuity, the swap rate SR^0 of Eq. (46) is used. The transition from the discrete sum of the red line to the continuous integral is the reason for the difference (in addition to the order of approximation of SR^0 of Eq. (46)).

When we calculate the continuous Annuity from the integral of (26), there is a difference with the standard discrete Annuity of (22), see Fig. 3. Notice that the Bonds term structure in the continuous limit is invariant at the pillar points, same as the discrete case, whereas the continuous Annuity does not coincide with the discrete one at any point. Differences are the ones to expect when comparing discrete sums and their continuous limits, in particular here bonds are convex functions of time.

Finally, in Figure 4, we show the zero rate $R^C(t)$ obtained by solving Eq. (24) with $B^{\text{cont}}(t)$ given in (47) obtained by using for $SR^0(t)$ Eq. (46). The red line is the discrete analytic R_i^D curve obtained by Eq.(21), for the EUR curve. In Figure 5, we show the respective zero rates for the USD curve.

In Tables 1 and 2 we show the differences between $R^C(t_i)$ and R_i^D for $i = 1, \ldots, n$, for the EUR and USD

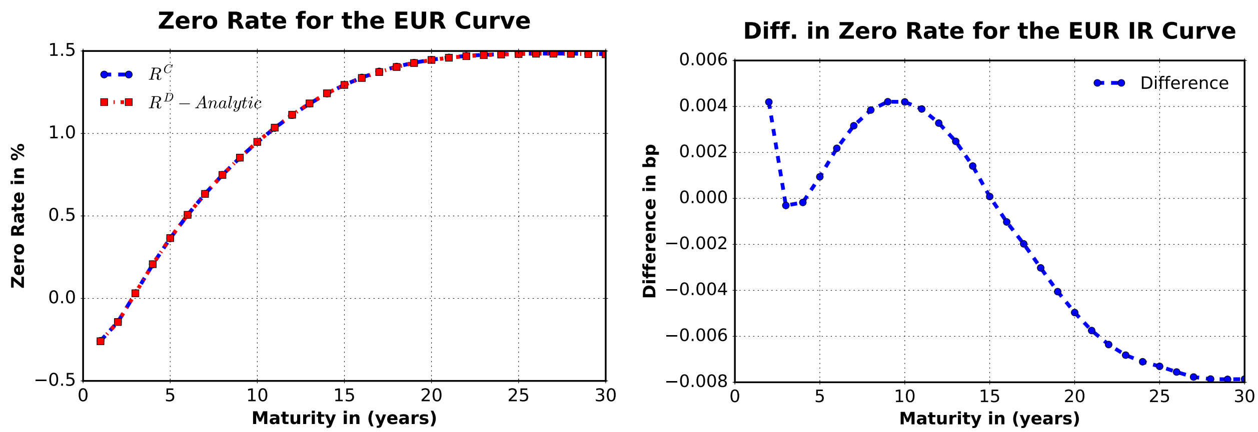

Figure 4: We plot the zero rate obtained by the procedure described in Section 2.2 for the EUR Curve. In particular we use in Eq. (24) the better approximation $SR^0(t) = F(t)\exp\{-(F(t)/2)\}/(1 + F^2(t)/24)$ obtained in Eq. (46), instead of using the simple $SR^0(t) = F(t)$.

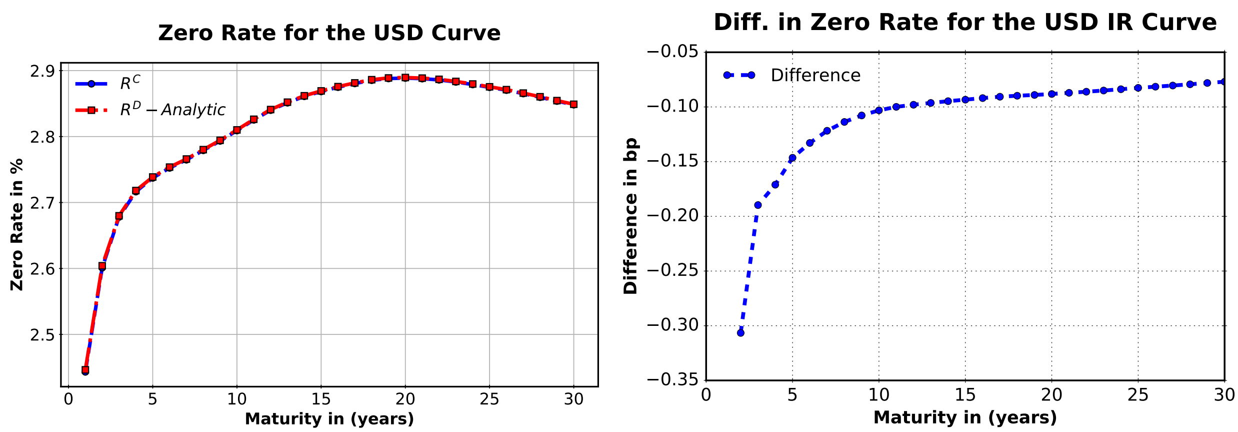

Figure 5: We plot the zero rate obtained by the procedure described in Section 2.2 for the USD Curve. In particular we use in Eq. (24) the better approximation $SR^0(t) = F(t)\exp\{-(F(t)/2)/(1 + F^2(t)/24)$ obtained in Eq. (46), instead of using the simple $SR^0(t) = F(t)$.

curves, respectively. Differences between these are less than a hundredth of a basis point.

2.3 Co-terminal Swap Rates

In this section we calculate analytically the Bond prices from the knowledge of the co-terminal Swap Rates. We denote the co-terminal Annuities spanning the time periods between T_j and T_n by A_{jn} as follows

$$A_{jn} = \tau_j B_j + \dots \tau_n B_n \quad , \tag{48}$$

$$A_{j+1,n} = \tau_{j+1} B_{j+1} + \dots \tau_n B_n \quad . \tag{49}$$

Their difference produces the consecutive bonds

$$\tau_j B_j = A_{jn} - A_{j+1,n} \quad , \tag{50}$$

Table 1: We show the zero rate R_i^C, second column, obtained using the continuous swap rates with Eq. (24), and the discrete swap rates R_i^D, third column, obtained using Eq.(21), for the EUR Curve. In particular we use in Eq. (24) the better approximation $SR^0(t) = F(t)\exp\{-(F(t)/2)\}/(1 + F^2(t)/24)$ obtained in Eq. (46), instead of using the simple $SR^0(t) = F(t)$.

Time (in years)	R_i^C (in %) Semi-Analytic	R_i^D(%) Analytic	$\lvert R_i^C - R_i^D \rvert$ (in bps) Diff
1	-0.259144	-0.259406	0.026235
2	-0.143014	-0.143065	0.005045
3	0.031315	0.031319	-0.000380
4	0.207520	0.207526	-0.000634
5	0.365825	0.365822	0.000339
6	0.506800	0.506785	0.001464
7	0.633639	0.633615	0.002396
8	0.748518	0.748488	0.003047
9	0.853433	0.853399	0.003393
10	0.949001	0.948967	0.003380
11	1.035459	1.035428	0.003073
12	1.113328	1.113304	0.002466
13	1.182345	1.182329	0.001682
14	1.243414	1.243408	0.000619
15	1.294246	1.294253	-0.000690
16	1.336426	1.336444	-0.001774
17	1.372463	1.372490	-0.002707
18	1.402995	1.403033	-0.003734
19	1.427085	1.427133	-0.004736
20	1.445654	1.445710	-0.005626
21	1.459034	1.459098	-0.006387
22	1.468414	1.468483	-0.006973
23	1.474593	1.474667	-0.007411
24	1.478694	1.478771	-0.007675
25	1.481620	1.481699	-0.007857
26	1.483401	1.483482	-0.008088
27	1.483425	1.483508	-0.008277
28	1.482475	1.482558	-0.008351
29	1.481074	1.481157	-0.008348
30	1.479821	1.479904	-0.008322

which will be used below.
With these, the basic co-terminal Swap equation:

$$B_j - SR_{jn}\left(\tau_{j+1}B_{j+1} + \ldots + \tau_n B_n\right) = B_n \quad , \quad j = 0, \ldots, n \quad , \tag{51}$$

takes the following form:

$$-\frac{1}{\tau_j}\Delta A_{jn} - SR_{jn}A_{j+1,n} = B_n \quad , \quad j = 0, \ldots, n \quad , \tag{52}$$

$$\Delta A_{jn} + \tau_j SR_{jn}A_{j+1,n} = -\tau_j B_n \quad , \quad j = 0, \ldots, n \quad . \tag{53}$$

In the above, $A_{nn} = \tau_n B_n$, $SR_{nn} = 0$, $A_{n+1,n} = 0$. For compact notation, we write $\tilde{SR}_{jn} = \tau_j SR_{jn}$.

Table 2: We show the zero rate R_i^C, second column, obtained using the continuous swap rates with Eq. (24), and the discrete swap rates R_i^D, third column, obtained using Eq.(21), for the USD Curve. In particular we use in Eq. (24) the better approximation $SR^0(t) = F(t)\exp\{-(F(t)/2)\}/(1+F^2(t)/24)$ obtained in Eq. (46), instead of using the simple $SR^0(t) = F(t)$. Differences $R_i^C - R_i^D$ are of the order of less than tenth of a basis point.

Time (in years)	$R_i^C(\%)$ Semi-Analytic	$R_i^D(\%)$ Analytic	$\lvert R_i^C - R_i^D\rvert$(bps) Diffs(bps)
1	2.443570	2.446331	-0.276053
2	2.602527	2.604106	-0.157871
3	2.678577	2.679849	-0.127185
4	2.716984	2.718104	-0.111933
5	2.737642	2.738662	-0.101996
6	2.752722	2.753669	-0.094734
7	2.765032	2.765923	-0.089127
8	2.779279	2.780128	-0.084909
9	2.793491	2.794311	-0.082031
10	2.809239	2.810036	-0.079772
11	2.825403	2.826189	-0.078551
12	2.840123	2.840906	-0.078282
13	2.851331	2.852111	-0.078003
14	2.861040	2.861816	-0.077663
15	2.868418	2.869192	-0.077371
16	2.874902	2.875671	-0.076890
17	2.880574	2.881339	-0.076486
18	2.885436	2.886200	-0.076337
19	2.888071	2.888834	-0.076300
20	2.888734	2.889495	-0.076039
21	2.887835	2.888590	-0.075560
22	2.886003	2.886754	-0.075065
23	2.882805	2.883549	-0.074480
24	2.878961	2.879698	-0.073747
25	2.874691	2.875421	-0.072921
26	2.870387	2.871110	-0.072214
27	2.865191	2.865906	-0.071542
28	2.859619	2.860326	-0.070742
29	2.853789	2.854488	-0.069854
30	2.848258	2.848948	-0.069014

Now multiply both sides of Eq. (53) by γ_j^c, a function still to be defined of the known co-terminal Swap Rates, such that $\Delta\gamma_j = \tilde{SR}_{jn}\gamma_j^c$. With this, for Eq. (53) we get

$$\gamma_j^c \,\Delta A_{jn} + \Delta\gamma_j^c A_{j+1,n} = -\tau_j\gamma_j^c B_n \quad , \tag{54}$$

with

$$\Delta\gamma_j^c = \tilde{SR}_{jn}\gamma_j^c \quad . \tag{55}$$

Now we can write the Swap Equation as a discrete differential equation

$$\Delta(\gamma_j^c A_{jn}) = -\tau_j\gamma_j^c B_n \quad . \tag{56}$$

Solving first for $\gamma^{\rm c}_j$, we have

$$\gamma^{\rm c}_{j+1} - \gamma^{\rm c}_j = \tilde{SR}_{jn}\gamma^{\rm c}_j \quad , \quad \gamma^{\rm c}_{j+1} = \gamma^{\rm c}_j(1+\tilde{SR}_{jn}) \quad . \tag{57}$$

Taking $\gamma^{\rm c}_0 = 1$, the solution to the above equation is

$$\gamma^{\rm c}_1 = \gamma^{\rm c}_0(1+\tau_0 SR_{0n}) \quad , \quad \gamma^{\rm c}_{n+1} = \prod_{j=0}^{n}(1+\tau_j SR_{jn}) \quad . \tag{58}$$

Notice here that $\gamma^{\rm c}_j$ is similar to the one for the spot-starting Swaps Eq. (18), with the difference being that one of them uses the first one uses the spot Swap rates SR_{0j}, whereas this one uses the co-terminal Swap Rates SR_{jn}.
Integrating Eq. (56), and using the fact that $A_{n+1,n} = 0$, we obtain

$$-\gamma^{\rm c}_0 A_{0n} = -B_n\sum_{j=0}^{n}\tau_j\gamma^{\rm c}_j \quad , \quad A_{0n} = B_n\left(\tau_0 + \sum_{j=1}^{n}\tau_j\gamma^{\rm c}_j\right) \quad . \tag{59}$$

The rest of the co-terminal Annuities can be calculated by same integration

$$\gamma^{\rm c}_j A_{jn} = A_{0n} - B_n\sum_{i=0}^{j-1}\tau_i\gamma^{\rm c}_i \quad , \quad \gamma^{\rm c}_j A_{jn} = B_n\sum_{i=j}^{n}\tau_i\gamma^{\rm c}_i \quad , \tag{60}$$

$$A_{jn} = \frac{B_n}{\gamma^{\rm c}_j}\sum_{i=j}^{n}\tau_i\gamma^{\rm c}_i \quad , \tag{61}$$

where the as of yet unknown terminal Bond enters into the formula for all Annuities. The terminal Bond factor in front of the sum is the main difference with the spot-starting Annuity formula Eq. (22).
We can find B_n from the boundary condition

$$1 = B_0 = \frac{A_{0n} - A_{1n}}{\tau_0} = \frac{B_n\sum_{i=0}^{n}\tau_i\gamma^{\rm c}_i - \frac{B_n}{\gamma^{\rm c}_1}\sum_{i=1}^{n}\tau_i\gamma^{\rm c}_i}{\tau_0} \quad , \tag{62}$$

$$1 = B_n + \frac{B_n}{\tau_0}\sum_{i=1}^{n}\tau_i\gamma^{\rm c}_i - \frac{B_n}{\tau_0\gamma^{\rm c}_1}\sum_{i=1}^{n}\tau_i\gamma^{\rm c}_i \quad , \tag{63}$$

$$B_n = \left[1 + \frac{1}{\tau_0}\left(1 - \frac{1}{\gamma^{\rm c}_1}\right)\sum_{i=1}^{n}\tau_i\gamma^{\rm c}_i\right]^{-1} \quad . \tag{64}$$

In the continuous limit, the above becomes

$$B_n = \frac{1}{1 + SR^0_{0n}\int_0^T {\rm d}t\,\gamma^{\rm c}(t)} \quad , \tag{65}$$

$$= \frac{1}{1 + SR^0_{0n}\int_0^T {\rm d}t\, e^{\int_0^t SR^0(u,T){\rm d}u}} \quad . \tag{66}$$

Compare this with Eq (26) for the spot-starting Bond.

$$B(T) = 1 - SR^0(T)\int_0^T e^{-\int_t^T SR^0(u){\rm d}u}{\rm d}t \quad , \tag{67}$$

The signs in front of the integral of the exponent are different, but the derivation is correct as can be seen at the end of this section.
The Bonds on the other hand can be calculated by taking the difference between the co-terminal Annuities

$$\tau_j B_j = A_{jn} - A_{j+1,n} \quad = \quad \frac{B_n}{\gamma_j^{\mathrm{c}}}\tau_j\gamma_j^{\mathrm{c}} + \frac{B_n}{\gamma_j^{\mathrm{c}}}\sum_{i=j+1}^{n}\tau_i\gamma_i^{\mathrm{c}} - \frac{B_n}{\gamma_{j+1}^{\mathrm{c}}}\sum_{i=j+1}^{n}\tau_i\gamma_i^{\mathrm{c}} \quad , \tag{68}$$

$$= \quad \tau_j B_n\left[1 + \frac{1}{\tau_j}\left(\frac{1}{\gamma_j^{\mathrm{c}}} - \frac{1}{\gamma_{j+1}^{\mathrm{c}}}\right)\sum_{i=j+1}^{n}\tau_i\gamma_i^{\mathrm{c}}\right] \quad , \tag{69}$$

resulting in

$$B_j = \frac{\left[1 + \frac{1}{\tau_j}\left(\frac{1}{\gamma_j^{\mathrm{c}}} - \frac{1}{\gamma_{j+1}^{\mathrm{c}}}\right)\sum_{i=j+1}^{n}\tau_i\gamma_i^{\mathrm{c}}\right]}{\left[1 + \left(1 - \frac{1}{\gamma_1^{\mathrm{c}}}\right)\sum_{i=1}^{n}\tau_i\gamma_i^{\mathrm{c}}\right]} \quad . \tag{70}$$

or explicitly

$$B_j = \frac{\left[1 + SR_{jn}\tau_{j+1} + SR_{jn}\sum_{i=j+2}^{n}\left(\tau_i\prod_{k=j+2}^{i}(1+\tau_k SR_{kn})\right)\right]}{\left[1 + \tau_1 SR_{0n} + SR_{0n}\sum_{i=2}^{n}\tau_i\prod_{j=2}^{i}(1+\tau_j SR_{jn})\right]} \quad .$$

With this equation we have now a full extracted the full set of Bonds from the knowledge of the complete set of Annual Swap Rates assumed observed in the market.
In the continuous limit the γ^{c} function Eq. (58) becomes

$$\gamma^{\mathrm{c}}(t) = \exp\left\{\int_0^t SR^0(u,T)\mathrm{d}u\right\} \quad . \tag{71}$$

This is similar to the case of the spot-starting Swap Rates, with the only difference in the $SR(0,t)$ and $SR(t,T)$ inside the integral.
Let us look at the continuous limit of this equation. From

$$B(t) - B(T) = SR^0(t,T)A^{\mathrm{cont}}(t,T) \quad . \tag{72}$$

From this we have $B(t) = -\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{cont}}(t,T)$. The basic Swap equation is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{cont}}(t,T) - B(T) = SR^0(t,T)A^{\mathrm{cont}}(t,T) \quad , \tag{73}$$

which is equivalent to

$$\mathrm{d}\left(e^{-\int_t^T SR^0(u,T)\mathrm{d}u}A^{\mathrm{cont}}(t,T)\right) = -B(T)e^{-\int_t^T SR^0(u,T)\mathrm{d}u}\mathrm{d}t \quad , \tag{74}$$

with solution

$$-e^{-\int_t^T SR^0(u,T)\mathrm{d}u}A^{\mathrm{cont}}(t,T) = -B(T)\int_t^T \mathrm{d}v e^{-\int_v^T SR^0(u,T)\mathrm{d}u} \quad , \tag{75}$$

$$A^{\mathrm{cont}}(t,T) = B(T)\int_t^T \mathrm{d}v\, e^{\int_t^v SR^0(u,T)\mathrm{d}u} \quad . \tag{76}$$

This is in agreement with what we get from Eq. (61) for the discrete case.

Based on the above calculation and on the Eq. (72), we can extract $B(T)$

$$B(t) - B(T) = SR^0(t,T)B(T)\int_t^T \mathrm{d}v\, e^{\int_t^v SR^0(u,T)\mathrm{d}u} \quad , \tag{77}$$

resulting in

$$B(T) = \frac{B(t)}{1 + SR^0(t,T)\int_t^T \mathrm{d}v\, e^{\int_t^v SR^0(u,T)\mathrm{d}u}} \quad . \tag{78}$$

This result is same as the one found in Eq. (64) the discrete case, which was obtained using $t = 0$.

2.4 Approximating Bond prices from Swap Rates

In this section we look into simplifying or approximating the analytic formula for the bond prices found previously as function of the Swap Rates

$$B_n = \prod_{j=1}^{n} \frac{1}{1+\tau_j SR_j} \sum_{i=1}^{n} \frac{\tau_i}{\tau_n} \prod_{k=1}^{i-1} (1+\tau_k SR_k) - \prod_{j=1}^{n-1} \frac{1}{1+\tau_j SR_j} \sum_{i=1}^{n-1} \frac{\tau_i}{\tau_n} \prod_{k=1}^{i-1} (1+\tau_k SR_k) \quad . \tag{79}$$

Initially, one can make an effort to get an approximate expression by calculating explicitly the bonds starting from the first one, the second etc. From the first Swap Rates relationship we have $1 - B_1 = SR_1\,(\tau_1 B_1)$, $B_1 = (1+\tau_1 SR_1)^{-1}$. Looking into the second spot-starting Swap,

$$1 - B_2 = SR_2 \sum_{i=1}^{2} \tau_i B_i \quad , \tag{80}$$

$$B_2 = \frac{1 - \tau_1\,(SR_2 - SR_1)}{(1+\tau_1 SR_1)\,(1+\tau_2 SR_2)} \quad . \tag{81}$$

One can think of B_2 in terms of a zeroth order bond plus a correction, on the amount $\tau_1(SR_2 - SR_1)$ discounted with B_2^0, (no correction if the curve is flat $SR_1 = SR_2$)

$$B_2 = B_2^0 - B_2^0\tau_1(SR_2 - SR_1) \quad , \quad B_2^0 = \frac{1}{(1+\tau_1 SR_1)\,(1+\tau_2 SR_2)} \quad . \tag{82}$$

The above equations suggest the following zeroth-order approximation for the bonds

$$\tilde{B}_n^0 = \prod_{i=1}^{n} \frac{1}{1+\tau SR_i} \quad . \tag{83}$$

The higher order corrections to the approximation Eq.(83) bond price, can be calculated exactly, as shown by Eq.(96) below. Further, the exact series equation for the correction, Eq.(96), can be approximated as in Eq.(103).

In Figures 6 we show a comparison of the zero-order approximation Eq.(83) and the analytic bond prices for the EUR and USD zero rates, respectively.

However, notice that to first order Taylor expansion the zero rates are same as the Swap Rates

$$R_1 = -\frac{\ln B_1}{\tau_1} = \frac{1}{\tau_1}\ln(1+\tau_1 SR_1) \approx SR_1 \quad . \tag{84}$$

For the second bond we have

$$R_2 = -\frac{1}{\tau_1+\tau_2}\ln B_2^0(1 - \tau_1(SR_2 - SR_1)) \approx SR_2 \quad . \tag{85}$$

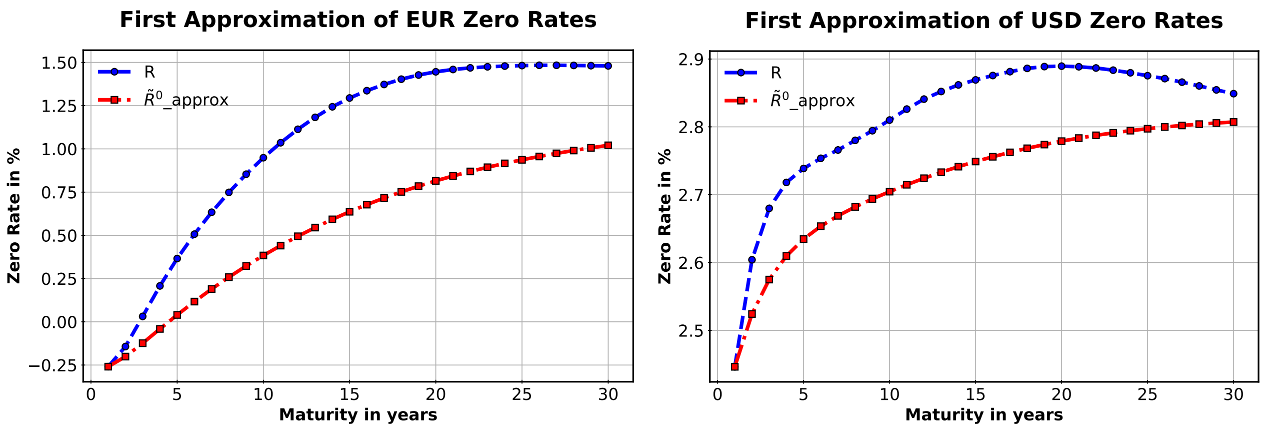

Figure 6: A plot of the exact zero rate R obtained from Eq.(21), blue line, and the approximate zero rate $\tilde{B}_n^0$ of (83), red line, are shown. The Bond price approximation, $\tilde{B}_n^1$ given by Eq.(86) is a better approximation then the $\tilde{B}_n^0$ one shown here, see Fig. 7.

This means that the yield curve looks very similar to the Swap-Rate curve, with minor differences due to Taylor approximation above.
In light of this, it is worth thinking of another approximation and rewrite the discount bond in this alternative way

$$B_2 \quad = \quad \frac{1}{(1+\tau_1 SR_2)(1+\tau_2 SR_2)}\left[1+\frac{\tau_1^2 SR_2(SR_2-SR_1)}{(1+\tau_1 SR_1)}\right] \quad . \tag{86}$$

The correction in the square brackets in this formula is of higher order than the one obtained in Eq.(82). Therefore, the above calculation suggests a better approximation to the zero rate than the previous one Eq. (83).

$$\tilde{B}_n^1 = \frac{1}{(1+\tau SR_n)^n} \quad . \tag{87}$$

In Figures 7 we show a comparison of the approximation Eq.(87) and the analytic zero rate R obtained from Eq.(21), for the EUR and USD zero rates, respectively.
The first approximation in (83) underestimates the zero rates. This comes from the fact that in a expansive economic period, the swap rates are essentially monotonically increasing. From (23), one can show rigorously that whenever the swap rates are monotonically increasing the first approximation underestimates the zero rate.
Often in various calculations in fixed income, the Annuity, to a first order approximation is obtained by approximating the yield curve with another flat one. One possible flat rate is the mean rate, namely

$$\bar{S} = \frac{1}{n}\sum_{i=1}^{n} SR_i \quad , \quad A_n \quad = \quad \sum_{i=1}^{n}\frac{\tau}{(1+\tau\bar{S})^i} = \frac{1}{\bar{S}}\left(1-\frac{1}{(1+\tau\bar{S})^n}\right) \quad . \tag{88}$$

Using the previous pages equation

$$B_j = \frac{1-SR_j A_{j-1}}{1+\tau_j SR_j} = \frac{1}{(1+\tau\bar{S})^j} \quad , \tag{89}$$

which is of course self-consistent with the Eq (88).
A similar computation can be done with the continuous swap rate. Namely, one can substitute the average of S^0 in below formula

$$A^{\text{cont}}(T) = \int_0^T e^{-\int_t^T S^0(u)\mathrm{d}u}dt \quad .$$

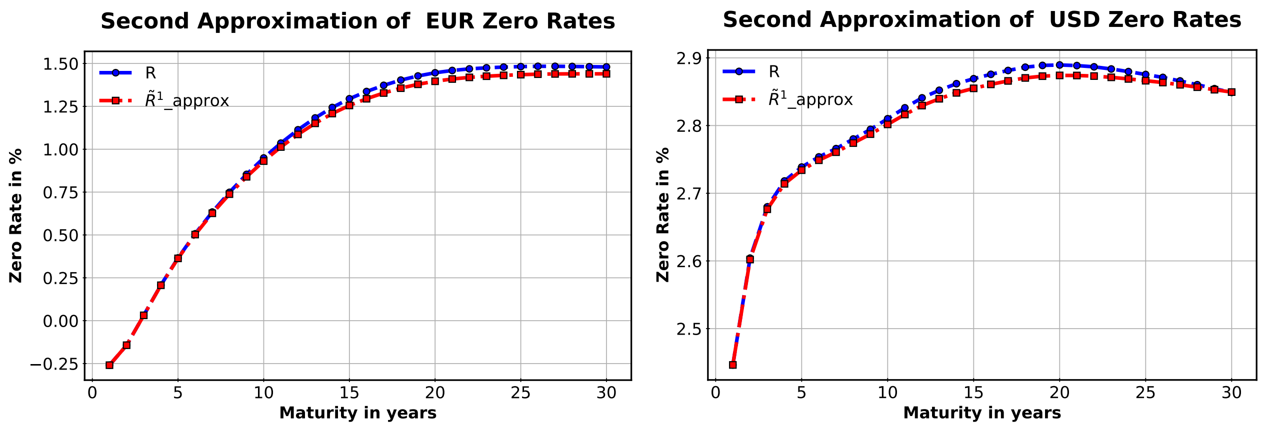

Figure 7: A plot of the exact zero rate R obtained from Eq.(21), blue line, and the approximate zero rate $\tilde{B}_n^1$ of (87), red line, are shown. The Bond price approximation, $\tilde{B}_n^1$ given by Eq.(87) is a better approximation then the $\tilde{B}_n^0$ one shown in Fig. 6.

Both in the discrete and continuous case, the average swap rate is not usually the best option. Indeed, from the analytic formula one can obtain a better approximation. Let us recall the annuity formula of Eq. (22)), namely

$$
\begin{aligned}
A_n &= \sum_{i=1}^{n} \tau_i \prod_{k=i}^{n} \frac{1}{1+\tau_k SR_k} \quad , \qquad (90) \\
&= \frac{\tau_n}{1+\tau_n SR_n} + \frac{\tau_{n-1}}{(1+\tau_n SR_n)(1+\tau_{n-1}SR_{n-1})} + \cdots + \frac{\tau_1}{(1+\tau_n SR_n)\cdots(1+\tau_1 SR_1)} \quad .
\end{aligned}
$$

It can be easily seen from the above formula that in order to compute the annuity for the nth year, the last swap rates are more important than the initial ones. For this reason, instead of the simple average, it is more convenient to make a weighted average skewed towards the tail of the sequence of the swap rates. In Figure 9, we take the average of the last 3/4 term in the sequence $SR_1, SR_2, \ldots, SR_n$. Namely, suppose we want to calculate the annuity for the n -th year. We take $\tau_i = 1$ for ease of calculations below. Let $k = \lfloor n/4 \rfloor$ and $m = \sum_{i=k}^{n} SR_i$. Then from the constant swap rate formula we have that

$$
A_n = \frac{1 - 1/(1+m)^n}{m} \quad , \qquad (91)
$$

In the following we will make a better approximation by using Eq.(82). As we already saw, the quantity $\tilde{B}_n^0$ and $\tilde{B}_n^1$ capture to a good extent the behavior of B_n.

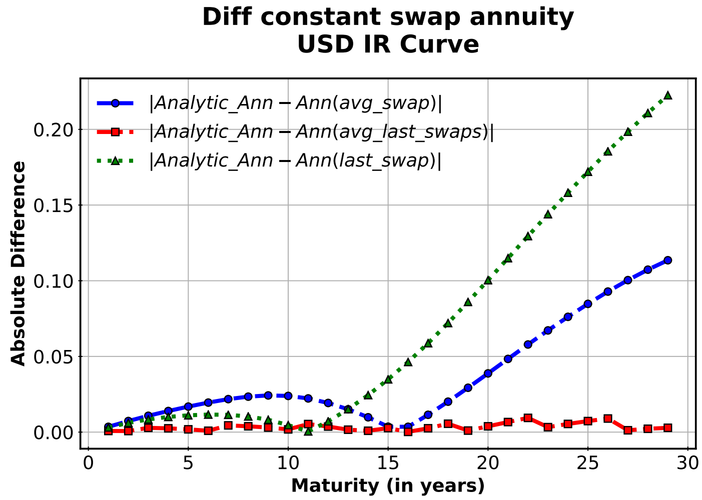

Figure 8: In the above graph we plot the absolute difference between the annuity computed analytically and three types of annuities computed by using flat rates. By $Ann(Avg\_swap)$ we mean the annuity computed by assuming that the swap rates are flat and equal to $\frac{1}{n}\sum_{i=1}^{n} SR_i$. By $Ann(avg\_last\_swaps)$, we mean the annuity computed by using as a flat swap rate, the swap rate weighted as in(91). As it can be seen the error is significantly smaller. By $Ann(last\_swap)$, we mean the annuity computed by using as a flat rate the rate of the last swap.

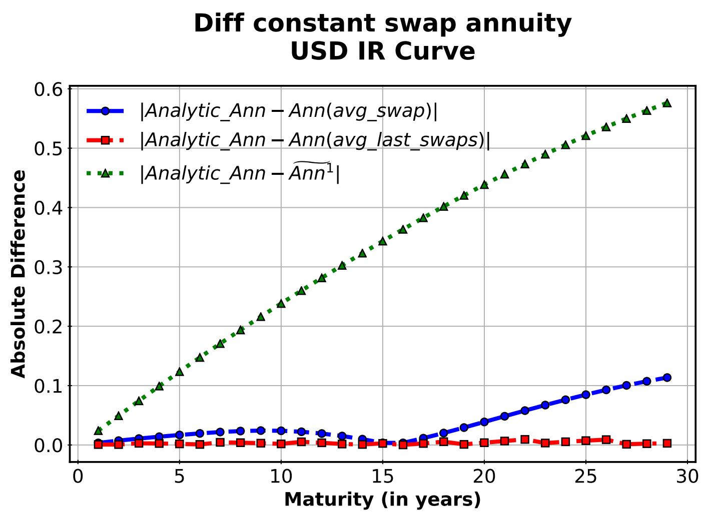

Figure 9: In the above graph we plot the absolute difference between the annuity computed analytically and three types of annuities computed by using flat rates. By $Ann(Avg\_swap)$ we mean the annuity computed by assuming that the swap rates are flat and equal to $\frac{1}{n}\sum_{i=1}^{n} SR_i$. By $Ann(avg\_last\_swaps)$, we mean the annuity computed by using as a flat swap rate, the swap rate weighted as in(91). As it can be seen the error is significantly smaller. By $\widetilde{Ann}$, we mean the annuity computed by using as as a bond price the ones obtained via (87). Notice that the error increases as maturity grows. This is due to the fact that the approximation of the yield curve via (87) is always bellow the exact yield curve (see Figure 7), thus the error increases over time. The error of $Ann(avg\_last\_swap)$ from the Analytic Annuity is approximately 40 times smaller than the error of $Ann(avg\_swap)$ from the Analytic Annuity.

Let us denote by $B_n^{\text{corr}} = B_n - \tilde{B}_n$. By using (23) one has that

$$
\begin{aligned}
B_n^{\text{corr},0} = B_n - \tilde{B}_n^0 &= \frac{1}{\tau_n}\left(\frac{1}{\gamma_n}\sum_{i=1}^{n}\tau_i\gamma_{i-1} - \frac{1}{\gamma_{n-1}}\sum_{i=1}^{n-1}\tau_i\gamma_{i-1}\right) - \frac{1}{\gamma_n} \quad , &(92)\\
&= \left(\frac{1}{\gamma_n}\sum_{i=1}^{n-1}\gamma_i - \frac{1}{\gamma_{n-1}}\sum_{i=1}^{n-1}\gamma_{i-1}\right) \quad , &(93)\\
&= \sum_{i=1}^{n-1}\frac{\gamma_{i-1}}{\gamma_{n-1}}\left(\frac{\gamma_i}{\gamma_n}\frac{\gamma_{n-1}}{\gamma_{i-1}} - 1\right) \quad , &(94)\\
&= \frac{1}{\gamma_{n-1}}\sum_{i=0}^{n-2}\Big(\frac{1+\tau SR_{i+1}}{1+\tau SR_n} - 1\Big)\gamma_i \quad , &(95)\\
&= \sum_{i=0}^{n-2}\tau(SR_{i+1} - SR_n)\frac{\gamma_i}{\gamma_n} \quad . &(96)
\end{aligned}
$$

In the continuous limit, a similar identity holds:

$$
B^{\text{corr}}(T) = \int_0^T \big(SR^0(t) - SR^0(T)\big)\, e^{-\int_t^T SR^0(u)\mathrm{d}u}\mathrm{d}t \quad . \tag{97}
$$

Indeed, the above is equivalent to

$$
\begin{aligned}
B^0(T) &= 1 - \int_0^T SR^0(x)e^{-\int_x^T SR^0(u)\mathrm{d}u}\mathrm{d}x &(98)\\
&= 1 - \int_0^T \partial_x e^{-\int_x^T SR^0(u)\mathrm{d}u}\mathrm{d}x \quad . &(99)
\end{aligned}
$$

To prove the above notice that

$$
\frac{\mathrm{d}}{\mathrm{d}x}\exp\Big(-\int_x^T SR^0(u)\mathrm{d}u\Big) = SR^0(x)\exp\Big(-\int_x^T SR^0(u)\mathrm{d}u\Big)
$$

and thus

$$
1 - \int_0^T SR^0(x)e^{-\int_x^T SR^0(u)\mathrm{d}u}\mathrm{d}x = 1 - e^{\int_x^T SR^0(u)\mathrm{d}u}\Big|_0^T = e^{\int_0^T SR^0(u)\mathrm{d}u} = B^0(T) \quad .
$$

Put in similar form

$$
\begin{aligned}
B(T) &= 1 - \int_0^T SR^0(T)e^{-\int_x^T SR^0(u)\mathrm{d}u}\mathrm{d}x \quad , &(100)\\
B^0(T) &= 1 - \int_0^T SR^0(x)e^{-\int_x^T SR^0(u)\mathrm{d}u}\mathrm{d}x \quad , &(101)
\end{aligned}
$$

In order to approximate B_n^{corr} to a rather good approximation it is sufficient to substitute SR_i with

$$
\bar{S} := \frac{1}{n}\sum_{i=1}^{n} SR_i \quad , \tag{102}
$$

when computing γ_i/γ_n. This corresponds to computing γ_i as if the interest rates were constant. Namely we approximate B_n^{corr} with

$$\tilde{B}_n^{\text{corr}} = \sum_{i=0}^{n-2} (SR_{i+1} - SR_n) \frac{(1+\bar{S})^i}{(1+\bar{S})^n} \quad . \tag{103}$$

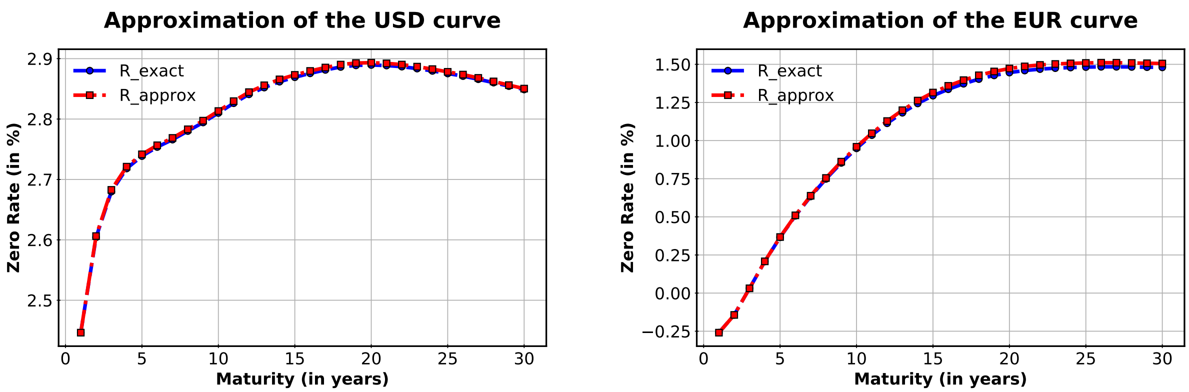

Figure 10: Plot of $-\log(B_t)/t$ and $-\log(\text{approx\_}B_t)/t$ (USD on the left and EUR on the right).

Table 3: USD curve approximation obtained by $\tilde{B}_n^0$, Eq.(83), plus $\tilde{B}_n^{\text{corr}}$, Eq.(103), versus the analytic zero rates, are given.

Maturity	R_exact (in %)	R_approx (in %)	R_exact - R_approx (in bp)
1	2.446331	2.446331	0.000000
2	2.604106	2.606214	-0.210844
3	2.679849	2.682720	-0.287138
4	2.718104	2.721143	-0.303900
5	2.738662	2.741628	-0.296651
6	2.753669	2.756561	-0.289155
7	2.765923	2.768762	-0.283870
8	2.780128	2.783036	-0.290808
9	2.794311	2.797334	-0.302252
10	2.810036	2.813253	-0.321717
11	2.826189	2.829641	-0.345197
12	2.840906	2.844572	-0.366610
13	2.852111	2.855903	-0.379161
14	2.861816	2.865706	-0.388981
15	2.869192	2.873122	-0.393084
16	2.875671	2.879630	-0.395934
17	2.881339	2.885314	-0.397588
18	2.886200	2.890178	-0.397865
19	2.888834	2.892751	-0.391723
20	2.889495	2.893293	-0.379876
21	2.888590	2.892227	-0.363617
22	2.886754	2.890205	-0.345073
23	2.883549	2.886776	-0.322636
24	2.879698	2.882686	-0.298817

25	2.875421	2.878164	-0.274355
26	2.871110	2.873616	-0.250601
27	2.865906	2.868151	-0.224483
28	2.860326	2.862305	-0.197837
29	2.854488	2.856199	-0.171071
30	2.848948	2.850409	-0.146122

Table 4: EUR curve approximation obtained by $\tilde{B}_n^0$, Eq.(83), plus $\tilde{B}_n^{\text{corr}}$, Eq.(103), versus the analytic zero rates, are given.

Maturity	R_exact (in %)	R_approx (in %)	R_exact - R_approx (in bp)
1	-0.259406	-0.259406	0.000000
2	-0.143065	-0.143148	0.008310
3	0.031319	0.031402	-0.008232
4	0.207526	0.208230	-0.070418
5	0.365822	0.367535	-0.171384
6	0.506785	0.509808	-0.302291
7	0.633615	0.638190	-0.457566
8	0.748488	0.754816	-0.632829
9	0.853399	0.861651	-0.825193
10	0.948967	0.959270	-1.030352
11	1.035428	1.047860	-1.243178
12	1.113304	1.127896	-1.459213
13	1.182329	1.199046	-1.671697
14	1.243408	1.262183	-1.877487
15	1.294253	1.314873	-2.061967
16	1.336444	1.358683	-2.223929
17	1.372490	1.396174	-2.368366
18	1.403033	1.427977	-2.494431
19	1.427133	1.453075	-2.594280
20	1.445710	1.472398	-2.668835
21	1.459098	1.486268	-2.717018
22	1.468483	1.495913	-2.742975
23	1.474667	1.502165	-2.749808
24	1.478771	1.506202	-2.743157
25	1.481699	1.508980	-2.728132
26	1.483482	1.510533	-2.705056
27	1.483508	1.510210	-2.670236
28	1.482558	1.508842	-2.628389
29	1.481157	1.506986	-2.582933
30	1.479904	1.505283	-2.537945

With this approximation the bond price has been reduced from

$$B_n = \prod_{j=1}^{n} \frac{1}{1+\tau_j SR_j} \sum_{i=1}^{n} \frac{\tau_i}{\tau_n} \prod_{k=1}^{i-1} (1+\tau_k SR_k) - \prod_{j=1}^{n-1} \frac{1}{1+\tau_j SR_j} \sum_{i=1}^{n-1} \frac{\tau_i}{\tau_n} \prod_{k=1}^{i-1} (1+\tau_k SR_k) \quad , \tag{104}$$

to

$$B_n \approx \prod_{i=1}^{n} \frac{1}{1+\tau SR_i} + \sum_{i=0}^{n-2} \frac{\tau_{i+1}(SR_{i+1} - SR_n)}{(1+\tau_n \bar{S})^{n-i}} \quad . \tag{105}$$

3 Bootstrapping and interpolating in presence of incomplete data

As already mentioned in the introduction, the available data from the market is partial. The challenge is to build a realistic curve which compatible with the market data. Thus, in building a proper yield curve we need to keep in mind several considerations:

i). The yield curve needs to reprice the market instruments that were used as inputs.

ii). The yield curve should not allow arbitrage, the forward rates flat-discrete (1m, 3m, or 6m ones) or instantaneous forward rates should remain positive.

iii). The interpolation method needs to be local. Small changes of market input should not change far away points in the curve. Changes should affect only the neighboring points which are most correlated with the change.

iv). Are the forwards continuous and stable? Stability can be defined by bps changes in forwards for 1bp change in one of of the input points.

Some other attractive features are:

a). The yield curve and the forward f are expected to be smooth (not only continuous).

b). Concavity of the yield curve is also a desirable property. Although there is no economics argument for concavity, it is often observed in market data and linked to the no-arbitrage in some specific situations.

The process of extrapolating the yield curve from market data is often called "bootstrapping and interpolating". Apart to the above qualitative properties that the curve should satisfy, an interesting question is how good can bootstrapping and interpolating process reconstruct a realistic yield curve. More precisely, given a realistic yield curve R one can calculate the implied swap rates SR_i at maturities $\{1, \dots, 5, 7, 10, 12, 15, 20, 25, 30\}$. Is it possible to reconstruct the original curve R by using only the swap rates at these specified maturities?
In order to extrapolate the complete yield curve from the swap rates, there are several methods used in the literature. In this paper we will concentrate on two groups of such techniques: the parametric and the variational technique. These will be discussed more in detail in the following.

3.1 Bootstrap and interpolate: the parametric method

The idea of the parametric method is to assume that the yield curve belongs to a specific class of functions in the parts where no market data is available. Usually the particular function in the class is determined by one or more parameters (thus the name "parametric"). More precisely, for every interval (a, b) where no market data is available one assumes that there exists a function R_α (defined by the parameter α) that matches the yield curve in that interval. The curve created in such a way needs to fit the market data. Usually this means that a series of linear conditions have to be satisfied. Thus one would like to tune the various parameters such that these conditions are satisfied.
In the following we give a series of examples where different types of classes are considered.

3.1.1 Swap Rates on quarterly Libor frequency - complete set of annual points

In this case we assume that we have a complete set of Swap Rates at the annual points $T_i = 1, 2, \dots, 30$. In this case we would need to fill the points for $B(t_j)$, where $T_{i-1} \le t_j \le T_i$, for example $j = 1, \cdots 4$ where $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, $t_4 = T_1$.
The most common way to build the yield curve is by assuming that the instantaneous forward rates are piecewise flat between the two swap maturity dates T_{n_i} and $T_{n_{i+1}}$, where $n_1 < n_2 \cdots$

$$f(0,t) = \begin{cases} f_1\ , & \text{for } 0 \le t \le T_1 \quad , \\ f_{i+1}\ , & \text{for } T_i \le t \le T_{i+1} \quad . \end{cases} \tag{106}$$

For $i = 1$ and $j = 1, \ldots, n$, where $n = 2$ for semi-annual Swaps and $n = 4$ for quarterly Swaps,

$$B(0, t_j) = \exp\left(-\int_0^{t_j} f(0, u)\mathrm{d}u\right) = \exp\left(-f_1 t_j\right) \quad . \tag{107}$$

Bond $B(0, T_1)$:

$$1 - \exp(-f_1 T_1) = S_1 \sum_{j=1}^{n} \tau_j \exp(-f_1 t_j) \quad . \tag{108}$$

In general we can solve this for f_1 by Newton-Raphson method. This way we extract also the other in-between bonds $B(0, t_j) = \exp\{-f_1 t_j\}$ for $j = 1, \ldots, n-1$.
For the ith forward rate proceed iteratively. We know all the forward rates up to maturity T_i. We assume the next forward rate is constant f_{i+1} out to maturity T_{i+1}. We now solve the following for f_{i+1}

$$\begin{aligned} 1 - B(0, T_{i+1}) &= S_{i+1} \sum_{j=1}^{n_{i+1}} \tau_j B(0, t_j) \quad , & (109) \\ &= S_{i+1} \left(\sum_{j=1}^{n_i} \tau_j B(0, t_j) + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(0, t_j) \right) \quad . & (110) \end{aligned}$$

$$B(0, T_{n_{i+1}}) + S_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(0, t_j) = 1 - S_{i+1} \sum_{j=1}^{n_i} \tau_j B(0, t_j) \quad , \tag{111}$$

where $n_i = ni$ and $n_{i+1} = n(i+1)$ for shorter notation.
For $j = n_i + 1, \ldots, n_{i+1}$ we have

$$\begin{aligned} B(0, t_j) &= \exp\left(-\int_0^{t_j} f(0, u)\mathrm{d}u\right) = B(0, T_{n_i}) \exp\left(-\int_{T_{n_i}}^{t_j} f(0, u)\mathrm{d}u\right) \quad , \\ &= B(0, T_{n_i}) \exp\Big(-f_{i+1}(t_j - T_{n_i})\Big) \quad . \end{aligned} \tag{112}$$

We now solve (111) for f_{i+1}

$$\begin{aligned} &\exp\left(-f_{i+1}(T_{n_{i+1}} - T_{n_i})\right) + S_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j \exp\left(-f_{i+1}(T_j - T_{n_i})\right) \\ &= B^{-1}(0, T_{n_i}) \left(B(0, T_0) - S_{i+1} \sum_{j=1}^{n_i} \tau_j B(0, T_j) \right) \quad . \end{aligned} \tag{113}$$

In the simple case of all τ_i being all equal to 0.25, the above equations can be solved semi-analytically. For the first year

$$1 - \exp\{-f_1\} = \frac{1}{n} S_1 \sum_{i=1}^{n_1} \exp\left\{-f_1 \frac{i}{n}\right\} = \frac{1}{n} S_1 \exp\left\{-\frac{f_1}{n}\right\} \frac{1 - \exp\{-f_1\}}{1 - \exp\left\{-\frac{f_1}{n}\right\}} \quad , \tag{114}$$

where $n = 4$ and $T_{n_1} = 1$. Canceling $(1 - \exp\{-f_1\})$ on both sides, we get

$$1 = \frac{S_1}{n} \frac{\exp\left\{-\frac{f_1}{n}\right\}}{1 - \exp\left\{-\frac{f_1}{n}\right\}} \quad , \tag{115}$$

resulting in

$$f_1 = n \ln\left(1 + \frac{S_1}{n}\right) \approx S_1 \quad . \tag{116}$$

We can now continue iteratively, as follows

$$1 - B(0, T_{n_{i+1}}) = S_{i+1} \sum_{j=1}^{n_{i+1}} \tau_j B(0, t_j) \quad , \tag{117}$$

$$1 - B(0, T_{n_{i+1}}) = S_{i+1} \left(\sum_{j=1}^{n_i} \tau_j B(0, t_j) + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(0, t_j) \right) \quad , \tag{118}$$

$$1 - B(0, T_{n_{i+1}}) = S_{i+1} \left(\frac{1 - B(0, T_{n_i})}{S_i} + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(0, t_j) \right) \quad . \tag{119}$$

We can split away the factor $Z_{i+1} = \exp\{-f_{i+1}/n\}$, where

$$\begin{aligned} B(0, T_{n_i+1}) &= B(0, T_{n_i}) Z_{i+1} \quad , & (120) \\ B(0, T_{n_i+2}) &= B(0, T_{n_i}) Z_{i+1}^2 \quad , & (121) \\ B(0, T_{n_{i+1}}) &= B(0, T_{n_i}) Z_{i+1}^n \quad . & (122) \end{aligned}$$

Substituting in Eq. (119)

$$1 - B(0, T_{n_i}) Z_{i+1}^n = S_{i+1} \left[\frac{1 - B(0, T_{n_i})}{S_i} + \frac{B(0, T_{n_i})}{n} \sum_{j=1}^{n} Z_{i+1}^j \right] \quad . \tag{123}$$

Equivalently

$$1 - (1 - B(0, T_{n_i})) \frac{S_{i+1}}{S_i} = B(0, T_{n_i}) Z_{i+1}^n + B(0, T_{n_i}) \frac{S_{i+1}}{n} Z_{i+1} \frac{(1 - Z_{i+1}^n)}{1 - Z_{i+1}} \quad . \tag{124}$$

This leads to the following equation:

$$Z_{i+1}^n + \frac{S_{i+1}}{n} \frac{Z_{i+1}(1 - Z_{i+1}^n)}{1 - Z_{i+1}} = \frac{1}{B(0, T_{n_i})} - \left(\frac{1}{B(0, T_{n_i})} - 1 \right) \frac{S_{i+1}}{S_i} \quad . \tag{125}$$

For $n = 1$, the equation reduces to first order

$$Z_{i+1} = \frac{1}{1 + S_{1+1}} \left[\frac{1}{B(0, T_{n_i})} - \left(\frac{1}{B(0, T_{n_i})} - 1 \right) \frac{S_{i+1}}{S_i} \right] \quad . \tag{126}$$

Notice that in the case when $n = 1$, this corresponds to the case discussed in Section 2.
Let us come back now to the case of $n = 4$. For ease of notation we denote by $B_T = B(0, T)$. From Eq. (121) and Eq. (123) in order to obtain the bond prices we need to solve the following n-th order polynomial:

$$g(z) = B_{kn-n}(1 + SR_{kn}) z^n + B_{kn-n} SR_{kn} z^{n-1} + \cdots + B_{kn-n} SR_{kn-n} z^1 = 1 - SR_{kn} \sum_{j=1}^{kn-k} B_j \quad , \tag{127}$$

where all SR_j in the above equation stand for short notation for SR_j/n.
In order for the above to make sense there should exist at least one solution $z>0$ and for an algorithm to be well-defined there should exist only one such solution. Indeed, there exists only one $z>0$ such $g(z)=0$. In order to see this fact notice that for $z>0$ one has that g is monotonically increasing as all the coefficients of the polynomial are positive. Given that $g(0)=0$ and $\lim_{z\to\infty} g(z)=+\infty$ one has that there exists only one solution.
When $n=4$ one has finer results on the structure of the roots.
From the above discussion we have that

$$B_{4k-4}(1+SR_{4k})z^4+B_{4k-4}SR_{4k}z^3+B_{4k-4}SR_{4k}z^2+B_{4k-4}SR_{4k}z+SR_{4k}\sum_{j=1}^{4k-4}B_{j/4}-1=0 \quad .$$

Given a generic 4th order equation $az^4+bz^3+cz^2+dz^1+e=0$, its roots can be calculated via

$$\begin{aligned} x_{1,2} &= -\frac{b}{4a}-S\pm\frac{1}{2}\left[-4S^2-2p+\frac{q}{S}\right]^{\frac{1}{2}} \quad , \\ x_{3,4} &= -\frac{b}{4a}+S\pm\frac{1}{2}\left[-4S^2-2p-\frac{q}{S}\right]^{\frac{1}{2}} \quad , \end{aligned}$$

where

$$p=\frac{8ac-3b^2}{8a^2} \quad , \quad q=\frac{b^3-4abc+8a^2d}{8a^3} \quad ,$$

$$S=\frac{1}{2}\left[-\frac{2}{3}\,p+\frac{1}{3a}\left(Q+\frac{\Delta_0}{Q}\right)\right]^{\frac{1}{2}} \quad , \quad Q=\left[\frac{\Delta_1+\left(\Delta_1^2-4\Delta_0^3\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{3}} \quad ,$$

and

$$\Delta_0=c^2-3bd+12ae \quad , \quad \Delta_1=2c^3-9bcd+27b^2e+27ad^2-72ace \quad .$$

In our specific case, we have that $a=B_{4k-4}(1+SR_{4k})$, $b=c=d=B_{4k-4}$ and $e=SR_{4k}\sum_{j=1}^{4k-4}B_j-1$.
Given that there exists only one positive root, we can find all the roots using the above explicit formulas and take the only real positive root. The two procedures are equivalent. Depending on the situation by the explicit formula we avoid convergence issues.
One can show that if $SR_{4k}\sum_{j=1}^{4k-4}B_j-1<0$, one has that the roots $\{z_1,z_2,z_3,z_4\}$ satisfy the following: $z_1>0$, $z_2<0$ and z_3,z_4 are complex with $z_3=\bar{z}_4$.
In order to see this let

$$f(z):=B_{4k-4}(1+SR_{4k})z^4+B_{4k-4}SR_{4k}z^3+B_{4k-4}SR_nz^2+B_{4k-4}SR_{4k}z^1+SR_{4k}\sum_{j=1}^{4k-4}B_j-1 \quad .$$

Given that $f(0)=SR_{4k}\sum_{j=1}^{4k-4}B_j-1$ and that $\lim_{z\to-\infty}f(z)=\lim_{z\to+\infty}f(z)=+\infty$, it is immediately obvious that there exist a positive root and a negative root. In order to show that the other roots are complex, let us consider the derivative $f'(z)=4az^3+3bz^2+2cz+d$.
We can now study the discriminant of the cubic polynomial $f'=4az^3+3bz^2+2cz+d$. The cubic discriminant of f' is $\Delta=(3b)^2(2c)^2-4(4a)(2c)^3-4(3b)^3d-27(4a)^2d^2+18(4a)(3b)(2c)d$. By plugging in the values of the a,b,c,d one has that $\Delta<0$ whenever $a>0$. It is well-known that whenever $\Delta<0$, there exists only one real root and the other two are complex. If there were more than two real solutions of $f=0$, there would be at least one local minimum and one local maximum and thus at each of them, one would have that $f'(z_{\min})=f'(z_{\max})$.

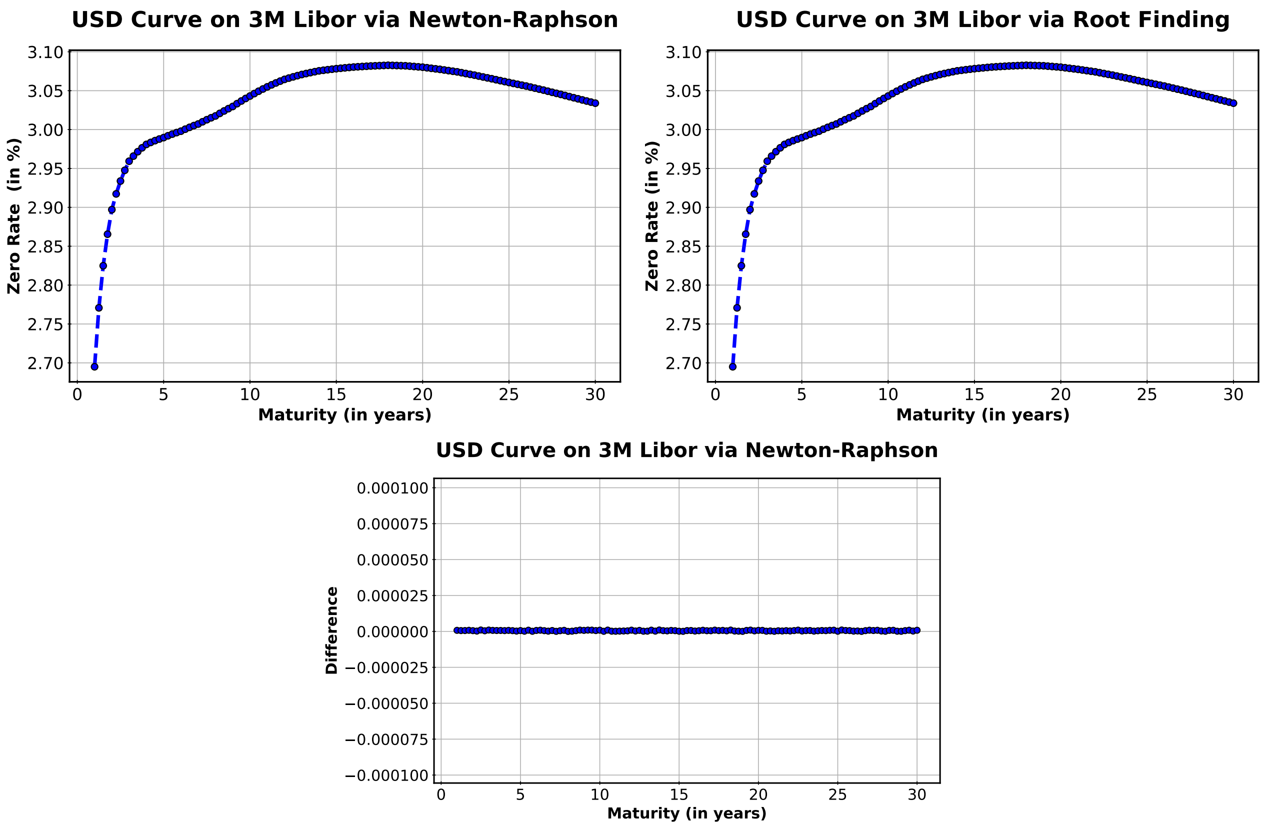

Figure 11: In the above graph we plot the zero rate calculated via the Newton-Raphson and root finding.

Table 5: Roots of the 4th grade polynomials

Time	root 1	root 2	root 3	root 4
1	(0.99252+0j)	(-0.99850+0j)	(4.49611e-06+0.99850j)	(4.49611e-06-0.99850j)
2	(0.99342+0j)	(-0.99965+0j)	(1.07009e-06+0.99965j)	(1.07009e-06-0.99965j)
3	(0.99318+0j)	(-0.99959+0j)	(1.29739e-06+0.99959j)	(1.29739e-06-0.99959j)
4	(0.99259+0j)	(-0.99923+0j)	(2.55919e-06+0.99923j)	(2.55919e-06-0.99923j)
5	(0.99235+0j)	(-0.99917+0j)	(2.83320e-06+0.99917j)	(2.83320e-06-0.99917j)
6	(0.99223+0j)	(-0.99919+0j)	(2.80393e-06+0.99919j)	(2.80393e-06-0.99919j)
7	(0.99234+0j)	(-0.99939+0j)	(2.14697e-06+0.99939j)	(2.14697e-06-0.99939j)
8	(0.99219+0j)	(-0.99932+0j)	(2.42440e-06+0.99932j)	(2.42440e-06-0.99932j)
9	(0.99221+0j)	(-0.99940+0j)	(2.13749e-06+0.99940j)	(2.13749e-06-0.99940j)
10	(0.99223+0j)	(-0.99947+0j)	(1.90687e-06+0.99947j)	(1.90687e-06-0.99947j)
11	(0.99256+0j)	(-0.99981+0j)	(6.58734e-07+0.99981j)	(6.58734e-07-0.99981j)
12	(0.99235+0j)	(-0.99962+0j)	(1.35199e-06+0.99962j)	(1.35199e-06-0.99962j)
13	(0.99232+0j)	(-0.99963+0j)	(1.34153e-06+0.99963j)	(1.34153e-06-0.99963j)
14	(0.99237+0j)	(-0.99969+0j)	(1.12760e-06+0.99969j)	(1.12760e-06-0.99969j)
15	(0.99270+0j)	(-1.00002+0j)	(-7.79993e-08+1.00002j)	(-7.79993e-08-1.00002j)
16	(0.99240+0j)	(-0.99974+0j)	(9.43580e-07+0.99974j)	(9.43580e-07-0.99974j)
17	(0.99242+0j)	(-0.99976+0j)	(8.49937e-07+0.99976j)	(8.49937e-07-0.99976j)
18	(0.99244+0j)	(-0.99979+0j)	(7.54879e-07+0.99979j)	(7.54879e-07-0.99979j)
19	(0.99275+0j)	(-1.00010+0j)	(-3.83046e-07+1.00010j)	(-3.83046e-07-1.00010j)

20	(0.99243+0j)	(-0.99979+0j)	(7.75324e-07+0.99979j)	(7.75324e-07-0.99979j)
21	(0.99242+0j)	(-0.99979+0j)	(7.64867e-07+0.99979j)	(7.64867e-07-0.99979j)
22	(0.99242+0j)	(-0.99979+0j)	(7.42754e-07+0.99979j)	(7.42754e-07-0.99979j)
23	(0.99263+0j)	(-1.00000+0j)	(-2.44442e-09+1.00000j)	(-2.44442e-09-1.00000j)
24	(0.99237+0j)	(-0.99975+0j)	(9.00368e-07+0.99975j)	(9.00368e-07-0.99975j)
25	(0.99234+0j)	(-0.99972+0j)	(1.01332e-06+0.99972j)	(1.01332e-06-0.99972j)
26	(0.99231+0j)	(-0.99970+0j)	(1.08051e-06+0.99970j)	(1.08051e-06-0.99970j)
27	(0.99263+0j)	(-1.00002+0j)	(-1.10430e-07+1.00002j)	(-1.10430e-07-1.00002j)
28	(0.99225+0j)	(-0.99965+0j)	(1.27234e-06+0.99965j)	(1.27234e-06-0.99965j)
29	(0.99223+0j)	(-0.99964+0j)	(1.30634e-06+0.99964j)	(1.30634e-06-0.99964j)
30	(0.99221+0j)	(-0.99963+0j)	(1.37120e-06+0.99963j)	(1.37120e-06-0.99963j)

3.1.2 Swap Rates on quarterly Libor frequency - incomplete set of annual points:

Here we consider the general case where not all Swap Rates SR_i for $i = 1, \dots n$ are known. Usually we will have knowledge of rates SR_1, SR_2, ..., SR_n at the points τ_1, τ_2, ..., τ_n which are not consecutive 1 year apart.
One of the first issues that we need to resolve therefore is the issue of filling the missing points. We will need to determine $SR(\tau_j)$ at τ_j points different from any of the given τ_i's above and then proceed to extract the discount factors and the zero rates $R(\tau)$.
Usually the liquid points are 1y, 2y, 3y, 4y, 5y, 7y, 10y, 15y, 20y, 30y. Filling out the missing information is usually done through an interpolation, and we will discuss several kinds of interpolations here.
One simple way to fill the missing data is to interpolate market input to the missing points. As an example, if the swap rates at the points 1y, 2y, ..., 6y and 10y are given, we can interpolate, say linearly, between 6y and 10y to obtain swap rates at the points 7y, 8y, 9y.
While this method is simple, it has a drawback in that it decouples the bootstrap method from the interpolation as has been pointed out previously [8]. The best way to construct the yield curve in fact is to keep the two processes inter-dependent. This methodology has been discussed at length in literature, see for instance Hagan and West among others. To illustrate the idea let us think of the example above. We are tasked with constructing the yield curve from the data available in the market for annual Swaps rates with some Swap rate data not liquid enough in the market to be used; for this example the nodes at 6y to 9y are missing. In this methodology, we express the unknown discount factors at the 6y to 9y nodes in terms of the last known bond B_6 and the last as of yet unknown bond B^*_{10}. Having expressed all the unknown bonds in terms of a single B^*_{10} bond, then through a zero root-finding we find all missing bonds such that we reprice the next available, the 10y Swap instrument. In our example here express $B_7 = f(B^*_{10})$, $B_8 = f(B^*_{10})$, $B_9 = f(B^*_{10})$ and substitute these in Eq. (128), then solve basic Swap Rate equation to find B^*_{10}. This is the only unknown in (128),

$$B^*(t, T_{10}) = \frac{B(t, T_0) - S_{10} \sum_{j=1}^{10} \tau_j B(t, T_j)}{1 + S_{10}\tau_{10}} \quad , \tag{128}$$

$$B^*(t, T_{10}) = \frac{B(t, T_0) - S_{10} \sum_{j=1}^{6} \tau_j B(t, T_j) + S_{10} \sum_{j=7}^{9} \tau_j f_j(B^*(t, T_{10}))}{1 + S_{10}\tau_{10}} \quad . \tag{129}$$

As mentioned in the previous paragraph, some of the discount factors in the sum on the right of Eq. (128) are unknown, but they all are functions of one and the same unknown bond $B^*(t, T_{10})$. The precise form of the function $f_j(B^*(t, T_{10}))$ that will define them, will depend on the interpolation scheme that we will use.
For interpolation schemes, the most widely used in literature are one of the following:

- Piecewise flat-constant forward rates. For T in the interval $[T_{i-1}, T_i]$ for the zero-rates we get

$$R(T)T = R_{i-1}T_{i-1} + \int_{T_{i-1}}^{T} f(u)\mathrm{d}u = R_{i-1}T_{i-1} + f \cdot (T - T_{i-1}) \quad . \tag{130}$$

Substituting the constant instantaneous forward rate

$$f(T) = \frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} \quad , \quad T \in [T_{i-1}, T_i] \quad , \tag{131}$$

for the zero-rates between these two points we get

$$\begin{aligned} R(T)T &= R_{i-1}T_{i-1} + (T - T_{i-1})\frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} \quad , & (132) \\ &= \frac{T - T_{i-1}}{T_i - T_{i-1}} R_i T_i + \frac{T_i - T}{T_i - T_{i-1}} R_{i-1} T_{i-1} \quad , & (133) \end{aligned}$$

thus the "linear R(T) T" naming.

In this case the bonds $B(T_j)$, for integer $j = 6, \ldots, 9$, we have

$$B(T_j) = \exp\{-R(T_j)T_j\} = \exp\left\{-\left[\frac{T_j - T_{i-1}}{T_i - T_{i-1}} R_i T_i + \frac{T_i - T_j}{T_i - T_{i-1}} R_{i-1} T_{i-1}\right]\right\} \quad . \tag{134}$$

The functions $f_j(B^*(T_i))$ that we were looking for, in this interpolation, are

$$B(T_j) = f_j(B_{i-1}, B_i^*) = \Big(B^*(T_i)\Big)^{\frac{T_j - T_{i-1}}{T_i - T_{i-1}}} \Big(B(T_{i-1})\Big)^{\frac{T_i - T_j}{T_i - T_{i-1}}} \quad . \tag{135}$$

All these functions depend on the unknown bond price B_i^*. Equation (128) then can be solved by Newton-Raphson for B_i^* and then all the unknown B_j's.

- Linear in R(T)

$$R(T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} R_i + \frac{T_i - T}{T_i - T_{i-1}} R_{i-1} \quad , \tag{136}$$

resulting in

$$f(T) \quad = \quad \frac{2T - T_{i-1}}{T_i - T_{i-1}} R_i + \frac{T_i - 2T}{T_i - T_{i-1}} R_{i-1} \quad . \tag{137}$$

Notice some important features pointed out in [HaganWest] [8]. In Eq. (136) as $T \to T_i$, $R(T_i^-) \to R_i$ and the rate loses memory of R_{i-1}. This is not the case for $f(T)$. This rate will hold memory of R_{i-1} as $T \to T_i^-$. And it will have information also from R_{i+1} as $T \to T_i^+$. Therefore the linear interpolation in $R(T)$ produces a linear interpolation in $f(T)$ but with jumps at the pillar points T_i. Notice also that $T_i - 2T$ becomes negative as $T \to T_i$. If the rate R_{i-1} is high and R_i is low, the second negative term will overtake the first positive term at some T, leading to negative forward rates. Thus the resulting yield curve has an arbitrage opportunity. Therefore a check needs to be put a place any time the scheme is being used in practice.

In this interpolation scheme the functions $f_j(B^*(T_i))$ that we were looking for, are

$$B(T_j) = f_j(B_{i-1}, B_i^*) = \Big(B^*(T_i)\Big)^{\frac{T_j - T_{i-1}}{T_i - T_{i-1}} \frac{T_j}{T_i}} \Big(B(T_{i-1})\Big)^{\frac{T_i - T_j}{T_i - T_{i-1}} \frac{T_j}{T_{i-1}}} \quad . \tag{138}$$

This is a slight modification of the formula obtained in the previous scheme Eq. (135).

- Linear on log of rates

$$\ln R(T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} \ln R_i + \frac{T_i - T}{T_i - T_{i-1}} \ln R_{i-1} \quad , \tag{139}$$

Electronic copy available at: https://ssrn.com/abstract=3489078

resulting in

$$R(T) = R_i^{\frac{T-T_{i-1}}{T_i-T_{i-1}}} R_{i-1}^{\frac{T_i-T}{T_i-T_{i-1}}} \quad . \tag{140}$$

This method has the drawback that it cannot work at negative rates.

In this interpolation scheme $f_j(B^*(T_i))$ will be given by

$$B(T_j) = f_j(B_{i-1}, B_i^*) = \exp\left\{ -T_j \left[\left(-\frac{1}{T_i} \ln B^*(T_i) \right)^{\frac{T_j-T_{i-1}}{T_i-T_{i-1}}} \left(-\frac{1}{T_{i-1}} \ln B(T_{i-1}) \right)^{\frac{T_i-T_j}{T_i-T_{i-1}}} \right] \right\} \tag{141}$$

- Linear on discount factors

For $T_{i-1} < T < T_i$,

$$B(0,T) = \frac{T-T_{i-1}}{T_i-T_{i-1}} B(0,T_i) + \frac{T_i-T}{T_i-T_{i-1}} B(0,T_{i-1}) \quad . \tag{142}$$

For the bonds at the missing T_j points this of course gives

$$B(T_j) = f_j(B_{i-1}, B_i^*) = \frac{T_j-T_{i-1}}{T_i-T_{i-1}} B^*(T_i) + \frac{T_i-T_j}{T_i-T_{i-1}} B(T_{i-1}) \quad . \tag{143}$$

Solving Eq. (129) for $B^*(T_i)$ we complete the task of the bootstrap and interpolation. For the zero-rates this interpolation scheme results in

$$R(T_j) = -\frac{1}{T_j} \ln\left[\frac{T_j-T_{i-1}}{T_i-T_{i-1}} e^{-R_i T_i} + \frac{T_i-T_j}{T_i-T_{i-1}} e^{-R_{i-1}T_{i-1}} \right] \quad . \tag{144}$$

In the following we describe a Hybrid method for interpolating the yield curve, that makes use of tension spline interpolation.

The cubic spline $f(x)$ interpolating a set of given points (x_i, f_i), $i = 1, \ldots, N$. Such spline is piecewise linear in its second derivative

$$f''(x) = \frac{x_{i+1}-x}{x_{i+1}-x_i} f_i'' + \frac{x-x_i}{x_{i+1}-x_i} f_{i+1}'' \quad , \tag{145}$$

where the second derivative is continuous at knot points, $\lim_{x\to x_i^-} f''(x) = \lim_{x\to x_i^+} f''(x)$. We supply these equations with $f''(x_1) = f''(x_N) = 0$. This type of interpolation is called *natural cubic spline*.

An improvement of the cubic spline is by applying tensile force at the end points, measured by σ. Formally this is accomplished by replacing Eq. (145) by

$$f''(x) - \sigma^2 f(x) = \frac{x_{i+1}-x}{x_{i+1}-x_i} (f_i'' - \sigma^2 f_i) + \frac{x-x_i}{x_{i+1}-x_i} (f_{i+1}'' - \sigma^2 f_{i+1}) \quad . \tag{146}$$

See figure of previous strengths of σ. At strong σ the interpolation approaches a linear Swap Rate interpolation. Small σ limit approaches the natural cubic spline.

Exponential tension spline interpolation [3] allows the degree of non-local behaviour controlled through the use of σ. Non-local effects are often viewed as a shortcoming of the cubic spline method, since perturbing the swap rate, f_i, can significantly effect swap rates at times far from t_i.

For a set of swap rates $\{f_i\}$ at maturities $\{t_i\}$, the interpolated tension spline, f, is defined on the interval $t \in [t_i, t_{i+1}]$ by:

$$f''(t) - \sigma^2 f(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i} (f_i'' - \sigma^2 f_i) + \frac{t-t_i}{t_{i+1}-t_i} (f_{i+1}'' - \sigma^2 f_{i+1}) \quad , \tag{147}$$

where f'' is the second derivative of the spline with respect to time. The natural cubic spline, which is piecewise linear in the second derivative, follows by taking $\sigma \to 0$.
Equation (147), is a second order ordinary differential equation with solution

$$
\begin{aligned}
f(t) \quad &= \quad \left(\frac{\sinh(\sigma(t_{i+1}-t))}{\sinh(\sigma h_i)} - \frac{t_{i+1}-t}{h_i}\right)\frac{f''_i}{\sigma^2} + \left(\frac{\sinh(\sigma(t-t_i))}{\sinh(\sigma h_i)} - \frac{t-t_i}{h_i}\right)\frac{f''_{i+1}}{\sigma^2} \\
&\quad + f_i\frac{t_{i+1}-t}{h_i} + f_{i+1}\frac{t-t_i}{h_i} \quad , \qquad (148)
\end{aligned}
$$

where $h_i = t_{i+1} - t_i$. The second derivatives, f''_i, are completely determined by requiring the first derivative, f_i, to be continuous across the time pegs, resulting in the three-term recurrence relation

$$
a_i f''_{i-1} + b_i f''_i + c_i f''_{i+1} = d_i \quad , \qquad (149)
$$

with

$$
\begin{aligned}
a_i &= \frac{1}{\sigma^2}\left(\frac{1}{h_{i-1}} - \frac{\sigma}{\sinh\sigma h_{i-1}}\right) \quad , & (150) \\
b_i &= \frac{1}{\sigma^2}\left(\frac{\sigma\cosh\sigma h_{i-1}}{\sinh\sigma h_{i-1}} - \frac{1}{h_{i-1}} + \frac{\sigma\cosh\sigma h_i}{\sinh\sigma h_i} - \frac{1}{h_i}\right) \quad , & (151) \\
c_i &= \frac{1}{\sigma^2}\left(\frac{1}{h_i} - \frac{\sigma}{\sinh\sigma h_i}\right) \quad , & (152) \\
d_i &= \frac{f_{i+1}-f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}} \quad . & (153)
\end{aligned}
$$

Recurrence relation (149) can be solved for f''_i by assuming natural boundary conditions $f''_0 = f''_{N-1} = 0$ at the boundary maturities. We solve the recurrence relation using the Thomas algorithm which is essentially an application of Gaussian elimination to the tridiagonal matrix structure obtained by rewriting (149) as the matrix system

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & \cdots & 0 \\
a_2 & b_2 & c_2 & 0 & \cdots & 0 \\
0 & a_3 & b_3 & c_3 & 0 & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & & & c_{N-3} \\
0 & 0 & 0 & \cdots & a_{N-2} & b_{N-2}
\end{bmatrix}
\begin{bmatrix}
f''_1 \\ f''_2 \\ f''_3 \\ . \\ . \\ f''_{N-3} \\ f''_{N-2}
\end{bmatrix}
=
\begin{bmatrix}
d_1 \\ d_2 \\ d_3 \\ . \\ . \\ d_{N-3} \\ d_{N-2}
\end{bmatrix}
\quad . \qquad (154)
$$

By defining the new coefficients

$$
c^*_i = \begin{cases} \frac{c_1}{b_1} \quad , & i = 1 \quad , \\ \frac{c_i}{b_i - c^*_{i-1}a_i} \quad , & i = 2, 3, \cdots k-1 \quad , \end{cases} \qquad (155)
$$

$$
d^*_i = \begin{cases} \frac{d_1}{b_1} \quad , & i = 1 \quad , \\ \frac{d_i - d^*_{i-1}a_i}{b_i - c^*_{i-1}a_i} \quad , & i = 2, 3, \cdots k-1 \quad , \end{cases} \qquad (156)
$$

the matrix system can be rewritten as

$$
\begin{bmatrix}
1 & c_1^* & 0 & 0 & \cdots & 0 \\
0 & 1 & c_2^* & 0 & \cdots & 0 \\
0 & 0 & 1 & c_3^* & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & c_{N-3}^* \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f_1'' \\ f_2'' \\ f_3'' \\ \cdot \\ \cdot \\ f_{N-3}'' \\ f_{N-2}''
\end{bmatrix}
=
\begin{bmatrix}
d_1^* \\ d_2^* \\ d_3^* \\ \cdot \\ \cdot \\ d_{N-3}^* \\ d_{N-2}^*
\end{bmatrix}
\quad , \tag{157}
$$

with solution

$$
f_i'' = d_i^* - c_i^* f_{i+1}'' \quad , \tag{158}
$$

where the f_i'' are solved in reverse beginning from $f_{N-2}'' = d_{N-2}^*$.

In the original version, the tension parameter has to be chosen by the user. Moreover, in the original version the yield curve does not necessarily reprice the market data. When pricing market data is imposed the curve is less smooth (see Figure 3.1.2). σ is increased the yield curve becomes smoother, however the forwards curve becomes less regular.

One approach to choose the correct value of σ would be to choose σ_o such that it gives the "most" regularity. In order to do this, we introduce a functional

$$
\mathcal{G}(\sigma) := \int_0^{T_N} |R'(t)|^2 dt + \int_0^{T_N} |R''(t)|^2 dt + \int_0^{T_N} |(R(t)t)'|^2 dt + \int_0^{T_N} |(R(t)t)''|^2 dt
$$

Given that all the formulas are explicit, by doing the above we obtain that $\sigma_o = 8.9$ (see Figure 12). In some sense this value compromises between the regularity of the zero rates and the regularity of the forwards.

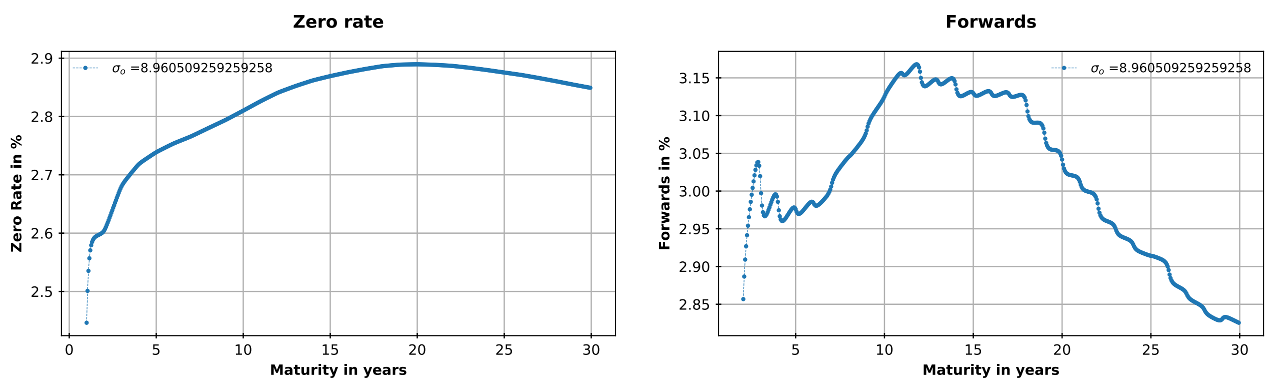

Figure 12: Interpolation via tension spline with σ_o settings.

Given by one of the functions, $R(T)$, above, the method works by guessing a value for the next zero rate R_i which is then used to price a swap with maturity T_i, where yield values $R(T)$ at the coupon payment maturities $T \in [R_{i-1}, R_i]$ are used. The bisection algorithm is used to find the zero rate, R_i, which fairly prices the swap given the fixed swap rate.

4 Bootstrapping through the continuous formula

The purpose of this section is to apply the continuous formula 160, in order to bootstrap the yield curve. By using the same reasoning as in Section 2.2, let us initially notice the following relation between the continuous

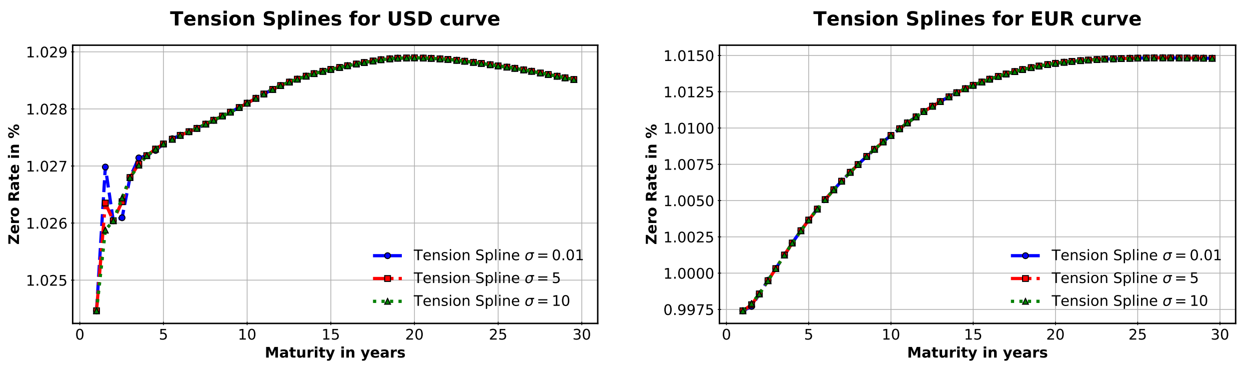

Figure 13: We plot the yield curves of USD and EUR obtained by interpolating via the tension spline technique. Because of the inflection point in the USD yield curve a spike is formed. By increasing the parameter σ the spike can be removed.

and discrete swap rate

$$S^0(t) := F(t)e^{-F(t)/2}/(1 + F(t)^2/24) \quad , \tag{159}$$

where $F(t)$ is the cubic interpolation to the discrete swap rates and m is the average discrete swap rate. After obtaining the continuous swap rate, we can obtain an explicit expression for B^{cont}

$$B^{\text{cont}}(t) = 1 - S^0(t) \int_0^t \exp\left\{ - \int_u^t S^0(w)\mathrm{d}w \right\} \mathrm{d}u \quad . \tag{160}$$

In the bellow graphs we show the difference between the continuous and the original yield curve. Namely, we take $S^1(i)$ the discrete swap rates with maturity in i years and frequency 1y for i in $\{1, 2, \ldots, 30\}$. Afterwards we take a subset of the swap rates which correspond to the maturities at $T := \{1, 2, 3, 4, 5, 7, 10, 12, 15, 20, 25, 30\}$. Afterwards we use a cubic interpolation to calculate F.

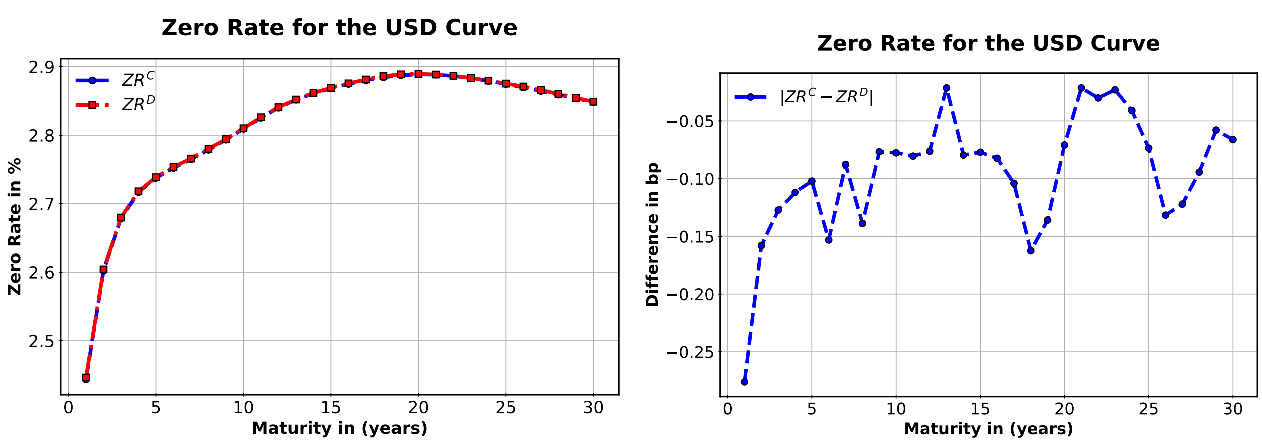

Figure 14: We plot the zero rate obtained by the procedure described in Section 4 for the USD Curve.

5 A variational non-parametric approach to bootstrapping

Differently from the parametric approach to bootstrapping, the variational approach works by postulating that the yield curve is the minima of a functional (to be defined). In particular, this approach allows us to start from

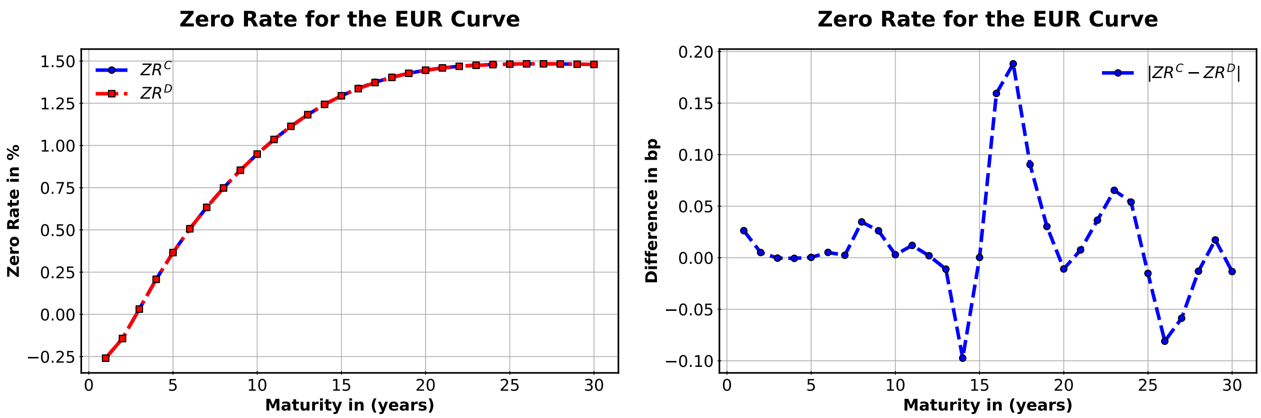

Figure 15: We plot the zero rate obtained by the procedure described in Section 4 for the EUR Curve.

the properties that are desirable for the curve and find a functional which penalizes the curve not satisfying those properties.
We will see that the most common parametric functions considered in the previous section can be reformulated in terms of the variational approach.
In order to apply the variational approach, one issue is to find a functional for which curve should be a minimizer of. One possible recipe to find such a functional is the following:

1. List the properties P_i that the rate curve should satisfy.

2. For each of the desired properties find a functional $\mathcal{F}_i$ such that it penalizes the curve not fulfilling the property P_i.

3. Consider the functional
$$\mathcal{F}(R) = \sum_i M_i \mathcal{F}_i(R) \quad ,$$
where R is the yield curve. The constants M_i can be thought as the relative strengths of the different terms. The choice of the constant should reflect the value we give to the property P_i. The higher the constant the more important we believe that property to be.

In order to make the above recipe more transparent, let us say we are interested the yield curve to satisfy the following properties:

(a) the curve can reprice the market data;

(b) the curve is smooth;

(c) the curve has positive forwards;

(d) the curve is concave.

Let us now list some functionals which penalize the above properties.

(a) In order to impose that the curve is smooth some good options are

$$\mathcal{F}_1(R) = \sum_{i=1}^{N} a_i \int_0^T \left| \frac{d^i}{ds^i} R(s) \right|^2 ds \qquad \text{for } N \geq 1 \text{ and } a_i \geq 0 \quad .$$

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(b) In order to impose that the curve has positive forwards one possible choice is

$$\mathcal{F}_2(R) = \int_0^T \chi((R(s)s)')ds = \int_0^T \chi(R(s)'s + R(s))ds \quad .$$

where

$$\chi(s) = \begin{cases} 1 & \text{if } s < 0 \quad , \\ 1 & \text{otherwise} \quad . \end{cases}$$

In practice the function χ is not good for minimization purposes due to its discontinuity. For this reason we prefer to use instead of χ the logistic function which is defined by $\sigma_n(s) = 1/\exp(nx)$. The logistic function is smooth and approximates χ for n sufficiently large.

(c) In order to impose that the curve is concave one possible choice is

$$\mathcal{F}_2(R) = \int_0^T \chi(R''(s))ds \quad .$$

For the same reasons we will use instead of χ the logistic function.

The optimal/desired curve is chosen the be the minima of the functional $\mathcal{F}$. The functional $\mathcal{F}$ can be thought as a way of quantifying "how much the yield curve in unsatisfactory". Thus among all possible yield curve we choose the one which is the least unsatisfactory.

5.1 Obtaining the some of the parametric bootstrapping methods via variational methods

Let us now make the connection to some of the parametric interpolation methods discussed above. In particular, we will see that most of the parametric models discussed can be viewed as continuous variational models.

5.1.1 Piecewise-flat

Consider the variational model defined on the spot forward rate:

$$\mathcal{F}(f) := \int_0^{T_N} |f(s)|^2 ds \tag{161}$$

subject to the condition of repricing the original bonds. This can be expressed as a linear system of equations

$$1 - B_i - SR_i(\tau_1 B_1 + \cdots + \tau_i B_i) = 0 \tag{162}$$

for every T_i.
Let us now show that minimizing this functional under the constrain corresponds to piecewise flat case. In order to do this we will use the Lagrange multiplier technique:

$$\int_0^{T_N} |f|^2 ds + \sum_{i=1}^{k} \lambda_{T_i} \left(1 - e^{-\int_0^{T_i} f(s)ds} - SR_i \sum_{j=0}^{T_i} e^{-\int_0^j f(s)ds} \right) \tag{163}$$

where in the above formula we used that

$$B_i = \exp\left\{ - \int_0^i f(s)ds \right\} \quad .$$

For simplicity denote $u(t) = \int_0^t f(s)ds$. Then (163) becomes,

$$\int_0^{T_N} |u'|^2 ds + \sum_{i=1}^{k} \left(1 - e^{-u(T_i)} - SR_i \sum_{j=0}^{T_i} e^{-u(j)} \right) \quad . \tag{164}$$

By doing variations in $t \mapsto u + \varepsilon h$ where h is a smooth function, we obtain that

$$\frac{d}{d\varepsilon} \left[\int_0^{T_N} |u' + \varepsilon h'|^2 ds + \sum_{i=1}^{k} \lambda_{T_i} \left(1 - e^{-u(T_i) + \varepsilon h(T_i)} - SR_{T_i} \sum_{j=0}^{T_i} e^{-u(j) + \varepsilon h(j)} \right) \right] = 0 \quad .$$

Thus

$$\left[\int_0^{T_N} u'(s) h'(s) ds + \sum_{i=1}^{k} \lambda_{T_i} \left(-h(T_i) e^{-u(T_i)} - SR_{T_i} \sum_{j=0}^{T_i} h(j) e^{-u(j)} \right) \right] = 0 \quad . \tag{165}$$

The above equation (up to considering the contributions from the boundary) one has that

$$\int_0^{T_N} u''(s) h(s) ds + \sum_{i=1}^{f} \lambda_{T_i} \int_0^{T_N} h(s) d\mu_i(s) = 0 \quad , \tag{166}$$

where

$$\mu_i = -e^{-u(s)} \delta_{T_i} - SR_{T_i} \sum_{j=0}^{T_i} e^{-u(j)} \delta_j$$

and δ_x is the Dirac delta measure at x.
It is not difficult to see that u can be integrated and that u is piecewise linear. This in particular implies that f is piecewise constant.

5.1.2 "Linear on Log rates"

A similar reasoning can be used for the "linear on the log rates". Indeed, define $R_l(t) = \log(R(t))$ and consider the functional

$$\mathcal{F}(f) := \int_0^{T_N} |R_l'|^2 ds \quad . \tag{167}$$

subject to the condition of repricing the original bonds. This can be expressed as a linear system of equations

$$1 - B_{T_i} - SR_{T_i}(\tau_1 B_1 + \cdots + \tau_{T_i} B_{T_i}) = 0 \quad , \tag{168}$$

for every T_i and τ_i is the daycount fraction between two consecutive payments. In order to simplify the notation we will assume that $\tau_i = 1$.
We will now show that R_l is linear.
We use again the Lagrange multiplier technique, thus the problem can be written as

$$\int_0^{T_N} |R_l'(s)|^2 ds + \sum_{i=1}^{N} \lambda_{T_i} \left(1 - e^{-e^{R_l(T_i)} T_i} - SR_{T_i} \sum_{j=0}^{T_i} e^{-e^{R_l(j)} j} \right) \quad , \tag{169}$$

where in the above formula we used that

$$B_i = \exp\left\{ -e^{R_l(i)} i \right\} \quad .$$

Then (169) becomes,

$$\int_0^{T_N} |u'|^2 ds + \sum_{i=1}^{N} \left(1 - e^{e^{R_l(T_i)}T_i} - SR_{T_i} \sum_{j=0}^{T_i} e^{-e^{R_l(j)}j} \right) \quad . \tag{170}$$

By doing variations in $t \mapsto u + \varepsilon h$ where h is a smooth function, we obtain that

$$\frac{d}{d\varepsilon} \left[\int_0^{T_N} \left| R_l' + \varepsilon h' \right|^2 ds + \sum_{i=1}^{N} \lambda_{T_i} \left(1 - e^{-e^{R_l(T_i)+\varepsilon h(T_i)}T_i} - SR_{T_i} \sum_{j=0}^{T_i} e^{R_l(j)+\varepsilon h(j)} j \right) \right] = 0 \quad .$$

Thus

$$\left[\int_0^{T_N} R_l'(s) h'(s) ds + \sum_{i=1}^{N} \lambda_{T_i} \left(-h(T_i) e^{-eR_l(T_i)T_i} e^{-R_l} - SR_{T_i} \sum_{j=0}^{T_i} h(j) e^{-e^{R_l(j)}j} \right) \right] = 0 \quad . \tag{171}$$

The above equation (up to considering the contributions from the boundary) one has that

$$\int_0^{T_N} R_l''(s) h(s) ds + \sum_{i=1}^{N} \lambda_{T_i} \int_0^{T_N} h(s) d\mu_i(s) = 0 \quad , \tag{172}$$

where

$$\mu_i = -e^{-e^{R_l(s)}s} \delta_{T_i} - SR_{T_i} \sum_{j=0}^{T_i} e^{-e^{R_l(s)}s} \delta_j \quad .$$

and δ_x is the Dirac delta measure at x. Thus as before one has that R_l is piecewise linear.

5.2 The variational discrete case

The full continuous functionals described in the beginning of Section 5 are not trivial to minimize explicitly. For this reason from now on we will work with the discrete version of it.

For simplicity, let us assume that we are given $\{S_1, S_2, S_3, S_4, S_5, S_{10}\}$ which correspond to $\{1, 2, 3, 4, 5, 10\}$ and suppose we would like to bootstrap the rates for $\{S_6, S_7, S_8, S_9\}$. Given $\{S_1, \cdots, S_5\}$, let us initially create the function $G : (S_6, \cdots, S_9) \to (B_1, \ldots, B_{10})$. This can be done through explicitly via (20). For any swap rates $S_6, \cdots, S_9$ the functional G constructs its implied annual bond prices. Notice that only $B_6, \ldots, B_{10}$ depend on $(S_6, \cdots, S_9)$. The missing data $(S_6, \ldots, S_9)$ will be defined such that they minimize $F(G(\cdot))$.

We now define the functional $\tilde{\mathcal{F}}$ as

$$\begin{aligned} \tilde{\mathcal{F}}(B_1, \ldots, B_{10}) = & \sum_{i=0}^{n} H(B_{i+1} - B_i) + M \sum_i (R_i - R_{i+1})^2 + M^2 \sum_i (R_i + R_{i+2} - 2R_{i+1})^2 \\ & + M \sum_{i=0}^{n} H(R_i + R_{i+2} - 2R_{i+1}) \quad , \end{aligned} \tag{173}$$

where $R_i = -\log(B_i)/i$ and H is the Heaviside function which is defined by

$$H(t) := \begin{cases} 1 & \text{if } t > 0 \quad , \\ 0 & \text{otherwise} \quad . \end{cases} \tag{174}$$

The term $R_{i+2} + R_i - 2R_i$ can be seen as the discrete second derivative. The above functional is not well-defined for numerical purposes, this is due to the fact that the Heaviside function has a jump in 0. For this reason we

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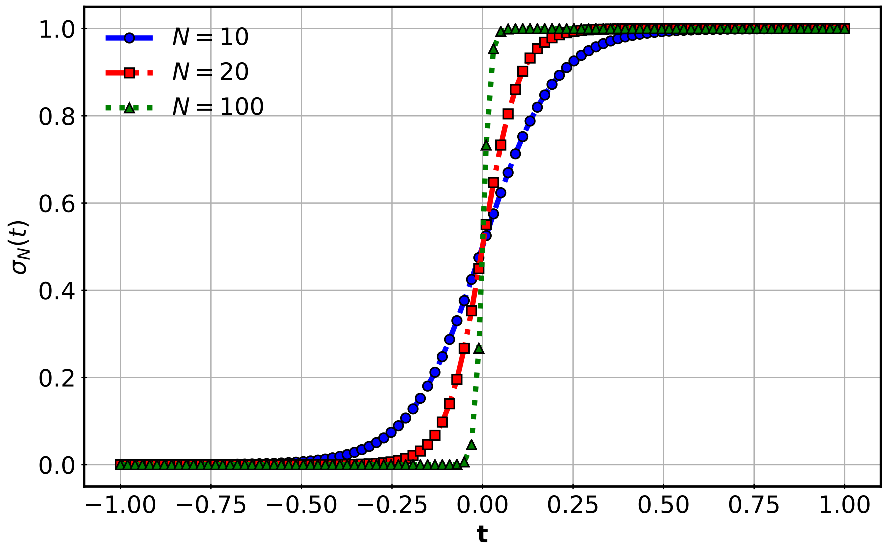

Figure 16: A plot of the logistic function with $N = 1000$

will substitute the Heaviside function with the logistic function which is defined by $\sigma_N(t) = 1/(1 + \exp(-Nt))$. Notice that for N sufficiently large σ_N approximates the Heaviside function.
The more convenient functional to be minimized is

$$\begin{aligned}\tilde{\mathcal{F}}(B_1, \ldots, B_{10}) = &\sum_{i=0}^{n} \sigma_N(B_{i+1} - B_i) + M \sum_i (R_i - R_{i+1})^2 + M^2 \sum_i (R_i + R_{i+2} - 2R_{i+1})^2 \\ &+ M \sum_{i=0}^{n} \sigma_N(R_i + R_{i+2} - 2R_{i+1}) \quad .\end{aligned} \tag{175}$$

The first term in the right hand side of the above is penalizing the forwards being negative. Indeed, whenever the forwards are negative, namely $B_{i+1} - B_i > 0$, one has that $\sigma_N(B_{i+1} - B_i) \sim 1$ and whenever $B_{i+1} - B_i < 0$ one has that $\sigma_N(B_{i+1} - B_i) \sim 0$. Because σ_N is only an approximation of H, one has if the other terms would not exist the functional would try to have forwards as high as possible. This holds true because $\sigma_N(B_{i+1} - B_i)$ is the smallest when $B_{i+1} - B_i$ is the highest. Because of the constrain to fit one has that the forwards are irregular. As we increase the constant M one sees that graphs are more and more regular.
In all the simulations in this section $M = 10^4$ and $N = 10^3$.
Given the swap rates at dates $\{1, 2, 3, 4, 5, 10, 12, 15, 20, 25, 30\}$, we can initially bootstrap the data at dates $\{6, 7, 8, 9\}$. Afterwards by using the market data and the previously bootstrapped data, we can iteratively bootstrap $\{11\}$, $\{13, 14\}$, $\{16, 17, 18, 19\}$, $\{21, 22, 23, 24\}$ and $\{26, 27, 28, 29\}$.
We now make the reconstruction test described in the beginning of Section 5. The test is done in the following way: from commonly available market data we obtain swap rates for $\{1, 2, \ldots, 30\}$; afterwards we reconstruct the curve by using only the swap rates with annual frequency at maturity $\{1, 2, 3, 4, 5, 10, 12, 15, 20, 25, 30\}$; than we compare our results with the original data. Given that in this way we obtain the zero coupon bonds at certain maturities, a further interpolation of the data is needed. Given that the data points is are frequent in our experiments the precise type of interpolation is not very relevant. We prefer a Hagan-West type of interpolation. The results are plotted in Figure 22 and Figure 23.
In Figures 17, 19, 20, 21 we show the interpolation with various methods and compare it to the variational method.

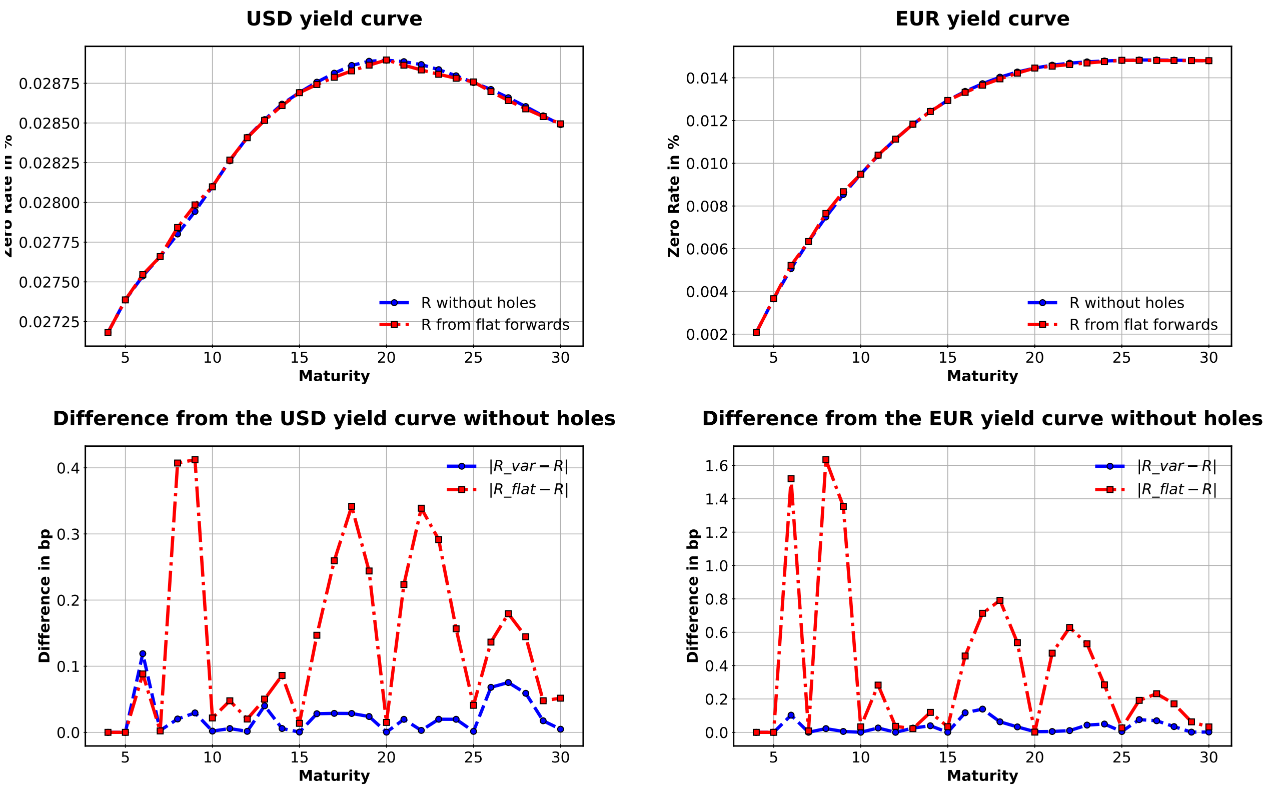

Figure 17: A plot of the piecewise constant on forward rates for USD and EUR curve

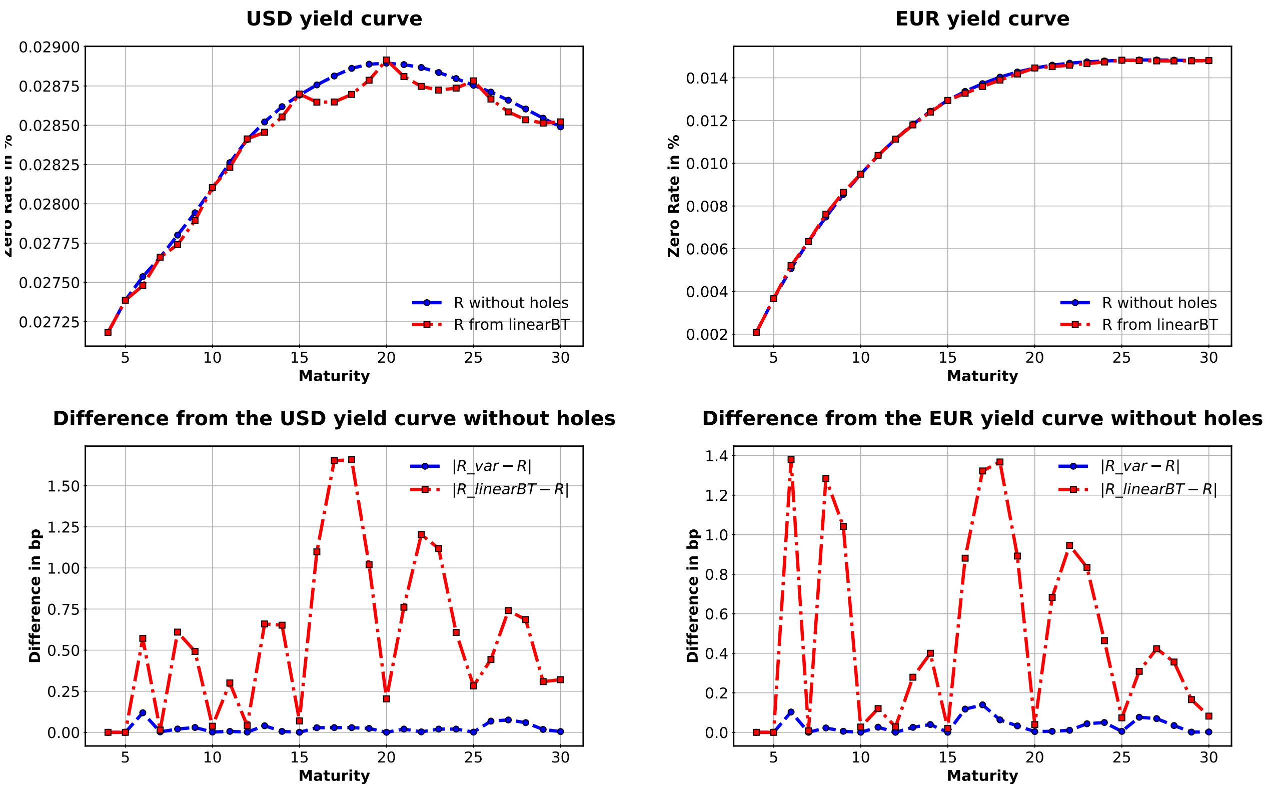

Figure 18: A plot of the piecewise linear on discount factors for USD and EUR curve

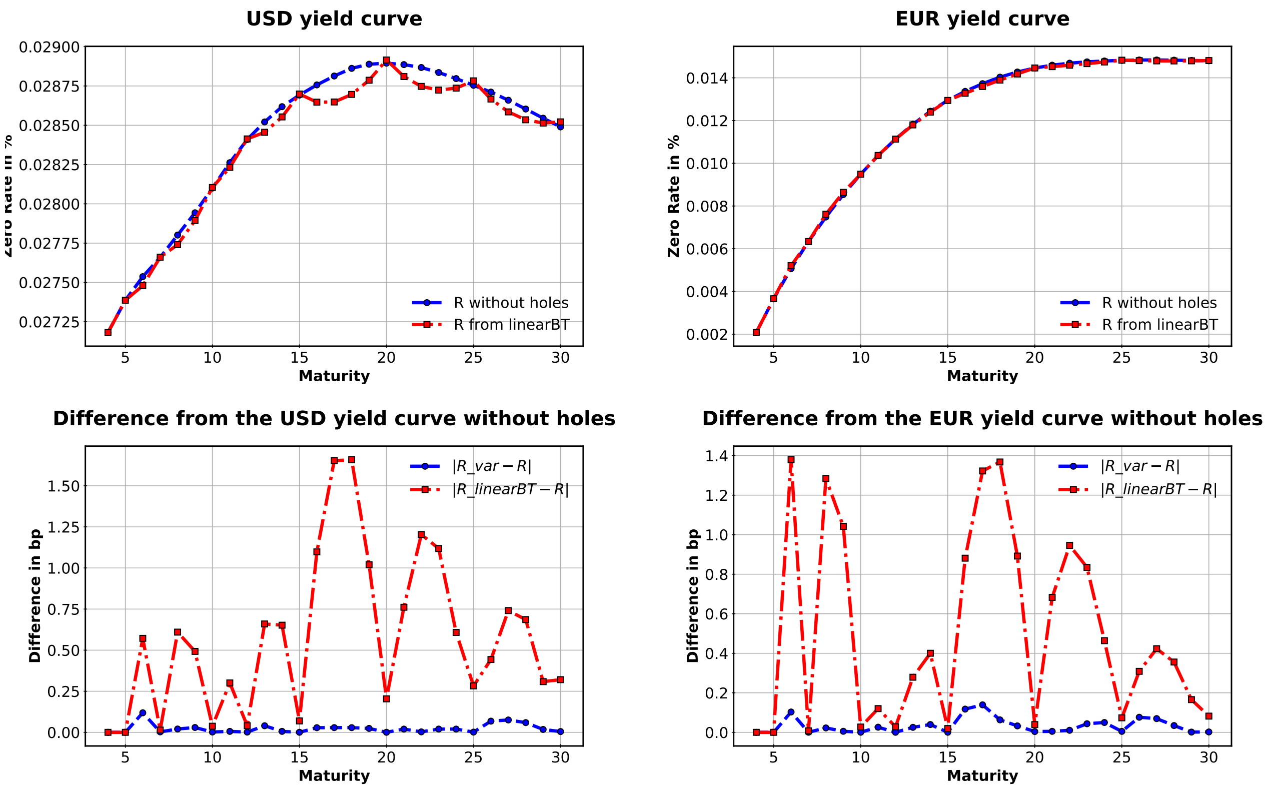

Figure 19: A plot of the piecewise linear on discount factors for USD and EUR curve

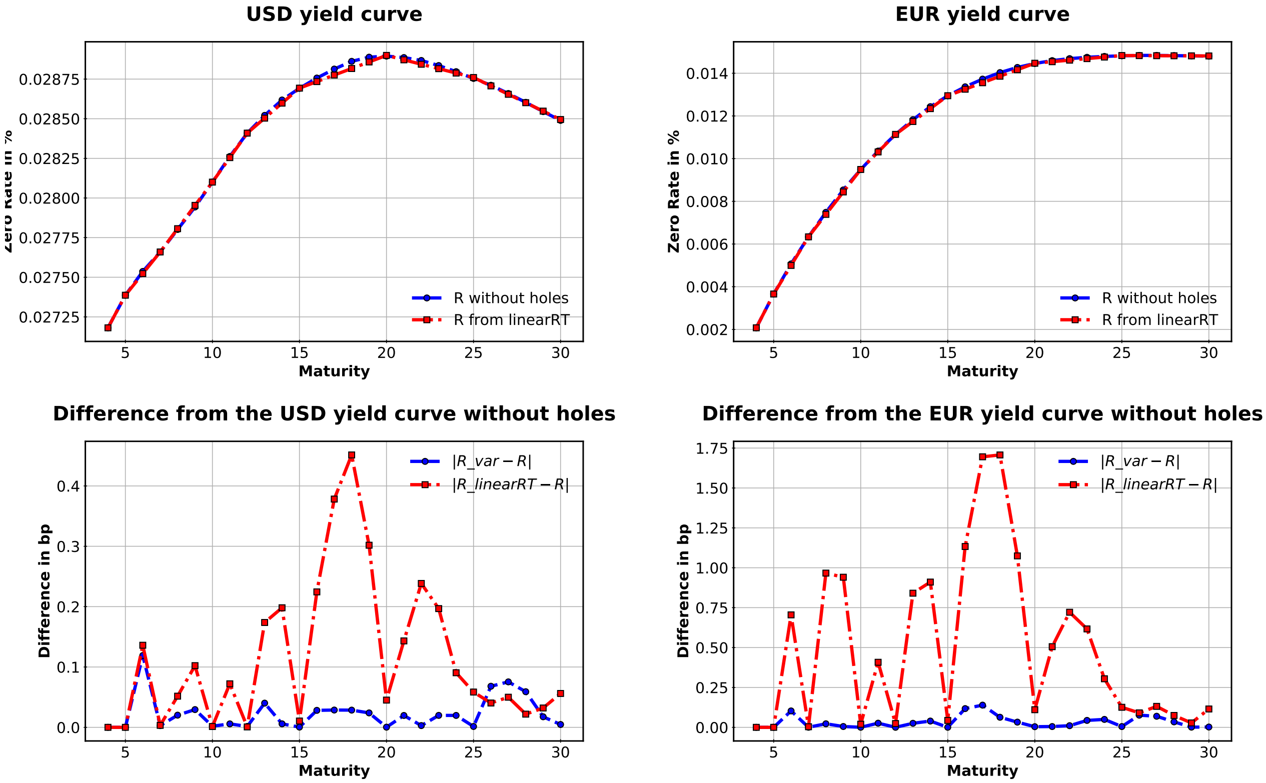

Figure 20: A plot of the piecewise linear in R for USD and EUR curve

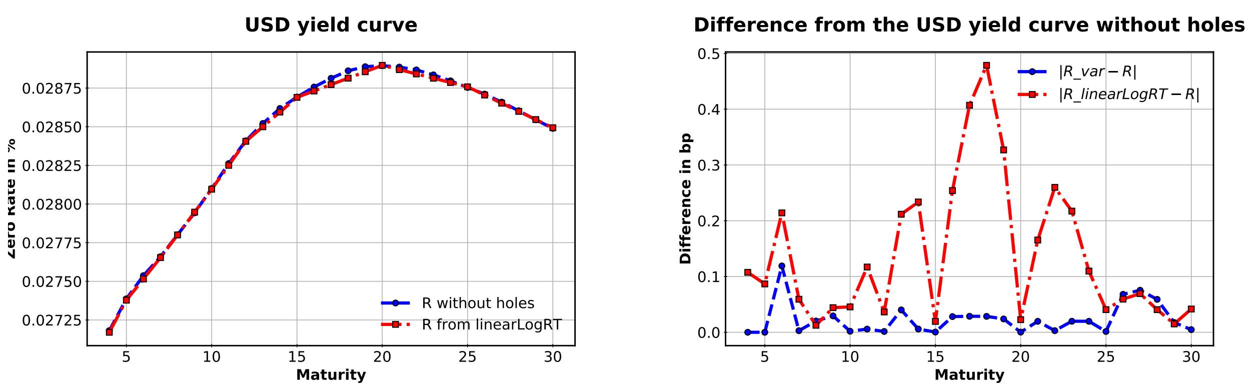

Figure 21: A plot of the piecewise linear in $\log(R)$ for USD curve. This method can not be applied for the EUR curve as the initial zero rate is negative.

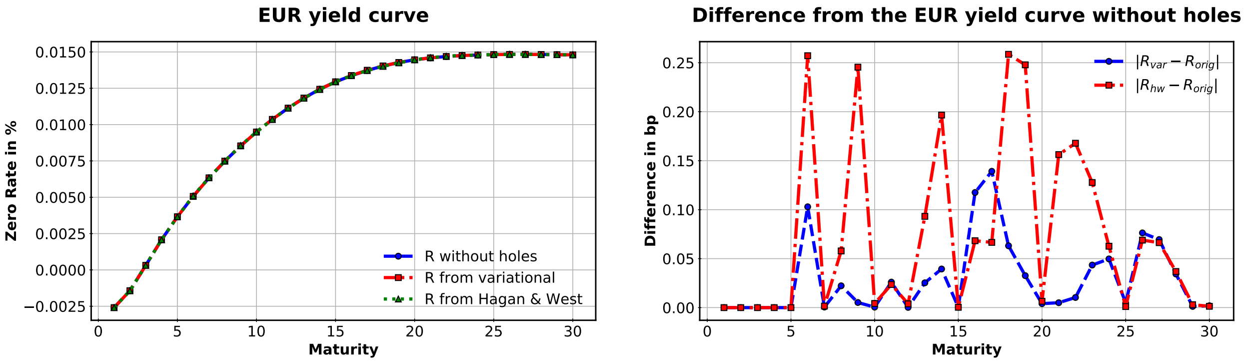

Figure 22: A comparison between the Variational interpolation and Hagan West for the EUR curve

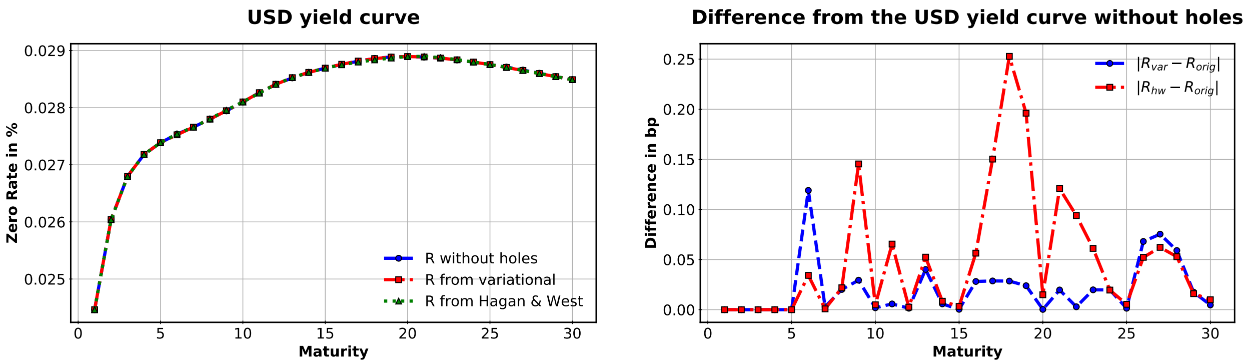

Figure 23: A comparison between the Variational interpolation and Hagan West for the USD curve

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