ECE 6320 Power System Control and Operation

Module 2: Economic Optimization Applications

Lecture 11: LP-OPF

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In This Lecture

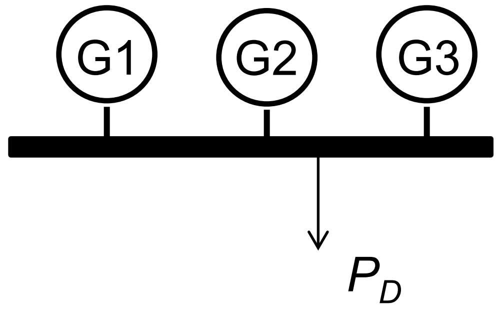
- Linear Programming (LP)
- LP-OPF
  - Theta Formulation
  - PTDF Formulation
- Sequential LP-OPF

Linear Programming

- **Example 1 a**
  - Consider a single-bus power system with 3 generating units.
  - All units have a capacity of 400MW, and cost functions:

$$C_1(P_{G1}) = 10P_{G1} \quad [\$/hr]$$
$$C_2(P_{G2}) = 18P_{G2} \quad [\$/hr]$$
$$C_3(P_{G3}) = 20P_{G3} \quad [\$/hr]$$

  - The system demand for a given hour is $P_D = 900$ MW.



Linear Programming

- We want to minimize the total operating cost for the given hour:

$$\min \quad C_T = 10P_{G1} + 18P_{G2} + 20P_{G3}$$

$$\text{Subject to:} \quad P_{G1} + P_{G2} + P_{G3} = P_D$$

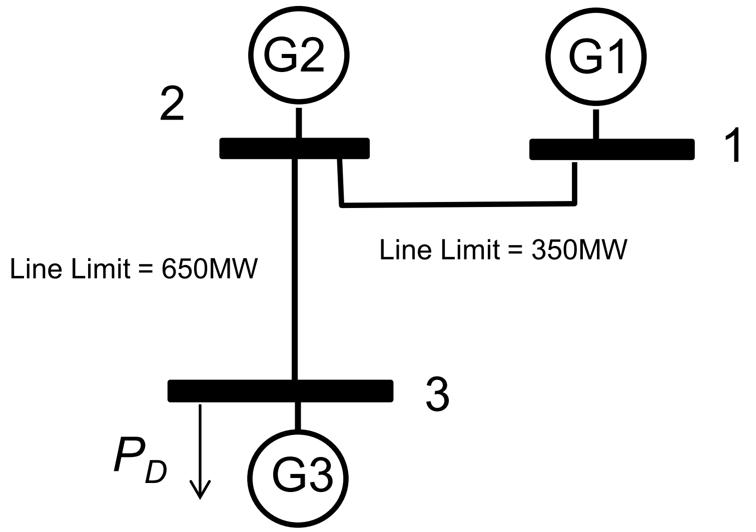
$$0 \le P_{Gi} \le P_{Gi}^{\max}$$

Gen	Capacity (MW)	Output (MW)
G1	400	400
G2	400	400
G3	400	100

- The generator incremental costs are constant.
- The obvious solution is to use the cheapest generator first (400 MW from G1), then the second cheapest generator (400MW from G2) and then 100MW from G3.
- The generation is "stacked".

Linear Programming

- **Example 1 b**
  - The same generators are now connected to a system with the following topology:
  - Adding transmission line constraints:



$$\min \quad C_T = 10P_{G1} + 18P_{G2} + 20P_{G3}$$

$$\text{Subject to:} \quad P_{G1} + P_{G2} + P_{G3} = P_D$$

$$P_{G1} \leq 350$$

$$P_{G1} + P_{G2} \leq 650$$

$$0 \leq P_{Gi} \leq P_{Gi}^{\max}$$

Gen	Capacity (MW)	Output (MW)
G1	400	350
G2	400	300
G3	400	250

Linear Programming

- For more complicated systems it becomes nontrivial to determine the optimal solutions. We need a systematic approach to solve these problems.
- The cases in the previous examples have a linear objective function and linear constraints. Hence we can use ***Linear Programming***.
  - One of the most widely-used optimization methods.
  - Used in many industries to solve large-scale optimization problems.

Linear Programming

- All LP problems have a linear objective function with linear constraints, and the requirement that all variables x be ≥ 0.
- LP Standard Form:

Decision Variables

$$\text{Minimize:} \quad \underbrace{\mathbf{c}}_{n\times 1}{}^{T} \underbrace{\mathbf{x}}_{n\times 1}$$

$$\text{Subject to:} \quad \underbrace{\mathbf{A}}_{m\times n} \underbrace{\mathbf{x}}_{n\times 1} = \underbrace{\mathbf{b}}_{m\times 1} \quad ; \qquad \mathbf{x} \geq 0$$

- For LP problems, $n \geq m$. Typically, $n >> m$.
- A vector $\mathbf{x}$ is said to be *feasible* if $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
- If $n = m$, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ (provided $\mathbf{A}$ is nonsingular) is the only solution; no minimization is involved.

Linear Programming

Converting Problems to LP Standard Form

1. Maximization: The standard form considers a minimization problem. If we encounter a maximization problem, simply change the sign of the objective function.

$$\text{Maximize:} \quad \underbrace{\mathbf{c}}_{n\times 1}{}^{T} \underbrace{\mathbf{x}}_{n\times 1} \quad = \quad \text{Minimize} \quad -\underbrace{\mathbf{c}}_{n\times 1}{}^{T} \underbrace{\mathbf{x}}_{n\times 1}$$

Linear Programming

Converting Problems to LP Standard Form

2. Inequality Constraints: Slack Variables

- The standard form does not include inequality constraints.
- One can convert inequality to equality constraints by introducing *slack variables*.

Example A:

The inequality constraint $x_i \le b_j, \;\; x_i \ge 0$ can be written as:

$x_i + x_s = b_j, \quad x_i,\, x_s \ge 0;$ where x_s is called the slack variable.

Example B:

The inequality constraint $50 \le x_i \le 100, \;\; x_i \ge 0$ can be written as:

$$x_i + x_{s1} = 100$$

$$x_i - x_{s2} = 50, \quad x_i,\, x_{s1}, x_{s2} \ge 0$$

Linear Programming

Converting Problems to LP Standard Form

3. Variable bounds: any variable with lower and upper bound constraints, $lb_i \le x_i \le ub_i$ can be converted to $0 \le x'_i \le x'^{\max}$ by means of a change of variable (subtract lb_i from each term):

$$lb_i - lb_i \le x_i - lb_i \le ub_i - lb_i$$

Linear Programming

LP formulation with bounds

- Since it is possible to convert problems to LP standard form, LP solvers allow direct modeling inequalities and bounds of variables.

$$\text{Minimize:} \quad \mathbf{c}^T \mathbf{x}$$

$$\text{Subject to:} \quad \mathbf{A}_{in}\mathbf{x} \le \mathbf{b}_{in}$$

$$\mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}$$

$$\mathbf{lb} \le \mathbf{x} \le \mathbf{ub}$$

Linear Programming Solutions

- Powerful optimization solvers have been developed to solve linear programming problems.
  - Gurobi
  - CEPLEX
  - PuLP
  - GLPK (*GNU* *L*inear *P*rogramming *K*it)
  - LP_Solve
- For our examples, we can use a LP solver that operates within Matlab: linprog.
- Can use python with PuLP, academic Gurobi, etc.

Linear Programming Solutions

- `x = linprog(f,A,b)`
  - Solves min $\mathbf{f}^T\mathbf{x}$ such that $\mathbf{Ax} \le \mathbf{b}$
- `x = linprog(f,A,b,Aeq,beq)`
  - Includes equality constraints $\mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq}$.
  - Set $\mathbf{A}$ = [] and $\mathbf{b}$ = [] if no inequalities exist.
- `x = linprog(f,A,b,Aeq,beq,lb,ub)`
  - Defines a set of lower and upper bounds on the decision variables, $\mathbf{x}$, so that the solution is always in the range $\mathbf{lb} \le \mathbf{x} \le \mathbf{ub}$.
  - Set $\mathbf{A}_{eq}$ = [] and $\mathbf{b}_{eq}$ = [] if no equalities exist.

Linear Programming Solutions

```
Example 1a
clear all
f = [10;18;20];         % Cost Vector
A= [];                  % LHS inequalities
b = [];                 % RHS inequalities
Aeq = [1 1 1];          % LHS equalities
beq = [900];            % RHS equalities
lb = [0 0 0]';          % Lower bounds
ub = [400 400 400]';    % Upper bounds
x0 = []                 % No Initial guess
x= linprog(f,A,b,Aeq,beq,lb,ub,x0)
Output
x =
   400.0000
   400.0000
   100.0000
```

$$\min \quad C_T = 10P_{G1} + 18P_{G2} + 20P_{G3}$$

$$\text{Subject to:} \quad P_{G1} + P_{G2} + P_{G3} = P_D$$

$$0 \le P_{Gi} \le P_{Gi}^{\max}$$

Linear Programming Solutions

$$\min \quad C_T = 10P_{G1} + 18P_{G2} + 20P_{G3}$$

$$\text{Subject to:} \quad P_{G1} + P_{G2} + P_{G3} = P_D$$

$$P_{G1} \le 350$$

$$P_{G1} + P_{G2} \le 650$$

$$0 \le P_{Gi} \le P_{Gi}^{\max}$$

Example 1b

```
clear all
f = [10;18;20];         % Cost Vector
A= [1 0 0;
    1 1 0];             % LHS inequalities
b = [350; 650];         % RHS inequalities
Aeq = [1 1 1];          % LHS equalities
beq = [900];            % RHS equalities
lb = [0 0 0]';          % Lower bounds
ub = [400 400 400]';    % Upper bounds
x0 = []                 % No Initial guess
x= linprog(f,A,b,Aeq,beq,lb,ub,x0)
```

Output

```
x =
  350.0000
  300.0000
  250.0000
```

Linear Programming OPF

Theta Formulation

- In the previous example (1b), we know how the flows are related to the injections of power.
- In general systems, we don't. We would have to calculate PTDFs.
- Instead we use what is called the ***Theta Formulation***, in which the bus voltage angles are explicitly included.
- We write equations of power balance at each bus
  - Recall that the active power flow in a line can be approximated as:

$$P_{jk} = \frac{1}{x_{jk}}(\theta_j - \theta_k)$$

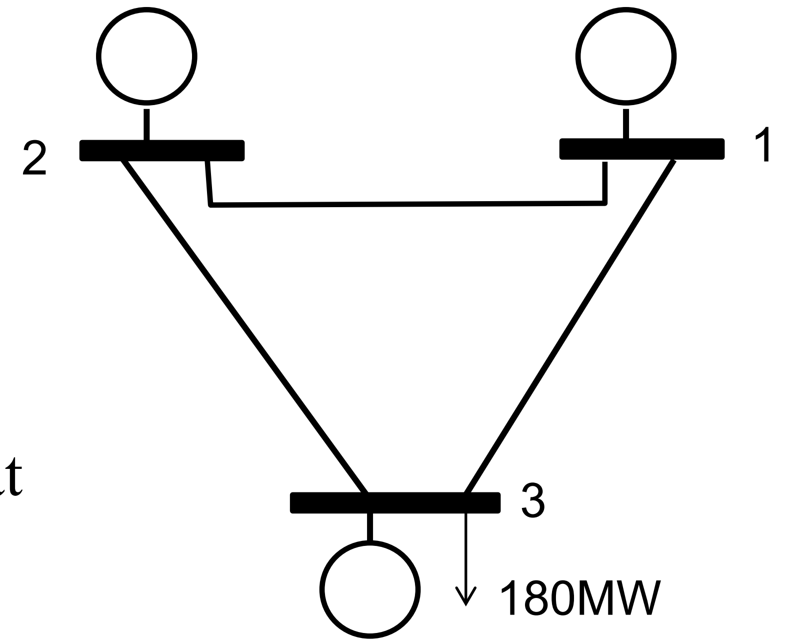
  - If x and the power are in p.u., then the angles are in radians.

Linear Programming OPF

Example 2

- Consider the following power system where all lines have reactance $x = 0.1$ p.u. and a limit of 100 MW $\quad x_{jk} = 0.1; \quad \frac{1}{x_{jk}} = 10$

Gen	Energy Cost	Capacity (P^{max})
	\$/MWh	MW
1	10	400
2	12	400
3	20	400



- The power balance equation at bus 1 is:

$$
\begin{aligned}
P_{G1} &= P_{12} + P_{13} \\
&= \tfrac{1}{x_{12}}(\theta_1 - \theta_2) + \tfrac{1}{x_{13}}(\theta_1 - \theta_3) \\
&= -10\theta_2 - 10\theta_3
\end{aligned}
$$

- We can write similar power balance equations at buses 2 and 3.

Linear Programming OPF

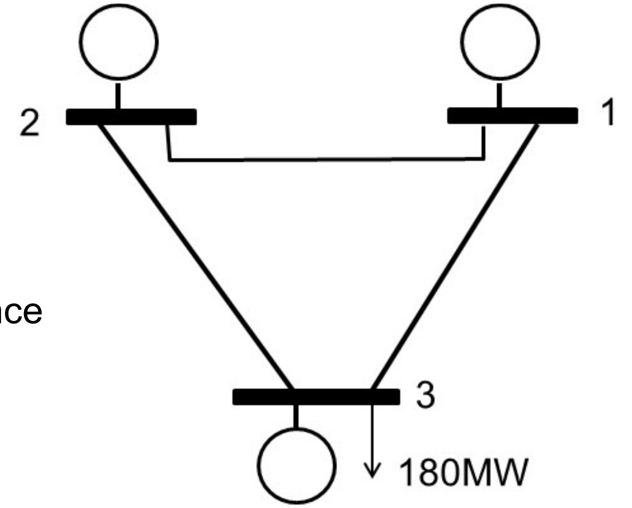
- LP OPF using theta formulation.

$$
\begin{aligned}
\min \quad & c_1 P_{G1} + c_2 P_{G2} + c_3 P_{G3} \\
s.t. \quad & -10\theta_2 - 10\theta_3 = P_{G1} \\
& 20\theta_2 - 10\theta_3 = P_{G2} \\
& -10\theta_2 + 20\theta_3 = P_{G3} - P_D \\
& -10\theta_2 \le P_{12}^{\max} \\
& -10\theta_3 \le P_{13}^{\max} \\
& 10(\theta_2 - \theta_3) \le P_{23}^{\max} \\
& 0 \le P_{Gi} \le P_i^{\max}, \quad \forall i
\end{aligned}
$$

Bus Power Balance Constraints

Line Limit Constraints

Generator Constraints



Linear Programming OPF

- LP OPF using theta formulation.

$$
\begin{aligned}
\min \quad & c_1 P_{G1} + c_2 P_{G2} + c_3 P_{G3} \\
s.t. \quad & -10\theta_2 - 10\theta_3 = P_{G1} \\
& 20\theta_2 - 10\theta_3 = P_{G2} \\
& -10\theta_2 + 20\theta_3 = P_{G3} - P_D \\
& -10\theta_2 \le P_{12}^{\max} \\
& -10\theta_3 \le P_{13}^{\max} \\
& 10(\theta_2 - \theta_3) \le P_{23}^{\max} \\
& 0 \le P_{Gi} \le P_i^{\max}, \quad \forall i
\end{aligned}
$$

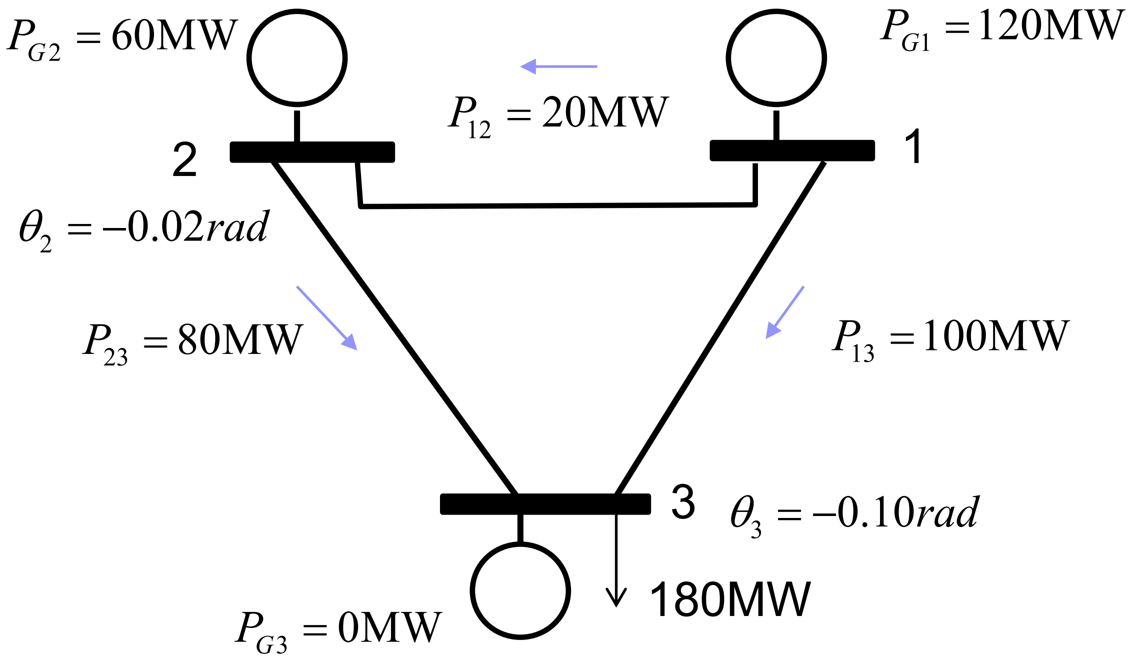
```
clear all
% PE1 PE2 PE3  t2  t3
f = [10 12 20  0   0]';       % Cost Vector
Aeq= [1 0  0  10  10;
      0 1  0 -20  10;
      0 0  1  10 -20];        % LHS Equalities
beq = [0  0  1.8];            % RHS Equalities
A = [ 0  0  0  -10  0;
      0  0  0   0 -10;
      0  0  0  10 -10];       % LHS inequaliries
b = [1 1 1];                  % RHS inequalities
lb = [0;0;0; -99;-99];        % Lower bounds
ub = [4;4;4;  99; 99];       % Upper bounds
x0 = [];                      % Initial guess
[x,fval,exitflag,output,lambda] =
linprog(f,A,b,Aeq,beq,lb,ub,x0)
```

Linear Programming OPF

```
fval =
    19.2000
```

Op. Cost = $1,920/hr

```
x =
     1.2000
     0.6000
     0.0000
    -0.0200
    -0.1000
```



Linear Programming OPF

PTDF Formulation

- In the Theta formulation, we need to include as many angle variables as buses in the system, and we need to include all the lines as constraints. This can slow down the solution of the OPF.
- We know that only a few lines in the system would be fully loaded (become binding constraints). An alternative is to incorporate only the constraints that become fully loaded. This can be achieved by using PTDFs.

LP Optimal Power Flow

- Using a vector of constant generator incremental costs **c**, the objective function becomes:

$$\min \quad \mathbf{c}^T \mathbf{P}_G$$

- s.t.

$$\sum_{j \in A_i} P_{ij} = P_{Gi} - P_{Di}, \qquad \forall \text{ bus } i \qquad \text{Bus power balance}$$

$$P_{jk} = \sum_{i=1}^{N} PTDF_{jk} \times (P_{Gi} - P_{Di}) \qquad \forall \text{ line } jk \qquad \text{Change in flows}$$

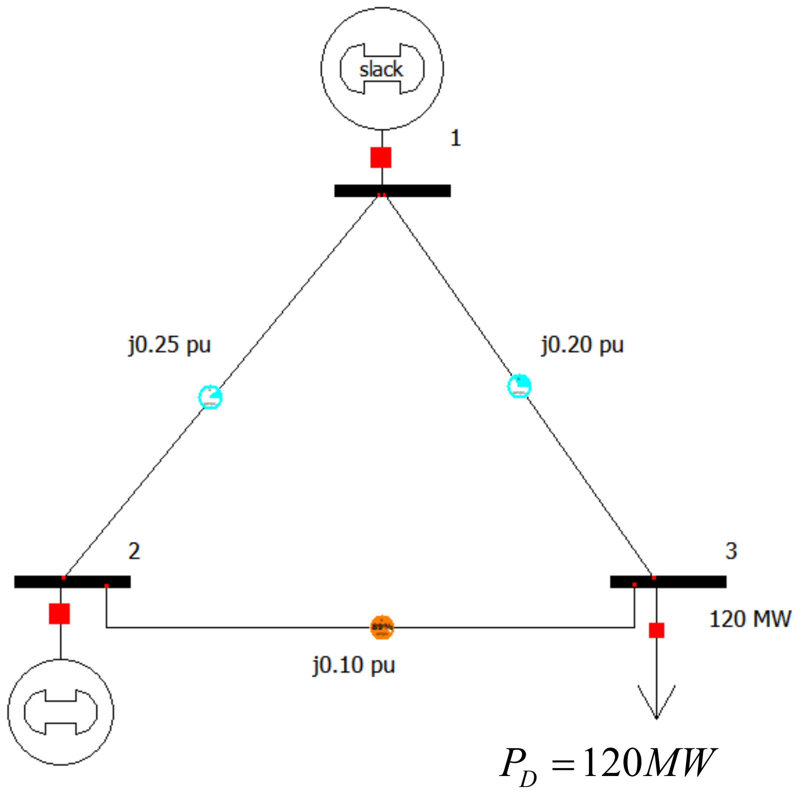
$$P_{Gi}^{\min} \le P_{Gi} \le P_{Gi}^{\max} \qquad \forall \text{ geneator } i \qquad \text{Generator limits}$$

$$-P_{jk}^{\max} \le P_{jk} \le P_{jk}^{\max} \qquad \forall \text{ line } jk \qquad \text{Line thermal limits}$$

Line flow as function of
PTDFs and net bus injections

Example 3

- Same power system as in the PTDF example in lecture 6, slide 11.
- Incremental costs of generators 1 and 2 are 10$/MWh and 12$/MWh, respectively, and they both have a capacity of 100MW.
- Assume all line limits are 75MW.



$$\mathbf{Y}_{bus} = j\begin{bmatrix} -9 & 4 & 5 \\ 4 & -14 & 10 \\ 5 & 10 & -15 \end{bmatrix}$$

Example 3

- Assume slack is bus 1. For transfers 2-1 and 3-1, the PTDFs are:

$$PTDF_{12,2\to1} = -0.5454 \qquad PTDF_{12,3\to1} = -0.3636$$
$$PTDF_{13,2\to1} = -0.4545 \qquad PTDF_{13,3\to1} = -0.6364$$
$$PTDF_{23,2\to1} = 0.4545 \qquad PTDF_{23,3\to1} = -0.3636$$

- Generator 1 is cheaper. Thus line 1-3 can be overloaded. We write the line 1-3 flow as a function of PTDFs and net injections:

$$P_{13} = PTDF_{13,2\to1} \times P_{G2} + PTDF_{13,3\to1} \times (-P_D),$$
$$P_{13} - PTDF_{13,2\to1} \times P_{G2} = -PTDF_{13,3\to1} \times P_D; \quad P_D = 120\text{MW}:$$
$$P_{13} - (-0.4545)P_{G2} = -(-0.6364) \times 120$$
$$P_{13} + 0.4545P_{G2} = 76.3636$$

Example 3

- The problem formulation is therefore:

$$\min \quad 10P_{G1} + 12P_{G2}$$

- Bus power balance:

$$P_{G1} = P_{12} + P_{13}$$
$$P_{G2} = P_{21} + P_{23} = -P_{12} + P_{23}$$
$$-P_{D3} = P_{31} + P_{32} \quad \text{or} \quad P_{13} + P_{23} = P_{D3}$$

- Change in line flow:

$$P_{13} + 0.4545P_{G2} = 76.3636$$

- Line limits:

$$-75 \le P_{jk} \le 75$$

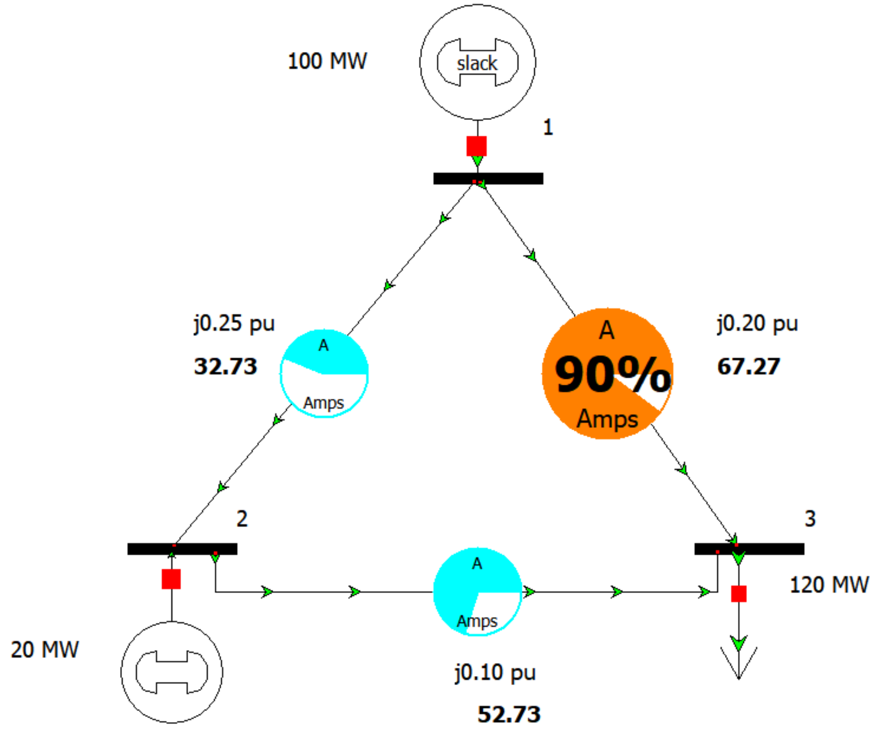
- Generator limits

$$0 \le P_{Gi} \le 100$$

Example 3

```
clear all
%   PG1  PG2  P12  P13  P23
f = [10   12   0    0    0 ]; % Cost Vector
A= [];                        % No LHS inequalities
b = [];                       % No RHS inequalities
Aeq = [1   0 -1 -1 0;
       0   1   1   0 -1;
       0   0   0   1   1;
       0 0.4545 0 1 0];  % LHS Equalities
beq = [0 0 120 76.3636];     % RHS Equalities
lb = [0 0 -75 -75 -75]';  % Lower bounds
ub = [100 100 75 75 75]'; % Upper bounds
x0 = []'                      % Initial guess
x= linprog(f,A,b,Aeq,beq,lb,ub,x0)

x= 100.0000
    20.0000
    32.7264
    67.2736
    52.7264
```



Example 3

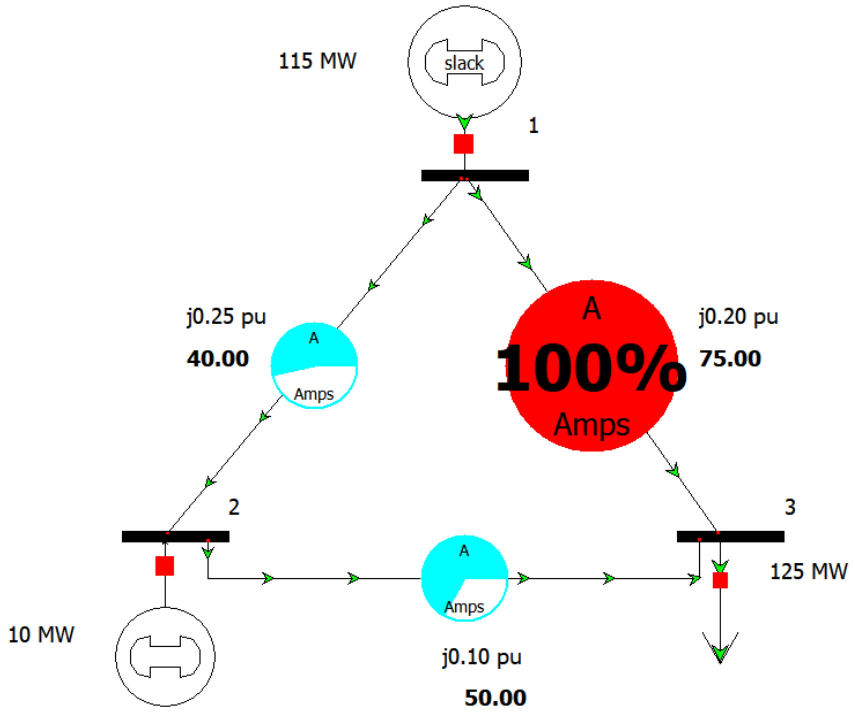
- Now increase the capacity of generator 1 to 120 MW and the demand to 125 MW.
- The line flow equation becomes:

$$P_D = 125\text{MW}:$$

$$P_{13} + 0.4545 P_{G2} = -(-0.6364)\times 125 = 79.5454$$

```
beq = [0 0 125 79.5454];      % RHS Equalities
ub = [120 100 75 75 75]';     % Upper bounds

x=
  114.9991
   10.0009
   39.9991
   75.0000
   50.0000
```



Sequential LP-OPF

- Minimize operating cost, taking into account realistic equality and inequality constraints.
- Equality constraints:
  - Bus real and reactive power balance
  - Generator voltage setpoints
  - Area MW interchange
  - Transmission line/transformer/interface flow limits

Sequential LP-OPF

- Inequality constraints
  - Transmission line/transformer/interface flow limits
  - Generator MW limits
  - Generator reactive power capability curves
  - Bus voltage magnitudes
- Available Controls
  - Generator MW outputs
  - Load MW demands
  - Phase shifters
  - Area Transactions

Sequential LP-OPF

- Solution iterates between
  - Solving a full ac power flow single solution
    - Enforces real/reactive power balance at each bus
    - Enforces generator reactive limits
    - System controls are assumed fixed
    - Takes into account non-linearities
  - Solving the LP problem.
    - Changes system controls to enforce linearized constraints while minimizing cost (or control change)