Seminar 4

Let $(K, *, \circ)$ be a field, then we define the operation between a scalar from K and a vector from V such that $\forall k \in K$ and $\forall v \in V$ we have $k \cup v \in V$. Then V is a **vector space** if $(V, \perp)$ is an Abelian group (where $\perp$ is the operation between vectors) and:

a $k \cup (v_1 \perp v_2) = k \cup v_1 \perp k \cup v_2$

b $(k_1 * k_2) \cup v = k_1 \cup v \perp k_2 \cup v$

c $(k_1 \circ k_2) \cup v = k_1 \cup (k_2 \cup v)$

d $e \cup v = v$, where e is the identity element.

But for a vector space, please remember that we talk about the operations on vectors and multiplication with scalars.

S is a **subspace** of V if: S is a stable subset and S is a vector space with respect to the same operation. We may also say that S is a subspace of V if $\forall k_1, k_2 \in K$ and $\forall x, y \in S \Rightarrow S \neq \emptyset$ and $k_1 x + k_2 y \in S$.

1. a $k(f_1 + f_2) = k[a_{10} + a_{20} + (a_{11} + a_{21})X + \cdots + (a_{1n} + a_{2n})X^n] = ka_{10} + ka_{11}X + \cdots + ka_{1n}X^n + ka_{20} + \cdots + ka_{2n}X^n = kf_1 + kf_2$

   b $(k_1 + k_2)f = (k_1 + k_2)a_0 + (k_1 + k_2)a_1 X + \cdots + (k_1 + k_2)a_n X^n = k_1 a_0 + k_1 a_1 X + \cdots + k_1 a_n X^n + k_2 a_0 + \cdots + k_2 a_n X^n = k_1 f + k_2 f$

   c $(k_1 k_2)f = k_1 k_2 a_0 + k_1 k_2 a_1 X + \cdots + k_1 k_2 a_n X^n = k_1(k_2 a_0 + k_2 a_1 X + \cdots + k_2 a_n X^n) = k_1(k_2 f)$

   d $1 \cdot f = 1 \cdot (a_0 + a_1 X + \cdots + a_n X^n) = a_0 + a_1 X + \cdots + a_n X^n = f$

2. a $\alpha(A + B) = \alpha A + \alpha B$

   b $(\alpha + \beta)A = (\alpha + \beta) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \alpha A + \beta A$

   c $(\alpha\beta)A = \begin{bmatrix} \alpha\beta a_{11} & \dots & \alpha\beta a_{1n} \\ \dots & \dots & \dots \\ \alpha\beta a_{m1} & \dots & \alpha\beta a_{mn} \end{bmatrix} = \alpha \begin{bmatrix} \beta a_{11} & \dots & \beta a_{1n} \\ \dots & \dots & \dots \\ \beta a_{m1} & \dots & \beta mn \end{bmatrix} = \alpha(\beta A)$

   d $1 \cdot A = 1 \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & mn \end{bmatrix} = A$

3. a $\forall x \in A : [k(f+g)](x) = k(f(x)+g(x)) = kf(x)+kg(x)$

   b $\forall x \in A : [(k_1+k_2)f](x) = (k_1+k_2)f(x) = k_1 f(x) + k_2 f(x)$

   c $[(k_1 \cdot k_2)f](x) = (k_1 \cdot k_2)f(x) = k_1 k_2 f(x) = [k_1(k_2 f)](x)$

   d $(1 \cdot f)(x) = 1 \cdot f(x) = f(x)$

   So, $\forall A \neq \emptyset$ and $\forall K$-field $\Rightarrow K^A$ is a vector space. Now, take $A = \mathbb{R} \neq \emptyset$ and $\mathbb{R}$ is a field $\Rightarrow \mathbb{R}^{\mathbb{R}}$ is a vector space.

4. a $kT(x \perp y) = kT(xy) = (xy)^k = x^k y^k = (kTx)(kTy) = (kTx) \perp (kTy)$

   b $(k_1+k_2)Tx = x^{k_1+k_2} = x^{k_1} \cdot x^{k_2} = (k_1 Tx) \perp (k_2 Tx)$

   c $(k_1 \cdot k_2)Tx = x^{k_1 \cdot k_2} = (x^{k_2})^{k_1} = k_1 T(k_2 Tx)$

   d $1Tx = x^1 = x$

   $\Rightarrow$ is a vector space.

5. For this exercise we will use that S is a subspace of V if $S \neq \emptyset$ and $\forall k_1, k_2 \in K$ and $\forall x, y \in S$, we have $k_1 x + k_2 y \in S$.

   (i) $\forall \alpha, \beta \in \mathbb{R}, \forall (0, y_1, z_1), (0, y_2, z_2) \in A \Rightarrow \alpha(0, y_1, z_1) + \beta(0, y_2, z_2) = (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in A \Rightarrow A$ is a subspace of $\mathbb{R}^3$.

   (ii) $\forall \alpha, \beta \in \mathbb{R}, \forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in B$.
   If $x_1 = x_2 = 0 \Rightarrow B$ is a subspace of $\mathbb{R}^3$.
   If $z_1 = z_2 = 0 \Rightarrow B$ is a subspace of $\mathbb{R}^3$.
   If $x_1 = z_2 = 0 \Rightarrow \alpha(0, y_1, z_1) + \beta(x_2, y_2, 0) = (\beta x_2, \alpha y_1 + \beta y_2, \alpha z_1) \notin B \Rightarrow B$ is NOT a subspace of $\mathbb{R}^3$.

   (iii) $\forall \alpha, \beta \in \mathbb{R}, \forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in C \Rightarrow \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \notin C$, as $x_1, x_2 \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$, but $\alpha x_1, \beta x_2 \in \mathbb{R}$, when they should be in $\mathbb{Z} \Rightarrow C$ is NOT a subspace of $\mathbb{R}^3$. If $\alpha, \beta \in \mathbb{Z} \Rightarrow C$ is a subspace.

   (iv) $\forall \alpha, \beta \in \mathbb{R}, \forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in D \Rightarrow \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in D$, as $x_1 + y_1 + z_1 = 0 \Rightarrow \alpha x_1 + \alpha y_2 + \alpha z_1 = 0$ (the same for β) and so $\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2 = 0$.

   (v) $\forall \alpha, \beta \in \mathbb{R}, \forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in E \Rightarrow \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \notin E$, as $x_1 + y_1 + z_1 = 1 \Rightarrow \alpha x_1 + \alpha y_1 + \alpha z_1 = \alpha$ (the same goes for β), so $\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2 = \alpha + \beta \neq 1 \Rightarrow E$ is NOT a subspace.

(vi) $\forall \alpha, \beta \in \mathbb{R}, \forall (x_1, x_1, x_1), (x_2, x_2, x_2) \in F \Rightarrow \alpha(x_1, x_1, x_1) + \beta(x_2, x_2, x_2) = (\alpha x_1 + \beta x_2, \alpha x_1 + \beta x_2, \alpha x_1 + \beta x_2) \in F \Rightarrow F$ is a subspace.

6. (i) $\forall x, y \in [-1, 1] \Rightarrow -1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Now, multiply those with $\alpha, \beta \in \mathbb{R} \Rightarrow -\alpha \leq \alpha x \leq \alpha$ and $-\beta \leq \beta y \leq \beta \Rightarrow -\alpha - \beta \leq \alpha x + \beta y \leq \alpha + \beta \Rightarrow [-\alpha - \beta, \alpha + \beta] \neq [-1, 1]$. So $[-1, 1]$ is NOT a subpace of $\mathbb{R}$.

   (ii) Take $(1, 0), (0, 1)$ in our set. Then $(1, 0) + (0, 1) = (1, 1)$ is not in our set, as $1^2 + 1^2 = 2 \leq 1$ is not true. $\Rightarrow A$ is NOT a subspace of $\mathbb{R}^2$.

   (iii) $\alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \beta \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} \alpha a + \beta d & \alpha b + \beta e \\ 0 & \alpha c + \beta f \end{bmatrix}$. If $a, d \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{Q} \Rightarrow \alpha a + \beta d \in \mathbb{Q}$ (analogues for the others) $\Rightarrow B$ is a subspace of $M_2(\mathbb{Q})$. If $a, d \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha a + \beta d \in \mathbb{R}$ (Analogues for the others) $\Rightarrow B$ is NOT a subspace of $M_2(\mathbb{R})$.

   (iv) $\alpha f_1 + \beta f_2$ is continuous, as f_1, f_2 continuous $\Rightarrow$ our set is a subspace of $\mathbb{R}^{\mathbb{R}}$.

7. (i) Take $k, l \leq n$ and $f = a_0 + a_1 X + \cdots + a_k X^k, g = b_0 + b_1 X + \cdots + b_l X^l$. Suppose $k < l$ and take $\alpha, \beta \in K$. Then $\alpha f + \beta g = (\alpha a_0 + \beta b_0) + \ldots (\alpha a_k + \beta b_k) X^k + \cdots + \beta b_l X^l \Rightarrow degree(\alpha f + \beta g) = l \leq n$, so our set is a subspace.

   (ii) Take $\alpha, \beta \in K$ and $f = a_0 + a_1 X + \ldots a_n X^n, g = b_0 + b_1 X + \cdots + b_n X^n \Rightarrow \alpha f + \beta g = (\alpha a_0 + \beta b_0) + \cdots + (\alpha a_n + \beta b_n) X^n \Rightarrow degree(\alpha f + \beta g) \leq n \Rightarrow$ our set is NOT a subspace.

8. $S = \{(x_1, x_2) \mid (x_1, x_2)$ - system solutions $\} \Rightarrow S \neq \emptyset$, as $(0, 0)$ is a solution for our system. Now, take $(x_1, x_2), (y_1, y_2) \in S$ and $\alpha, \beta \in K \Rightarrow \alpha(x_1, x_2) + \beta(y_1, y_2) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$ should be a solution $\iff a_{11}(\alpha x_1 + \beta y_1) + a_{12}(\alpha x_2 + \beta y_2) = \alpha(a_{11} x_1 + a_{12} x_2) + \beta(a_{11} y_1 + a_{22} y_2) = 0$, as $a_{11} x_1 + a_{12} x_2 = 0$ ((x_1, x_2) is a solution). And the same goes for the other equation $\Rightarrow S$ is a subspace for $\mathbb{R}^2$.