Seminar W8 - 911

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Linear systems

Kronecker–Capelli thm:

S is a system, M its matrix

S compatible $\Leftrightarrow \operatorname{rank}(M) = \operatorname{rank}(\bar{M})$

$\Updownarrow$

Rouché's thm:

S system, M its matrix

Δ_{max} is a principal minor

S compatible $\Leftrightarrow$ $\forall$ characteristic determinant is zero

$\hookrightarrow$ ch. det.:

$$\begin{vmatrix} \Delta_{max} & \\ \text{line from } M & \end{vmatrix}$$

column of free terms

To find the solutions

- Decide which are the principal equations (and keep them) and throw out the secondary equations
- Decide which are the principal variables and you regard the secondary ones as parameters
- Solve the resulting system

Gaussian elimination

6.5 (i)
$$\begin{cases} 2x + 2y + 3z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right) \sim$$

$L_2 \leftarrow L_2 - 2L_1$, $L_3 \leftarrow L_3 + L_1$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right) \sim$$

$L_3 \leftarrow L_3 - \frac{1}{4} L_2$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 0 & \frac{1}{4} & \frac{11}{4} \end{array}\right)$$

$L_3 \leftarrow 4L_3$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

Gaussian elimination:

You stop here and switch to the system

- Solve the resulting system with Cramer's rule

8.3 (iii) Use Rouché's theorem to check if the system is compatible and if so, solve it

$$\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$$

$\Rightarrow \text{rank } M = 2 \Rightarrow \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ is a principal minor

We write the characteristic minors

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} =$$

$$= 0$$

You stop here and switch to the system

$$\begin{cases} x-y=1 \\ 4y+3z=1 \\ z=11 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z=11 \\ y=-8 \\ x=-7 \end{cases}$$

Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 0 & 1 & 11 \end{array}\right) \sim$$

$$\underset{\sim}{L_2 \leftarrow L_2 - 3L_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 0 & -32 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\underset{\sim}{L_2 \leftarrow \frac{1}{4}L_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right) \sim$$

$$\underset{\sim}{L_1 \leftarrow L_1 + L_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\Rightarrow x = -7$$
$$y = -8$$
$$z = 11$$

8.5 (ii) $\begin{cases} 2x+5y+z=7 \\ x+2y-z=3 \end{cases}$

$$- \ 1 \ 1 \ 2 \ 3 \ 1$$

$$= 0$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} + 4 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= \overset{1}{\cancel{3}} + 3 - 4 - 4 = -\overset{4}{\cancel{2}} \neq 0$$

Rouché's thm $\Rightarrow$ S is incompatible

8.2(c) Use the K-C theorem on:

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$$

Sol.:

$$\overline{M} = \begin{pmatrix} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \underbrace{\begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix}}_{-5} + (-1)^{1+2} \cdot 1 \cdot \overbrace{\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix}}^{6} + (-1)^{1+3} \cdot 1 \cdot \underbrace{\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix}}_{=-8} = -19 \neq 0$$

$\Rightarrow \operatorname{rank} \overline{M} = 3 = \operatorname{rank} M$

Kronecker-Capelli $\Rightarrow$ the system is compatible

Principal unknowns: x_1, x_2, x_3

Secondary unknowns: $\underline{x_4}$

$$\begin{cases} x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_2 + L_3}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x + 2y - z = 3 \\ y + 3z = 1 \end{cases}$$

$$\Rightarrow \begin{cases} y = 1 - 3z \\ x + 2 - 6z - z = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 7z = 1 \\ y = 1 - 3z \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 1 - 3z \\ x = 1 + 7z \end{cases} \Leftrightarrow \begin{cases} x = 1 + 7z \\ y = 1 - 3z \\ z = z \end{cases}$$

Secondary unknowns: $\underset{\alpha}{\overset{x_4}{\|}}$

$$\begin{cases} x_1+2x_2+x_3 = 5+2\alpha \\ 2x_1+x_2-2x_3 = 1-\alpha \\ 2x_1-3x_2+x_3 = 3-2\alpha \end{cases}$$

$$\Delta = -19$$

$$x_1 = \frac{\Delta x_1}{\Delta}, \quad x_2 = \frac{\Delta x_2}{\Delta}, \quad x_3 = \frac{\Delta x_3}{\Delta}$$

$$\Delta x_1 = \begin{vmatrix} 5+2\alpha & 2 & 1 \\ 1-\alpha & 1 & -2 \\ 3-2\alpha & -3 & 1 \end{vmatrix} =$$

$$= -38$$

$$\Delta x_2 = \begin{vmatrix} 1 & 5+2\alpha & 1 \\ 2 & 1-\alpha & -2 \\ 2 & 3-2\alpha & 1 \end{vmatrix}$$

$$\Delta x_3 = \begin{vmatrix} 1 & 1 & 5+2\alpha \\ 2 & 1 & 1-\alpha \\ 2 & -3 & 3-2\alpha \end{vmatrix}$$

$$\iff \begin{cases} x = 1+7z \end{cases} \iff \begin{cases} z = z \end{cases}$$

$$\boxed{S = (1,1,0) + \langle (7,-3,1) \rangle}$$

8.7. $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}$

$(a \in \mathbb{R})$

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - aL_1}{\sim}} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a^2 & 1-a & 1-a^2 \\ 0 & 1-a & a-1 & a^2-a \end{array}\right) \sim$$

$$\overset{L_3 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \\ 0 & 1-a^2 & 1-a & 1-a^2 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 - (1+a) \cdot L_2}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \\ 0 & 0 & a^2-a & 0 \end{array}\right)$$

$1-a^2-(1+a)\cdot(a^2-a)=$

$=-(a^2-1)(a+1)$

$$\sim\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 0 & a(a-1) & -(a-1)(a+1)^2 \end{array}\right)$$

(i) $a=1$

$$\Rightarrow\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x+y+z=1$

$\Rightarrow z=1-x-y$

(ii) $a=0$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\Rightarrow$ System is incompatible

(iii) $a\neq 0,1$

$$\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 0 & a(a-1) & -(a-1)(a+1)^2 \end{array}\right)$$

$$\begin{cases} x+ay+z=a \\ (1-a)y+(a-1)z=a(a-1) \\ a(a-1)\cdot z=-(a-1)(a+1)^2 \end{cases}$$

$$\Rightarrow z=-\frac{(a+1)^2}{a}$$

$y = \cdots$

$4 = \cdots$

$\Rightarrow$ compatible

In applying Steinitz's thm. to complete a basis, you still you need to check if what you have is lin. indep.