Seminar 6

We say that $v_1, v_2, \dots v_n$ are **linearly independent** if

$$a_1v_1 + a_2v_2 + \dots a_nv_n = 0 \iff a_1 = a_2 = \dots = a_n = 0$$

or $\begin{vmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{vmatrix} \neq 0.$

We say that B is a **base** if the vectors in B are linearly independent and the vectors in B generate the whole space.

1. (i) $a_1v_1 + a_2v_2 + a_3v_3 = 0 \iff \begin{cases} a_1 + 2a_2 + a_3 = 0 \\ a_1 + a_2 + 5a_3 = 0 \\ a_2 + 2a_3 = 0 \end{cases}$

   From the last equation we have $a_2 = -2a_3$ so our system becomes $\begin{cases} a_1 - 4a_3 + a_3 = 0 \\ -a_1 - 2a_3 + 5a_3 = 0 \end{cases} \Rightarrow a_1 - 3a_3 = 0 \Rightarrow a_1 = 3a_3 \Rightarrow S = \{(3a, -2a, a) \mid a \in \mathbb{R}\} \Rightarrow v_1, v_2, v_3$ are linearly dependent.

   (ii) $a_1v_1 + a_2v_2 = 0 \iff \begin{cases} a_1 + 2a_2 = 0 \\ -a_1 + a_2 = 0 \\ a_2 = 0 \end{cases} \Rightarrow a_1 = 0 \Rightarrow v_1, v_2$ are linearly independent.

2. (i) $a_1v_1 + a_2v_2 + a_3v_3 = 0 \iff \begin{cases} a_1 - a_2 + 3a_3 = 0 \\ 2a_2 + a_3 = 0 \\ 2a_1 + a_2 + a_3 = 0 \end{cases} \Rightarrow$ By simple computations we get that $a_1 = a_2 = a_3 = 0 \Rightarrow v_1, v_2, v_3$ are linearly independent.

   (ii) $\begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = -192 \neq 0 \Rightarrow v_1, v_2, v_3, v_4$ are linearly independent.

3. $\begin{vmatrix} 1 & a & 0 \\ a & 1 & 1 \\ 1 & 0 & a \end{vmatrix} = a(\sqrt{2}-a)(\sqrt{2}+a) = 0$ (if this is 0, the vectors are dependent, if not, they are independent) $\Rightarrow a = 0$ or $a = \sqrt{2}$ or $a = -\sqrt{2} \Rightarrow$ for v_1, v_2, v_3 to be linearly independet $a \in \mathbb{R} \setminus \{-\sqrt{2}, 0, \sqrt{2}\}$.

4. $a_1v_1+a_2v_2+a_3v_3 = 0 \iff \begin{cases} a_1 + 2a_2 = 0 \\ -2a_1 + a_2 + aa_3 = 0 \\ a_2 + a_3 = 0 \\ -a_1 + 2a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = -a_3 \\ a_1 = 2a_3 \\ -4a_3 - a_3 + aa_3 = 0 \end{cases} \Rightarrow$ $(a-5)a_3 = 0 \Rightarrow a \in \mathbb{R} \setminus \{5\}$. (Here, a_3 could not be 0, as all of them would have been 0, which means that the vectors would have been linearly independent).

5. (i) $\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow (v_1, v_2, v_3)$ linealry independent.
   $\forall u = (u_1, u_2, u_3) \in \mathbb{R}^3, \exists! a_1, a_2, a_3 \in \mathbb{R}$ such that

   $$a_1v_1 + a_2v_2 + a_3v_3 = u \Rightarrow \begin{cases} a_1 - a_2 + a_3 = u_1 \\ a_1 + a_3 = u_2 \\ 2a_2 + a_3 = u_3 \end{cases} \Rightarrow$$

   By simple computations we get that $\begin{cases} a_1 = 3u_2 - u_3 - 2u1 \\ a_2 = u_2 - u_1 \\ a_3 = u_3 - 2u_2 + 2u_1 \end{cases} \Rightarrow$ (v_1, v_2, v_3) generates $\mathbb{R}^3 \Rightarrow (v_1, v_2, v_3)$ is a base.

   (ii) We have to solve all three systems $a_1v_1 + a_2v_2 + a_3v_3 = e_1$ and $a_1v_1 + a_2v_2 + a_3v_3 = e_2$ and $a_1v_1 + a_2v_2 + a_3v_3 = e_3$. In other words, to find a_1, a_2, a_3 in each case. $\Rightarrow \begin{cases} e_1 = 2v_1 + v_2 - 2v_3 \\ e_2 = 3v_1 + v_2 - 2v_3 \\ e_3 = -v_1 + v_3 \end{cases}$

   (iii) In (e_1, e_2, e_3) we have the coordinates for u as $(1, -1, 2)$. So, $a_1v_1 + a_2v_2 + a_3v_3 = (1, -1, 2)$, and by solving the system, we find $a_1 = -7, a_2 = -2, a_3 = 6 \Rightarrow$ In the base (v_1, v_2, v_3), u has the coordinates $(-7, -2, 6)$.

6. We know that (E_1, E_2, E_3, E_4) is a base in $M_2(\mathbb{R})$ and so, the coordinates of B in the base are $(2, 1, 1, 0)$.

   For the second one we have to solve the system $a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = 0 \Rightarrow a_1 = a_2 = a_3 = a_4 = 0 \Rightarrow A_1, A_2, A_3, A_4$ are linearly independent. Then $\forall A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}), \exists! a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = A \Rightarrow \begin{cases} a_1 = a - b \\ a_2 = b - c \\ a_3 = c - d \\ a_4 = d \end{cases}$

   $\Rightarrow < A_1, A_2, A_3, A_4 >= M_2(\mathbb{R})$ so it is a base of $M_2(\mathbb{R})$. Then, the coordinates of B is this base are $(1, 0, 1, 0)$.

7. We know that E is a base in $\mathbb{R}_2[X]$ and so, the coordinates of f in E are (a_0, a_1, a_2).

   For the second one, we have

   $$\alpha_1 \cdot 1 + \alpha_2 \cdot (X - a) + \alpha_3 \cdot (X - a^2) = 0 \iff \begin{cases} \alpha_1 - a\alpha_2 + a^2\alpha_3 = 0 \\ \alpha_2 - 2\alpha_3 = 0 \\ \alpha_3 = 0 \end{cases}$$

   $\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow B$ has linearly independent vectors.

   $\forall f = b_0 + b_1X + b_2X^2 \in \mathbb{R}_2[X], \exists! \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

   $$f = \alpha_1 \cdot 1 + \alpha_2 \cdot (X - a) + \alpha_3 \cdot (X - a^2) \Rightarrow \begin{cases} \alpha_1 = b_0 + ab_1 - a^2b_2 \\ \alpha_2 = b_1 + 2ab_2 \\ \alpha_3 = b_2 \end{cases}$$

   $\Rightarrow < B >= \mathbb{R}_2[X] \Rightarrow B$ is a base of $\mathbb{R}_2[X]$. And so, the coordinates of B in this base are $(a_0 + aa_1 - a^2a_2, a_1 + 2aa_2, a_2)$.

8. $\mathbb{Z}_2^3 = \{(\hat{0}, \hat{0}, \hat{0}), (\hat{1}, \hat{0}, \hat{0}), (\hat{0}, \hat{1}, \hat{0}), (\hat{0}, \hat{0}, \hat{1}), (\hat{1}, \hat{1}, \hat{0}), (\hat{1}, \hat{0}, \hat{1}), (\hat{0}, \hat{1}, \hat{1}), (\hat{1}, \hat{1}, \hat{1})\}$.

   So, $\mid \mathbb{Z}_2^3 \mid= 2^3$.

   A pair $(z_1, z_2, z_3) \in \mathbb{Z}_2^3$ is a base $\iff$ z_1, z_2, z_3 are linearly independent.

   Take $z_1 \in \mathbb{Z}_2^3 \setminus \{(\hat{0}, \hat{0}, \hat{0})\} \Rightarrow z_1$ is a part of the base $\Rightarrow z_1$ can be chosen in $2^3 - 1$ ways. If $z_2, z_3 \in \mathbb{Z}_2^3 \Rightarrow z_1, z_2, z_3$ linearly independent $\iff z_2 \in \mathbb{Z}_2^3 \setminus < z_1 >$ and $z_3 \in \mathbb{Z}_2^3 \setminus < z_1, z_2 >$. So, z_2 can be chosen in

$(2^3-1)-1$ ways and z_3 in $((2^3-2)-1)-1$ ways. Hence, the number of bases of $\mathbb{Z}_2^3$ is $(2^3-1)(2^3-2)(2^3-4)=148$.