Seminar 11

$\forall v \in V : [v]_B = T_{BB'} \cdot [v]_{B'}$ and $T_{BB'}^{-1} = T_{B'B}$.
$[f]_{B'} = T_{BB'}^{-1} \cdot [f]_B \cdot T_{BB'}$.
$f(v) = \lambda \cdot v$, where λ is the eigenvalue and v is the eigenvector.
$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda \cdot Tr(A) + det(A)$.

1. We want to determine $T_{BB'}$. So, we compute the vectors in B as a linear combination of vectors in B'.

   $v_1 = a_1 v_1 + a_2 v_2 + a_3 v_3 = (a_1 - a_2, a_1, a_3) = (1, 0, 1) \Rightarrow a_1 = 0$, $a_2 = -1$ and $a_3 = 1$.

   $v_2 = (a_1 - a_2, a_1, a_3) = (0, 1, 1) \Rightarrow a_1 = 1$, $a_2 = 1$ and $a_3 = 1$.

   $v_3 = (a_1 - a_2, a_1, a_3) = (1, 1, 1) \Rightarrow a_1 = 1$, $a_2 = 0$ and $a_3 = 1$.

   Hence, $T_{BB'} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (on the columns). Now, we need $T_{B'B}$ which is actually $T_{BB'}^{-1}$. And by simple computations, we get that $T_{B'B} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix}$.

   Now, we have to find $[u]_{B'}$, which is $(2, 0, -1) = (a_1 - a_2, a_1, a_3) \Rightarrow a_1 = 0$, $a_2 = -2$ and $a_3 = -1$. And, for $[u]_B$ we use the formula $[u]_B = T_{BB'} \cdot [u]_{B'} = \begin{bmatrix} -3 & -2 & -3 \end{bmatrix}$.

   We could also use $[u]_{B'} = T_{B'E} \cdot [u]_E$.

2. $v_1' = -3v_1 + 2v_2$ and $v_2' = -5v_1 + 3v_2$. So, $T_{B'B} = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$. And, we know that $T_{B'B} = T_{BB'}^{-1}$, hence $T_{BB'} = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$.

   Then, $[g]_B = T_{B'B}^{-1} \cdot [g]_{B'} \cdot T_{B'B} = \begin{bmatrix} -20 & -31 \\ 13 & 20 \end{bmatrix}$.

   Hence, $[f + g]_B = [f]_B + [g]_B = \begin{bmatrix} -19 & -30 \\ 12 & -19 \end{bmatrix}$.

   For $[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$. We compute $[f]_{B'} = T_{BB'}^{-1} \cdot [f]_B \cdot T_{BB'} = \begin{bmatrix} 8 & 13 \\ -5 & -8 \end{bmatrix}$.

Hence, $[f \circ g]_{B'} = \begin{bmatrix} 9 & -13 \\ 5 & 9 \end{bmatrix}$.

3. (i) $f(e_1) = (3,2)$ and $f(e_2) = (3,4) \Rightarrow A = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = 0 \iff (\lambda - 3)(\lambda - 4) = 6 \iff \lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda_1 = 1$ and $\lambda_2 = 6$.
Take $\begin{bmatrix} 3-\lambda & 3 \\ 2 & 4-\lambda \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
For $\lambda_1 = 1 \Rightarrow 2x_1 + 3x_2 = 0 \Rightarrow x_1 = -\frac{3}{2}x_2 \Rightarrow V(1) = \{(-\frac{3}{2}x_2, x_2) \mid x_2 \in \mathbb{R}\} =< (\frac{3}{2}, 1) >$.
For $\lambda_2 = 6 \Rightarrow 2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow V(6) = \{(x_2, x_2) \mid x_2 \in \mathbb{R}\} =< (1,1) >$.

(ii) As $dim(\mathbb{R}^2) = 2$, where $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $\lambda_1 \neq \lambda_2 \Rightarrow B =< (\frac{3}{2}, 1), (1,1) >$ is a basis and $[f]_B = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

4. $\begin{vmatrix} 3-\lambda & 1 & 0 \\ -4 & -1-\lambda & 0 \\ -4 & -8 & -2-\lambda \end{vmatrix} = 0 \iff (2+\lambda)[(\lambda+1)(3-\lambda) - 4] = 0 \Rightarrow \lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 1$.

For $\lambda_1 = -2 \Rightarrow \begin{bmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -4 & -8 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O_3 \Rightarrow \begin{cases} 5x_1 + x_2 = 0 \\ -4x_1 + x_2 = 0 \\ -4x_1 - 8x_2 = 0 \end{cases} \Rightarrow$
$x_1 = -2x_2 \Rightarrow V(-2) = \{(-2x_2, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} =< (-2,1,0), (0,0,1) >$.
For $\lambda_2 = \lambda_3 = 1 \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ -4 & -8 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O_3 \Rightarrow \begin{cases} 2x_1 + x_2 = 0 \\ -4x_1 - 8x_2 = 0 \end{cases} \Rightarrow$
$-2x_1 = x_2$ and $x_3 = 4x_1 \Rightarrow V(1) = \{(x_1, -2x_1, 4x_1) \mid x_1 \in \mathbb{R}\} =< (1, -2, 4) >$.

5. $\begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0 \iff (\lambda - 1)(\lambda + 1)(\lambda^2 + 1) = 0 \Rightarrow \lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = i$ and $\lambda_4 = -i$.

For $\lambda_1 = 1$ we have the system $\begin{cases} -x_1 + x_4 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow x_1 = x_4$ and $x_2 = x_3 \Rightarrow V(1) = \{(x_1, x_2, x_2, x_1) \mid x_1, x_2 \in \mathbb{R}\} =< (1,0,0,1), (0,1,1,0) >$.

For $\lambda_2 = -1$ we have the system $\begin{cases} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow x_1 = -x_4$ and $x_2 = -x_3 \Rightarrow V(-1) = \{(x_1, x_2, -x_2, -x_1) \mid x_1, x_2 \in \mathbb{R}\} =< (1,0,0,-1), (0,1,-1,0) >$.

For $\lambda_3 = i$ we have the system $\begin{cases} x_1 - ix_4 = 0 \\ x_2 - ix_3 = 0 \end{cases} \Rightarrow x_1 = ix_4$ and $x_2 = ix_3 \Rightarrow V(i) = \{(ix_4, ix_3, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\} =< (i,0,0,1), (0,i,1,0) >$.

For $\lambda_4 = -i$ we have the system $\begin{cases} ix_1 + x_4 = 0 \\ ix_2 + x_3 = 0 \end{cases} \Rightarrow -ix_1 = x_4$ and $-ix_2 = x_3 \Rightarrow V(-i) = \{(x_1, x_2, -ix_2, -ix_1) \mid x_1, x_2 \in \mathbb{R}\} =< (1,0,0,-i), (0,1,-i,0) >$.

6. $\begin{vmatrix} x-\lambda & 0 & y \\ 0 & x-\lambda & 0 \\ y & 0 & x-\lambda \end{vmatrix} = 0 \iff (x-\lambda)(x-\lambda-y)(x-\lambda+y) = 0 \Rightarrow$ $\lambda_1 = x$, $\lambda_2 = x - y$ and $\lambda_3 = x + y$.

For $\lambda_1 = x$ we have the system $\begin{cases} yx_3 = 0 \\ yx_1 = 0, y \neq 0 \end{cases} \Rightarrow x_1 = x_3 = 0 \Rightarrow$ $V(x) = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\} =< (0,1,0) >$.

For $\lambda_2 = x - y$ we have the system $\begin{cases} yx_1 + yx_3 = 0 \\ yx_2 = 0, y \neq 0 \end{cases} \Rightarrow x_1 = -x_3$ and $x_2 = 0 \Rightarrow V(x-y) = \{(-x_3, 0, x_3) \mid x_3 \in \mathbb{R}\} =< (-1,0,1) >$.

For $\lambda_3 = x + y$ we have the system $\begin{cases} -yx_1 + yx_3 = 0 \\ -yx_2 = 0, y \neq 0 \end{cases} \Rightarrow x_1 = x_3$ and $x_2 = 0 \Rightarrow V(x+y) = \{(x_1, 0, x_1) \mid x_1 \in \mathbb{R}\} =< (1,0,1) >$.

7. (i) $p(\lambda) = det(A - \lambda I_2)$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2 - \lambda(a+d) + (ad - bc)$.
Now, $p(0) = det(A - 0 \cdot I_2) = det(A) = 0^2 - 0 \cdot (a+d) + ad - bc$.
As λ_1, λ_2 are eigenvalues $\Rightarrow p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1 \cdot \lambda_2$. Also, $p(0) = (0 - \lambda_1)(0 - \lambda_2) = \lambda_1 \cdot \lambda_2 = det(A)$.
Hence, $\lambda_1 + \lambda_2 = a + d = Tr(A)$.

(ii) $\lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1 \cdot \lambda_2 = \lambda^2 - \lambda \cdot Tr(A) + det(A) = 0 \Rightarrow \Delta = (Tr(A))^2 - 4det(A)$. If $0 \leq \Delta \Rightarrow \exists \lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$.

If $\exists \lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow 0 \leq \Delta$.

(iii) For A to be aroot of $p(\lambda)$, $p(A) = O_2 \iff A^2 - A \cdot Tr(A) + I_2 \cdot det(A) = O_2 \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d) \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc) \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = O_2$, which, by simple computations, we get that is true.

8. $det(A - iI_2) = 0 = p(i)$, where $p(i) = (i - \lambda_1)(i - \lambda_2) = -1 - i \cdot Tr(A) + det(A) = 0 \Rightarrow det(A) = 1 + i \cdot Tr(A)$.

   Now $det(A - 2I_2) = 4 - 2Tr(A) + det(A)$, so $det(A - 2I_2) = 4 - 2Tr(A) + 1 + iTr(A) = 5 + (i-2)Tr(A)$.

   From $det(A) = 1 + iTr(A)$, we have that $det(A), Tr(A), 1 \in \mathbb{R} \Rightarrow Tr(A) = 0$.

   Hence, $det(A - 2I_2) = 5$.