Seminar 5

$V = A \oplus B$ if $V = A + B$ and $A \cap B = \{0\}$. Or $\forall v \in V, \exists! s \in S, t \in T$ such that $v = s + t$.

$f : A \to B$ **endomorphism** if $A = B$ and f homomorphism.

$ker(f) = \{x \in R \mid f(x) = 0\}$ and $Im(f) = \{f(x) \mid x \in R\}$.

1. (i) $< 1, X, X^2 >= \{a + bX + cX^2 \mid a, b, c \in \mathbb{R}\} = \mathbb{R}_2[X]$.

   (ii) $< \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} >= \{a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b, c, d \in \mathbb{R}\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}\} = M_2(\mathbb{R})$.

2. (i) $(0, a, b) = (0, a, 0) + (0, 0, b) = a \cdot (0, 1, 0) + b \cdot (0, 0, 1) \Rightarrow A =< (0, 1, 0), (0, 0, 1) >$.

   (ii) $a + b + c = 0 \Rightarrow a = -b - c = -(b + c) \Rightarrow (-(b + c), b, c) = (-b, b, -0) + (-c, 0, c) = b(-1, 1, 0) + c(-1, 0, 1) \Rightarrow B =< (-1, 1, 0), (-1, 0, 1) >$.

   (iii) $(a, a, a) = a(1, 1, 1) \Rightarrow C =< (1, 1, 1) >$.

3. $S =< (-1, 1, 0), (-1, 0, 1) >\Rightarrow s_1 = (-1, 1, 0)$ and $s_2 = (-1, 0, 1)$.

   $T =< (1, 1, 1) >\Rightarrow t = (1, 1, 1)$.

   From *Seminar4*, we know that S, T are subspaces of $\mathbb{R}^{\mathbb{R}}$. To prove that $\mathbb{R}^3 = S \oplus T$, we prove that $S + T = \mathbb{R}^3$ and $S \cap T = \{0_3\}$.

   $\forall v \in \mathbb{R}^3, \exists! s \in S, \exists! t \in T$ such that $v = s + t \iff (v_1, v_2, v_3) = a \cdot s_1 + b \cdot s_2 + c \cdot t \iff (v_1, v_2, v_3) = (-a, a, 0) + (-b, 0, b) + (c, c, c) \iff v_1 = -a - b + c, v_2 = a + c, v_3 = b + c \Rightarrow a = -\frac{1}{3}v_1 + \frac{2}{3}v_2 - \frac{1}{3}v_3$, $b = -\frac{1}{3}v_1 - \frac{1}{3}v_2 + \frac{2}{3}v_3$ and $c = \frac{1}{3}(v_1 + v_2 + v_3)$, so they are unique.

4. Remember: $f : \mathbb{R} \to \mathbb{R}, f$-odd $\Rightarrow \forall x \in \mathbb{R}, f(-x) = -f(x)$. $f : \mathbb{R} \to \mathbb{R}, f$-even $\Rightarrow f(-x) = f(x)$.

   $S \neq \emptyset$, as $\theta(x) = 0 \in S$ and $t \neq \emptyset$, as $f(x) = -x \in T$.

   Take $f, g \in S, a, b \in \mathbb{R} \Rightarrow (af + bg)(-x) = (af)(-x) + (bg)(-x) = -af(x) - bg(x) = -(af + bg)(x) \in S \Rightarrow S \leq \mathbb{R}^{\mathbb{R}}$.

   Take $f, g \in T, a, b \in \mathbb{R} \Rightarrow (af + bg)(-x) = (af)(-x) + (bg)(-x) = af(x) + bg(x) = (af + bg)(x) \in T \Rightarrow T \leq \mathbb{R}^{\mathbb{R}}$.

Take $f : \mathbb{R} \to \mathbb{R}, g \in S, h \in T$, as $f(x) = g(x) + h(x)$. Then $f(-x) = g(-x) + h(-x) = -g(x) + h(x) \Rightarrow g(x) = \frac{1}{2}(f(x) + f(-x)) \in S$ and $h(x) = \frac{1}{2}(f(x) - f(-x)) \in R$. So, g, h are unique functions, with which we can write any function $f : \mathbb{R} \to \mathbb{R}$. Now, for the intersection: if $f(-x) = -f(x)$ and $f(-x) = f(x) \Rightarrow f(x) = -f(x) \Rightarrow f(x) = \theta(x)$. So $S \cap T = \{\theta(x) = 0\}$.

5. $f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2) = (x_1 + y_1, x_1 - y_1) + (x_2 + y_2, x_2 - y_2) = f(x_1, y_1) + f(x_2, y_2) \Rightarrow f$ endomorphism.

   $g((x_1, y_1) + (x_2, y_2)) = g(x_1 + x_2, y_1 + y_2) = (2x_1 + 2x_2 - y_1 - y_2, 4x_1 + 4x_2 - 2y_1 - 2y_2) = g(x_1, y_1) + g(x_2, y_2) \Rightarrow g$ endomorphism.

   $h((x_1, y_1, z_1) + (x_2, y_2, z_2)) = h(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2 - y_1 - y_2, y_1 + y_2 - z_1 - z_2, z_1 + z_2 - x_1 - x_2) = h(x_1, y_1, z_1) + h(x_2, y_2, z_2) \Rightarrow h$ endomorphism.

6. (i) $f(x, y) = (ax + by, cx + dy)$
   $f(x_1 + x_2, y_1 + y_2) = (ax_1 + ax_2 + by_1 + by_2, cx_1 + cx_2 + dy_1 + dy_2) = (ax_1 + by_1, cx_1 + dy_1) + (ax_2 + by_2, cx_2 + dy_2) = f(x_1, y_1) + f(x_2, y_2) \Rightarrow f$ endomorphism.

   (ii) $g(x, y) = (a + x, b + y)$
   For $a = b = 0 \Rightarrow g(x, y) = (x, y)$ - endomorphism of $\mathbb{R}^2$. But $\forall a, b \in \mathbb{R}^* \Rightarrow g(x_1 + x_2, y_1 + y_2) = (a + x_1 + x_2, b + y_1 + y_2) = (a + x_1, b + y_1) + (x_2, y_2) = g(x_1, y_1) + (x_2, y_2) \Rightarrow g$ is NOT an endomorphism.

7. $ker(f) = \{(x, y) \mid (x + y, x - y) = (0, 0)\} \Rightarrow x + y = 0$ and $x - y = 0 \Rightarrow x = y$ and $2y = 0 \Rightarrow x = y = 0 \Rightarrow ker(f) = \{(0, 0)\}$.

   $Im(f) = \{(x + y, x - y) \mid x, y \in \mathbb{R}\} = \{(x, x) + (y, -y) \mid x, y \in \mathbb{R}\} = \{x(1, 1) + y(1, -1) \mid x, y \in \mathbb{R}\} \Rightarrow Im(f) =< (1, 1), (1, -1) >$.

   $ker(g) = \{(x, y) \mid (2x - y, 4x - 2y) = (0, 0)\} \Rightarrow 2x - y = 0$ and $4x - 2y = 0 \Rightarrow 2x = y$. So, take $x = a \in \mathbb{R} \Rightarrow y = 2a \in \mathbb{R} \Rightarrow ker(g) = \{(a, 2a) \mid a \in \mathbb{R}\} =< (1, 2) >$

   $Im(g) = \{(2a - b, 4a - 2b) \mid x, y \in \mathbb{R}\} = \{(2a, 4a) + (-b, -2b) \mid x, y \in \mathbb{R}\} = \{a(2, 4) + b(-1, -2) \mid x, y \in \mathbb{R}\} \Rightarrow Im(g) =< (2, 4), (-1, -2) >$

   $ker(h) = \{(x, y, z) \mid (x - y, y - z, z - x) = (0, 0, 0)\} \Rightarrow x - y = 0, y - z = 0, z - x = 0 \Rightarrow x = y = z \Rightarrow ker(h) = \{(x, x, x) \mid x \in \mathbb{R}\} =< (1, 1, 1) >$

$Im(h) = \{(a-b, b-c, c-a) \mid a,b,c \in \mathbb{R}\} = \{(a,0,a) + (-b,b,0) + (0,-c,c) \mid a,b,c \in \mathbb{R}\} = \{a(1,0,-1) + b(-1,1,0) + c(0,-1,1) \mid a,b,c \in \mathbb{R}\} \Rightarrow Im(h) =< (1,0,-1), (-1,1,0), (0,-1,1) >$.

8. $S \neq \emptyset$, as $f(0) = 0 \in S$.

   $\forall x, y \in S \Rightarrow x + y = f(x) + f(y) = f(x+y) \in S$, as f is an endomorphism.

   $\forall a \in K, \forall x \in S \Rightarrow ax = af(x) = f(ax) \in S$, as f is an endomorphism.

   So, $S \leq V$.