Seminar 10

$[f]_E = [f(e_1) \mid f(e_2) \mid f(e_3)]$.

If we have the bases $B = e \cdot S$ and $B' = e' \cdot T$, then $[f]_{BB'} = T^{-1} \cdot [f]_{ee'} \cdot S$.

$[f]_E \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ for a basis in $ker(f)$.

$dim(Im(f)) = dim(f(e)) = dim(< f(e_1), f(e_2), f(e_3), f(e_4) >) =$ maximum number of linearly independent vectors in $[f]_E = rank([f]_E)$.

f is an automorphism $\iff det([f]_E) \neq 0$ and $[2f]_E = 2[f]_E$.

1. We use $[f]_E = [f(e_1)f(e_2)f(e_3)]$. So, we compute $\begin{cases} f(e_1) = f(1,0,0) = (1,0,2) \\ f(e_2) = f(0,1,0) = (1,1,1) \\ f(e_3) = f(0,0,1) = (0,-1,1) \end{cases}$

   Hence, $[f]_E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$.

2. $\begin{cases} f(v_1) = f(1,1,0) = (1,-1) \\ f(v_2) = f(0,1,1) = (1,0) \\ f(v_3) = f(1,0,1) = (0,-1) \end{cases}$

   So, $[f]_{BE'} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$.

   From $f(v_1) = (1,-1)$, we get $(1,-1) = a_1v_1' + a_2v_2' = (a_1 + a_2, a_1 - 2a_2) \Rightarrow a_2 = \frac{2}{3}$ and $a_1 = \frac{1}{3}$.

   From $f(v_2) = (1,0)$, we get $(1,0) = a_1v_1' + a_2v_2' = (a_1 + a_2, a_1 - 2a_2) \Rightarrow a_2 = \frac{1}{3}$ and $a_1 = \frac{2}{3}$.

   From $f(v_3) = (0,-1)$, we get $(0,-1) = a_1v_1' + a_2v_2' = (a_1 + a_2, a_1 - 2a_2) \Rightarrow a_2 = \frac{1}{3}$ and $a_1 = -\frac{1}{3}$.

   Hence, $[f]_{BB'} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

3. (i) $v = (1,4,1,-1) = e_1 + 4e_2 + e_3 - e_4 \Rightarrow f(v) = f(e_1) + 4f(e_2) + f(e_3) - f(e_4) = (1,-1,2,1) + 4(1,1,1,2) + (-3,1,-5,-4) - (2,4,1,5) = (0,0,0,0) \Rightarrow v \in ker(f)$.

$v' \in Im(f) \iff \exists v$ such that $f(v) = v'$. So, $v' = af(e_1) + bf(e_2) + cf(e_3) + df(e_4) \Rightarrow \begin{cases} a+b-3c+2d=2 \\ -a+b+c+4d=-2 \\ 2a+b-5c+d=4 \\ a+2b-4c+5d=2 \end{cases}$.

By solving the system, we get that $c, d \in \mathbb{R}$, $b = c - 3d$ and $a = 2 + 2c + d$. Hence, there is a v such that $f(v) = v'$.

(ii) We use $[f]_E \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. By solving the system here, we get that $x_4 = a$, $x_3 = b$, $x_2 = b - 3a$ and $x_1 = 2b + a \iff <(1,-3,0,1),(2,1,1,0)>= ker(f) \Rightarrow dim = 2$.

We use $dim(Im(f)) = rank[f]_E$ and we know that $rank[f]_E = 2 \Rightarrow dim(Im(f)) = 2$ and $Im(f) =< (1,1,-3,2),(-1,1,1,4) >$.

(iii) $\begin{cases} f(1,0,0,0) = (1,-1,2,1) = (x,-x,2x,x) \\ f(0,1,0,0) = (1,1,1,2) = (y,y,y,2y) \\ f(0,0,1,0) = (-3,1,-5,-4) = (-3z,z,-5z,-4z) \\ f(0,0,0,1) = (2,4,1,5) = (2t,4t,t,5t) \end{cases}$

$\Rightarrow f(x,y,z,t) = (x+y-3z+2t, -x+y+z+4t, 2x+y-5z+t, x+2y-4z+5t)$.

4. $det[f]_B = 1 \neq 0 \Rightarrow f$ is an automorphism $\Rightarrow [2f]_B = 2[f]_B = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$.

$\begin{cases} f(v_1) = (1,-1) \\ f(v_2) = (2,-1) \end{cases} \Rightarrow \begin{cases} a_1x + b_1y = 1 \\ a_2x + b_2y = -1 \end{cases}$.

From $x = 1, y = 2 \Rightarrow \begin{cases} a_1 + 2b_1 = 1 \\ a_2 + 2b_2 = -1 \end{cases} \Rightarrow a_1 = -1$ and $a_2 = -1$.

From $x = 1, y = 3 \Rightarrow \begin{cases} a_1 + 3b_1 = 2 \\ a_2 + 3b_2 = -1 \end{cases} \Rightarrow b_1 = 1$ and $b_2 = 0$.

Hence, $f(x,y) = (y - x, -x)$.

$\begin{cases} g(v_1') = (-7,5) \\ g(v_2') = (-13,7) \end{cases} \Rightarrow \begin{cases} a_1x + b_1y = -7 \\ a_2x + b_2y = 5 \end{cases}$.

From $x = 1, y = 0 \Rightarrow a_1 = -7$ and $a_2 = 5$. From $x = 2, y = 1 \Rightarrow b_1 = 1$ and $b_2 = -3 \Rightarrow g(x, y) = (y - 7x, 5x - 3y)$.

Now, we compute $(f + g)(x, y) = f(x, y) + g(x, y) = (y - x, -x) + (y - 7x, 5x - 3y)$. And we apply this to the vectors v_1, v_2. So, $(f + g)(v_1) = (-4, -2)$ and $(f + g)(v_2) = (-2, -5) \Rightarrow [f + g]_B = \begin{bmatrix} -4 & -2 \\ -2 & -5 \end{bmatrix}$.

In the end, we compute $(f \circ g)(x, y) = f(g(x, y)) = (12x - 4y, -y + 7x)$ and we apply this to the vectors v'_1, v'_2. So, $(f \circ g)(v'_1) = (12, 7)$ and $(f \circ g)(v'_2) = (20, 13) \Rightarrow [f \circ g]_{B'} = \begin{bmatrix} 12 & 20 \\ 7 & 13 \end{bmatrix}$.

5. $f(e_1) = (cos(\alpha), sin(\alpha))$ and $f(e_2) = (-sin(\alpha), cos(\alpha))$.

   So, $[f]_E = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}$.

   We compute $det([f]_E) = cos^2(\alpha) + sin^2(\alpha) = 1 \neq 0 \Rightarrow f$ is an automorphism.

6. $dim_{\mathbb{Z}_2}(V) = 2 \Rightarrow \mid V \mid = 2^2 = 4$ and $\mid M_2(\mathbb{Z}_2) \mid = 2^4 = 16$.

   As $End_{\mathbb{Z}_2}(V)$ is isomorphic to $M_2(\mathbb{Z}_2) \Rightarrow \mid End_{\mathbb{Z}_2}(V) \mid = 2^4$.