Seminar 7

The dimension of a space is given by the number of vectors in its (canonical) base.

If $f : V \to V$, then $dim(V) = dim(ker(f)) + dim(Im(f))$.

If $ker(f) = \{0\}$, then $dim(ker(f)) = 0$.

If $A \subseteq B$, then $dim(A) \leq dim(B)$.

We have: $dim(A) + dim(B) = dim(A + B) + dim(A \cap B)$.

1. $A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} = \{(x, y, 0) \mid x, y \in \mathbb{R}\} = \{(x, 0, 0) + (0, y, 0) \mid x, y \in \mathbb{R}\} = \{x(1, 0, 0)+y(0, 1, 0) \mid x, y \in \mathbb{R}\} =< (1, 0, 0), (0, 1, 0) >$ which is a base if the vectors are linearly independent $\iff a(1, 0, 0)+ b(0, 1, 0) = (0, 0, 0) \iff (a, b, 0) = (0, 0, 0) \Rightarrow a = b = 0$ (true) $\Rightarrow< (1, 0, 0), (0, 1, 0) >$ is a base of A. So, $dim(A) = 2$.

   $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} = \{(x, y, z) \in \mathbb{R}^3 \mid z = -x - y\} = \{(x, y, -x - y) \mid x, y \in \mathbb{R}\} = \{(x, 0, -x) + (0, y, -y) \mid x, y \in \mathbb{R}\} = \cdots =< (1, 0, -1), (0, 1, -1) >$ which is a base if the vectors are linearly independent $\iff a(1, 0, -1) + b(0, 1, -1) = (0, 0, 0) \Rightarrow a = b = 0 \Rightarrow$ is a base of B. So, $dim(B) = 2$.

   $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\} = \{(x, x, x) \mid x \in \mathbb{R}\} = \cdots =$ $< (1, 1, 1) >$ which is a basis, as we have one vector (so linearly independent). So, $dim(C) = 1$.

2. (i) S is a subspace of K^n if $S \neq \emptyset$ (which is true, as $(0, 0, \ldots, 0) \in S$) and $\forall a, b \in K, \forall x, y \in S \Rightarrow ax + by \in S$. So, $ax + by = a(x_1, \ldots, x_n) + b(y_1, \ldots, y_n) = \cdots = (ax_1 + by_1, \ldots, ax_n + by_n)$, which is in S if $ax_1 + by_1 + \cdots + ax_n + by_n = 0 \iff a(x_1 + \ldots x_n) + b(y_1 + \ldots y_n) = a \cdot 0 + b \cdot 0 = 0 \Rightarrow S$ is a subspace of K^n.

   (ii) From $x_1 + \cdots + x_n = 0 \Rightarrow x_n = -x_1 - x_2 \cdots - x_{n-1}$.
   So, $S = \{(x_1, \ldots, x_{n-1}, -x_1 - \cdots - x_{n-1}) \mid x_1, \ldots, x_{n-1} \in K\} = \{(x_1, 0, \ldots, 0, -x_1)+\cdots+(0, 0, \ldots, 0, x_{n-1}, -x_{n-1}) \mid x_1, \ldots, x_{n-1} \in K\} = \cdots =< (1, 0, \ldots, 0, -1), \ldots, (0, 0, \ldots, 0, 1, -1) >$ which is a basis if the vectors are linearly independent (you can prove this yourselves). So, $dim(S) = n - 1$.

3. We know that $(\mathbb{C}, +)$ is an Abelian group. Now, for $\mathbb{C}$ to be a vector space, we need to see if the 4 conditions hold:

   (a) $(k_1 + k_2)z = k_1z + k_2z$ (true)

(b) $k(z_1 + z_2) = kz_1 + kz_2$ (true)

(c) $(k_1 k_2)z = k_1(k_2 z)$ (true)

(d) $1 \cdot z = z$ (true)

Now, $\forall z \in \mathbb{C}, \exists a, b \in \mathbb{R}$ such that $z = a \cdot 1 + b \cdot i \Rightarrow \mathbb{C} =< 1, i >$ which is a base, as 1 and i are linearly independent and $dim(\mathbb{C}) = 2$.

4. f is an $\mathbb{R}$-linear map if $\forall a, b \in \mathbb{R}, \forall x, y \in \mathbb{R}^3 : f(ax + by) = af(x) + bf(y)$.

   So, $f(a(x_1, x_2, x_3) + b(y_1, y_2, y_3)) = f(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) = \cdots = a(x_2, -x_1) + b(y_2, -y_1) = af(x) + bf(y)$.

   $ker(f) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = (0, 0)\} = \{(x, y, z) \in \mathbb{R}^3 \mid (y, -x) = (0, 0)\} = \{(0, 0, z) \mid z \in \mathbb{R}\} =< (0, 0, 1) >$.

   So, $dim(ker(f)) = 1$.

   $Im(f) = \{(y, -x) \in \mathbb{R}^2 \mid f(x, y, z) = (y, -x)\} = \{(y, 0) + (0, -x) \mid x, y \in \mathbb{R}\} =< (1, 0), (0, -1) >$. So, $dim(Im(f)) = 2$.

5. $ker(f) = \{(x, y, z) \in \mathbb{R}^3 \mid (-y+5z, x, y-5z) = (0, 0, 0)\} \Rightarrow -y+5z = 0$ and $x = 0$ and $y - 5z = 0 \Rightarrow x = 0$ and $y = 5z \Rightarrow ker(f) = \{(0, 5z, z) \mid z \in \mathbb{R}\} =< (0, 5, 1) >$, with $dim(ker(f)) = 1$.

   $Im(f) = \{(-y+5z, x, y-5z) \in \mathbb{R}^3 \mid f(x, y, z) = (-y, 0, y)+(5z, 0, -5z)+(0, x, 0)\} =< (-1, 0, 1), (5, 0, -5), (0, 1, 0) >$, but $(5, 0, -5) = -5(-1, 0, 1)$ (i.e. they are not linearly independent) so a base for $Im(f) =< (-1, 0, 1), (0, 1, 0) >$, with $dim(Im(f)) = 2$.

6. For A we need to find a third vector in the base, which is linearly independent with the other two. So, $a(1, 0, 0) + b(0, 1, 0) + c(x, y, z) = (0, 0, 0) \iff a = b = c = 0 \Rightarrow a + cx = 0$ and $b + cy = 0$ and $cz = 0 \Rightarrow x, y, z \in \mathbb{R}$ (not all zero) $\Rightarrow (x, y, z) = (0, 0, 1)$.

   For B the same as above $\Rightarrow a(1, 0, -1) + b(0, 1, -1) + c(x, y, z) = (0, 0, 0) \iff a = b = c = 0 \Rightarrow a + cx = 0$ and $b + cy = 0$ and $-a-b+cz = 0 \Rightarrow c(x+y+z) = 0 \Rightarrow x+y+z \neq 0 \Rightarrow (x, y, z) = (1, 1, 0)$.

   For C we need to find two vectors in the base, which are linearly independent with the third one. So, we can add the vectors $(a, b, c) = (1, 1, 0)$ and $(x, y, z) = (1, 0, 1)$.

7. $dim(S)+dim(U) = dim(S \cap U)+dim(S+U) = dim(T \cap U)+dim(T+U) = dim(T) + dim(U) \Rightarrow dim(S) = dim(T)$. As $S \subseteq T$, we know $dim(S) \leq dim(T)$. So, if their dimensions are equal $\Rightarrow S = T$.

8. First, rewrite S as a generated subset and T as a set.

   $S =< (0, 1, 0), (0, 0, 1) >$ and $T = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$ (you can simply find those).

   Now, $S \cap T = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } x - y + z = 0\} = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } y = z\} =< (0, 1, 1) >$, with $dim(S \cap T) = 1$.

   As, $S, T \subseteq \mathbb{R}^3 \Rightarrow S + T \subseteq \mathbb{R}^3 \Rightarrow dim(S + T) \leq dim(\mathbb{R}^3) = 3$.

   From $dim(S) + dim(T) = dim(S \cap T) + dim(S + T)$ and $dim(S) = dim(T) = 2 \Rightarrow 2 + 2 = 1 + dim(S + T) \Rightarrow dim(S + T) = 4 - 1 = 3 = dim(\mathbb{R}^3) \Rightarrow S + T = \mathbb{R}^3$.