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B5.7. $K^{mn} \xrightarrow{\sim} M_{m,n}(K)$

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 2 & 6 \\ 7 & 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 5 & 0 & 1 & 2 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

$M_{m,n}(K) \to$ Basis: $\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & & 0 & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \dots \\ & & 0 & \end{pmatrix}$ ...

$$\varphi : M_{m,n}(K) \to K^{mn}$$

$$A = (a_{ij})_{i,j} \longmapsto \begin{pmatrix} \left((a_{1,j})_{j=\overline{1,n}}\right) \\ \left((a_{2,j})_{j=\overline{1,n}}\right) \\ \vdots \\ \left((a_{m,j})_{j=\overline{1,n}}\right) \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

B5.8.: $K_{n-1}[X] = \{p \in K[X] \mid \deg p \leq n-1\} \simeq K^n$

$$K_{n-1}[X] \longrightarrow K^n$$

$$\begin{aligned} 1 &\longmapsto (1,0,0,\dots,0) \\ X &\longmapsto (0,1,0,\dots,0) \\ X^2 &\longmapsto (0,0,1,\dots,0) \\ &\vdots \\ X^{n-1} &\longmapsto (0,0,0,\dots,1) \end{aligned}$$

By definition $\varphi(X^k) = e_{k+1}$, $k = 0, \dots, n-1$

7.7. V K-v.s.

$S, T, U \leq_K V$

$\dim(S \cap U) = \dim(T \cap U)$

$\dim(S+U) = \dim(T+U)$

Prove that if $S \subseteq T \Rightarrow S = T$

Proof: 2nd dimension theorem:

$\dim(S+U) = \dim S + \dim U - \dim(S \cap U)$

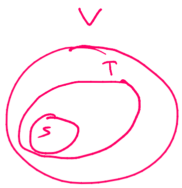
$\dim(T+U) = \dim T + \dim U - \dim(T \cap U)$

$\dim S = \dim(S+U) + \dim(S \cap U) - \dim U =$

$= \dim(T+U) + \dim(T \cap U) - \dim U =$

$= \dim T$

$$\left.\begin{array}{l}\dim S = \dim T \\ S \subseteq T\end{array}\right\} \Rightarrow S = T$$



$$\left.\begin{array}{l}S \leq_K V \\ T \leq_K V\end{array}\right\} S \leq_K T$$

Field = corp comutativ $\longrightarrow$ corp, corps, $\boxed{\text{Körper}}$

Division ring (skew field) = corp

$$\mathbb{H} = \left\{ a + bi + cj + dk \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{R} \\ i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \end{array} \right\}$$

the division ring of quaternions (not a field!)

2.2. A, B sets

$|A| = n, \quad |B| = m$

(i) # relations $r = (A, B, R)$

(ii) # relations $r = (A, A, R)$

Proof: Number of relations = Number of possible graphs

$R \in \mathcal{P}(A \times B)$

Number of possible graphs = Number of possible choices of whether $(a,b) \in R$ or $(a,b) \notin R$ $\forall a \in A, \forall b \in B$

$$\| \quad 2^{|A \times B|} = 2^{mn}$$

(ii) $\# = 2^{n^2}$

V K-v.s.

$X \subseteq V$, finite set

X basis: $\begin{cases} \text{(i) } X \text{ lin indep} \\ \text{(ii) } V = \langle X \rangle \end{cases}$

X basis : { (i) X ... [illegible]

(ii) $V = \langle X \rangle$

(X system of generators)

If X basis for $V \Rightarrow \dim V = |X|$

$X = \{v_1, v_2, \dots, v_n\}$

$\forall v \in V, \exists! \alpha_1, \alpha_2, \dots, \alpha_n :$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Remark: If for $v \in V$, $\exists \alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ s.t.

$v = \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n$

$\Rightarrow (\alpha_1 - \beta_1) \cdot v_1 + (\alpha_2 - \beta_2) \cdot v_2 + \dots + (\alpha_n - \beta_n) \cdot v_n = 0$

If $\exists i \in \{1, \dots, n\}$ s.t. $\alpha_i \neq \beta_i$, then $v_1, v_2, \dots, v_n$ linearly dependent

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- If $\dim V = n$ and

$X = \{v_1, v_2, \dots, v_n\}$, then:

(i) X basis

$\Updownarrow$

(ii) X lin. indep.

$\Updownarrow$

(iii) X system of generators

V k - v.s.

$(v_1, \dots, v_n)$ basis for V, $\forall v \in V$

$\Rightarrow \exists! \alpha_1, \alpha_2, \dots, \alpha_n \in k$:

$v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

2.4. r, s, t ?

- "<", ">" on $\mathbb{R}$: $\cancel{r}$, $\cancel{s}$, t
- | on $\mathbb{N}$: r, $\cancel{s}$, t
- | on $\mathbb{Z}$: r, $\cancel{s}$, t
- $\perp$ on "lines in space" : $\cancel{r}$, s, $\cancel{t}$
- || on "lines in" r s t

- $\perp$ on "lines in space" ~~r~~, s, ~~t~~
- $\parallel$ on "lines in space" r, s, t
- $\equiv$ on "triangles in the plane" r, s, t
- $\sim$ on "triangles in the plane" r, s, t

4.p. Let $a_{11}, a_{12}, a_{21}, a_{22} \in K$

$$S = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{cases} a_{11}x_1 + a_{12}x_2 = 0 \\ a_{21}x_1 + a_{22}x_2 = 0 \end{cases} \right\}$$

Prove that $S \subseteq \mathbb{R}^2$

Let $v_x = (x_1, x_2)$, $v_y = (y_1, y_2)$

$v_x, v_y \in S$; $(0,0) \in S \Rightarrow S \neq \emptyset$

Let $\alpha, \beta \in K$. We will prove that $\alpha v_x + \beta v_y = S$

$\alpha v_x + \beta v_y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$

We must show that

$$\begin{cases} a_{11} \cdot (\alpha x_1 + \beta y_1) + a_{12} \cdot (\alpha x_2 + \beta y_2) = 0 \\ a_{21} \cdot (\alpha x_1 + \beta y_1) + a_{22} \cdot (\alpha x_2 + \beta y_2) = 0 \end{cases}$$

$$a_{11} \cdot (\alpha x_1 + \beta y_1) + a_{12}(\alpha x_2 + \beta y_2) =$$

$$= \alpha \cdot \underbrace{(a_{11}x_1 + a_{12}x_2)}_{=0,\text{ because } (x_1, x_2) \in S} + \beta \cdot \underbrace{(a_{11}y_1 + a_{12}y_2)}_{=0,\text{ because } (y_1, y_2) \in S}$$

$$= 0$$

$$a_{21} \cdot (\alpha x_1 + \beta y_1) + a_{22} \cdot (\alpha x_2 + \beta y_2) =$$

$$= \alpha \cdot \underbrace{(a_{21}x_1 + a_{22}x_2)}_{=0,\ (x_1, x_2) \in S} + \beta \cdot \underbrace{(a_{21}y_1 + a_{22}y_2)}_{=0,\ (y_1, y_2) \in S}$$

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38. $(\mathbb{Z}_n, +) \simeq (U_n, \cdot)$?

$$f : \mathbb{Z}_n \longrightarrow U_n$$

$$\hat{k} \longmapsto \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad , \quad k \in \{0, \ldots, n-1\}$$

$$f(\hat{k} + \hat{l}) = f(\widehat{k+l}) = \cos\frac{2(k+l)\pi}{n} + i\sin\frac{2(k+l)\pi}{n} = f(\hat{k}) \cdot f(\hat{l})$$

f bijective, we can define

$$g : U_n \longrightarrow \mathbb{Z}_n$$

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \mapsto \hat{k}$$

5.3. :

$S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$

$T = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$

$\mathbb{R}^3 = S \oplus T$

$S \cap T = \{(x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x+y+z = 0 \\ x = y = z \end{cases} \} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z = 0\} =$

$= \{(0,0,0)\}$

To show that $\mathbb{R}^3 = S + T$, it suffices to show that $\forall v \in V, \exists v_s \in S, \exists v_t \in T$:

$$v = v_s + v_T$$

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We assume that we found the decomposition.

$$u = (x,y,z) = \underbrace{(a,b,c)}_{\in S} + \underbrace{(d,d,d)}_{\in T}$$

$$a+b+c=0$$

$$\Rightarrow \begin{cases} x = a+d \\ y = b+d \\ z = c+d \end{cases} \Rightarrow d = \frac{x+y+z}{3}$$

$$\Rightarrow a = x - \frac{x+y+z}{3}$$

$$b = y - \frac{x+y+z}{3}$$

$$c = z - \frac{x+y+z}{3}$$

Let $u = (x,y,z) \in \mathbb{R}^3$. We see that

$$(x,y,z) = \left(x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}, z - \frac{x+y+z}{3}\right) + \left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right)$$

Then $u_S = \left(x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}, z - \frac{x+y+z}{3}\right)$

$u_T = \left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right)$

We can see that $u_S \in S$ and $u_T \in T$

$\Rightarrow \mathbb{R}^3 = S + T$

Because $\mathbb{R}^3 = S+T$ and $S \cap T = 0 \Rightarrow$

$\Rightarrow \mathbb{R}^3 = S \oplus T$

7.1. $B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

basis and dimension?

Complete the basis to a basis of $\mathbb{R}^3$

$B = \{(x, y, -x-y) \mid x, y \in \mathbb{R}\}$

Basis for B:

$b = ((1, 0, -1), (0, 1, -1))$

$\dim B = 2$

Steinitz's thm $\Rightarrow \forall \beta = (v_1, v_2, v_3)$ basis of $\mathbb{R}^3$

$\exists \; \beta' = ((1, 0, -1), (0, 1, -1), v)$ basis of $\mathbb{R}^3$

$v \in \{v_1, v_2, v_3\}$

$\mathbb{C}$

Basis for ${}_{\mathbb{R}}\mathbb{C}$: $\quad B = (1, i)$

$b_2 = (1+i, 1-i)$

$\dim_{\mathbb{R}} \mathbb{C} = 2$

$\Rightarrow \mathbb{C} \cong \mathbb{R}^2$

$a \cdot (1+i) + b \cdot (1-i) = 0$

$\Rightarrow a + ai + b - bi = 0$

$\Rightarrow a = -b, \; a = b \Rightarrow b = -b = 0 \Rightarrow a = 0$

$\Rightarrow 1+i$ and $1-i$ are lin. indep in ${}_{\mathbb{R}}\mathbb{C} \Rightarrow$ basis