Seminar W9 - 911

24 November 2020 13:12

9.1. Compute (by applying elementary operations) the rank of the matrix:

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \sim$$

$$\underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rank} A = 3$$

9.3. Compute (by applying elementary transf.) the inverse of the matrix:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$(A \mid I_n) \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} (I_n \mid B)$$

$$\rightarrow B = A^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\overset{L_2 \leftarrow -\frac{1}{3}L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right)$$

$$\overset{L_3 \leftarrow L_3/9}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \underset{L_2 \leftarrow L_2 - 2L_3}{\overset{L_1 \leftarrow L_1 - 2L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

1 0 0 2 1
0 0 0

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & \frac{-2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

9.4. $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

Find the inverse using elementary transformations

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow -\frac{1}{5} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12 L_1}{\sim}$$

$$\begin{pmatrix} 0 & 0 & 1 & \bigg| & \frac{5}{9} & -\frac{1}{9} & \frac{1}{9} \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & \bigg| & \frac{?}{9} & -\frac{?}{9} & \frac{?}{9} \end{pmatrix}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \underset{\sim}{L_3 \leftarrow 5L_3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_2}{\overset{L_2 \leftarrow L_2 - \frac{3}{5} L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$L_1 \leftarrow L_1 - 4L_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix}$$

9.6. $X = (u_1, u_2, u_3)$

$u_1 = (1, 0, 4, 3)$, $u_2 = (0, 2, 3, 1)$, $u_3 = (0, 4, 6, 2)$

Determine $\dim \langle X \rangle$ and a basis of $\langle X \rangle$

$$\begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 6 & 2 \end{pmatrix} \overset{L_3 \leftarrow L_3 - 2L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim \langle X \rangle = 2$

A basis for $\langle X \rangle$: $\big((1, 0, 4, 3), (0, 2, 3, 1)\big)$

9.8. Determine the dimension of the subspaces $S, T, S+T$ and $S \cap T$ of $\mathbb{R}^4$ and a basis for the first three of them.

$$S = \langle (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) \rangle$$

$$T = \langle (2, 5, -6, -5), (-1, 2, -7, -3) \rangle$$

<u>Sol.</u> :

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \underset{L_2 \leftarrow L_2 - 3L_1}{\overset{L_3 \leftarrow L_3 + L_1}{\sim}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 2 & 0 & -3 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 + \frac{2}{5} L_2}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \end{pmatrix} \sim$$

$L_3 \leftarrow 5 L_3$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ \end{pmatrix}$$

$\to \dim S = 3$

$\to$ basis for S

$((1, 2, -1, -2), (0, -5, 4, 7),$

For T:

$$\begin{pmatrix} 2 & 5 & -6 & -5 \\ -1 & 2 & -7 & -3 \end{pmatrix} \overset{L_2 \leftarrow L_2 + \frac{1}{2} \cdot L_1}{\sim}$$

$$\sim \begin{pmatrix} 2 & 5 & -6 & -5 \\ 0 & \frac{9}{2} & -10 & -\frac{11}{2} \end{pmatrix} \overset{L_2 \leftarrow 2L_2}{\sim}$$

$$\sim \begin{pmatrix} 2 & 5 & -6 & -5 \\ 0 & 9 & -20 & -11 \end{pmatrix}$$

$\Rightarrow \dim T = 2$

A basis: $((2,5,-6,-5),(0,9,-20,-11))$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & 8 & -1 \\ 2 & 5 & -6 & -5 \end{pmatrix} \overset{L_4 \leftarrow L_4 - 2L_1}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & 8 & -1 \\ \end{pmatrix}$$

$$\begin{pmatrix} 0 & -5 & 4 & 7 \\ 0 & 0 & 8 & -1 \end{pmatrix} \quad \big((1,2,-1,-2),\ (0,-5,4,7),\ (0,0,8,-1)\big)$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & 8 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 9 & -20 & -11 \end{pmatrix} \underset{L_2 \leftrightarrow L_4}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 8 & -1 \\ 0 & -5 & 4 & 7 \\ 0 & 9 & -20 & -11 \end{pmatrix}$$

$L_4 \leftarrow L_4 + 5L_2$

$L_5 \leftarrow L_5 - 9L_2$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 8 & -1 \\ 0 & 0 & -16 & 2 \\ 0 & 0 & 16 & -2 \end{pmatrix} \sim$$

$L_5 \leftarrow L_5 - 2L_3$

$L_4 \leftarrow L_4 + 2L_3$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 8 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim(S+T) = 3$. A basis:

$$\begin{pmatrix} 0 & 0 & 8 & -1 \\ 2 & 5 & -6 & -5 \\ 0 & 9 & -20 & -11 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 8 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 9 & -20 & -11 \end{pmatrix}$$

$$\dim(S \cap T) = \dim S + \dim T - \dim(S+T) =$$

$$= 3 + 2 - 3 = 2$$

$$\Rightarrow \dim(S \cap T) = 2$$

$\Rightarrow \dim(S+T) = 3$. A basis:

$\big((1,2,-1,-2),\ (0,1,-4,-1),\ (0,0,8,-1)\big)$