Mathematical Analysis – Exam

Feb. 4, 2021

Please give complete explanations/proofs. Unjustified answers will not be marked. Good luck!

1. (3 p)

a) Let $x_n = \ln(n+2) - \ln(n+1),\ n \in \mathbb{N}$.

   i) Study if the sequence (x_n) is monotone, bounded and convergent.

   ii) Find $\lim\limits_{n\to\infty} ((3n+1)\cdot x_n)$ and $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{4}+\ldots+\frac{1}{3n-2}}{\ln(n+1)}$.

   iii) Study if the series $\sum\limits_{n\geq 1} x_n$ is convergent or divergent.

b) Let (x_n) be a sequence in $[0,2)$ and suppose that $\sum\limits_{n\geq 1} x_n$ is convergent. Is the series $\sum\limits_{n\geq 1} \dfrac{x_n}{4-x_n^2}$ convergent as well?

2. (1 p) Let $f : (0,\infty)\times[0,\infty) \to \mathbb{R},\ f(x,y) = \dfrac{e^{x-y} - \cos(x-y)}{x+y}$. Does f have a limit at 0_2?

3. (2.5 p) Let $f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = x^3 + 3xy^2 + 6xy$.

a) Find the gradient $\nabla f(x,y)$ and the Hessian matrix $H_f(x,y)$ of f at $(x,y)\in\mathbb{R}^2$.

b) Find the stationary points of f and then classify them (as local minimum points, local maximum points, or points that are not local extremum points).

c) Study whether the obtained local extremum points (if any) are in fact global extremum points.

4. (2.5 p)

a) Let $f : [0,\infty) \to \mathbb{R},\ f(x) = \dfrac{x^2}{4+x^6}$. Study the improper integrability of f on its domain and, in case f is improperly integrable, determine the improper integral $\displaystyle\int_0^\infty f(x)\,dx$.

b) Let M be the subset of $\mathbb{R}^2$ bounded by the parabola $y = x^2$ and the lines $y = 0$ and $x = 1$. Compute $\displaystyle\iint_M e^{x^3}\,dx\,dy$.