Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Mathematical Analysis - Lecture Notes
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Lecture 4

Example 1. (i) *The harmonic series:* $\sum_{n\geq 1}\frac{1}{n}$ is divergent with sum ∞.

Follows from lecture 3, Rmk. 6.

The generalized harmonic series: Let $\alpha \in \mathbb{R}$. Then

$$\sum_{n\geq 1}\frac{1}{n^\alpha} = \begin{cases} \text{convergent}, & \text{if } \alpha > 1,\\ \text{divergent}, & \text{if } \alpha \leq 1.\end{cases}$$

In particular, $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

(ii) $\sum_{n=0}^{\infty}\frac{1}{n!}$ is convergent with sum e. Thus, Euler's number can be equivalently defined via this series.

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Denote for $n\in\mathbb{N}_0$ the partial sums by $s_n = 1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}$.

$$\left(1+\frac{1}{n}\right)^n \overset{\text{Binomial Thm}}{=} \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \dots + \frac{n(n-1)\cdot\ldots\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{n}\right)}_{\leq 1} + \dots + \frac{1}{n!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n}\right)}_{\leq 1} \leq s_n$$

Fix $n\in\mathbb{N}$. Take $m\in\mathbb{N}$ s.t. $m\geq n$. Then

$$\left(1+\frac{1}{m}\right)^m = 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\dots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{m}\right)+\dots+$$
$$+\frac{1}{m!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{m-1}{m}\right)$$
$$\geq 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\dots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{m}\right)$$

Letting $m\to\infty$, $e \geq s_n$

$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq s_n \leq e,\ \forall n\in\mathbb{N} \quad \Rightarrow \lim_{n\to\infty} s_n = e.$

Theorem 1 (The n^{th} Term Test). *If the series $\sum_{n\geq 1} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.*

Pf: $\sum_{n\geq 1} x_n$ convergent $\Rightarrow$ the sequence (s_n) of partial sums is convergent

$\Rightarrow \lim_{n\to\infty}\underbrace{(s_n - s_{n-1})}_{x_n} = 0$, $\forall n\geq 2$

Remark 1.

- (x_n) $\begin{cases}\text{divergent} \\ \text{OR} \\ \text{convergent, but } \lim_{n\to\infty} x_n \neq 0\end{cases}$ $\Rightarrow \sum_{n\geq 1} x_n$ divergent
- $\lim_{n\to\infty} x_n = 0 \not\Rightarrow \sum_{n\geq 1} x_n$ convergent (e.g. $\sum_{n\geq 1}\frac{1}{n}$, $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$, $\frac{1}{n}\to 0$)

Series with nonnegative terms

Let (x_n) be a sequence of in $\mathbb{R}$. Consider the series $\sum_{n\geq 1} x_n$ and the sequence (s_n) of partial sums.

$\sum_{n\geq 1} x_n$ is convergent $\underset{\not\Leftarrow}{\Rightarrow}$ (s_n) bounded

e.g. $\sum_{n\geq 1}(-1)^n$, $s_n = \begin{cases} -1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$ $\Rightarrow$ (s_n) bounded, but not convergent

$\Rightarrow \sum_{n\geq 1}(-1)^n$ is divergent.

A series $\sum_{n\geq 1} x_n$ is with *nonnegative (positive) terms* if $\forall n \in \mathbb{N}, x_n \geq 0$ ($x_n > 0$).
Assume that $\sum_{n\geq 1} x_n$ is a series with nonnegative terms.

$\sum_{n\geq 1} x_n$ is convergent $\Leftrightarrow$ (s_n) is bounded

$s_{n+1} = s_n + x_{n+1} \geq s_n$ $\Rightarrow$ (s_n) is increasing, (s_n) bd. below

if (s_n) is bd. above $\Rightarrow \sum_{n\geq 1} x_n$ is convergent

unbounded $\Rightarrow \lim_{n\to\infty} s_n = \infty$, $\sum_{n\geq 1} x_n$ is divergent, $\sum_{n=1}^{\infty} x_n = \infty$.

Hence, series with nonnegative terms always have a sum in $[0, \infty) \cup \{\infty\}$:

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n.$$

Theorem 2 (First Comparison Test). *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be series with nonnegative terms satisfying*

$$\exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, x_n \leq y_n.$$

Then:

(i) if $\sum_{n\geq 1} y_n$ is convergent, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\sum_{n\geq 1} x_n$ is divergent, then $\sum_{n\geq 1} y_n$ is divergent.

Pf: (i) We may assume $n_0 = 1$. Denote the partial sums of the two series by

$s_n = x_1 + x_2 + \ldots + x_n$ and $\tilde{s}_n = y_1 + y_2 + \ldots + y_n$, $n \in \mathbb{N}$.

$\left.\begin{array}{l} \sum_{n\geq 1} y_n \text{ is convergent} \Rightarrow (\tilde{s}_n) \text{ is bd. above} \\ s_n \leq \tilde{s}_n, \ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow (s_n)$ is bd. above $\Rightarrow \sum_{n\geq 1} x_n$ is convergent.

(ii) is the contrapositive of (i) $\quad (p \to q \Leftrightarrow \neg q \to \neg p)$

Example 2. Let $\alpha \in \mathbb{R}$, $\alpha \leq 1$. Then $\sum_{n\geq 1} \frac{1}{n^\alpha}$ is divergent.

Take $x_n = \frac{1}{n}$, $y_n = \frac{1}{n^\alpha}$, $n \in \mathbb{N}$

$\left.\begin{array}{l} \text{Then } x_n = \frac{1}{n} \leq \frac{1}{n^\alpha} = y_n, \ \forall n \in \mathbb{N} \\ \sum_{n\geq 1} \frac{1}{n} \text{ is divergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \frac{1}{n^\alpha}$ is divergent

Theorem 3 (Second Comparison Test). *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms and $\sum_{n\geq 1} y_n$ a series with positive terms. Suppose $\exists L = \lim\limits_{n\to\infty} \dfrac{x_n}{y_n} \in [0,\infty)\cup\{\infty\}$. Then:*

(i) for $L \in (0,\infty)$: $\sum_{n\geq 1} x_n$ is convergent if and only if $\sum_{n\geq 1} y_n$ is convergent (equivalently, $\sum_{n\geq 1} x_n$ is divergent if and only if $\sum_{n\geq 1} y_n$ is divergent).

(ii) for $L = 0$: if $\sum_{n\geq 1} y_n$ is convergent, then $\sum_{n\geq 1} x_n$ is convergent (equivalently, if $\sum_{n\geq 1} x_n$ is divergent, then $\sum_{n\geq 1} y_n$ is divergent).

(iii) for $L = \infty$: if $\sum_{n\geq 1} x_n$ is convergent, then $\sum_{n\geq 1} y_n$ is convergent (equivalently, if $\sum_{n\geq 1} y_n$ is divergent, then $\sum_{n\geq 1} x_n$ is divergent).

Example 3. $\displaystyle\sum_{n\geq 1} \frac{1}{n^2 - \ln n + \sin n}$ is convergent.

Take $x_n = \dfrac{1}{n^2}$, $y_n = \dfrac{1}{n^2 - \ln n + \sin n}$, $n \in \mathbb{N}$

Then $\dfrac{x_n}{y_n} = \dfrac{n^2 - \ln n + \sin n}{n^2} \to 1 \in (0,\infty)$

$\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$ is convergent

$\overset{\text{S.C.T.}}{\Longrightarrow}$ the given series is convergent

Theorem 4 (Ratio Test, d'Alembert). *Let $\sum_{n\geq 1} x_n$ be a series with positive terms. Then:*

Particularly useful for series involving the factorial function

(i) if $\exists q \in (0,1), \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\dfrac{x_{n+1}}{x_n} \leq q$, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\dfrac{x_{n+1}}{x_n} \geq 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(iii) assuming $\exists L = \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} \in [0,\infty)\cup\{\infty\}$, we have:

(a) if $L < 1$, then $\sum_{n\geq 1} x_n$ is convergent.

(b) if $L > 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(c) if $L = 1$, the test gives no information.

Example 4. (i) $\displaystyle\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}$ is convergent.

Denote $x_n = \dfrac{(n!)^2}{(2n)!}$, $n \in \mathbb{N}$

$$\frac{x_{n+1}}{x_n} = \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{2(2n+1)} \to \frac{1}{4} < 1.$$

By the Ratio Test, the given series is convergent

(ii) $\displaystyle\sum_{n\geq 1} \frac{1}{n}$ is divergent, $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$ is convergent, yet $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}$.

Theorem 5 (Root Test, Cauchy). *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms. Then:*

Particularly useful for series where n appears as an exponent in the terms of the series

(i) if $\exists q \in [0,1), \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\sqrt[n]{x_n} \leq q$, then $\sum_{n\geq 1} x_n$ is convergent.

(ii) if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\sqrt[n]{x_n} \geq 1$, then $\sum_{n\geq 1} x_n$ is divergent.

(iii) assuming $\exists L = \lim\limits_{n\to\infty} \sqrt[n]{x_n} \in [0,\infty)\cup\{\infty\}$, we have:

(a) if $L < 1$, then $\sum_{n\geq 1} x_n$ is convergent.

(b) *if* $L > 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

(c) *if* $L = 1$, *the test gives no information.*

Example 5. $\displaystyle\sum_{n\geq 1} \frac{n^\alpha}{(1+\beta)^n}$, where $\alpha \in \mathbb{N}$ and $\beta > 0$, is convergent.

$$\sqrt[n]{\frac{n^\alpha}{(1+\beta)^n}} = \frac{(\sqrt[n]{n})^\alpha}{1+\beta} \longrightarrow \frac{1}{1+\beta} < 1.$$ By the Root Test, the given series is convergent.

Theorem 6 (Raabe's Test)**.** *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms. Then:*

(i) *if* $\exists q > 1, \exists n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0, n\left(\frac{x_n}{x_{n+1}} - 1\right) \geq q$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

(ii) *if* $\exists n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0, n\left(\frac{x_n}{x_{n+1}} - 1\right) \leq 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

(iii) *assuming* $\exists L = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) \in \overline{\mathbb{R}}$, *we have:*

(a) *if* $L > 1$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

(b) *if* $L < 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

(c) *if* $L = 1$, *the test gives no information.*

Example 6. Let $a > 0$. Then

$$\sum_{n\geq 1} \frac{n!}{a(a+1)\cdot \ldots \cdot (a+n)} = \begin{cases} \text{divergent}, & \text{if } a \in (0,1], \\ \text{convergent}, & \text{if } a > 1. \end{cases}$$

Take $x_n = \frac{n!}{a(a+1)\cdot \ldots \cdot (a+n)}$, $n \in \mathbb{N}$

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{a(a+1)\cdot \ldots \cdot (a+n+1)} \cdot \frac{a(a+1)\cdot \ldots \cdot (a+n)}{n!} = \frac{n+1}{a+n+1} \xrightarrow{n\to\infty} 1$$

$\Rightarrow$ the Ratio Test is inconclusive

$$n\left(\frac{x_n}{x_{n+1}} - 1\right) = n\left(\frac{a+n+1}{n+1} - 1\right) = n \cdot \frac{a}{n+1} \xrightarrow{n\to\infty} a$$

By Raabe's Test, $\sum_{n\geq 1} x_n$ is convergent if $a > 1$ and divergent if $a \in (0,1)$.

If $a = 1$, $x_n = \frac{1}{n+1}$ and $\sum_{n\geq 1} \frac{1}{n+1}$ is divergent.

Series with arbitrary terms

Definition 1. A series $\sum_{n\geq 1} x_n$ is called *alternating* if either

$$x_n = (-1)^{n+1}|x_n|, \forall n \in \mathbb{N} : x_1 \geq 0,\, x_2 \leq 0,\, x_3 \geq 0, \ldots$$

or

$$x_n = (-1)^n|x_n|, \forall n \in \mathbb{N} : x_1 \leq 0,\, x_2 \geq 0,\, x_3 \leq 0, \ldots$$

Example 7. (i) $\displaystyle\sum_{n\geq 1} (-1)^{n+1} \frac{n}{n+1}$ is divergent. (apply the n^{th} Term Test)

(ii) $\sum_{n\geq 1} \cos(n\pi)$ is divergent. $\quad \left(\cos(n\pi) = (-1)^n \right)$

Theorem 7 (Alternating Series Test, Leibniz)**.** *Let $\sum_{n\geq 1} x_n$ be an alternating series. If the sequence $(|x_n|)$ is decreasing, then $\sum_{n\geq 1} x_n$ is convergent if and only if $\lim_{n\to\infty} x_n = 0$.*

Definition 2. We say that a series $\sum_{n\geq 1} x_n$ is

- *absolutely convergent* if the series $\sum_{n\geq 1} |x_n|$ is convergent.
- *semi-convergent* (or *conditionally convergent*) if it is convergent, but not absolutely convergent.

Theorem 8. *Let $\sum_{n\geq 1} x_n$ be an absolutely convergent series. Then $\sum_{n\geq 1} x_n$ is convergent.*

Pf: $0 \leq x_n + |x_n| \leq 2|x_n|, \ \forall n \in \mathbb{N}$

$\sum_{n\geq 1} |x_n|$ is conv. $\Rightarrow \sum_{n\geq 1} 2|x_n|$ is conv $\overset{\text{F.C.T.}}{\Longrightarrow} \sum_{n\geq 1} (x_n + |x_n|)$ is conv.

$\sum_{n\geq 1} |x_n|$ is conv $\Rightarrow \sum_{n\geq 1} (-|x_n|)$ is conv

$\Rightarrow \sum_{n\geq 1} \underbrace{\left((x_n + |x_n|) + (-|x_n|) \right)}_{x_n}$ is conv $\Rightarrow \sum_{n\geq 1} x_n$ is conv.

Remark 2. If $\sum_{n\geq 1} x_n$ is with nonnegative terms, then absolute convergence and convergence are equivalent. However, in general, convergence does not imply absolute convergence (i.e., there exist semi-convergent series).

Example 8. *The alternating generalized harmonic series:* Let $\alpha \in \mathbb{R}$.

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^\alpha} = \begin{cases} \text{divergent}, & \text{if } \alpha \leq 0, \\ \text{semi-convergent}, & \text{if } \alpha \in (0,1], \\ \text{absolutely convergent}, & \text{if } \alpha > 1. \end{cases}$$

(apply the n^{th} Term Test)

$\left(\sum_{n\geq 1} \frac{1}{n^\alpha} \text{ is convergent} \right)$

$\alpha \in (0,1]$: • $\sum_{n\geq 1} \frac{1}{n^\alpha}$ is divergent $\Rightarrow \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^\alpha}$ is not absolutely convergent $(*)$

• $\left(\frac{1}{n^\alpha}\right)_{n\in\mathbb{N}}$ is a decreasing sequence $\left(\frac{1}{(n+1)^\alpha} < \frac{1}{n^\alpha}, \ n \in \mathbb{N} \right)$

$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$

By the Alternating Series Test, $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^\alpha}$ is convergent $(**)$

$\overset{(*),(**)}{\Longrightarrow} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^\alpha}$ is semi-convergent.