Seminar 5

(1)

Ex 1: Study if the following series are convergent or divergent:

a) $\sum_{n\geq 1} \sin n$

$(\sin n)$ is divergent. By the n^{th} Term Test, the given series is divergent.

b) $\sum_{n\geq 1} \operatorname{arctg} n$

$\lim_{n\to\infty} \operatorname{arctg} n = \frac{\pi}{2} \neq 0$. By the n^{th} Term Test, the given series is divergent

c) $\sum_{n\geq 1} \frac{5^{n/2}}{n\cdot 2^n}$

$\lim_{n\to\infty} \frac{5^{n/2}}{n\cdot 2^n} = \lim_{n\to\infty} \frac{1}{n}\cdot\left(\frac{\sqrt{5}}{2}\right)^n \overset{\frac{\sqrt{5}}{2}>1}{=} \infty$. By the n^{th} Term Test, the given series is divergent.

(2)

d) $\sum_{n\geq 1} \frac{e^n}{n+3^n}$

$$\left.\begin{array}{l} 0 \leq \frac{e^n}{n+3^n} \leq \left(\frac{e}{3}\right)^n, \forall n\in\mathbb{N} \\ \sum_{n\geq 1}\left(\frac{e}{3}\right)^n \text{ is convergent } \left(\frac{e}{3}\in(0,1)\right) \end{array}\right\} \overset{\text{F.C.T.}}{\Longrightarrow} \text{the given series is convergent}$$

e) $\sum_{n\geq 1} \frac{\sqrt{n+1}}{1+2+\dots+n}$

$\frac{\sqrt{n+1}}{1+2+\dots+n} = \frac{\sqrt{n+1}}{\frac{n(n+1)}{2}} = \frac{2}{n\sqrt{n+1}}$

$$\left.\begin{array}{l} \lim_{n\to\infty} \frac{\frac{2}{n\sqrt{n+1}}}{\frac{1}{n\sqrt{n}}} = 2 \in (0,\infty) \\ \sum_{n\geq 1}\frac{1}{n\sqrt{n}} \text{ is convergent} \end{array}\right\} \overset{\text{S.C.T.}}{\Longrightarrow} \text{the given series is convergent}$$

(3)

f) $\sum_{n\geq 1} \frac{(n+1)^n}{n^{n+2}}$

$\frac{(n+1)^n}{n^{n+2}} = \left(\frac{n+1}{n}\right)^n\cdot\frac{1}{n^2} = \left(1+\frac{1}{n}\right)^n\cdot\frac{1}{n^2}$

$$\left.\begin{array}{l} \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n\cdot\cancel{\frac{1}{n^2}}}{\cancel{\frac{1}{n^2}}} = e \in (0,\infty) \\ \sum_{n\geq 1}\frac{1}{n^2} \text{ is convergent} \end{array}\right\} \overset{\text{S.C.T.}}{\Longrightarrow} \text{the given series is convergent}$$

g) $\sum_{n\geq 1} \left(\frac{1}{2}\right)^{\ln n}$

$$\left.\begin{array}{l} \frac{1}{2^{\ln n}} = \frac{1}{n^{\ln 2}}, \forall n\in\mathbb{N} \\ \ln 2 < 1 \end{array}\right\} \Rightarrow \text{the given series is divergent}$$

$2 < e$

$\frac{1}{2} > \frac{1}{e}$

$$\left.\begin{array}{l} \left(\frac{1}{2}\right)^{\ln n} \geq \frac{1}{n}, \forall n\in\mathbb{N} \\ \sum_{n\geq 1}\frac{1}{n} \text{ is divergent} \end{array}\right\} \overset{\text{F.C.T}}{\Longrightarrow} \text{the given series is divergent}$$

(4)

h) $\sum_{n\geq 1} \frac{2^n\cdot n!}{n^n}$

$x_n = \frac{2^n\cdot n!}{n^n}, n\in\mathbb{N}$ $\quad \frac{x_{n+1}}{x_n} = \frac{2^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n\cdot n!} = 2\cdot\left(\frac{n}{n+1}\right)^n = 2\cdot\left[\left(1-\frac{1}{n+1}\right)^{-(n+1)}\right]^{-\frac{n}{n+1}} \to \frac{2}{e} < 1$

By the Ratio Test, the given series is convergent.

i) $\sum_{n\geq 1} \frac{n^2}{2^{n^2}}$

$x_n = \frac{n^2}{2^{n^2}}, n\in\mathbb{N}$ $\quad \frac{x_{n+1}}{x_n} = \frac{(n+1)^2}{2^{(n+1)^2}}\cdot\frac{2^{n^2}}{n^2} = \left(1+\frac{1}{n}\right)^2\cdot 2^{\cancel{n^2}-\cancel{n^2}-2n-1} \to 0 < 1$

By the Ratio Test, the given series is convergent.

(5)

j) $\sum_{n\geq 1} \left(1+\frac{1}{n}\right)^{-n^2}$

$x_n = \left(1+\frac{1}{n}\right)^{-n^2}, n\in\mathbb{N}$, $\quad \sqrt[n]{x_n} = \left(1+\frac{1}{n}\right)^{-n} \to \frac{1}{e} < 1$

By the Root Test, the given series is convergent.

k) $\sum_{n\geq 2} (2-\sqrt{e})(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e})$

$\forall k\in\mathbb{N}$, $e < \left(1+\frac{1}{k}\right)^{k+1}$

$\forall k\geq 2$, $e < \left(1+\frac{1}{k-1}\right)^{k}$

$\sqrt[k]{e} < 1+\frac{1}{k-1} \Rightarrow 2-\sqrt[k]{e} > 2-\left(1+\frac{1}{k-1}\right) = 1-\frac{1}{k-1} = \frac{k-2}{k-1}$

$$\left.\begin{array}{l} \forall n\geq 3,\ (2-\sqrt{e})(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e}) \geq (2-\sqrt{e})\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\ldots\cdot\frac{\cancel{n-2}}{n-1} = \frac{2-\sqrt{e}}{n-1} \\ \sum_{n\geq 2}\frac{1}{n-1} \text{ is divergent} \end{array}\right\} \overset{\text{F.C.T.}}{\Longrightarrow} \text{the given series is divergent}$$

(6)

Ex 2: $\quad x_n = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}, n\in\mathbb{N}$

Study if the following series are convergent or divergent: $\sum_{n\geq 1} x_n$, $\sum_{n\geq 1}\frac{x_n}{n}$

$x_n^2 = \frac{1^2\cdot 3^2\cdot\ldots\cdot(2n-1)^2}{2^2\cdot 4^2\cdot\ldots\cdot(2n)^2} = \underbrace{\frac{1\cdot 3}{2^2}}_{<1}\cdot\underbrace{\frac{3\cdot 5}{4^2}}_{<1}\cdot\ldots\cdot\underbrace{\frac{(2n-3)(2n-1)}{(2n-2)^2}}_{<1}\cdot\frac{2n-1}{(2n)^2} < \underbrace{\frac{2n-1}{2n}}_{<1}\cdot\frac{1}{2n} < \frac{1}{2n}$

$\forall k\in\mathbb{N}$, $\frac{k(k+2)}{(k+1)^2} = \frac{k^2+2k}{k^2+2k+1} < 1$

$0 < x_n < \frac{1}{\sqrt{2n}}, \forall n\in\mathbb{N} \Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow$ the n^{th} Term Test is inconclusive

$\forall n\in\mathbb{N}$, $x_{n+1} = x_n\cdot\frac{2n+1}{2n+2}$ $\qquad \frac{x_{n+1}}{x_n} = \frac{2n+1}{2n+2} \to 1 \Rightarrow$ the Ratio Test is inconclusive

$n\left(\frac{x_n}{x_{n+1}}-1\right) = n\left(\frac{2n+2}{2n+1}-1\right) = \frac{n}{2n+1} \to \frac{1}{2} < 1 \Rightarrow$ by Raabe's Test, the series $\sum_{n\geq 1} x_n$ is divergent

(7)

Alternatively, we can show that $x_n \geq \frac{1}{n+1}$, $\forall n\in\mathbb{N}$, using mathematical induction (HW)

Apply the F.C.T. to conclude that $\sum_{n\geq 1} x_n$ is divergent

$\sum_{n\geq 1}\frac{x_n}{n}$:

$n\left(\frac{\frac{x_n}{n}}{\frac{x_{n+1}}{n+1}} - 1\right) = n\left(\frac{x_n}{x_{n+1}}\cdot\frac{n+1}{n} - 1\right) = n\left(\frac{2n+2}{2n+1}\cdot\frac{n+1}{n} - 1\right) = \cancel{n}\cdot\frac{\cancel{2n^2}+4n+2-\cancel{2n^2}-n}{\cancel{n}(2n+1)} = \frac{3n+2}{2n+1} \to \frac{3}{2} > 1$

By Raabe's Test, the series $\sum_{n\geq 1}\frac{x_n}{n}$ is convergent.

$\forall n\in\mathbb{N}$, $0 < x_n < \frac{1}{\sqrt{2n}}$ $\quad |:n$

$$\left.\begin{array}{l} 0 < \frac{x_n}{n} < \frac{1}{\sqrt{2}}\cdot\frac{1}{n\sqrt{n}} \\ \sum_{n\geq 1}\frac{1}{n^{3/2}} \text{ is convergent} \end{array}\right\} \overset{\text{F.C.T.}}{\Longrightarrow} \sum_{n\geq 1}\frac{x_n}{n} \text{ is convergent}$$

(8)

Ex 3: Let $\sum_{n\geq 1} x_n$ be a convergent series with nonnegative terms. Study which of the following series are convergent:

a) $\sum_{n\geq 1}\frac{x_n}{1+x_n}$

$$\left.\begin{array}{l} 0 \leq \frac{x_n}{1+x_n} \leq x_n \\ \sum_{n\geq 1} x_n \text{ is conv.} \end{array}\right\} \overset{\text{F.C.T.}}{\Longrightarrow} \sum_{n\geq 1}\frac{x_n}{1+x_n} \text{ is convergent.}$$

b) $\sum_{n\geq 1} x_n^2$

$\sum_{n\geq 1} x_n$ is conv $\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \exists n_0\in\mathbb{N}$ s.t. $\forall n\in\mathbb{N}, n\geq n_0, x_n \leq 1$

$$\left.\begin{array}{l} \Rightarrow n\geq n_0,\ x_n^2 \leq x_n \\ \sum_{n\geq 1} x_n \text{ is conv.} \end{array}\right\} \overset{\text{F.C.T.}}{\Longrightarrow} \sum x_n^2 \text{ is conv.}$$

c) $\sum_{n\geq 1}\sqrt{x_n}$

$x_n = \frac{1}{n^2}, n\in\mathbb{N}$ $\qquad \sum_{n\geq 1}\frac{1}{n^2}$ is convergent

$\sqrt{x_n} = \frac{1}{n}$ $\qquad \sum_{n\geq 1}\frac{1}{n}$ is divergent

$\sum_{n\geq 1}\sqrt{x_n}$ might not be convergent.

(9)

d) $\sum_{n\geq 1}\frac{\sqrt{x_n}}{n}$

$G\left(x_n,\frac{1}{n^2}\right) \leq A\left(x_n,\frac{1}{n^2}\right)$

$$\left.\begin{array}{l} \frac{\sqrt{x_n}}{n} = \sqrt{\frac{x_n}{n^2}} \leq \frac{x_n+\frac{1}{n^2}}{2}, \forall n\in\mathbb{N} \\ \sum_{n\geq 1} x_n \text{ is conv},\ \sum_{n\geq 1}\frac{1}{n^2} \text{ is conv} \Rightarrow \sum_{n\geq 1}\frac{x_n+\frac{1}{n^2}}{2} \text{ is conv.} \end{array}\right\} \overset{\text{F.C.T.}}{\Longrightarrow} \sum_{n\geq 1}\frac{\sqrt{x_n}}{n} \text{ is convergent}$$