Homework 4

Seminar 11

Exercise 11.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$$

Study the continuity and the partial differentiablilty of f at 0_2.

Solution We have

$$\forall (x,y) \in \mathbb{R}^2 \setminus 0_2, \quad 0 \leq \left| \frac{xy}{\sqrt{x^2+y^2}} - f(0_2) \right| = \frac{|x|}{\sqrt{x^2+y^2}} |y| \leq |y|,$$

from where, using the Squeeze Theorem, we get $\lim\limits_{(x,y)\to 0_2} f(x,y) = f(0_2)$. This shows that f is continuous at 0_2.
IMPORTANT: $(x,y) \in \mathbb{R}^2 \setminus 0_2$ means ($x \neq 0$ or $y \neq 0$), so, xy might be 0. Be careful how you bound $|f(x,y) - f(0_2)|$ so that you avoid the possibility of the denominator being 0.
Since

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 0 \qquad \text{and} \qquad \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 0$$

it follows that f is partially differentiable at 0_2.

Exercise 11.2. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{2x+y}\cos(3z)$. Find the gradient and the Hessian matrix of f at $(0,0,\pi/6)$.

Solution For $(x,y,z) \in \mathbb{R}^3$,

$$\frac{\partial f}{\partial x}(x,y,z) = 2e^{2x+y}\cos(3z), \quad \frac{\partial f}{\partial y}(x,y,z) = e^{2x+y}\cos(3z), \quad \frac{\partial f}{\partial z}(x,y,z) = -3e^{2x+y}\sin(3z),$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 4e^{2x+y}\cos(3z), \quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = e^{2x+y}\cos(3z), \quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = -9e^{2x+y}\cos(3z),$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = 2e^{2x+y}\cos(3z) = \frac{\partial^2 f}{\partial x \partial y}(x,y,z), \quad \frac{\partial^2 f}{\partial z \partial x}(x,y,z) = -6e^{2x+y}\sin(3z) = \frac{\partial^2 f}{\partial x \partial z}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z \partial y}(x,y,z) = -3e^{2x+y}\sin(3z) = \frac{\partial^2 f}{\partial y \partial z}(x,y,z).$$

Then,

$$\frac{\partial f}{\partial x}(0,0,\pi/6) = 0, \quad \frac{\partial f}{\partial y}(0,0,\pi/6) = 0, \quad \frac{\partial f}{\partial z}(0,0,\pi/6) = -3,$$

so $\nabla f(0,0,\pi/6) = (0,0,-3)$. Because

$$\frac{\partial^2 f}{\partial x^2}(0,0,\pi/6) = 0, \quad \frac{\partial^2 f}{\partial y^2}(0,0,\pi/6) = 0, \quad \frac{\partial^2 f}{\partial z^2}(0,0,\pi/6) = 0,$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0,\pi/6) = 0 = \frac{\partial^2 f}{\partial x \partial y}(0,0,\pi/6), \quad \frac{\partial^2 f}{\partial z \partial x}(0,0,\pi/6) = -6 = \frac{\partial^2 f}{\partial x \partial z}(0,0,\pi/6),$$
$$\frac{\partial^2 f}{\partial z \partial y}(0,0,\pi/6) = -3 = \frac{\partial^2 f}{\partial y \partial z}(0,0,\pi/6).$$

we have

$$H_f(0,0,\pi/6) = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & -3 \\ -6 & -3 & 0 \end{pmatrix}.$$

Seminar 12

Exercise 12.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3xy$. Find the local extremum points of f and specify their type.

Solution For $(x,y) \in \mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 3y \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = 3y^2 - 3x.$$

To find the stationary points, we solve the system of equations

$$\begin{cases} y = x^2 \\ x = y^2. \end{cases}$$

Then $y = y^4$, that is, $y(1-y)(1+y+y^2) = 0$. Thus, $y \in \{0,1\}$ and we obtain the stationary points $(0,0)$ and $(1,1)$. We now classify these points. To this end, we first determine the second order partial derivatives of f. For $(x,y) \in \mathbb{R}^2$,

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 6y, \quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = -3 = \frac{\partial^2 f}{\partial x \partial y}(x,y).$$

Note that

$$H_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

is indefinite. This can be justified either because $\Delta_2 = -9 < 0$ or, directly, because for the associated quadratic form $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(h_1,h_2) = -6h_1h_2$ we have $\Phi(1,1) = -6 < 0 < 6 = \Phi(-1,1)$. Hence, $(0,0)$ is not a local extremum point of f.
Note that

$$H_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

is positive definite. This can be justified either because $\Delta_1 = 6 > 0$ and $\Delta_2 = 36 - 9 = 27 > 0$ or, directly, because the associated quadratic form $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(h_1,h_2) = 6h_1^2 - 6h_1h_2 + 6h_2^2$ satisfies $\Phi(h_1,h_2) = 6(h_1 - h_2/2)^2 + 9h_2^2/2 > 0$ for all $(h_1,h_2) \in \mathbb{R}^2 \setminus \{0_2\}$. Hence, $(1,1)$ is a local minimum point of f.

Exercise 12.2. Let $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $f(x,y) = x(y^2 + \ln^2 x)$. Find the local extremum points of f and specify their type.

Solution For $(x,y) \in (0,\infty) \times \mathbb{R}$,

$$\frac{\partial f}{\partial x}(x,y) = y^2 + \ln^2 x + 2\ln x \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = 2xy.$$

To find the stationary points, we solve the system of equations

$$\begin{cases} y^2 + \ln^2 x + 2\ln x = 0 \\ xy = 0. \end{cases}$$

Then $y = 0$, so, $\ln^2 x + 2\ln x = 0$, from where $x \in \{1, e^{-2}\}$ and we obtain the stationary points $(1, 0)$ and $(e^{-2}, 0)$. We now classify these points. To this end, we first determine the second order partial derivatives of f. For $(x, y) \in (0, \infty) \times \mathbb{R}$,

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{2}{x}\ln x + \frac{2}{x}, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2x, \quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 2y = \frac{\partial^2 f}{\partial x \partial y}(x,y).$$

Note that

$$H_f(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive definite. This can be justified either because $\Delta_1 = 2 > 0$ and $\Delta_2 = 4 > 0$ or, directly, because the associated quadratic form $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(h_1, h_2) = 2h_1^2 + 2h_2^2$ satisfies $\Phi(h_1, h_2) > 0$ for all $(h_1, h_2) \in \mathbb{R}^2 \setminus \{0_2\}$. Hence, $(1, 0)$ is a local minimum point of f.
Note that

$$H_f(e^{-2},0) = \begin{pmatrix} -2e^2 & 0 \\ 0 & 2e^{-2} \end{pmatrix}$$

is indefinite. This can be justified either because $\Delta_2 = -4 < 0$ or, directly, because for the associated quadratic form $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(h_1, h_2) = -2e^2 h_1^2 + 2e^{-2} h_2^2$ we have $\Phi(1, 0) = -2e^2 < 0 < 2e^{-2} = \Phi(0, 1)$. Hence, $(e^{-2}, 0)$ is not a local extremum point of f.

Seminar 13

Exercise 13.1. Let $\alpha, \beta \in \mathbb{R}$ and $f : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{x^\alpha \arctan x}{1 + x^\beta}$. Study the improper integrability f on its domain.

Solution First note that f is continuous and $f(x) \geq 0$ for all $x \geq 1$.
We try to find $p \in \mathbb{R}$ so that there exists $L = \lim\limits_{x\to\infty} \dfrac{x^{p+\alpha} \arctan x}{1 + x^\beta}$ $(\in (0, \infty)$ preferably).
Case 1. $\beta \leq 0$
For $p = -\alpha$,

$$L = \begin{cases} \pi/2 & \text{if } \beta < 0 \\ \pi/4 & \text{if } \beta = 0. \end{cases}$$

Hence, $L \in (0, \infty)$, so f is improperly integrable on $[1, \infty)$ if and only if $p > 1$, that is, $\alpha < -1$.

Case 2. $\beta > 0$
For $p = \beta - \alpha$, $L = \pi/4 \in (0, \infty)$. Hence, f is improperly integrable on $[1, \infty)$ if and only if $p > 1$, that is, $\alpha < \beta - 1$.