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Seminar 10

Ex 1: Let $A\subset\mathbb{R}, A\neq\emptyset$, $f:A\to\mathbb{R}$. We say that f is Lipschitz if $\exists\, L\geq 0$ s.t.

$$|f(x)-f(y)|\leq L\cdot|x-y|,\quad \forall x,y\in A.$$

The number L from above is called a Lipschitz constant for f and we say that f is L-Lipschitz if we wish to emphasize this constant.

a) Prove that if f is Lipschitz, then f is continuous.

Let $c\in A$.

Let $(x_n)\subseteq A$ with $x_n\to c$.

$\forall n\in\mathbb{N}$, $|f(x_n)-f(c)|\leq L\cdot\underbrace{|x_n-c|}_{\xrightarrow{n\to\infty} 0}$. By the Squeeze Thm, $f(x_n)\to f(c)$. Applying the Seq. Charact. of Cont., f is cont at c.

$\Rightarrow$ f is cont.

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b) If A is an interval, f is cont on A, diff on $\operatorname{int} A$ with f' bounded by $L\geq 0$, then f is L-Lipschitz.

Let $x,y\in A$.

- $x=y$: $|f(x)-f(y)|=0=L\cdot|x-y|$
- $x>y$: By MVT, $\exists\, c\in(y,x)$ s.t. $|f(x)-f(y)|=\underbrace{|f'(c)|}_{\leq L}\cdot|x-y|\leq L\cdot|x-y|$
- $x<y$: Similarly.

$\Rightarrow$ f is L-Lipschitz.

c) Prove that the function $f:[0,1]\to\mathbb{R}$, $f(x)=\sqrt{x}$ is not Lipschitz.
(Thus, $\exists$ continuous functions that are not Lipschitz.)

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Suppose that f is Lipschitz $\Rightarrow$ $\exists\, L\geq0$ s.t. $\forall x,y\in[0,1]$, $|\sqrt{x}-\sqrt{y}|\leq L|x-y|$

$\forall n\in\mathbb{N}$, $\left|\sqrt{\frac{1}{n^2}}-\sqrt{\frac{1}{(2n)^2}}\right|\leq L\cdot\left|\frac{1}{n^2}-\frac{1}{(2n)^2}\right|=L\left(\frac1n-\frac1{2n}\right)\left(\frac1n+\frac1{2n}\right)=L\cdot\frac{1}{2n}\cdot\frac{3}{2n}$

$\left|\frac1n-\frac1{2n}\right|=\frac1{2n}$

$\Rightarrow$ $\forall n\in\mathbb{N}$, $1\leq L\cdot\frac{3}{2n}$ $\qquad n\to\infty$ $\Rightarrow$ contr.

$\Rightarrow$ f is not Lipschitz

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Ex 2: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}e^{-1/x^2}, & x\neq0\\ 0, & x=0\end{cases}$

Prove that f is infinitely diff. Does the equality $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ hold for $x\neq0$?

$$\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{e^{-\frac{1}{x^2}}}{x}=\lim_{x\to0}\frac{\frac1x}{e^{\frac{1}{x^2}}}=0 \Rightarrow f \text{ is diff at } 0 \text{ and } f'(0)=0$$

$$f'(x)=\begin{cases}e^{-\frac{1}{x^2}}\cdot\frac{2}{x^3}, & x\neq0\\ 0, & x=0\end{cases}$$

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$$\lim_{x\to0}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to0}\frac{e^{-\frac{1}{x^2}}\cdot\frac{2}{x^3}}{x}=\lim_{x\to0}\frac{\frac{2}{x^4}}{e^{\frac{1}{x^2}}}=0$$

$$f''(x)=\begin{cases}e^{-\frac{1}{x^2}}\left(\frac{4}{x^6}-\frac{6}{x^4}\right), & x\neq0\\ 0, & x=0\end{cases}$$

$$f^{(n)}(x)=\begin{cases}e^{-\frac{1}{x^2}}\cdot P\left(\frac1x\right), & x\neq0\\ 0, & x=0\end{cases}$$, where P is a polynomial function

$(n\in\mathbb{N})$

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$\Rightarrow$ f is infinitely diff on $\mathbb{R}$ and $f^{(n)}(0)=0$, $\forall n\in\mathbb{N}$.

$\forall x\in\mathbb{R}$, $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=0$, but for $x\neq0$, $f(x)=e^{-1/x^2}>0$.

Ex 3: $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq0_2\\ 0, & (x,y)=0_2\end{cases}$

By Seminar 9, Ex. 3.a), we know that f is continuous.

Prove that f is partially diff., but its partial derivatives are not continuous.

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$\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{x^{\cancel{2}}\sin\frac{1}{|x|}}{\cancel{x}}=0$ $\Rightarrow$ f is part. diff w.r.t. x at 0_2, $\frac{\partial f}{\partial x}(0,0)=0$

$\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=0$ $\Rightarrow$ f is part. diff w.r.t. y at 0_2, $\frac{\partial f}{\partial y}(0,0)=0$

$\Rightarrow$ f is part. diff at 0_2.

Since f is also part diff on $\mathbb{R}^2\setminus\{0_2\}$, it follows that f is part. diff.

For $(x,y)\in\mathbb{R}^2\setminus\{0_2\}$:

$$\frac{\partial f}{\partial x}(x,y)=2x\cdot\sin\frac{1}{\sqrt{x^2+y^2}}-\cancel{(x^2+y^2)}\cos\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{\cancel{(x^2+y^2)}\sqrt{x^2+y^2}}\cdot x$$

$$=2x\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{x}{\sqrt{x^2+y^2}}\cdot\cos\frac{1}{\sqrt{x^2+y^2}}$$; $\frac{\partial f}{\partial y}(x,y)=2y\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{y}{\sqrt{x^2+y^2}}\cdot\cos\frac{1}{\sqrt{x^2+y^2}}$

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$a^k=\left(\frac1k,0\right)$, $\forall k\in\mathbb{N}$, $\quad a^k\to0_2$

$\frac{\partial f}{\partial x}(a^k)=\frac2k\cdot\sin k-\cos k$ has no limit as $k\to\infty$

$b^k=\left(0,\frac1k\right)$, $\forall k\in\mathbb{N}$, $\quad b^k\to0_2$

$\frac{\partial f}{\partial y}(b^k)=\frac2k\sin k-\cos k$ has no limit as $k\to\infty$

$\Rightarrow$ neither partial derivative is cont at 0_2.

Ex 4: $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\frac{x^3y}{x^2+y^2}, & (x,y)\neq0_2\\ 0, & (x,y)=0_2\end{cases}$

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a) Prove that f is continuously partially diff ($f\in C^1(\mathbb{R}^2)$)

$\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=0$ $\Rightarrow$ f is part. diff w.r.t. x at 0_2, $\frac{\partial f}{\partial x}(0,0)=0$

$\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=0$ $\Rightarrow$ f is part. diff w.r.t. y at 0_2, $\frac{\partial f}{\partial y}(0,0)=0$

$\Rightarrow$ f is part. diff at 0_2

Since f is also part. diff on $\mathbb{R}^2\setminus\{0_2\}$, we have that f is part. diff.

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For $(x,y)\in\mathbb{R}^2\setminus\{0_2\}$:

$$\frac{\partial f}{\partial x}(x,y)=\frac{3x^2y(x^2+y^2)-x^3y\cdot2x}{(x^2+y^2)^2}=\frac{x^2y(x^2+3y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{x^3(x^2+y^2)-x^3y\cdot2y}{(x^2+y^2)^2}=\frac{x^3(x^2-y^2)}{(x^2+y^2)^2}$$

$\forall(x,y)\in\mathbb{R}^2\setminus\{0_2\}$, $\left|\frac{\partial f}{\partial x}(x,y)-\underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0}\right|=\left|\frac{x^2y(x^2+3y^2)}{(x^2+y^2)^2}\right|=|y|\frac{\overbrace{x^4+2x^2y^2+y^4}^{(x^2+y^2)^2}+x^2y^2-y^4}{(x^2+y^2)^2}$

$$=|y|\cdot\left(1+\frac{y^2(x^2-y^2)}{(x^2+y^2)^2}\right)=|y|\cdot\left(1+\frac{y^2}{x^2+y^2}\cdot\frac{x^2-y^2}{x^2+y^2}\right)\leq2|y|\xrightarrow{(x,y)\to0_2}0$$

$\Rightarrow$ $\frac{\partial f}{\partial x}$ is cont at 0_2

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$\forall(x,y)\in\mathbb{R}^2\setminus\{0_2\}$, $\left|\frac{\partial f}{\partial y}(x,y)-\underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}\right|=\left|\frac{x^3(x^2-y^2)}{(x^2+y^2)^2}\right|=|x|\cdot\underbrace{\frac{x^2}{x^2+y^2}}_{\leq1}\cdot\underbrace{\frac{|x^2-y^2|}{x^2+y^2}}_{\leq1}\leq|x|\xrightarrow{(x,y)\to0_2}0$

$\Rightarrow$ $\frac{\partial f}{\partial y}$ is cont at 0_2

$\Rightarrow$ $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ are cont on $\mathbb{R}^2$

$\Rightarrow$ $f\in C^1(\mathbb{R}^2)$

b) Prove that f is twice partially diff w.r.t. (x,y) and w.r.t. (y,x) at 0_2, $\frac{\partial^2 f}{\partial y\partial x}(0,0)\neq\frac{\partial^2 f}{\partial x\partial y}(0,0)$

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$\lim_{y\to0}\frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y-0}=0$ $\Rightarrow$ f is twice part. diff w.r.t. (x,y) at 0_2, $\frac{\partial^2 f}{\partial y\partial x}(0,0)=0$

$\lim_{x\to0}\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x-0}=\lim_{x\to0}\frac{\frac{x^5}{x^4}}{x}=1$ $\Rightarrow$ f is twice part. diff w.r.t. (y,x) at 0_2, $\frac{\partial^2 f}{\partial x\partial y}(0,0)=1$