Seminar 11 ①

Ex 1: For the following functions $f:\mathbb{R}^2\to\mathbb{R}$, study the partial differentiability at 0_2:

a) $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq 0_2\\ 0, & (x,y)=0_2\end{cases}$

By Seminar 9, Ex. 2d, we know that f has no limit at 0_2 $\Rightarrow$ f is not cont at 0_2

$\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}=0 \to f$ is part diff w.r.t. x at 0_2 $\left(\dfrac{\partial f}{\partial x}(0,0)=0\right)$

$\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0}=0 \to f$ is part. diff w.r.t y at 0_2 $\left(\dfrac{\partial f}{\partial y}(0,0)=0\right)$

$\Rightarrow f$ is part diff at 0_2

②

b) $f(x,y)=\begin{cases}\dfrac{x^4-y^4}{2(x^4+y^4)}, & (x,y)\neq 0_2\\ 0, & (x,y)=0_2\end{cases}$

Is f cont at 0_2?

Take $a_k=(\frac{1}{k},0)$, $k\in\mathbb{N}$, $\lim\limits_{k\to\infty}a_k=0_2$

$f(a_k)=f(\frac{1}{k},0)=\frac{1}{2}\to\frac{1}{2}\neq f(0_2)$ $\Big\}\Rightarrow f$ is not cont at 0_2

$\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}}{x}=\lim\limits_{x\to 0}\dfrac{1}{2x}\ \nexists \Rightarrow f$ is not part. diff. w.r.t x at 0_2 $\Rightarrow f$ is not part diff at 0_2

$\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0}=\lim\limits_{y\to 0}\dfrac{-1}{2y}\ \nexists \Rightarrow f$ is not part. diff. w.r.t. y at 0_2.

③

c) $f(x,y)=|x|$

f is not part diff w.r.t. x at 0_2 $\Rightarrow f$ is not part diff at 0_2

f is part diff. w.r.t. y at 0_2 (in fact on $\mathbb{R}^2$ and $\dfrac{\partial f}{\partial y}=0$)

↑ f does not depend on y!

Ex 2: Let $r>0$, $f:B(0_2,r)\to\mathbb{R}$, $f(x,y)=2\ln\dfrac{r\sqrt{8}}{r^2-x^2-y^2}$

$$\boxed{\begin{gathered}(x,y)\in B(0_2,r)\\ \Updownarrow\\ x^2+y^2<r^2\\ r^2-x^2-y^2>0\end{gathered}}$$

Prove that $\forall (x,y)\in B(0_2,r)$, $\dfrac{\partial^2 f}{\partial x^2}(x,y)+\dfrac{\partial^2 f}{\partial y^2}(x,y)=e^{f(x,y)}$

For $(x,y)\in B(0_2,r)$, $\dfrac{\partial f}{\partial x}(x,y)=2\cdot\dfrac{\cancel{r^2-x^2-y^2}}{\cancel{r\sqrt{8}}}\cdot\cancel{r\sqrt{8}}\cdot\left(-\dfrac{1}{(r^2-x^2-y^2)^{\cancel{2}}}\right)\cdot(-2x)=\dfrac{4x}{r^2-x^2-y^2}$

$\dfrac{\partial f}{\partial y}(x,y)=\dfrac{4y}{r^2-x^2-y^2}$

④

$\dfrac{\partial^2 f}{\partial x^2}(x,y)=4\cdot\dfrac{r^2-x^2-y^2-x\cdot(-2x)}{(r^2-x^2-y^2)^2}=4\cdot\dfrac{r^2+x^2-y^2}{(r^2-x^2-y^2)^2}$, $\dfrac{\partial^2 f}{\partial y^2}(x,y)=4\cdot\dfrac{r^2-x^2+y^2}{(r^2-x^2-y^2)^2}$

$\dfrac{\partial^2 f}{\partial x^2}(x,y)+\dfrac{\partial^2 f}{\partial y^2}(x,y)=\dfrac{8r^2}{(r^2-x^2-y^2)^2}$

$e^{f(x,y)}=e^{2\ln\frac{r\sqrt{8}}{r^2-x^2-y^2}}=e^{\ln\frac{8r^2}{(r^2-x^2-y^2)^2}}=\dfrac{8r^2}{(r^2-x^2-y^2)^2}$

Ex 3: Let $g:\mathbb{R}^2\to\mathbb{R}$, $g(x,y)=x^2+y^2$, $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function

Prove that $\forall (x,y)\in\mathbb{R}^2$, $y\,\dfrac{\partial(f\circ g)}{\partial x}(x,y)-x\,\dfrac{\partial(f\circ g)}{\partial y}(x,y)=0$

⑤

$f\circ g:\mathbb{R}^2\to\mathbb{R}$, $(f\circ g)(x,y)=f(g(x,y))=f(x^2+y^2)$

For $(x,y)\in\mathbb{R}^2$, $\dfrac{\partial(f\circ g)}{\partial x}(x,y)=f'(x^2+y^2)\cdot 2x$, $\dfrac{\partial(f\circ g)}{\partial y}(x,y)=f'(x^2+y^2)\cdot 2y$

$y\cdot f'(x^2+y^2)\cdot 2x-x\cdot f'(x^2+y^2)\cdot 2y=0$

$$\begin{tikzcd}\mathbb{R} \arrow[r, "f"] & \mathbb{R} \\ & \mathbb{R}^2 \arrow[u, "h"'] \arrow[ul, "g"]\end{tikzcd}$$

$h=f\circ g$ $\quad$ $f:\mathbb{R}\to\mathbb{R}$

$h:\mathbb{R}^2\to\mathbb{R}$

⑥

Ex 4: Let $g:\mathbb{R}\to\mathbb{R}^2$, $g(t)=(3t,t^3)$. Suppose that $f:\mathbb{R}^2\to\mathbb{R}$ is C^1 near $(6,8)$, $\dfrac{\partial f}{\partial x}(6,8)=-5$, $\dfrac{\partial f}{\partial y}(6,8)=1$. Find $(f\circ g)'(2)$.

$g=(g_1,g_2)$, $g_1,g_2:\mathbb{R}\to\mathbb{R}$, $g_1(t)=3t$ $\quad g_2(t)=t^3$; $g_1'(t)=3$, $g_2'(t)=3t^2$

$f\circ g:\mathbb{R}\to\mathbb{R}$ $\qquad g(2)=(6,8)$

$(f\circ g)'(2)=\underbrace{\dfrac{\partial f}{\partial x}(g(2))}_{=-5}\cdot g_1'(2)+\underbrace{\dfrac{\partial f}{\partial y}(g(2))}_{=1}\cdot g_2'(2)=-5\cdot 3+1\cdot 3\cdot 2^2=-15+12=-3.$

⑦

Ex 5 Find the gradient and the Hessian matrix of the following functions at the indicated point:

a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2y^3$ at $(1,1)$

For $(x,y)\in\mathbb{R}^2$, $\dfrac{\partial f}{\partial x}(x,y)=2xy^3$ $\quad\dfrac{\partial f}{\partial y}(x,y)=3x^2y^2$

$\dfrac{\partial f}{\partial x}(1,1)=2$ $\quad\dfrac{\partial f}{\partial y}(1,1)=3$ $\quad\nabla f(1,1)=(2,3)\in\mathbb{R}^2$

$\dfrac{\partial^2 f}{\partial x^2}(x,y)=2y^3$ $\quad\dfrac{\partial^2 f}{\partial y^2}(x,y)=6x^2y$ $\quad\dfrac{\partial^2 f}{\partial y\partial x}(x,y)=6xy^2=\dfrac{\partial^2 f}{\partial x\partial y}(x,y)$

$\dfrac{\partial^2 f}{\partial x^2}(1,1)=2$ $\quad\dfrac{\partial^2 f}{\partial y^2}(1,1)=6$ $\quad\dfrac{\partial^2 f}{\partial y\partial x}(1,1)=6=\dfrac{\partial^2 f}{\partial x\partial y}(1,1)$

$H_f(1,1)=\begin{pmatrix}2 & 6\\ 6 & 6\end{pmatrix}$

⑧

b) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y,z)=e^{xyz}$ at 0_3

For $(x,y,z)\in\mathbb{R}^3$, $\dfrac{\partial f}{\partial x}(x,y,z)=e^{xyz}\cdot yz$ $\qquad\nabla f(0_3)=0_3$

$\dfrac{\partial^2 f}{\partial x^2}(x,y,z)=e^{xyz}(yz)^2$

$\dfrac{\partial^2 f}{\partial y\partial x}(x,y,z)=z\left(e^{xyz}+y\cdot e^{xyz}\cdot xz\right)$

$H_f(0_3)=0_{3,3}$ (the zero matrix)

⑨

Ex 6: Let $f,g:\mathbb{R}^n\to\mathbb{R}$ be two partially diff. functions.

Prove that $\forall c\in\mathbb{R}^n$, $\nabla(f\cdot g)(c)=f(c)\cdot\nabla g(c)+g(c)\nabla f(c)$.

$f=f(x_1,\dots,x_n)$, $g=g(x_1,\dots,x_n)$, $f\cdot g=(f\cdot g)(x_1,\dots,x_n)$

Let $c\in\mathbb{R}^n$

$\nabla(f\cdot g)(c)=\left(\dfrac{\partial(f\cdot g)}{\partial x_1}(c),\dots,\dfrac{\partial(f\cdot g)}{\partial x_n}(c)\right)$

Let $j\in\{1,\dots,n\}$. Then $\dfrac{\partial(fg)}{\partial x_j}(c)=\dfrac{\partial f}{\partial x_j}(c)\cdot g(c)+f(c)\cdot\dfrac{\partial g}{\partial x_j}(c)$

$\nabla(fg)(c)=\left(g(c)\dfrac{\partial f}{\partial x_1}(c)+f(c)\dfrac{\partial g}{\partial x_1}(c),\dots,g(c)\dfrac{\partial f}{\partial x_n}(c)+f(c)\dfrac{\partial g}{\partial x_n}(c)\right)=$

⑩

$=\left(g(c)\dfrac{\partial f}{\partial x_1}(c),\dots,g(c)\dfrac{\partial f}{\partial x_n}(c)\right)+\left(f(c)\dfrac{\partial g}{\partial x_n}(c),\dots,f(c)\dfrac{\partial g}{\partial x_n}(c)\right)$

$=g(c)\cdot\nabla f(c)+f(c)\nabla g(c)$

Determine $\nabla(fg)(0,\pi,1)$ for $f,g:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z)=xz+\cos y$, $g(x,y,z)=y\sin x-2z$

$\nabla(fg)(0,\pi,1)=f(0,\pi,1)\cdot\nabla g(0,\pi,1)+g(0,\pi,1)\cdot\nabla f(0,\pi,1)=-(\pi,0,-2)-2(1,0,0)=(-\pi-2,0,2)$

For $(x,y,z)\in\mathbb{R}^3$, $\dfrac{\partial f}{\partial x}(x,y,z)=z$, $\dfrac{\partial f}{\partial y}(x,y,z)=-\sin y$, $\dfrac{\partial f}{\partial z}(x,y,z)=x$

$\dfrac{\partial g}{\partial x}(x,y,z)=y\cos x$, $\dfrac{\partial g}{\partial y}(x,y,z)=\sin x$, $\dfrac{\partial g}{\partial z}(x,y,z)=-2$

$\nabla f(0,\pi,1)=(1,0,0)$, $\nabla g(0,\pi,1)=(\pi,0,-2)$, $f(0,\pi,1)=-1$, $g(0,\pi,1)=-2$