# Lecture 2

## Sequences of real numbers

**Definition 1.** Let $m \in \mathbb{Z}$. A *sequence* in $\mathbb{R}$ is a function $x : \{n \in \mathbb{Z} \mid n \geq m\} \to \mathbb{R}$. We usually write $x_n$ instead of $x(n)$.

Notation: $(x_n)_{n \geq m}$, $(x_n)_{n=m}^{\infty}$.
In general, we consider $m = 1$ and use the notation $(x_n)_{n \geq 1}$, $(x_n)_{n \in \mathbb{N}}$, or $(x_n)$.

**Example 1.** (i) Let $\alpha \in \mathbb{R}$, $x_n = \alpha$, $n \in \mathbb{N}$ – the sequence constantly equal to $\alpha$.

(ii) The Fibonacci sequence: $(x_n)$ defined recursively by

$$x_1 = 1, \quad x_2 = 1, \quad \text{and} \quad x_{n+1} = x_n + x_{n-1} \quad \text{for } n \in \mathbb{N}, n \geq 2.$$

**Remark 1.** A sequence $(x_n)$ should not be confused with the set of its values $\{x_n \mid n \in \mathbb{N}\}$.

$x_n = (-1)^n$, $n \in \mathbb{N}$, i.e., $(x_n) = (-1, 1, -1, 1, \ldots)$, while $\{(-1)^n \mid n \in \mathbb{N}\} = \{-1, 1\}$.

**Definition 2.** A sequence $(x_n)$ in $\mathbb{R}$ is said to be *bounded below* (*bounded above*, *bounded*, *unbounded*) if the set of its values $\{x_n \mid n \in \mathbb{N}\}$ is bounded below (bounded above, bounded, unbounded).

**Remark 2.** $(x_n)$ is:

bounded below $\iff$ $\exists \alpha \in \mathbb{R}$ s.t. $x_n \geq \alpha$, $\forall n \in \mathbb{N}$

bounded above $\iff$ $\exists \alpha \in \mathbb{R}$ s.t. $x_n \leq \alpha$, $\forall n \in \mathbb{N}$

bounded $\iff$ $\exists \alpha \in \mathbb{R}$ s.t. $|x_n| \leq \alpha$, $\forall n \in \mathbb{N}$

unbounded $\iff$ $\forall \alpha \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $|x_n| > \alpha$.

**Definition 3.** A sequence $(x_n)$ in $\mathbb{R}$ is

- *increasing (decreasing)* if $\forall n \in \mathbb{N}$, $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$).
- *strictly increasing (strictly decreasing)* if $\forall n \in \mathbb{N}$, $x_n < x_{n+1}$ ($x_n > x_{n+1}$).
- *monotone (strictly monotone)* if it is either increasing or decreasing (if it is either strictly increasing or strictly decreasing).

**Example 2.** Let $\alpha \in \mathbb{R}$ and $x_n = \alpha^n$, $n \in \mathbb{N}$.

$$(x_n) \text{ is } \begin{cases} \text{increasing for } \alpha = 0 \text{ or } \alpha \geq 1 \text{ (strictly increasing for } \alpha > 1) \\ \text{decreasing for } \alpha \in [0,1] \text{ (strictly decreasing for } \alpha \in (0,1)) \\ \text{neither increasing, nor decreasing for } \alpha < 0. \end{cases}$$

**Limit of a sequence**

**Definition 4.** A sequence $(x_n)$ in $\mathbb{R}$ is said to *have a limit* (in $\overline{\mathbb{R}}$) if there exists $x \in \overline{\mathbb{R}}$ such that

$$\forall V \in \mathcal{V}(x), \exists n_V \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq n_V \text{ we have } x_n \in V. \tag{1}$$

(every neighborhood of $x$ contains all terms of $(x_n)$ except a finite number).

**Remark 3.** A sequence in $\mathbb{R}$ cannot have two distinct limits.

If $x, y \in \overline{\mathbb{R}}$ with $x \neq y$, $\exists\, U \in \mathcal{V}(x)$, $\exists\, V \in \mathcal{V}(y)$ s.t. $U \cap V = \emptyset$

**Definition 5.** If a sequence $(x_n)$ in $\mathbb{R}$ has a limit, then the unique $x \in \overline{\mathbb{R}}$ satisfying (1) is called the *limit* of $(x_n)$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$. Sometimes we also say that $(x_n)$ *tends to* $x$.

**Proposition 1.** *Let $(x_n)$ be a sequence in $\mathbb{R}$. Then*

$$\lim_{n\to\infty} x_n = x \in \mathbb{R} \iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq n_\varepsilon \text{ we have } |x_n - x| < \varepsilon.$$

$$\lim_{n\to\infty} x_n = \infty\ (-\infty) \iff \forall a \in \mathbb{R}, \exists n_a \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq n_a \text{ we have } x_n > a\ (x_n < a).$$

Pf (Sketch) $|x_n - x| < \varepsilon \iff -\varepsilon < x_n - x < \varepsilon \iff x_n \in (x-\varepsilon, x+\varepsilon)$

$x_n > a$ $(x_n < a)$ $\iff$ $x_n \in (a, \infty)$ $(x_n \in (-\infty, a))$

**Definition 6.** A sequence $(x_n)$ in $\mathbb{R}$ is called

- *convergent* if it has a finite limit. In this case we also say that $(x_n)$ *converges* to $\lim\limits_{n\to\infty} x_n \in \mathbb{R}$.
- *divergent* if it is not convergent (i.e., it has no limit or the limit is infinite).

**Example 3.** Let $\alpha \in \mathbb{R}$ and $x_n = \alpha^n$, $n \in \mathbb{N}$.

$$(x_n) \text{ is } \begin{cases} \text{convergent for } \alpha \in (-1, 1] \\ \text{divergent for } \alpha \leq -1 \text{ or } \alpha > 1 \end{cases} \qquad \lim_{n\to\infty} x_n = \begin{cases} \nexists & \text{for } \alpha \leq -1 \\ 0 & \text{for } \alpha \in (-1,1) \\ 1 & \text{for } \alpha = 1 \\ \infty & \text{for } \alpha > 1 \end{cases}$$

**Remark 4.** For the behavior of a sequence w.r.t. its convergence/divergence, a finite number of terms of the sequence is irrelevant.

**Relation monotony - boundedness**

$\forall n \in \mathbb{N}, \begin{cases} x_n \geq x_1 \\ x_n \leq x_1 \end{cases}$

Every increasing (decreasing) sequence is bounded below (above).

**Relation convergence - boundedness**

**Theorem 1.** *Every convergent sequence $(x_n)$ in $\mathbb{R}$ is bounded.*

Pf: Let $x = \lim\limits_{n\to\infty} x_n$. Then $\exists\, n_1 \in \mathbb{N}$ s.t. $\forall n \geq n_1$, $|x_n - x| < 1$

$\Rightarrow \forall n \geq n_1$, $|x_n| \leq |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$

Take $a = \max\{|x_1|, \dots, |x_{n_1 - 1}|, 1 + |x|\}$. Then $\forall n \in \mathbb{N}$, $|x_n| \leq a$.

**Remark 5.** Bounded sequences are not always convergent. However, for monotone sequences, convergence and boundedness agree.

**Theorem 2.** *Let $(x_n)$ be a monotone sequence in $\mathbb{R}$. Then*

*(i)* *$(x_n)$ has a limit in $\overline{\mathbb{R}}$.*

*(ii)* *if $(x_n)$ is increasing, then $\lim\limits_{n\to\infty} x_n = \sup\limits_{n\in\mathbb{N}} x_n$, so $(x_n)$ is convergent if and only if $(x_n)$ is bounded above.*

*(iii)* *if $(x_n)$ is decreasing, then $\lim\limits_{n\to\infty} x_n = \inf\limits_{n\in\mathbb{N}} x_n$, so $(x_n)$ is convergent if and only if $(x_n)$ is bounded below.*

Pf: Suppose $(x_n)$ is increasing.

Case 1: $(x_n)$ bd. above

$\{x_n \mid n\in\mathbb{N}\} \neq \phi$, bd. above $\Rightarrow \exists\, x = \sup_{n\in\mathbb{N}} x_n \in \mathbb{R}$

Let $\varepsilon > 0$. Then $x-\varepsilon < x \Rightarrow x-\varepsilon \notin$ ub $(\{x_n \mid n\in\mathbb{N}\}) \Rightarrow \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $x-\varepsilon < x_{n_\varepsilon}$

$\Rightarrow \forall n \geq n_\varepsilon,\ x-\varepsilon < x_{n_\varepsilon} \leq x_n$ (since $(x_n)$ is increasing)

$\Rightarrow \forall n \geq n_\varepsilon,\ x-\varepsilon < x_n \leq x < x+\varepsilon$

$\Updownarrow$

$|x_n - x| < \varepsilon$

Case 2: $(x_n)$ not bd. above $\Rightarrow \sup_{n\in\mathbb{N}} x_n = \infty$ and $\forall a\in\mathbb{R}, \exists\, n_a \in \mathbb{N}$ s.t. $x_{n_a} > a$.

$\Rightarrow \forall a\in\mathbb{R}, \exists\, n_a\in\mathbb{N}$ s.t. $\forall n \geq n_a,\ x_n \geq x_{n_a} > a$ (since $(x_n)$ is increasing)

$\Rightarrow \forall a\in\mathbb{R}, \exists\, n_a\in\mathbb{N}$ s.t. $\forall n\geq n_a,\ x_n > a \Rightarrow \lim_{n\to\infty} x_n = \infty$

**Limit theorems**

**Proposition 2.** *Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}$, $x_n \leq y_n$.*

*(i) If $(x_n)$ and $(y_n)$ are convergent, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.*

*(ii) If $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} y_n = \infty$.*

*(iii) If $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.*

**Theorem 3** (Squeeze Theorem)**.** *Let $(x_n), (y_n)$, and $(z_n)$ be sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. Suppose that $(x_n)$ and $(z_n)$ are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l \in \mathbb{R}$. Then $(y_n)$ is also convergent and $\lim_{n\to\infty} y_n = l$.*

**Theorem 4** (Stolz-Cesàro)**.** *Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that*

*(i) $(y_n)$ is strictly increasing and $\lim_{n\to\infty} y_n = \infty$,*

*(ii) $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L \in \overline{\mathbb{R}}$.*

*Then $\lim_{n\to\infty} \frac{x_n}{y_n} = L$.*

**Example 4.** $\lim_{n\to\infty} \frac{1!+2!+\ldots+n!}{n!} = 1$.

Take $x_n = 1!+2!+\ldots+n!$, $y_n = n!$, $n\in\mathbb{N}$

$(y_n)$ strictly incr., $y_n \to \infty$

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \frac{(n+1)!}{(n+1)!-n!} = \frac{(n+1)!}{n!\,[(n+1)-1]} = \frac{n+1}{n} \to 1$$

$\overset{\text{S.-C.}}{\Rightarrow} \lim_{n\to\infty} \frac{x_n}{y_n} = 1$

**Consequences of the Stolz-Cesàro Theorem**

**Corollary 1.** *If $\lim_{n\to\infty} x_n = x \in \overline{\mathbb{R}}$, then $\lim_{n\to\infty} \frac{x_1+x_2+\ldots+x_n}{n} = x$.*

Pf: Take $a_n = x_1+x_2+\ldots+x_n$, $b_n = n$, $n\in\mathbb{N}$

$(b_n)$ strictly incr, $b_n \to \infty$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{x_{n+1}}{1} = x_{n+1} \to x$$

$\overset{\text{S.-C.}}{\Rightarrow} \frac{a_n}{b_n} \to x$, i.e., $\frac{x_1+x_2+\ldots+x_n}{n} \to x$

**Remark 6.** The converse implication in Corollary 1 does not hold.

Take $x_n = (-1)^n$, $n \in \mathbb{N}$ divergent

$$\frac{x_1+x_2+\ldots+x_n}{n} = \begin{cases} 0 & \text{for } n \text{ even} \\ -\frac{1}{n} & \text{for } n \text{ odd} \end{cases} \Rightarrow \forall n \in \mathbb{N},\ -\frac{1}{n} \leq \frac{x_1+x_2+\ldots+x_n}{n} \leq 0 \Rightarrow \lim_{n\to\infty} \frac{x_1+\ldots+x_n}{n} = 0.$$

**Corollary 2.** *If* $\forall n \in \mathbb{N}$, $x_n > 0$ *and* $\lim\limits_{n\to\infty} x_n = x \in [0,\infty) \cup \{\infty\}$, *then* $\lim\limits_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = x$.

*Proof.* Apply Corollary 1 for the sequence defined by $y_n = \ln x_n$, $n \in \mathbb{N}$. □

**Corollary 3.** *If* $\forall n \in \mathbb{N}$, $x_n > 0$ *and* $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = L \in [0,\infty) \cup \{\infty\}$, *then* $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = L$.

*Proof.* Apply Corollary 2 for the sequence defined by $y_1 = x_1$ and $y_n = \frac{x_n}{x_{n-1}}$, $n \in \mathbb{N}$, $n \geq 2$, taking into account that $\sqrt[n]{x_n} = \sqrt[n]{x_1 \cdot \frac{x_2}{x_1} \cdot \ldots \cdot \frac{x_n}{x_{n-1}}}$, $n \in \mathbb{N}$, $n \geq 2$. □

**The number $e$**

Define the sequences $e_n = \left(1+\frac{1}{n}\right)^n$ and $e'_n = \left(1+\frac{1}{n}\right)^{n+1}$ for $n \in \mathbb{N}$. Then $(e_n)$ is strictly increasing, while $(e'_n)$ is strictly decreasing:

Take $a_1 = 1$, $a_2 = \ldots = a_{n+1} = 1 + \frac{1}{n}$, where $n \in \mathbb{N}$

$G(a_1, a_2, \ldots, a_{n+1}) < A(a_1, a_2, \ldots, a_{n+1})$ (we have strict inequality because the numbers $a_i$ are not all equal)

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} < \frac{1+n\left(1+\frac{1}{n}\right)}{n+1} = \frac{1+n+1}{n+1} = 1 + \frac{1}{n+1} \Rightarrow \underbrace{\left(1+\frac{1}{n}\right)^n}_{e_n} < \underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{e_{n+1}} \Rightarrow (e_n) \text{ is strictly incr.}$$

$\left(\left(1-\frac{1}{n}\right)^n\right)$ is also strictly increasing $\left(a_1 = 1,\ a_2 = \ldots = a_{n+1} = 1 - \frac{1}{n}\right)$

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \frac{1}{\left(\frac{n}{n+1}\right)^{n+1}} = \frac{1}{\left(\frac{n+1-1}{n+1}\right)^{n+1}} = \frac{1}{\left(1-\frac{1}{n+1}\right)^{n+1}} \Rightarrow (e'_n) \text{ is strictly decreasing}$$

$\Rightarrow 2 = e_1 < e_n < e'_n < e'_1 = 4$, $\forall n \in \mathbb{N}$, $n \geq 2$

$\Rightarrow (e_n)$ converges to some real nr. btw 2 and 4. We denote its limit by $e$ (Euler's number, $e \simeq 2.71$). Note that $(e'_n)$ also converges to $e$ and we have

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}, \quad \forall n \in \mathbb{N}$$

**Remark 7.** (i) Another approach to define the number $e$ is via the series $\sum\limits_{k\geq 0} \frac{1}{k!}$. We will see that these two approaches are equivalent.

(ii) One can prove that if $(x_n)$ is a sequence in $\mathbb{R}$ such that $x_n \neq 0$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} x_n = 0$, then $\lim\limits_{n\to\infty} (1+x_n)^{\frac{1}{x_n}} = e$.

**Limit laws**

$x + \infty = \infty + x = \infty,\ \forall x \in \mathbb{R},$

$x + (-\infty) = (-\infty) + x = -\infty,\ \forall x \in \mathbb{R},$

$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty,$

$$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, \text{ if } x \in (0, \infty) \\ -\infty, \text{ if } x \in (-\infty, 0), \end{cases}$$

$$x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} -\infty, \text{ if } x \in (0, \infty) \\ \infty, \text{ if } x \in (-\infty, 0), \end{cases}$$

$\infty \cdot \infty = \infty, \quad (-\infty) \cdot (-\infty) = \infty, \quad \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty,$

$\dfrac{x}{\infty} = \dfrac{x}{-\infty} = 0,\ \forall x \in \mathbb{R},$

$\dfrac{1}{0+} = \infty, \quad \dfrac{1}{0-} = -\infty,$

$$x^{\infty} = \begin{cases} \infty, \text{ if } x \in (1, \infty) \\ 0, \text{ if } x \in [0, 1), \end{cases}$$

$$x^{-\infty} = \begin{cases} 0, \text{ if } x \in (1, \infty) \\ \infty, \text{ if } x \in (0, 1), \end{cases}$$

$$(\infty)^{x} = \begin{cases} \infty, \text{ if } x \in (0, \infty) \\ 0, \text{ if } x \in (-\infty, 0), \end{cases}$$

$\infty^{\infty} = \infty, \quad \infty^{-\infty} = 0.$

**Not defined**

$\infty + (-\infty), \quad (-\infty) + \infty,$

$0 \cdot \infty, \quad \infty \cdot 0, \quad 0 \cdot (-\infty), \quad (-\infty) \cdot 0,$

$\dfrac{\infty}{\infty}, \quad \dfrac{-\infty}{-\infty}, \quad \dfrac{\infty}{-\infty}, \quad \dfrac{-\infty}{\infty},$

$1^{\infty}, \quad 0^{0}, \quad \infty^{0}, \quad 1^{-\infty}.$

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$\left(\left(1-\frac{1}{n}\right)^n\right)$ str. increasing:

Take $a_1 = 1, \quad a_2 = \dots = a_{n+1} = 1 - \frac{1}{n}$

$G(a_1, a_2, \dots, a_{n+1}) < A(a_1, a_2, \dots, a_{n+1})$

$$\sqrt[n+1]{1 \cdot \left(1-\frac{1}{n}\right)^n} < \frac{1 + n\left(1-\frac{1}{n}\right)}{n+1} = \frac{1+n-1}{n+1} = 1 - \frac{1}{n+1}$$

$$\Rightarrow \left(1-\frac{1}{n}\right)^n < \left(1 - \frac{1}{n+1}\right)^{n+1}$$