Ex 1: abs. conv, semi-conv, or div

a) $\sum_{n\geq 1} \frac{\sin n}{n^2}$

$$\left.\begin{array}{l} \left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2},\ n\in\mathbb{N} \\ \sum_{n\geq 1}\frac{1}{n^2} \text{ is conv} \end{array}\right\} \overset{\text{E.C.T.}}{\Longrightarrow} \sum \frac{\sin n}{n^2} \text{ is abs. conv.}$$

b) $\sum_{n\geq 1} \frac{\sqrt[3]{n}}{n+1} \overbrace{\cos(n\pi)}^{(-1)^n}$

$$\left|\frac{\sqrt[3]{n}}{n+1}\cdot(-1)^n\right| = \frac{\sqrt[3]{n}}{n+1}$$

$$\left.\begin{array}{l} \lim_{n\to\infty} \dfrac{\frac{\sqrt[3]{n}}{n+1}}{\frac{1}{n^{2/3}}} = \lim_{n\to\infty} \dfrac{n}{n+1} = 1 \in (0, \infty) \\ \sum_{n\geq 1} \frac{1}{n^{2/3}} \text{ is div} \end{array}\right\} \overset{\text{SCT}}{\Longrightarrow}$$

$\Rightarrow$ the given series is not absolutely convergent.

We check if $\left(\frac{\sqrt[3]{n}}{n+1}\right)$ is decreasing:

$$\frac{\sqrt[3]{n+1}}{n+2}\cdot\frac{n+1}{\sqrt[3]{n}} = \sqrt[3]{\frac{n+1}{n}}\cdot\frac{n+1}{n+2} = \left(1+\frac{1}{n}\right)^{\frac{1}{3}}\cdot\frac{n+1}{n+2} \overset{?}{\le} 1$$

$$\left(1+\frac{1}{n}\right)^{\frac{1}{3}} \le \frac{n+2}{n+1}$$

$$1+\frac{1}{n} \le \left(1+\frac{1}{n+1}\right)^{3}$$

$$1+\frac{1}{n} \le 1+\frac{2}{n+1} \le \left(1+\frac{1}{n+1}\right)^{2} \le \left(1+\frac{1}{n+1}\right)^{3}, \quad \forall n\in\mathbb{N}.$$

$\Rightarrow \left(\frac{\sqrt[3]{n}}{n+1}\right)$ is decreasing

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}}{n+1} = 0$$

By the Alternating Series Test, the given series is convergent

$\Rightarrow$ the given series is semi-convergent

c) $\displaystyle\sum_{n\geq 1} \frac{a^n}{1+a^{2n}}$, $a\in\mathbb{R}$

$$\lim_{n\to\infty} \frac{a^n}{1+a^{2n}} = \begin{cases} 0, & a\in(-1,1)\cup(1,\infty)\cup(-\infty,-1) \\ \frac{1}{2}, & a=1 \\ \nexists, & a=-1 \end{cases}$$

$$\boxed{\lim_{n\to\infty} \frac{1+a^{2n}}{1+a^{2n+2}} = \lim_{n\to\infty} \frac{\cancel{a^{2n}}\left(\frac{1}{a^{2n}}+1\right)}{\cancel{a^{2n}}\left(\frac{1}{a^{2n}}+a^2\right)} = \frac{1}{a^2} \quad (a>1)}$$

$a=-1$, $\frac{(-1)^n}{2}$

By the n^{th} Term Test, the given series is divergent if $a\in\{-1,1\}$.

$$\left|\frac{a^n}{1+a^{2n}}\right| = \frac{|a|^n}{1+a^{2n}}$$

If $a=0$, the given is also conv.

If $a\in\mathbb{R}\setminus\{-1,0,1\}$:

$$\frac{|a|^{\cancel{n+1}}}{1+a^{2n+2}}\cdot\frac{1+a^{2n}}{\cancel{|a|^n}} = |a|\cdot\frac{1+a^{2n}}{1+a^{2n+2}} \longrightarrow \begin{cases} |a|, & a\in(-1,0)\cup(0,1) \\ \frac{|a|}{a^2}=\frac{1}{|a|}, & a\in(1,\infty)\cup(-\infty,-1) \end{cases}$$

< 1

By the Ratio Test, the given series is abs. conv. if $a \in \mathbb{R} \setminus \{-1, 0, 1\}$

$\Rightarrow$ the given series is abs. conv if $a \in \mathbb{R} \setminus \{-1, 1\}$ and it is divergent if $a \in \{-1, 1\}$.

d) $\sum_{n \geq 1} (-1)^{n+1} \cdot \frac{x_1 + x_2 + \ldots + x_n}{n}$, (x_n) decreasing, $x_1 > 0$, $x_n \geq 0$, $\forall n \in \mathbb{N}$, $n \geq 2$, $\lim_{n \to \infty} x_n = 0$

Let $y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$, $n \in \mathbb{N}$, $y_n > 0$, $\forall n \in \mathbb{N}$

$$\frac{\frac{1}{2}}{1} + \frac{\frac{1}{2}}{2} + \frac{\frac{1}{2}}{3} + \ldots + \underbrace{\frac{\frac{1}{2}}{n}}_{\to 0} + \ldots \xrightarrow{n \to \infty} \infty$$

$x_1 = \frac{1}{2}$, $x_2 = \ldots = x_n = 0$

$|(-1)^{n+1} y_n| = y_n$, $n \in \mathbb{N}$

$\lim_{n \to \infty} y_n = 0$ (apply Stolz-Cesàro)

$$\left.\begin{array}{l} y_n \geq \frac{x_1}{n}, \forall n \in \mathbb{N} \\ x_1 > 0, \sum_{n \geq 1} \frac{1}{n} \text{ div.} \end{array}\right\} \overset{FCT}{\Rightarrow} \sum_{n \geq 1} y_n \text{ is div.} \Rightarrow \text{the given series is not abs. conv.}$$

$$y_{n+1} - y_n = \frac{x_1+x_2+\dots+x_{n+1}}{n+1} - \frac{x_1+x_2+\dots+x_n}{n} =$$

$$= \frac{x_{n+1}}{n+1} + (x_1+\dots+x_n)\cdot\underbrace{\left(\frac{1}{n+1} - \frac{1}{n}\right)}_{-\frac{1}{n(n+1)}} = \frac{x_{n+1}}{n+1} - \frac{x_1+\dots+x_n}{n(n+1)} =$$

$$= \frac{n x_{n+1} - (x_1+\dots+x_n)}{n(n+1)} \leq 0, \quad \forall n \in \mathbb{N}$$

$\forall i \in \{1, \dots, n\}$, $x_i \geq x_{n+1}$ ((x_n) decr.)

$\Rightarrow x_1 + \dots + x_n \geq n \cdot x_{n+1}$

$\Rightarrow (y_n)$ is decr.

$\Rightarrow$ the given series is semi-convergent

$$\lim_{n\to\infty} y_n = 0$$

By the Alternating Series Test, the given series is convergent

Ex 2 : Find A' ;

a) $A = [1,2) \cup \{3\}$ $A' = [1,2]$

b) $A = \mathbb{Q}$ $A' = \overline{\mathbb{R}}$

c) $A = (-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$ $A' = [-\sqrt{2}, \sqrt{2}]$

Ex 3 Study the existence of the limit of Dirichlet's function

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

at every acc. point of its domain. Then study its continuity and determine the type of its discontinuities.

$\mathbb{R}' = \overline{\mathbb{R}}$

Let $c \in \mathbb{R}$. Every real nr. is the limit of a str. incr. sq of rationals and of a str. incr. sq. of irrationals.

$\exists (x_n) \subseteq \mathbb{Q}, \; x_n \to c, \; x_n < c, \; \forall n \in \mathbb{N}$

$\exists (y_n) \subseteq \mathbb{R} \setminus \mathbb{Q}, \; y_n \to c, \; y_n < c, \; \forall n \in \mathbb{N}$

$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$f(x_n) = 1, \quad f(y_n) = 0$

$\downarrow \qquad \downarrow$

$1 \qquad 0$

By the Seq. Charact. of Limits, f has no (left-hand) limit at c

$\Rightarrow f$ is discont. at c and c is a discontinuity point of the second kind

f is nowhere continuous.

$c = \infty \quad x_n = 2^n, \; y_n = 2^n + \sqrt{2}, \; n \in \mathbb{N} \quad x_n \to \infty, \; y_n \to \infty$

$f(x_n) = 1, \; f(y_n) = 0$

$\Rightarrow f$ has no limit at ∞.

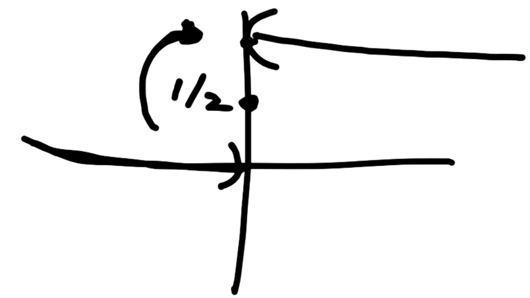
$c = -\infty$ as before

Ex 4: Find a function $f:\mathbb{R}\to\mathbb{R}$ that is nowhere cont, but $|f|$ is cont. on $\mathbb{R}$.

$$f:\mathbb{R}\to\mathbb{R}, \quad f(x)=\begin{cases} 1, & x\in\mathbb{Q} \\ -1, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Ex 5: Study the continuity of the following functions and determine the type of their discontinuities:

a) $f:\mathbb{R}\to\mathbb{R}$, $$f(x)=\lim_{n\to\infty}\frac{e^{nx}}{1+e^{nx}} = 1-\lim_{n\to\infty}\frac{1}{1+e^{nx}} = \begin{cases} 1, & x>0 \\ 0, & x<0 \\ \frac{1}{2}, & x=0 \end{cases}$$



f is cont on $(-\infty,0)\cup(0,\infty)$

0 is a jump discontinuity point

b) $g: \mathbb{R} \to \mathbb{R}$, $\quad g(x) = \begin{cases} \frac{1}{x} \cdot \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

g cont. on $\mathbb{R} \setminus \{0\}$

$x_n = \dfrac{1}{2n\pi} \overset{>0}{\longrightarrow} 0$, $\quad g(x_n) = 0 \to 0$

$y_n = \dfrac{1}{2n\pi + \frac{\pi}{2}} \overset{>0}{\longrightarrow} 0$, $\quad g(y_n) = 2n\pi + \frac{\pi}{2} \to \infty$

$\sin\left(\frac{1}{y_n}\right) = 1$

$\Rightarrow$ g has no right-hand limit at 0

$\Rightarrow$ 0 is a discontinuity point of the second kind.

Rmk: considering only the sequence (y_n) would have been enough to conclude that 0 is a discontinuity point of the second kind.

Ex 6: $a, b \in \mathbb{R}$, $a < b$, $f: [a,b] \to \underbrace{[a,b]}$ cont

$\Rightarrow \exists x_0 \in [a,b]$ s.t. $f(x_0) = x_0$

Take $g: [a,b] \to \mathbb{R}$, $g(x) = f(x) - x$

$g(a) \geq 0$, $g(b) \leq 0$

- $g(a) = 0 \Rightarrow f(a) = a$
- $g(b) = 0 \Rightarrow f(b) = b$
- $g(a) > 0$ and $g(b) < 0 \xRightarrow{\text{I.V.Thm}} \exists c \in (a,b)$ s.t. $g(c) = 0$

$\uparrow$ g cont

$\Updownarrow$

$f(c) = c$.