Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Mathematical Analysis - Homework Solutions
Computer Science, Academic Year: 2020/2021

Homework 3

Seminar 7

Exercise 7.1. Find the n^{th} derivative ($n \in \mathbb{N}$) of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x \sin x$.

Solution Using the Leibniz formula, we obtain

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \quad f^{(n)}(x) = \sum_{k=0}^{n} C_n^k e^x \sin\left(x + \frac{k\pi}{2}\right) = e^x \sum_{k=0}^{n} C_n^k \sin\left(x + \frac{k\pi}{2}\right).$$

One could further show that $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$, $f^{(n)}(x) = 2^{n/2} e^x \sin\left(x + \frac{n\pi}{4}\right)$.

Exercise 7.2. Compute the following limits:

a) $\displaystyle\lim_{x\to\infty} \frac{x + \ln x}{x \ln x}$, b) $\displaystyle\lim_{\substack{x\to 0\\ x>0}} x \ln \sin x$, c) $\displaystyle\lim_{\substack{x\to 0\\ x>0}} (\sin x)^x$.

Solution a) $\displaystyle\lim_{x\to\infty} \frac{x + \ln x}{x \ln x} = \lim_{x\to\infty} \left(\frac{1}{\ln x} + \frac{1}{x}\right) = 0.$

b) Applying L'Hôpital's rule, we get

$$\lim_{\substack{x\to 0\\ x>0}} x \ln \sin x = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln \sin x}{\frac{1}{x}} = \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}} = \lim_{\substack{x\to 0\\ x>0}} \frac{-x}{\sin x} x \cos x = 0.$$

c) As $\displaystyle\lim_{\substack{x\to 0\\ x>0}} \ln(\sin x)^x = \lim_{\substack{x\to 0\\ x>0}} x \ln \sin x = 0$, we have $\displaystyle\lim_{\substack{x\to 0\\ x>0}} (\sin x)^x = \lim_{\substack{x\to 0\\ x>0}} e^{\ln(\sin x)^x} = 1$.

Exercise 7.3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 5x + 1$. Find the third Taylor polynomial $T_3(x)$ of f at 1.

Solution Because f is itself a polynomial function of degree 3, it follows by Lecture 6, Remark 5, that $T_3(x) = f(x)$, $\forall x \in \mathbb{R}$. Note also that one can write $f(x) = 4 + 2(x-1) + (x-1)^3$, $\forall x \in \mathbb{R}$.

Seminar 8

Exercise 8.1. Prove that the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = 1/x^2$, can be expanded as a Taylor series around 1 on $[1,2)$ and find the corresponding Taylor series expansion.

Solution One shows by induction that $f^{(k)}(x) = (-1)^k \dfrac{(k+1)!}{x^{k+2}}$, $\forall k \in \mathbb{N}$, $\forall x > 0$. Note that $f(1) = 1$ and $f^{(k)}(1) = (-1)^k (k+1)!$, $\forall k \in \mathbb{N}$.
Let $n \in \mathbb{N}$. The n^{th} Taylor polynomial of f at 1 is $T_n : \mathbb{R} \to \mathbb{R}$,

$$T_n(x) = 1 - 2(x-1) + \ldots + (-1)^n (n+1)(x-1)^n.$$

Let $x \in [1,2)$. Applying the Taylor-Lagrange Theorem, we find a number c between 1 and x such that $\dfrac{1}{x^2} = T_n(x) + R_n(x)$, where $R_n(x) = \dfrac{(-1)^{n+1}(n+2)!}{c^{n+3}(n+1)!}(x-1)^{n+1} = \dfrac{(-1)^{n+1}(n+2)}{c^{n+3}}(x-1)^{n+1}$.
IMPORTANT: Note that c may also depend on n.

Then, since $c \geq 1$, we have

$$|R_n(x)| = \frac{n+2}{c^{n+3}}(x-1)^{n+1} \leq (n+2)(x-1)^{n+1}.$$

Because $x - 1 \in [0,1)$, it follows that $\lim_{n\to\infty}(n+2)(x-1)^{n+1} = 0$. By the Squeeze Theorem, $\lim_{n\to\infty} R_n(x) = 0$.
Thus, f can be expanded as a Taylor series around 1 on $[1,2)$ and

$$\frac{1}{x^2} = 1 + \sum_{n=1}^{\infty}(-1)^n(n+1)(x-1)^n, \quad \forall x \in [1,2).$$

Exercise 8.2. Let $z \in \mathbb{R}^n$, $r > 0$, and $\varepsilon \in (0,2]$. Prove that if $x, y \in \overline{B}(z,r)$ such that $\|x-y\| \geq \varepsilon r$, then $\left\|z - \dfrac{x+y}{2}\right\| \leq r\sqrt{1 - \dfrac{\varepsilon^2}{4}}$.

Solution Applying the parallelogram law for the points $z-x$ and $z-y$, we obtain

$$\|2z-(x+y)\|^2 + \|x-y\|^2 = 2(\|z-x\|^2 + \|z-y\|^2) \leq 2(r^2+r^2) = 4r^2.$$

Now using the fact that $\|x-y\| \geq \varepsilon r$, we get

$$4\left\|z - \frac{x+y}{2}\right\|^2 = \|2z-(x+y)\|^2 \leq (4-\varepsilon^2)r^2.$$

The desired inequality follows after dividing by 4 and then taking the square root.

Seminar 9

Exercise 9.1. In each the following cases, study if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at 0_2:

a) $f(x,y) = \begin{cases} \dfrac{xy + x^2 y \ln(x^2+y^2)}{x^2+y^2}, & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2, \end{cases}$

b) $f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$

Solution a) Consider the sequence (a^k) defined by $a^k = \left(\dfrac{1}{k}, \dfrac{1}{k}\right)$, $k \in \mathbb{N}$. Note that $\lim_{k\to\infty} a^k = 0_2$ and

$$f(a^k) = \frac{\dfrac{1}{k^2} + \dfrac{1}{k^3}\ln\dfrac{2}{k^2}}{\dfrac{1}{k^2} + \dfrac{1}{k^2}} = \frac{1}{2} + \frac{\ln 2 - 2\ln k}{2k}.$$

Then $\lim_{k\to\infty} f(a^k) = 1/2 \neq f(0_2)$, so f is not continuous at 0_2.

b) Let $x, y \in \mathbb{R}$. Then $x^4 + y^4 = (x^2+y^2)^2 - 2x^2y^2 \geq (x^2+y^2)^2 - x^4 - y^4$, so $x^4+y^4 \geq (x^2+y^2)^2/2$. Using this inequality, we get

$$\forall (x,y) \in \mathbb{R}^2 \setminus 0_2, \quad 0 \leq \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} \leq \frac{2e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2}.$$

As $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{2e^{-\frac{1}{t}}}{t^2} = 0$, one can deduce that $\lim\limits_{(x,y)\to 0_2} \dfrac{2e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2} = 0$, so by the Squeeze Theorem, $\lim\limits_{(x,y)\to 0_2} f(x,y) = 0 = f(0_2)$. This shows that f is continuous at 0_2.

IMPORTANT – Only for the students who learned on their own how to pass to polar coordinates and decided to use this method. This is not a topic covered in class, you don't need to know this for the exam. –

Passing to polar coordinates needs to be done with care! It is not correct to conclude that, after writing $f(x,y) = f(r\cos\alpha, r\sin\alpha)$, if $\lim\limits_{r\to 0} f(r\cos\alpha, r\sin\alpha) = 0$ for all α, then $\lim\limits_{(x,y)\to 0_2} f(x,y) = 0$.
As a counterexample, consider Example 1 in Lecture 8.
However, one can conclude the following:

- If for two distinct values of α one obtains different values for $\lim\limits_{r\to 0} f(r\cos\alpha, r\sin\alpha)$, then f has no limit at 0_2.
- If $f(r\cos\alpha, r\sin\alpha) = g(r)h(\alpha)$, where g, h are functions of one real variable such that h is bounded and $\lim\limits_{r\to 0} g(r) = 0$, then $\lim\limits_{(x,y)\to 0_2} f(x,y) = 0$.

Exercise 9.2. Find the second order partial derivatives of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(x\sin y)$, b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (1+x^2)ye^z$.

Solution a) For $(x,y) \in \mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x,y) = \cos(x\sin y)\sin y, \quad \frac{\partial f}{\partial y}(x,y) = \cos(x\sin y)x\cos y,$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin(x\sin y)\sin^2 y, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = -x\left(x\sin(x\sin y)\cos^2 y + \cos(x\sin y)\sin y\right),$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y) = \cos y\left(\cos(x\sin y) - x\sin y\sin(x\sin y)\right) = \frac{\partial^2 f}{\partial x\partial y}(x,y).$$

b) For $(x,y,z) \in \mathbb{R}^3$,

$$\frac{\partial f}{\partial x}(x,y,z) = 2xye^z, \quad \frac{\partial f}{\partial y}(x,y,z) = (1+x^2)e^z, \quad \frac{\partial f}{\partial z}(x,y,z) = (1+x^2)ye^z,$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2ye^z, \quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = (1+x^2)ye^z,$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y,z) = 2xe^z = \frac{\partial^2 f}{\partial x\partial y}(x,y,z), \quad \frac{\partial^2 f}{\partial z\partial x}(x,y,z) = 2xye^z = \frac{\partial^2 f}{\partial x\partial z}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z\partial y}(x,y,z) = (1+x^2)e^z = \frac{\partial^2 f}{\partial y\partial z}(x,y,z).$$