Homework Assignments

- Homework 1 (due October 19, 2020, 8 pm): 2 exercises out of 1.1-1.3, 2 exercises out of 2.1-2.3, exercise 3.1, and 1 exercise out of 3.2-3.3.
- Homework 2 (due November 9, 2020, 8 pm): 1 exercise out of 4.1-4.2, exercise 4.3, exercises 5.1-5.2, exercise 6.1, and 1 exercise out of 6.2-6.3.
- Homework 3 (due December 7, 2020, 8 pm): exercises 7.1-7.3, exercises 8.1-8.2, and exercises 9.1-9.2.
- Homework 4 (due January 11, 2021, 8 pm): exercises 11.1-11.2, 1 exercise out of 12.1-12.2, and exercise 13.1.

Exercises are added to homework assignments on a weekly basis. The solutions to the exercises corresponding to a homework assignment will be published after the deadline.

Submission instructions
The solutions can be written by hand, scanned, and then sent by e-mail to the following addresses according to your group (repeaters should send the file to the address corresponding to the group they chose to attend the seminar with):

- 911 and 915: *lorinczi@math.ubbcluj.ro*
- 912, 913, 914, and 917: *examen.analiza.reala@gmail.com*
- 916: *adrian.viorel2020@gmail.com*

IMPORTANT:

- E-mail subject: HWi - Surname Forename - Group. (Note that $i \in \{1, 2, 3, 4\}$.)
- File name: Surname-Forename-Group.pdf.
- Do not include more exercises than required.
- Late submissions are not accepted.
- Send a single pdf file (not a link to the file, attach it). If you forget to attach the file or attach a wrong or an incomplete file, you will not be allowed to resubmit the homework after the deadline. It is the student's responsibility to attach the correct file.

Seminar 1

Exercise 1.1. Prove that for every $n \in \mathbb{N}$, $n \geq 2$, we have $\displaystyle\sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}$.

Exercise 1.2. Let $x > 0$ and $n \in \mathbb{N}$. Use the inequality $G(x_1, \ldots, x_m) \leq A(x_1, \ldots, x_m)$ for some appropriate choice of $m \in \mathbb{N}$ and of real numbers $x_1, \ldots x_m > 0$ to deduce that:

a) $\dfrac{x^n}{1 + x + \ldots + x^{2n}} \leq \dfrac{1}{2n+1}$;

b) $1 + (n+1)x \leq (1+x)^{n+1}$.

Exercise 1.3. Prove that for every $n \in \mathbb{N}$ with $n \geq 2$ and for any numbers $x_1, x_2, \ldots, x_n \in [-1, \infty)$ all of the same sign, we have

$$(1 + x_1)(1 + x_2) \cdot \ldots \cdot (1 + x_n) \geq 1 + x_1 + x_2 + \ldots + x_n \quad \text{(the generalized Bernoulli inequality).}$$

Seminar 2

Exercise 2.1. For each set A_i from below find $\text{lb}(A_i)$ and $\text{ub}(A_i)$ (as subsets of $\mathbb{R}$), $\min(A_i)$ and $\max(A_i)$ (if they exist), and $\inf(A_i)$ and $\sup(A_i)$ (in $\overline{\mathbb{R}}$):

$$A_1 = [-8, \pi) \cap \mathbb{Z}, \qquad A_3 = \left\{ x + \frac{1}{x} \mid x \in \mathbb{R}, x < 0 \right\},$$
$$A_2 = \{2^m + n! \mid m, n \in \mathbb{N}\}, \qquad A_4 = \left\{ \frac{n}{1 - n^2} \mid n \in \mathbb{N}, n \geq 2 \right\}.$$

Exercise 2.2. Find two sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that the following conditions are simultaneously met:

(i) one of the sets is unbounded (but not an interval) and the other is finite;

(ii) $\sup A = \inf B = 2 \in A$;

(iii) for every $a \in A$ and every $b \in B$, there exists $c \in \mathbb{R}$ with $a < c < b$.

Is it possible to choose B the finite set?

Exercise 2.3. Decide which of the following sets are neighborhoods of 0. Justify.

$$A_1 = (-1, 0] \cup \{1\}, \qquad A_3 = \mathbb{R},$$
$$A_2 = \left[1 - \frac{3}{2}, 1 + \frac{3}{2}\right] \cup (3, 4), \qquad A_4 = \mathbb{R} \setminus \mathbb{Q}.$$

Seminar 3

Exercise 3.1. Find the limit (as $n \to \infty$) of the sequence whose general term x_n, $n \in \mathbb{N}$, is given below:

a) $\dfrac{n + \sin(n^2)}{\cos(n) - 3n}$, b) $(n^2 + n)^{-\frac{n}{n+1}}$, c) $\left(1 + \dfrac{1}{n^3 + 2n^2}\right)^{n - n^3}$, d) $\dfrac{1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!}{(n+1)!}$,

e) $\sqrt[n]{1 + 2 + \ldots + n}$, f) $n\left(\left(1 + \frac{1}{n}\right)^{1 + \frac{1}{n}} - 1\right)$.

Exercise 3.2. For $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ such that $a_n \leq b_n$ and $\lim\limits_{n \to \infty} (b_n - a_n) = 0$. Suppose, in addition, that $\forall n \in \mathbb{N}$, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. By the Nested Interval Property, $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$. Can $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$ contain more than one point?

Exercise 3.3. Let (x_n) be a sequence in $\mathbb{Z}$. If (x_n) is convergent, is it eventually constant (i.e., $\exists n_0 \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$ with $m, n \geq n_0$ we have $x_m = x_n$)?

Seminar 4

Exercise 4.1. Define the sequence (x_n) by $x_1 \in (0, 1)$ and $x_{n+1} = x_n - x_n^2$, $n \in \mathbb{N}$. Prove that the sequence (x_n) converges and find its limit. Then study the convergence of the sequence $(n \cdot x_n)$ (for this, one could apply the Stolz-Cesàro Theorem for the sequences (a_n) and (b_n) defined, for $n \in \mathbb{N}$, by $a_n = n$ and $b_n = 1/x_n$, respectively).

Exercise 4.2 (Koch snowflake)**.** Define a sequence (S_n) of polygons such that S_1 is an equilateral triangle of side length 1 and, for $n \in \mathbb{N}$, S_{n+1} is obtained from S_n by adding to the middle third of each side an equilateral triangle pointing outwards (and removing this middle third). Denote by a_n the area of S_n. Determine the sequence (a_n) and study if it is convergent.

Exercise 4.3. Find the sum of the following series and specify whether they are convergent or divergent:
a) $\sum_{n\geq 1} \left(-\frac{\pi}{4}\right)^n$, b) $\sum_{n\geq 0} \frac{2^{3n}}{5^{n-1}}$, c) $\sum_{n\geq 1} \frac{1}{4n^2-1}$, d) $\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$, e) $\sum_{n\geq 1} \frac{3n-2}{2^n}$.

Seminar 5

Exercise 5.1. Study if the following series are convergent or divergent:
a) $\sum_{n\geq 1} \left(1-\frac{1}{n}\right)^n$, b) $\sum_{n\geq 1} \sin\frac{1}{n^{5/4}}$, c) $\sum_{n\geq 1} \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$, d) $\sum_{n\geq 1} \frac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}$, e) $\sum_{n\geq 1} \frac{n^3 5^n}{2^{3n+1}}$,
f) $\sum_{n\geq 1} \frac{2\cdot 5\cdot \ldots \cdot (3n-1)}{3\cdot 6\cdot \ldots \cdot (3n)}$.

Exercise 5.2. Let (x_n) and (y_n) be two sequences of positive numbers. Suppose that the series $\sum_{n\geq 1} \frac{x_n}{y_n}$ and $\sum_{n\geq 1} y_n$ are both convergent. Is the series $\sum_{n\geq 1} \sqrt{x_n}$ convergent as well?

Seminar 6

Exercise 6.1. Study if the following series are absolutely convergent, semi-convergent or divergent:
a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n+1}}$, b) $\sum_{n\geq 1} \frac{n}{n^2+1}\cos(n\pi)$.

Exercise 6.2. Let $f, g : [0,1] \to \mathbb{R}$ be continuous such that $f(x) = g(x)$, $\forall x \in [0,1]\cap\mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0,1]$. Is it enough solely to assume that f and g are continuous on $[0,1] \setminus \{\alpha\}$ for some $\alpha \in [0,1] \cap (\mathbb{R}\setminus\mathbb{Q})$?

Exercise 6.3. Find all continuous functions $f : \mathbb{R} \to \mathbb{N}$ and all continuous functions $f : \mathbb{N} \to \mathbb{R}$.

Seminar 7

Exercise 7.1. Find the n^{th} derivative $(n \in \mathbb{N})$ of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x \sin x$.

Exercise 7.2. Compute the following limits:
a) $\lim_{x\to\infty} \frac{x+\ln x}{x\ln x}$, b) $\lim_{\substack{x\to 0\\ x>0}} x\ln\sin x$, c) $\lim_{\substack{x\to 0\\ x>0}} (\sin x)^x$.

Exercise 7.3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 5x + 1$. Find the third Taylor polynomial $T_3(x)$ of f at 1.

Seminar 8

Exercise 8.1. Prove that the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = 1/x^2$, can be expanded as a Taylor series around 1 on $[1,2)$ and find the corresponding Taylor series expansion.

Exercise 8.2. Let $z \in \mathbb{R}^n$, $r > 0$, and $\varepsilon \in (0,2]$. Prove that if $x, y \in \overline{B}(z,r)$ such that $\|x-y\| \geq \varepsilon r$, then $\left\|z - \frac{x+y}{2}\right\| \leq r\sqrt{1-\frac{\varepsilon^2}{4}}$.

Seminar 9

Exercise 9.1. In each the following cases, study if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at 0_2:

a) $f(x,y) = \begin{cases} \dfrac{xy + x^2 y \ln(x^2+y^2)}{x^2+y^2}, & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2, \end{cases}$

b) $f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$

(One could show first that $x^4 + y^4 \geq (x^2+y^2)^2/2$ for all $x, y \in \mathbb{R}$.)

Exercise 9.2. Find the second order partial derivatives of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(x \sin y)$, b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (1+x^2)ye^z$.

Seminar 10

No homework.

Seminar 11

Exercise 11.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$$

Study the continuity and the partial differentiablilty of f at 0_2.

Exercise 11.2. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{2x+y}\cos(3z)$. Find the gradient and the Hessian matrix of f at $(0, 0, \pi/6)$.

Seminar 12

Exercise 12.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3xy$. Find the local extremum points of f and specify their type.

Exercise 12.2. Let $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $f(x,y) = x(y^2 + \ln^2 x)$. Find the local extremum points of f and specify their type.

Seminar 13

Exercise 13.1. Let $\alpha, \beta \in \mathbb{R}$ and $f : [1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{x^\alpha \arctan x}{1+x^\beta}$. Study the improper integrability f on its domain.