Seminar 1

We use the following notation: $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Principle of Mathematical Induction: Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a property defined for any $n \in \mathbb{N}$, $n \geq n_0$. Suppose that:

i) $P(n_0)$ is true;

ii) if $P(k)$ is true for some $k \in \mathbb{N}$, $k \geq n_0$, then $P(k+1)$ is also true.

Then $P(n)$ is true, $\forall n \in \mathbb{N}$, $n \geq n_0$.

Exercise 1. Prove that for every $n \in \mathbb{N}$, $n \geq 4$, we have $n! \geq 2^n$.

Solution We use mathematical induction. Consider the following statement for $n \in \mathbb{N}$ with $n \geq 4$:

$$P(n) : \text{“}\, n! \geq 2^n \,\text{”}.$$

$P(4)$ is clearly valid since $4! = 24 \geq 16 = 2^4$. Now let $k \in \mathbb{N}$ such that $k \geq 4$ and assume that $P(k)$ is valid. Then

$$(k+1)! = k!(k+1) \geq 2^k(k+1) \geq 2^{k+1},$$

so $P(k+1)$ is valid as well. We can conclude that $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$.

Alternatively, note that for $n \in \mathbb{N}$, $n \geq 4$,

$$(n-1)! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \geq 1 \cdot 2 \cdot \ldots \cdot 2 = 2^{n-2},$$

from where $n! = (n-1)! \cdot n \geq 2^{n-2} \cdot 2^2 = 2^n$.

Exercise 2. Prove that for every $n \in \mathbb{N}$ we have $4 \displaystyle\sum_{m=1}^{n} m^3 = n^2(n+1)^2$.

Solution We prove the identity using mathematical induction. Consider the following statement for $n \in \mathbb{N}$:

$$P(n) : \text{“}\, 4 \sum_{m=1}^{n} m^3 = n^2(n+1)^2 \,\text{”}.$$

One can easily verify that $P(1)$ is true. Now let $k \in \mathbb{N}$ and assume that $P(k)$ is valid. Then

$$4 \sum_{m=1}^{k+1} m^3 = 4(k+1)^3 + 4 \sum_{m=1}^{k} m^3 = 4(k+1)^3 + k^2(k+1)^2 = (k+1)^2(4(k+1)+k^2) = (k+1)^2(k+2)^2,$$

hence $P(k+1)$ is valid as well. We can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Exercise 3. Prove that for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $m_n^2 \leq n < (m_n+1)^2$.

Solution We use mathematical induction. Consider the following statement for $n \in \mathbb{N}$:

$$P(n) : \text{“}\, \exists m_n \in \mathbb{N} \text{ such that } m_n^2 \leq n < (m_n+1)^2 \,\text{”}.$$

For $n = 1$ we take $m_1 = 1$, so $P(1)$ is true. Now let $k \in \mathbb{N}$ and assume that $P(k)$ is valid. We distinguish the following two cases:

(i) $k+1<(m_k+1)^2$: Take $m_{k+1}=m_k$. Then

$$m_{k+1}^2=m_k^2\le k<k+1<(m_k+1)^2=(m_{k+1}+1)^2;$$

(ii) $k+1\ge(m_k+1)^2$: Take $m_{k+1}=m_k+1$. Then

$$m_{k+1}^2=(m_k+1)^2\le k+1<(m_k+1)^2+1=m_{k+1}^2+1<(m_{k+1}+1)^2.$$

Thus, $P(k+1)$ is also true. We can conclude that $P(n)$ is true for all $n\in\mathbb{N}$.

Exercise 4. Prove that for every $n\in\mathbb{N}$ with $n\ge 2$ and for any real numbers $a_1,a_2,\ldots,a_n>0$ satisfying $a_1\cdot a_2\cdot\ldots\cdot a_n=1$, we have $a_1+a_2+\ldots+a_n\ge n$.

Solution We use mathematical induction. Consider the following statement for $n\in\mathbb{N}$ with $n\ge 2$:

$$P(n): \text{"}\forall a_1,a_2,\ldots,a_n>0 \text{ such that } a_1\cdot a_2\cdot\ldots\cdot a_n=1 \text{ we have } a_1+a_2+\ldots+a_n\ge n\text{"}.$$

We verify that $P(2)$ is true. Let $a_1,a_2>0$ with $a_1\cdot a_2=1$. Then $a_2=1/a_1$, so

$$a_1+a_2-2=a_1+\frac{1}{a_1}-2=\frac{a_1^2+1-2a_1}{a_1}=\frac{(a_1-1)^2}{a_1}\ge 0.$$

Now let $k\in\mathbb{N}$ such that $k\ge 2$ and assume that $P(k)$ is valid. Let $a_1,a_2,\ldots,a_{k+1}>0$ such that $a_1\cdot a_2\cdot\ldots\cdot a_{k+1}=1$. We can suppose that $a_1\le a_2\le\ldots\le a_{k+1}$ (otherwise we reorder these numbers). Then

$$a_1\le 1 \quad\text{and}\quad a_{k+1}\ge 1. \tag{1}$$

Note that $a_1\cdot a_2\cdot\ldots\cdot a_{k+1}=a_2\cdot\ldots\cdot a_k\cdot(a_{k+1}\cdot a_1)=1$ and we can apply $P(k)$ for the k numbers $a_2,\ldots,a_k,a_{k+1}\cdot a_1$ to get that $a_2+\ldots+a_k+a_{k+1}\cdot a_1\ge k$. Therefore,

$$\begin{aligned}
a_1+a_2+\ldots+a_{k+1}&=a_2+\ldots+a_k+a_{k+1}\cdot a_1-a_{k+1}\cdot a_1+a_{k+1}+a_1\\
&\ge k+a_{k+1}(1-a_1)+a_1-1+1\\
&=k+1+(1-a_1)(a_{k+1}-1)\\
&\ge k+1 \quad\text{by (1)}.
\end{aligned}$$

Thus, $P(k+1)$ is also true. We can conclude that $P(n)$ is true for all $n\in\mathbb{N}$, $n\ge 2$.

Exercise 5. Given $n\in\mathbb{N}$ with $n\ge 2$ and the real numbers $x_1,x_2,\ldots,x_n>0$, denote

$$\begin{aligned}
H(x_1,\ldots,x_n)&=\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}} &&\text{(the harmonic mean)},\\
G(x_1,\ldots,x_n)&=\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n} &&\text{(the geometric mean)},\\
A(x_1,\ldots,x_n)&=\frac{x_1+x_2+\ldots+x_n}{n} &&\text{(the arithmetic mean)}.
\end{aligned}$$

Prove that $H(x_1,\ldots,x_n)\le G(x_1,\ldots,x_n)\le A(x_1,\ldots,x_n)$.

Solution We apply the previous exercise taking $a_i=\dfrac{x_i}{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}}$, where $i\in\{1,\ldots,n\}$. Note that

$$a_1\cdot a_2\cdot\ldots\cdot a_n=\frac{x_1}{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}}\cdot\frac{x_2}{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}}\cdot\ldots\cdot\frac{x_n}{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}}=1.$$

Thus,

$$\frac{x_1+x_2+\ldots+x_n}{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}}=a_1+a_2+\ldots+a_n\ge n.$$

This means that $\dfrac{x_1+x_2+\ldots+x_n}{n} \geq \sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}$, i.e., $G(x_1,\ldots,x_n)\leq A(x_1,\ldots,x_n)$. We apply now this inequality for $1/x_i$ instead of x_i, where $i\in\{1,\ldots,n\}$. Then

$$\frac{1}{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}} = G(1/x_1,\ldots,1/x_n) \leq A(1/x_1,\ldots,1/x_n) = \frac{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}{n}.$$

Inverting the above fractions we obtain $H(x_1,\ldots,x_n)\leq G(x_1,\ldots,x_n)$.

Exercise 6. Prove that:

a) for every $x\in[-1,\infty)$ and every $n\in\mathbb{N}$ we have

$$(1+x)^n \geq 1+nx \quad \text{(Bernoulli's inequality);}$$

b) for every $y\in\mathbb{R}$ and every even $m\in\mathbb{N}$ we have $(1+y)^m\geq 1+my$.

Solution a) We use mathematical induction. Consider the following statement for $n\in\mathbb{N}$:

$$P(n): \text{“}\forall x\in[-1,\infty) \text{ we have } (1+x)^n\geq 1+nx\text{”}.$$

The fact that $P(1)$ is true is immediate. Now let $k\in\mathbb{N}$ and assume that $P(k)$ is valid. Let $x\in[-1,\infty)$. Then

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1+(k+1)x+kx^2 \geq 1+(k+1)x.$$

Thus, $P(k+1)$ is also true. We can conclude that $P(n)$ is true for all $n\in\mathbb{N}$.

b) Let $y\in\mathbb{R}$ and $m\in\mathbb{N}$ be even. Then there exists $n\in\mathbb{N}$ such that $m=2n$ and we have

$$\begin{aligned}(1+y)^m = \left[(1+y)^2\right]^n = \left(1+2y+y^2\right)^n &\geq 1+n(2y+y^2) \quad \text{using Bernoulli's inequality}\\ &\geq 1+2ny\\ &= 1+my.\end{aligned}$$

Recall that the *absolute value* of $x\in\mathbb{R}$, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0.\end{cases}$$

Exercise 7. Let $x,y\in\mathbb{R}$. Prove that:

a) $|x+y|\leq|x|+|y|$ (the triangle inequality);

b) $||x|-|y||\leq|x-y|$.

Solution a) Follows from $(x+y)^2 = x^2+y^2+2xy \leq x^2+y^2+2|x||y| = (|x|+|y|)^2$.

b) Using item a), we get $|x|-|y| = |x-y+y|-|y| \leq |x-y|+|y|-|y| = |x-y|$. Switching the roles of x and y we have $|y|-|x|\leq|y-x|=|x-y|$. Hence, $||x|-|y||\leq|x-y|$.