Homework 1

Seminar 1

Exercise 1.1. Prove that for every $n \in \mathbb{N}$, $n \geq 2$, we have $\displaystyle\sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}$.

Solution We use mathematical induction. Consider the following statement for $n \in \mathbb{N}$ with $n \geq 2$:

$$P(n) : \text{“} \sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n} \text{”}.$$

One can easily verify that $P(2)$ is true. Now let $k \in \mathbb{N}$ such that $k \geq 2$ and assume that $P(k)$ is valid. Then

$$\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}} = \frac{1}{\sqrt{k+1}} + \sum_{m=1}^{k} \frac{1}{\sqrt{m}} > \frac{1}{\sqrt{k+1}} + \sqrt{k} = \frac{1+\sqrt{k(k+1)}}{\sqrt{k+1}} > \frac{1+k}{\sqrt{k+1}} = \sqrt{k+1},$$

hence $P(k+1)$ is valid as well. We can conclude that $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 2$.

Exercise 1.2. Let $x > 0$ and $n \in \mathbb{N}$. Use the inequality $G(x_1, \dots, x_m) \leq A(x_1, \dots, x_m)$ for some appropriate choice of $m \in \mathbb{N}$ and of real numbers $x_1, \dots x_m > 0$ to deduce that:

a) $\dfrac{x^n}{1 + x + \ldots + x^{2n}} \leq \dfrac{1}{2n+1}$;

b) $1 + (n+1)x \leq (1+x)^{n+1}$.

Solution a) Take $m = 2n+1$ and $x_i = x^{i-1}$ for $i \in \{1, \dots, 2n+1\}$.

b) Take $m = n+1$, $x_1 = 1 + (n+1)x$ and $x_i = 1$ for $i \in \{2, \dots, n+1\}$.

Exercise 1.3. Prove that for every $n \in \mathbb{N}$ with $n \geq 2$ and for any numbers $x_1, x_2, \dots, x_n \in [-1, \infty)$ all of the same sign, we have

$$(1+x_1)(1+x_2) \cdot \ldots \cdot (1+x_n) \geq 1 + x_1 + x_2 + \ldots + x_n \quad \text{(the generalized Bernoulli inequality).}$$

Solution We use mathematical induction. Consider the following statement for $n \in \mathbb{N}$ with $n \geq 2$:

$$\begin{aligned} P(n) : \text{“}&\forall x_1, x_2, \dots, x_n \geq -1 \text{ all of the same sign we have} \\ &(1+x_1)(1+x_2) \cdot \ldots \cdot (1+x_n) \geq 1 + x_1 + x_2 + \ldots + x_n \text{”}. \end{aligned}$$

We verify that $P(2)$ is true. Let $x_1, x_2 \geq -1$ of the same sign. Then

$$(1+x_1)(1+x_2) = 1 + x_1 + x_2 + x_1x_2 \geq 1 + x_1 + x_2.$$

Now let $k \in \mathbb{N}$ such that $k \geq 2$ and assume that $P(k)$ is valid. Let $x_1, x_2, \dots, x_{k+1} \geq -1$ all of the same sign. Then

$$\begin{aligned} (1+x_1)(1+x_2) \cdot \ldots \cdot (1+x_k)(1+x_{k+1}) &\geq (1 + x_1 + x_2 + \ldots + x_k)(1+x_{k+1}) \\ &= 1 + x_1 + x_2 + \ldots + x_k + x_{k+1} + x_1x_{k+1} + x_2x_{k+1} + \ldots + x_kx_{k+1} \\ &\geq 1 + x_1 + x_2 + \ldots + x_k + x_{k+1}. \end{aligned}$$

Thus, $P(k+1)$ is also true. We can conclude that $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 2$.

Seminar 2

Exercise 2.1. For each set A_i from below find $\text{lb}(A_i)$ and $\text{ub}(A_i)$ (as subsets of $\mathbb{R}$), $\min(A_i)$ and $\max(A_i)$ (if they exist), and $\inf(A_i)$ and $\sup(A_i)$ (in $\overline{\mathbb{R}}$):

$$A_1 = [-8, \pi) \cap \mathbb{Z}, \qquad A_3 = \left\{ x + \frac{1}{x} \mid x \in \mathbb{R}, x < 0 \right\},$$

$$A_2 = \{2^m + n! \mid m, n \in \mathbb{N}\}, \qquad A_4 = \left\{ \frac{n}{1-n^2} \mid n \in \mathbb{N}, n \geq 2 \right\}.$$

Solution

- A_1: $A_1 = \{-8, -7, \ldots, 2, 3\}$, so $\text{lb}(A_1) = (-\infty, -8]$, $\min A_1 = -8 = \inf A_1$, $\text{ub}(A_1) = [3, \infty)$, $\max A_1 = 3 = \sup A_1$.

- A_2: Since
$$2^m + n! \geq 3, \quad \text{for all } m, n \in \mathbb{N},$$
with equality for $m = n = 1$, we have $\text{lb}(A_2) = (-\infty, 3]$, $\min A_2 = 3 = \inf A_2$. The set A_2 is not bounded above, so $\text{ub}(A_2) = \emptyset$, no maximum, $\sup A_2 = \infty$.

- A_3: Since
$$x + \frac{1}{x} + 2 = \frac{(x+1)^2}{x} \leq 0, \quad \text{for all } x \in \mathbb{R}, x < 0,$$
with equality for $x = -1$, we have $\text{ub}(A_3) = [-2, \infty)$, $\max A_3 = -2 = \sup A_3$. The set A_3 is not bounded below, so $\text{lb}(A_3) = \emptyset$, no minimum, $\inf A_3 = -\infty$.

- A_4: Since
$$\frac{n}{1-n^2} = \frac{n-1+1}{1-n^2} = -\frac{1}{n+1} + \frac{1}{1-n^2} \geq -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}, \quad \text{for all } n \in \mathbb{N}, n \geq 2,$$
with equality for $n = 2$, we have $\text{lb}(A_4) = (-\infty, -2/3]$, $\min A_4 = -2/3 = \inf A_4$. Note that $\frac{n}{1-n^2} < 0$ for all $n \in \mathbb{N}$, $n \geq 2$, so $0 \in \text{ub}(A_4)$. To see that $\sup A_4 = 0$, it is enough to note that $\lim\limits_{n\to\infty} \frac{n}{1-n^2} = 0$. Then $\text{ub}(A_4) = [0, \infty)$, no maximum.

Exercise 2.2. Find two sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that the following conditions are simultaneously met:

(i) one of the sets is unbounded (but not an interval) and the other is finite;

(ii) $\sup A = \inf B = 2 \in A$;

(iii) for every $a \in A$ and every $b \in B$, there exists $c \in \mathbb{R}$ with $a < c < b$.

Is it possible to choose B the finite set?

Solution Take, e.g., $A = \{2\}$ and $B = (2, 3) \cup (4, \infty)$. Then, for every $b \in B$, $c = (2+b)/2 = 1+b/2$ satisfies $2 < c < b$.
The set B cannot be chosen to be finite. Indeed, in this case, $\inf B = \min B$. Since $\sup A \in A$, we have $\sup A = \max A$. Hence $\min B = \max A$ and condition (iii) does not hold.

Exercise 2.3. Decide which of the following sets are neighborhoods of 0. Justify.

$$A_1 = (-1, 0] \cup \{1\}, \qquad A_3 = \mathbb{R},$$

$$A_2 = \left[1 - \frac{3}{2}, 1 + \frac{3}{2}\right] \cup (3, 4), \qquad A_4 = \mathbb{R} \setminus \mathbb{Q}.$$

Solution

- A_1: False because $\nexists \varepsilon > 0$ so that $(-\varepsilon, \varepsilon) \subseteq A_1$.
- A_2: True because $(-1/2, 1/2) \subseteq A_2$.
- A_3: True because $(-\varepsilon, \varepsilon) \subseteq A_3$ for all $\varepsilon > 0$.
- A_4: False because, by the Density Property of $\mathbb{Q}$ in $\mathbb{R}$, between any two irrational numbers there exists a rational one, so A_4 contains no intervals. Moreover, $0 \notin A_4$.

Seminar 3

Exercise 3.1. Find the limit (as $n \to \infty$) of the sequence whose general term x_n, $n \in \mathbb{N}$, is given below:
a) $\dfrac{n + \sin(n^2)}{\cos(n) - 3n}$, b) $(n^2+n)^{-\frac{n}{n+1}}$, c) $\left(1 + \dfrac{1}{n^3+2n^2}\right)^{n-n^3}$, d) $\dfrac{1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!}{(n+1)!}$,
e) $\sqrt[n]{1+2+\ldots+n}$, f) $n\left(\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1\right)$.

Solution a) $x_n = \dfrac{1 + \frac{\sin(n^2)}{n}}{-3 + \frac{\cos(n)}{n}}, n \in \mathbb{N}$. Since $\forall n \in \mathbb{N}$, $-\dfrac{1}{n} \leq \dfrac{\sin(n^2)}{n} \leq \dfrac{1}{n}$, by the Squeeze Theorem, $\lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{n} = 0$. Similarly, $\lim\limits_{n\to\infty} \dfrac{\cos(n)}{n} = 0$, and hence $\lim\limits_{n\to\infty} x_n = -\dfrac{1}{3}$.

b) $\lim\limits_{n\to\infty} x_n = 0$ (case ∞^{-1}).

c) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \left[\left(1 + \dfrac{1}{n^3+2n^2}\right)^{n^3+2n^2}\right]^{\frac{n-n^3}{n^3+2n^2}} = e^{-1} = \dfrac{1}{e}$.

d) Define, for $n \in \mathbb{N}$, $a_n = 1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n!$ and $b_n = (n+1)!$. Then the sequence (b_n) is strictly increasing and $\lim\limits_{n\to\infty} b_n = \infty$. In addition,

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{(n+1)(n+1)!}{(n+2)! - (n+1)!} = \frac{(n+1)(n+1)!}{(n+1)!(n+2-1)} = 1.$$

By the Stolz-Casàro Theorem, $\lim\limits_{n\to\infty} x_n = 1$.

e) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n}\,\sqrt[n]{n+1}}{\sqrt[n]{2}} = 1$.

f) Since $\sqrt[n]{1 + \dfrac{1}{n}} \leq 1 + \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$ (see Seminar 3), we get

$$1 + \frac{1}{n} \leq \left(1 + \frac{1}{n}\right)^{1+\frac{1}{n}} \leq \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n^2}\right) = 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3},$$

from where, subtracting 1 and then multiplying by n, we get $1 \leq x_n \leq 1 + \dfrac{1}{n} + \dfrac{1}{n^2}$. By the Squeeze Theorem, $\lim\limits_{n\to\infty} x_n = 1$.

Exercise 3.2. For $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ such that $a_n \leq b_n$ and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Suppose, in addition, that $\forall n \in \mathbb{N}$, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. By the Nested Interval Property, $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

Can $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$ contain more than one point?

Solution No, because supposing that there exist $x, y \in \mathbb{R}$ with $x \neq y$ such that $x, y \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, we get that $\forall n \in \mathbb{N}$, $|x - y| \leq b_n - a_n$, so, $0 < |x - y| \leq 0 = \lim_{n\to\infty} (b_n - a_n)$, a contradiction.

Exercise 3.3. Let (x_n) be a sequence in $\mathbb{Z}$. If (x_n) is convergent, is it eventually constant (i.e., $\exists n_0 \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$ with $m, n \geq n_0$ we have $x_m = x_n$)?

Solution Yes, (x_n) will eventually be constant. Denote $\lim_{n\to\infty} x_n = x$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$ we have $|x_n - x| < 1/2$. Hence, for all $m, n \in \mathbb{N}$ with $m, n \geq n_0$ we have

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| < \frac{1}{2} + \frac{1}{2} = 1.$$

Since $x_m, x_n \in \mathbb{Z}$, it follows that $x_m = x_n$.