Lecture 3

Subsequences

Definition 1. Let (x_n) be a sequence in $\mathbb{R}$. A *subsequence* of (x_n) is a sequence (y_k) in $\mathbb{R}$ given by $y_k = x_{n_k}$, $k \in \mathbb{N}$, where (n_k) is a strictly increasing sequence in $\mathbb{N}$.

Example 1. $(x_n) = (2^n) = (2, 4, 8, \ldots)$

$(2^{2k}) = (2^2, 2^4, 2^6, \ldots) = (4, 16, 64, \ldots)$ - a subsequence of (x_n)

$(4, 16, 2, 64, \ldots)$ - not a subsequence : we must keep the order of the terms in (x_n)

$(2, 4, 4, 8, \ldots)$ - not a subsequence : we are not allowed to repeat terms

Proposition 1. *Let (x_n) be a sequence in $\mathbb{R}$ that has a limit (in $\overline{\mathbb{R}}$). Then any subsequence (x_{n_k}) of (x_n) has the same limit, i.e., $\lim\limits_{k\to\infty} x_{n_k} = x$.*

Remark 1. If a sequence has two subsequences that have different limits, then the sequence has no limit.

$x_n = (-1)^n$, $n \in \mathbb{N}$; $x_{2k} = 1$, $x_{2k-1} = -1$, $k \in \mathbb{N}$

Theorem 1 (Bolzano-Weierstrass)**.** *A bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Remark 2. In fact, one can show that every sequence in $\mathbb{R}$ has a monotone subsequence. This, together with the equivalence of convergence and boundedness for monotone sequences, yields the Bolzano-Weierstrass theorem.

Application: An analysis of insertion sort

Growth of functions

Definition 2. Let $f, g : \mathbb{N} \to [0, \infty)$. We say that f is *big-O* of g if there exist $c, n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, n \geq n_0, f(n) \leq c\, g(n).$$

Notation: $f(n) = O(g(n))$.

Remark 3. (i) If $f(n) = O(g(n))$, then g is an asymptotic upper bound of f up to a constant. We can also say that f is asymptotically at most g.

(ii) $f(n) = O(f(n))$.

Example 2. (i) $f(n) = 3n^3 + 2n^2 + 5n + 7$, $n \in \mathbb{N}$.

$3n^3 + 2n^2 + 5n + 7 \leq 4n^3$, $\forall n \geq 4$ $\Rightarrow$ $f(n) = O(n^3)$, $f(n) = O(n^4)$, but $f(n) \neq O(n^2)$

In general, any polynomial function of degree k is $O(n^p)$, $\forall p \geq k$

(ii) $f(n) = \log_b n$, $b > 1$, $n \in \mathbb{N}$.

$\log_b n = \dfrac{\log_2 n}{\log_2 b}$ $\Rightarrow$ we simply write $f(n) = O(\log n)$ without specifying the base because the base b only changes the value of $\log_b n$ by a constant

(iii) $f(n) = 3n \log_2 n + n \log_2(\log_2 n) + 1$, $n \in \mathbb{N}$, $n \geq 2$. $f(n) = O(n \log n)$

Proposition 2. *Let $f : \mathbb{N} \to [0, \infty)$, $g : \mathbb{N} \to (0, \infty)$ and suppose $\exists L = \lim_{n\to\infty} \frac{f(n)}{g(n)} \in [0, \infty) \cup \{\infty\}$. Then $f(n) = O(g(n))$ if and only if $L \in [0, \infty)$.*

Pf $\boxed{\Rightarrow}$ $f(n) = O(g(n)) \Rightarrow \exists\, c, n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $f(n) \leq c \cdot g(n)$

$$\Updownarrow$$

$$\frac{f(n)}{g(n)} \leq c$$

$\Rightarrow L \leq c \Rightarrow L \in [0, \infty)$

$\boxed{\Leftarrow}$ $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L \in [0, \infty) \Rightarrow \exists\, n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $\frac{f(n)}{g(n)} \leq L + 1$

$$\Updownarrow$$

$$f(n) \leq (L+1)\, g(n)$$

$\Rightarrow f(n) = O(g(n))$

Example 3. $f(n) = \frac{7n^4 + n^3 - n^2 + 1}{5n^2 - 4}$, $n \in \mathbb{N}$. $f(n) = O(n^2)$ (more generally, $f(n) = O(n^p)$, $\forall p \geq 2$)

$\lim_{n\to\infty} \frac{f(n)}{n^2} = \frac{7}{5} \in [0, \infty)$.

Definition 3. Let $f : \mathbb{N} \to [0, \infty)$ and $g : \mathbb{N} \to (0, \infty)$. We say that f is *little-o* of g if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
Notation: $f(n) = o(g(n))$.

Remark 4. (i) $f(n) = o(g(n)) \iff \forall c > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq n_0, f(n) < c\, g(n)$. This condition says that f is asymptotically less than g.

(ii) $f(n) = o(g(n)) \implies f(n) = O(g(n))$.

(iii) $f(n) \neq o(f(n))$.

Example 4. (i) $n^2 = o(n^3)$.

(ii) $n^\alpha = o((1 + \beta)^n)$, $\alpha \in \mathbb{N}$, $\beta > 0$.

(iii) $\log_b n = o(n)$, $b > 1$.

Definition 4. Let $f, g : \mathbb{N} \to [0, \infty)$. We say that f is *big-Theta* of g if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.
Notation: $f(n) = \Theta(g(n))$.

Remark 5. (i) This condition says that f and g have the same growth rate (or the same order).

(ii) $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$.

Example 5. $\log_b n = \Theta(\log n)$, $b > 1$.

Proposition 3. *Let $f, g : \mathbb{N} \to (0, \infty)$ and suppose $\exists L = \lim_{n\to\infty} \frac{f(n)}{g(n)} \in [0, \infty) \cup \{\infty\}$. Then:*

(i) if $L = 0$, then $f(n) = o(g(n))$, hence $f(n) = O(g(n))$.

(ii) if $L \in (0, \infty)$, then $f(n) = \Theta(g(n))$.

(iii) if $L = \infty$, then $g(n) = o(f(n))$, hence $g(n) = O(f(n))$.

Example 3. (revisited) $f(n) = \frac{7n^4 + n^3 - n^2 + 1}{5n^2 - 4}$, $n \in \mathbb{N}$. $f(n) = \Theta(n^2)$

Insertion sort

Let A be an array containing n numbers ($n \in \mathbb{N}$): $A[1, \dots, n]$.

```
for i = 1 to n − 1 do
    key ← A[i + 1]          // insert A[i + 1] into the ordered array A[1, . . . , i]
    j ← i
    while j > 0 and A[j] > key do
        A[j + 1] ← A[j]
        j ← j − 1
    end
    A[j + 1] ← key
end
```

Example 6. $A = \langle 5, 2, 4, 6, 1, 3 \rangle$.

$\overset{2}{\cancel{5}}\quad \cancel{2}^{5}\quad 4\quad 6\quad 1\quad 3 \qquad key = 2$

$2\quad \cancel{5}^{4}\quad \cancel{4}^{5}\quad 6\quad 1\quad 3 \qquad key = 4$

$2\quad 4\quad 5\quad 6\quad 1\quad 3 \qquad key = 6$

${}^{1}\cancel{2}\quad \cancel{4}^{2}\quad \cancel{5}^{4}\quad \cancel{6}^{5}\quad \cancel{1}^{6}\quad 3 \qquad key = 1$

$1\quad 2\quad \cancel{4}^{3}\quad \cancel{5}^{4}\quad \cancel{6}^{5}\quad \cancel{3}^{6} \qquad key = 3$

$1\quad 2\quad 3\quad 4\quad 5\quad 6$

I stands for insertion

Proposition 4. *The average number of comparisons $C_I(n)$ done by insertion sort is $\Theta(n^2)$.*

Proof. Let $i \in \{1, \dots, n-1\}$ and suppose that $A[1, \dots, i]$ is ordered and we insert $A[i+1]$. There are $i+1$ possible positions where $A[i+1]$ can be placed. The probability that $A[i+1]$ will be placed in any given position is $1/(i+1)$ assuming that all positions are equally likely.
Depending on the position where $A[i+1]$ must be placed, we distinguish the following cases:

(i) on the first position: i comparisons

(ii) on the second position (i.e., between $A[1]$ and $A[2]$): i comparisons

in general, on the j-th position (i.e., between $A[j-1]$ and $A[j]$), where $j \in \{2, \dots, i\}$:

$i-j+2$ comparisons

$$1\ 2\ \dots\ (j-1)\ \underbrace{j\ \dots\ i}_{i-j+1}$$

(iii) on the $(i+1)$-th position (i.e., at the end of the array):

1 comparison

Average number of comparisons i^{th} stage $= \frac{1}{i+1}\left(i + i + (i-1) + \dots + 2 + 1\right)$

$$= \frac{i}{i+1} + \frac{1}{\cancel{i+1}} \cdot \frac{\cancel{(i+1)}\, i}{2} = \frac{i}{i+1} + \frac{i}{2}$$

$$C_I(n) = \sum_{i=1}^{n-1} \left(\frac{i}{i+1} + \frac{i}{2} \right) = \sum_{i=1}^{n-1} \left(\frac{i+1-1}{i+1} + \frac{i}{2} \right) = \sum_{i=1}^{n-1} \left(1 - \frac{1}{i+1} + \frac{i}{2} \right) = n-1 + \frac{n(n-1)}{4} - \sum_{i=1}^{n-1} \frac{1}{i+1}$$

$$= \frac{n^2}{4} + \frac{3n}{4} - 1 - \sum_{i=1}^{n-1} \frac{1}{i+1}$$

$$\left(1+\frac{1}{i}\right)^{i} < e < \left(1+\frac{1}{i}\right)^{i+1}, \quad \forall i \in \mathbb{N}$$

$$i \ln\left(1+\frac{1}{i}\right) < 1 \Rightarrow \ln \frac{i+1}{i} < \frac{1}{i} \Rightarrow \ln(i+1) - \ln i < \frac{1}{i} \Rightarrow \sum_{i=1}^{n-1} \frac{1}{i+1} = \sum_{i=2}^{n} \frac{1}{i} > \ln(n+1) - \ln 2$$

$$1 < (i+1) \ln\left(1+\frac{1}{i}\right) \Rightarrow \frac{1}{i+1} < \ln \frac{i+1}{i} = \ln(i+1) - \ln i \Rightarrow \sum_{i=1}^{n-1} \frac{1}{i+1} < \ln n$$

$$\frac{n^2}{4} + \frac{3n}{4} - \ln n - 1 < C_I(n) = \frac{n^2}{4} + \frac{3n}{4} - 1 - \sum_{i=1}^{n-1} \frac{1}{i+1} < \frac{n^2}{4} + \frac{3n}{4} - \ln(n+1) + \ln 2 - 1$$

$$\Rightarrow C_I(n) = \Theta(n^2)$$

□

Remark 6. Consider the sequence (γ_n) defined by $\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$, $n \in \mathbb{N}$. The above reasoning shows that (γ_n) is bounded below by 0 and strictly decreasing. Hence, it is convergent. Its limit is called Euler's constant and is denoted by $\gamma \simeq 0.577$.

Exercise 1. *Compute the number of needed comparisons in the worse case (i.e., when the array is ordered in a decreasing way).*

Series of real numbers

Definition 5. Let (x_n) be a sequence in $\mathbb{R}$. We attach to (x_n) the sequence (s_n) given by

$$s_n = x_1 + x_2 + \ldots + x_n, \quad n \in \mathbb{N}.$$

The pair $((x_n), (s_n))$ is called the *series* with terms x_n.
Notation: $\sum_{n\geq 1} x_n$ or $\sum x_n$.
For $n \in \mathbb{N}$, the number s_n is called the n^{th} *partial sum* of the series. If the sequence (s_n) of partial sums converges (diverges), we say that the series $\sum_{n\geq 1} x_n$ is *convergent (divergent)*. If (s_n) has a limit, we say that the series $\sum_{n\geq 1} x_n$ *has a sum*. In this case we call $\lim_{n\to\infty} s_n \in \overline{\mathbb{R}}$ the *sum* of the series $\sum_{n\geq 1} x_n$ and we denote it by $\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$.

Remark 7. We also consider series of the form $\sum_{n\geq m} x_n$ generated by a sequence $(x_n)_{n\geq m}$, where $m \in \mathbb{Z}$. Note that for any $k \in \mathbb{N}$, $\sum_{n\geq m} x_n$ has a sum if and only if $\sum_{n\geq m+k} x_n$ has a sum. In this case we have

$$\sum_{n=m}^{\infty} x_n = x_m + x_{m+1} + \ldots x_{m+k-1} + \sum_{n=m+k}^{\infty} x_n.$$

Proposition 5. *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be convergent series and let $c \in \mathbb{R}$. Then*

(i) $\sum_{n\geq 1}(x_n + y_n)$ is convergent and

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

(ii) $\sum_{n\geq 1}(c\, x_n)$ is convergent and

$$\sum_{n=1}^{\infty}(c\, x_n) = c \sum_{n=1}^{\infty} x_n.$$

Example 7. (i) *The geometric series:* Let $q \in \mathbb{R}$. By convention, we set $q^0 = 1$ even for $q = 0$.

$$\sum_{n\geq 0} q^n = \begin{cases} \text{divergent with no sum,} & \text{if } q \leq -1, \\ \text{convergent with sum } 1/(1-q), & \text{if } q \in (-1, 1), \\ \text{divergent with sum } \infty, & \text{if } q \geq 1. \end{cases}$$

Denote for $n \in \mathbb{N}_0$ the partial sums by

$$s_n = 1 + q + \ldots + q^n = \begin{cases} \frac{1-q^{n+1}}{1-q}, & q \neq 1 \\ n+1, & q = 1 \end{cases}$$

- $q \leq -1 \Rightarrow (s_n)$ has no limit
- $q \in (-1,1) \Rightarrow \lim_{n\to\infty} s_n = \frac{1}{1-q}$
- $q \geq 1 \Rightarrow \lim_{n\to\infty} s_n = \infty$

(ii) *Telescoping series:* Let (x_n) be a sequence in $\mathbb{R}$. The series $\sum_{n\geq 1}(x_n - x_{n+1})$ is called a *telescoping series*. This series is convergent if and only if (x_n) is convergent. In this case,

$$\sum_{n=1}^{\infty}(x_n - x_{n+1}) = x_1 - \lim_{n\to\infty} x_n.$$

E.g., $\displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)}$ is a telescoping series.

$$\frac{1}{n(n+1)} = \frac{1+n-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \ \forall n \in \mathbb{N}$$

Denote for $n \in \mathbb{N}$ the partial sums by $s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

$$\Rightarrow s_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \ldots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} s_n = 1$$

$$\Rightarrow \sum_{n\geq 1} \frac{1}{n(n+1)} \text{ is convergent and } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$