(1)

Seminar 14:

Ex1: Compute the following multiple integrals:

a) $A=[0,2]\times[1,2]$, $\iint_A (x-3y^2)\,dx\,dy = \int_0^2\int_1^2 (x-3y^2)\,dy\,dx = -12$

Inner integral: $\int_1^2 (x-3y^2)\,dy = (xy-y^3)\Big|_{y=1}^{y=2} = 2x-8-x+1 = x-7$

Outer integral: $\int_0^2 (x-7)\,dx = \left(\frac{x^2}{2}-7x\right)\Big|_0^2 = 2-14=-12$

b) $A=[0,\pi/2]\times[0,\pi/4]$, $\iint_A(\sin x+\sin y)\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/4}(\sin x+\sin y)\,dy\,dx = \frac{\pi}{4}(3-\sqrt{2})$

(2)

Inner integral: $\int_0^{\pi/4}(\sin x+\sin y)\,dy = (y\cdot\sin x-\cos y)\Big|_{y=0}^{y=\frac{\pi}{4}} = \frac{\pi}{4}\cdot\sin x-\frac{\sqrt{2}}{2}+1$

Outer integral: $\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{4}\sin x+1-\frac{\sqrt{2}}{2}\right)dx = \left(-\frac{\pi}{4}\cos x+\left(1-\frac{\sqrt{2}}{2}\right)x\right)\Big|_0^{\frac{\pi}{2}} = \left(1-\frac{\sqrt{2}}{2}\right)\frac{\pi}{2}+\frac{\pi}{4} = \frac{\pi}{4}(3-\sqrt{2})$

c) $A=[0,1]\times[0,1]$, $\iint_A \min\{x,y\}\,dx\,dy = \int_0^1\int_0^1 \min\{x,y\}\,dy\,dx = \frac{1}{3}$

$\min\{x,y\} = \begin{cases} x, & x\le y\\ y, & x>y\end{cases}$

Inner integral: $\int_0^1\min\{x,y\}\,dy = \int_0^x y\,dy+\int_x^1 x\,dy = \frac{y^2}{2}\Big|_{y=0}^{y=x}+xy\Big|_{y=x}^{y=1} = \frac{x^2}{2}+x-x^2 = x-\frac{x^2}{2}$

(3)

Outer integral: $\int_0^1\left(x-\frac{x^2}{2}\right)dx = \left(\frac{x^2}{2}-\frac{x^3}{6}\right)\Big|_0^1 = \frac{1}{2}-\frac{1}{6}=\frac{1}{3}$

d) $A=[1,2]\times[0,\pi]$, $\iint_A y\cos(xy)\,dx\,dy = \int_0^\pi\int_1^2 y\cos(xy)\,dx\,dy = -2$

Inner integral: $\int_1^2 y\cos(xy)\,dx = \sin(xy)\Big|_{x=1}^{x=2} = \sin(2y)-\sin y$

Outer integral: $\int_0^\pi(\sin(2y)-\sin y)\,dy = \left(-\frac{1}{2}\cos(2y)+\cos y\right)\Big|_0^\pi = -\frac{1}{2}-1+\frac{1}{2}-1=-2$

e) $A=[1,e]\times[1,9]$, $\iint_A\frac{\ln\sqrt{x}}{xy}\,dx\,dy = \int_1^e\frac{\ln\sqrt{x}}{x}\,dx\cdot\int_1^9\frac{1}{y}\,dx = \frac{1}{4}\cdot 2\cdot\ln 3 = \frac{\ln 3}{2}$.

$A=[a_1,b_1]\times[a_2,b_2]$, $f(x,y)=g(x)\cdot h(y) \Rightarrow \iint_A f(x,y)\,dx\,dy = \int_{a_1}^{b_1} g(x)\,dx\cdot\int_{a_2}^{b_2}h(y)\,dy$

(4)

$\int_1^e\frac{\ln\sqrt{x}}{x}\,dx = \frac{1}{2}\int_1^e\frac{\ln x}{x}\,dx = \frac{1}{2}\int_1^e \ln x\cdot(\ln x)'\,dx = \frac{1}{4}\ln^2 x\Big|_1^e = \frac{1}{4}$

$\int_1^9\frac{1}{y}\,dy = \ln y\Big|_1^9 = \ln 9 = 2\ln 3$

f) $A=[1,2]\times[1,2]\times[1,2]$, $\iiint_A\frac{1}{(x+y+z)^3}\,dx\,dy\,dz = \int_1^2\int_1^2\int_1^2\frac{1}{(x+y+z)^3}\,dz\,dy\,dx = \ln\left(\frac{4}{5}\sqrt{\frac{8}{5}}\right)$

Inner integral: $\int_1^2\frac{1}{(x+y+z)^3}\,dz = -\frac{1}{2(x+y+z)^2}\Big|_{z=1}^{z=2} = -\frac{1}{2}\left(\frac{1}{(x+y+2)^2}-\frac{1}{(x+y+1)^2}\right)$

Intermediate integral: $\int_1^2\frac{1}{2}\left(\frac{1}{(x+y+1)^2}-\frac{1}{(x+y+2)^2}\right)dy = \frac{1}{2}\left(-\frac{1}{x+y+1}+\frac{1}{x+y+2}\right)\Big|_{y=1}^{y=2} = \frac{1}{2}\left(\frac{1}{x+4}-\frac{1}{x+3}+\frac{1}{x+2}-\frac{1}{x+3}\right)$

(5)

Outer integral: $\int_1^2\frac{1}{2}\left(\frac{1}{x+4}+\frac{1}{x+2}-\frac{2}{x+3}\right)dx = \frac{1}{2}\ln\frac{(x+4)(x+2)}{(x+3)^2}\Big|_1^2 = \frac{1}{2}\left(\ln\frac{6\cdot 4}{5^2}-\ln\frac{5\cdot 3}{4^2}\right)$

$= \frac{1}{2}\ln\frac{\cancel{6}^{2}\cdot 4\cdot 4^2}{5^2\cdot 5\cdot\cancel{3}} = \frac{1}{2}\ln\left(2\cdot\frac{4^3}{5^3}\right) = \ln\left(\frac{4}{5}\sqrt{\frac{8}{5}}\right)$

g) $A=\underbrace{[0,1]\times\dots\times[0,1]}_{n\text{ times}}$, $\int\cdots\int_A e^{x_1+\dots+x_n}\,dx_1\dots dx_n = \int\cdots\int_A e^{x_1}\cdot\ldots\cdot e^{x_n}\,dx_1\dots dx_n$

$= \left(\int_0^1 e^x\,dx\right)^n = (e-1)^n$

(6)

Def: A set $M\subseteq\mathbb{R}^2$ is called

- simple w.r.t. the y-axis if $\exists\, a,b\in\mathbb{R}$, $a<b$, $\exists\,\varphi_1,\varphi_2:[a,b]\to\mathbb{R}$ cont, $\varphi_1\le\varphi_2$ s.t.
  $M=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\}$
- simple w.r.t. the x-axis if $\exists\, c,d\in\mathbb{R}$, $c<d$, $\exists\,\psi_1,\psi_2:[c,d]\to\mathbb{R}$, $\psi_1\le\psi_2$ s.t.
  $M=\{(x,y)\in\mathbb{R}^2 \mid c\le y\le d,\ \psi_1(y)\le x\le\psi_2(y)\}$

Thm: $M\subseteq\mathbb{R}^2$, $f:M\to\mathbb{R}$ cont

If $M$ is simple w.r.t. the y-axis, then $f$ is Riemann integrable on $M$ and

$$\iint_M f(x,y)\,dx\,dy = \int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

(7)

If $M$ is simple w.r.t. the x-axis, the $f$ is Riemann integrable on $M$ and

$$\iint_M f(x,y)\,dx\,dy = \int_c^d\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

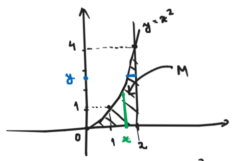
Ex2: Let $M$ be the subset of $\mathbb{R}^2$ bounded by the parabola $y=x^2$ and the lines $x=2$ and $y=0$.

a) Express $M$ as a simple set first w.r.t. the y-axis and then w.r.t. the x-axis.

b) Compute $\iint_M xy\,dx\,dy$ in two ways

c) Compute $\iint_M x\sin((4-y)^2)\,dx\,dy$

(8)




a) $M=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ 0\le y\le x^2\}$
simple w.r.t. the y-axis

$M=\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 4,\ \sqrt{y}\le x\le 2\}$
simple w.r.t. the x-axis

b) • $\iint_M xy\,dx\,dy = \int_0^2\int_0^{x^2} xy\,dy\,dx = \frac{16}{3}$

Inner integral: $\int_0^{x^2} xy\,dy = x\cdot\frac{y^2}{2}\Big|_{y=0}^{y=x^2} = \frac{x^5}{2}$ ; Outer integral: $\int_0^2\frac{x^5}{2}\,dx = \frac{x^6}{12}\Big|_0^2 = \frac{2^6}{12} = \frac{16}{3}$

(9)

• $\iint_M xy\,dx\,dy = \int_0^4\int_{\sqrt{y}}^2 xy\,dx\,dy = \frac{16}{3}$

Inner integral: $\int_{\sqrt{y}}^2 xy\,dx = y\cdot\frac{x^2}{2}\Big|_{x=\sqrt{y}}^{x=2} = 2y-\frac{y^2}{2}$

Outer integral: $\int_0^4\left(2y-\frac{y^2}{2}\right)dy = \left(y^2-\frac{y^3}{6}\right)\Big|_0^4 = 16-\frac{\cancel{64}^{32}}{\cancel{6}_{3}} = 16\left(1-\frac{2}{3}\right) = \frac{16}{3}$

c) $\iint_M x\sin((4-y)^2)\,dx\,dy = \int_0^4\int_{\sqrt{y}}^2 x\sin((4-y)^2)\,dx\,dy = \frac{1}{4}(1-\cos 16)$

Inner integral: $\int_{\sqrt{y}}^2 x\sin((4-y)^2)\,dx = \sin((4-y)^2)\cdot\frac{x^2}{2}\Big|_{x=\sqrt{y}}^{x=2} = \sin((4-y)^2)\cdot\frac{4-y}{2}$

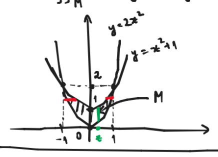
(10)

Outer integral: $\int_0^4\sin((4-y)^2)\cdot\frac{4-y}{2}\,dy = \frac{1}{4}\cos((4-y)^2)\Big|_0^4 = \frac{1}{4}(1-\cos 16)$

Ex 3: Let $M$ be the subset of $\mathbb{R}^2$ bounded by the parabolas $y=2x^2$ and $y=x^2+1$.

a) Express $M$ as a simple set w.r.t. the y-axis. Is $M$ simple w.r.t. the x-axis?

b) $\iint_M(x+2y)\,dx\,dy$




a) $M=\{(x,y)\in\mathbb{R}^2 \mid -1\le x\le 1,\ 2x^2\le y\le x^2+1\}$
simple w.r.t. the y-axis

M cannot be expressed as a simple set w.r.t. the x-axis because not every parallel line to the x-axis intersects M along a compact interval.

(11)

b) $\iint_M(x+2y)\,dx\,dy = \int_{-1}^1\int_{2x^2}^{x^2+1}(x+2y)\,dy\,dx$

Inner integral: $\int_{2x^2}^{x^2+1}(x+2y)\,dy = (xy+y^2)\Big|_{y=2x^2}^{y=x^2+1} = x(x^2+1)+(x^2+1)^2-2x^3-4x^4$
$= x^3+x+x^4+2x^2+1-2x^3-4x^4$
$= -3x^4-x^3+2x^2+x+1$

Outer integral: $\int_{-1}^1(-3x^4-x^3+2x^2+x+1)\,dx = \dots$