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Exercise no. 1.

Define the sequence (x_n) by $x_1 \in (0,1)$ and $x_{n+1} = x_n - x_n^2$, $n \in \mathbb{N}$. Prove that the sequence (x_n) - converges and find its limit. Then study the convergence of the sequence $(n \cdot x_n)$.

PROOF:

We want to check if (x_n) is monotone

$$x_{n+1} - x_n = \cancel{x_n} - x_n^2 - \cancel{x_n}$$

$$x_{n+1} - x_n = -x_n^2 < 0 \Rightarrow x_{n+1} < x_n \Rightarrow$$

(x_n) - is strictly decreasing (1)

$p(n): x_n \in (0,1), \forall n \geq 1$

I. We verify if $p(1)$ is true

$p(1): x_1 \in (0,1)$ (True)

II. We assume that $p(k)$ is true and we want to prove $p(k+1)$ is also true, $\forall k \geq 1$

$p(k): x_k \in (0,1), \forall k \geq 1$

$p(k+1): x_{k+1} = x_k - x_k^2$

$x_{k+1} = x_k(1 - x_k)$

$$\left.\begin{array}{l} x_k \in (0,1) \\ 1 - x_k \in (0,1), \forall x_k \in (0,1) \end{array}\right\} \Rightarrow$$

$\Rightarrow x_k(1-x_k) \in (0,1), \forall k \geq 1 \Rightarrow p(k+1)$ is true $\Rightarrow$

$\Rightarrow p(n)$ is true, $\forall n \geq 1 \Rightarrow x_n \in (0,1) \Rightarrow$

$\Rightarrow x_n$ - is bounded ②

From ①, ② $\xRightarrow[\text{Weierstrass}]{\text{Theoreme}}$ x_n - is convergent $\Rightarrow$

$\lim_{n\to\infty} x_n \in \mathbb{R}$

Because we know that x_n is convergent we can say that $\lim_{n\to\infty} x_n = l \in \mathbb{R}$

$$x_{n+1} = x_n - x_n^2$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \lim_{n\to\infty} x_n^2$$

$$l = l - l^2 \Rightarrow l^2 = 0 \Rightarrow l = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} x_n = 0$$

We study the convergence of the sequence $(n \cdot x_n)$ by calculating the limit.

$$\lim_{n \to \infty} n \cdot x_n = 0 \cdot \infty$$

$$a_n = n$$

$$b_n = \frac{1}{x_n}$$

$$\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{x_n}}{n}$$

$n \to \infty$

n - strictly increasing

$$L = \lim_{n \to \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n}$$

$$\lim_{n \to \infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{\cancel{n} + 1 - \cancel{n}} = \lim_{n \to \infty} \left(\frac{{}^{x_n)}1}{x_{n+1}} - \frac{{}^{x_{n+1})}1}{x_n} \right)$$

$$= \lim_{n \to \infty} \frac{x_n - x_{n+1}}{x_n \cdot x_{n+1}} = \lim_{n \to \infty} \frac{\cancel{x_n} - \cancel{x_n} + \cancel{x_n^2}\; x_n}{\underset{1}{\cancel{x_n}}(x_n - x_n^2)}$$

$$x_{n+1} = x_n - x_n^2$$

$$= \lim_{n \to \infty} \frac{x_n}{x_n - x_n^2} = \lim_{n \to \infty} \frac{\cancel{x_n}}{\cancel{x_n}(1 - x_n)}$$

$$= \lim_{n \to \infty} \frac{1}{1 - x_n} = \frac{1}{1 - 0} = 1 \xrightarrow[\text{Cesaro}]{\text{Stolz}} \lim_{n \to \infty} \frac{b_n}{a_n} = 1$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{\lim_{n \to \infty} \frac{b_n}{a_n}} = \frac{1}{1} \Rightarrow \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \Rightarrow (x_n \cdot n)$$

is convergent

Exercise no 3.

Find the sum of the following series and specify whether they are convergent or divergent:

a) $\sum_{n\geq 1}\left(-\frac{3}{4}\right)^n$

$$S_k=\sum_{i=1}^{k}\left(-\frac{3}{4}\right)^i=\underbrace{\left(-\frac{3}{4}\right)^1+\left(-\frac{3}{4}\right)^2+\ldots+\left(-\frac{3}{4}\right)^k}_{\hookrightarrow k \text{ terms}}$$

$$\left.\begin{aligned} b_1&=-\frac{3}{4}\\ q&=-\frac{3}{4}\end{aligned}\right\}\Rightarrow S_k=-\frac{3}{4}\cdot\frac{\left(-\frac{3}{4}\right)^k-1}{-\frac{3}{4}-1}$$

$$S_k=-\frac{3}{4}\cdot(-1)\cdot\frac{\left(-\frac{3}{4}\right)^k-1}{\frac{3}{4}+1}$$

$$S_k=\frac{3}{4}\cdot\frac{\left(-\frac{3}{4}\right)^k-1}{\frac{3}{4}+1}$$

$$\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^n=\lim_{k\to\infty}S_k=\lim_{k\to\infty}\frac{3}{4}\cdot\frac{\left(-\frac{3}{4}\right)^k-1}{\frac{3}{4}+1}$$

$\left(-\frac{3}{4}\right)^k\to 0$

$$-\frac{3}{4}\in(-1,1)$$

$$=\frac{3}{4}\cdot\frac{-1}{\frac{3}{4}+1}=\frac{-\frac{3}{4}}{\frac{3}{4}+1}\in\mathbb{R}\Rightarrow \text{the series is convergent}$$

b) $\sum\limits_{n=0}^{\infty} \frac{2^{3n}}{5^{n-1}} = \sum\limits_{n=0}^{\infty} \frac{2^{3n}}{\frac{5^n}{5}} = 5 \cdot \sum\limits_{n=0}^{\infty} \frac{2^{3n}}{5^n}$

$= 5 \cdot \sum\limits_{n=0}^{\infty} \left(\frac{8}{5}\right)^n$

$S_k = \sum\limits_{i=1}^{k} \left(\frac{8}{5}\right)^i = \overbrace{\left(\frac{8}{5}\right)^0 + \left(\frac{8}{5}\right)^1 + \ldots + \left(\frac{8}{5}\right)^k}^{k+1 \text{ terms}}$

$S_k = 1 \cdot \frac{\left(\frac{8}{5}\right)^{k+1} - 1}{\frac{8}{5} - 1} = \frac{\left(\frac{8}{5}\right)^{k+1} - 1}{\frac{3}{5}}$

$S_k = \frac{5}{3}\left(\left(\frac{8}{5}\right)^{k+1} - 1\right)$

$5 \cdot \sum\limits_{n=0}^{\infty} \left(\frac{8}{5}\right)^n = 5 \cdot \lim\limits_{k \to \infty} S_k = 5 \cdot \lim\limits_{k \to \infty} \frac{5}{3}\left(\overbrace{\left(\frac{8}{5}\right)^{k+1}}^{\to\infty} - 1\right)$

$5 \cdot \sum\limits_{n=0}^{\infty} \left(\frac{8}{5}\right)^n = \infty \notin \mathbb{R} \Rightarrow$ the series is divergent

c) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

$2n+1 - (2n-1) = 2$

We rewrite the sum:

$$\frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)}$$

$$S_k = \sum_{i=1}^{k} \frac{(2i+1)-(2i-1)}{(2i-1)(2i+1)}$$

$$S_k = \sum_{i=1}^{k} \left(\frac{1}{2i-1} - \frac{1}{2i+1} \right)$$

$$S_k = \left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \dots - \cancel{\frac{1}{2k-1}} + \cancel{\frac{1}{2k+1}} - \frac{1}{2k+1} \right)$$

$$S_k = 1 - \frac{1}{2k+1}$$

$$\frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(2n+1)-(2n-1)}{(2n+1)(2n-1)} = \frac{1}{2} \lim_{k \to \infty} S_k$$

$$= \frac{1}{2} \cdot \lim_{k \to \infty} \left(1 - \overbrace{\frac{1}{2k+1}}^{\to 0} \right) = \frac{1}{2} \in \mathbb{R}$$ => the series is convergent

d) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

$$S_k = \sum_{i=1}^{k} \ln\left(1+\frac{1}{i}\right) = \sum_{i=1}^{k} \ln\left(\frac{i+1}{i}\right)$$

$$S_k = \sum_{i=1}^{k} \left[\ln(i+1) - \ln i\right]$$

$$S_k = \cancel{\ln 2} - \underbrace{\ln 1}_{0} + \cancel{\ln 3} - \cancel{\ln 2} + \dots + \cancel{\ln k} - \cancel{\ln(k-1)} + \ln(k+1) - \cancel{\ln k}$$

$$S_k = \ln(k+1)$$

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+1) = \infty \notin \mathbb{R}$$

$\Downarrow$

the series is divergent

e) $\sum_{n=1}^{\infty} \frac{3n-2}{2^n} = \sum_{n=1}^{\infty} \left(\frac{3n}{2^n} + \frac{-2}{2^n} \right)$

$= 3 \cdot \sum_{n=1}^{\infty} \frac{n}{2^n} + (-1) \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

$S_{k_1} = \sum_{i=1}^{k} \frac{i}{2^i} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k}$

$S_{k_1} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{2}{2^3} + \dots + \frac{1}{2^k} + \frac{k-1}{2^k}$

$S_{k_1} = \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \right) + \frac{1}{2} \left(\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{k-1}{2^k} \right)$

$S_{k_1} = \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \right) + \frac{1}{2} \left(\underbrace{\frac{1}{2} + \frac{2}{2^2} + \dots + \frac{k}{2^k}}_{S_{k_1}} - \frac{k}{2^k} \right)$

$S_{k_1} = \underbrace{\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \right)}_{s} + \frac{1}{2} S_{k_1} - \frac{k}{2^{k+1}}$

$S_{k_1} - \frac{1}{2} S_{k_1} = s - \frac{k}{2^{k+1}}$

$s = \frac{1}{2} \cdot \frac{\left(\frac{1}{2}\right)^k - 1}{\frac{1}{2} - 1} = \frac{1}{2} \cdot (-2) \left(\left(\frac{1}{2}\right)^k - 1 \right) = \left(1 - \left(\frac{1}{2}\right)^k \right)$

$\frac{1}{2} S_{k_1} = \left(1 - \left(\frac{1}{2}\right)^k \right) - \frac{k}{2^{k+1}} \,\Big|\, \cdot 2 \Rightarrow S_{k_1} = 2\left(\left(1 - \left(\frac{1}{2}\right)^k\right) - \frac{k}{2^{k+1}} \right)$

$\sum_{n=1}^{\infty} \frac{n}{2^n} = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \left(2\left(1 - \left(\frac{1}{2}\right)^k\right) - \frac{2k}{2^{k+1}} \right) = 2 \Rightarrow$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{2^n} = 2$

$\frac{2k}{2^{k+1}} \to 0$, because the exponential function grows faster then the polinomial function

$$S_{k_2} = (-1)\sum_{i=1}^{k} \left(\frac{1}{2}\right)^{i-1}$$

$$S_{k_2} = \underbrace{\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \dots + \left(\frac{1}{2}\right)^{k-1}}_{\hookrightarrow\ k \text{ terms}}$$

$$S_{k_2} = 1 \cdot \frac{\left(\frac{1}{2}\right)^k - 1}{\frac{1}{2} - 1}$$

$$S_{k_2} = 2\left(1 - \left(\frac{1}{2}\right)^k\right)$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} 2\left(1 - \left(\frac{1}{2}\right)^{k}\right) = 2 \Rightarrow$$

$$\Rightarrow (-1)\cdot\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = -2 \quad (2)$$

From (1), (2) $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{2^n} = 3\cdot\sum_{n=1}^{\infty}\frac{n}{2^n} + (-1)\cdot\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$

$= 3\cdot 2 - 2 \quad = 4 \in \mathbb{R} \Rightarrow$ the sum is convergent

Seminar 5

Exercise 5.1.

Study if the following series are convergent or divergent

a) $\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^n$

We use the n^{th} term test:

We take $x_n = \left(1-\frac{1}{n}\right)^n$ and we calculate the limit as $n \to \infty$. If $\lim_{n\to\infty} x_n \neq 0 \Rightarrow$

$\sum_{n\geq 1} x_n$ - diverges

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\underbrace{\left[\left(1+\left(-\frac{1}{n}\right)\right)^{-n}\right]}_{e}^{\left(-\frac{1}{n}\right)\cdot n}$$

$$= e^{\lim_{n\to\infty} n\cdot\left(-\frac{1}{n}\right)}$$

$$= e^{-1} = \frac{1}{e} \neq 0 \Rightarrow \lim_{n\to\infty} x_n \neq 0 \Rightarrow$$

$\Rightarrow \sum_{n\geq 1} x_n$ - diverges

e) $\sum\limits_{n\geq 1} \sin \frac{1}{n^{\frac{5}{4}}} = \sum\limits_{n\geq 1} \sin \frac{1}{n\sqrt[4]{n}}$

We apply the Second Comparison Test

Take $x_n = \sin \frac{1}{n^{\frac{5}{4}}}$, $y_n = \frac{1}{n^{\frac{5}{4}}}$, $n \in \mathbb{N}$

Then $L = \lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{\sin \frac{1}{n\sqrt[4]{n}}}{\frac{1}{n\sqrt[4]{n}}} = 1$ (1)

We know from the generalized harmonic series that: $\sum\limits_{n\geq 1} \frac{1}{n^{\alpha}}$ - convergent if $\alpha > 1$ $\Rightarrow$

$\Rightarrow \sum\limits_{n\geq 1} y_n = \sum\limits_{n\geq 1} \frac{1}{n^{\frac{5}{4}}}$ - convergent because $\frac{5}{4} > 1$ (2)

From (1), (2) $\Rightarrow$ Because $L = 1 \in (0, \infty)$ and

$\sum\limits_{n\geq 1} y_n$ is convergent $\Rightarrow \sum\limits_{n\geq 1} x_n$ - is convergent $\Rightarrow$

$\Rightarrow \sum\limits_{n\geq 1} \sin \frac{1}{n^{\frac{4}{5}}}$ - is convergent

c) $\sum_{n\geq 1} \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$

We apply the Second Comparison Test

Take $x_n = \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$, $y_n = \frac{1}{\sqrt{n}}$, $n \in \mathbb{N}$

Then $L = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n\sqrt[3]{n}+2} \cdot \sqrt{n}$

$$L = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(\sqrt[3]{n}+\underbrace{\frac{2}{n}}_{\to 0})} = \frac{1}{\infty} = 0 \quad (1)$$

We know from the generalized harmonic series that: $\sum_{n\geq 1} \frac{1}{n^\alpha}$ - divergent if $\alpha \leq 1$ $\Rightarrow$

$\Rightarrow \sum_{n\geq 1} y_n = \sum_{n\geq 1} \frac{1}{n^{\frac{1}{2}}}$ - divergent because $\frac{1}{2} \leq 1$ (2)

From (1), (2) $\Rightarrow$ Because $L=0$ and $\sum_{n\geq 1} y_n$ is divergent $\Rightarrow \sum_{n\geq 1} x_n$ is divergent $\Rightarrow$

$\Rightarrow \sum_{n\geq 1} \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$ - divergent

d) $\sum\limits_{n\geq 1} \dfrac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}$

We apply Ratio Test

Denote $x_n = \dfrac{n!}{3\cdot 5 \cdot \ldots \cdot (2n+1)}$, $n \in \mathbb{N}$

$\exists\, L = \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} \in [0, \infty) \cup \{\infty\}$

$$L = \lim_{n\to\infty} \frac{\overset{n+1}{\cancel{(n+1)!}}}{\underset{1}{\cancel{3\cdot 5 \cdots (2n+1)}}\cdot(2n+3)} \cdot \frac{\overset{1}{\cancel{3\cdot 5\cdot \ldots \cdot (2n+1)}}}{\underset{1}{\cancel{n!}}}$$

$$L = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

$L = \frac{1}{2} < 1 \Rightarrow \sum\limits_{n\geq 1} x_n$ - convergent $\Rightarrow$

$\Rightarrow \sum\limits_{n\geq 1} \dfrac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}$ - convergent

e) $\sum_{n\geq 1} \frac{n^3 5^n}{2^{3n+1}}$

We apply the Ratio Test

Denote $x_n = \frac{n^3 \cdot 5^n}{2^{3n+1}}$, $n \in \mathbb{N}$

$\exists L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0, \infty) \cup \{\infty\}$

$$L = \lim_{n\to\infty} \frac{(n+1)^3 \cdot \cancel{5^{n+1}}^{\,1}}{2^{3n+4}} \cdot \frac{2^{3n+1}}{n^3 \cdot \cancel{5^n}_{1}}$$

$$L = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{\cancel{2^{3n}}^{1} \cdot \cancel{2^1}}{\cancel{2^{3n}}_{1} \cdot \cancel{2^4}\, 2^3}$$

$$L = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{8} = \frac{1}{8} \Rightarrow$$

$\Rightarrow L = \frac{1}{8} < 1 \Rightarrow \sum_{n\geq 1} x_n$ - convergent $\Rightarrow$

$\Rightarrow \sum_{n\geq 1} \frac{n^3 5^n}{2^{3n+1}}$ - convergent

f) $\sum\limits_{n\geq 1} \frac{2\cdot 5\cdot \ldots \cdot (3n-1)}{3\cdot 6\cdot \ldots \cdot 3n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$$

$$x_{n+1} = \frac{[2\cdot 5\cdot \ldots \cdot (3n-1)](3n+2)}{[3\cdot 6\cdot \ldots \cdot 3n](3n+3)} = x_n \cdot \frac{3n+2}{3n+3}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+3}{3n+2} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{3n}+3-\cancel{3n}-2}{3n+2}\right)$$

$$= \lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} < 1 \xRightarrow{\text{Raabe's Test}} \sum_{n\geq 1} \frac{2\cdot 5\cdot \ldots \cdot (3n-1)}{3\cdot 6\cdot \ldots \cdot 3n}$$

$\Downarrow$

is divergent

Exercise 5.2.

Let (x_n) and (y_n) be two sequences of positive numbers. Suppose that the series $\sum_{n\geq 1} \frac{x_n}{y_n}$ and $\sum_{n\geq 1} y_n$ are both convergent. Is the series

$\sum_{n\geq 1} \sqrt{x_n}$ convergent as well?

We have the Second Comparison Test which says the following:

Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms

and $\sum_{n\geq 1} y_n$ a series with positive terms. Suppose

$\exists\, L = \lim_{n\to\infty} \frac{x_n}{y_n} \in [0, \infty) \cup \{\infty\}$. Then we

have 3 cases: $L \in (0, \infty)$, $L = 0$ and $L = \infty$

But we know that $\sum_{n\geq 1} \frac{x_n}{y_n}$ - converges

and by the n^{th} term test we know that if

a series $\sum_{n\geq 1} x_n$ - converges, then $\lim_{n\to\infty} x_n = 0 \Rightarrow$

$\Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = 0 \Rightarrow$ we are in the case for SCT where $L = 0$

We know that:

for $L=0$: if $\sum_{n\geq 1} y_n$ is convergent then $\sum_{n\geq 1} x_n$ is convergent

And because we actually know $L=0$ and

$\sum_{n\geq 1} y_n$ - convergent $\Rightarrow \sum_{n\geq 1} x_n$ is also convergent

We know that $\sum_{n\geq 1} y_n$ and $\sum_{n\geq 1} \frac{x_n}{y_n}$ are convergent series. Then.

$\sum_{n\geq 1} \left(y_n + \frac{x_n}{y_n}\right)$ is convergent

We apply the mean inequality for $\frac{x_n}{y_n}$ and y_n:

$$G\left(\frac{x_n}{y_n}, y_n\right) \leq A\left(\frac{x_n}{y_n}, y_n\right)$$

$$\sqrt{\frac{x_n}{\cancel{y_n}} \cdot \cancel{y_n}} \leq \frac{\frac{x_n}{y_n} + y_n}{2}$$

$$\sqrt{x_n} \leq \frac{1}{2}\left(\frac{x_n}{y_n} + y_n\right) \quad (1)$$

Because $\sum_{n\geq 1} \left(\frac{x_n}{y_n} + y_n\right)$ is convergent $\Rightarrow$

$\sum_{n\geq 1} \frac{1}{2}\left(\frac{x_n}{y_n} + y_n\right)$ is also convergent (2)

From (1), (2) $\overset{F.C.T.}{\Longrightarrow} \sum_{n\geq 1} \sqrt{x_n}$ is convergent

Exercise 6.1. Study if the following series are absolutely convergent, semi - convergent or divergent:

a) $$\sum_{n\geq 1} \underbrace{\frac{(-1)^{n+1}}{n\sqrt{n+1}}}_{x_n}$$

$$|x_n| = \left|\frac{(-1)^{n+1}}{n\sqrt{n+1}}\right| = \frac{|(-1)^{n+1}|}{|n\sqrt{n+1}|}$$

$$|x_n| = \frac{1}{n\sqrt{n+1}}$$

$$\frac{1}{n\sqrt{n+1}} \leq \frac{1}{n\sqrt{n}} \quad 1$$

$$\sum_{n\geq 1} \frac{1}{\sqrt{n^3}} = \sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}} \text{ - convergent } 2$$

From 1, 2 $\xRightarrow{F.C.T.}$ $\sum_{n\geq 1} \frac{1}{n\sqrt{n+1}}$ - convergent $\Rightarrow$

$\Rightarrow \sum_{n\geq 1} \frac{(-1)}{n\sqrt{n+1}}$ - absolutely convergent

e) $\sum\limits_{n\geq 1} \frac{n}{n^2+1} \cdot \cos(n\pi)$

$$\sum_{n\geq 1} \underbrace{\frac{n}{n^2+1} \cdot (-1)^n}_{x_n}$$

$$|x_n| = \left|\frac{n}{n^2+1} \cdot (-1)^n\right|$$

$$|x_n| = \frac{n}{n^2+1} = \sum_{n\geq 1} |x_n| = \sum_{n\geq 1} \underbrace{\frac{n}{n^2+1}}_{y_n}$$

$\sum\limits_{n\geq 1} \underbrace{\frac{1}{n}}_{z_n}$ - divergent (1)

S.C.T.

$$\lim_{n\to\infty} \frac{y_n}{z_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \in (0,\infty) \Rightarrow$$

$\sum\limits_{n\geq 1} y_n$ - divergent $\Leftrightarrow$ $\sum\limits_{n\geq 1} z_n$ - divergent (2)

From (1), (2) $\Rightarrow$ $\sum\limits_{n\geq 1} y_n$ - divergent $\Rightarrow$

$\Rightarrow$ $\sum\limits_{n\geq 1} x_n$ - is not absolutely convergent (I)

$$x_n = \frac{n}{n^2+1} \cdot (-1)^n$$

$|x_n|$ - decreasing (1)

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{n^2+1}\cdot(-1)^n = \lim_{n\to\infty} \frac{\cancel{n}^{1}}{\cancel{n^2}_{n}(1+\underbrace{\frac{1}{n}}_{0})}\cdot(-1)^n$$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot(-1)^n \in \{-1,1\} = 0 \Rightarrow \lim_{n\to\infty} x_n = 0 \quad (2)$$

Leibniz

From (1), (2) $\Rightarrow \sum_{n\geq 1} x_n$ - is convergent (II)

From (I), (II) $\Rightarrow \sum_{n\geq 1} x_n$ is semi-convergent

Exercise 6.2. Let $f, g : [0,1] \to \mathbb{R}$ be continuous such that $f(x) = g(x)$, $\forall x \in [0,1] \cap \mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0,1]$. Is it enough solely to assume that f and g are continuous on $[0,1] \setminus \{\alpha\}$ for some $\alpha \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$?

PROOF:

Let $y \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$ be irrational.
Let (x_n) be a sequence of rationals in $[0,1]$ converging to y. Since f and g are continuous and equal on $\mathbb{Q}$,

$$f(y) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(y)$$

Which means it is enough solely to assume that f and g are continuous on $[0,1] \setminus \{y\}$ for some $y \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$.