Bledea Mihaela Alexandra
911

3.

$f : \mathbb{R}^2 \to \mathbb{R}, \; f(x,y) = x^3 + 3xy^2 + 6xy$

a)

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 + 6y$$

$$\frac{\partial f}{\partial y}(x,y) = 6xy + 6x$$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$$

$$\nabla f(x,y) = (3x^2 + 3y^2 + 6y, \; 6xy + 6x)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = 6x$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = 6y + 6$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = 6x$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = 6y + 6$$

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 6y+6 \\ 6y+6 & 6x \end{pmatrix}$$

Bledea Mihaela
Alexandra
911

a)

$$\nabla f(x,y) = (3x^2+3y^2+6y,\ 6xy+6x)$$
$$\nabla f(x,y) = (0,0) \quad \Big\} \Rightarrow$$

$$\Rightarrow \begin{cases} 3x^2+3y^2+6y=0 \\ 6xy+6x=0 \end{cases} \Leftrightarrow \begin{cases} 3x^2+3y^2+6y=0 \\ 6x(y+1)=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3x^2+3y^2+6y=0 \\ x=0 \vee y=-1 \end{cases}$$

I. $x=0 \Rightarrow 3y^2+6y=0 \Leftrightarrow 3y(y+2)=0 \Leftrightarrow y=0 \vee y=-2$

$$\Rightarrow \begin{cases} x=0 \\ \wedge \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ \wedge \\ y=-2 \end{cases}$$

II. $y=-1 \Rightarrow 3x^2+3-6=0 \Leftrightarrow 3x^2=3 \Leftrightarrow x=1 \vee x=-1 \Rightarrow$

$$\Rightarrow \begin{cases} x=1 \\ \wedge \\ y=-1 \end{cases} \vee \begin{cases} x=-1 \\ \wedge \\ y=-1 \end{cases}$$

I, II $\Rightarrow \{(0,0), (0,-2), (1,-1), (-1,-1)\} \rightarrow$ the set of stationary points.

Bledea Mihaela Alexandra
911

1. $(0,0)$, $H_f(0,0) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$

$\phi_c = (h_1 \cdot h_2) \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$\phi_c = (6h_2 \, 6h_1) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$\phi_c = 6h_1h_2 + 6h_1h_2$

$\phi_c = 12h_1h_2$

We take $v_1 = (-1,1)$ and $v_2 = (1,1)$

$$\left.\begin{array}{l} \phi_c(-1,1) = -12 < 0 \\ \phi_c(1,1) = 12 > 0 \end{array}\right\} \Rightarrow H_f(0,0) \text{ - indefinite } \Rightarrow$$

$\Rightarrow (0,0)$ - is not a local extremum point

2. $(0,-2)$, $H_f(0,-2) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$

$\phi_c = (h_1 \, h_2) \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$\phi_c = (-6h_2 \; -6h_1) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$\phi_c = -6h_1 \cdot h_2 - 6h_1 \cdot h_2$

$\phi_c = -12h_1h_2$

We take $v_1 = (1,1)$ and $v_2 = (-1,1)$

Bedea Mihaela Alexandra
311

$$\left.\begin{array}{l}\Phi_c(1,1) = -12 < 0 \\ \Phi_c(-1,1) = 12 > 0\end{array}\right\} \Rightarrow H_f(0,-2) \text{ - indefinite } \Rightarrow$$

$\Rightarrow (0,-2)$ - is not a local extremum point.

3. $(1,-1)$, $H_f(1,-1) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$\Delta_1 = 6 > 0$, $\Delta_2 = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0 \Rightarrow H_f$ - positive definite $\Rightarrow$

$\Rightarrow (1,-1)$ - local minimum point

4. $(-1,-1)$, $H_f(-1,-1) = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$

$\Delta_1 = -6 < 0$, $\Delta_2 = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 > 0 \Rightarrow H_f$ - negative definite $\Rightarrow$

$\Rightarrow (-1,-1)$ - local maximum point

c). Because $(1,-1)$ is the unique local minimum point and $(-1,-1)$ is the unique local maximum point $\Rightarrow$ $(-1,-1)$ and $(1,-1)$ are global extremum points.

1.

Bendea Mihaela
Alexandra
911

a) $x_n = \ln(n+2) - \ln(n+1),\ n \in \mathbb{N}$

$x_{n+1} - x_n = \ln(n+3) - \ln(n+2) - \ln(n+2) + \ln(n+1)$

$x_{n+1} - x_n = \ln(n+3) - \ln(n+2)^2 + \ln(n+1)$

$x_{n+1} - x_n = \ln \dfrac{n+3}{(n+2)^2} + \ln(n+1)$

$x_{n+1} - x_n = \ln \dfrac{(n+3)(n+1)}{(n+2)^2}$

iii)

$x_1 = \ln 3 - \ln 2$

$x_2 = \ln 4 - \ln 3$

$x_3 = \ln 5 - \ln 4$

$\vdots$

$x_{n-1} = \ln(n+1) - \ln n$

$x_n = \ln(n+2) - \ln(n+1)$ $\quad |+$

$\Rightarrow x_1 + x_2 + \dots + x_n = \ln(n+2) - \ln 2$

$\cancel{\sum_{n \geq 1} x_n = \sum_{n \geq 1} (\ln(n+2) - \ln(n+1))}$

$\cancel{= \ln 3 - \ln 2 + \ln 4 - \ln 3 + \dots + \ln(n+1) - \ln n + \ln(n+2) - \ln(n+1)}$

$\cancel{= \ln(n+2) - \ln 2}$

Bledea Mihaela

Bledea Mihaela
Alexandra
311

$$\sum_{n\geq 1} x_n = \sum_{n\geq 1} (\ln(n+2) - \ln(n+1))$$

$$S_k = \sum_{i=1}^{k} (\ln(i+2) - \ln(i+1))$$

$$= \cancel{\ln 3} - \ln 2 + \cancel{\ln 4} - \cancel{\ln 3} + \dots + \ln\cancel{(k+1)} - \cancel{\ln k} + \ln(k+2) - \ln\cancel{(k+1)}$$

$$= \ln(k+2) - \ln 2$$

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty}(\ln(k+2) - \ln 2) = \infty \notin \mathbb{R} \Rightarrow$$

$\Rightarrow$ the series is divergent

i) $x_n = \ln(n+2) - \ln(n+1)$

If $\lim\limits_{n\to\infty} x_n \in \mathbb{R} \Rightarrow x_n$ - convergent

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (\ln(n+2) - \ln(n+1))$$

$$= \lim_{n\to\infty} \ln \frac{n+2}{n+1} = \ln\left(\lim_{n\to\infty} \frac{n+2}{n+1}\right) = \ln 1 = 0 \in \mathbb{R} \Rightarrow$$

$\Rightarrow x_n$ - convergent $\Rightarrow x_n$ - bounded

$x_n = \ln \frac{n+2}{n+1} \rightarrow$ strictly increasing because the base is $e > 0 \Rightarrow$

$\Rightarrow x_n$ - is monotone

ii)

Bledea Mihaela Alexandra
911

$$\lim_{n\to\infty} ((3n+1)\cdot x_n)$$

$$= \lim_{n\to\infty} (3n+1)\cdot \ln \frac{n+2}{n+1} = \infty \cdot 0$$

$$= \lim_{n\to\infty} \left(3n \cdot \ln \frac{n+2}{n+1} + \ln \frac{n+2}{n+1} \right)$$

$$= 3 \cdot \lim_{n\to\infty} n \cdot \ln \frac{n+2}{n+1} + \lim_{n\to\infty} \frac{n+2}{n+1}$$

$$= 3 \cdot \lim_{n\to\infty} n \cdot \ln \frac{n+2}{n+1}$$

$f : [0, \infty) \to \mathbb{R}$, $f(x) = \ln \frac{x+2}{x+1} \cdot x$

$n+1 \neq 0 \Rightarrow n \neq 1$

$n+2 \neq 0 \Rightarrow n \neq -2$

$\frac{n+2}{n+1} > 0$

$\ln \frac{x+2}{x+1}$ - strictly ~~ascending~~ increasing

x - strictly increasing, $\forall x \in [0, \infty)$

$\Rightarrow$

$$\lim_{x\to\infty} x \cdot \ln \frac{x+2}{x+1}$$

$\Rightarrow f$ - strictly increasing

$$\lim_{x\to\infty} x \cdot \ln \frac{x+2}{x+1} = \lim_{x\to\infty} \frac{\ln \frac{x+2}{x+1}}{\frac{1}{x}} = \frac{0}{0} \overset{\text{l'Hospital}}{=}$$

$$= \lim_{x\to\infty} \frac{\left(\ln \frac{x+2}{x+1}\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty} \frac{\frac{-1}{(x+1)(x+2)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{(x+1)(x+2)}$$

Bledea Mihaela

Bledea Mihaela
Alexandra
911

$$\ln\left(\frac{x+2}{x+1}\right)' = \frac{1}{\frac{x+2}{x+1}} \cdot \left(\frac{x+2}{x+1}\right)'$$

$$= \frac{x+1}{x+2} \cdot \frac{x+1-(x+2)}{(x+1)^2}$$

$$= \frac{\cancel{x+1}}{x+2} \cdot \frac{(-1)}{\cancel{(x+1)^2}\,(x+1)} = \frac{-1}{(x+1)(x+2)}$$

$$= \lim_{x\to\infty} \frac{x^2}{(x+1)(x+2)} = \underline{1}$$

From the fact that we used f to compute $\lim\limits_{x\to\infty} f(x)$ is like that we could apply l'Hospital

$$\Rightarrow 3 \cdot \underbrace{\lim_{n\to\infty} n \cdot \ln\frac{n+2}{n+1}}_{= 1} = 3$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{4}+\ldots+\frac{1}{3n-2}}{\ln(n+1)}$$

$b_n = \ln(n+1) \to \infty,\ n\to\infty$

b_n - strictly increasing

$$a_n = 1+\frac{1}{4}+\ldots+\frac{1}{3n-2}$$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\cancel{1}+\cancel{\frac{1}{4}}+\ldots+\cancel{\frac{1}{3n-2}}+\frac{1}{3n+1}-\cancel{1}-\cancel{\frac{1}{4}}-\ldots-\cancel{\frac{1}{3n-2}}}{\ln(n+2)-\ln(n+1)}$$

Bledea Mihaela
Alexandra
911

$$\left.\begin{array}{l} = \dfrac{1}{3n+1} \cdot \dfrac{1}{\ln\left(\frac{n+2}{n+1}\right)} = \dfrac{1}{(3n+1)\ln\left(\frac{n+2}{n+1}\right)} \\ \text{from the previous limit we know that} \\ \lim\limits_{n \to \infty} (3n+1)\ln\left(\dfrac{n+2}{n+1}\right) = 3 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{1}{3} \in [0, \infty) \cup \{\infty\} \Rightarrow$$

$$\Rightarrow \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{3}$$

Bodea Mihaela
Alexandra
911

e) $x_n \in [0,2)$

$\sum_{n\ge 1} x_n$ - convergent

$\sum_{n\ge 1} \frac{x_n}{4-x_n^2}$ - convergent?

$$\sum_{n\ge 1} \frac{x_n}{(2-x_n)(2+x_n)}$$

$$\frac{x_n}{(2-x_n)(2+x_n)} = \frac{x_n}{(2-x_n)} \cdot \frac{1}{\underbrace{(2+x_n)}_{\ge 1}} \le \frac{x_n}{2-x_n}$$

$$\frac{x_n}{\underbrace{2-x_n}_{\in(0,2]}} \ge x_n$$

if $2-x_n \in [1,2] \Rightarrow \left.\begin{array}{l}\frac{x_n}{2-x_n} \le x_n \\ x_n\text{-convergent}\end{array}\right\} \overset{F.C.T.}{\Rightarrow} \frac{x_n}{2-x_n}$ - is convergent

but if $(2-x_n) \in (0,1)$ $\frac{x_n}{2-x_n} \ge x_n$

$$\frac{x_n+2-2}{2-x_n} = \frac{x_n-2+2}{2-x_n} = \frac{-(2-x_n)+2}{2-x_n}$$

$= \frac{2}{2-x_n} - 1$ $\quad 0 \le x_n < 2 \ |\cdot(-1) \Leftrightarrow -2 < -x_n \le 0 \ |+2$

$0 < 2-x_n \le 2 \Leftrightarrow \frac{1}{2} \le \frac{1}{2-x_n} < 0 \ |\cdot 2$

$1 \le \frac{2}{2-x_n} < 0$ (False) $\Rightarrow$ we can't bound it

Bledea Mihaela
Alexandra
911

because we cant bound it $\Rightarrow$

$$\Rightarrow \frac{x_n}{2-x_n} \text{ - is divergent } \Rightarrow$$

$$\left.\frac{x_n}{4-x_n^2} < \frac{x_n}{2-x_n}\right\} \overset{F.C.T.}{\Rightarrow} \sum_{n\geq 1} \frac{x_n}{4-x_n^2} \text{ - divergent}$$

Bledea Mihaela
Alexandra 211

2. $f:(0,\infty)\times[0,\infty)$, $f(x,y)=\dfrac{e^{x-y}-\cos(x-y)}{x+y}$

$a_n=(\frac{1}{n},0)$

$$\lim_{n\to\infty} f(a_n)=\lim_{n\to\infty}\frac{e^{\frac{1}{n}}-\cos\frac{1}{n}}{\frac{1}{n}}=\frac{0}{0}\overset{\text{l'Hospital}}{=}$$

$$=\lim_{n\to\infty}\frac{\left(-\frac{1}{n^2}\right)\cdot e^{\frac{1}{n}}+\sin\frac{1}{n}}{-\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{\to 1}\cdot(-n)+\lim_{n\to\infty}e^{\frac{1}{n}}$$

$$=1\cdot(-\infty)+1=-\infty \quad (1)$$

$b_n=(\frac{1}{n},\frac{1}{n})$

$$\lim_{n\to\infty} f(b_n)=\lim_{n\to\infty}\frac{e^0-\cos 0}{\frac{2}{n}}=\lim_{n\to\infty}\frac{1-1}{\frac{2}{n}}=0 \quad (2)$$

$$(1),(2)\Rightarrow \lim_{n\to\infty} f(a_n)\neq\lim_{n\to\infty} f(b_n)\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

Bledea Mihaela Alexandra
911

4.

a) $f : [0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2}{4+x^6}$

f - improperly integrable on $[0, \infty)$ if

$\exists \lim_{t \to \infty} \int_0^t \frac{x^2}{4+x^6} dx$

$$J = \int_0^t \frac{x^2}{4+x^6} dx = \int_0^t \frac{x^2}{4+(x^3)^2} dx$$

$$x^3 = u \Rightarrow 3x^2 dx = du \quad \begin{vmatrix} u = 0 \\ u = t^3 \end{vmatrix}$$

$$J = \frac{1}{3} \int_0^{t^3} \frac{3 \cdot x^2}{4+(x^3)^2} dx = \frac{1}{3} \int_0^{t^3} \frac{1}{2^2+u^2} du$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot \arctan \frac{u}{2} \Big|_0^{t^3} = \frac{1}{6} \Big(\arctan \frac{t^3}{2} - \underbrace{\arctan 0}_{\shortparallel \atop 0} \Big)$$

$$= \frac{1}{6} \cdot \arctan \frac{t^3}{2}$$

$$\lim_{t \to \infty} \frac{1}{6} \cdot \arctan \frac{t^3}{2} = \frac{1}{6} \cdot \underbrace{\arctan \infty}_{\shortparallel \atop \frac{\pi}{2}} = \frac{\pi}{12} \in \mathbb{R} \Rightarrow$$

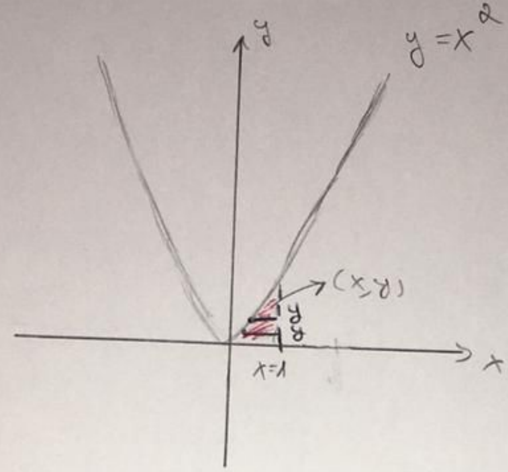
$\Rightarrow f$ is improperly integrable on $[0, \infty)$

$$\int_0^\infty \frac{x^2}{4+x^6} dx = \frac{\pi}{12}$$

Bedea Mihaela
Alexandra
911

e) $y = x^2$, $y = 0$, $x = 1$.

~~$M \subseteq \mathbb{R}^2$ simple set wrt~~



$M = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le x^2\}$ - with respect to y axis

$M = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\}$ - with respect to x axis

$$\iint_M e^{x^3} dx dy$$

$$J = \int_0^{x^2} \int_0^1 e^{x^3} dx dy$$