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1.1. Prove that for every $n \in \mathbb{N}$, $n \geq 2$ we have

$$\sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}$$

$$P(n): \sum_{m=1}^{n} \frac{1}{\sqrt{m}} > \sqrt{n}$$

$$P(2): \sum_{m=1}^{2} \frac{1}{\sqrt{m}} > \sqrt{2} \iff \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2} \Big| \cdot \sqrt{2} \iff$$

$$\sqrt{2} + 1 > 2 \iff \sqrt{2} > 1 \Rightarrow$$

$\Rightarrow P(2)$ - is true (1)

Suppose that $P(k)$ is true and prove $P(k+1)$ is true as well

$$P(k): \sum_{m=1}^{k} \frac{1}{\sqrt{m}} > \sqrt{k} \;\Big|+ \frac{1}{\sqrt{k+1}} \iff \underbrace{\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}}}_{A} > \underbrace{\sqrt{k} + \frac{1}{\sqrt{k+1}}}_{B}$$

$$P(k+1): \underbrace{\sum_{m=1}^{k+1} \frac{1}{\sqrt{m}}}_{A} > \underbrace{\sqrt{k+1}}_{C}$$

$\Rightarrow$

If we have $A > B$ and we want to prove $A \geq C$ then it is enough to prove $B > C$

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \;\Big| \cdot \sqrt{k+1}$$

$$\sqrt{k^2+k} + \cancel{1} > k + \cancel{1}$$

$$\sqrt{k^2+k} > k \;\big|\,()^2, \; k \in \mathbb{N}$$

$$k^2 \cancel{+} k > \cancel{k^2} \Rightarrow k > 0 \; (A), \; \forall k \in \mathbb{N} \Big\} \Rightarrow$$

$\Rightarrow P(k+1)$ - is true (2)

1.3. Prove that for every $n \in \mathbb{N}$ with $n \geq 2$ and for any numbers $x_1, x_2, \dots, x_n \in [-1, \infty)$ all of the same sign, we have

$$(1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_n) \geq 1+x_1+x_2+\ldots+x_n$$

$P(n): (1+x_1)\cdot(1+x_2)\cdot\ldots\cdot(1+x_n) \geq 1+x_1+x_2+\ldots+x_n$
$\forall n \in \mathbb{N}, n \geq 2, \forall x_1, x_2, \dots, x_n \in [-1, \infty)$, all of the same sign

$P(2): (1+x_1)\cdot(1+x_2) \geq 1+x_1+x_2$

$P(2): \cancel{1}+\cancel{x_1}+\cancel{x_1}+x_1\cdot x_2 \geq \cancel{1}+\cancel{x_1}+\cancel{x_2} \iff$

$x_1 \cdot x_2 \geq 0$ (T) because x_1 and x_2 have the same sign $\Rightarrow P(2)$ - is true (1)

$P(k): (1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_k) \geq 1+x_1+x_2+\ldots+x_k \;\Big|\cdot(1+x_{k+1}) \geq 0$

$x_{k-1} \in [-1, +\infty)$

$P(k+1): \underbrace{(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_k)(1+x_{k+1})}_{A} \geq \underbrace{1+x_1+x_2+\ldots+x_k+x_{k+1}}_{C}$

$P(k): \underbrace{(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_k)(1+x_{k+1})}_{A} \geq \underbrace{(1+x_{k+1})(1+x_1+\ldots+x_k)}_{B}$

If we have $A \geq B$ and we want to prove $A \geq C$ then it is enough to prove $B \geq C$

$\cancel{1+x_1+x_2+\ldots+x_k}+x_{k+1}(1+x_1+x_2+\ldots+x_k) \geq \cancel{1+x_1+x_2+\ldots+x_k}+x_{k+1}$

$\cancel{x_{k+1}}+x_1\cdot x_{k+1}+\overset{(=)}{\ldots}+x_{k+1}\cdot x_k \geq \cancel{x_{k+1}}$

(=)

$x_1 \cdot x_{k+1} + x_2 \cdot x_{k+1} + \dots + x_k \cdot x_{k+1} \geq 0$ (T) because

$x_1, x_2, \dots, x_{k+1}$ all have the same sign $\Rightarrow$

$\Rightarrow P(k+1)$ - is true (2)

From (1), (2) $\Rightarrow P(n)$ is true, $\forall n \in \mathbb{N}$, $n \geq 2$

$\forall x_1, x_2, \dots, x_n \in [-1, \infty)$

Exercise 2.1.

For each set A_i from below find $lb(A_i)$ and $ub(A_i)$ (as subsets of $\overline{\mathbb{R}}$), $\min(A_i)$ and $\max(A_i)$ (if they exist), and $\inf(A_i)$ and $\sup(A_i)$ (in $\overline{\mathbb{R}}$)

1) $A_1 = [-8, 4) \cap \mathbb{Z} \Rightarrow A_1 = \{-8, -7, \dots, 2, 3\}$

$lb(A_1) = (-\infty, -8]$

$\inf(A_1) = -8$

$\min(A_1) = -8$

$ub(A_1) = [3, +\infty)$

$\sup(A_1) = 3$

$\max(A_1) = 3$

2) $A_2 = \{2^m + n! \mid m, n \in \mathbb{N}\}$

$m, n \in \mathbb{N} \Rightarrow m, n \geq 1$

For $m = 1, n = 1$ we have $2^1 + 1! = 3 \rightarrow$ this is the first element $\Rightarrow lb(A_2) = (-\infty, 3]$

$\inf(A_2) = 3$

$\min(A_2) = 3$

$ub(A_2) = \emptyset$

$\sup(A_2) = +\infty$

no maximum

3) $A_3 = \left\{ x + \frac{1}{x} \mid x \in \mathbb{R},\ x < 0 \right\}$

$f : (-\infty, 0) \to \mathbb{R},\ f(x) = x + \frac{1}{x}$

$\operatorname{Im} f = A_3$

$f'(x) = \left(x + \frac{1}{x}\right)' = x' + \left(\frac{1}{x}\right)' = 1 - \frac{1}{x^2}$

$f'(x) = 0 \iff 1 - \frac{1}{x^2} = 0 \iff \frac{1}{x^2} = 1 \iff x = 1 \vee x = -1$

$-\frac{1}{2}$

x	$-\infty$		-1		0
$f'(x)$		$+ + + + + + +$	0	$- - - - -$	
$f(x)$	$\lim\limits_{-\infty} f(x)$	$\nearrow$	$f(-1)$	$\searrow$	$l_s(0)$

$f'\left(-\frac{1}{2}\right) = 1 - \frac{1}{\left(-\frac{1}{2}\right)^2} = 1 - \frac{1}{\frac{1}{4}} = 1 - 4 = -3 < 0$

$f'(-2) = 1 - \frac{1}{(-2)^2} = \frac{4}{1} - \frac{1}{4} = \frac{3}{4} > 0$

(1) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to \infty} \left(x + \frac{1}{x}\right) = -\infty + \left(\frac{1}{-\infty}\right)^{\to 0} = -\infty$

(2) $\lim\limits_{\substack{x \to 0 \\ x < 0}} f(x) = \lim\limits_{\substack{x \to 0 \\ x < 0}} \left(x + \frac{1}{x}\right) = 0 + \left(\frac{1}{0_-}\right)^{\to -\infty} = -\infty$

(3) $f(-1) = -1 + \frac{1}{-1} = -2$

f - is a continuous function on $(-\infty, 0)$ - being a composition of elementary functions and continuous functions (4)

From (1), (2), (3), (4) $\Rightarrow \operatorname{Im} f = (-\infty, -2] = A_3 \Rightarrow$

$$\Rightarrow \begin{cases} lb(A_3) = \emptyset \\ \inf(A_3) = -\infty \\ \text{no minimum} \end{cases}$$

$$\wedge$$

$$\begin{cases} ub(A_3) = [-2 + \infty) \\ \sup(A_3) = -2 \\ \max(A_3) = -2 \in A_3 \end{cases}$$

4) $A_4 = \left\{ \frac{n}{1-n^2} \,\middle|\, n \in \mathbb{N}, n \geq 2 \right\}$

$n = 2: \quad \frac{2}{1-2^2} = -\frac{2}{3}$

$n = 3: \quad \frac{3}{1-3^2} = -\frac{3}{8}$

$n = 4: \quad \frac{4}{1-16} = -\frac{4}{15}$

$-\frac{2}{3} < -\frac{3}{8} < -\frac{4}{15}$

We have here a strictly ascending function

$\Rightarrow lb(A_4) = (-\infty, -\frac{2}{3}]$

$\inf(A_4) = -\frac{2}{3}$

$\min(A_4) = -\frac{2}{3} \in A_4$

$\sup(A_4) = 0$

We suppose that there $\exists\, a \in ub(A_4)$, $a < 0$

such that $\frac{n}{1-n^2} < a$, $\forall n \in \mathbb{N}, n \geq 2$

$\frac{n}{1-n^2} < a \;\Big|\; \cdot (1-n^2) < 0$

$n > a(1-n^2)$

$n > a - an^2$

$n + an^2 > a \iff n^2 a - a + n > 0$

$a < 0 \Rightarrow b = -a$, $b \in ub(A_4)$

We put b instead of a in the inequality

$-bn^2 + b + n > 0$

n	$-\infty$		n_1		n_2		$+\infty$
$-bn^2 + b + n$		$+ + + +$	0	$- - -$	0	$+ + + +$	

$-bn^2 + b + n = 0$

$\Delta = b^2 - 4 \cdot (-b) + 1$

$\Delta = b^2 + 4b > 0 \Rightarrow$ the equation has two solutions ②

$$n_{1,2} = \frac{-b \pm \sqrt{b^2 + 4b}}{-2}$$

$$n_1 = \frac{\cancel{-}(b + \sqrt{b^2 + 4b})}{\cancel{-}2} \Rightarrow n_1 = \frac{b + \sqrt{b^2 + 4b}}{2}$$

$$n_2 = \frac{b - \sqrt{b^2 + 4b}}{2}$$

①

From ①, ② $\Rightarrow$ the inequality won't be true for any n from $\mathbb{N}$ $\Rightarrow$

$\Rightarrow$ that's a contradiction $\Rightarrow$

$\Rightarrow \sup(A_4) = 0$

$ub(A_4) = [0, +\infty)$

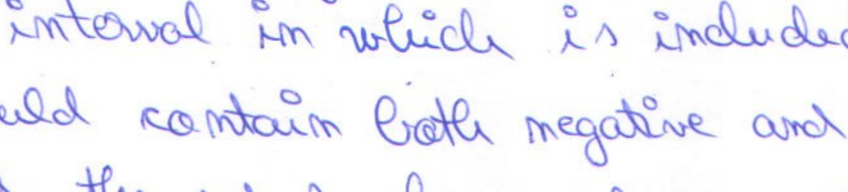
no maximum because $\sup(A_4) \notin A_4$

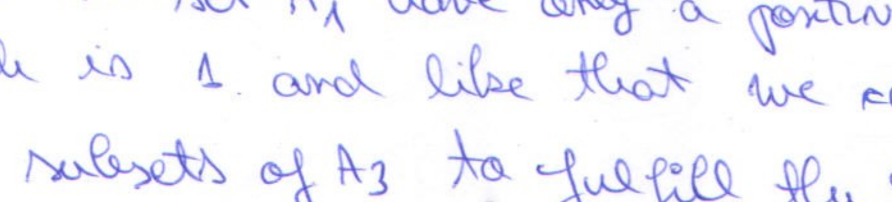
Exercise 2.3. Decide which of the following sets are neighborhoods of 0. Justify.

1) $A_1 = (-1, 0] \cup \{1\}$

$A_1 \notin \mathcal{V}(0)$ because for a set to be a neighborhood of 0, it should firstly be an interval in which is included 0 and should contain both negative and positive numbers and the set A_1 have only a positive number which is 1 and like that we can't create any subsets of A_3 to fulfill the requirements.

2) $A_2 = \left[1 - \frac{3}{2}, 1 + \frac{3}{2}\right] \cup (3, 4)$





We take $\varepsilon = -\frac{1}{2} \Rightarrow \left.\begin{array}{l} \left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq A_1 \\ 0 \in \left(-\frac{1}{2}, \frac{1}{2}\right) \end{array}\right\} \Rightarrow A_2 \in \mathcal{V}(0)$

3) $A_3 = \mathbb{R}$

We take $\varepsilon = 1 \Rightarrow \left.\begin{array}{l}(-1;1) \subseteq A_3 \\ 0 \in (-1,1)\end{array}\right\} \Rightarrow A_3 \in \mathcal{V}(0)$

4) $A_4 = \mathbb{R} \setminus \mathbb{Q}$

A_4 is not a neighborhood of 0 because between any irrational numbers we can find a rational numbers

Density of Rational numbers

If $x, y \in \mathbb{R} \setminus \mathbb{Q}$ $x < y$, then $\exists r \in \mathbb{Q}$ such that

$x < r < y$, $r \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow r \in A_4$, $(x, y) \in A_4 \Rightarrow$

$\Rightarrow A_4$ - doesn't contain any intervals $\Rightarrow$

$A_4 \notin \mathcal{V}(0)$

Exercise 3.1. Find the limit (as $n \to \infty$) of the sequence whose general term x_n, $n \in \mathbb{N}$, is given below:

a) $$\lim_{n\to\infty} \frac{n+\sin(n^2)}{\cos(n)-3n} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{\sin(n^2)}{n}\right)}{\cancel{n}\left(\frac{\cos(n)}{n}-3\right)}$$

$$\left.\begin{array}{l} \sin n \in [-1,1] \Rightarrow \sin(n^2) \in [-1,1] \\ \frac{1}{n} \to 0 \text{ when } n \to \infty \end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{\sin(n^2)}{n} = 0$$

$$\left.\begin{array}{l} \cos n \in [-1,1] \\ \frac{1}{n} \to 0 \text{ when } n \to \infty \end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{\cos(n)}{n} = 0$$

$$\lim_{n\to\infty} \frac{1+\overbrace{\frac{\sin(n^2)}{n}}^{\to 0}}{\underbrace{\frac{\cos(n)}{n}}_{\downarrow\, 0}-3} = -\frac{1}{3}$$

e) $\lim\limits_{n \to \infty} (n^2+n)^{-\frac{n}{n+1}} = l_1^{\,l_2}$ (1)

we distribute the limit to the base and to the exponent

$l_1 = \lim\limits_{n \to \infty} (n^2+n) = \infty^2 + \infty = \infty$ (2)

$l_2 = \lim\limits_{n \to \infty} \left(-\frac{n}{n+1}\right) = \lim\limits_{n \to \infty} \frac{-\cancel{n}}{\cancel{n}(1+\underbrace{\frac{1}{n}}_{0})} \Rightarrow l_2 = -1$ (3)

From (1), (2), (3) $\Rightarrow \lim\limits_{n \to \infty} (n^2+n)^{-\frac{n}{n+1}} = \infty^{-1} = \frac{1}{\infty} = 0$

c) $\lim_{n\to\infty}\left(1+\frac{1}{n^3+2n^2}\right)^{\overset{l_2}{n-n^3}} = l_1$ (1)

We distribute the limit to the base and to the exponent

$l_1 = \lim_{n\to\infty}\left(1+\frac{1}{n^3+2n^2}\right) = 1 + \lim_{n\to\infty}\frac{1}{n^3(1+\underbrace{\frac{2}{n}}_{0})}$

$l_1 = 1 + \lim_{n\to\infty}\left(\frac{1}{n^3}\right)^{\to 0} = 1$ (2)

$l_2 = \lim_{n\to\infty}(n-n^3) = \lim_{n\to\infty} n^3\left(\left(\frac{1}{n^2}\right)^{\to 0}-1\right)$

$l_2 = -\infty$ (3)

From (1), (2), (3) $\Rightarrow$ We can apply Euler

$\lim_{n\to\infty}\left(1+\frac{1}{n^3+2n^2}\right)^{n-n^3} = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{n^3+2n^2}\right)^{n^3+2n^2}}_{e}\right]^{\frac{1}{n^3+2n^2}\cdot(n-n^3)}$

$= e^{\underbrace{\lim_{n\to\infty}\frac{n-n^3}{n^3+2n^2}}_{l_3}}$

$l_3 = \lim_{n\to\infty}\frac{\cancel{n^3}\left(\left(\frac{1}{n^2}\right)^{\to 0}-1\right)}{\cancel{n^3}(1+\underbrace{\frac{2}{n}}_{0})} = \frac{-1}{1} = -1$

$\Rightarrow e^{-1} = \frac{1}{e}$

the result

d) $\lim\limits_{n\to\infty} \dfrac{1\cdot 1! + 2\cdot 2! + \dots + n\cdot n!}{(n+1)!}$

$a_n = 1\cdot 1! + 2\cdot 2! + \dots + n\cdot n!$

$b_n = (n+1)!$

b_n - strictly increasing

$\lim\limits_{n\to\infty} b_n = +\infty$

$$L = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

$$= \lim_{n\to\infty} \frac{\cancel{1\cdot 1!} + \cancel{2\cdot 2!} + \dots + \cancel{n\cdot n!} + (n+1)(n+1)! - \cancel{1\cdot 1!} - \dots - \cancel{n\cdot n!}}{(n+2)! - (n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)!}{(n+2)(n+1)! - (n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}\,(n+1)}{\cancel{(n+1)!}\,(n+2-1)} = \lim_{n\to\infty} \frac{n+1}{n+1} = 1 \overset{\text{Stolz}}{\underset{\text{Cesaro}}{=}} $$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

e) $\lim\limits_{n\to\infty} \sqrt[n]{1+2+\dots+n}$

$y_n = 1+2+\dots+n > 0 \quad \forall n \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{1+2+\dots+n+n+1}{1+2+\dots+n}$$

$a_n = 1+2+\dots+(n+1)$

$b_n = 1+2+\dots+n$

b_n - strictly increasing

$\lim\limits_{n\to\infty} b_n = +\infty$

$$L = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\cancel{1+2}+\dots+\cancel{(n+1)}+(n+2)-\cancel{1}-\cancel{2}-\dots-\cancel{(n+1)}}{\cancel{1+2}+\dots+(n+1)-\cancel{1}-\cancel{2}-\dots-\cancel{n}}$$

$$L = \lim_{n\to\infty} \frac{n+2}{n+1} = \underline{1}$$

Stolz Cesaro $\Rightarrow$ Cor. 3 $\quad \lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{y_{n+1}}{y_n} = \underline{1}$

f) $x_n = n\left(\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1\right)$

$1+\frac{1}{n} = 1 + n\cdot\frac{1}{n^2} \le \left(1+\frac{1}{n^2}\right)^n$ By applying Bernoulli's for $x = \frac{1}{n^2}$

$$1+\frac{1}{n} \le \left(1+\frac{1}{n^2}\right)^n \Big|^{\frac{1}{n}} \iff \left(1+\frac{1}{n}\right)^{\frac{1}{n}} \le 1+\frac{1}{n^2} \Big| \cdot \left(1+\frac{1}{n}\right) \iff$$

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{\frac{1}{n}} \le \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right) \iff$$

$$\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} \le 1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3} \quad (1)$$

$$0 < \frac{1}{n} \Big| +1 \iff 1 < \frac{1}{n}+1 \Rightarrow 1+\frac{1}{n} < \left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} \quad (2)$$

We use (1), (2) and we obtain:

$$1+\frac{1}{n} < \left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} \le 1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3} \Big| -1$$

$$\frac{1}{n} < \left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1 \le \frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3} \Big| \cdot n$$

$$\left.\begin{array}{l} 1 < n\left(\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1\right) \le 1+\frac{1}{n}+\frac{1}{n^2} \\ 1 \xrightarrow{n\to\infty} 1, \quad 1+\frac{1}{n}+\frac{1}{n^2} \xrightarrow{n\to\infty} 1 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ By applying Squeeze Theorem we have $\lim_{n\to\infty} x_n = 1$

Exercise 3.2.

For $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ such that $a_n \leq b_n$ and $\lim_{n \to \infty} (b_n - a_n) = 0$. Suppose, in addition, that $\forall n \in \mathbb{N}$, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. By the Nested Interval Property $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

Can $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contain more than one point?

Proof.

We consider a nested family of intervals $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, where for each n we have $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, then $a := \sup_n a_n$ and $b := \sup_n b_n$, both exist and $\bigcap_{n=1}^{\infty} I_n = [a, b] \neq \emptyset$

We also know that $\lim_{n \to \infty} [b_n - a_n] = 0$

We see that for all $m, n \in \mathbb{N}$ we have $a_n \leq b_m$, and therefore $\sup\{a_n : n \in \mathbb{N}\} \leq b_m$, $m \in \mathbb{N}$.

Therefore $a := \sup\{a_n : n \in \mathbb{N}\} \leq b := \inf\{b_n : n \in \mathbb{N}\}$.

This means $[a, b] \subset [a_n, b_n]$, for all $n \in \mathbb{N}$ so that

$$\bigcap_{n=1}^{\infty} I_n \supset [a, b] \neq \emptyset$$

If $x < a$ then $x < a_n$ for some n and if $x > b$ then $x > b_m$ for some m. In either case $x \notin [a_n, b_n]$ for some n and hence is not the intersection of all $[a_n, b_n]$. Therefore $\bigcap_{n=1}^{\infty} I_n = [a, b] \neq \emptyset$

Finally $b - a \leq b_n - a_n$ so that if $b_n - a_n$ is arbitrarily small then $b - a = 0$ and $[a, b] = \{a\} = \{b\}$