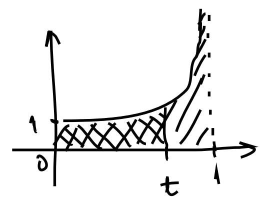
Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Mathematical Analysis - Lecture Notes
Computer Science, Academic Year: 2020/2021

Lecture 11

Improper integrals

$f:[0,1) \to \mathbb{R}$, $f(x) = \frac{1}{\sqrt{1-x^2}}$

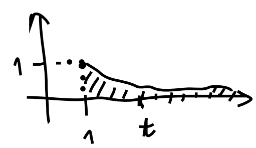


$x=1$ vertical asymptote of f

$t \in [0,1)$, $f|_{[0,t]}$, $A_t = \int_0^t \frac{1}{\sqrt{1-x^2}} dx = \arcsin t$ – the area under the graph of $f|_{[0,t)}$

So we can define the area under the graph of f as $A = \lim\limits_{\substack{t \to 1 \\ t<1}} A_t = \frac{\pi}{2}$

$f:[1,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$



$t \in [1,\infty)$, $A_t = \int_1^t \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^t = 1 - \frac{1}{t}$, so $A = \lim\limits_{t \to \infty} A_t = 1$.

Definition 1. Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$ with $a < b$ and $f:[a,b) \to \mathbb{R}$ continuous. We say that f is *improperly integrable on* $[a,b)$ if

$$\exists \lim_{\substack{t \to b \\ t<b}} \int_a^t f(x)dx \in \mathbb{R}.$$

In this case this limit is called the *improper integral of* f *on* $[a,b)$.
Notation: $\displaystyle\int_a^b f(x)dx$.
Alternative notation if $b \in \mathbb{R}$: $\displaystyle\int_a^{b-0} f(x)dx$.

Definition 2. Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$ with $a < b$ and $f:(a,b] \to \mathbb{R}$ continuous. We say that f is *improperly integrable on* $(a,b]$ if

$$\exists \lim_{\substack{t \to a \\ t>a}} \int_t^b f(x)dx \in \mathbb{R}.$$

In this case this limit is called the *improper integral of* f *on* $(a,b]$.
Notation: $\displaystyle\int_a^b f(x)dx$.
Alternative notation if $a \in \mathbb{R}$: $\displaystyle\int_{a+0}^b f(x)dx$.

Definition 3. Let $a, b \in \overline{\mathbb{R}}$ with $a < b$ and $f : (a, b) \to \mathbb{R}$ continuous. We say that f is *improperly integrable on* (a, b) if there exists $c \in (a, b)$ such that f is improperly integrable both on $(a, c]$ and on $[c, b)$. In this case the *improper integral of* f *on* (a, b) is defined as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Alternative notation:

$$\text{if } a, b \in \mathbb{R} : \int_{a+0}^{b-0} f(x)dx,$$
$$\text{if } a \in \mathbb{R}, b = \infty : \int_{a+0}^{\infty} f(x)dx,$$
$$\text{if } a = -\infty, b \in \mathbb{R} : \int_{-\infty}^{b-0} f(x)dx.$$

Remark 1. The above definition does not depend on the choice of $c \in (a, b)$.

Example 1. (i) Let $f : (1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x(\ln x)^2}$.

Let $t \in (1,2]$ Then $\displaystyle\int_t^2 \frac{1}{x(\ln x)^2}dx = -\frac{1}{\ln x}\Big|_t^2 = \frac{1}{\ln t} - \frac{1}{\ln 2} \xrightarrow{t \downarrow 1} \infty$

$\Rightarrow$ f not impr int on $(1,2]$

Since f is not impr int on one of the intervals $(1,2]$ and $[2,\infty)$

$\Rightarrow$ f is not impr int on $(1,\infty)$

(ii) Let $f : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x} - \dfrac{2}{2x-1}$.

Let $t \in [1,\infty)$ Then $\displaystyle\int_1^t f(x)dx = (\ln x - \ln(2x-1))\Big|_1^t = \ln t - \ln(2t-1) =$

$\displaystyle = \ln\frac{t}{2t-1} \xrightarrow{t\to\infty} \ln\frac{1}{2}$

Let $s, u \in [1,\infty)$

$\displaystyle\int_1^s \frac{1}{x}dx = \ln x\Big|_1^s = \ln s \xrightarrow{s\to\infty} \infty$ We cannot split the integrand

$\displaystyle\int_1^u \frac{2}{2x-1}dx = \ln(2u-1) \xrightarrow{u\to\infty} \infty$

Remark 2. Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$ with $a < b$ and $f : [a, b) \to \mathbb{R}$ continuous. Sometimes, even if the limit $\lim\limits_{\substack{t \to b \\ t<b}} \int_a^t f(x)dx$ does not exist or it exists, but is infinite, the expression $\int_a^b f(x)dx$ is called an improper integral which is said to be *convergent* if f is improperly integrable on $[a, b)$ and *divergent* otherwise.
In a similar way one defines convergence and divergence for improper integrals in the cases considered in Definitions 2, 3.

Remark 3. Improper integrability can be defined for more general functions than continuous ones, namely for functions $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is a nonempty interval, that are locally Riemann integrable (that is, for any $a, b \in I$ with $a < b$, $f \in \mathcal{R}[a, b]$). However, to simplify the discussion, we will only consider continuous functions.

Example 2. (i) Let $a, b \in \mathbb{R}$ with $a < b$, $p \in \mathbb{R}$ and $f : [a, b) \to \mathbb{R}$, $f(x) = \dfrac{1}{(b-x)^p}$.

f cont

Let $t \in [a, b)$ Then
$$\int_a^t \frac{1}{(b-x)^p}\,dx = \begin{cases} -\ln(b-x)\Big|_a^t, & p = 1 \\ -\dfrac{(b-x)^{1-p}}{1-p}\Big|_a^t, & p \neq 1 \end{cases} = \begin{cases} \ln(b-a) - \ln(b-t), & p = 1 \\ \dfrac{(b-a)^{1-p} - (b-t)^{1-p}}{1-p}, & p \neq 1 \end{cases}$$

$$\lim_{\substack{t \to b \\ t < b}} \int_a^t \frac{1}{(b-x)^p}\,dx = \begin{cases} \infty, & p \geq 1 \\ \dfrac{(b-a)^{1-p}}{1-p}, & p < 1 \end{cases} \Rightarrow f \text{ impr int on } [a, b) \Leftrightarrow p < 1$$

(ii) Let $a > 0$, $p \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^p}$.

f cont

Let $t \in [a, \infty)$ Then
$$\int_a^t \frac{1}{x^p}\,dx = \begin{cases} \ln x\Big|_a^t, & p = 1 \\ \dfrac{x^{1-p}}{1-p}\Big|_a^t, & p \neq 1 \end{cases} = \begin{cases} \ln t - \ln a, & p = 1 \\ \dfrac{t^{1-p} - a^{1-p}}{1-p}, & p = 1 \end{cases}$$

$$\lim_{t \to \infty} \int_a^t \frac{1}{x^p}\,dx = \begin{cases} \infty, & p \leq 1 \\ \dfrac{1}{(p-1)\,a^{p-1}}, & p > 1 \end{cases} \Rightarrow f \text{ is impr int on } [a, \infty) \Leftrightarrow p > 1$$

There exists a close connection between the theory of of improper integrals and of series of real numbers.

Theorem 1 (Integral Test for Convergence of Series)**.** *Let $m \in \mathbb{N}$ and $f : [m, \infty) \to [0, \infty)$ continuous and decreasing. Then f is improperly integrable on $[m, \infty)$ if and only if the series $\sum_{n \geq m} f(n)$ is convergent.*

Example 3. (The generalized harmonic series) $\sum_{n \geq 1} \dfrac{1}{n^\alpha}$, where $\alpha \in \mathbb{R}$.

$\alpha \leq 0$: $\lim_{n \to \infty} \frac{1}{n^\alpha} \neq 0$ by the n^{th} Term Test, the series is div

$\alpha > 0$. $f : [1, \infty) \to [0, \infty)$, $f(x) = \frac{1}{x^\alpha}$ cont, decr

f is impr int on $[1, \infty) \Leftrightarrow \alpha > 1$ By the Integral Test, $\sum_{n \geq 1} \frac{1}{n^\alpha}$ is conv $\Leftrightarrow \alpha > 1$

$$\Rightarrow \sum_{n \geq 1} \frac{1}{n^\alpha} = \begin{cases} \text{convergent} & \text{if } \alpha > 1 \\ \text{divergent} & \text{if } \alpha \leq 1 \end{cases}$$

Sometimes we cannot easily evaluate an improper integral of a given function or we do not need its precise value and are only interested to know if the function is improperly integrable or not. As with series of real numbers, we give in the following certain tests that can be used in such a situation.

Theorem 2 (Comparison Test for Improper Integrals)**.** *Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$ with $a < b$ and $f, g : [a, b) \to \mathbb{R}$ be continuous functions satisfying*

$$\exists c \in [a, b) \text{ such that } \forall x \in [c, b),\ 0 \le f(x) \le g(x). \tag{1}$$

(i) If g is improperly integrable on $[a, b)$, then f is improperly integrable on $[a, b)$.

(ii) If f is not improperly integrable on $[a, b)$, then g is not improperly integrable on $[a, b)$.

Remark 4. If f and g in the above theorem are nonnegative continuous functions satisfying instead of (1) the following condition

$$\exists \alpha, \beta > 0, \exists c \in [a, b) \text{ such that } \forall x \in [c, b),\ \alpha g(x) \le f(x) \le \beta g(x),$$

then f is improperly integrable on $[a, b)$ if and only if g is improperly integrable on $[a, b)$.

Example 4. (i) Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{e^x + x}$.

Take $g : [0, \infty) \to \mathbb{R}$, $g(x) = \frac{1}{e^x} = e^{-x}$; f, g are cont

$\forall x \ge 0$, $0 \le f(x) \le g(x)$

Let $t \in [0, \infty)$. Then $\int_0^t g(x)\,dx = \int_0^t e^{-x}\,dx = -e^{-x}\Big|_0^t = 1 - e^{-t} \Rightarrow \lim_{t \to \infty} \int_0^t g(x)\,dx = 1$

$\Rightarrow$ g impr. int on $[0, \infty)$ $\Rightarrow$ f impr. int on $[0, \infty)$

(ii) Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{x}{1 + x^2 \cos^2 x}$.

Take $g : [0, \infty) \to \mathbb{R}$, $g(x) = \frac{x}{1 + x^2}$; f, g cont

$\forall x \ge 0$, $0 \le g(x) \le f(x)$

Let $t \in [0, \infty)$. Then $\int_0^t g(x)\,dx = \int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2} \ln(1+x^2)\Big|_0^t = \frac{1}{2}\ln(1+t^2)$

$\Rightarrow \lim_{t \to \infty} \int_0^t g(x)\,dx = \infty \Rightarrow$ g is not impr. int on $[0, \infty)$ $\Rightarrow$ f is not impr. int on $[0, \infty)$

Theorem 3. *Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b) \to [0, \infty)$ continuous and $p \in \mathbb{R}$ such that $\exists L = \lim\limits_{\substack{x \to b \\ x < b}} (b - x)^p f(x) \in [0, \infty) \cup \{\infty\}$. Then:*

(i) if $p < 1$ and $L < \infty$, then f is improperly integrable on $[a, b)$.

(ii) if $p \ge 1$ and $L > 0$, then f is not improperly integrable on $[a, b)$.

Pf : (i) $L = \lim\limits_{\substack{x \to b \\ x < b}} (b - x)^p f(x) < \infty \Rightarrow \exists c \in [a, b)$ s.t

$\forall x \in [c, b)$, $(b-x)^p f(x) < L+1$.

$\Rightarrow \forall x \in [c,b)$, $0 \leq f(x) < \dfrac{L+1}{(b-x)^p}$

Take $g: (a,b) \to \mathbb{R}$, $g(x) = \dfrac{L+1}{(b-x)^p}$

$p<1 \Rightarrow g$ is impr. int on $[a, b)$ (by Example 2.(i)).

By Thm 2.(i), f is impr. int on (a,b).

(ii) Let $r \in (0,L)$. $L = \lim\limits_{\substack{x\to b\\ x<b}} (b-x)^p f(x) \Rightarrow \exists\, c \in [a,b)$ s.t.

$\forall x \in [c,b)$, $r < (b-x)^p f(x) \Rightarrow \forall x \in [c,b)$, $0 < \dfrac{r}{(b-x)^p} < f(x)$

Take $h: [a,b) \to \mathbb{R}$, $h(x) = \dfrac{r}{(b-x)^p}$

$p \geq 1 \Rightarrow h$ is not impr. int. on $[a,b)$ (by Example 2.(i))

By Thm 2, (ii), f is not impr. int on $[a,b)$.

Example 5. Let $f : [0, 1) \to [0, \infty)$, $f(x) = \dfrac{1}{\sqrt[4]{1 - x^4}}$.

f cont.

We try to find $p \in \mathbb{R}$ s.t. $\exists\, L = \lim\limits_{\substack{x\to 1\\ x<1}} (1-x)^p f(x)$ ($\in (0,\infty)$ preferably)

$$L = \lim_{\substack{x\to 1\\ x<1}} \frac{(1-x)^p}{\sqrt[4]{1-x^4}} = \lim_{\substack{x\to 1\\ x<1}} \frac{(1-x)^p}{\sqrt[4]{(1-x)(1+x)(1+x^2)}}$$

For $p = \frac{1}{4}$, $L = \frac{1}{\sqrt{2}}$. Since $p<1$ and $L<\infty$, then f is impr. int. on $[0,1)$.

Theorem 4. *Let $a, b \in \mathbb{R}$ with $a < b$, $f : (a, b] \to [0, \infty)$ continuous and $p \in \mathbb{R}$ such that $\exists L = \lim\limits_{\substack{x\to a\\ x>a}} (x - a)^p f(x) \in [0, \infty) \cup \{\infty\}$. Then:*

(i) if $p < 1$ and $L < \infty$, then f is improperly integrable on $(a, b]$.

(ii) if $p \geq 1$ and $L > 0$, then f is not improperly integrable on $(a, b]$.

Theorem 5. *Let $a \in \mathbb{R}$, $f : [a, \infty) \to [0, \infty)$ continuous and $p \in \mathbb{R}$ such that $\exists L = \lim_{x\to\infty} x^p f(x) \in [0, \infty) \cup \{\infty\}$. Then:*

(i) if $p > 1$ and $L < \infty$, then f is improperly integrable on $[a, \infty)$.

(ii) if $p \leq 1$ and $L > 0$, then f is not improperly integrable on $[a, \infty)$.

Example 6. Let $f : [1, \infty) \to [0, \infty)$, $f(x) = \dfrac{\sqrt{x^2+1}}{1+\sqrt[3]{x^4-1}}$.

f cont

We try to find $p \in \mathbb{R}$ s.t. $\exists L = \lim_{x\to\infty} x^p f(x)$ ($\in (0,\infty)$ preferably)

$$L = \lim_{x\to\infty} x^p \cdot \frac{\sqrt{x^2+1}}{1+\sqrt[3]{x^4-1}} = \lim_{x\to\infty} x^p \cdot \frac{x\overbrace{\sqrt{1+\frac{1}{x^2}}}^{\to 1}}{x^{\frac{4}{3}}\Big(\underbrace{\frac{1}{x^{4/3}}}_{\to 0} + \underbrace{\sqrt[3]{1-\frac{1}{x^4}}}_{\to 1}\Big)}$$

For $p = \frac{1}{3}$, $L = 1$. Since $p \leq 1$ and $L > 0 \Rightarrow f$ is not impr. int on $[1, \infty)$.