Seminar 3

Ex 1: Let $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, $A \neq \emptyset$. Prove that:

a) if A is bd. above, $x = \sup A \iff \begin{cases} x \in ub(A) \\ \exists (a_n) \subseteq A \text{ s.t. } \lim\limits_{n\to\infty} a_n = x \end{cases}$

b) if A is bd. below, $x = \inf A \iff \begin{cases} x \in lb(A) \\ \exists (a_n) \subseteq A \text{ s.t. } \lim\limits_{n\to\infty} a_n = x \end{cases}$

(1)

a) $\boxed{\Rightarrow}$ $x = \sup A \Rightarrow x \in ub(A)$ (by def)

Let $n \in \mathbb{N}$. Because $x - \frac{1}{n} < x \Rightarrow x - \frac{1}{n} \notin ub(A) \Rightarrow \exists\, a_n \in A$ s.t. $a_n > x - \frac{1}{n}$

$\forall n \in \mathbb{N}$, $\underbrace{x - \frac{1}{n}}_{\to x} < a_n \le x$. By the Squeeze Thm., $\lim\limits_{n\to\infty} a_n = x$.

$\boxed{\Leftarrow}$ Let $x' \in ub(A)$. Then $\left.\begin{array}{l} a_n \le x',\ \forall n \in \mathbb{N} \\ \lim\limits_{n\to\infty} a_n = x \end{array}\right\} \Rightarrow x \le x'$

Because $x \in ub(A)$, this shows that $x = \sup A$.

Ex 2: Find the limit (as $n \to \infty$) of the sequence whose general term x_n, $n \in \mathbb{N}$, is given below: (2)

a) $x_n = \left(\sin \frac{\pi}{7}\right)^n$ $\quad \lim\limits_{n\to\infty} x_n = 0$ because $\sin\frac{\pi}{7} \in (0,1)$

b) $x_n = \dfrac{\alpha n^3 + \beta n^2 + \gamma n + 1}{n^2 - n + 1}$, $\alpha, \beta, \gamma \in \mathbb{R}$

$$= \frac{\alpha n + \beta + \frac{\gamma}{n} + \frac{1}{n^2}}{1 - \frac{1}{n} + \frac{1}{n^2}}$$

case 1: $\alpha \neq 0$ $\quad \lim\limits_{n\to\infty} x_n = \begin{cases} \infty, & \alpha > 0 \\ -\infty, & \alpha < 0 \end{cases}$

case 2: $\alpha = 0$ $\quad \lim\limits_{n\to\infty} x_n = \beta$

(3)

c) $x_n = \dfrac{e^n - 2^n}{\pi^n - 3^n} = \dfrac{e^n\left(1 - \left(\frac{2}{e}\right)^n\right)}{\pi^n\left(1 - \left(\frac{3}{\pi}\right)^n\right)} = \underbrace{\left(\frac{e}{\pi}\right)^n}_{\to 0} \cdot \underbrace{\dfrac{1 - \left(\frac{2}{e}\right)^n}{1 - \left(\frac{3}{\pi}\right)^n}}_{\to 1} \to 0$

d) $x_n = \left(1 + \frac{1}{n}\right)^{\frac{3n}{n+1}}$ $\quad \lim\limits_{n\to\infty} x_n = 1$

e) $x_n = \left(\dfrac{n^2+n+1}{n^2+1}\right)^{\frac{2n^2+n+1}{n+1}} = \left[\underbrace{\left(1 + \frac{n}{n^2+1}\right)^{\frac{n^2+1}{n}}}_{\to e}\right]^{\underbrace{\frac{n}{n^2+1}\cdot\frac{2n^2+n+1}{n+1}}_{\to 2}} \to e^2$

(4)

f) $x_n = \sqrt{n}\left(\sqrt{n} - \sqrt{n+3}\right) = \sqrt{n}\,\dfrac{n-(n+3)}{\sqrt{n}+\sqrt{n+3}} = -3\,\dfrac{\sqrt{n}}{\sqrt{n}+\sqrt{n+3}} = -3\cdot\dfrac{1}{1+\sqrt{1+\frac{3}{n}}} \to -\dfrac{3}{2}$

g) $x_n = \dfrac{2^n}{n!}$

$0 < \dfrac{2^n}{n!} = \dfrac{2}{1}\cdot\dfrac{2}{2}\cdot\underbrace{\dfrac{2}{3}\cdot\ldots\cdot\dfrac{2}{n}}_{\le\left(\frac{2}{3}\right)^{n-2}} \le \underbrace{2\cdot\left(\frac{2}{3}\right)^{n-2}}_{\to 0}$. By the Squeeze Thm., $\lim\limits_{n\to\infty} x_n = 0$.

h) $x_n = \dfrac{n^\alpha}{(1+\beta)^n}$, $\alpha \in \mathbb{N}$, $\beta > 0$

(5)

For $n \ge 2\alpha$,

$(1+\beta)^n = \sum\limits_{k=0}^{n} \binom{n}{k}\beta^k > \binom{n}{\alpha+1}\beta^{\alpha+1} = \dfrac{n(n-1)\cdot\ldots\cdot(n-\alpha)}{(\alpha+1)!}\beta^{\alpha+1} \ge \dfrac{\left(\frac{n}{2}\right)^{\alpha+1}}{(\alpha+1)!}\beta^{\alpha+1}$

$n \ge \alpha + 1$
$n - \alpha \ge \frac{n}{2}$
$\frac{n}{2} \ge \alpha$
$n \ge 2\alpha$
$\Updownarrow$
$n \ge \alpha + 1$

$\binom{n}{k} = C_n^k = \dfrac{n!}{k!(n-k)!} = \dfrac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}$; $n - i \ge \frac{n}{2}$, $\forall i \in \{0,1,\ldots,\alpha\}$

$\Rightarrow (1+\beta)^n > \dfrac{\beta^{\alpha+1}}{2^{\alpha+1}\cdot(\alpha+1)!}\cdot n^{\alpha+1}$

$0 < x_n = \dfrac{n^\alpha}{(1+\beta)^n} < \dfrac{2^{\alpha+1}\cdot(\alpha+1)!}{\beta^{\alpha+1}}\cdot\dfrac{n^\alpha}{n^{\alpha+1}} = \underbrace{\dfrac{2^{\alpha+1}\cdot(\alpha+1)!}{\beta^{\alpha+1}}\cdot\dfrac{1}{n}}_{\to 0}$

By the Squeeze Thm, $\lim\limits_{n\to\infty} x_n = 0$.

(6)

i) $x_n = \dfrac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$, $p \in \mathbb{N}$

$a_n = 1^p + 2^p + \ldots + n^p$, $b_n = n^{p+1}$, $n \in \mathbb{N}$

(b_n) str. incr., $b_n \to \infty$

$\dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = \dfrac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} = \dfrac{n^p + \ldots + 1}{\underbrace{\sum\limits_{k=0}^{p+1}\binom{p+1}{k} n^k}_{n^{p+1} + \binom{p+1}{p} n^p + \ldots + 1} - n^{p+1}} \to \dfrac{1}{p+1}$

$\binom{p+1}{p} = C_{p+1}^{p} = \dfrac{(p+1)!}{p!\cdot 1!} = p+1$

S.-C. $\Rightarrow \lim\limits_{n\to\infty} \underbrace{\dfrac{a_n}{b_n}}_{= x_n} = \dfrac{1}{p+1}$.

(7)

j) $x_n = \sqrt[n]{n}$

(Cor. 3 from Lecture 2): $\left(\forall n \in \mathbb{N},\ a_n > 0 \text{ and } \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = L \in [0,\infty) \cup \{\infty\}\right) \Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{a_n} = L$

Take $a_n = n$, $n \in \mathbb{N}$. Then $\left.\begin{array}{l} \frac{a_{n+1}}{a_n} = \frac{n+1}{n} \to 1 \\ a_n > 0,\ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty}\underbrace{\sqrt[n]{a_n}}_{=\sqrt[n]{n} = x_n} = 1$

k) $x_n = \sqrt[n]{n!}$

(Cor. 2 from Lecture 2): $(\forall n \in \mathbb{N},\ a_n > 0 \text{ and } \lim\limits_{n\to\infty} a_n = L \in [0,\infty) \cup \{\infty\}) \Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n} = L$.

Take $a_n = n$, $n \in \mathbb{N}$. Then $\left.\begin{array}{l} a_n \to \infty \\ a_n > 0,\ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty}\underbrace{\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}}_{=\sqrt[n]{n!} = x_n} = \infty$

(8)

l) $x_n = \dfrac{\sqrt[n]{n!}}{n} = \sqrt[n]{\dfrac{n!}{n^n}}$

Take $a_n = \dfrac{n!}{n^n}$, $n \in \mathbb{N}$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{(n+1)^{n+1}}\cdot\dfrac{n^n}{n!} = \left(\dfrac{n}{n+1}\right)^n = \left[\underbrace{\left(1 - \dfrac{1}{n+1}\right)^{-(n+1)}}_{\to e}\right]^{\underbrace{-\frac{n}{n+1}}_{\to -1}} \to \dfrac{1}{e}$

$a_n > 0$, $\forall n \in \mathbb{N}$

$\Rightarrow \lim\limits_{n\to\infty}\underbrace{\sqrt[n]{a_n}}_{= x_n} = \dfrac{1}{e}$

m) $x_n = \sqrt[n]{\sin^2(n) + 2\cos^2(n)}$

$1 \le x_n = \sqrt[n]{1 + \cos^2(n)} \le \underbrace{\sqrt[n]{2}}_{\to 1}$. By the Squeeze Thm., $\lim\limits_{n\to\infty} x_n = 1$;

(9)

n) $x_n = \sin\left(\pi\sqrt{n^2+1}\right)$

$\sin(\pi n) = 0$, $n \in \mathbb{N}$

$\sqrt{n^2+1} - n = \dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n} \Rightarrow \sqrt{n^2+1} = n + \dfrac{1}{\sqrt{n^2+1}+n}$

$\sin\left(\pi\sqrt{n^2+1}\right) = \sin\left(\pi n + \dfrac{\pi}{\sqrt{n^2+1}+n}\right) = \underbrace{\sin(\pi n)}_{=0}\cos\dfrac{\pi}{\sqrt{n^2+1}+n} + \underbrace{\cos(\pi n)}_{=(-1)^n}\cdot\sin\dfrac{\pi}{\sqrt{n^2+1}+n}$

$\sin(a+b) = \sin a\cdot\cos b + \cos a\cdot\sin b$

$\underbrace{-\sin\dfrac{\pi}{\sqrt{n^2+1}+n}}_{\to 0} \le x_n \le \underbrace{\sin\dfrac{\pi}{\sqrt{n^2+1}+n}}_{\to 0}$ $\quad \left(0 < \sin\dfrac{\pi}{\sqrt{n^2+1}+n} < \underbrace{\dfrac{\pi}{\sqrt{n^2+1}+n}}_{\to 0}\right)$

By the Squeeze Thm, $\lim\limits_{n\to\infty} x_n = 0$.

(10)

o) $x_n = n\left(\sqrt[n]{e} - 1\right)$, $\sqrt[n]{e} \to 1$

$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)$

$1 + \frac{1}{n} = 1 + n\cdot\frac{1}{n^2} \le \left(1+\frac{1}{n^2}\right)^n$ using Bernoulli's ineq.

$\sqrt[n]{1+\frac{1}{n}} \le 1 + \frac{1}{n^2}$

$1 + \frac{1}{n} < \sqrt[n]{e} < \left(1+\frac{1}{n}\right)\sqrt[n]{1+\frac{1}{n}} \le \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right) = 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}$ $\quad \big|\, -1$

$\frac{1}{n} < \sqrt[n]{e} - 1 < \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}$ $\quad \big|\cdot n$

$1 < n\left(\sqrt[n]{e} - 1\right) < \underbrace{1 + \frac{1}{n} + \frac{1}{n^2}}_{\to 1}$. By the Squeeze Theorem, $\lim\limits_{n\to\infty} x_n = 1$.

Exercise 3. Decide whether for an arbitrary sequence (x_n) in $\mathbb{R}$ the next statements hold true:

a) if (x_n) converges, then $(|x_n|)$ converges.

b) if $(|x_n|)$ converges, then (x_n) converges.

Solution a) The statement is true. Denote $x = \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $n \geq n_\varepsilon$,

$$||x_n| - |x|| \leq |x_n - x| < \varepsilon.$$

Hence the sequence $(|x_n|)$ converges and $\lim_{n\to\infty} |x_n| = |x|$.

b) The statement is, in general, false. Take $x_n = (-1)^n$, $n \in \mathbb{N}$. Then $|x_n| = 1$, $\forall n \in \mathbb{N}$, so the sequence $|x_n|$ converges to 1, yet the sequence (x_n) is divergent.