Seminar 12

$\emptyset \neq A \subseteq \mathbb{R}^n$ open, $f \in C^2(A) \Rightarrow H_f(c)$ symmetric, $\forall c \in A$

Algorithm

1. Determine the first order partial derivatives of f
2. Determine the stationary points of f: $c \in A$, $\nabla f(c) = 0_n$

   If f has no stationary points $\Rightarrow$ f has no local extremum points
3. Determine the second order partial derivatives of f
4. Evaluate $H_f(c)$ at each stationary point $c \in A$ of f

If $H_f(c)$ is
- positive definite $\Rightarrow$ c is a local minimum point of f
- negative definite $\Rightarrow$ c is a local maximum point of f
- indefinite $\Rightarrow$ c is not a local extremum point of f

(1)

Let $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$ be a symmetric 2×2 matrix of real nr. Denote $\Delta_2 = c_{11}c_{22} - c_{12}^2$. (2)

C is
- pos. def (neg. def) $\Leftrightarrow$ $c_{11} > 0$ and $\Delta_2 > 0$ ($c_{11} < 0$ and $\Delta_2 > 0$)
- pos. semidefinite (neg. semidefinite) $\Leftrightarrow$ $c_{11} \geq 0$, $c_{22} \geq 0$ and $\Delta_2 \geq 0$ ($c_{11} \leq 0$, $c_{22} \leq 0$ and $\Delta_2 \geq 0$)
- indefinite $\Leftrightarrow$ $\Delta_2 < 0$

Thm (Sylvester)

$C = (c_{ij})_{1 \leq i,j \leq n}$ symmetric, $\Delta_k = \det(c_{ij})_{1 \leq i,j \leq k}$, $k \in \{1, \dots, n\}$

C is
- pos. def $\Leftrightarrow$ $\Delta_k > 0$, $\forall k \in \{1, \dots, n\}$
- neg. def $\Leftrightarrow$ $(-1)^k \Delta_k > 0$, $\forall k \in \{1, \dots, n\}$

Ex 1: For the following functions, find the local extremum points and specify their type: (3)

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 - 3x + y^2$

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = 3x^2 - 3$, $\frac{\partial f}{\partial y}(x,y) = 2y$

$\begin{cases} 3x^2 - 3 = 0 \Rightarrow x \in \{-1, 1\} \\ 2y = 0 \Rightarrow y = 0 \end{cases}$ Stationary point: $(-1,0)$, $(1,0)$

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial^2 f}{\partial x^2}(x,y) = 6x$ $\quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2$ $\quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0 = \frac{\partial^2 f}{\partial x \partial y}(x,y)$

$H_f(-1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$: $\Delta_2 = -12 < 0 \Rightarrow H_f(-1,0)$ indefinite

(OR: using Sylvester's Thm: $\Delta_2 < 0 \Rightarrow H_f(-1,0)$ is not pos def, not neg. def.
$\phi: \mathbb{R}^2 \to \mathbb{R}$, $\phi(h_1,h_2) = -6h_1^2 + 2h_2^2$; $\phi(1,0) = -6 < 0 < 2 = \phi(0,1) \Rightarrow H_f(-1,0)$ is indefinite)

$\Rightarrow (-1,0)$ is not a local extremum point of f (4)

$H_f(1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$: $\Delta_2 = 12 > 0$, $\Delta_1 = 6 > 0 \Rightarrow H_f(1,0)$ is pos. def. $\Rightarrow (1,0)$ is a local min. point of f

b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 3xy^2 - 15x - 12y$

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15$, $\frac{\partial f}{\partial y}(x,y) = 6xy - 12$

$\begin{cases} x^2 + y^2 - 5 = 0 \\ xy = 2 \Rightarrow y \neq 0,\ x = \frac{2}{y} \end{cases}$ $\quad \frac{4}{y^2} + y^2 - 5 = 0$, $y^4 - 5y^2 + 4 = 0$, $(y^2-1)(y^2-4) = 0$

1. $y^2 = 1$: $y = 1, x = 2$; $y = -1, x = -2$
2. $y^2 = 4$: $y = 2, x = 1$; $y = -2, x = -1$

Stationary points: $(2,1)$, $(-2,-1)$, $(1,2)$, $(-1,-2)$

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial^2 f}{\partial x^2}(x,y) = 6x$ $\quad \frac{\partial^2 f}{\partial y^2}(x,y) = 6x$ $\quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 6y = \frac{\partial^2 f}{\partial x \partial y}(x,y)$ (5)

$H_f(2,1) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$: $\Delta_2 = 144 - 36 = 108 > 0$, $\Delta_1 = 12 > 0 \Rightarrow H_f(2,1)$ is pos. def $\Rightarrow (2,1)$ is a local min. point of f

$H_f(-2,-1) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix}$: $\Delta_2 = 108 > 0$, $\Delta_1 = -12 < 0 \Rightarrow H_f(-2,-1)$ is neg. def $\Rightarrow (-2,-1)$ is a local max. point of f

$H_f(1,2) = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix}$: $\Delta_2 = -108 < 0 \Rightarrow H_f(1,2)$ is indefinite $\Rightarrow (1,2)$ is not a local extremum point of f

($\phi: \mathbb{R}^2 \to \mathbb{R}$, $\phi(h_1,h_2) = 6h_1^2 + 24h_1h_2 + 6h_2^2$; $\phi(-1,1) = -12 < 0 < 6 = \phi(1,0) \Rightarrow H_f(1,2)$ is indefinite)

$H_f(-1,-2) = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix}$: $\Delta_2 = -108 < 0 \Rightarrow H_f(-1,-2)$ is indefinite $\Rightarrow (-1,-2)$ is not a local extremum point of f

c) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4 + y^4 - 4(x-y)^2$ (6)

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = 4x^3 - 8(x-y)$, $\frac{\partial f}{\partial y}(x,y) = 4y^3 + 8(x-y)$

$\begin{cases} x^3 - 2(x-y) = 0 \\ y^3 + 2(x-y) = 0 \end{cases} \overset{\text{"+"}}{\Rightarrow} x^3 + y^3 = 0$

$(x+y)(x^2 - xy + y^2) = 0 \Leftrightarrow x = -y$

$x^2 - xy + y^2 = \left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4} = 0 \Leftrightarrow x = y = 0$

$-y^3 - 2 \cdot (-2y) = 0$

$y^3 = 4y \quad y \in \{0, -2, 2\}$ $\quad$ Stationary points: $(0,0)$, $(2,-2)$, $(-2,2)$

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 8$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 8$ $\quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = 8 = \frac{\partial^2 f}{\partial x \partial y}(x,y)$

$H_f(0,0) = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}$: $\Delta_2 = 0 \Rightarrow H_f(0,0)$ is not pos. def, is not neg def, is not indefinite (7)

($c_{11} = -8 \leq 0$, $c_{22} = -8 \leq 0 \Rightarrow H_f(0,0)$ is neg semidefinite
OR: using Sylvester's Thm: $\Delta_2 = 0 \Rightarrow H_f(0,0)$ is not pos def, not neg def.
$\phi: \mathbb{R}^2 \to \mathbb{R}$, $\phi(h_1,h_2) = -8h_1^2 + 16h_1h_2 - 8h_2^2 = -8(h_1 - h_2)^2 \leq 0$, $\forall (h_1,h_2) \in \mathbb{R}^2$
$\Rightarrow H_f(0,0)$ neg. semidefinite
$\phi(1,1) = 0 \Rightarrow H_f(0,0)$ is not neg. definite)

$f(0,0) = 0$ $\qquad f(x,y) = x^4 + y^4 - 4(x-y)^2$

$f(x,0) = x^4 - 4x^2 = x^2(x^2 - 4) < 0 \quad \forall x \in (-2,2) \setminus \{0\}$

$f(\frac{1}{n}, 0) = \frac{1}{n^2}\left(\frac{1}{n^2} - 4\right) < \underset{f(0,0)}{\underbrace{0}}$, $\forall n \in \mathbb{N}$ $\Rightarrow (0,0)$ is not a local minimum point of f

$f(x,x) = 2x^4 > 0$, $\forall x \neq 0$ (8)

$f(\frac{1}{n}, \frac{1}{n}) = \frac{2}{n^4} > 0 = f(0,0)$, $\forall n \in \mathbb{N}$ $\Rightarrow (0,0)$ is not a local max. point of f

$\Rightarrow (0,0)$ is not a local extremum point of f

$H_f(2,-2) = \begin{pmatrix} 40 & 8 \\ 8 & 40 \end{pmatrix}$: $\Delta_2 = 40^2 - 8^2 > 0$, $\Delta_1 = 40 > 0 \Rightarrow H_f(2,-2)$ is pos def $\Rightarrow (2,-2)$ is a local min. point

$H_f(-2,2) = H_f(2,-2) \Rightarrow (-2,2)$ is a local min. point of f

d) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2(1+xy) + xy$

For $(x,y,z) \in \mathbb{R}^3$, $\frac{\partial f}{\partial x}(x,y,z) = z^2y + y$, $\frac{\partial f}{\partial y}(x,y,z) = z^2x + x$ $\quad \frac{\partial f}{\partial z}(x,y,z) = 2z(1+xy)$

$\begin{cases} y(z^2+1) = 0 \Rightarrow y = 0 \\ x(z^2+1) = 0 \Rightarrow x = 0 \\ 2z(1+xy) = 0 \end{cases}$ (9)

$\Rightarrow z = 0$

Stationary point: $(0,0,0)$

For $(x,y,z) \in \mathbb{R}^3$, $\frac{\partial^2 f}{\partial x^2}(x,y,z) = 0$, $\frac{\partial^2 f}{\partial y^2}(x,y,z) = 0$, $\frac{\partial^2 f}{\partial z^2}(x,y,z) = 2(1+xy)$

$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = z^2+1$, $\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = 2yz$, $\frac{\partial^2 f}{\partial z \partial y}(x,y,z) = 2xz$

$H_f(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$: $\Delta_1 = 0 \underset{\text{Sylvester's Thm}}{\Longrightarrow} H_f(0,0,0)$ is not pos. def, not neg def

$\phi: \mathbb{R}^3 \to \mathbb{R}$, $\phi(h_1,h_2,h_3) = 2h_1h_2 + 2h_3^2$; $\phi(-1,1,0) = -2 < 0 < 2 = \phi(0,0,1) \Rightarrow$ (10)

$\Rightarrow H_f(0,0,0)$ is indefinite $\Rightarrow (0,0,0)$ is not a local extremum point of f.

Ex 2 Let $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (x^2 - y)(x^2 - 3y)$

a) Prove that 0_2 is a stationary point of f. Is 0_2 a local min. point of f?

For $(x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = 4x(x^2 - 2y)$, $\frac{\partial f}{\partial y}(x,y) = -4x^2 + 6y$

$\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0 \Rightarrow 0$ is a stationary point of f

$f(0,0) = 0$ (11)

Suppose $x \neq 0$ is fixed

$(x^2 - y)(x^2 - 3y) < 0 \Leftrightarrow (y - x^2)(y - \frac{x^2}{3}) < 0 \Leftrightarrow y \in \left(\frac{x^2}{3}, x^2\right)$

$f(\frac{1}{n}, \frac{2}{3n^2}) < 0 = f(0,0)$, $\forall n \in \mathbb{N}$ $\Rightarrow (0,0)$ is not a local min. point of f

$\frac{2}{3n^2} \in \left(\frac{1}{3n^2}, \frac{1}{n^2}\right)$

b) Prove that the restriction to any line through 0_2 attains a local min. at 0_2

$A_c = \{(x,y) \in \mathbb{R}^2 \,/\, y = c \cdot x\}$, $c \in \mathbb{R}$

$f(x, cx) = (x^2 - cx)(x^2 - 3cx) = x^2(x-c)(x-3c)$

$c = 0$: $f(x,0) = x^4 \geq 0 = f(0,0)$, $\forall x \in \mathbb{R}$ (12)

$c \neq 0$: if $x \in (-|c|, |c|) \Rightarrow (x-c)(x-3c) \geq 0 \Rightarrow f(x, cx) \geq 0$

$\|(x, cx)\| = |x|\sqrt{1+c^2}$

Let $(x,y) \in B(0_2, |c|\sqrt{1+c^2}) \cap A_c$. Then $y = cx$ and $\sqrt{x^2+y^2} < |c|\sqrt{1+c^2}$

$\sqrt{x^2+y^2} = \sqrt{x^2 + c^2x^2} = |x|\sqrt{1+c^2}$

$\Rightarrow |x| < |c| \Rightarrow x \in (-|c|, |c|)$

$f(x,y) = f(x, cx) \geq 0 = f(0,0)$

$\Rightarrow (0,0)$ is a local min point of $f|_{A_c}$