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1.

a) $x_1 \in (0, 2)$, $x_{n+1} = \sqrt{2 + x_n}$, $n \in \mathbb{N}$

$$x_{n+1} - x_n = \sqrt{2 + x_n} - x_n$$

$p(n): x_n \in (0, 2), \forall n \in \mathbb{N}$

I. We verify if $p(1)$ is true:

$p(1): x_1 \in (0, 2)$ (True)

II. We assume that $p(k)$ is true and we prove $p(k+1)$

$p(k+1): x_{k+1} = \sqrt{2 + x_k}$, $x_{k+1} \in (0, 2)$

$p(k): x_k = \sqrt{2 + x_{k-1}}$ $x_k \in (0, 2)$

$$x_k \in (0, 2) \Rightarrow 0 < x_k < 2 \,|+2 \Rightarrow 2 < 2 + x_k < 4 \,|\sqrt{\ } \Rightarrow$$

$$\sqrt{2} < \sqrt{2 + x_k} < 2 \Rightarrow \sqrt{2 + x_k} \in (\sqrt{2}, 2) \subset (0, 2) \text{ (True)} \Rightarrow$$

$\Rightarrow p(n)$ is True $\Rightarrow x_n \in (0, 2), \forall n \in \mathbb{N}$ - bounded (1)

$$x_{n+1} - x_n = \sqrt{2 + x_n} - x_n$$

$$x_n \in (0, 2) \Rightarrow 0 < x_n < 2 \,|+2 \Rightarrow 2 < 2 + x_n < 4 \,|\sqrt{\ } \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow \sqrt{2} < \sqrt{2 + x_n} < 2 \\ &0 < x_n < 2 \end{aligned}\right\} \Rightarrow x_n < \sqrt{2 + x_n} \Rightarrow \sqrt{2 + x_n} - x_n > 0, \forall n \in \mathbb{N} \Rightarrow$$

$\Rightarrow x_n$ - monotone (2)

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$①, ② \overset{\text{Weierstrass}}{\Longrightarrow} (x_n)$ - convergent, $\forall n \in \mathbb{N}$

We know that x_n - convergent $\Rightarrow \exists! \, l = \lim\limits_{n \to \infty} x_n$ ①

$L = \sqrt{2+L} \Rightarrow L = \frac{2+L}{\sqrt{2+L}}$

$L = \sqrt{2+L} \; |(\,)^2 \Rightarrow L^2 = 2+L \Rightarrow L^2 - L = 2 \Leftrightarrow$

$\Leftrightarrow L^2 - L - 2 = 0$

$\Delta = 1 + 8$

$\Delta = 9$

$L_{1,2} = \frac{1 \pm 3}{2} \Rightarrow L_1 = 1 \vee L_2 = -1$ ③.

Because the sequence (x_n) is strictly increasing and it's bounded in $(0, 2) \Rightarrow L \neq -1$ ②

$①, ②, ③ \Rightarrow L = \underline{1}$

ii) $A = \{x_n : n \in \mathbb{N}\}$

We know that $x_n \in (0, 2)$, $\forall n \in \mathbb{N}$

So the $\inf A = 0$, and we don't have a minimum because $0 \notin (0, 2)$, $0 \notin A$

So the $\sup A = 2$, and we don't have a maximum because $2 \notin (0, 2)$, $2 \notin A$

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a)

$a > 0$, $y_m = \dfrac{a^m (a \cdot m^2 + a \cdot m + 1)}{3^m \cdot m^3 (m + |2 - a|)^3}$, $m \in \mathbb{N}$

i) $a = 3$, $\sum_{m \geq 1} y_m = ?$

$y_n = \dfrac{\cancel{3^m} (3 \cdot m^2 + 3 \cdot m + 1)}{\cancel{3^m} \cdot m^3 (m + |-1|)^3}$, $m \in \mathbb{N}$

$y_m = \dfrac{3 \cdot m^2 + 3m + 1}{m^3 (m + 1)^3}$, $m \in \mathbb{N}$

$y_m = \dfrac{3 \cdot m^2 + 3m + 1}{m^3 (m^3 + 3m^2 + 3m + 1)}$

$S_k = \sum_{i=1}^{k} y_k = \sum_{i=1}^{k} \dfrac{3i^2 + 3i + 1}{i^3 (i^3 + 3 \cdot i^2 + 3 \cdot i + 1)}$

$= \sum_{i=1}^{k} \dfrac{i^3 + 3i^2 + 3 \cdot i + 1 - i^3}{i^3 (i^3 + 3i^2 + 3i + 1)}$

$= \sum_{i=1}^{k} \dfrac{1}{i^3} - \sum_{i=1}^{k} \dfrac{1}{i^3 + 3i^2 + 3i + 1}$

$= \sum_{i=1}^{k} -\left(\dfrac{1}{(i+1)^3} - \dfrac{1}{i^3}\right) = -\left(\dfrac{1}{1^3} - \lim_{k \to \infty}\left(-\dfrac{1}{k^3}\right)\right) = -(-1 - 0) = \underline{1}$

$\rightarrow$ telescoping series

$\Rightarrow \sum_{m \geq 1} y_m = S_k = \underline{1}$

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iii) $\sum_{n\geq 1} y_n \quad , y_n = \frac{a^n(a\cdot n^2 + a\cdot n + 1)}{3^n \cdot n^3 (n + |2-a|)^3}$

I. for $a = 3$, $\sum_{n\geq 1} y_n = 1$ - convergent

II. $a < 3$, $a \in (0, 3)$

III. $a > 3$, $a \in (3, \infty)$

For $a < 3 \Rightarrow a^n < 3^n$

$\sum_{n\geq 1} y_n$ - series with nonnegative terms

$\sum_{n\geq 1} \frac{1}{n^3}$ - series with positive terms, $\forall n \in \mathbb{N}$

$\hookrightarrow$ convergent

We apply the second comparison test:

$$\lim_{n\to\infty} \frac{a^n(a\cdot n^2 + a\cdot n + 1)}{3^n \cdot n^3 (n + |2-a|)^3} : \frac{1}{n^3}$$

$$\left.\begin{aligned} &= \lim_{n\to\infty} \frac{a^n \cdot \cancel{n^3}(a\cdot n^2 + a\cdot n + 1)}{3^n \cdot \cancel{n^3} (n + |2-a|)^3} \\ &a^n < 3^n \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} \frac{y_n}{\frac{1}{n^3}} = 0$$

$\frac{n^3}{3^n} \to 0, n \to \infty$ $\quad \sum_{n\geq 1} \frac{1}{n^3}$ - convergent $\Big\} \Rightarrow$

The exponential function grows faster than the polynomial

S.C.T.

$\Rightarrow \sum_{n\geq 1} y_n$ - convergent, $\forall n \in \mathbb{N}$, for $a \in (0, 3)$

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III. $a > 3$, $a \in (3, \infty)$

$\sum_{n\geq 1} y_n$ - series with nonnegative terms

$\sum_{n\geq 1} \frac{1}{n}$ - series with positive terms

We apply the second comparison test:

$$\lim_{n\to\infty} \frac{a^n \cdot (a\cdot n^2 + a\cdot n+1)}{3^n \cdot n^3 (n+|2-a|)^3} : \frac{1}{n}$$

$$\left.\begin{matrix}\lim_{n\to\infty} \frac{a^n \cdot n \cdot (a\cdot n^2 + a\cdot n+1)}{3^n \cdot n^3 (n+|2-a|)^3} \\ a^n > 3^n\end{matrix}\right\} \Rightarrow \lim_{n\to\infty} \frac{y_n}{\frac{1}{n}} = \infty$$

$\sum_{n\geq 1} \frac{1}{n}$ - divergent $\Rightarrow$

$\overset{S.C.T.}{\Rightarrow} \sum_{n\geq 1} y_n$ - divergent, $\forall n \in \mathbb{N}$, for $a \in (3, \infty)$

I. for $a \in (0, 3] \Rightarrow \sum_{n\geq 1} y_n$ - convergent

II for $a \in (3, \infty) \Rightarrow \sum_{n\geq 1} y_n$ - divergent

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2.

$$f(x) = \begin{cases} \operatorname{arctg} \frac{x^2+1}{|x|}, & x \neq 0 \\ \frac{\pi}{2}, & x = 0 \end{cases}$$

f - continuous at $0 \Leftrightarrow \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} f(x) = f(0)$

$$\lim_{\substack{x\to 0\\ x<0}} f(x) = \lim_{\substack{x\to 0\\ x<0}} \operatorname{arctg} \frac{x^2+1}{|x|} = \lim_{\substack{x\to 0\\ x<0}} \operatorname{arctg} \frac{x^2+1}{-x}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \operatorname{arctg} \frac{0+1}{-0} = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$\lim_{\substack{x\to 0\\ x>0}} \operatorname{arctg} \frac{x^2+1}{|x|} = \lim_{\substack{x\to 0\\ x>0}} \operatorname{arctg} \frac{x^2+1}{x} = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$

$\Rightarrow$

$\Rightarrow \lim\limits_{\substack{x\to 0\\ x>0}} f(x) \neq \lim\limits_{\substack{x\to 0\\ x<0}} f(x) \Rightarrow f$ is not continuous at 0

f - differentiable at $0 \Leftrightarrow \lim\limits_{\substack{x\to 0\\ x<0}} \frac{f(x)-f(0)}{x-0} = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0}$

$$\lim_{\substack{x\to 0\\ x<0}} \frac{\operatorname{arctg} \frac{x^2+1}{-x} - \frac{\pi}{2}}{x} = \frac{-\pi}{0_-} = \frac{\pi}{0} = \infty$$

$$\lim_{\substack{x\to 0\\ x>0}} \frac{\operatorname{arctg} \frac{x^2+1}{x} - \frac{\pi}{2}}{x} = \frac{0}{0} \overset{\text{l'Hopital}}{=} \lim_{\substack{x\to 0\\ x>0}} \frac{\left(\operatorname{arctg} \frac{x^2+1}{x}\right)'}{x'}$$

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$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{1}{\left(\frac{x^2+1}{x}\right)^2 + 1} \cdot \left(\frac{x^2+1}{x}\right)'$$

$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{x^2 - 1}{x^4 + 3x^2 + 1} = \frac{-1}{1} = -1$$

$$\lim_{\substack{x \to 0 \\ x < 0}} \frac{f(x) - f(0)}{x - 0} \neq \lim_{\substack{x \to 0 \\ x > 0}} \frac{f(x) - f(0)}{x - 0} \Rightarrow f \text{ - not differentiable at } 0$$

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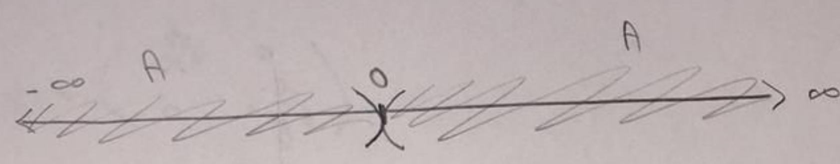
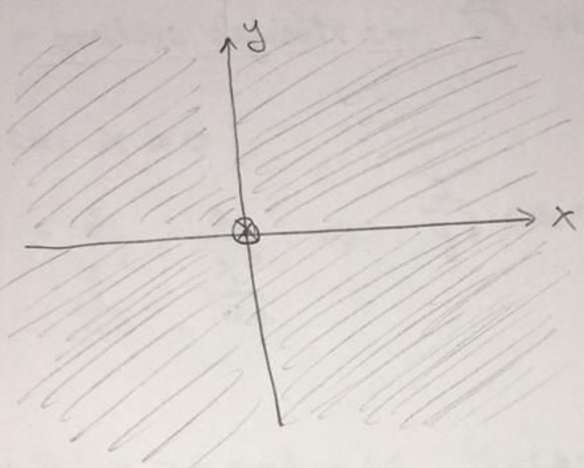
3. $A = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y \neq 0\}$, $f : A \to \mathbb{R}$,

$f(x,y) = \frac{1}{x} - \frac{1}{y} + x - 4y$

a)





int $A = A$ => there are no points in A that are not interior points of A

b)
$$\frac{\partial f}{\partial x}(x,y) = -\frac{1}{x^2} + 1$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{y^2} - 4$$

$$\frac{\partial f^2}{\partial x^2}(x,y) = \frac{2}{x^3}$$

$$\frac{\partial f}{\partial y^2}(x,y) = -\frac{2}{y^3}$$

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$$\frac{\partial f}{\partial x \partial y}(x, y) = 0$$

$$\frac{\partial f}{\partial y \partial x}(x, y) = 0$$

$$H f(x, y) = \begin{pmatrix} \frac{2}{x^3} & 0 \\ 0 & -\frac{2}{y^3} \end{pmatrix}$$

$H f(x, y)$ - positive definite $\Leftrightarrow$ $\frac{2}{x^3} > 0$ and $\frac{4}{-x^3 \cdot y^3} > 0$

$$\left.\begin{aligned} &\frac{2}{x^3} > 0 \Leftrightarrow x > 0 \\ &\frac{4}{-x^3 \cdot y^3} = -\underbrace{\frac{2}{y^3}}_{<0} \cdot \underbrace{\frac{2}{x^3}}_{>0} \Rightarrow \frac{2}{y^3} < 0 \Leftrightarrow y < 0 \end{aligned}\right\} \Rightarrow$$

$P = \{(x, y) \in A \mid x > 0$ and $y < 0$, $H f(x, y)$ - positive definite$\}$

$$\left.\begin{aligned} &H f(x, y) \text{ - negative definite} \Leftrightarrow \frac{2}{x^3} < 0 \text{ and } \frac{4}{-x^3 \cdot y^3} > 0 \\ &\frac{2}{x^3} < 0 \Leftrightarrow x < 0 \\ &\frac{4}{-x^3 \cdot y^3} > 0 \Leftrightarrow \underbrace{\frac{2}{x^3}}_{<0} \cdot \underbrace{\left(-\frac{2}{y^3}\right)}_{<0} > 0 \Rightarrow \frac{2}{y^3} > 0 \Leftrightarrow y > 0 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow N = \{(x, y) \in A \mid x < 0$ and $y > 0$, $H f(x, y)$ - negative definite$\}$

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$H_f(x,y)$ - indefinite $\Leftrightarrow \frac{2}{x^3} < 0$ and $\frac{-4}{x^3 \cdot y^3} < 0$ or

$\frac{2}{x^3} > 0$ and $\frac{-4}{x^3 \cdot y^3} < 0$ or $\frac{2}{x^3} = 0$ (which is impossible)

or $\frac{-4}{x^3 \cdot y^3} = 0$ (which is impossible) $\Rightarrow$

$\Rightarrow$ We have two cases:

I. $\frac{2}{x^3} < 0$ and $\frac{-4}{x^3 \cdot y^3} < 0 \Rightarrow \frac{2}{x^3} < 0 \Leftrightarrow x < 0$

$$-\underbrace{\frac{2}{y^3}}_{<0} \cdot \underbrace{\frac{2}{x^3}}_{<0} < 0 \Rightarrow \frac{2}{y^3} < 0 \Leftrightarrow y < 0$$

II. $\frac{2}{x^3} > 0$ and $\frac{-4}{x^3 \cdot y^3} < 0 \Rightarrow \frac{2}{x^3} > 0 \Leftrightarrow x > 0$

$$-\underbrace{\frac{2}{y^3}}_{>0} \cdot \underbrace{\frac{2}{x^3}}_{>0} < 0 \Rightarrow \frac{1}{y^3} > 0 \Leftrightarrow y > 0$$

I, II $\Rightarrow J = \{(x,y) \in A \mid (x > 0$ and $y > 0)$ or $(x < 0$ and $y < 0)$, $H_f(x,y)$ indefinite$\}$

a) $\nabla f(x,y) = \left(-\frac{1}{x^2}+1,\ \frac{1}{y^2}-4\right)$
$\nabla f(x,y)=0$ $\Bigg\}$ =>

$\Rightarrow \begin{cases} -\frac{1}{x^2}+1=0 \\ \frac{1}{y^2}-4=0 \end{cases} \Leftrightarrow \begin{cases} x^2=1 \\ y^2=\frac{1}{4} \end{cases} \Leftrightarrow \begin{cases} x\in\{-1,1\} \\ y\in\{-\frac{1}{2},\frac{1}{2}\} \end{cases} \Rightarrow$

$\{(-1,-\frac{1}{2}),(1,-\frac{1}{2}),(-1,\frac{1}{2}),(1,\frac{1}{2})\}$
↳ the set of stationary points

From subpoint b, we know that:

1) when $x>0$ and $y>0$, $H_f(x,y)$ - positive definite =>
=> $(1,-\frac{1}{2})$ - local minimum point

2) when $x<0$ and $y>0$, $H_f(x,y)$ - negative definite =>
=> $(-1,\frac{1}{2})$ - local maximum point

3) when $x<0$ and $y<0$ or $x>0$ and $y>0$, $H_f(x,y)$ - indefinite

I. $H_f(-1,-\frac{1}{2}) = \begin{pmatrix} -2 & 0 \\ 0 & 16 \end{pmatrix}$

$\phi_c = (h_1\ h_2)\begin{pmatrix} -2 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$\phi_c = (-h_1\ \ 16h_2)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \Rightarrow \phi_c = -h_1^2+16h_2^2$

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We take $v=(1,1)$ and $v=(1,0,1)$

$\Phi_c(1,1) = -1+16 = 15>0$

$\Phi_c(1,0,1) = -1+0,16 = -0,84 < 0$ $\Big\} \Rightarrow H_f(x,y)$ - indefinite

$\Rightarrow (-1,-\frac{1}{2})$ - ~~(…)~~ is not a local extremum point

$(1,\frac{1}{2})$ - is not a local extremum point because $H_f(1,\frac{1}{2})$ indefinite

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e) $\iint_A y \cdot e^y \cdot \sin(xy)\, dx\, dy$, $A = [0,1] \times [0,\pi]$

$$\mathcal{I} = \int_0^{\pi} \int_0^1 y \cdot e^y \cdot \sin(x \cdot y)\, dx\, dy$$

$$\mathcal{I}_1 = \int_0^1 y \cdot e^y \cdot \sin(xy)\, dx$$

$$\mathcal{I}_1 = y \cdot e^y \int_0^1 \sin(xy)\, dx$$

$$xy = t \Rightarrow y \cdot dx = dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = 1 \Rightarrow t = y$$

$$\mathcal{I}_1 = y \cdot e^y \int_0^y \frac{\sin t}{y}\, dt = e^y \int_0^y \sin t\, dt$$

$$\mathcal{I}_1 = e^y \cdot (-\cos t)\Big|_0^y \Rightarrow \mathcal{I}_1 = e^y \cdot (-\cos y) + e^y$$

$$\mathcal{I}_1 = e^y (1 - \cos y)$$

$$\mathcal{I} = \int_0^{\pi} e^y (1 - \cos y)\, dy$$

$$\mathcal{I} = \int_0^{\pi} e^y dy - \int_0^{\pi} e^y \cdot \cos y\, dy$$

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$$I = e^y \Big|_0^{\pi} - \underbrace{\int_0^{\pi} e^y \cdot \cos y \, dy}_{I_2}$$

$$I_2 = \int_0^{\pi} e^y \cdot \cos y \, dy$$

$$f = e^y \Rightarrow f' = e^y$$
$$g' = \cos y \Rightarrow g = \sin y$$

$$I_2 = e^y \cdot \sin y \Big|_0^{\pi} - \int_0^{\pi} e^y \sin y \, dy$$

$$I_2 = - \underbrace{\int_0^{\pi} e^y \sin y \, dy}_{I_3}$$

$$I_3 = \int_0^{\pi} e^y \cdot \sin y \, dy$$

$$f = e^y \Rightarrow f' = e^y$$
$$(\cancel{g' = \sin}) \; g' = \sin y \Rightarrow g = -\cos y$$

$$I_3 = e^y(-\cos y)\Big|_0^{\pi} + \underbrace{\int_0^{\pi} e^y \cdot \cos y \, dy}_{I_2}$$

$$-2I_2 = -e^y \cos y \Big|_0^{\pi}$$

$$-2I_2 = e^{\pi} + 1 \Rightarrow I_2 = \frac{e^{\pi}+1}{-2}$$

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$$y = e^{\ddot{o}} - \frac{2}{1} + \frac{e^{\ddot{o}}+1}{2}$$

$$y = e^{\ddot{o}} + \frac{e^{\ddot{o}}-1}{2}$$

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4.

a) $\alpha \in \mathbb{R}$, $f: [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^{\alpha} \cdot \sqrt{1+x^4}}$

f - continuous

$p \in \mathbb{R}$ such that $\exists L = \lim\limits_{x \to \infty} x^p \cdot f(x)$

$$\left.\begin{array}{l} L = \lim\limits_{x \to \infty} \dfrac{x^p}{x^{\alpha} \cdot \sqrt{1+x^4}} \\ p = \alpha \end{array}\right\} \Rightarrow L = 1$$

I. $\alpha > 1$, $L = 1$ $\Rightarrow$ f - improper integrable on $[1, \infty)$

II. $\alpha \leq 1$, $L = 1$ $\Rightarrow$ f - is not improper integrable on $[1, \infty)$