Seminar 9

(1)

Ex 1: Determine if $(x^k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ is convergent or not. If the sequence is convergent, find also its limit:

a) $n=2$, $x^k = \left(\frac{1}{k}, \frac{k^2+4k}{2k^2+1}\right)$

$\lim\limits_{k\to\infty} \frac{1}{k} = 0$, $\lim\limits_{k\to\infty} \frac{k^2+4k}{2k^2+1} = \frac{1}{2}$ $\Rightarrow$ (x^k) conv. and $\lim\limits_{k\to\infty} x^k = (0, \frac{1}{2})$

b) $n=2$, $x^k = \left(\left(-\frac{1}{2}\right)^k, (-1)^k\right)$

$((-1)^k)_{k\in\mathbb{N}}$ is div $\Rightarrow$ (x^k) is div.

c) $n=2$, $x^k = \left(\sin k, \frac{1}{k^2}\right)$

$(\sin k)_{k\in\mathbb{N}}$ is div $\Rightarrow$ (x^k) is div

(2)

d) $n=2$, $x^k = \left(\left(\frac{\sqrt{k}}{1+\sqrt{k}}\right)^k, \frac{1^1+2^2+\dots+k^k}{k^k}\right)$

$\lim\limits_{k\to\infty}\left(\frac{\sqrt{k}}{1+\sqrt{k}}\right)^k = \lim\limits_{k\to\infty}\left[\left(1-\frac{1}{1+\sqrt{k}}\right)^{-(1+\sqrt{k})}\right]^{-\frac{k}{1+\sqrt{k}}}$ $e^{-\infty} = 0$

$a_k = 1^1+2^2+\dots+k^k$, $b_k = k^k$, $k\in\mathbb{N}$

(b_k) str. incr., $b_k \to \infty$

$\frac{a_{k+1}-a_k}{b_{k+1}-b_k} = \frac{(k+1)^{k+1}}{(k+1)^{k+1}-k^k} = \frac{1}{1-\underbrace{\left(\frac{k}{k+1}\right)^k}_{\to \frac{1}{e}}\cdot\underbrace{\frac{1}{k+1}}_{\to 0}} \to 1$

S.-C.

$\Rightarrow \lim\limits_{k\to\infty}\frac{1^1+2^2+\dots+k^k}{k^k} = 1$

$\Rightarrow$ (x^k) conv. and $\lim\limits_{k\to\infty} x^k = (0,1)$

(3)

e) $n=3$, $x^k = \left(\underbrace{\frac{2^k}{k!}}_{\to 0}, \underbrace{\frac{1-4k^7}{k^7+12k}}_{\to -4}, \underbrace{\frac{\sqrt{k}}{e^{3k}}}_{\to 0}\right)$ $\Rightarrow$ (x^k) conv. and $\lim\limits_{k\to\infty} x^k = (0,-4,0)$

f) $n=3$, $x^k = (e^{-k}\cos k, e^{-k}\sin k, k)$

(k) div $\Rightarrow$ (x^k) div.

g) $n=4$, $x^k = \left(\frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}}, \frac{1}{\sqrt[k]{k!}}, (e^k+k)^{\frac{1}{k}}, \frac{\alpha^k}{k}\right)$, where $\alpha \geq 0$ is fixed

$\lim\limits_{k\to\infty}\frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}} = \lim\limits_{k\to\infty}\left(\frac{2}{2+\frac{1}{k}}\right)^{2k} = \lim\limits_{k\to\infty}\frac{1}{\left(1+\frac{1}{2k}\right)^{2k}} = \frac{1}{e}$

$\lim\limits_{k\to\infty}\frac{1}{\sqrt[k]{k!}} = 0$ $\left(\lim\limits_{k\to\infty}\sqrt[k]{k!} = \infty\right)$

(4)

$\lim\limits_{k\to\infty}(e^k+k)^{\frac{1}{k}} = e$

$x>0$, $(e^x+x)^{\frac{1}{x}} = e^{\frac{\ln(e^x+x)}{x}}$

$\lim\limits_{x\to\infty}\frac{\ln(e^x+x)}{x} \overset{L'H}{=} \lim\limits_{x\to\infty}\frac{1}{e^x+x}\cdot(e^x+1) = 1$

$\Rightarrow \lim\limits_{x\to\infty}(e^x+x)^{\frac{1}{x}} = e$

$e^k \leq e^k+k \leq 2e^k$, $\forall k\in\mathbb{N}$

$e \leq (e^k+k)^{\frac{1}{k}} \leq \underbrace{\sqrt[k]{2}}_{\to 1}\cdot e$

$\lim\limits_{k\to\infty}\frac{\alpha^k}{k} = \begin{cases} 0, & \alpha\in[0,1] \\ \infty, & \alpha > 1 \end{cases}$

If $\alpha\in[0,1]$, (x^k) is conv. and $\lim\limits_{k\to\infty} x^k = \left(\frac{1}{e}, 0, e, 0\right)$

If $\alpha > 1$, (x^k) is div.

(5)

Ex 2: Study if $f:\mathbb{R}^n\setminus\{0_n\}\to\mathbb{R}$ has a limit at 0_n:

a) $n=2$, $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$

$\lim\limits_{(x,y)\to(0,0)}\frac{\sin(x^2+y^2)}{x^2+y^2} = 1$ because $\lim\limits_{t\downarrow 0}\frac{\sin t}{t} = 1$

(Let $(a^k) \subset \mathbb{R}^2\setminus\{0_2\}$ s.t. $a^k \to 0_2$, $a^k = (a_1^k, a_2^k)$

$\Updownarrow$

$a_1^k \to 0$ and $a_2^k \to 0$

$\Downarrow$

$(a_1^k)^2+(a_2^k)^2 \to 0$

$\lim\limits_{t\downarrow 0}\frac{\sin t}{t} = 1 \Rightarrow \lim\limits_{k\to\infty}\frac{\sin((a_1^k)^2+(a_2^k)^2)}{(a_1^k)^2+(a_2^k)^2} = 1$.

Apply the Seq. Charact. of Limits)

b) $n=2$, $f(x,y) = \frac{\sin(x^2)}{x^2+y^2}$

$a^k = \left(0, \frac{1}{k}\right)$, $b^k = \left(\frac{1}{k}, 0\right)$, $k\in\mathbb{N}$

$\lim\limits_{k\to\infty} a^k = 0_2 = \lim\limits_{k\to\infty} b^k$

$\lim\limits_{k\to\infty} f(a^k) = 0 \neq 1 = \lim\limits_{k\to\infty} f(b^k)$

By the Seq. Charact. of Limits, f has no limit at 0_2

(6)

c) $n=2$, $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$

$\forall (x,y)\in\mathbb{R}^2\setminus\{0_2\}$, $0 \leq \left|\frac{x^3+y^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| = \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1}|x| + \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1}\cdot|y| \leq |x|+|y|$

$\lim\limits_{(x,y)\to(0,0)}(|x|+|y|) = 0$ ($g:\mathbb{R}^2\to\mathbb{R}$, $g(x,y) = |x|+|y|$ is cont. $\Rightarrow$ $\exists \lim\limits_{(x,y)\to(0,0)} g(x,y)$ and is equal to $g(0,0)$)

By the Squeeze Thm, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

d) $f(x_1,x_2,\dots,x_n) = \frac{x_1x_2\dots x_n}{(x_1)^2+(x_2)^2+\dots+(x_n)^2}$

$n=1$, $f(x_1) = \frac{x_1}{(x_1)^2} = \frac{1}{x_1}$ f has no limit at 0

(7)

$n=2$, $f(x_1,x_2) = \frac{x_1x_2}{(x_1)^2+(x_2)^2}$

$a^k = \left(0, \frac{1}{k}\right)$, $b^k = \left(\frac{1}{k}, \frac{1}{k}\right)$, $k\in\mathbb{N}$, $\lim\limits_{k\to\infty} a^k = 0_2 = \lim\limits_{k\to\infty} b^k$

$\lim\limits_{k\to\infty} f(a^k) = 0 \neq \frac{1}{2} = \lim f(b^k)$

$\Rightarrow$ f has no limit at 0_2

$n\geq 3$

$\forall (x_1,x_2,\dots,x_n)\setminus\{0_n\}$, $0 \leq \left|\frac{x_1x_2\dots x_n}{(x_1)^2+(x_2)^2+\dots+(x_n)^2}\right| = \frac{|x_1x_2|}{(x_1)^2+(x_2)^2+\dots+(x_n)^2}\cdot|x_3\dots x_n|$

$\leq \underbrace{\frac{|x_1x_2|}{(x_1)^2+(x_2)^2}}_{\leq 1/2}|x_3\dots x_n| \leq \underbrace{\frac{1}{2}|x_3|\dots|x_n|}_{\xrightarrow{(x_1,\dots,x_n)\to 0_n} 0}$

$2|x_1x_2| \leq (x_1)^2+(x_2)^2$

$0 \leq (|x_1|-|x_2|)^2$

$\Rightarrow \lim\limits_{(x_1,\dots,x_n)\to 0_n} f(x_1,\dots,x_n) = 0$.

(8)

Ex 3 Study if $f:\mathbb{R}^2\to\mathbb{R}$ is continuous at 0_2:

a) $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases}$

$\forall (x,y)\in\mathbb{R}^2\setminus\{0_2\}$, $0 \leq |f(x,y)-f(0_2)| = \left|(x^2+y^2)\cdot\sin\frac{1}{\sqrt{x^2+y^2}}\right| = (x^2+y^2)\cdot\underbrace{\left|\sin\frac{1}{\sqrt{x^2+y^2}}\right|}_{\leq 1} \leq \underbrace{x^2+y^2}_{\xrightarrow{(x,y)\to 0_2} 0}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0_2)$ $\Rightarrow$ f is cont. at 0_2.

b) $p,q\in\mathbb{N}$, $f(x,y) = \begin{cases} \frac{x^py^q}{x^2-xy+y^2}, & (x,y)\neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases}$

(9)

$p=q=1$, $(x,y)\neq 0_2$, $f(x,y) = \frac{xy}{x^2-xy+y^2}$

$a^k = \left(\frac{1}{k}, \frac{1}{k}\right)$, $k\in\mathbb{N}$, $\lim\limits_{k\to\infty} a^k = 0_2$

$\lim\limits_{k\to\infty} f(a^k) = \lim\limits_{k\to\infty}\frac{\frac{1}{k^2}}{\frac{1}{k^2}-\frac{1}{k^2}+\frac{1}{k^2}} = 1 \neq f(0_2)$

$\Rightarrow$ f is not cont at 0_2.

$x^2-xy+y^2 = \underbrace{\frac{1}{2}(x-y)^2+\frac{1}{2}(x^2+y^2)}_{>0 \text{ if } (x,y)\neq 0_2}$

$0 \leq x^2-2|xy|+y^2 \leq x^2-|xy|-xy+y^2$

$|xy| \leq x^2-xy+y^2$

$\frac{|xy|}{x^2-xy+y^2} \leq 1$

$p+q \geq 3$

$\forall (x,y)\in\mathbb{R}^2\setminus\{0_2\}$, $0 \leq |f(x,y)-f(0_2)| = \left|\frac{x^py^q}{x^2-xy+y^2}\right| = \underbrace{\frac{|xy|}{x^2-xy+y^2}}_{\leq 1}\cdot|x|^{p-1}\cdot|y|^{q-1} \leq \underbrace{|x|^{p-1}\cdot|y|^{q-1}}_{\xrightarrow{(x,y)\to(0,0)} 0}$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0_2) \Rightarrow f$ is cont at 0_2

(10)

Ex 4: Find the second order partial derivatives of:

a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = \sin(xy)$

For $(x,y)\in\mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = y\cos(xy)$, $\frac{\partial f}{\partial y}(x,y) = x\cos(xy)$

$\frac{\partial^2 f}{\partial x^2}(x,y) = y^2(-\sin(xy))$, $\frac{\partial^2 f}{\partial y^2}(x,y) = x^2(-\sin(xy))$, $\frac{\partial^2 f}{\partial y\partial x}(x,y) = \cos(xy)+y\cdot(-\sin(xy))\cdot x = \frac{\partial^2 f}{\partial x\partial y}(x,y)$

b) $f:(0,\infty)\times(0,\infty)\to\mathbb{R}$, $f(x,y) = x^y$

For $(x,y)\in\mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = yx^{y-1}$, $\frac{\partial f}{\partial y}(x,y) = x^y\cdot\ln x$

$\frac{\partial^2 f}{\partial x^2}(x,y) = y(y-1)x^{y-2}$, $\frac{\partial^2 f}{\partial y^2}(x,y) = (\ln x)^2x^y$, $\frac{\partial^2 f}{\partial y\partial x}(x,y) = x^{y-1}+y\cdot x^{y-1}\cdot\ln x = x^{y-1}(1+y\ln x)$

$\parallel$

$\frac{\partial^2 f}{\partial x\partial y}(x,y) = yx^{y-1}\cdot\ln x + x^y\cdot\frac{1}{x} = x^{y-1}(1+y\ln x)$

(11)

$\frac{\partial^2 f}{\partial y\partial x}(x,y) = \left(yx^{y-1}\right)'_y \underset{\substack{\uparrow \\ \text{product rule}}}{=} 1\cdot x^{y-1} + y\cdot x^{y-1}\cdot\ln x$

c) $f:\underbrace{(\mathbb{R}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}}_{D}\to\mathbb{R}$, $f(x,y,z) = \frac{z^2e^y}{x}$

For $(x,y,z)\in D$, $\frac{\partial f}{\partial x}(x,y,z) = -\frac{z^2e^y}{x^2}$, $\frac{\partial f}{\partial y}(x,y,z) = \frac{z^2e^y}{x}$, $\frac{\partial f}{\partial z}(x,y,z) = \frac{2ze^y}{x}$

$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2\frac{z^2e^y}{x^3}$, $\frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{z^2e^y}{x}$, $\frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{2e^y}{x}$

$\frac{\partial^2 f}{\partial y\partial x}(x,y,z) = -\frac{z^2e^y}{x^2} = \frac{\partial^2 f}{\partial x\partial y}(x,y,z)$, $\frac{\partial^2 f}{\partial z\partial x}(x,y,z) = -\frac{2ze^y}{x^2} = \frac{\partial^2 f}{\partial x\partial z}(x,y,z)$, $\frac{\partial^2 f}{\partial z\partial y}(x,y,z) = \frac{2ze^y}{x}$

$\parallel$

$\frac{\partial^2 f}{\partial y\partial z}(x,y,z)$