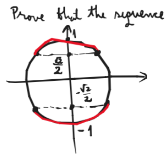
Seminar 4

(1)

Ex. 1: Prove that the sequence $(\sin n)$ has no limit



For $k\in\mathbb{N}$, consider $I_k = \left[\frac{\pi}{4}+2k\pi, \frac{3\pi}{4}+2k\pi\right]$

$J_k = \left[\frac{5\pi}{4}+2k\pi, \frac{7\pi}{4}+2k\pi\right]$

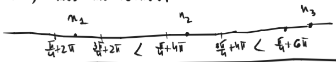
$\text{length}(I_k) = \text{length}(J_k) = \frac{\pi}{2} > 1$

$\Rightarrow \exists\, n_k \in I_k \cap \mathbb{N},\ \exists\, m_k \in J_k \cap \mathbb{N}$

$\Rightarrow \sin n_k \geq \frac{\sqrt{2}}{2},\quad \sin m_k \leq -\frac{\sqrt{2}}{2}$

$\Rightarrow (\sin n_k), (\sin m_k)$ are two subsequences of $(\sin n)$ that cannot have the same limit

$\Rightarrow (\sin n)$ has no limit.



(2)

Ex. 2: In each of the following cases, study if the sequence (x_n) is bounded, monotone and convergent (if possible, find also its limit)

a) $x_n = \frac{(-1)^n}{n}$, $n\in\mathbb{N}$. Does the sequence $\left(\frac{1}{x_n}\right)$ have a limit?

$-1 \leq x_n \leq \frac{1}{2}$, $\forall n\in\mathbb{N}$ $\Rightarrow$ (x_n) bd.

$x_{2k} = \frac{1}{2k}$, $x_{2k-1} = -\frac{1}{2k-1}$, $k\in\mathbb{N}$; $\left.\begin{array}{l}\forall k\in\mathbb{N},\ x_{2k-1} < x_{2k}\\ x_{2k} > x_{2k+1}\end{array}\right\} \Rightarrow (x_n)$ is not monotone **(not even eventually monotone)**

$\underset{\downarrow 0}{-\frac{1}{n}} \leq x_n \leq \underset{\downarrow 0}{\frac{1}{n}}$, $\forall n\in\mathbb{N}$. By the Squeeze Thm, (x_n) conv. and $\lim_{n\to\infty} x_n = 0$

$y_n = \frac{1}{x_n} = (-1)^n\cdot n$, $n\in\mathbb{N}$; $\left.\begin{array}{l} y_{2k} = 2k \to \infty\\ y_{2k-1} = -(2k-1) \to -\infty\end{array}\right\} \Rightarrow (y_n)$ has no limit, is divergent

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b) $x_n = (-1)^n + \frac{n+1}{n}$, $n\in\mathbb{N}$

$x_n = (-1)^n + 1 + \frac{1}{n}$

$0 < x_n < 3$, $\forall n\in\mathbb{N}$ $\Rightarrow$ (x_n) bd.

$x_{2k} = 1+1+\frac{1}{2k} = 2+\frac{1}{2k}$

$x_{2k-1} = -1+1+\frac{1}{2k-1} = \frac{1}{2k-1}$, $k\in\mathbb{N}$; $\left.\begin{array}{l}\forall k\in\mathbb{N},\ x_{2k-1} < x_{2k}\\ x_{2k} > x_{2k+1}\end{array}\right\} \Rightarrow (x_n)$ not monotone **(not even eventually monotone)**

$\left.\begin{array}{l} x_{2k} \to 2\\ x_{2k-1} \to 0\end{array}\right\}$ as $k\to\infty$ $\Rightarrow (x_n)$ has no limit, is divergent

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c) $x_n = \frac{n!}{n^n}$, $n\in\mathbb{N}$

$0 < x_n = \underbrace{\frac{n}{n}}_{=1}\cdot\underbrace{\frac{n-1}{n}\cdot\ldots\cdot\frac{2}{n}}_{\leq 1}\cdot\frac{1}{n} \leq \underset{\downarrow 0}{\frac{1}{n}} \leq 1$. By the Squeeze Thm, (x_n) is conv and $\lim_{n\to\infty} x_n = 0$

(x_n) bd

$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n < 1$, $\forall n\in\mathbb{N}$ $\Rightarrow$ (x_n) is strictly decreasing.

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d) $x_1 \in (0,1)$, $x_{n+1} = \frac{2x_n+1}{3}$

We show that $x_n\in(0,1)$, $\forall n\in\mathbb{N}$, using mathematical induction:

- $x_1 \in (0,1)$
- Let $k\in\mathbb{N}$ and suppose $x_k\in(0,1)$. Then $0 < x_{k+1} = \frac{2x_k+1}{3} < \frac{2\cdot 1+1}{3} = 1$ $\Rightarrow$ $x_{k+1}\in(0,1)$

$\Rightarrow x_n \in (0,1)$, $\forall n\in\mathbb{N}$

$x_{n+1} - x_n = \frac{2x_n+1}{3} - x_n = \frac{1-x_n}{3} > 0$, $\forall n\in\mathbb{N}$ $\Rightarrow$ (x_n) strictly increasing

$\Rightarrow (x_n)$ convergent. Let $l = \lim_{n\to\infty} x_n \in\mathbb{R}$

$l = \frac{2l+1}{3} \Rightarrow l = 1$.

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e) $a>0$, $x_1>0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $n\in\mathbb{N}$

Clearly, $x_n > 0$, $\forall n\in\mathbb{N}$

$x_2 = \frac{1}{2}\left(x_1+\frac{a}{x_1}\right) \geq \sqrt{x_1\cdot\frac{a}{x_1}} = \sqrt{a}$

$A\left(x_1, \frac{a}{x_1}\right) \geq G\left(x_1, \frac{a}{x_1}\right)$

$x_{n+1} \geq \sqrt{x_n\cdot\frac{a}{x_n}} = \sqrt{a}$

$x_n \geq \sqrt{a}$, $\forall n\geq 2$

$x_{n+1} - x_n = \frac{x_n}{2} + \frac{a}{2x_n} - x_n = \frac{a-x_n^2}{2x_n} \leq 0$, $\forall n\geq 2$

$\Rightarrow (x_n)_{n\geq 2}$ decreasing

$\Rightarrow (x_n)$ convergent. Let $l = \lim_{n\to\infty} x_n \in\mathbb{R}$ $\Rightarrow l\geq\sqrt{a} > 0$

$l = \frac{1}{2}\left(l+\frac{a}{l}\right) \Rightarrow \frac{l}{2} = \frac{a}{2l} \Rightarrow l^2 = a \Rightarrow l = \sqrt{a}$

$\min\{x_1,\sqrt{a}\} \leq x_n \leq \max\{x_1, x_2\}$, $\forall n\in\mathbb{N}$

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Ex. 3: Find the sum of the following series:

a) $\sum_{n\geq 2}\left(-\frac{5}{9}\right)^n \underset{\uparrow}{=} \frac{\left(-\frac{5}{9}\right)^2}{1-\left(-\frac{5}{9}\right)} = \frac{25}{81}\cdot\frac{1}{1+\frac{5}{9}} = \frac{25}{\cancel{81}_{9}}\cdot\frac{\cancel{9}}{14} = \frac{25}{126}$.

$-5/9 \in (-1,1)$

$q\in(-1,1)$

$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

$\sum_{n=m}^{\infty} q^n = \frac{1}{1-q} - \underbrace{(1+q+\ldots+q^{m-1})}_{\frac{1-q^m}{1-q}}$

$= \frac{q^m}{1-q}$

b) $\sum_{n\geq 1} 2^{1-2n}$

$2^{1-2n} = 2\cdot\frac{1}{2^{2n}} = 2\cdot\left(\frac{1}{4}\right)^n$

$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$ $\qquad \sum_{n=1}^{\infty} 2^{1-2n} = 2\cdot\frac{1}{3}$

c) $\sum_{n\geq 2}\ln\left(1-\frac{1}{n^2}\right)$

For $n\geq 2$, $\ln\left(1-\frac{1}{n^2}\right) = \ln\frac{n^2-1}{n^2} = \ln(\underbrace{n^2-1}_{(n-1)(n+1)}) - \ln n^2 = \ln(n-1) + \ln(n+1) - 2\ln n$

$= \left(\ln(n+1) - \ln n\right) - \left(\ln n - \ln(n-1)\right)$

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For $n\geq 2$, denote the partial sums by

$s_n = (\ln 3 - \cancel{\ln 2}) - (\ln 2 - \underbrace{\ln 1}_{=0}) + (\cancel{\ln 4} - \ln 3) - (\cancel{\ln 3} - \ln 2) + \ldots + (\ln(n+1) - \ln n) - (\cancel{\ln(n)} - \ln(n-1))$

$= \ln(n+1) - \ln n - \ln 2 = \ln\left(1+\frac{1}{n}\right) - \ln 2 \longrightarrow -\ln 2$ $\quad \left(0 < \ln\left(1+\frac{1}{n}\right) < \underset{\downarrow 0}{\frac{1}{n}},\ \forall n\in\mathbb{N}\right)$

$\left(1+\frac{1}{n}\right)^n < e$

$\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right) = -\ln 2$

d) $\sum_{n\geq 1}\frac{1}{n(n+1)(n+2)}$

$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\cdot\frac{(n+2)-n}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$, $\forall n\in\mathbb{N}$

$n\in\mathbb{N}$, $s_n = \frac{1}{2}\left(\frac{1}{1\cdot 2} - \cancel{\frac{1}{2\cdot 3}} + \ldots + \cancel{\frac{1}{n(n+1)}} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) \longrightarrow \frac{1}{4}$

$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

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e) $n\in\mathbb{N}$, $\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\cdot\frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)} = \frac{1}{3}\left(\frac{1}{3n-2} - \underbrace{\frac{1}{3n+1}}_{3(n+1)-2}\right)$

f) $n\in\mathbb{N}$, $\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} = \left(\sqrt{n+2} - \sqrt{n+1}\right) - \left(\sqrt{n+1} - \sqrt{n}\right)$

g) $\sum_{n\geq 1}\frac{n+1}{2^n}$

$n\in\mathbb{N}$, $s_n = \frac{2}{2} + \frac{3}{2^2} + \ldots + \frac{n+1}{2^n}$

$\frac{1}{2}s_n = \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}}$

$s_n - \frac{1}{2}s_n = 1 + \frac{1}{2^2} + \ldots + \frac{1}{2^n} - \frac{n+1}{2^{n+1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} - \frac{1}{2} - \frac{n+1}{2^{n+1}} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - \frac{1}{2} - \underset{\to 0}{\frac{n+1}{2^{n+1}}} \longrightarrow \frac{3}{2}$

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$\Rightarrow \lim_{n\to\infty} s_n = 3$

$\sum_{n=1}^{\infty}\frac{n+1}{2^n} = 3$

$s_n = \frac{2}{2} + \frac{3}{2^2} + \ldots + \frac{n+1}{2^n}$ $\qquad$ Sum each line

$$s_n = \left.\begin{array}{r} \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \\ \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \\ + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \\ \vdots \\ \frac{1}{2^n} \end{array}\right\}(n+1)\text{ lines}$$