Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Mathematical Analysis - Lecture Notes
Computer Science, Academic Year: 2020/2021

Lecture 5

Real-valued functions of one real variable

Limits of functions

Definition 1. Let $A \subseteq \mathbb{R}$ and $c \in \overline{\mathbb{R}}$. We say that c is an

- *accumulation point* (or a *cluster point* or a *limit point*) of A if

$$\forall V \in \mathcal{V}(c), \quad V \cap (A \setminus \{c\}) \neq \emptyset.$$

  The set of all accumulation points of A is called the *derived set* of A and is denoted by A'.

- *isolated point* of A if $c \in A \setminus A'$.

Remark 1. Let $A \subseteq \mathbb{R}$.

(i) Accumulation points of A may or may not belong to A.

(ii) $a \in A \setminus A' \iff \exists V \in \mathcal{V}(a)$ such that $V \cap A = \{a\}$.

Theorem 1 (Sequential characterization of accumulation points)**.** *Let $A \subseteq \mathbb{R}$ and $c \in \overline{\mathbb{R}}$. Then $c \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{c\}$ such that $\lim\limits_{n\to\infty} x_n = c$.*

Example 1. (i) $A = \{0, 1\}$. $A' = \emptyset$, $0, 1$ – isolated points

In general, a finite set has no acc. points

(ii) $A = (0, 1) \cup (1, 2)$. $A' = [0, 2]$

A has no isolated points

(iii) $A = \mathbb{N}$. $A' = \{\infty\}$

All points in A are isolated.

(iv) $A = [0, 1] \cap \mathbb{Q}$. $A' = [0, 1]$ (by the DP of $\mathbb{Q}$ in $\mathbb{R}$)

A has no isolated points

(v) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. $A' = \{0\}$

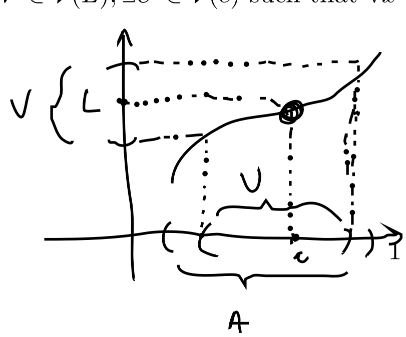
All points in A are isolated

In what follows we consider $A \subseteq \mathbb{R}$, $A \neq \emptyset$.

Definition 2. Let $f : A \to \mathbb{R}$ and $c \in A'$. We say that f *has a limit at* c if there exists $L \in \overline{\mathbb{R}}$ such that

$$\forall V \in \mathcal{V}(L), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap (A \setminus \{c\}) \text{ we have } f(x) \in V. \tag{1}$$

$c, L \in \mathbb{R}$



Remark 2. f cannot have two distinct limits at c.

Definition 3. If $f : A \to \mathbb{R}$ has a limit at $c \in A'$, then the unique $L \in \overline{\mathbb{R}}$ satisfying (1) is called *the limit of f at c.*
Notation: $\lim_{x\to c} f(x)$ or $f(x) \to L$ as $x \to c$ (read as "$f(x)$ approaches L as x approaches c").

Remark 3. (i) Limits of functions are not considered at isolated points of the domain, only at accumulation points (which may or may not belong to the domain).

(ii) The limit of a function $f : A \to \mathbb{R}$ at a given $c \in A'$ depends only on the values of f "near" c, while the values of the f "away" from c are irrelevant.

Theorem 2 (Sequential characterization of limits, Heine)**.** *Let $f : A \to \mathbb{R}$, $c \in A'$, $L \in \overline{\mathbb{R}}$. Then*

$$\lim_{x\to c} f(x) = L \iff \forall \text{ sequence } (x_n) \text{ in } A \setminus \{c\} \text{ with } \lim_{n\to\infty} x_n = c \text{ we have } \lim_{n\to\infty} f(x_n) = L.$$

Example 2. $f : (0,\infty) \to \mathbb{R}$, $f(x) = \sin \frac{1}{x}$ has no limit at 0. $\leftarrow$ $0 \notin (0,\infty)$, but $0 \in (0,\infty)'$

Take $x_n = \frac{1}{2\pi n}$, $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, $n \in \mathbb{N}$

$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$, but $\lim_{n\to\infty} f(x_n) = 0 \neq 1 = \lim_{n\to\infty} f(y_n)$

Since by Theorem 2, limits of functions can be characterized using limits of sequences, limit theorems for functions can be derived from corresponding ones for sequences. We solely include below the following results.

Theorem 3. *Let $f, g : A \to \mathbb{R}$ and $c \in A'$. Suppose $\exists U \in \mathcal{V}(c)$ such that $f(x) \le g(x)$, $\forall x \in U \cap (A \setminus \{c\})$.*

(i) If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$.

(ii) If $\lim_{x\to c} f(x) = \infty$, then $\lim_{x\to c} g(x) = \infty$.

(iii) If $\lim_{x\to c} g(x) = -\infty$, then $\lim_{x\to c} f(x) = -\infty$.

Remark 4. Strict inequalities are not maintained: $1 < \frac{x+1}{x}, \forall x > 0$, but $\lim_{x\to\infty} \frac{x+1}{x} = 1$.

Theorem 4 (Squeeze Theorem for functions)**.** *Let $f, g, h : A \to \mathbb{R}$ and $c \in A'$. Suppose $\exists U \in \mathcal{V}(c)$ such that $f(x) \le g(x) \le h(x)$, $\forall x \in U \cap (A \setminus \{c\})$. If $L \in \mathbb{R}$ is the limit of both f and h at c, then L is also the limit of g at c.*

One-sided limits

Definition 4. Let $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$. If c is an accumulation point of $A \cap (-\infty, c)$ and $f|_{A\cap(-\infty,c)}$ has a limit at c, then we call this limit the *left-hand limit of f at c* and we write

$$\lim_{\substack{x\to c\\ x<c}} f(x) \quad \text{or} \quad \lim_{x\to c^-} f(x) \quad \text{or} \quad \lim_{x\nearrow c} f(x).$$

In a similar way one defines the *right-hand limit of f at c* when considering the set $A \cap (c, \infty)$. In this case we write

$$\lim_{\substack{x\to c\\ x>c}} f(x) \quad \text{or} \quad \lim_{x\to c^+} f(x) \quad \text{or} \quad \lim_{x\searrow c} f(x).$$

These two limits are called *one-sided limits of f at c.*

Remark 5. It may happen that none of the one-sided limit exists, only one of them exists or both of them exist and are different.

Theorem 5 (Characterization of limits using one-sided limits)**.** *Let $f : A \to \mathbb{R}$, $L \in \overline{\mathbb{R}}$ and let $c \in \mathbb{R}$ be an accumulation point of both the sets $A \cap (-\infty, c)$ and $A \cap (c, \infty)$. Then*

$$\lim_{x \to c} f(x) = L \Longleftrightarrow \lim_{\substack{x \to c \\ x < c}} f(x) = L = \lim_{\substack{x \to c \\ x > c}} f(x).$$

Remark 6. If $c \in \mathbb{R}$ is an accumulation point of both the sets $A \cap (-\infty, c)$ and $A \cap (c, \infty)$, then the usual limit $\lim\limits_{x \to c} f(x)$ is also called the *two-sided limit of f at c.*

Continuous functions

Definition 5. Let $f : A \to \mathbb{R}$ and $c \in A$. We say that f is *continuous at c* if

$$\forall V \in \mathcal{V}\left(f(c)\right), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap A \text{ we have } f(x) \in V.$$

In this case we call c a *continuity point of f*. If f fails to be continuous at c, then we say that f is *discontinuous at c* and that c is a *discontinuity point of f*.
If B is a subset of A, we say that f is *continuous on B* if it is continuous at every point of B. If f is continuous on A, then f is simply called continuous.

Remark 7. (i) An important difference between the notions of limit and continuity is that the point c is now assumed to belong to A (and not necessarily to A') so that $f(c)$ makes sense. Recall that limits are only considered at points in A' (which may or may not belong to A).
(ii) If $c \in A \cap A'$, then f is continuous at c if and only if $\lim\limits_{x \to c} f(x) = f(c)$.
(iii) If c is an isolated point of A, then $\exists U \in \mathcal{V}(c)$ such that $U \cap A = \{c\}$. Thus, f is continuous at c.

Theorem 6 (Sequential characterization of continuity)**.** *Let $f : A \to \mathbb{R}$ and $c \in A$. Then f is continuous at c if and only if for every sequence (x_n) in A with $\lim\limits_{n \to \infty} x_n = c$ we have $\lim\limits_{n \to \infty} f(x_n) = f(c)$.*

Remark 8. Sums, products, quotients and compositions of continuous functions (when defined) are continuous.

Classifying discontinuities

Definition 6. Let $f : A \to \mathbb{R}$ and $c \in A$ be a discontinuity point of f. We say that c is a

- *discontinuity point of the first kind* of f (or that f has a *discontinuity of the first kind* at c): if the one-sided limits of f at c both exist and are finite.
- *discontinuity of the second kind* of f: if it is not of the first kind.

Remark 9. If c is a discontinuity of the first kind, then it is either a

- *jump discontinuity*: if the one-sided limits are not equal.
- *removable discontinuity*: if the one-sided limits are equal, but are not equal to $f(c)$.

Example 3. (i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$ discontinuity of the second kind at 0

(ii) $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$, $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$ jump discontinuity at 0

(iii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$ removable discontinuity at 0

Continuous functions on intervals

Definition 7. A function $f : A \to \mathbb{R}$ is called *bounded* if the set $f(A)$ is bounded. We say that f

- *attains its maximum* if there exists $\overline{x} \in A$ such that $\forall x \in A$, $f(x) \leq f(\overline{x})$.
- *attains its minimum* if there exists $\underline{x} \in A$ such that $\forall x \in A$, $f(\underline{x}) \leq f(x)$.

In this case $\overline{x}$ is called a *maximum point* for f and $\underline{x}$ is called a *minimum point* for f.

Theorem 7 (Maximum-Minimum Theorem, Weierstrass)**.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is bounded. Moreover, f attains both its maximum and minimum.*

Remark 10. (i) The function f may be unbounded if

- the interval is unbounded: $f : [0, \infty) \to \mathbb{R}$, $f(x) = x$
- the interval is not closed: $f : (0, 1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$
- f is not continuous: $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(ii) A maximum (minimum) point is not necessarily unique.

Theorem 8 (Intermediate Value Theorem, Bolzano-Darboux)**.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be continuous. If $\gamma \in \mathbb{R}$ satisfies $f(a) < \gamma < f(b)$ or $f(b) < \gamma < f(a)$, then there exists $c \in (a,b)$ such that $f(c) = \gamma$.*

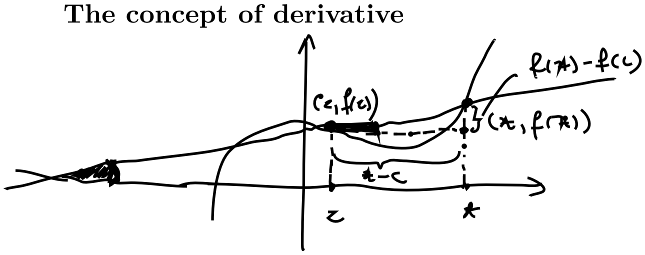
Remark 11. (i) Location of Roots: If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) \cdot f(b) < 0$, then $\exists c \in (a,b)$ such that $f(c) = 0$.

(ii) If I is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval (possibly degenerated into a singleton).

(iii) If $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is a closed bounded interval (possibly degenerated into a singleton).

Differentiation of functions

The concept of derivative



$\frac{f(x) - f(c)}{x - c} \to$ slope of the line that connects the points $(c, f(c))$ and $(x, f(x))$

Definition 8. Let $f : A \to \mathbb{R}$ and $c \in A \cap A'$. We say that f *has a derivative at c* if

$$\exists \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \in \overline{\mathbb{R}}.$$

In this case, the above limit is called the *derivative of f at c* and is denoted by $f'(c)$. If $f'(c) \in \mathbb{R}$, then f is said to be *differentiable at c*.
If B is a subset of A, we say that f is *differentiable on B* if it is differentiable at every point of B. In this case, the function $f' : B \to \mathbb{R}$, $x \in B \mapsto f'(x)$ is called the *derivative of f on B*. If f is differentiable on A, then f is simply called differentiable.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$\forall c \in \mathbb{R}$, $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c$

$\Rightarrow f$ is diff. on $\mathbb{R}$, $f'(x) = 2x$, $\forall x \in \mathbb{R}$

Theorem 9. *Let $f : A \to \mathbb{R}$ and $c \in A \cap A'$. If f is differentiable at c, then f is also continuous at c.*

<u>Pf</u>: $f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x-c)$, $\forall x \in A \setminus \{c\}$ $\Rightarrow \lim_{x \to c} f(x) = f(c)$

Remark 12. Let $f : A \to \mathbb{R}$ and $c \in A \cap A'$.

$$f \text{ has a derivative at } c \quad \overset{(i)}{\not\Rightarrow} \quad f \text{ is continuous at } c$$
$$\overset{(ii)}{\not\Leftarrow}$$

(i) sgn has a derivative at 0, $\mathrm{sgn}'(0) = \infty$:

$\lim_{\substack{x \to 0 \\ x < 0}} \frac{\mathrm{sgn}(x) - \mathrm{sgn}(0)}{x - 0} = \lim_{\substack{x \to 0 \\ x < 0}} \frac{-1}{x} = \infty$; $\lim_{\substack{x \to 0 \\ x > 0}} \frac{\mathrm{sgn}(x) - \mathrm{sgn}(0)}{x - 0} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{1}{x} = \infty$

sgn is discontinuous at 0.

(ii) $|\cdot|$ is cont, but has no derivative at 0: $\lim_{\substack{x \to 0 \\ x < 0}} \frac{|x| - |0|}{x - 0} = -1$, $\lim_{\substack{x \to 0 \\ x > 0}} \frac{|x| - |0|}{x - 0} = 1$

Remark 13 (Weierstrass, Hardy). There exist functions that are continuous on $\mathbb{R}$, but nowhere differentiable: $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad \text{where } a \in (0,1), ab \geq 1, b > 1.$$

Definition 9. Let $f : A \to \mathbb{R}$ and $c \in A$. If c is an accumulation point of $A \cap (-\infty, c)$, then f *has a left-hand derivative at* c if

$$\exists \lim_{\substack{x \to c \\ x < c}} \frac{f(x) - f(c)}{x - c} \in \overline{\mathbb{R}}.$$

In this case, the above left-hand limit is called the *left-hand derivative of* f *at* c and is denoted by $f_l'(c)$. If $f_l'(c) \in \mathbb{R}$, then f is said to be *left-hand differentiable at* c.
In a similar way one defines the *right-hand derivative of* f *at* c, denoted by $f_r'(c)$, and the *right-hand differentiability of* f *at* c.

Remark 14. If $A = [a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, then the differentiability of $f : A \to \mathbb{R}$ at a is actually the right-hand differentiability of f at a and the differentiability of f at b is actually the left-hand differentiability of f at b.

Theorem 10. *Let $c \in A \cap A'$ and $f, g : A \to \mathbb{R}$ two functions that are differentiable at c.*

(i) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$.

(ii) (Sum rule) The function $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.

(iii) (Product rule) The function fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv) (Quotient rule) If $g(c) \neq 0$, then the function f/g (defined on some neighborhood of c) is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Theorem 11 (Chain rule). *Let $I, J \subseteq \mathbb{R}$ be intervals, $c \in I$, $f : I \to J$ and $g : J \to \mathbb{R}$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f : I \to \mathbb{R}$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))f'(c)$.*

Theorem 12 (Inverse Function Theorem). *Let $I, J \subseteq \mathbb{R}$ be intervals, $c \in I$ and let $f : I \to J$ be invertible. If f is differentiable at c, $f'(c) \neq 0$ and $f^{-1} : J \to I$ is continuous at $f(c)$, then f^{-1} is differentiable at $f(c)$ and*

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}.$$

Theorem 13 (L'Hôpital's rule for left-hand limits). *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$ and let $f, g : (a, b) \to \mathbb{R}$ be differentiable. In addition, suppose that the following conditions are satisfied:*

(i) $\forall x \in (a, b),\ g'(x) \neq 0.$

(ii) $\lim\limits_{\substack{x \to b \\ x<b}} f(x) = L = \lim\limits_{\substack{x \to b \\ x<b}} g(x)$, *where* $L \in \{-\infty, 0, \infty\}$.

(iii) $\exists \lim\limits_{\substack{x \to b \\ x<b}} \dfrac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$.

Then $\lim\limits_{\substack{x \to b \\ x<b}} \dfrac{f(x)}{g(x)} = \lim\limits_{\substack{x \to b \\ x<b}} \dfrac{f'(x)}{g'(x)}$.

Remark 15. The result for right-hand limits is similar.

Theorem 14 (L'Hôpital's rule for two-sided limits). *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$, $c \in (a, b)$ and let $f, g : (a, b) \to \mathbb{R}$ be differentiable on $(a, b) \setminus \{c\}$. In addition, suppose that the following conditions are satisfied:*

(i) $\forall x \in (a, b) \setminus \{c\},\ g'(x) \neq 0.$

(ii) $\lim\limits_{x \to c} f(x) = L = \lim\limits_{x \to c} g(x)$, *where* $L \in \{-\infty, 0, \infty\}$.

(iii) $\exists \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$.

Then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$.

Example 5. (i) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = \lim\limits_{x \to 0} \dfrac{\sin x}{2x} = \lim\limits_{x \to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$

(ii) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^x}{1} = 1$

(iii) $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} = \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0$

(iv) $\lim\limits_{\substack{x \to 0 \\ x>0}} \dfrac{\ln x}{\ln(\sin x)} = \lim\limits_{\substack{x \to 0 \\ x>0}} \dfrac{1/x}{\frac{1}{\sin x} \cdot \cos x} = \lim\limits_{\substack{x \to 0 \\ x>0}} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x} = 1$