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Exercise 7.1.

Find the n^{th} derivate ($n \in \mathbb{N}$) of the function

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x \cdot \sin x$

$$f(x) = e^x \cdot \sin x$$

$$f'(x) = e^x \cdot \sin x + e^x \cdot \cos x \Rightarrow f'(x) = e^x(\sin x + \cos x)$$

$$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$$

$$f''(x) = 2 \cdot e^x \cdot \cos x$$

$$f'''(x) = 2e^x \cdot \cos x + 2 \cdot e^x \cdot (-\sin x)$$

$$f'''(x) = 2e^x(\cos x - \sin x)$$

$$f^{(4)}(x) = 2e^x(\cos x - \sin x) + 2e^x(-\sin x - \cos x)$$

$$f^{(4)}(x) = -4e^x \sin x$$

$$f^{(5)}(x) = -4e^x \sin x + (-4) \cdot e^x \cdot \cos x$$

$$f^{(5)}(x) = -4 \cdot e^x(\sin x + \cos x)$$

We found a pattern:

$$f^{(4)}(x) = -4 \cdot f(x)$$

$$f^{(5)}(x) = -4 \cdot f'(x)$$

$$f^{(6)}(x) = -4 \cdot f''(x)$$

$$f^{(7)}(x) = -4 \cdot f'''(x)$$

$$f^{(8)}(x) = -4 \cdot f^{(4)}(x) = 16 \cdot f(x)$$

$$f^{(4k)}(x) = (-4)^k \cdot f(x) = (-4)^k \cdot e^x \cdot \sin x$$

$$f^{(4k+1)}(x) = (-4)^k \cdot f'(x) = (-4)^k \cdot e^x(\sin x + \cos x)$$

$$f^{(4k+2)}(x) = (-4)^k \cdot f''(x) = (-4)^k \cdot 2 \cdot e^x \cdot \cos x \quad , \forall k \in \mathbb{N}_0$$

$$f^{(4k+3)}(x) = (-4)^k \cdot f'''(x) = (-4)^k \cdot 2 \cdot e^x(\cos x - \sin x)$$

So we can write that:

$$f^{(n)}(x) = \begin{cases} (-4)^k \cdot e^x \cdot \sin x & , n = 4k \\ (-4)^k \cdot e^x(\sin x + \cos x) & , n = 4k+1 \\ (-4)^k \cdot 2e^x \cdot \cos x & , n = 4k+2 \\ (-4)^k \cdot 2 \cdot e^x(\cos x - \sin x) & , n = 4k+3 \end{cases} \quad , \forall k \in \mathbb{N}_0$$

We will prove this generalisation by using Mathematical Induction.

$$p(k):\ f^{(n)}(x)=\begin{cases}(-4)^k\cdot e^x\cdot\sin x, & n=4k\\ (-4)^k\cdot e^x(\sin x+\cos x), & n=4k+1\\ (-4)^k 2e^x\cos x, & n=4k+2\\ (-4)^k\cdot 2\cdot e^x(\cos x-\sin x), & n=4k+3\end{cases},\ k\in\mathbb{N}_0$$

We check if $p(0)$ is true:

$$p(0):\ f^{(n)}(x)=\begin{cases}(-4)^0\cdot e^x\cdot\sin x, & n=0\\ (-4)^0\cdot e^x(\sin x+\cos x), & n=1\\ (-4)^0 2e^x\cos x, & n=2\\ (-4)^0\cdot 2\cdot e^x(\cos x-\sin x), & n=3\end{cases}$$

$$p(0):\ f^{(n)}(x)=\begin{cases}e^x\cdot\sin x, & n=0\\ e^x(\sin x+\cos x), & n=1\\ 2e^x\cdot\cos x, & n=2\\ 2\cdot e^x\cdot(\cos x-\sin x), & n=3\end{cases}\Rightarrow$$

$\Rightarrow p(0)$ is valid (I)

We assume that $p(s)$ is true and we have to prove that $p(s+1)$ is also true.

$$p(s):\ f^{(n)}(x)=\begin{cases}(-4)^s\cdot e^x\cdot\sin x, & n=4s\\ (-4)^s\cdot e^x(\sin x+\cos x), & n=4s+1\\ (-4)^s 2\cdot e^x\cdot\cos x, & n=4s+2\\ (-4)^s 2\cdot e^x(\cos x-\sin x), & n=4s+3\end{cases},\ s\in\mathbb{N}_0$$

In order to find $P(n+1)$ we have to compute:

$f^{(4(n+1))}$, $f^{(4(n+1)+1)}$, $f^{(4(n+1)+2)}$, $f^{(4(n+1)+3)}$ equivalent to $f^{(m)}(x)$ for $m = \overline{4n+4,\ 4n+7}$

$$f^{(4(n+1))}(x) = f^{(4n+4)}(x) = \left(f^{(4n+3)}(x)\right)'$$

$$= \left((-4)^n \cdot 2 \cdot e^x \cdot (\cos x - \sin x)\right)'$$

$$= (-4)^n \cdot 2 \cdot e^x \cdot (-\sin x - \cos x) + (-4)^n \cdot e^x (\cos x - \sin x) \cdot 2$$

$$= (-4)^n \cdot 2 \cdot e^x (-\sin x - \cos x + \cos x - \sin x)$$

$$= (-4)^n \cdot e^x \cdot (-4) \cdot \sin x$$

$$= (-4)^{n+1} \cdot e^x \cdot \sin x \quad (1)$$

$$f^{(4(n+1)+1)}(x) = f^{(4n+5)}(x) = \left(f^{(4n+4)}\right)'$$

$$= \left((-4)^{n+1} \cdot e^x \cdot \sin x\right)' = (-4)^{n+1} \cdot e^x \cdot \cos x + (-4)^{n+1} \cdot e^x \cdot \sin x$$

$$= (-4)^{n+1} \cdot e^x (\sin x + \cos x) \quad (2)$$

$$y^{(4(n+1)+2)}_{(x)} = y^{(4n+6)}_{(x)} = \left(y^{(4n+5)}_{(x)}\right)'$$

$$= \left((-4)^{n+1} \cdot e^x (\sin x + \cos x)\right)'$$

$$= (-4)^{n+1} \cdot e^x (\cos x - \sin x) + (-4)^{n+1} \cdot e^x (\cos x + \sin x)$$

$$= (-4)^{n+1} \cdot e^x \cdot 2 \cdot \cos x$$

$$= (-4)^{n+1} \cdot 2 \cdot e^x \cdot \cos x \quad (3)$$

$$y^{(4(n+1)+3)}_{(x)} = y^{(4n+7)}_{(x)} = \left(y^{(4n+6)}_{(x)}\right)'$$

$$= \left((-4)^{n+1} \cdot 2 \cdot e^x \cdot \cos x\right)'$$

$$= (-4)^{n+1} \cdot 2 \cdot e^x \cdot (-\sin x) + (-4)^{n+1} \cdot 2 \cdot e^x \cdot \cos x$$

$$= (-4)^{n+1} \cdot 2 \cdot e^x (\cos x - \sin x) \quad (4)$$

$$(1), (2), (3), (4) \Rightarrow P(n+1): y^{(m)}_{(x)} = \begin{cases} (-4)^{n+1} e^x \sin x, & m = 4(n+1) \\ (-4)^{n+1} \cdot e^x (\sin x + \cos x), & m = 4(n+1)+1 \\ (-4)^{n+1} \cdot 2 \cdot e^x \cdot \cos x, & m = 4(n+1)+2 \\ (-4)^{n+1} \cdot 2 \cdot e^x \cdot (\cos x - \sin x), & m = 4(n+1)+3 \end{cases}$$

$n \in \mathbb{N}_0$

- is also valid (II)

$$(I), (II) \Rightarrow P(m): y^{(m)}_{(x)} = \begin{cases} (-4)^k \cdot e^x \cdot \sin x, & m = 4k \\ (-4)^k \cdot e^x (\sin x + \cos x), & m = 4k+1 \\ (-4)^k \cdot 2 \cdot e^x \cos x, & m = 4k+2 \\ (-4)^k \cdot 2 \cdot e^x (\cos x - \sin x), & m = 4k+3 \end{cases}$$

is valid $\forall k \in \mathbb{N}_0$

Exercise 1.2. Compute the following limits:

a) $\lim\limits_{x\to\infty} \frac{x+\ln x}{x\cdot\ln x} = \lim\limits_{x\to\infty}\left(\frac{\cancel{x}}{\cancel{x}\cdot\ln x} + \frac{\cancel{\ln x}}{x\cdot\cancel{\ln x}}\right)$

$= \lim\limits_{x\to\infty}\left(\frac{1}{\ln x} + \frac{1}{x}\right) = \frac{1}{\ln\infty} + \frac{1}{\infty} = \frac{1}{\infty} + \frac{1}{\infty} = 0+0=0$

e) $\lim\limits_{\substack{x\to 0\\ x>0}} x\cdot\ln(\sin x) = \infty\cdot 0$ so we rewrite the

limit as: $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{\ln(\sin x)}{\frac{1}{x}} = \frac{\infty}{\infty} \overset{\text{l'Hospital}}{=} \lim\limits_{\substack{x\to 0\\ x>0}} \frac{(\ln(\sin x))'}{\left(\frac{1}{x}\right)'}$

$= \lim\limits_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}} = \lim\limits_{\substack{x\to 0\\ x>0}}\left(-\cos x\cdot\frac{x^2}{\sin x}\right)$

$= \lim\limits_{\substack{x\to 0\\ x>0}}(-\cos x)\cdot\left(\frac{x}{\sin x}\right)\cdot x$

$= \lim\limits_{\substack{x\to 0\\ x>0}}(-\cos x)\cdot\left(\frac{\cancel{x}}{\frac{\sin x}{x}\cdot\cancel{x}}\right)\cdot x$

$\underset{1}{}$

$= \lim\limits_{\substack{x\to 0\\ x>0}}(-\cos x)\cdot x^1 = -(\cos 0)\cdot 0 = (-1)\cdot 0 = 0$

c) $\lim\limits_{\substack{x \to 0 \\ x > 0}} (\sin x)^x = \lim\limits_{\substack{x \to 0 \\ x > 0}} e^{\ln(\sin x)^x}$

$= \lim\limits_{\substack{x \to 0 \\ x > 0}} e^{x \cdot \ln(\sin x)} = e^{\underbrace{\lim\limits_{\substack{x \to 0 \\ x > 0}} x \cdot \ln(\sin x)}_{l}}$

$l = \lim\limits_{\substack{x \to 0 \\ x > 0}} x \cdot \ln(\sin x) = 0$ (We know that is the => result from b)

$= e^0 = \underline{1} \Rightarrow \lim\limits_{\substack{x \to 0 \\ x > 0}} (\sin x)^x = \underline{1}$

Exercise 7.3.

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 5x + 1$. Find the third Taylor polynomial $T_3(x)$ of f at 1

SOLUTION:

We know that:

$$T_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

In our case $n = 3$ and $x_0 = 1$

$$T_3(x) = f(1) + \sum_{k=1}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$T_3(x) = f(1) + \frac{f'(1)}{1!}\cdot(x-1) + \frac{f''(1)}{2!}\cdot(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$f'(x) = (x^3 - 3x^2 + 5x + 1)' \Rightarrow f'(x) = 3x^2 - 6x + 5$$

$$f''(x) = (3x^2 - 6x + 5)' = 6x - 6$$

$$f'''(x) = (6x - 6)' \Rightarrow f'''(x) = 6$$

$$f(1) = 1 - 3 + 5 + 1 = 4$$

$$f'(1) = 3 - 6 + 5 = 2$$

$$f''(1) = 0$$

$$f'''(1) = 6$$

$$T_3(x) = 4 + 2(x-1) + 0\cdot(x-1)^2 + \frac{\cancel{6}^{\,1}}{\cancel{6}_{\,1}}\cdot(x-1)^3$$

$$T_3(x) = 4 + 2x - 2 + (x-1)^3$$

$$T_3(x) = 4 + 2x - 2 + (x-1)(x^2 - 2x + 1)$$

$T_3(x) = 4 + 2x - 2 + (x^3 - 2x^2 + x - x^2 + 2x - 1)$

$T_3(x) = 4 + 2x - 2 + x^3 - 3x^2 + 3x - 1$

$T_3(x) = x^3 - 3x^2 + 5x + \underline{1}$

Exercise 8.1

Prove that the function $f:(0,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{x^2}$, can be expended as a Taylor series around 1 on $[1,2]$ and find the corresponding Taylor series expansion.

SOLUTION:

f can be expended as a Taylor series around x_0 and on $J \iff \lim_{n\to\infty} R_n(x)=0,\ \forall x\in J$

In this case

$$f(x)=f(x_0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$f(x)=\frac{1}{x^2}$$

$$f'(x)=\left(\frac{1}{x^2}\right)'=(x^{-2})'=-2\cdot x^{-3}=-2\cdot\frac{1}{x^3}$$

$$f''(x)=-2\cdot\left(\frac{1}{x^3}\right)'=(-2)\cdot(-3)\cdot x^{-4}=6\cdot\frac{1}{x^4}$$

$$f'''(x)=-24\cdot\frac{1}{x^5}$$

$$f^{(4)}(x)=120\,\frac{1}{x^6}$$

$$f^{(n)}(x)=(-1)^n\cdot(n+1)!\,\frac{1}{x^{n+2}}$$

$p(n): f^{(n)}(x) = (-1)^n \cdot (n+1)! \cdot \frac{1}{x^{n+2}}, \forall n \in \mathbb{N}$

We check if $p(1)$ is true:

$p(1): f'(x) = (-1)^1 \cdot 2! \cdot \frac{1}{x^3}$

$f'(x) = -2 \cdot \frac{1}{x^3}$ (True) (1)

We assume that $p(k)$ is true and prove that $p(k+1)$ is also true:

$p(k): f^{(k)}(x) = (-1)^k (k+1)! \frac{1}{x^{k+2}}$

$p(k+1): f^{(k+1)}(x) = (-1)^{k+1} (k+2)! \frac{1}{x^{k+3}}$

$f^{(k)}(x) = (-1)^k (k+1)! \frac{1}{x^{k+2}} \Big|'$

$f^{(k+1)}(x) = (-1)^k \cdot (k+1)! \cdot (x^{-(k+2)})'$

$f^{(k+1)}(x) = (-1)^k \cdot (k+1)! \cdot (-1) \cdot (k+2) \cdot \frac{1}{x^{k+3}}$

$f^{(k+1)}(x) = (-1)^{k+1} \cdot (k+2)! \cdot \frac{1}{x^{k+3}}$ (True) (2)

(1), (2) $\Rightarrow$ We can conclude that that $p(n)$ is true, $\forall n \in \mathbb{N}$

By Taylor - Lagrange Theorem

$\forall x \in [1,2]$ and $x_0 = 1 \Rightarrow \exists\, c \in (1, x)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

$$f^{(n+1)}(c) = (-1)^{n+1} \cdot (n+2)! \, \frac{1}{x^{n+3}}$$

$$R_n(x) = \frac{(-1)^{n+1} \cdot \cancelto{(n+2)}{(n+2)!} \cdot c^{-(n+3)}}{\cancelto{1}{(n+1)!}}(x-1)^{n+1}, \quad x \in [1,2], \; c \in (1, x)$$

$$R_n(x) = (-1)^{n+1} \cdot (n+2) \cdot c^{-(n+3)} \cdot (x-1)^{n+1}$$

$$|R_n(x)| = (n+2) \cdot c^{-(n+3)} \cdot (x-1)^{n+1}$$

$$|R_n(x)| = \frac{(n+2) \cdot (x-1)^{n+1}}{c^{n+3}}, \quad x \in [1, 2], \; c \in (1, x)$$

$$x \in [1,2] \Rightarrow 1 \leq x \leq 2 \;\Big|\, -1 \Rightarrow 0 \leq x-1 \leq 1 \;\Big|^{\wedge\, n+1}$$

$$\Leftrightarrow 0 \leq (x-1)^{n+1} \leq 1 \Rightarrow |R_n(x)| = \frac{(n+2) \cdot (x-1)^{n+1}}{c^{n+3}} < \frac{n+2}{c^{n+3}} \Rightarrow$$

$$\Rightarrow -\frac{n+2}{c^{n+3}} < \frac{(n+2) \cdot (x-1)^{n+1}}{c^{n+3}} < \frac{n+2}{c^{n+3}}$$

$$\lim_{n\to\infty} \frac{n+2}{c^{n+3}}$$

$a_n = n+2$

$b_n = c^{n+3}$

b_n - strictly increasing

$$\lim_{n\to\infty} b_n = +\infty$$

$$L = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\cancel{n}+3-\cancel{n}-2}{c^{n+4}-c^{n+3}}$$

$$L = \lim_{n\to\infty} \frac{1}{c^{n+4}\left(1-\frac{1}{c}\right)} = \frac{1}{\infty\left(1-\frac{1}{c}\right)}$$

$c\in(1,2) \Rightarrow c^{n+4} \to \infty$ when $n\to\infty$

$c\in(1,2) \Rightarrow 1<c<2 \Leftrightarrow \frac{1}{2}<\frac{1}{c}<1 \,\Big|\cdot(-1)$

$$-1 < -\frac{1}{c} < -\frac{1}{2} \,\Big|+1 \Leftrightarrow 0 < 1-\frac{1}{c} < \frac{1}{2} \Rightarrow$$

$$\Rightarrow 1-\frac{1}{c} > 0 \Rightarrow L = \frac{1}{\infty\cdot\underbrace{\left(1-\frac{1}{c}\right)}_{\neq 0}} = 0$$

$$-\frac{n+2}{c^{n+3}} < \frac{(n+2)\cdot(x-1)^{n+1}}{c^{n+3}} < \frac{n+2}{c^{n+3}}$$

$n\to\infty$ $\qquad$ $n\to\infty$ $\qquad$ 0

Squeeze $\Longrightarrow$ Theoreme

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0 \Longrightarrow$$

$\Rightarrow$ φ can be expended as a Taylor series around 1 on $[1, 2]$

Exercise 8.2.

Let $z \in \mathbb{R}^m$, $r > 0$, and $\varepsilon \in (0, 2]$. Prove that if $x, y \in \bar{B}(z, r)$ such that $\|x - y\| \geq \varepsilon \cdot r$, then $\|z - \frac{x+y}{2}\| \leq \sqrt{1 - \frac{\varepsilon^2}{4}} \cdot r$

PROOF:

$\bar{B}(z, r) = \{a \in \mathbb{R}^m \mid \|a - z\| \leq r\}$

$x \in \bar{B}(z, r) \Rightarrow \|x - z\| \leq r$

$y \in \bar{B}(z, r) \Rightarrow \|y - z\| \leq r$

We know that:

$\|x - y\| \geq \varepsilon \cdot r \mid \hat{}^2 \Rightarrow \|x - y\|^2 \geq \varepsilon^2 \cdot r^2 \Leftrightarrow$

$\Leftrightarrow \|x - y\|^2 = \sum_{i=1}^{m} (x_i - y_i)^2 \geq \varepsilon^2 \cdot r^2$

$$\left.\begin{aligned}
&\|x - z\| \leq r \mid \hat{}^2 \Leftrightarrow \|x - z\|^2 = \sum_{i=1}^{m} (x_i - z_i)^2 \leq r^2 \mid \cdot(-1) \\
&\|y - z\| \leq r \mid \hat{}^2 \Leftrightarrow \|y - z\|^2 = \sum_{i=1}^{m} (y_i - z_i)^2 \leq r^2 \mid \cdot(-1) \\
&\hline \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow -\left(\sum_{i=1}^{m}(x_i-z_i)^2+\sum_{i=1}^{m}(y_i-z_i)^2\right)\geq -2r^2 \;\Big|\cdot\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow -\frac{1}{2}\left(\sum_{i=1}^{m}(x_i-z_i)^2+\sum_{i=1}^{m}(y_i-z_i)^2\right)\geq -r^2 \quad (1)$$

$$\|x-y\|^2=\sum_{i=1}^{m}(x_i-y_i)^2\geq \varepsilon^2\cdot r^2 \;\Big|\cdot\frac{1}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{4}\sum_{i=1}^{m}(x_i-y_i)^2\geq \frac{\varepsilon^2\cdot r^2}{4} \quad (2)$$

$$(1)+(2) \Rightarrow \frac{1}{4}\sum_{i=1}^{m}(x_i-y_i)^2-\frac{1}{2}\left(\sum_{i=1}^{m}(x_i-z_i)^2+\sum_{i=1}^{m}(y_i-z_i)^2\right)\geq \frac{\varepsilon^2 r^2}{4}-r^2 \;\Big|\cdot(-1)$$

$$\Leftrightarrow \frac{1}{2}\left(\sum_{i=1}^{m}(x_i-z_i)^2+\sum_{i=1}^{m}(y_i-z_i)^2\right)-\frac{1}{4}\sum_{i=1}^{m}(x_i-y_i)^2\leq r^2\left(1-\frac{\varepsilon^2}{4}\right)$$

$$\frac{1}{2}\|x-z\|^2+\frac{1}{2}\|y-z\|^2-\frac{1}{4}\|x-y\|^2\leq r^2\left(1-\frac{\varepsilon^2}{4}\right) \quad (3)$$

$$=\sum_{i=1}^{m}(2z_i-x_i-y_i)^2=\sum_{i=1}^{m}(4z_i^2+x_i^2+y_i^2-4x_iz_i-4y_iz_i+2x_iy_i)$$

$$=\sum_{i=1}^{m}(4z_i^2+2x_i^2+2y_i^2-4x_iz_i-4y_iz_i+2x_iy_i-x_i^2-y_i^2)$$

$$=\sum_{i=1}^{m}(2(2z_i^2+x_i^2-2x_iz_i+y_i^2-2y_iz_i)-(x_i^2-2x_iy_i+y_i^2))$$

$$=\sum_{i=1}^{m}(2((z_i^2-2x_iz_i+x_i)+(z_i^2-2y_iz_i+y_i^2))-(x_i^2-2x_iy_i+y_i^2))$$

$$=\sum_{i=1}^{m}(2(z_i-x_i)^2+2(z_i-y_i)^2-(x_i-y_i)^2)$$

$$\|z - \frac{x+y}{2}\|^2 = \sum_{i=1}^{m} \left(z_i - \frac{x_i}{2} - \frac{y_i}{2}\right)^2$$

$$= \sum_{i=1}^{m} \left(z_i^2 + \frac{x_i^2}{4} + \frac{y_i^2}{4} - x_i z_i - y_i z_i + \frac{x_i y_i}{2}\right)$$

$$= \sum_{i=1}^{m} \left(z_i^2 + \frac{2x_i^2}{4} + \frac{2y_i^2}{4} - x_i z_i - y_i z_i + \frac{x_i y_i}{2} - \frac{x_i^2}{4} - \frac{y_i^2}{4}\right)$$

$$= \sum_{i=1}^{m} \left(\left(\frac{1}{2} z_i^2 - x_i z_i + \frac{x_i^2}{2} + \frac{1}{2} z_i^2 - y_i z_i + \frac{y_i^2}{2}\right) - \left(\frac{x_i^2}{4} - \frac{x_i y_i}{2} + \frac{y_i^2}{4}\right)\right)$$

$$= \sum_{i=1}^{m} \left(\frac{1}{2}(z_i^2 - 2x_i z_i + x_i^2) + \frac{1}{2}(z_i^2 - 2y_i z_i + y_i^2) - \frac{1}{4}(x_i^2 - 2x_i y_i + y_i^2)\right)$$

$$= \frac{1}{2} \sum_{i=1}^{m} (z_i^2 - 2x_i z_i + x_i^2) + \frac{1}{2} \sum_{i=1}^{m} (z_i^2 - 2y_i z_i + y_i^2) - \frac{1}{4} \sum_{i=1}^{m} (x_i^2 - 2x_i y_i + y_i^2)$$

$$= \frac{1}{2} \sum_{i=1}^{m} (z_i - x_i)^2 + \frac{1}{2} \sum_{i=1}^{m} (z_i^2 - y_i)^2 - \frac{1}{4} \sum_{i=1}^{m} (x_i - y_i)^2$$

$$= \frac{1}{2} \|z - x\|^2 + \frac{1}{2} \|z - y\|^2 - \frac{1}{4} \|x - y\|^2$$

$$\left.\begin{array}{l} \|z - x\|^2 = \|x - z\|^2 \\ \|z - y\|^2 = \|y - z\|^2 \end{array}\right\} \Rightarrow \|z - \frac{x+y}{2}\|^2 =$$

$$= \frac{1}{2} \|x - z\|^2 + \frac{1}{2} \|y - z\|^2 - \frac{1}{4} \|x - y\|^2 \overset{(1)}{\Longrightarrow}$$

$$\Rightarrow \frac{1}{2} \|x - z\|^2 + \frac{1}{2} \|y - z\|^2 - \frac{1}{4} \|x - y\|^2 \le r^2 \left(1 - \frac{\varepsilon}{4}\right)^2 \quad |\sqrt{\ }$$

$$\sqrt{\frac{1}{2} \|x - z\|^2 + \frac{1}{2} \|y - z\|^2 - \frac{1}{4} \|x - y\|^2} \le r \sqrt{1 - \frac{\varepsilon^2}{4}} \quad (I)$$

We know that:

$$\frac{1}{2}\|x-z\|^2+\frac{1}{2}\|y-z\|^2-\frac{1}{4}\|x-y\|^2=\|z-\frac{x+y}{2}\|^2 \Rightarrow$$

$$\Rightarrow \|z-\frac{x+y}{2}\| = \sqrt{\frac{1}{2}\|x-z\|^2+\frac{1}{2}\|y-z\|^2-\frac{1}{4}\|x-y\|^2} \quad \text{II}$$

I, II $\Rightarrow \|z-\frac{x+y}{2}\| \leq r\cdot\sqrt{1-\frac{\varepsilon^2}{4}}$ (True) if

$x, y \in \bar{B}(z,r)$ with $\|x-y\| \geq \varepsilon\cdot r$

Exercise 9.1.

In each of the following cases, study if the function $f:\mathbb{R}^2 \to \mathbb{R}$ is continuous at 0_2

a) $f(x,y) = \begin{cases} \dfrac{xy + x^2 \cdot y \cdot \ln(x^2+y^2)}{x^2+y^2}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2 \end{cases}$

<u>SOLUTION</u>:

f is continuous at $(0,0) \Leftrightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0_2$

If $(x,y) \neq 0_2$, we consider $a^k = \left(\frac{1}{k}, \frac{1}{k}\right)$, $\lim\limits_{k\to\infty} a^k = 0_2$, $k \in \mathbb{N}$

$$\lim_{k\to\infty} f(a^k) = \lim_{k\to\infty} \frac{\frac{1}{k}\cdot\frac{1}{k} + \frac{1}{k^2}\cdot\frac{1}{k}\cdot\ln\left(\frac{2}{k^2}\right)}{\frac{2}{k^2}}$$

$$= \lim_{k\to\infty} \frac{\cancel{\frac{1}{k^2}}\left(1 + \frac{1}{k}\cdot\ln 2 - \frac{1}{k}\cdot\ln k^2\right)}{2\cdot\cancel{\frac{1}{k^2}}}$$

$$= \lim_{k\to\infty} \frac{1 + \frac{1}{k}\cdot\ln 2 - \frac{1}{k}\cdot\ln k^2}{2}$$

$$= \lim_{k\to\infty} \frac{1 + \overbrace{\frac{1}{k}\cdot\ln 2}^{\to 0} - 2\cdot\overbrace{\frac{1}{k}\cdot\ln k}^{\to 0}}{2} = \frac{1+0-2\cdot 0}{2} = \frac{1}{2} \neq 0_2 \Rightarrow$$

$\Rightarrow f$ is not continuous at 0_2

b) $f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2 \end{cases}$

$$x^4+y^4 \overset{?}{\geq} \frac{(x^2+y^2)^2}{2}, \forall x,y \in \mathbb{R}$$

$$\Leftrightarrow \quad 2x^4+2y^4 \geq x^4+2x^2y^2+y^4$$

$$\Leftrightarrow \quad x^4+y^4 \geq 2x^2y^2 \Leftrightarrow x^4-2x^2y^2+y^4 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \quad (x^2-y^2) \geq 0 \text{ (True)}, \forall x,y \in \mathbb{R}$$

$\cancel{x = r\cos t}$

$\cancel{y = r\sin t}$

$\cancel{r>0}$

$\cancel{t\in[0,2\pi)}$

$$\cancel{\lim_{r\to 0} \frac{e^{-\frac{1}{r^2(\sin^2 t+\cos^2 t)}}}{(r^2\cdot\cos^2 t+r^2\cdot\sin^2 t)^2} = \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{}}$$

We know that $x^4+y^4 \geq \dfrac{(y^2+x^2)^2}{2} \Rightarrow$

$$\frac{2}{(x^2+y^2)^2} \geq \frac{1}{x^4+y^4} \;\Big|\cdot\left(e^{-\frac{1}{x^2+y^2}}\right) \Rightarrow$$

$$2\cdot\frac{e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2} \geq \frac{e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2} \geq 0 \quad (1)$$

We want to compute: $\lim\limits_{(x,y)\to O_2} \dfrac{2\cdot e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2}$

$x = r\cdot\cos\alpha$
$y = r\cdot\sin\alpha$
$r > 0$
$\alpha\in[0,2\pi)$

$$\lim_{r\to 0}\frac{2\cdot e^{-\frac{1}{r^2\cdot\cos^2\alpha+r^2\cdot\sin^2\alpha}}}{(r^2\cos^2\alpha+r^2\cdot\sin^2\alpha)^2}$$

$$=\lim_{r\to 0}\frac{2\cdot e^{-\frac{1}{r^2}}}{r^4} = \lim_{r\to 0}\frac{2}{e^{\frac{1}{r^2}}\cdot r^4} = \lim_{r\to 0}\frac{2\cdot\frac{1}{r^4}}{e^{\frac{1}{r^2}}} = \frac{\infty}{\infty}$$

$$\overset{\text{l'Hospital}}{=} \lim_{r\to 0}\frac{2\cdot(r^{-4})'}{(e^{\frac{1}{r^2}})'} = 2\cdot\lim_{r\to 0}\frac{\overset{2}{\cancel{-4}}\cdot\cancel{r^{-5}}\ r^{-2}}{e^{\frac{1}{r^2}}\underset{1}{\cancel{(-2)}}\cdot\underset{1}{\cancel{r^{-3}}}}$$

$$=4\cdot\lim_{r\to 0}\frac{r^{-2}}{e^{\frac{1}{r^2}}} = \frac{\infty}{\infty} \overset{\text{l'Hospital}}{=}$$

$$=4\cdot\lim_{r\to 0}\frac{\overset{1}{\cancel{-2}}\cdot\cancel{r^{-3}}}{e^{\frac{1}{r^2}}\cdot\underset{1}{\cancel{(-2)}}\cdot\cancel{r^{-3}}} = 4\cdot\lim_{r\to 0}\frac{1}{e^{\frac{1}{r^2}}} = \frac{1}{e^{\infty}} = 0 \quad (2)$$

$$① \cdot ② \Rightarrow 2 \cdot \frac{e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2} \geq \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} \geq 0$$

$\searrow 0 \swarrow$ $\Rightarrow$

Squeeze $\Rightarrow$ Theorem $\lim_{(x,y)\to 0} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = 0$

f continuous at $0_2 \Leftrightarrow \lim_{(x,y)\to 0_2} f(x,y) = f(0,0) = 0_2$ $\Rightarrow$

$\Rightarrow f$ - continuous at 0_2

Exercise 9.2.:

Find the second order partial derivates of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(x \cdot \sin y)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial(\sin(x \cdot \sin y))}{\partial x} \overset{\substack{x\text{ - variable} \\ y\text{ - constant}}}{=}$$

$$= \sin y \cdot \cos(x \cdot \sin y)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial(\sin(x \cdot \sin y))}{\partial y} \overset{\substack{y\text{ - variable} \\ x\text{ - constant}}}{=}$$

$$= x \cos y \cdot \cos(x \cdot \sin y)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 f}{\partial x \partial x}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)$$

$$= \frac{\partial}{\partial x}\left(\sin y \cdot \cos(x \cdot \sin y)\right) = -\sin^2 y \cdot \sin(x \cdot \sin y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial^2 f}{\partial y \partial y}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)$$

$$= \frac{\partial}{\partial y}\left(x \cdot \cos y \cdot \cos(x \cdot \sin y)\right)$$

$$= x\left(-\sin y \cdot \cos(x \cdot \sin y) + x \cdot \cos^2 y \cdot (-\sin(x \cdot \sin y))\right)$$

$$\frac{\partial^2 y}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y}\right)(x,y) = \frac{\partial}{\partial x}\left(\cos(x \cdot \sin y) \cdot x \cdot \cos y\right)$$

$$= \cos y\,(-\sin(x \cdot \sin y) \cdot \sin y \cdot x + \cos(x \cdot \sin y))$$

$$\frac{\partial^2 y}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right)(x,y) = \frac{\partial}{\partial y}\left(\cos(x \cdot \sin y) \cdot \sin y\right)$$

$$= -\sin(x \cdot \sin y) \cdot \cos y \cdot \sin y + \cos(x \cdot \sin y) \cdot \cos y$$

b) $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (1+x^2)\cdot y\cdot e^z$

x - variabile
y - constant
z - constant

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{\partial((1+x^2)\cdot y\cdot e^z)}{\partial x} =$$

$$= 2x\cdot y\cdot e^z$$

y - variabile
x - constant
z - constant

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial((1+x^2)\cdot y\cdot e^z)}{\partial y} =$$

$$= (1+x^2)\cdot e^z$$

z - variabile
x - constant
y - constant

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial((1+x^2)\cdot y\cdot e^z)}{\partial z} =$$

$$= (1+x^2)\cdot y\cdot e^z$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = \frac{\partial}{\partial y}\left((1+x^2)\cdot e^z\right) = 0$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \frac{\partial}{\partial x}(2x\cdot y\cdot e^z) = 2\cdot y\cdot e^z$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = \frac{\partial}{\partial z}((1+x^2)\cdot y\cdot e^z)$$

$$= (1+x^2)y\cdot e^z$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = \frac{\partial}{\partial x}((1+x^2)\cdot e^z)$$

$$= 2x \cdot e^z$$

$$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = \frac{\partial}{\partial x}((1+x^2)\cdot e^z \cdot y)$$

$$= 2x \cdot e^z \cdot y$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \frac{\partial}{\partial y}(2x \cdot y \cdot e^z)$$

$$= 2x \cdot e^z$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = \frac{\partial}{\partial y}((1+x^2)\cdot y \cdot e^z))$$

$$= (1+x^2)\cdot e^z$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \frac{\partial}{\partial z}(2x \cdot e^z \cdot y)$$

$$= 2x \cdot e^z \cdot y$$

$$\frac{\partial^2 f}{\partial z \cdot \partial y}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = \frac{\partial}{\partial z}((1+x^2)\cdot e^z)$$

$$= (1+x^2)\cdot e^z$$