Seminar 7 (1)

Ex1 Prove that $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sqrt[3]{x}$ is not diff. at 0, although its derivative at 0 exists.

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\sqrt[3]{x}-0}{x}=\lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}}=\infty\notin\mathbb{R}$$

$\Rightarrow$ f is not diff. at 0, but f has a derivative at 0 and $f'(0)=\infty$.

Ex2: How many times is the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$ differentiable?

$$\left.\begin{aligned}&\lim_{\substack{x\to 0\\ x>0}}\frac{f(x)-f(0)}{x-0}=\lim_{\substack{x\to 0\\ x>0}}\frac{x^2}{x}=\lim_{\substack{x\to 0\\ x>0}}x=0\\ &\lim_{\substack{x\to 0\\ x<0}}\frac{f(x)-f(0)}{x-0}=\lim_{\substack{x\to 0\\ x<0}}\frac{-x^2}{x}=\lim_{\substack{x\to 0\\ x<0}}(-x)=0\end{aligned}\right\}\Rightarrow \text{f is diff at 0 and } f'(0)=0$$

(2)

f is diff on $\mathbb{R}$

$x>0$, $f'(x)=2x$

$x<0$, $f'(x)=-2x$

$\forall x\in\mathbb{R}$, $f'(x)=2|x|$ - not diff at 0

f is only once diff

Ex3: Find the n^{th} derivative ($n\in\mathbb{N}$) of the following functions:

a) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=(\sin x-\cos x)^2+\sin 2x=\sin^2 x+\cos^2 x\cancel{-2\sin x\cos x}+\cancel{2\sin x\cos x}=1$

$f^{(n)}(x)=0$, $\forall x\in\mathbb{R}$, $\forall n\in\mathbb{N}$

b) $f:(-1,\infty)\to\mathbb{R}$, $f(x)=\ln(1+x)$

$f'(x)=\frac{1}{1+x}$, $f''(x)=-\frac{1}{(1+x)^2}$, $f'''(x)=2\frac{1}{(1+x)^3}$

(3)

$n\in\mathbb{N}$

$P(n)$: " $f^{(n)}(x)=(-1)^{n+1}\cdot\frac{(n-1)!}{(1+x)^n}$, $\forall x>-1$ ".

$P(1)$ true

Let $k\in\mathbb{N}$ and suppose $P(k)$ true

$$f^{(k)}(x)=(-1)^{k+1}\cdot\frac{(k-1)!}{(1+x)^k}$$

$$f^{(k+1)}(x)=(-1)^{k+1}\cdot(k-1)!\cdot(-k)\cdot\frac{1}{(1+x)^{k+1}}=(-1)^{k+2}\cdot\frac{k!}{(1+x)^{k+1}}$$

$\Rightarrow P(k+1)$ true

$\Rightarrow P(n)$ true, $\forall n\in\mathbb{N}$

(4)

c) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$

$f'(x)=\cos x=\sin\left(x+\frac{\pi}{2}\right)$

$f''(x)=\cos\left(x+\frac{\pi}{2}\right)=\sin(x+\pi)$

$\vdots$

$f^{(n)}(x)=\sin\left(x+\frac{n\pi}{2}\right)$

$\cos x=\sin\left(x+\frac{\pi}{2}\right)$

$\sin x=-\cos\left(x+\frac{\pi}{2}\right)$

d) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cos x$

$f'(x)=-\sin x=\cos\left(x+\frac{\pi}{2}\right)$

$f''(x)=-\sin\left(x+\frac{\pi}{2}\right)=\cos(x+\pi)$

$\vdots$

$f^{(n)}(x)=\cos\left(x+\frac{n\pi}{2}\right)$

(5)

e) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{2x}\cdot x^3$

The product rule $(g\cdot h)'=g'\cdot h+gh'$ can be generalised as follows:

Let $I\subseteq\mathbb{R}$ interval, $n\in\mathbb{N}$, $g,h:I\to\mathbb{R}$ n-times diff. Then

$$\forall x\in I,\quad (g\cdot h)^{(n)}(x)=\sum_{k=0}^{n}C_n^k\, g^{(n-k)}(x)\cdot h^{(k)}(x)\quad\text{(the Leibniz formula)}$$

Pf. By mathematical induction.

$g,h:\mathbb{R}\to\mathbb{R}$, $g(x)=e^{2x}$, $h(x)=x^3$

$$\forall n\in\mathbb{N},\quad g^{(n)}(x)=2^n e^{2x},\quad h^{(n)}(x)=\begin{cases}3x^2, & n=1\\ 6x, & n=2\\ 6, & n=3\\ 0, & n\geq 4\end{cases},\quad x\in\mathbb{R}$$

(6)

$$f^{(n)}(x)=(g\cdot h)^{(n)}(x)=C_n^0 2^n e^{2x}x^3+C_n^1 2^{n-1}e^{2x}\cdot 3x^2+C_n^2 2^{n-2}e^{2x}\cdot 6x+C_n^3 2^{n-3}e^{2x}\cdot 6$$

Ex4: Compute $\lim\limits_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x}$. Then determine $\lim\limits_{n\to\infty}n\left(e-\left(1+\frac{1}{n}\right)^n\right)$.

$$\left((1+x)^{\frac{1}{x}}\right)'=\left(e^{\ln(1+x)^{\frac{1}{x}}}\right)'=\left(e^{\frac{\ln(1+x)}{x}}\right)'=e^{\frac{\ln(1+x)}{x}}\cdot\left(\frac{\ln(1+x)}{x}\right)'=(1+x)^{\frac{1}{x}}\cdot\frac{\frac{1}{1+x}\cdot x-\ln(1+x)}{x^2}$$

$$\lim_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x}=-\lim_{x\to 0}(1+x)^{\frac{1}{x}}\cdot\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}=\frac{e}{2}.$$

$$\lim_{x\to 0}\frac{1-\frac{1}{1+x}-\ln(1+x)}{x^2}=\lim_{x\to 0}\frac{\frac{1}{(1+x)^2}-\frac{1}{1+x}}{2x}=\lim_{x\to 0}\frac{\overbrace{1-(1+x)}^{-x}}{2x(1+x)^2}=\lim_{x\to 0}\frac{-1}{2(1+x)^2}=-\frac{1}{2}$$

(7)

$$\lim_{n\to\infty}n\left(e-\left(1+\frac{1}{n}\right)^n\right)=\lim_{n\to\infty}\frac{e-\left(1+\frac{1}{n}\right)^{\frac{1}{1/n}}}{\frac{1}{n}}=\frac{e}{2}$$

$\frac{1}{n}\to 0$, $\frac{1}{n}\neq 0$, $\forall n\in\mathbb{N}$

use the Seq. Charact. of Limits

Ex5: Let $a,b\in\mathbb{R}$, $a<b$, $f:[a,b]\to\mathbb{R}$ cont on $[a,b]$, diff. on (a,b).

Prove that $\exists\, c\in(a,b)$ s.t. $(c-a)(c-b)f'(c)=a+b-2c$

Take $g:[a,b]\to\mathbb{R}$, $g(x)=e^{f(x)}\underbrace{(x-a)(x-b)}_{x^2-(a+b)x+ab}$

g is cont $[a,b]$, diff on (a,b)

$$g'(x)=e^{f(x)}\cdot f'(x)(x-a)(x-b)+e^{f(x)}\cdot(2x-a-b)=e^{f(x)}\Big((x-a)(x-b)\cdot f'(x)-(a+b-2x)\Big),$$

$\forall x\in(a,b)$

(8)

$$\left.\begin{aligned}&g(a)=g(b)\\ &g\text{ cont on }[a,b]\\ &g\text{ diff on }(a,b)\end{aligned}\right\}\overset{\text{Rolle's Thm}}{\Longrightarrow}\ \exists\, c\in(a,b)\text{ s.t. } g'(c)=0$$

$\Updownarrow$

$(c-a)(c-b)f'(c)=a+b-2c$

Taylor polynomials : $I\subseteq\mathbb{R}$ int, $x_0\in I$, $n\in\mathbb{N}_0$, $f:I\to\mathbb{R}$ n-times diff. at x_0

n^{th} Taylor polynomial of f at x_0 : $T_n:\mathbb{R}\to\mathbb{R}$

$$T_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Remainder function : $R_n:I\to\mathbb{R}$

$R_n(x)=f(x)-T_n(x)$

Thm (Taylor-Lagrange) $I\subseteq\mathbb{R}$ int, $n\in\mathbb{N}_0$, $f:I\to\mathbb{R}$ $(n+1)$-times diff

$\forall x,x_0\in I$, $x\neq x_0$, $\exists$ c strictly btw x and x_0 s.t.

$$f(x)=\underbrace{f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{T_n(x)}+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

(9)

Ex6 Let $f:(0,\infty)\to\mathbb{R}$, $f(x)=\sqrt[3]{x}$. Find the second Taylor polynomial $T_2(x)$ of f at 1 and the remainder term $R_2(x)$ of the corresponding Taylor formula in the Lagrange form. If $x\in[0.9,1.1]$, find an upper bound for $|R_2(x)|$.

$x>0$ $\quad f'(x)=\frac{1}{3}\cdot x^{-\frac{2}{3}}$, $f''(x)=-\frac{2}{9}x^{-\frac{5}{3}}$, $f'''(x)=\frac{10}{27}x^{-\frac{8}{3}}$

$f(1)=1$, $f'(1)=\frac{1}{3}$, $f''(1)=-\frac{2}{9}$

$$T_2:\mathbb{R}\to\mathbb{R},\quad T_2(x)=1+\frac{\frac{1}{3}}{1!}(x-1)+\frac{-\frac{2}{9}}{2!}(x-1)^2=1+\frac{1}{3}(x-1)-\frac{1}{9}(x-1)^2$$

For $x>0$, $\exists$ c btw 1 and x s.t. $R_2(x)=\frac{\frac{10}{27}\cdot c^{-\frac{8}{3}}}{3!}\cdot(x-1)^3=\frac{5}{81}\cdot c^{-8/3}\cdot(x-1)^3$

$|R_2(x)|=\frac{5}{81}\cdot c^{-\frac{8}{3}}\cdot|x-1|^3$

If $x\in[0.9,1.1]\Rightarrow|x-1|\leq 0.1$; $c>0.9\Rightarrow c^{-\frac{8}{3}}<(0.9)^{-\frac{8}{3}}$

$\Rightarrow|R_2(x)|<\frac{5}{81}\cdot(0.9)^{-8/3}\cdot(0.1)^3<(0.1)^4$

(10)

Ex7: Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cos x$. Find the second Taylor polynomial $T_2(x)$ of f at 0 and the remainder term $R_2(x)$ of the corresponding Taylor formula in the Lagrange form. Then show that $\forall x\in\mathbb{R}$, $1-\frac{x^2}{2}\leq\cos x$

$x\in\mathbb{R}$, $f'(x)=-\sin x$, $f''(x)=-\cos x$, $f'''(x)=\sin x$

$f(0)=1$, $f'(0)=0$ $\quad f''(0)=-1$

$T_2:\mathbb{R}\to\mathbb{R}$, $T_2(x)=1-\frac{x^2}{2}$

For $x\in\mathbb{R}$, $\exists$ c btw 0 and x s.t. $R_2(x)=\frac{\sin c}{3!}\cdot x^3$

$$\cos x=1-\frac{x^2}{2}+\frac{\sin c}{6}x^3$$

Case 1 $\quad|x|\leq\pi$, $-\pi\leq x\leq\pi$

$\bullet\ x\in[0,\pi]$: $\left.\begin{aligned}&\sin c\geq 0\\ &x\geq 0\end{aligned}\right\}\Rightarrow\frac{\sin c}{6}x^3\geq 0$

$\bullet\ x\in[-\pi,0)$: $\left.\begin{aligned}&\sin c\leq 0\\ &x<0\end{aligned}\right\}\Rightarrow\frac{\sin c}{6}\cdot x^3\geq 0$

$\Rightarrow\cos x\geq 1-\frac{x^2}{2}$

Case 2 : $|x|>\pi\Rightarrow x^2>\pi^2>9$

$1-\frac{x^2}{2}<1-\frac{9}{2}=-\frac{7}{2}<\cos x$

$\Rightarrow\forall x\in\mathbb{R}$, $1-\frac{x^2}{2}\leq\cos x$.