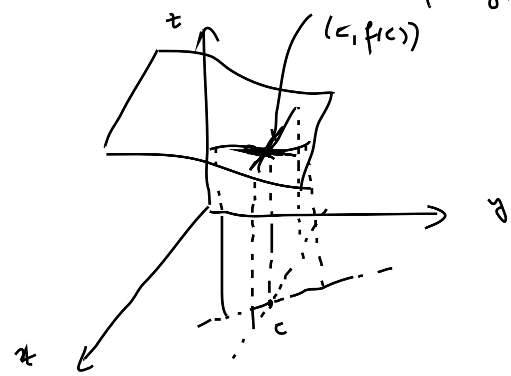
Lecture 10

Directional derivatives

$f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\qquad$ $z = f(x,y)$ a surface in $\mathbb{R}^3$



In the following we consider $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$.

Definition 1. Let $f : A \to \mathbb{R}$, $c \in \operatorname{int} A$ and $v \in \mathbb{R}^n$ a unit vector (that is, $\|v\| = 1$). We say that f is *differentiable in the direction v at c* if

$$\exists \lim_{t \to 0} \frac{f(c+tv) - f(c)}{t} \in \mathbb{R}.$$

In this case, the above limit is called the *directional derivative of f in the direction v at c* and is denoted by $f'(c; v)$.

Remark 1. For $v = e^j$, $j \in \{1, \dots, n\}$, we obtain the partial derivative of f w.r.t. x_j.

Theorem 1. *Let $f : A \to \mathbb{R}$, $c \in \operatorname{int} A$ and suppose that f is C^1 near c. Then $\forall v \in \mathbb{R}^n$ with $\|v\| = 1$, f is differentiable in the direction v at c and $f'(c; v) = \langle \nabla f(c), v \rangle$.*

Pf. $f = f(x_1, \dots, x_n)$, $c = (c_1, \dots, c_n)$. Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ with $\|v\| = 1$

$c \in \operatorname{int} A \Rightarrow \exists\, \varepsilon > 0$ s.t. $c + tv \in A$, $\forall t \in (-\varepsilon, \varepsilon)$

Define $g : (-\varepsilon, \varepsilon) \to A$, $g(t) = c + tv$. Then
- $g(0) = c$
- $g = (g_1, \dots, g_n)$

$g_i : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $g_i(t) = c_i + t \cdot v_i$, $\forall i \in \{1, \dots, n\}$

$f \circ g : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $(f \circ g)(t) = f(c + tv)$

By the Chain Rule, $f \circ g$ is diff at 0,

$$\left.\begin{aligned} (f \circ g)'(0) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underbrace{g(0)}_{=c}) \underbrace{g_i'(0)}_{=v_i} = \langle \nabla f(c), v \rangle \\ f'(c, v) &= \lim_{t \to 0} \frac{f(c+tv) - f(c)}{t} = \lim_{t \to 0} \frac{(f \circ g)(t) - (f \circ g)(0)}{t} = (f \circ g)'(0) \end{aligned}\right\} \Rightarrow f'(c, v) = \langle \nabla f(c), v \rangle$$

Example 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2. \end{cases}$

By Example 1 in Lecture 8, f has no limit at 0_2

$\Rightarrow$ f is not cont at 0_2

Let $v = (v_1, v_2) \in \mathbb{R}^2$ with $\|v\| = 1$

$$\lim_{t\to 0} \frac{f(0_2 + tv) - f(0_2)}{t} = \lim_{t \to 0} \frac{f(tv_1, tv_2)}{t} = \lim_{t\to 0} \frac{1}{t} \cdot \frac{t^2 v_1^2 \cdot t \cdot v_2}{t^4 \cdot v_1^4 + t^2 v_2^2} =$$

$$= \lim_{t \to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} = \begin{cases} \dfrac{v_1^2}{v_2}, & v_2 \neq 0 \\ 0, & v_2 = 0 \end{cases}$$

$$\Rightarrow f'(0_2; v) = \begin{cases} \dfrac{v_1^2}{v_2}, & v_2 \neq 0 \\ 0, & v_2 = 0 \end{cases} \quad \Rightarrow \frac{\partial f}{\partial x}(0_2) = 0, \quad \frac{\partial f}{\partial y}(0_2) = 0 \qquad \nabla f(0_2) = 0_2$$

if $v_1 \neq 0$ and $v_2 \neq 0$ $\Rightarrow$ $f'(0_2; v) \neq 0$ $\Rightarrow$ $f'(0_2; v) \neq \langle \nabla f(0_2), v \rangle$

This does not contradict Thm. 1 because the hypothesis that f is C^1 near 0_2 is not satisfied (f is not even continuous at 0_2)

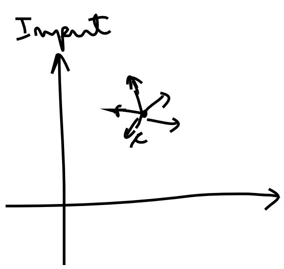
Remark 2. Let $f : A \to \mathbb{R}$ and $c \in \operatorname{int} A$.

$$f \text{ differentiable in every direction at } c \quad \not\Rightarrow \quad f \text{ continuous at } c.$$

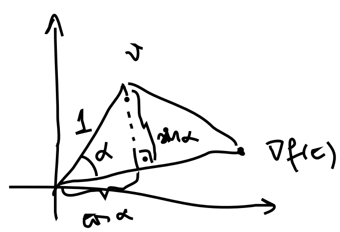
The gradient (revisited)

Let $A \subseteq \mathbb{R}^2$, $A \neq \emptyset$, $f : A \to \mathbb{R}$. Take $c \in \operatorname{int} A$ and suppose that f is C^1 near c and that c is not a stationary point for f.

Problem: In which direction should we move away from c to get the maximal increase for f?



$$\max_{\substack{v \in \mathbb{R}^2 \\ \|v\|=1}} f'(c; v) = \max_{\substack{v \in \mathbb{R}^2 \\ \|v\|=1}} \langle \nabla f(c), v \rangle$$



α = the angle btw the vectors $\nabla f(c)$ and v

Suppose $\|\nabla f(c)\| \geq \|v\| = 1$

$$\|\nabla f(c) - v\|^2 = \|\nabla f(c)\|^2 + \overbrace{\|v\|^2}^{1} - 2 \langle \nabla f(c), v \rangle$$

$$\|$$

$$\sin^2 \alpha + (\|\nabla f(c)\| - \cos\alpha)^2 =$$

$$= \underbrace{\sin^2\alpha + \cos^2\alpha}_{1} + \|\nabla f(c)\|^2 - 2\cos\alpha \, \|\nabla f(c)\|$$

$$\Rightarrow \langle \nabla f(c), v \rangle = \underbrace{\|\nabla f(c)\|}_{} \cdot \cos\alpha$$

c fixed $\Rightarrow$ $\|\nabla f(c)\|$ is fixed

$f'(c; v)$ is • maximal if $\cos\alpha = 1 \Rightarrow \alpha = 0 \Rightarrow v = \frac{1}{\|\nabla f(c)\|}\nabla f(c)$

• minimal if $\cos\alpha = -1 \Rightarrow \alpha = \pi \Rightarrow v = -\frac{1}{\|\nabla f(c)\|}\nabla f(c)$

$\Rightarrow$ we get steepest increase in the direction of the gradient and steepest decrease in the opposite direction of the gradient

$v = \frac{1}{\|\nabla f(c)\|}\cdot \nabla f(c)$, $f'(c,v) = \langle \nabla f(c), v\rangle = \langle \nabla f(c), \frac{1}{\|\nabla f(c)\|}\nabla f(c)\rangle = \frac{1}{\|\nabla f(c)\|}\overbrace{\langle \nabla f(c), \nabla f(c)\rangle}^{\|\nabla f(c)\|^2}$

$= \|\nabla f(c)\|$

Riemann integrals

In the following we consider $a, b \in \mathbb{R}$ with $a < b$.

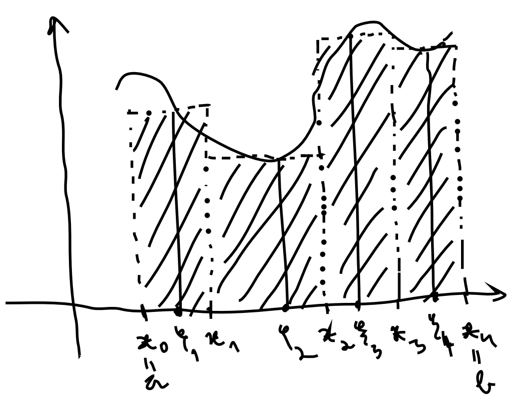
Definition 2. A *partition* of the interval $[a, b]$ is a finite ordered set $P = (x_0, x_1, \dots, x_n)$ of real numbers such that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. The intervals $[x_{i-1}, x_i]$ $(i = 1, \dots, n)$ are called subintervals of the partition P.
The *norm* of P is $\|P\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}$ (i.e., the length of the largest subinterval of the partition P).
Suppose that, for each $i = 1, \dots, n$, ξ_i has been chosen in each subinterval $[x_{i-1}, x_i]$ and denote $\xi = (\xi_1, \dots, \xi_n)$. Then (P, ξ) is called a *tagged partition* of $[a, b]$.

Definition 3. Let $f : [a, b] \to \mathbb{R}$ and (P, ξ) a tagged partition of $[a, b]$. Then the sum

$$\sigma(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is called the *Riemann sum* of f w.r.t. the tagged partition (P, ξ).

$f : [a,b] \to (0,\infty)$



the sum of the areas of n rectangles whose bases are the subintervals $[x_{i-1}, x_i]$ and whose heights are $f(\xi_i)$

Definition 4. Let $f : [a, b] \to \mathbb{R}$. We say that f is *Riemann integrable* on $[a, b]$ if there exists $I \in \mathbb{R}$ such that

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 \text{ s.t. } \forall (P, \xi) \text{ tagged partition of } [a, b] \text{ with } \|P\| < \delta, |\sigma(f, P, \xi) - I| < \varepsilon. \quad (1)$$

The family of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.
If $f \in \mathcal{R}[a, b]$, then $I \in \mathbb{R}$ satisfying (1) is uniquely determined and called the *Riemann integral* (or *definite integral*) of f on $[a, b]$.
Notation: $\int_a^b f(x)dx = \int_a^b f = I.$

Remark 3. (i) If $f : [a, b] \to [0, \infty)$ and $f \in \mathcal{R}[a, b]$, then $\mathcal{A} = \int_a^b f$ is the area under the graph of f (and above the Ox axis).

(ii) If $f : [a, b] \to \mathbb{R}$ is constantly equal to $M \in \mathbb{R}$, then $f \in \mathcal{R}[a, b]$ and $\int_a^b f = M(b - a)$.

(iii) If $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.

(iv) If $f : [a, b] \to \mathbb{R}$ is monotone, then $f \in \mathcal{R}[a, b]$.

(v) If $f \in \mathcal{R}[a, b]$, then f is bounded.

Theorem 2. *Let $a, b \in \mathbb{R}$, $a < b$, $f, g \in \mathcal{R}[a, b]$ and $\alpha \in \mathbb{R}$. Then*

(i) $f + g \in \mathcal{R}[a, b]$ *and* $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(ii) $(\alpha f) \in \mathcal{R}[a, b]$ *and* $\int_a^b (\alpha f) = \alpha \int_a^b f$.

(iii) $(f \cdot g) \in \mathcal{R}[a, b]$.

(iv) $|f| \in \mathcal{R}[a, b]$.

(v) If $f \le g$, *then* $\int_a^b f \le \int_a^b g$.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. Then*

$$f \in \mathcal{R}[a, b] \iff f|_{[a,c]} \in \mathcal{R}[a, c] \text{ and } f|_{[c,b]} \in \mathcal{R}[c, b].$$

In this case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Theorem 4 (First Fundamental Theorem of Calculus)**.** *Let $f \in \mathcal{R}[a, b]$. Define $F : [a, b] \to \mathbb{R}$,*

$$F(t) = \int_a^t f.$$

Then F is continuous. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

Pf (Continuity of F)

$f \in \mathcal{R}(a,b) \Rightarrow f$ bounded $\Rightarrow \exists M > 0$ s.t. $\forall x \in [a,b]$, $-M \le f(x) \le M$

Let $t, s \in [a,b]$, $t < s$. Then

$$F(s) - F(t) = \int_a^s f - \int_a^t f = \int_t^s f$$

$$\Rightarrow -M(s-t) = \int_t^s -M \le \int_t^s f \le \int_t^s M = M(s-t) \Rightarrow |F(s) - F(t)| \le M(s-t)$$

$\Rightarrow \forall t, s \in [a,b]$, $|F(s) - F(t)| \le M|s-t|$

Let $t \in [a,b]$. Take an arbitrary sequence $(t_n) \subseteq [a,b]$ s.t. $t_n \to t$

$$\forall n \in \mathbb{N}, \quad 0 \le |F(t_n) - F(t)| \le M \underbrace{|t_n - t|}_{\xrightarrow{n \to \infty} 0} \Rightarrow F(t_n) \to F(t)$$

$\Rightarrow F$ cont. at t

Example 2. Take $f : [-1, 1] \to \mathbb{R}$, $f(x) = \begin{cases} -1, & \text{if } x \in [-1, 0), \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x \in (0, 1]. \end{cases}$

$F : [-1, 1] \to \mathbb{R}$, $F(t) = \int_{-1}^{t} f(x)\,dx$

$$\left.\begin{aligned} &t \in [-1, 0], \quad F(t) = \int_{-1}^{t} (-1)\,dx = (-1)(t+1) = -t-1 \\ &t \in (0, 1], \quad F(t) = \int_{-1}^{0} (-1)\,dx + \int_{0}^{t} 1\,dx = (-1)\cdot 1 + 1 \cdot t = t - 1 \end{aligned}\right\} \Rightarrow \forall\, t \in [-1, 1],\ F(t) = |t| - 1$$

F not diff at 0

Theorem 5 (Second Fundamental Theorem of Calculus,)**.** *Let $f \in \mathcal{R}[a, b]$. If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f (i.e., $F'(x) = f(x)$ for all $x \in [a, b]$), then*

$$\int_a^b f = F(b) - F(a) \quad \text{(the Leibniz-Newton formula).}$$