Bledea Mihaela Alexandra

Exercise 11.1:

Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , \text{ if } (x,y) = 0_2 \\ 0 & , \text{ if } (x,y) \neq 0_2 \end{cases}$$

Study the continuity and the partial differentiability of f at 0_2.

<u>SOLUTION:</u>

f is continuous at $(0,0) \iff \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$$\begin{aligned} x &= r \cdot \cos t \\ y &= r \cdot \sin t \end{aligned} \quad , r > 0, \ t \in [0, 2\pi)$$

$$x^2 + y^2 = r^2 \cdot \cos^2 t + r^2 \cdot \sin^2 t$$

$$x^2 + y^2 = r^2 (\underbrace{\cos^2 t + \sin^2 t}_{=1}) \Rightarrow x^2 + y^2 = r^2$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r \cdot \cos t \cdot r \cdot \sin t}{\sqrt{r^2}}$$

$$= \lim_{r \to 0} \frac{\cancel{r^2}^{\,r} \cdot \cos t \cdot \sin t}{\cancel{r}} = \lim_{r\to 0} r \cdot \underbrace{\cos t \cdot \sin t}_{\in [-1,1]} = 0 \Rightarrow$$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0 \Rightarrow f$ is continuous at 0_2

In order to study if f is partial differentiable at 0_2 we need to compute the following limits:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0}$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{x \to 0} \frac{0 - 0}{x - 0} = 0 \in \mathbb{R} \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{y \to 0} \frac{0 - 0}{y - 0} = 0 \in \mathbb{R} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ f is partially differentiable with respect to x and y at 0_2.

Exercise 19.2:

Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{2x+y} \cdot \cos(3z)$. Find the gradient and the Hessian matrix of f at $(0,0,\frac{\pi}{6})$

<u>SOLUTION:</u>

$$\frac{\partial f}{\partial x}(x,y,z) = 2 \cdot e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial f}{\partial z}(x,y,z) = -3e^{2x+y} \cdot \sin(3z)$$

$$\nabla f\left(0,0,\frac{\pi}{6}\right) = \left(\frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right), \frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right), \frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right)\right) \quad (1)$$

$$\left.\begin{aligned}
&\frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right) = 2 \cdot e^{2\cdot 0+0} \cdot \cos\left(\overset{1}{\cancel{3}} \cdot \frac{\pi}{\underset{2}{\cancel{6}}}\right) = 2\cos\frac{\pi}{2} = 0 \\
&\frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right) = e^{2\cdot 0+0} \cdot \cos\left(\cancel{3} \cdot \frac{\pi}{\underset{2}{\cancel{6}}}\right) = \cos\frac{\pi}{2} = 0 \\
&\frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right) = -3 \cdot e^{2\cdot 0+0} \cdot \sin\left(\frac{\overset{1}{\cancel{3}} \cdot \pi}{\underset{2}{\cancel{6}}}\right) = -3 \cdot \sin\frac{\pi}{2} = -3
\end{aligned}\right\} \quad (2)$$

$$(1), (2) \Rightarrow \nabla f\left(0,0,\frac{\pi}{6}\right) = (0,0,-3)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z)$$

$$= 4\, e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = -9\, e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = 2 \cdot e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = 2 \cdot e^{2x+y} \cdot \cos(3z)$$

$$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = -6\, e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = -6 \cdot e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = -3\, e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial^2 f}{\partial z \partial y}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = -3 \cdot e^{2x+y} \cdot \sin(3z)$$

$$\frac{\partial^2 f}{\partial x^2}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial^2 f}{\partial y^2}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial^2 f}{\partial z^2}\left(0,0,\frac{\pi}{6}\right) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}\left(0,0,\frac{\pi}{6}\right) = 0 = \frac{\partial^2 f}{\partial y \partial x}\left(0,0,\frac{\pi}{6}\right)$$

$$\frac{\partial^2 f}{\partial y \partial z}\left(0,0,\frac{\pi}{6}\right) = -3 = \frac{\partial^2 f}{\partial z \partial y}\left(0,0,\frac{\pi}{6}\right)$$

$$\frac{\partial^2 f}{\partial x \partial z}\left(0,0,\frac{\pi}{6}\right) = -6 = \frac{\partial^2 f}{\partial z \partial x}$$

$$H_f\left(0,0,\frac{\pi}{6}\right) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}\left(0,0,\frac{\pi}{6}\right) & \frac{\partial^2 f}{\partial x \partial y}\left(0,0,\frac{\pi}{6}\right) & \frac{\partial^2 f}{\partial x \partial z}\left(0,0,\frac{\pi}{6}\right) \\ \frac{\partial^2 f}{\partial y \partial x}\left(0,0,\frac{\pi}{6}\right) & \frac{\partial^2 f}{\partial y^2}\left(0,0,\frac{\pi}{6}\right) & \frac{\partial^2 f}{\partial y \partial z}\left(0,0,\frac{\pi}{6}\right) \\ \frac{\partial^2 f}{\partial z \partial x}\left(0,0,\frac{\pi}{6}\right) & \frac{\partial^2 f}{\partial z \partial y}\left(0,0,\frac{\pi}{6}\right) & \frac{\partial^2 f}{\partial z^2}\left(0,0,\frac{\pi}{6}\right) \end{pmatrix}$$

$$H_f\left(0,0,\frac{\pi}{6}\right) = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & -3 \\ -6 & -3 & 0 \end{pmatrix}$$

Exercise 12.1.

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3xy$. Find the local extremum points for f and specify their type.

SOLUTION:

$$\left.\begin{array}{l} \nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right) \\ \frac{\partial f}{\partial x}(x,y) = 3x^2 - 3y \\ \frac{\partial f}{\partial y}(x,y) = 3y^2 - 3x \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \nabla f(x,y) = (3x^2 - 3y, 3y^2 - 3x) \\ \nabla f(0,0) = (0,0) \end{array}\right\} \Leftrightarrow \begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Leftrightarrow$$

$$\cancel{\Leftrightarrow \begin{cases} x \in \{-1, 1\} \\ y \in \{-1, 1\} \end{cases}}$$

$$\Leftrightarrow \begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = y \\ y^2 - x = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = y \\ x^4 - x = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x(x^3 - 1) = 0 \Leftrightarrow x(x-1)(x^2 - x + 1) = 0 \Leftrightarrow$$

$$\left.\begin{array}{l} \text{I. } x = 0 \Rightarrow y = 0 \\ \text{II. } x = 1 \Rightarrow y = 1 \\ \text{III. } x^2 - x + 1 = 0 \text{, but this equation doesn't} \\ \text{have real solutions} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (x,y) \in \{(0,0), (1,1)\}$$

$S = \{(0,0), (1,1)\}$ -> the set of the stationary points

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = 6x$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = -3$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = 6y$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = -3$$

$$H_f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$$H_f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$$H_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

$\Delta_1 = 6 > 0$, $\Delta_2 = 36+9 = 45 > 0$ $\overset{\text{Sylvester}}{\underset{\text{Theorem}}{\Rightarrow}}$ H_f - positive definite $\Rightarrow$

$\Rightarrow (1,1)$ - local minimum point

$$Hf(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

$\Delta_1 = 0$, $\Delta_2 = -9$, but because $\Delta_1 = 0 \Rightarrow$ inconclusive

$$\phi_c = (h_1, h_2) \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_1 \end{pmatrix}$$

$$\phi_c = (-3h_2, -3h_1) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\phi_c = -3h_2h_1 - 3h_1h_2$$

$$\phi_c = -6h_1h_2$$

We take $v_1 = (-1,1)$ and $v_2 = (1,1)$

$$\left.\begin{aligned} \phi_c(-1,1) &= 6 > 0 \\ \phi_c(1,1) &= -6 < 0 \end{aligned}\right\} \Rightarrow Hf(0,0) \text{ - indefinite} \Rightarrow$$

$\Rightarrow (0,0)$ - is not a local extremum point

Exercise 13.1:

Let $\alpha, \beta \in \mathbb{R}$ and $f: [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{x^{\alpha} \cdot \arctan x}{1+x^{\beta}}$

Study the improper integrability f on its domain

SOLUTION:

We will apply Theorem 5 from lecture 11

$$L = \lim_{x \to \infty} x^{p} \cdot \frac{x^{\alpha} \cdot \arctan x}{1+x^{\beta}}$$

$$L = \underbrace{\lim_{x \to \infty} \arctan x}_{=\frac{\pi}{2}} \cdot \lim_{x \to \infty} \frac{x^{p+\alpha}}{1+x^{\beta}}$$

$$L = \frac{\pi}{2} \cdot \lim_{x \to \infty} \frac{x^{p+\alpha}}{1+x^{\beta}}$$

I. $\beta > 0 \Rightarrow$ when $x \to \infty \Rightarrow x^{\beta} \to \infty$

We chose $p = \beta - \alpha$

$$L = \frac{\pi}{2} \cdot \lim_{x \to \infty} \frac{x^{\beta}}{1+x^{\beta}} = \frac{\pi}{2} \cdot \lim_{x \to \infty} \frac{\cancel{x^{\beta}}}{\cancel{x^{\beta}}(1+\underbrace{\frac{1}{x^{\beta}}}_{0})} = \frac{\pi}{2}$$

I.1) $p = \beta - \alpha > 1$, $L = \frac{\pi}{2} < \infty \Rightarrow f$ is improperly integrable on $[1, \infty)$

I.2) $p = \beta - \alpha \leq 1$, $L = \frac{\pi}{2} > 0 \Rightarrow f$ is not improperly integrable on $[1, \infty)$

II. $\beta = 0 \Rightarrow$ when $x \to \infty \Rightarrow x^{\beta} = 1$

We chose $p = -\alpha$

$$L = \frac{\pi}{2} \cdot \lim_{x \to \infty} \frac{x^0}{1 + x^0} = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{2 \cdot 2} = \frac{\pi}{4}$$

II. 1) $p = -\alpha > 1$, $L = \frac{\pi}{4} < \infty \Rightarrow f$ is improperly integrable on $[1, \infty)$

II. 2) $p = -\alpha \leq 1$, $L = \frac{\pi}{4} > 0 \Rightarrow f$ is not improperly integrable on $[1, \infty)$

III. $\beta < 0 \Rightarrow$ when $x \to \infty \Rightarrow x^{\beta} = 0$

We chose $p = -\alpha$

$$L = \frac{\pi}{2} \cdot \lim_{x \to \infty} \frac{x^0}{1 + \underbrace{x^{\beta}}_{0}} = \frac{\pi}{2}$$

III. 1) $p = -\alpha > 1$, $L = \frac{\pi}{2} < \infty \Rightarrow f$ is improperly integrable on $[1, \infty)$

III. 2) $p = -\alpha \leq 1$, $L = \frac{\pi}{2} > 0 \Rightarrow f$ is not improperly integrable on $[1, \infty)$