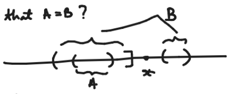
Seminar 2 :

(1)

Ex1: $B\subseteq\mathbb{R}, B\neq\phi$, bd., $A\subseteq B, A\neq\phi$

Prove that $\inf B\le\inf A\le\sup A\le\sup B$

If, in addition, $\inf A=\inf B$ and $\sup A=\sup B$, does it follow that $A=B$?



$\left.\begin{array}{l}B \text{ bd.}\\ A\subseteq B\end{array}\right\}\Rightarrow A$ bd.

$A,B\neq\phi$, bd. $\Rightarrow$ both A and B have an inf. and a sup. in $\mathbb{R}$.

$\forall a\in A$, $\inf A\le a\le\sup A\Rightarrow\inf A\le\sup A$.

$A\subseteq B\Rightarrow ub(A)\supseteq ub(B)$

$lb(A)\supseteq lb(A)$

$\inf B\in lb(B)\subseteq lb(A)\Rightarrow\inf B\le\inf A$

$\sup B\in ub(B)\subseteq ub(A)\Rightarrow\sup A\le\sup B$

$A=(0,2)$, $B=[0,2]$

$A\subsetneq B$

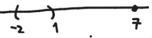
$\inf A=0=\inf B=\min B$

$\sup A=2=\sup B=\max B$

(2)

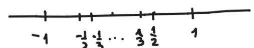
Ex2: Find $lb(A_i)$, $ub(A_i)$ (as subsets of $\mathbb{R}$), $\min A_i$, $\max A_i$ (if they exist), $\inf A_i$, $\sup A_i$ in $\overline{\mathbb{R}}$

- $A_1=(-2,1)\cup\{7\}$



$lb(A_1)=(-\infty,-2]$, $ub(A_1)=[7,\infty)$, no min, $\max A_1=7=\sup A_1$, $\inf A_1=-2$

- $A_2=\left\{\frac{1}{n}\,/\,n\in\mathbb{Z}\setminus\{0\}\right\}$



$lb(A_2)=(-\infty,-1]$, $\min A_2=-1=\inf A_2$

$ub(A_2)=[1,\infty)$, $\max A_2=1=\sup A_2$

- $A_3=\{x^2\,/\,x\in\mathbb{Z}\}=\{0,1,2^2,3^2,\dots\}$

$lb(A_3)=(-\infty,0]$, $\min A_3=0=\inf A_3$

$ub(A_3)=\phi$, no max, $\sup A_3=\infty$

(3)

- $A_4=\left\{\frac{n}{n+1}\,/\,n\in\mathbb{N}\right\}=\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},\dots\right\}$

$\forall n\in\mathbb{N}$, $\frac{n}{n+1}=\frac{n+1-1}{n+1}=1-\frac{1}{n+1}\ge1-\frac{1}{2}=\frac{1}{2}$ with equality for $n=1$

$lb(A_4)=(-\infty,\frac{1}{2}]$, $\min A_4=\frac{1}{2}=\inf A_4$

$\forall n\in\mathbb{N}$, $\frac{n}{n+1}<1\Rightarrow1\in ub(A_4)\Rightarrow\sup(A_4)\le1$.

We show that $\sup A_4=1$. Suppose that $\sup A_4<1$. Take $\varepsilon=1-\sup A_4>0$

$\overset{AP}{\Rightarrow}\exists n\in\mathbb{N}$ s.t. $n>\frac{1}{\varepsilon}$, i.e. $\frac{1}{n}<\varepsilon$

$\Rightarrow\sup A_4\ge\frac{n}{n+1}=1-\frac{1}{n+1}>1-\frac{1}{n}>1-\varepsilon=1-(1-\sup A_4)=\sup A_4$, a contr.

$\Rightarrow\sup A_4=1$

$ub(A_4)=[1,\infty)$, no max.

(4)

- $A_5=\left\{\frac{m}{m+n}\,/\,m,n\in\mathbb{N}\right\}$

$\forall m,n\in\mathbb{N}$, $0<\frac{m}{m+n}<1\Rightarrow0\in lb(A_5)$, $1\in ub(A_5)$

$\Downarrow$ $\inf A_5\ge0$ $\quad$ $\Downarrow$ $\sup A_5\le1$

$A_4\subseteq A_5\Rightarrow\underbrace{\sup A_4}_{=1}\le\sup A_5\le1\Rightarrow\sup A_5=1$

$A=\left\{\frac{1}{1+n}\,/\,n\in\mathbb{N}\right\}$ $\quad\inf A=0$ (see Lecture 1)

$A\subseteq A_5\Rightarrow0\le\inf A_5\le\underbrace{\inf A}_{=0}\Rightarrow\inf A_5=0$

$lb(A_5)=(-\infty,0]$, $ub(A_5)=[1,\infty)$, no min, no max.

(5)

- $A_6=\{x\in\mathbb{Q}\,/\,x^2\le2\}$

$\forall x\in A_6$, $-\sqrt{2}\le x\le\sqrt{2}\Rightarrow\sqrt{2}\in ub(A_6)$, $-\sqrt{2}\in lb(A_6)$

$\Downarrow$ $\sup A_6\le\sqrt{2}$

We show that $\sup A_6=\sqrt{2}$. Suppose $\sup A_6<\sqrt{2}$. By the Density Property of $\mathbb{Q}$ in $\mathbb{R}$, $\exists q\in\mathbb{Q}$ s.t. $\sup A_6<q<\sqrt{2}$.

$\underset{1\in A_6}{1}\le\sup A_6<q<\sqrt{2}\Rightarrow\left.\begin{array}{l}q^2<2\\ q\in\mathbb{Q}\end{array}\right\}\Rightarrow q\in A_6$

$q>\sup A_6$, a contr.

$\Rightarrow\sup A_6=\sqrt{2}$

$\inf A_6=-\sqrt{2}$

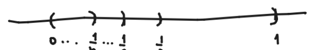
$ub(A_6)=[\sqrt{2},\infty)$, $lb(A_6)=(-\infty,-\sqrt{2}]$

no max, no min

Rmk: Since $\sqrt{2}\notin\mathbb{Q}$, this shows that a Supremum Property for $\mathbb{Q}$ does not hold.

(6)

Ex3: Is $\bigcap\limits_{n=1}^{\infty}(0,\frac{1}{n})=\phi$? What can be said about the NIP when dropping the assumption that the intervals are closed?



Suppose $\exists x\in\bigcap\limits_{n=1}^{\infty}(0,\frac{1}{n})$

$\Rightarrow x>0$

$\overset{AP}{\Rightarrow}\exists n_0\in\mathbb{N}$ s.t. $n_0>\frac{1}{x}$, i.e., $\frac{1}{n_0}<x\Rightarrow x\notin(0,\frac{1}{n_0})\Rightarrow x\notin\bigcap\limits_{n=1}^{\infty}(0,\frac{1}{n})$, a contr.

$\Rightarrow\bigcap\limits_{n=1}^{\infty}(0,\frac{1}{n})=\phi$.

This shows that the NIP is no longer true when removing the assumption that the intervals are closed.

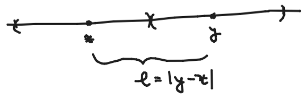
(7)

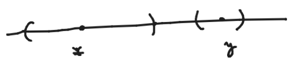
Ex4: Which sets are neighborhoods of 0?

- $A_1=[-1,1]\cup\{2\}$: true because $(-\frac{1}{2},\frac{1}{2})\subseteq A_1$
- $A_2=(-1,1)\cap\mathbb{Q}$: false because btw. any two rational numbers we find an irrational one: $x,y\in\mathbb{Q}$, $x<y$, $z=x+(y-x)\cdot\frac{\sqrt{2}}{2}\in\mathbb{R}\setminus\mathbb{Q}$, $x<z<y$
A_2 does not contain intervals
- $A_3=(-1,0)\cup(0,1)$: false because $0\notin A_3$
- $A_4=\{0\}\cup\{\frac{1}{n}\,/\,n\in\mathbb{Z}\setminus\{0\}\}$: false because it doesn't contain intervals
- $A_5=\bigcap\limits_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}]$: false because $A_5=\{0\}$, so it doesn't contain intervals.

(8)

Ex5: Prove that if $x,y\in\mathbb{R}$, $x\neq y$, $\exists U\in\mathcal{V}(x)$, $\exists V\in\mathcal{V}(y)$ s.t. $U\cap V=\phi$.





$U=(x-\frac{\ell}{2},x+\frac{\ell}{2})$, $V=(y-\frac{\ell}{2},y+\frac{\ell}{2})$

Suppose $U\cap V\neq\phi$ and take $z\in U\cap V$.

Then $\begin{cases}|z-x|<\frac{\ell}{2}\\ |z-y|<\frac{\ell}{2}\end{cases}$

$|y-x|=|y-z+z-x|\le|y-z|+|z-x|<\frac{\ell}{2}+\frac{\ell}{2}=\ell=|y-x|$, a contr.

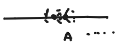
$\Rightarrow U\cap V=\phi$

(9)

Ex6: $A\subseteq\mathbb{R}$, $A\neq\phi$, $\alpha\in\mathbb{R}$, $\alpha\in lb(A)$

Prove that $\inf A=\alpha\Leftrightarrow\forall V\in\mathcal{V}(\alpha)$, $V\cap A\neq\phi$

$\boxed{\Rightarrow}$



Suppose $\exists V\in\mathcal{V}(\alpha)$ s.t. $V\cap A=\phi$

$\Downarrow$

$\exists\varepsilon>0$ s.t. $(\alpha-\varepsilon,\alpha+\varepsilon)\subseteq V$

Then $\alpha+\varepsilon\in lb(A)$ (otherwise, $\exists a\in A$, $a<\alpha+\varepsilon$, $\alpha\le a<\alpha+\varepsilon\Rightarrow a\in V\Rightarrow a\in V\cap A$, a contr.)

$\Rightarrow\alpha=\inf A\ge\alpha+\varepsilon$, a contr.

$\Rightarrow\forall V\in\mathcal{V}(\alpha)$, $V\cap A\neq\phi$.

(10)

$\boxed{\Leftarrow}$ $\alpha\in lb(A)\Rightarrow\inf A\ge\alpha$. Suppose $\inf A>\alpha$. Then we write $\inf A=\alpha+\varepsilon$, where $\varepsilon>0$

$(\alpha-\varepsilon,\alpha+\varepsilon)\in\mathcal{V}(\alpha)\Rightarrow\exists a\in A$, $a\in(\alpha-\varepsilon,\alpha+\varepsilon)$

$\Rightarrow\alpha+\varepsilon=\inf A\le a<\alpha+\varepsilon$, a contr.

$\Rightarrow\inf A=\alpha$