Homework 2

Seminar 4

Exercise 4.1. Define the sequence (x_n) by $x_1 \in (0,1)$ and $x_{n+1} = x_n - x_n^2$, $n \in \mathbb{N}$. Prove that the sequence (x_n) converges and find its limit. Then study the convergence of the sequence $(n \cdot x_n)$ (for this, one could apply the Stolz-Cesàro Theorem for the sequences (a_n) and (b_n) defined, for $n \in \mathbb{N}$, by $a_n = n$ and $b_n = 1/x_n$, respectively).

Solution We show by induction that $x_n \in (0,1)$ for all $n \in \mathbb{N}$. By hypothesis, $x_1 \in (0,1)$. Suppose that $x_k \in (0,1)$ for some $k \in \mathbb{N}$. Then $x_{k+1} = x_k(1 - x_k) \in (0,1)$. Thus, we can conclude that $x_n \in (0,1)$ for all $n \in \mathbb{N}$.
Since $x_{n+1} - x_n = -x_n^2 < 0$ for all $n \in \mathbb{N}$, the sequence (x_n) is strictly decreasing. Being also bounded, it follows that it is convergent. Let $x = \lim\limits_{n\to\infty} x_n \in \mathbb{R}$. Then $x = x - x^2$, which implies $x = 0$.
Observe that $\lim\limits_{n\to\infty} 1/x_n = \infty$ since $\lim\limits_{n\to\infty} x_n = 0$ and $x_n > 0$ for all $n \in \mathbb{N}$. In addition, the sequence $(1/x_n)$ is strictly increasing as the sequence (x_n) is strictly decreasing.
Define, for $n \in \mathbb{N}$, $a_n = n$ and $b_n = 1/x_n$. Then $nx_n = a_n/b_n$ and

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{n+1-1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \frac{x_n x_{n+1}}{x_n - x_{n+1}} = \frac{x_n(x_n - x_n^2)}{x_n^2} = 1 - x_n \to 1.$$

By the Stolz-Casàro Theorem, the sequence (nx_n) is convergent and $\lim\limits_{n\to\infty}(nx_n) = 1$.

Exercise 4.2 (Koch snowflake)**.** Define a sequence (S_n) of polygons such that S_1 is an equilateral triangle of side length 1 and, for $n \in \mathbb{N}$, S_{n+1} is obtained from S_n by adding to the middle third of each side an equilateral triangle pointing outwards (and removing this middle third). Denote by a_n the area of S_n. Determine the sequence (a_n) and study if it is convergent.

Solution (Sketch) Note that $a_1 = \dfrac{\sqrt{3}}{4}$, $a_2 = a_1 + 3 \cdot \dfrac{\sqrt{3}}{4} \cdot \left(\dfrac{1}{3}\right)^2$, $a_3 = a_2 + 3 \cdot 4 \cdot \dfrac{\sqrt{3}}{4} \cdot \left(\dfrac{1}{3^2}\right)^2$ and, in general,

$$a_{n+1} = a_n + 3 \cdot 4^{n-1} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^n}\right)^2 = a_n + \frac{3}{4} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{4}{9}\right)^n \quad \text{for all } n \in \mathbb{N}.$$

Hence, $a_n = \dfrac{\sqrt{3}}{4}\left(1 + \dfrac{3}{4}\displaystyle\sum_{k=1}^{n-1}\left(\dfrac{4}{9}\right)^k\right)$ for all $n \in \mathbb{N}$, so $\lim\limits_{n\to\infty} a_n = \dfrac{\sqrt{3}}{4}\left(1 + \dfrac{3}{4} \cdot \dfrac{4/9}{1 - 4/9}\right) = \dfrac{2\sqrt{3}}{5}$.

Exercise 4.3. Find the sum of the following series and specify whether they are convergent or divergent:
a) $\displaystyle\sum_{n\geq 1}\left(-\frac{\pi}{4}\right)^n$, b) $\displaystyle\sum_{n\geq 0}\frac{2^{3n}}{5^{n-1}}$, c) $\displaystyle\sum_{n\geq 1}\frac{1}{4n^2-1}$, d) $\displaystyle\sum_{n\geq 1}\ln\left(1+\frac{1}{n}\right)$, e) $\displaystyle\sum_{n\geq 1}\frac{3n-2}{2^n}$.

Solution (Sketch) a) $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{\pi}{4}\right)^n = \frac{-\pi/4}{1+\pi/4} = -\frac{\pi}{4+\pi} \in \mathbb{R}$, so the series is convergent.
b) Since for all $n \in \mathbb{N}_0$, $\dfrac{2^{3n}}{5^{n-1}} = 5\left(\dfrac{8}{5}\right)^n$ and $\dfrac{8}{5} > 1$, the series is divergent with sum ∞.

c) Since for all $n \in \mathbb{N}$,

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \cdot \frac{2n+1-(2n-1)}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$
$$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2(n+1)-1}\right),$$

we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \in \mathbb{R}$, so the series is convergent.

d) Since for all $n \in \mathbb{N}$,

$$\ln\left(1+\frac{1}{n}\right) = \ln\frac{n+1}{n} = \ln(n+1) - \ln n,$$

we get that the series is divergent with sum ∞.

e) Recall that $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2^n} = 3$ (see Seminar 4). Moreover, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ and $\displaystyle\frac{3n-2}{2^n} = 3\frac{n+1}{2^n} - 5\frac{1}{2^n}$ for all $n \in \mathbb{N}$. Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{2^n} = 3 \cdot 3 - 5 \cdot 1 = 4$. The series is convergent.

Seminar 5

Exercise 5.1. Study if the following series are convergent or divergent:
a) $\displaystyle\sum_{n\geq 1} \left(1-\frac{1}{n}\right)^n$, b) $\displaystyle\sum_{n\geq 1} \sin\frac{1}{n^{5/4}}$, c) $\displaystyle\sum_{n\geq 1} \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$, d) $\displaystyle\sum_{n\geq 1} \frac{n!}{3 \cdot 5 \cdot \ldots \cdot (2n+1)}$, e) $\displaystyle\sum_{n\geq 1} \frac{n^3 5^n}{2^{3n+1}}$,
f) $\displaystyle\sum_{n\geq 1} \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3 \cdot 6 \cdot \ldots \cdot (3n)}$.

Solution a) $\displaystyle\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \frac{1}{e} \neq 0$, so, by the n^{th} Term Test, the given series is divergent.

b) Since $\displaystyle 0 < \sin\frac{1}{n^{5/4}} < \frac{1}{n^{5/4}}$, $\forall n \in \mathbb{N}$, and the series $\displaystyle\sum_{n\geq 1} \frac{1}{n^{5/4}}$ is convergent, by the First Comparison Test, the given series is convergent.

c) Since $\displaystyle\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n\sqrt[3]{n}+2}}{\frac{1}{n^{5/6}}} = 1 \in (0,\infty)$ and the series $\displaystyle\sum_{n\geq 1} \frac{1}{n^{5/6}}$ is divergent, by the Second Comparison Test, the given series is divergent.

d) Denote $\displaystyle x_n = \frac{n!}{3 \cdot 5 \cdot \ldots \cdot (2n+1)}$ for $n \in \mathbb{N}$. Then $\displaystyle\frac{x_{n+1}}{x_n} = \frac{n+1}{2n+3} \to \frac{1}{2} < 1$, by the Ratio Test, the given series is convergent.

e) Since $\displaystyle\sqrt[n]{\frac{n^3 5^n}{2^{3n+1}}} = \frac{5\sqrt[n]{n^3}}{2^{3+1/n}} \to \frac{5}{8} < 1$, by the Root Test, the given series is convergent.

f) Denote $\displaystyle x_n = \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3 \cdot 6 \cdot \ldots \cdot (3n)}$ for $n \in \mathbb{N}$. Then

$$n\left(\frac{x_n}{x_{n+1}} - 1\right) = n\left(\frac{3n+3}{3n+2} - 1\right) = \frac{n}{3n+2} \to \frac{1}{3} < 1,$$

so, by Raabe's Test, the given series is divergent.

Exercise 5.2. Let (x_n) and (y_n) be two sequences of positive numbers. Suppose that the series $\sum_{n\geq 1} \frac{x_n}{y_n}$ and $\sum_{n\geq 1} y_n$ are both convergent. Is the series $\sum_{n\geq 1} \sqrt{x_n}$ convergent as well?

Solution Note that

$$\sqrt{x_n} = \sqrt{\frac{x_n}{y_n}\cdot y_n} \leq \frac{1}{2}\left(\frac{x_n}{y_n} + y_n\right), \quad \forall n \in \mathbb{N}. \tag{5.1}$$

Since the series $\sum_{n\geq 1} \frac{x_n}{y_n}$ and $\sum_{n\geq 1} y_n$ are both convergent, we get that the series $\sum_{n\geq 1} \frac{1}{2}\left(\frac{x_n}{y_n} + y_n\right)$ is convergent. Using (5.1) and the First Comparison Test, it follows that $\sum_{n\geq 1} \sqrt{x_n}$ is convergent.

Seminar 6

Exercise 6.1. Study if the following series are absolutely convergent, semi-convergent or divergent:
a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n+1}}$, b) $\sum_{n\geq 1} \frac{n}{n^2+1}\cos(n\pi)$.

Solution a) Since $\left|\frac{(-1)^{n+1}}{n\sqrt{n+1}}\right| = \frac{1}{n\sqrt{n+1}} \leq \frac{1}{n\sqrt{n}}$ for all $n \in \mathbb{N}$ and $\sum_{n\geq 1} \frac{1}{n\sqrt{n}}$ is convergent, by the First Comparison Test, the given series is absolutely convergent.

b) Note that $\frac{n}{n^2+1}\cos(n\pi) = \frac{n}{n^2+1}(-1)^n$ for all $n \in \mathbb{N}$. Moreover, $\left|\frac{n}{n^2+1}(-1)^n\right| = \frac{n}{n^2+1}$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = 1 \in (0,\infty)$ and $\sum_{n\geq 1} \frac{1}{n}$ is divergent, using the Second Comparison Test, it follows that $\sum_{n\geq 1} \frac{n}{n^2+1}$ is divergent, so the given series is not absolutely convergent.

Because $\frac{n+1}{(n+1)^2+1}\,\frac{n^2+1}{n} = \frac{n^3+n^2+n+1}{n^3+2n^2+2n} < 1$ for all $n \in \mathbb{N}$, we get that the sequence $\left(\frac{n}{n^2+1}\right)$ is decreasing. In addition, $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. By the Alternating Series Test, the given series is convergent. Therefore, we conclude that it is semi-convergent.

Exercise 6.2. Let $f, g : [0,1] \to \mathbb{R}$ be continuous such that $f(x) = g(x)$, $\forall x \in [0,1]\cap\mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0,1]$. Is it enough solely to assume that f and g are continuous on $[0,1]\setminus\{\alpha\}$ for some $\alpha \in [0,1]\cap(\mathbb{R}\setminus\mathbb{Q})$?

Solution Let $c \in [0,1]$. Then there exists a sequence (x_n) in $[0,1]\cap\mathbb{Q}$ such that $x_n \to c$. Since f and g are continuous at c, we get that $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$, respectively. But $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. Since a sequence in $\mathbb{R}$ cannot have two distinct limits, we obtain $f(c) = g(c)$. Thus, we can conclude that $f = g$.

It is not enough to assume that f and g are continuous on $[0,1]\setminus\{\alpha\}$ for some $\alpha \in [0,1]\cap(\mathbb{R}\setminus\mathbb{Q})$. Take, e.g., $f, g : [0,1] \to \mathbb{R}$,

$$f(x) = 0, \forall x \in [0,1] \qquad \text{and} \qquad g(x) = \begin{cases} 1, & \text{if } x = \alpha, \\ 0, & \text{if } x \in [0,1]\setminus\{\alpha\}. \end{cases}$$

Then $f(x) = g(x)$, $\forall x \in [0,1]\cap\mathbb{Q}$ and both f and g are continuous on $[0,1]\setminus\{\alpha\}$. However $f(\alpha) = 0 \neq 1 = g(\alpha)$.

Exercise 6.3. Find all continuous functions $f : \mathbb{R} \to \mathbb{N}$ and all continuous functions $f : \mathbb{N} \to \mathbb{R}$.

Solution a) Note that a constant function $f:\mathbb{R}\to\mathbb{N}$ is continuous. We show that f must be constant. Suppose to the contrary that there exist $m,n\in f(\mathbb{R})\subseteq\mathbb{N}$ with $m\neq n$. Without loss of generality, we can assume that $m<n$. Let $\gamma\in\mathbb{R}$ with $m<\gamma<n$. By the Intermediate Value Theorem, $\gamma\in f(\mathbb{R})$. Thus, $[m,n]\subseteq f(\mathbb{R})$, so $f(\mathbb{R})$ is not a subset of $\mathbb{N}$, a contradiction. Hence, f is constant.

b) Every function $f:\mathbb{N}\to\mathbb{R}$ is continuous because all point in $\mathbb{N}$ are isolated.