Seminar 8

(1)

Taylor series: $I \subseteq \mathbb{R}$ int, $f: I \to \mathbb{R}$ infinitely diff

$x_0 \in I$, $x \in \mathbb{R}$, $\quad \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \left(f(x_0) + \frac{f'(x_0)}{1!}\cdot(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots \right)$

Taylor series of f around x_0

n^{th} partial sum is n^{th} Taylor polynomial of f at x_0, $T_n(x)$

If $J \subseteq I$, $J \neq \emptyset$ s.t. $\forall x \in J$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$, then we say that f can be expanded as a Taylor series around x_0 on J.

$\Updownarrow$

$\forall x \in J$, $\lim_{n\to\infty} R_n(x) = 0$, where $R_n(x) = f(x) - T_n(x)$

(2)

Ex 1 Prove that the following functions can be expanded as a Taylor series around 0 on J and find the corresponding Taylor series expansion:

a) $f: (-1,\infty) \to \mathbb{R}$, $f(x) = \ln(1+x)$, $J = [0,1]$.

$f^{(k)}(x) = (-1)^{k-1} \cdot \frac{(k-1)!}{(1+x)^k}$, $\forall x \in (-1,\infty)$, $\forall k \in \mathbb{N}$

$f(0) = 0$, $f^{(k)}(0) = (-1)^{k-1}(k-1)!$, $k \in \mathbb{N}$

Let $n \in \mathbb{N}$

$T_n(x) = f(0) + \frac{f'(0)}{1!}\cdot x + \dots + \frac{f^{(n)}(0)}{n!}x^n = x - \frac{1}{2}x^2 + \dots + \frac{(-1)^{n-1}\cdot \cancel{(n-1)!}}{n\cancel{!}}x^n$

$= x - \frac{x^2}{2} + \dots + (-1)^{n-1}\cdot\frac{x^n}{n}$, $\quad x \in \mathbb{R}$

Let $x \in [0,1]$. Then $\exists\, c$ btw 0 and x s.t. $\ln(1+x) = T_n(x) + R_n(x)$, where

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ $\qquad$ Careful: c may also depend on n

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$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{(-1)^n \cdot \cancel{n!}}{(1+c)^{n+1}}\cdot\frac{1}{(n+1)\cancel{!}}x^{n+1} = (-1)^n \cdot \frac{1}{n+1}\cdot\left(\frac{x}{1+c}\right)^{n+1}$

$|R_n(x)| = \frac{1}{n+1}\cdot\left(\underbrace{\frac{x}{1+c}}_{\leq 1}\right)^{n+1} \leq \frac{1}{n+1} \xrightarrow{n\to\infty} 0$

By the Squeeze Thm, $\lim_{n\to\infty} R_n(x) = 0$.

$\Rightarrow f$ can be expanded as a Taylor series around 0 on $[0,1]$

$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{x^n}{n}$, $\forall x \in [0,1]$

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b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$, $J = \mathbb{R}$

$f^{(k)}(x) = \cos\left(x + \frac{k\pi}{2}\right)$, $\forall x \in \mathbb{R}$, $\forall k \in \mathbb{N}$

$f(0) = 1$, $f^{(k)}(0) = \cos\frac{k\pi}{2} \Rightarrow f^{(2p+1)}(0) = 0$, $f^{(2p)}(0) = (-1)^p$, $p \in \mathbb{N}_0$

Let $n \in \mathbb{N}_0$

Case 1: $n = 2k$, $k \in \mathbb{N}_0$

$T_{2k}(x) = 1 - \frac{1}{2!}x^2 + \dots + \frac{(-1)^k}{(2k)!}x^{2k}$, $x \in \mathbb{R}$

Case 2: $n = 2k+1$, $k \in \mathbb{N}_0$

$T_{2k+1}(x) = T_{2k}(x)$, $x \in \mathbb{R}$

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Let $x \in \mathbb{R}$. Then $\exists\, c$ btw. 0 and x s.t. $\cos x = T_n(x) + R_n(x)$, where

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{\cos\left(c + \frac{(n+1)\pi}{2}\right)}{(n+1)!}\cdot x^{n+1}$

$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$

Let $N \in \mathbb{N}$ s.t. $N+1 > |x|$.

For $n \geq N$, $\frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^N}{N!}\cdot\frac{|x|}{N+1}\cdot\dots\cdot\frac{|x|}{n+1} \leq \frac{|x|^N}{N!}\cdot\left(\underbrace{\frac{|x|}{N+1}}_{<1}\right)^{n-N+1} \xrightarrow{n\to\infty} 0$

$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$

$\Rightarrow f$ can be expanded as a Taylor series around 0 on $\mathbb{R}$ and $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$, $\forall x \in \mathbb{R}$

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c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, $J = \mathbb{R}$

$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot x^{2n+1}$, $\forall x \in \mathbb{R}$

Ex 2: Let $x, y \in \mathbb{R}^n$. Denote $\alpha = \langle x, y\rangle$, $\beta = \|x\|$, $\gamma = \|y\|$

$x = (x_1, x_2, \dots, x_n)$
$y = (y_1, y_2, \dots, y_n)$
$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$
$\|y\| = \sqrt{\langle y, y\rangle}$

a) Determine $\langle x+y, y\rangle$, $\langle x, 2x-3y\rangle$, $\|x-y\|$ in terms of α, β, γ

$\langle x+y, y\rangle = \langle x, y\rangle + \langle y, y\rangle = \alpha + \gamma^2$

$\langle x, 2x-3y\rangle = 2\langle x, x\rangle - 3\langle x, y\rangle = 2\beta^2 - 3\alpha$

$\|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\langle x, x-y\rangle - \langle y, x-y\rangle} = \sqrt{\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle} = \sqrt{\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle}$
$= \sqrt{\beta^2 - 2\alpha + \gamma^2}$

b) $n = 3$, $x = (-1, 2, 3)$, $y = (-2, 1, -3)$

i) Compute α, β, γ: $\alpha = \langle x, y\rangle = (-1)(-2) + 2\cdot 1 + 3\cdot(-3) = 2 + 2 - 9 = -5$

$\beta = \|x\| = \sqrt{1+4+9} = \sqrt{14}$

$\gamma = \|y\| = \sqrt{14}$

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ii) Find all $r > 0$ s.t. $B(x, r)$ does not contain the vector y

$B(x, r) = \{z \in \mathbb{R}^3 / \|z - x\| < r\}$

$y \notin B(x, r) \Leftrightarrow \|y - x\| \geq r$

$\|y - x\| = \sqrt{(-2+1)^2 + (1-2)^2 + (-3-3)^2} = \sqrt{1+1+36} = \sqrt{38} \Rightarrow r \in (0, \sqrt{38}]$

iii) Find all $t \in \mathbb{R}$ s.t. $\overline{B}(x, 5)$ contains the vector $(1, -1, t)$

$(1, -1, t) \in \overline{B}(x, 5) \Leftrightarrow \|\underbrace{(1,-1,t) - x}\| \leq 5$ $\qquad x = (-1, 2, 3)$

$\sqrt{2^2 + (-3)^2 + (t-3)^2} = \sqrt{13 + (t-3)^2} \leq 5$

$13 + (t-3)^2 \leq 25$

$(t-3)^2 \leq 12$

$t - 3 \in [-2\sqrt{3}, 2\sqrt{3}]$

$t \in [3 - 2\sqrt{3}, 3 + 2\sqrt{3}]$

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Ex 4: Let $x, y \in \mathbb{R}^n$. Prove that

a) $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$

b) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ $\quad$ (the parallelogram law)

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$

$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle$

$\overset{\text{"}-\text{"}}{\Rightarrow}$ a)

$\overset{\text{"}+\text{"}}{\Rightarrow}$ b)

Ex 5: Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if $\langle x, y\rangle = 0$

Prove that given $x, y \in \mathbb{R}^n$, TFAE:

a) x and y are orthogonal

b) $\|x+y\| = \|x-y\|$

c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\boxed{a) \Leftrightarrow b)}$

$\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$ (*)

$\langle x, y\rangle = 0 \overset{(*)}{\Leftrightarrow} \|x+y\|^2 = \|x-y\|^2 \Leftrightarrow \|x+y\| = \|x-y\|$

$\boxed{b) \Leftrightarrow c)}$

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (**)

$\|x+y\| = \|x-y\| \Leftrightarrow \|x+y\|^2 = \|x-y\|^2 \overset{(**)}{\Leftrightarrow}$

$\Leftrightarrow 2\|x+y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\Leftrightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

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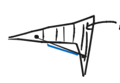
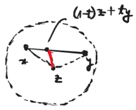
Ex 6 A set $A \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in A$, $\forall t \in [0,1]$, $(1-t)x + ty \in A$

$\mathbb{R}^2$



A not convex

Prove that $\forall z \in \mathbb{R}^n$, $\forall r > 0$, $B(z, r)$ and $\overline{B}(z, r)$ are convex



Let $x, y \in B(z, r) \Rightarrow \|x - z\| < r$, $\|y - z\| < r$ ($\leq$, $\leq$)

Let $t \in [0,1]$

$\|(1-t)x + ty - z\| = \|(1-t)x + ty - ((1-t)z + tz)\|$

$= \|(1-t)(x-z) + t(y-z)\|$ $\qquad \Rightarrow (1-t)x + ty \in B(z, r)$

triangle inequality $\to \leq \|(1-t)(x-z)\| + \|t(y-z)\|$

$= (1-t)\underbrace{\|x-z\|}_{<r\ (\leq)} + t\underbrace{\|y-z\|}_{<r\ (\leq)} < r$ ($\leq$) $\qquad \Rightarrow B(z, r)$ is convex

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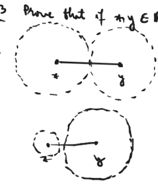
Ex 3 Prove that if $x, y \in \mathbb{R}^n$, $x \neq y$, $\exists\, U \in \mathcal{V}(x)$, $\exists\, V \in \mathcal{V}(y)$ s.t. $U \cap V = \emptyset$.

$\mathbb{R}^2$



$r = \frac{\|y - x\|}{2}$

Take $U = B(x, r)$, $V = B(y, r)$

Suppose $\exists\, z \in U \cap V$. Then

$\|z - x\| < r$, $\|z - y\| < r$

$\Rightarrow \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| < r + r = \|x - y\|$, a contr.

$\Rightarrow U \cap V = \emptyset$.