Seminar 13 ①

Substitution: $a, b \in \mathbb{R}$, $a < b$, $I \subseteq \mathbb{R}$ int

$g : [a,b] \to I$ diff, $g' \in \mathcal{R}[a,b]$

$f : I \to \mathbb{R}$ cont

$$\int_a^b (f\circ g)(x)\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integration by parts: $a, b \in \mathbb{R}$, $a < b$, $f, g : [a,b] \to \mathbb{R}$ diff, $f', g' \in \mathcal{R}[a,b]$

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx$$

Ex1: Study the improper integrability of the following functions on their domains and, in case they are improperly integrable, determine the corresponding improper integrals: ②

a) $f : [1,2) \to \mathbb{R}$, $f(x) = \dfrac{1}{x(x-2)}$

$$\frac{1}{x(x-2)} = \frac{x-(x-2)}{x(x-2)}\cdot\frac{1}{2} = \frac{1}{2}\left(\frac{1}{x-2} - \frac{1}{x}\right)$$

f cont

Let $t \in [1,2)$. Then $\displaystyle\int_1^t \frac{1}{x(x-2)}\,dx = \frac{1}{2}\int_1^t \left(\frac{1}{x-2} - \frac{1}{x}\right)dx = \frac{1}{2}\left(\ln|x-2| - \ln x\right)\Big|_1^t =$

$$= \frac{1}{2}\left(\ln(2-t) - \ln t\right) = \frac{1}{2}\ln\frac{2-t}{t}$$

$$\lim_{\substack{t\to 2\\ t<2}} \int_1^t \frac{1}{x(x-2)}\,dx = \lim_{\substack{t\to 2\\ t<2}} \frac{1}{2}\ln\frac{2-t}{t} = -\infty \Rightarrow f \text{ is not imp. int. on } [1,2).$$

b) $f : (-\infty, 0] \to \mathbb{R}$, $f(x) = x e^{-x^2}$ ③

f cont

Let $t \in (-\infty, 0]$. Then $\displaystyle\int_t^0 x e^{-x^2}\,dx = \frac{1}{2}\int_{t^2}^0 e^{-u}\,du = -\frac{1}{2}e^{-u}\Big|_{t^2}^0 = -\frac{1}{2}\left(1 - e^{-t^2}\right) \xrightarrow{t\to-\infty} -\frac{1}{2}$

$x^2 = u$

$2x\,dx = du$

$\Rightarrow f$ is improperly integrable on $(-\infty, 0]$ and $\displaystyle\int_{-\infty}^0 f(x)\,dx = -\frac{1}{2}$.

c) $f : [0,\infty) \to \mathbb{R}$, $f(x) = e^{-x}\cdot\sin x$

f cont

Let $t \in [0,\infty)$. Then $\displaystyle\int_0^t e^{-x}\sin x\,dx = -\int_0^t (e^{-x})'\cdot\sin x\,dx = -e^{-x}\cdot\sin x\Big|_0^t + \int_0^t e^{-x}\cdot\cos x\,dx$

$$= -e^{-t}\sin t - \int_0^t (e^{-x})'\cos x\,dx = -e^{-t}\sin t - e^{-x}\cdot\cos x\Big|_0^t + \int_0^t e^{-x}\cdot(-\sin x)\,dx$$ ④

$\Rightarrow 2\displaystyle\int_0^t e^{-x}\sin x\,dx = -e^{-t}\sin t - e^{-t}\cos t + 1$

$\Rightarrow \displaystyle\int_0^t e^{-x}\sin x\,dx = -\frac{1}{2}e^{-t}(\sin t + \cos t) + \frac{1}{2} \xrightarrow{t\to\infty} \frac{1}{2}$

$\Rightarrow f$ is imp. int on $[0,\infty)$ and $\displaystyle\int_0^\infty f(x)\,dx = \frac{1}{2}$.

d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{1+x^2}$

f cont

We study the improper integrability of f on $(-\infty, 0]$ and on $[0,\infty)$. ⑤

$[0,\infty)$: Let $t \in [0,\infty)$. Then $\displaystyle\int_0^t \frac{1}{1+x^2}\,dx = \arctan t \xrightarrow{t\to\infty} \frac{\pi}{2}$

$\Rightarrow f$ is imp. int. on $[0,\infty)$ and $\displaystyle\int_0^\infty f(x)\,dx = \frac{\pi}{2}$.

$(-\infty, 0]$: As f is even, f is impr. int on $(-\infty,0]$ as well and $\displaystyle\int_{-\infty}^0 f(x)\,dx = \frac{\pi}{2}$

$\Rightarrow f$ is impr. int on $\mathbb{R}$ and $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = \pi$

(See Lecture 11) ⑥

Thm1: $a, b \in \mathbb{R}$, $a < b$, $f : [a,b) \to [0,\infty)$ cont

$p \in \mathbb{R}$ s.t. $\exists\, L = \displaystyle\lim_{\substack{x\to b\\ x<b}} (b-x)^p f(x)$. Then

(i) if $p < 1$ and $L < \infty \Rightarrow f$ is impr. int on $[a,b)$

(ii) if $p \geq 1$ and $L > 0 \Rightarrow f$ is not impr. int. on $[a,b)$.

Thm2: $a, b \in \mathbb{R}$, $a < b$, $f : (a,b] \to [0,\infty)$ cont

$p \in \mathbb{R}$ st. $\exists\, L = \displaystyle\lim_{\substack{x\to a\\ x>a}} (x-a)^p f(x)$. Then

(i) if $p < 1$ and $L < \infty \Rightarrow f$ is impr. int on $(a,b]$

(ii) if $p \geq 1$ and $L > 0 \Rightarrow f$ is not impr. int. on $(a,b]$

Thm 3: $a \in \mathbb{R}$, $f : [a,\infty) \to [0,\infty)$ cont ⑦

$p \in \mathbb{R}$ st. $\exists\, L = \displaystyle\lim_{x\to\infty} x^p f(x)$. Then

(i) if $p > 1$ and $L < \infty \Rightarrow f$ is impr. int on $[a,\infty)$

(ii) if $p \leq 1$ and $L > 0 \Rightarrow f$ is not impr. int on $[a,\infty)$.

Ex2: Study the improper integrability of the following functions:

a) $f : [0, \pi/2) \to \mathbb{R}$, $f(x) = \dfrac{1}{\cos x}$

f cont

$\forall x \in [0, \pi/2)$, $f(x) > 0$

We try to find $p \in \mathbb{R}$ st. $\exists\, L = \displaystyle\lim_{\substack{x\to\pi/2\\ x<\pi/2}} \left(\frac{\pi}{2} - x\right)^p\cdot f(x)$ $\left(\in (0,\infty) \text{ preferably}\right)$

$L = \displaystyle\lim_{\substack{x\to\pi/2\\ x<\pi/2}} \frac{\left(\frac{\pi}{2} - x\right)^p}{\cos x}$ ⑧

For $p = 1$, $\displaystyle\lim_{\substack{x\to\pi/2\\ x<\pi/2}} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{\substack{x\to\pi/2\\ x<\pi/2}} \frac{-1}{-\sin x} = 1$

$p \geq 1$, $L > 0 \Rightarrow f$ is not impr. int on $[0, \pi/2)$.

b) $f : (1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{\ln x}{x\sqrt{x^2-1}}$

f cont

$\forall x > 1$, $f(x) \geq 0$

We study the improper integrability of f on $(1,2]$ and $[2,\infty)$: ⑨

$(1,2]$: $p \in \mathbb{R}$, $L = \displaystyle\lim_{\substack{x\to 1\\ x>1}} (x-1)^p \frac{\ln x}{x\sqrt{x^2-1}} = \lim_{\substack{x\to 1\\ x>1}} (x-1)^p\cdot\frac{\ln x}{x\sqrt{(x-1)(x+1)}}$

For $p = \frac{1}{2}$, $L = 0$

$p < 1$, $L < \infty \Rightarrow f$ is impr. int on $(1,2]$.

$[2,\infty)$: $p \in \mathbb{R}$, $L = \displaystyle\lim_{x\to\infty} x^p\cdot\frac{\ln x}{x\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}\cdot x^{p-2}\cdot\ln x$

$p - 2 < 0 \Rightarrow L = 0 < \infty$

$p > 1$ $\quad \frac{3}{2} \in (1,2)$

$\dfrac{\ln x}{x^\alpha}$, $\alpha > 0$

$\downarrow x\to\infty$

0

For $p = \frac{3}{2}$, $L = 0$ ⑩

$p > 1$, $L < \infty \Rightarrow f$ is impr. int on $[2,\infty)$.

$\Rightarrow f$ is impr. int on $(1,\infty)$.

d) $f : [1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x^\alpha(1+x^2)^\beta}$, where $\alpha, \beta \in \mathbb{R}$

f cont

$\forall x \geq 1$, $f(x) \geq 0$

$p \in \mathbb{R}$, $L = \displaystyle\lim_{x\to\infty} x^p\cdot\frac{1}{x^\alpha(1+x^2)^\beta} = \lim_{x\to\infty} \frac{x^{p-\alpha}}{x^{2\beta}\left(\frac{1}{x^2}+1\right)^\beta}$

For $p = \alpha + 2\beta$, $L = 1 \in (0,\infty)$

$\Rightarrow f$ is impr. int on $[1,\infty) \Leftrightarrow$

$\Leftrightarrow \alpha + 2\beta > 1$.

Integral Test for Convergence of Series ⑪

$m \in \mathbb{N}$, $f : [m,\infty) \to [0,\infty)$ cont + decr. Then

f impr. int on $[m,\infty) \Leftrightarrow \displaystyle\sum_{n\geq m} f(n)$ is conv

Ex3: Use the Integral Test to study if the following series are convergent or divergent:

a) $\displaystyle\sum_{n\geq 2} \frac{1}{n(\ln n)^2}$

$f : [2,\infty) \to [0,\infty)$, $f(x) = \dfrac{1}{x(\ln x)^2}$

f cont, decr.

Let $t \in [2,\infty)$. Then $\displaystyle\int_2^t \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln x}\Big|_2^t = -\frac{1}{\ln t} + \frac{1}{\ln 2} \xrightarrow{t\to\infty} \frac{1}{\ln 2}$ ⑫

$\Rightarrow f$ is impr. int on $[2,\infty)$

By the Integral Test, the given series is conv.

b) $\displaystyle\sum_{n\geq 1} \frac{\ln n}{n^{3/2}}$

$f : [2,\infty) \to [0,\infty)$, $f(x) = \dfrac{\ln x}{x^{3/2}}$

f cont

$f'(x) = \dfrac{\frac{1}{x}\cdot x^{3/2} - \ln x\cdot\frac{3}{2}\cdot x^{1/2}}{x^3} = \dfrac{\sqrt{x}\left(1 - \frac{3}{2}\ln x\right)}{x^3}$

$1 - \frac{3}{2}\ln x \leq 0 \Leftrightarrow \frac{3}{2}\ln x \geq 1 \Leftrightarrow \ln x \geq \frac{2}{3}$

$\Leftrightarrow x \geq e^{2/3}$ $\quad 2 > e^{2/3}$

$\Rightarrow f' \leq 0$ $\quad 8 > e^2$

$\Rightarrow f$ decr.

$p \in \mathbb{R}$, $L = \displaystyle\lim_{x\to\infty} x^p\cdot\frac{\ln x}{x^{3/2}} = \lim_{x\to\infty} x^{p-3/2}\cdot\ln x$ ⑬

$p - \frac{3}{2} < 0 \Rightarrow L = 0 < 0$

$p > 1$ $\quad 1.2 \in \left(1, \frac{3}{2}\right)$

For $p = 1.2$, $L = 0$

$p > 1$, $L < \infty \Rightarrow f$ is impr. int on $[2,\infty)$

By the Integral Test, $\displaystyle\sum_{n\geq 2} \frac{\ln n}{n^{3/2}}$ is conv, so the given series is conv.

Exercise 3.c Use the Integral Test to study if the series $\sum_{n\geq 1} \frac{n^2}{1+n^3}$ is convergent or divergent.

Solution Consider $f : [1,\infty) \to [0,\infty)$, $f(x) = \frac{x^2}{1+x^3}$. This function is continuous. One can check that $f|_{[2,\infty)}$ is decreasing (please do this).

Let $t \in [2,\infty)$. Then $\int_2^t f(x)dx = \frac{1}{3}\big(\ln(1+t^3) - 2\ln 3\big)$, so $\lim_{t\to\infty} \int_2^t f(x)dx = \infty$. Hence, f is not improperly integrable on $[2,\infty)$ and, by the Integral Test, the series $\sum_{n\geq 2} \frac{n^2}{1+n^3}$ is divergent, so the given series is divergent.