PROPOSITIONAL LOGIC

Syntactic Approach

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AXIOMATIC (FORMAL) SYSTEM:

$$P = (\Sigma_P, F_P, A_P, R_P)$$

- $\Sigma_P = Var\_propos \cup Connectives \cup \{(,)\}$ - vocabulary

  $Var\_propos = \{p_1, p_2, \ldots\}$ - a ***set of propositional variables***

  $Connectives = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus, \uparrow, \downarrow\}$;

- $F_P =$ ***the set of well formed formulas,***

- $A_p = \{A1, A2, A3\}$ - the ***set of axioms***

  $A1$: $U \rightarrow (V \rightarrow U)$

  $A2$: $((U \rightarrow (V \rightarrow Z)) \rightarrow ((U \rightarrow V) \rightarrow (U \rightarrow Z))$

  $A3$: $(U \rightarrow V) \rightarrow (\neg V \rightarrow \neg U)$ (modus tollens)

- $R_P = \{mp\}$ - the set of ***inference (deduction) rules*** containing ***modus ponens rule***.

  notation: $U, U \rightarrow V \vdash_{mp} V$

  with the meaning: "from the facts U and $U \rightarrow V$ we *deduce (infer)* V".

DEDUCTION

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Definition: Let $U_1,U_2,..,U_n$ be propositional formulas, called ***hypotheses*** and V be a formula, called ***conclusion***. V **is deducible (derivable, inferable) from** $U_1,...,U_n$ and we denote by $U_1,..,U_n \vdash V$, if there exists a sequence $(f_1, f_2, ..., f_m)$ of formulas such that $f_m = V$ and $\forall i \in \{1,..., m\}$ we have a) or b) or c).

a) $f_i \in A_P$ (axiom);

b) $f_i \in \{U_1,...,U_n\}$ (hypothesis formula);

c) $f_{i1}, f_{i2} \vdash_{mp} f_i, \quad i1 < i \text{ and } i2 < i$

(formula f_i is inferred using *modus ponens* rule from two existing formulas)

The sequence $(f_1, f_2, ..., f_m)$ is called the ***deduction of*** V from $U_1,U_2,..,U_n$.

Definition: A formula $U \in F_P$, such that $\emptyset \vdash U$ (or $\vdash U$) is called ***theorem***.

Remark: The theorems are the formulas derivable (inferable) only from the axioms and using modus ponens as inference rule.

PROPERTIES OF DERIVABILITY RELATION

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Theorem:

Let R, S be sets of propositional formulas and U, V, Z, be formulas.

The derivability (syntactic consequence) relation has the properties:

1. ***monotonicity*:**
   if $R \vdash U$ and $R \subseteq S$ then $S \vdash U$;
2. ***cut*:**
   if $S \vdash V_j$, $\forall j \in \{1,\dots,n\}, n \in \mathbf{N}$ and $S \cup \{V_1, V_2, \dots, V_n\} \vdash U$ then $S \vdash U$;
3. ***transitivity*:**
   if $S \vdash U$ and $\{U\} \vdash V$ then $S \vdash V$;
4. ***conjunction in conclusions (right ”and”)*:**
   if $S \vdash U$ and $S \vdash V$ then $S \vdash U \wedge V$;
5. ***disjunction in premises (left ”or”)*:**
   if $S \cup \{U\} \vdash Z$ and $S \cup \{V\} \vdash Z$ then $S \cup \{U \vee V\} \vdash Z$;

EXAMPLE 1 OF REASONING MODELING: *PARTY*

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H1: Mary will go to the party if Lucy will go and George will not go.

H2: If John will go to the party then Lucy will go too.

H3: If John is in town he will go to the party.

H4: George is sick and can't go to the party.

H5: Yesterday John has returned in town from Paris.

C(**onclusion**): Will Mary go to the party?

We have to check whether the following deduction holds.

$$H1, H2, H3, H4, H5 \vdash C$$

Notations for the propositional variables:

M – Mary will go to the party

L – Lucy will go to the party

G – George will go to the party

J – John will go to the party

Jt – John is in town

Propositional formulas:

H1: $L \wedge \neg G \to M$

H2: $J \to L$

H3: $Jt \to J$

H4: $\neg G$

H5: Jt

C: M

EXAMPLE 1 - BUILDING THE DEDUCTION

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The definition of the deduction and the axiomatic system are used.

$$U, U \rightarrow V \vdash_{mp} V$$

f1=H1: $L \wedge \neg G \rightarrow M$ (hypothesis)

f2=H2: $J \rightarrow L$ (hypothesis)

f3=H3: $Jt \rightarrow J$ (hypothesis)

f4=H4: $\neg G$ (hypothesis)

f5=H5: Jt (hypothesis)

The deduction (inference) process:

f5,f3 $\vdash_{mp} J$: **f6** *(modus ponens is applied)*

f6,f2 $\vdash_{mp} L$: **f7** *(modus ponens is applied)*

f4,f7 $\vdash L \wedge \neg G$: **f8** *(conjunction in conclusions)*

f8,f1 $\vdash_{mp} M$: **f9** =C *(modus ponens is applied)*

The sequence of formulas: (f1, f2, f3, f4, f5, f6, f7, f8, f9) is the ***deduction*** of C from the hypotheses therefore, based on the hypotheses , **Mary will go to the party.**

EXAMPLE 2

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Prove that: $\neg p \vee q, p \vee r, \neg q \vdash r$ holds using **the definition of deduction.**

We build the sequence (f1,f2,f3, f4, f5,f6,f7) of formulas as follows:

$U, U \to V \vdash_{mp} V$

$U \to V \equiv \neg U \vee V$

f1: $\neg p \vee q \equiv p \to q$

f2: $p \vee r \equiv \neg p \to r$

f3: $\neg q$

f4: $(p \to q) \to (\neg q \to \neg p)$ - axiom A3 (modus tollens)

f1,f4 $\vdash_{mp} \neg q \to \neg p$

f5: $\neg q \to \neg p$

f3,f5 $\vdash_{mp} \neg p$

f6: $\neg p$

f2,f6 $\vdash_{mp} r$

f7: r

f7 = r was proved from the hypotheses (f1,f2,f3) using modus ponens and A3.

PROPERTIES OF PROPOSITIONAL LOGIC

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Soundness theorem (*syntactic validity* implies *semantic validity*):

If $\vdash U$ **then** $\models U$ (A theorem is a tautology).

Completeness theorem (*semantic validity* implies *syntactic validity*):

If $\models U$ **then** $\vdash U$ (A tautology is a theorem).

Theorem of soundness and completeness for propositional logic:

$\vdash U$ **if and only if** $\models U$.

Consequences of this theorem are the following **properties**:

- **Propositional logic is *non-contradictory***: we can't have simultaneously

  $\vdash U$ and $\vdash \neg U$.

- **Propositional logic is *coherent***: not every propositional formula is a theorem.
- **Propositional logic is *decidable***: we can always decide whether a propositional formula is a theorem or not. The truth table method is a decision method.

COMPACTNESS PROPERTY

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Theorem (compactness 1)

An *infinite set* of propositional formulas has a model ***if and only if*** each of its subsets has a *finite* model.

Theorem (compactness 2)

A propositional formula V is a logical consequence of an *infinite set* of propositional formulas S (S $\models$ V)

if and only if

there exists a *finite subset* of S: $\{U_1, U_2, ..., U_n\} \subset S$ such that $U_1, U_2, ..., U_n \models$ V.

COMPACTNESS PROPERTY (CONTD.)

Theorem

Let $S=\{U_1,U_2,\ldots,U_m,\ldots\}$ *be an infinite set of propositional formula.*

1. *S is inconsistent* ***if and only if***

$$\exists k \in \mathrm{N}^*, \text{ such that } \{U_1,U_2,\ldots,U_k\} \text{ is inconsistent.}$$

2. *S is consistent* ***if and only if***

   $\{U_1\}$ *is consistent* ***and***

   $\{U_1,U_2\}$ *is consistent* ***and***

   ...

   $\{U_1,U_2,\ldots,U_m\}$ *is consistent* ***and***

   ...

Remarks:

- An infinite set of propositional formulas is inconsistent *if and only if* it has an inconsistent finite subset. **This can be proved in a finite number of steps.**
- An infinite set of propositional formulas is consistent *if and only if* all its subsets (an infinite number) are consistent. **This can't be proved in a finite number of steps.**

PROPOSITIONAL INFERENCE RULES

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	Inference rule	Theorem
Addition	$U \vdash U \vee V$	$\vdash U \to U \vee V$
Simplification	$U \wedge V \vdash U$ $U \wedge V \vdash V$	$\vdash U \wedge V \to U$ $\vdash U \wedge V \to V$
Modus ponens	$U, U \to V \vdash V$	$\vdash U \wedge (U \to V) \to U$
Modus tollens	$U \to V \vdash \neg V \to \neg U$ $\neg V, U \to V \vdash \neg U$	$\vdash (U \to V) \to (\neg V \to \neg U)$ $\vdash \neg V \wedge (U \to V) \to \neg U$
Syllogism	$U \to V, V \to Z \vdash U \to Z$ $U, U \to V, V \to Z \vdash Z$	$\vdash (U \to V) \wedge (V \to Z) \to (U \to Z)$ $\vdash U \wedge (U \to V) \wedge (V \to Z) \to Z$
Resolution	$U \vee V, \neg U \vee Z \vdash V \vee Z$	$\vdash (U \vee V) \wedge (\neg U \vee Z) \to (V \vee Z)$

WHICH RULE OF INFERENCE IS USED IN EACH ARGUMENT BELOW?

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H. Alice is a Math major. (p) C. Therefore, Alice is either a Math major or a CSI major. ($p \vee q$)	*Addition* $p \vdash p \vee q$
H. Jerry is a Math major and a CSI major ($p \wedge q$) C. Therefore, Jerry is a Math major. (p)	*Simplification* $p \wedge q \vdash p$
H1. If it is rainy, then the pool will be closed. ($p \rightarrow q$) H2. It is rainy. (p) C. Therefore, the pool is closed. (q)	*Modus ponens* $p, p \rightarrow q \vdash q$
H1. If it snows today, the university will close ($p \rightarrow q$) H2. The university is not closed today. ($\neg q$) C. Therefore, it did not snow today. ($\neg p$)	*Modus tollens* $\neg q, p \rightarrow q \vdash \neg p$
H1. If I go swimming, then I will stay in the sun too long ($p \rightarrow q$) H2. If I stay in the sun too long, then I will sunburn. ($q \rightarrow r$) C. Therefore, if I go swimming, then I will sunburn. ($p \rightarrow r$)	*Syllogism* $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$
H1. I go swimming or eat an ice cream ($p \vee q$) H2. I do not go swimming ($\neg p$) C. Therefore, I eat an ice cream (q)	*Resolution* $p \vee q, \neg p \vdash q$

FALLACIES

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Fallacy: mistaken belief based on unsound argument.

- **Fallacy of affirming the conclusion**

H1. If it is Saturday I go to the swimming pool. $(p \rightarrow q)$

H2. I go to the swimming pool. (q)

C. Is it Saturday? (Can we infer p?)

$p \rightarrow q, q \not\vdash p$ **because** $\not\models (p \rightarrow q) \wedge q \rightarrow p$

- **Fallacy of denying the hypothesis**

H1. If it is Saturday I go to the swimming pool. $(p \rightarrow q)$

H2. It is not Saturday. $(\neg p)$

C. I do not go to the swimming pool. (Can we infer $(\neg q)$?)

$p \rightarrow q, \neg p \not\vdash \neg q$ **because** $\not\models (p \rightarrow q) \wedge \neg p \rightarrow \neg q$

FALLACIES (contd.)

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- Fallacy of affirming a disjunct

  H1: I am at home or I am in the city. $(p \vee q)$

  H2: I am at home. (p)

  C: I am not in the city. (Can we infer $(\neg q)$?)

  $p \vee q, p \nvdash \neg q$ **because** $\nvDash (p \vee q) \wedge p \to \neg q$

- Fallacy of denying a conjunct

  H1: I cannot be both at home and in the city. $\neg(p \wedge q)$

  H2: I am not at home. ($\neg p$)

  C: I am in the city. (Can we infer (q)?)

  $\neg(p \wedge q), \neg p \nvdash q$ **because** $\nvDash \neg(p \wedge q) \wedge \neg p \to q$

THEOREM OF DEDUCTION AND ITS REVERSE

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Theorem of deduction:

If $U_1,\ldots,U_{n-1},U_n \vdash V$, **then** $U_1,\ldots,U_{n-1} \vdash U_n \to V$.

Reverse of the theorem of deduction:

If $U_1,\ldots,U_{n-1} \vdash U_n \to V$ **then** $U_1,\ldots,U_{n-1},U_n \vdash V$.

By applying "n" times the theorem of deduction and its reverse we obtain:

$U_1,\ldots,U_{n-1},U_n \vdash V$ **if and only if**

$U_1,\ldots,U_{n-1} \vdash U_n \to V$ **if and only if**

$U_1,\ldots,U_{n-2} \vdash U_{n-1} \to (U_n \to V)$ **if and only if**

...

$U_1 \vdash U_2 \to (..U_{n-1} \to (U_n \to V)...)$ **if and only if**

$\vdash U_1 \to (U_2 \to (\ldots \to (U_n \to V)...))$

1. $\vdash U \to ((U \to V) \to V)$

2. $\vdash (U \to V) \to ((V \to Z) \to (U \to Z))$

3. $\vdash (U \to (V \to Z)) \to (V \to (U \to Z))$ ***permutation of the premises*** law

4. $\vdash (U \to (V \to Z)) \to (V \wedge U \to Z))$ ***reunion of the premises*** law

5. $\vdash (U \wedge V \to Z) \to (U \to (V \to Z))$ ***separation of the premises*** law

Proof

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2. $\vdash (U \to V) \to ((V \to Z) \to (U \to Z))$

We begin with the deduction:

$U, U \to V, V \to Z \vdash Z$ **(syllogism)**

Application of the theorem of deduction ==> $U \to V, V \to Z \vdash U \to Z$

Application of the theorem of deduction ==>

$$U \to V \vdash (V \to Z) \to (U \to Z)$$

Application of the theorem of deduction ==>

$$\vdash (U \to V) \to ((V \to Z) \to (U \to Z))$$

EXAMPLE 4 – THEOREM OF DEDUCTION

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Using the theorem of deduction and its reverse prove:

$$\vdash (p \to r) \to ((p \wedge r \to q) \to (p \to q))$$

Step1: We apply the **reverse of the theorem of deduction** to obtain the initial deduction.

The premise of the main implication is moved from right to the left.

if $\vdash (p \to r) \to ((p \wedge r \to q) \to (p \to q))$ **then**

$p \to r \vdash (p \wedge r \to q) \to (p \to q)$ **then**

$p \to r, p \wedge r \to q \vdash p \to q$ **then**

$p \to r, p \wedge r \to q, p \vdash q$

EXAMPLE 4 (CONTD.)

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Step2: We prove the deduction obtained at Step1:

$$p \to r, p \wedge r \to q, p \vdash q \,,$$

building the sequence of formulas: (f1, f2, f3, f4, f5, f6):

f1: p --- premise (hypothesis)

f2: $p \to r$ --- premise

f1, f2 $\vdash_{mp} r$

f3: r

f4: f1 $\wedge$ f3= $p \wedge r$ (conjunction of the conclusions)

f5: $p \wedge r \to q$ --- premise

f4, f5 $\vdash_{mp} q$

f6: q

The sequence (f1, f2, f3, f4, f5, f6) is the deduction of q from

the premises $p \to r, p \wedge r \to q, p$.

EXAMPLE 4 (CONTD.)

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Step2: We prove the deduction obtained at Step1:

$$p \to r, p \wedge r \to q, p \vdash q \,,$$

building the sequence of formulas: (f1, f2, f3, f4, f5, f6):

f1: p --- premise (hypothesis)

f2: $p \to r$ --- premise

f1, f2 $\vdash_{mp} r$

f3: r

f4: f1 $\wedge$ f3= $p \wedge r$ (conjunction of the conclusions)

f5: $p \wedge r \to q$ --- premise

f4, f5 $\vdash_{mp} q$

f6: q

The sequence (f1, f2, f3, f4, f5, f6) is the deduction of q from the premises $p \to r, p \wedge r \to q, p$.

EXAMPLE 4 (CONTD.)

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Step3: We begin with the deduction $p \to r, p \wedge r \to q, p \vdash q$ proved at **Step2** and we apply three times the theorem of deduction.

There are 3!=6 such possibilities (to move the premises to the right side of the meta-symbol $\vdash$) and we prove 6 theorems: T1, T2, T3, T4, T5, T6.

The premises are moved to the right-hand side of '$\vdash$' in the following order:

$$p,\ p \wedge r \to q,\ p \to r$$

if $p \to r, p \wedge r \to q, p \vdash q$ **then**

$p \to r, p \wedge r \to q \vdash p \to q$ **then**

$p \to r \vdash (p \wedge r \to q) \to (p \to q)$ **then**

$\vdash T1 = (p \to r) \to ((p \wedge r \to q) \to (p \to q))$ --- the theorem to be proved.

The following theorems can be also proved:

$$\vdash T2 = (p \wedge r \to q) \to ((p \to r) \to (p \to q))$$

$$\vdash T3 = (p \wedge r \to q) \to (p \to ((p \to r) \to q))$$

$$\vdash T4 = p \to ((p \wedge r \to q) \to ((p \to r) \to q))$$

$$\vdash T5 = (p \to r) \to (p \to ((p \wedge r \to q) \to q))$$

$$\vdash T6 = p \to ((p \to r) \to ((p \wedge r \to q) \to q))$$

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DECISION PROBLEMS IN PROPOSITIONAL LOGIC

1. Is a propositional formula a tautology/theorem)?

$$\overset{?}{\models} V \qquad \text{or} \qquad \overset{?}{\vdash} V$$

2. Is a propositional formula a logical/syntactic consequence of a set of hypotheses?

$$U_1,\ldots,U_n \overset{?}{\models} V \qquad \text{or} \qquad U_1,\ldots,U_n \overset{?}{\vdash} V$$

In order to solve these two decision problems, **theorem proving methods** are applied.

CLASSIFICATION OF PROOF METHODS

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semantic *versus* syntactic methods	**direct *versus* refutation methods**
semantic methods • the truth table method; • the semantic tableaux method; • the CNF- conjunctive normal form.	***direct methods***: they use directly the formula to be proved • the truth table method; • the CNF- conjunctive normal form; • the definition of deduction; • the theorem of deduction and its reverse; • the sequent calculus method.
syntactic methods • the definition of deduction; • the theorem of deduction and its reverse; • the resolution method; • the sequent calculus method.	***refutation methods***: they model the "reductio ad absurdum" (proof by contradiction) and they use the negation of the formula to be proved • the semantic tableaux method; • the resolution method.