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PROPOSITIONAL LOGIC - SYNTAX -

The ***syntax*** introduces the entities used to define well-formed propositional formulas.

- $\Sigma_P = Var\_propos \cup \{F,T\} \cup Connectives \cup \{(,)\}$ is the ***vocabulary***;
- $Var\_propos = \{p, q, r, ...\}$ is a finite ***set of propositional variables***;
- ***Connectives*** $= \{\neg$ (***negation***), $\wedge$ (***conjunction***), $\vee$ (***disjunction***), $\rightarrow$ (***implication***), $\leftrightarrow$ (***equivalence***)$\}$.

The ***negation*** is a unary connective and all the others are binary connectives.

The decreasing order of precedence of the connectives is as follows:

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow .$$

- F_P is the ***set of well-formed formulas*** built using the propositional variables, the connectives and the parentheses (to avoid ambiguity).

example: $(p \rightarrow \neg q) \wedge (r \vee q \leftrightarrow p) \wedge s$

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Semantics of propositional logic

- **Logical propositions are models of propositional assertions from natural language, which can be true or false.**
- **The aim of the *semantics* is to give a *meaning* (to assign a truth value) to the propositional formulas**
- **The *semantic domain* is the set of *truth values*: $\{F$ (false), T (true)$\}$, satisfy the relations:**

$$\neg F = T, \neg T = F.$$

- **New connectives $\uparrow$ ("*nand*"), $\downarrow$ ("*nor*"), $\oplus$ ("*xor*") are introduced:**

$$p \uparrow q := \neg(p \wedge q), \quad p \downarrow q := \neg(p \vee q), \quad p \oplus q := \neg(p \leftrightarrow q)$$

- **These new connectives are used in the design of logic circuits.**
- **The semantics of the connectives are provided by the following *truth tables*:**

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \uparrow q$	$p \downarrow q$	$p \oplus q$
T	T	F	T	T	T	T	F	F	F
T	F	F	F	T	F	F	T	F	T
F	T	T	F	T	T	F	T	F	T
F	F	T	F	F	T	T	T	T	F

Truth tables

- A conjunction is true exactly when both its operands are true.

  As a generalization, the conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ is true exactly when all its n operands are true.

- A disjunction ("*inclusive or*") is false only when both its operands are false.

  As a generalization, the disjunction $p_1 \vee p_2 \vee \ldots \vee p_n$ is false only when all its n operands are false.

- The implication $p \rightarrow q$ is false only when the hypothesis p is true and the conclusion q is false (*true* cannot imply *false*).

- The equivalence $p \leftrightarrow q$ is true only when p and q have the same truth value.

- The connective $\oplus$ ("*exclusive or*") is the negation of equivalence and it is true only when one operand is true and the other one is false.

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Stylistic variants in English for logical connectives

$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
A and B Both A and B A, but B A, although B A as well as B A, B A, also B	A or B Either A or B A unless B	If A, then B If A, B A is a sufficient condition for B A is sufficient for B In case A, B Provided that A, then B B provided that A B is necessary for A A only if B B if A	A if and only if B A is equivalent to B A is necessary and sufficient for B A just in case B

Interpretation of a propositional formula

Definition

An ***interpretation*** of a formula $U(p_1, p_2, \ldots, p_n) \in F_P$ is a function

$$i : \{p_1, p_2, \ldots, p_n\} \to \{F, T\}$$

that can be extended to $i : F_P \to \{F, T\}$ using the following relations:

$$i(\neg p) = \neg i(p), \qquad i(p \wedge q) = i(p) \wedge i(q), \qquad i(p \vee q) = i(p) \vee i(q)$$

$$i(p \to q) = i(p) \to i(q), \qquad i(p \leftrightarrow q) = i(p) \leftrightarrow i(q)$$

- A formula $U(p_1, p_2, \ldots, p_n) \in F_P$ has $\mathbf{2^n}$ ***interpretations***.
- Interpretations assign truth values to propositional variables and using the semantics of the connectives evaluate formulas assigning truth values to them.
- The semantics is compositional, meaning that the truth value of a formula is obtained from the truth values of its subformulas.

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Semantic concepts

Let $U(p_1, p_2, ..., p_n)$ be a propositional formula.

1. An interpretation ***i*** which evaluates the formula U as *true* is called a ***model*** of U.

   $i:\{p_1,...,p_n\} \to \{T,F\}$ such that i(U)=T.

2. An interpretation ***i*** which evaluates the formula U as *false* is called an ***anti-model*** of U:

   $i:\{p_1,...,p_n\} \to \{T,F\}$ such that i(U)=F.

3. A ***formula*** U is called ***consistent (satisfiable)*** if it has a model:

   $\exists i:\{p_1,...,p_n\} \to \{T,F\}$ such that i(U)=T.

4. The ***formula*** U is called ***valid (tautology)*** and we use the notation: $\models U$, if U is evaluated as true in all interpretations:

   $\forall i:\{p_1,...,p_n\} \to \{T,F\}, i(U) = T.$ All interpretations of U are ***models*** for U.

5. The ***formula*** U is called ***inconsistent*** if U does not have any model,

   U is evaluated as false in all interpretations: $\forall i:\{p_1,...,p_n\} \to \{T,F\}, i(U) = F.$

6. The ***formula*** U is called ***contingent*** if U is consistent, but is not valid.

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Semantic concepts (contd.)

The ***logical consequence*** notion is a generalization of the ***tautology*** notion:

Definition: The formula V is a ***logical consequence*** of the formula *U*,

notation: $U \models V$,

if $\forall i : F_P \rightarrow \{T, F\}$ such that ***i***(U)=T, we have ***i***(V)=T.

Definition: The formulas U and V are ***logically equivalent***, notation: $U \equiv V$,

if they have identical truth tables.

Note:

"$\models$" and "$\equiv$" are ***meta-symbols*** used to express ***logical relations*** between formulas.

Problems in propositional logic

- Check the ***validity*** / ***consistency*** / ***inconsistency*** property of a propositional formula;

- Find the ***models*** and ***anti-models*** of a consistent formula

- Check the ***logical equivalence*** and ***logical consequence*** relations between two propositional formulas

- Check the ***logical consequence*** relation between a ***set of premises (hypotheses****) and a* ***conclusion****.*

Example 1. Build the truth tables of the formulas:

$U(p,q,r) = (\neg p \vee q) \wedge (r \vee p)$, $V(p,q,r) = (\neg p \wedge r) \vee (q \wedge r) \vee (q \wedge p)$,

$W(p,q,r) = (p \uparrow (\neg p \wedge q)) \vee r$, $Z(p,q,r) = p \wedge ((\neg q \vee r) \downarrow q)$.

	p	q	r	$\neg p \vee q$	$r \vee p$	$U(p,q,r)$	$V(p,q,r)$	$W(p,q,r)$	$Z(p,q,r)$
i1	T	T	T	T	T	T	T	T	F
i2	T	T	F	T	T	T	T	T	F
i3	T	F	T	F	T	F	F	T	F
i4	T	F	F	F	T	F	F	T	F
i5	F	T	T	T	T	T	T	T	F
i6	F	T	F	T	F	F	F	T	F
i7	F	F	T	T	T	T	T	T	F
i8	F	F	F	T	F	F	F	T	F

- i1,i2,i5,i7 are ***models*** of U and i3,i4,i6,i8 are ***anti-models*** of U
- $W(p,q,r)$ is a ***tautology***, all its 8 interpretations are also its models
- $Z(p,q,r)$ is ***inconsistent***, it is evaluated as false in all its 8 interpretations
- $U \equiv V$, U and V are ***logically equivalent*** because they have identical truth tables.
- $U \models \neg p \vee q$ (***logical consequence***) because in all interpretations (i1,i2,i5,i7) which evaluate the formula U as true, the formula $\neg p \vee q$ is also evaluated as true.

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Logical equivalences

Simplification laws:	Commutative laws:
$\neg\neg U \equiv U$ and $U \to U \equiv T$	$U \wedge V \equiv V \wedge U$
$U \wedge \neg U \equiv F$ and $U \vee \neg U \equiv T$	$U \vee V \equiv V \vee U$
$T \wedge U \equiv U$ and $F \vee U \equiv U$	
$U \to T \equiv T$ and $U \to F \equiv \neg U$	**Distributive laws:**
$T \to U \equiv U$ and $F \to U \equiv T$	$U \wedge (V \vee Z) \equiv (U \wedge V) \vee (U \wedge Z)$
$U \leftrightarrow T \equiv U$ and $U \leftrightarrow F \equiv \neg U$	$U \vee (V \wedge Z) \equiv (U \vee V) \wedge (U \vee Z)$
$U \oplus T \equiv \neg U$ and $U \oplus F \equiv U$	
$U \leftrightarrow U \equiv T$ and $U \oplus U \equiv F$	

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Logical equivalences (contd.)

Idempotency laws:

$$U \wedge U \equiv U$$
$$U \vee U \equiv U$$

Absorption laws:

$$U \wedge (U \vee V) \equiv U$$
$$U \vee (U \wedge V) \equiv U$$

De Morgan laws:

$$\neg(U \wedge V) \equiv \neg U \vee \neg V$$
$$\neg(U \vee V) \equiv \neg U \wedge \neg V$$

Associative laws:

$$(U \wedge V) \wedge Z \equiv U \wedge (V \wedge Z)$$
$$(U \vee V) \vee Z \equiv U \vee (V \vee Z)$$

Logical equivalences (contd.)

--- Definitions of the connectives ---

$U \to V \equiv \neg U \vee V$	$U \to V \equiv \neg(U \wedge \neg V)$
$U \to V \equiv U \leftrightarrow (U \wedge V)$	$U \to V \equiv V \leftrightarrow (U \vee V)$
$U \leftrightarrow V \equiv (U \to V) \wedge (V \to U)$	$U \oplus V \equiv \neg(U \to V) \vee \neg(V \to U)$
$U \leftrightarrow V \equiv (U \vee V) \to (U \wedge V)$	
$U \vee V \equiv \neg(\neg U \wedge \neg V)$	$U \wedge V \equiv \neg(\neg U \vee \neg V)$
$U \vee V \equiv \neg U \to V$	$U \wedge V \equiv \neg(U \to \neg V)$
$\neg U \equiv U \uparrow U$	$\neg U \equiv U \downarrow U$
$U \vee V \equiv (U \uparrow U) \uparrow (V \uparrow V)$	$U \wedge V \equiv (U \downarrow U) \downarrow (V \downarrow V)$
$U \vee V \equiv (U \downarrow V) \downarrow (U \downarrow V)$	$U \wedge V \equiv (U \uparrow V) \uparrow (U \uparrow V)$

- A set of connectives is **functionally complete** if there is no truth table that can not be expressed as a formula involving only these connectives. All the other connectives can be expressed using the connectives from the set.

- The following sets of connectives are **functionally complete**.

  1. $\{\neg, \wedge\}$; 2. $\{\neg, \vee\}$; 3. $\{\neg, \rightarrow\}$; 4. $\{\uparrow\}$;
  5. $\{\downarrow\}$; 6. $\{\oplus, \wedge\}$; 7. $\{\oplus, \vee\}$; 8. $\{\oplus, \rightarrow\}$;

- **<u>The duality principle</u>**:

  For every logical equivalence $U \equiv V$ containing only the connectives: $\neg, \wedge, \vee, \uparrow, \downarrow, \leftrightarrow, \otimes$ there is another logical equivalence $U' \equiv V'$, where U', V' are formulas obtained from U, V by interchanging the connectives $(\wedge, \vee)$, $(\uparrow, \downarrow)$, $(\leftrightarrow, \otimes)$ and the truth values (T, F).

- Notice that some of the above laws are pairs of ***dual logical equivalences***.

- **Dual connectives**: $(\wedge, \vee)$, $(\uparrow, \downarrow)$, $(\leftrightarrow, \otimes)$.

- **Dual truth values**: (T,F).

Sets of propositional formulas

Definition:

- The set $\{U_1,U_2,\ldots,U_n\}$ is called ***consistent*** if

  the formula $U_1 \wedge U_2 \wedge \ldots \wedge U_n$ is consistent:

  $\exists i : F_P \to \{T, F\}$ such that $i(U_1 \wedge U_2 \wedge \ldots \wedge U_n)=T$.

- The set $\{U_1,U_2,\ldots,U_n\}$ is called ***inconsistent*** if

  the formula $U_1 \wedge U_2 \wedge \ldots \wedge U_n$ is inconsistent:

  $\forall i : F_P \to \{T, F\}, i(U_1 \wedge U_2 \wedge \ldots \wedge U_n) = F.$

- The formula V is a ***logical consequence*** of the set $\{U_1,U_2,\ldots,U_n\}$ of formulas, **notation**: $U_1,U_2,\ldots,U_n \models V$, if

  $\forall i : F_P \to \{T, F\}$ such that $i(U_1 \wedge U_2 \wedge \ldots \wedge U_n)=T$, we have $i(V)=T$.

The formulas $U_1,U_2,\ldots,U_n$ are called ***premises, hypotheses, facts***, and

V is called ***conclusion.***

Theorems (semantic results)

Theorem 1: Let $U_1, U_2, ..., U_n, U, V$ be propositional formulas.

- $\models U$ *if and only if* $\neg U$ is inconsistent.
- $U \models V$ *if and only if* $\models U \rightarrow V$ *if and only if* $\{U, \neg V\}$ is inconsistent.
- $U \equiv V$ *if and only if* $\models U \leftrightarrow V$.
- $U_1, U_2, ..., U_n \models V$ *if and only if* $\models U_1 \wedge U_2 \wedge ... \wedge U_n \rightarrow V$ *if and only if* the set $\{U_1, U_2, ..., U_n, \neg V\}$ is inconsistent.

Theorem 2: Let $S = \{U_1, U_2, ..., U_n\}$ be a set of propositional formulas.

1. *If* S is a consistent set, *then* $\forall j, 1 \leq j \leq n, S - \{U_j\}$ is a consistent set.
2. *If* S is a consistent set *and* V is a valid formula, *then* $S \cup \{V\}$ is consistent.
3. *If* S is an inconsistent set, *then* $\forall V \in F_P$, $S \cup \{V\}$ is inconsistent.
4. *If* S is an inconsistent set *and* U_j is valid, where $1 \leq j \leq n$, *then* $S - \{U_j\}$ is inconsistent.

Example

A client describes the requirements of a software application:

R_1. ***If*** **condition** A **is satisfied** ***then*** **condition** B **must also be satisfied.**

R_2. ***If*** **conditions** B ***and*** C **are satisfied,** ***then*** D **must also be satisfied.**

R_3. ***If*** **condition** D **is satisfied** ***then*** **condition** A **is** ***not*** **satisfied.**

R_4. ***If*** **condition** C **is satisfied** ***then*** A **must also be satisfied.**

R_5. ***If*** A **is satisfied** ***then*** D ***or*** C **are satisfied.**

R_6. C **is satisfied** ***if neither*** B ***nor*** A **are satisfied.**

R_7. B **is** ***not*** **satisfied** ***if*** C **is** ***not*** **satisfied.**

Are these requirements simultaneously satisfiable?

In order to answer the question we have to check the ***consistency/inconsistency*** **of the formula:**

$$U = R_1 \wedge R_2 \wedge R_3 \wedge R_4 \wedge R_5 \wedge R_6 \wedge R_7 .$$

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Example (contd.)

R_1. *If* condition A is satisfied *then* condition B must also be satisfied.

R_2. *If* conditions B *and* C are satisfied, *then* D must also be satisfied.

R_3. *If* condition D is satisfied *then* condition A is *not* satisfied.

R_4. *If* condition C is satisfied *then* A must also be satisfied.

R_5. *If* A is satisfied *then* D *or* C are satisfied.

R_6. C is satisfied *if neither* B *nor* A are satisfied.

R_7. B is *not* satisfied *if* C is *not* satisfied.

$R_1.\ A \rightarrow B \equiv \neg A \vee B$

$R_2.\ B \wedge C \rightarrow D \equiv \neg B \vee \neg C \vee D$

$R_3.\ D \rightarrow \neg A \equiv \neg D \vee \neg A$

$R_4.\ C \rightarrow A \equiv \neg C \vee A$

$R_5.\ A \rightarrow C \vee D \equiv \neg A \vee C \vee D$

$R_6.\ \neg A \wedge \neg B \rightarrow C \equiv A \vee B \vee C$

$R_7.\ \neg C \rightarrow \neg B \equiv C \vee \neg B$

$$U = R_1 \wedge R_2 \wedge R_3 \wedge R_4 \wedge R_5 \wedge R_6 \wedge R_7 .$$

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Example (contd.) – Truth table

	A	B	C	D	R_1	R_2	R_3	R_4	R_5	R_6	R_7	U
i_1	T	T	T	T	T	T	F	T	T	T	T	F
i_2	T	T	T	F	T	F	T	T	T	T	T	F
i_3	T	T	F	T	T	T	F	T	T	T	F	F
i_4	T	T	F	F	T	T	T	T	F	T	F	F
i_5	T	F	T	T	F	T	F	T	T	T	T	F
i_6	T	F	T	F	F	T	T	T	T	T	T	F
i_7	T	F	F	T	F	T	F	T	T	T	T	F
i_8	T	F	F	F	F	T	T	T	F	T	T	F
i_9	F	T	T	T	T	T	T	F	T	T	T	F
i_{10}	F	T	T	F	T	F	T	F	T	T	T	F
i_{11}	F	T	F	T	T	T	T	T	T	T	F	F
i_{12}	F	T	F	F	T	T	T	T	T	T	F	F
i_{13}	F	F	T	T	T	T	T	F	T	T	T	F
i_{14}	F	F	T	F	T	T	T	F	T	T	T	F
i_{15}	F	F	F	T	T	T	T	T	T	F	T	F
i_{16}	F	F	F	F	T	T	T	T	T	F	T	F

The column of U contains only the truth value F, so U is evaluated as false in all 16 interpretations, therefore U is an inconsistent formula.

The conclusion is that *the requirements are contradictory, they cannot be satisfied simultaneously by the software application.*

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Normal forms - *definitions*

A **literal** is a propositional variable or its negation.	$p, \neg q, r$
A **clause** is a **disjunction** of a finite number of literals.	$p, \neg p \vee q, r \vee q \vee s$
A **cube** is a **conjunction** of a finite number of literals.	$q, p \wedge \neg q, r \wedge s \wedge p$
A formula is in **disjunctive normal form** (DNF), if it is written as a disjunction of cubes: $\bigvee_{i=1}^{p} (\bigwedge_{j=1}^{q} l_{ij})$	$p \vee \neg q \vee r$ - 3 unit cubes $p \wedge q$ - DNF, 1 cube $p \vee (q \wedge r) \vee (\neg p \wedge \neg r \wedge s)$ - 3 cubes
A formula is in **conjunctive normal form** (CNF), if it is written as a conjunction of clauses: $\bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m} l_{ij})$	$p \vee \neg q \vee r$ - CNF, 1 clause $p \wedge q$ - 2 unit clauses $p \wedge (q \vee r) \wedge (\neg p \vee \neg r \vee s)$ - 3 clauses

Property

Let $\{ l_1, l_2 ..., l_n \}$ be a set of literals.
The following sentences are equivalent:
a) The clause $\vee_{i=1}^{n} l_i$ is a tautology;
b) The cube $\wedge_{i=1}^{n} l_i$ is inconsistent;
c) The set $\{ l_1, l_2 ..., l_n \}$ of literals contains at least one pair of opposite literals:
$\exists i, j \in \{1,..,n\}$ such that $l_j = \neg l_i$.
	Examples
	➢ the ***clause*** $U = \underline{p} \vee q \vee r \vee \underline{\neg p}$ is a ***tautology*** $(U \equiv T)$, because $p, \neg p$ are opposite literals.
	➢ the ***cube*** $V = \underline{p} \wedge q \wedge r \wedge \underline{\neg p}$ is an ***inconsistent*** formula $(V \equiv F)$, because $p, \neg p$ are opposite literals.

Normalization algorithm

Aim: to transform a formula into another logically equivalent formula, having a certain character of "normal" or "canonical" form.

Transformations which **preserve the logical equivalence** are applied:

Step1: The formulas of " $X \to Y$ " type are replaced by the equivalent form $\neg X \vee Y$.

The formulas of " $X \leftrightarrow Y$ " type are replaced by the equivalent form

$$(\neg X \vee Y) \wedge (\neg Y \vee X) .$$

Step2: De Morgan laws are applied ==> push negations in until they apply only to propositional variables

Multiple negations are eliminated by the reduction rule: $\neg\neg X \equiv X$.

Step3: The distribution laws are applied.

Normal forms – *theoretical results*

Theorem 3:

Every propositional formula admits an equivalent **CNF** and an equivalent **DNF**.

Theorem 4:

1. A formula in **CNF** is a **tautology** if and only if **all its clauses are tautologies**.
2. A formula in **DNF** is **inconsistent** if and only if **all its cubes are inconsistent**.

Remarks:

- The first part of the above theorem provides a **direct method** to prove that a formula is a tautology.
- **DNF** of a propositional formula **provides all the models of that formula**, finding all the interpretations that evaluate the cubes as true.
- **CNF** of a propositional formula **provides all the anti-models of that formula**, finding all the interpretations that evaluate, one by one, the clauses as false.

Dual concepts: clause-cube, DNF-CNF.

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Example

Write the equivalent **DNF** of the formula and find its models.

$$X = (p \wedge q \to r) \to (p \to r) \wedge q$$

We apply the normalization algorithm:

$$X = (p \wedge q \overset{1}{\to} r) \overset{2}{\to} (p \overset{3}{\to} r) \wedge q \qquad \text{(replace } \overset{2}{\to} \text{ using } U \to V \equiv \neg U \vee V \text{)}$$

$$\equiv \neg(p \wedge q \overset{1}{\to} r) \vee (p \overset{3}{\to} r) \wedge q \qquad \text{(replace } \overset{1,3}{\to} \text{ using } U \to V \equiv \neg U \vee V \text{)}$$

$$\equiv \neg(\neg(p \wedge q) \vee r) \vee (\neg p \vee r) \wedge q \qquad \text{(apply de Morgan's law)}$$

$$\equiv (p \wedge q \wedge \neg r) \vee (\neg p \vee r) \wedge q \qquad \text{(apply distribution of } \wedge \text{ over } \vee \text{)}$$

$$\equiv (p \wedge q \wedge \neg r) \vee (\neg p \wedge q) \vee (r \wedge q) \qquad \text{– DNF with 3 cubes}$$

The **models** of X are the interpretations that evaluate one by one the **cubes** of DNF as true.

Example – *models of a formula*

$$\textbf{DNF(X)} = (p \wedge q \wedge \neg r) \vee (\neg p \wedge q) \vee (r \wedge q)$$

Cube: $p \wedge q \wedge \neg r$ provides one model:

i1:{p,q,r}->{T,F}, i1(p)=T, i1(q)=T, i1(r)=F

Cube: $\neg p \wedge q$ provides 2 models:

i2:{p,q,r}->{T,F}, i2(p)=F, i2(q)=T, i2(r)=T

i3:{p,q,r}->{T,F}, i3(p)=F, i3(q)=T, i3(r)=F

Cube: $r \wedge q$ provides 2 models:

i4:{p,q,r}->{T,F}, i4(p)=T, i4(q)=T, i4(r)=T

i5:{p,q,r}->{T,F}, i5(p)=F, i5(q)=T, i5(r)=T

We notice that i2 =i5.

The models of X are the interpretations: i1,i2,i3,i4.

i1(X) = i2(X) = i3(X) = i4(X) = T

All the other four interpretations evaluate the formula X as false,

they are anti-models of X.