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First-order (predicate) Logic

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TRANSFORMATION OF NATURAL LANGUAGE SENTENCES INTO PREDICATE FORMULAS

1. **All** Computer Science (CS) students are smart.
2. **There is** someone who studies at Babes-Bolyai University (BBU) and is smart.
3. If x and y are **nonnegative integers** and x is **greater than** y, then x^2 is **greater than** y^2.
4. In a plane if a line x **is parallel** to a constant line d then **all** the lines **perpendicular** to x are also **perpendicular** to d.
5. **Every** child loves **anyone** who gives the child **any** present.

The axioms that define the natural numbers

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a₁. Every natural number has a unique immediate successor.

existence: $(\forall x)(\exists y)equal(y, successor(x))$

unicity: $(\forall x)(\forall y)(\forall z)(equal(y, successor(x)) \wedge equal(z, successor(x)) \to equal(y, z))$

a₂. The number 0 *is not* the immediate successor of a natural number.

$\neg(\exists x)equal(0, succesor(x))$

a₃. Every natural number, except 0, has a unique immediate predecessor.

existence: $(\forall x)(\exists y)(\neg equal(0, x) \wedge equal(y, predecessor(x)))$

unicity: $(\forall x)(\forall y)(\forall z)(equal(y, predecessor(x)) \wedge equal(z, predecessor(x)) \to equal(y, z))$

unary functions: *successor, predecessor* ;

binary predicate: *equal*.

The predicate ***equal*** is defined by the following axioms:

$(\forall x)equal(x, x)$ – **reflexivity**

$(\forall x)(\forall y)(equal(x, y) \to equal(y, x))$ – **symmetry**

$(\forall x)(\forall y)(\forall z)(equal(x, y) \wedge equal(y, z) \to equal(x, z))$ – **transitivity**

The equality of the successors and the predecessors of two equal numbers:

$(\forall x)(\forall y)(equal(x, y) \to equal(successor(x), successor(y)))$

$(\forall x)(\forall y)(equal(x, y) \to equal(predecessor(x), predecessor(y)))$

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$\mathrm{Pr} = (\Sigma_{\mathrm{Pr}}, F_{\mathrm{Pr}}, A_{\mathrm{Pr}}, R_{\mathrm{Pr}})$

$\Sigma_{\mathrm{Pr}} = Var \cup Const \cup (\bigcup_{j=1}^{n} F_j) \cup (\bigcup_{j=1}^{m} P_j) \cup Connectives \cup Quantifiers$ - ***vocabulary***

- ***Var*** is the set of ***variable*** symbols $\{x, y, z, \ldots\}$;
- ***Const*** is the set of ***constants*** $\{a, b, c, \ldots\}$;
- $F_i = \{f \mid f : D^i \to D\}$ is the set of ***function symbols*** of arity "i"
- $P_i = \{p \mid p : D^i \to \{T, F\}\}$ is the set of ***predicate symbols*** of arity "i"
- ***Connectives*** $= \{\neg, \wedge, \vee, \to, \leftrightarrow\}$;
- ***Quantifiers*** $= \{\forall (universal\ quantifier),\ \exists (existential\ quantifier)\}$
  - ***ATOMS*** is the set of atomic formulas (atoms):
    - $T, F \in ATOMS$
    - if $P \in \mathrm{P}_k$ and $t_1, \ldots, t_k \in TERMS$ then $P(t_1, \ldots, t_k) \in ATOMS$

    examples: $T, F, P(x, y, a), Q(f(x), a), R(a, g(f(x), y)$
  - ***Literal*** **-** an atom or its negation.

    examples: $P(f(x), a, y, g(x, b))$, $\neg Q(x, a, g(x))$

AXIOMATIC SYSTEM (CONTD.)

$$\mathrm{Pr} = (\Sigma_{\mathrm{Pr}}, F_{\mathrm{Pr}}, A_{\mathrm{Pr}}, R_{\mathrm{Pr}})$$

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- F_{Pr} is the ***set of well-formed formulas***
- $A_{\mathrm{Pr}} = \{A_1, A_2, A_3, A_4, A_5\}$ is the ***set of axioms***
  - $A_1 : U \to (V \to U)$
  - $A_2 : ((U \to (V \to Z)) \to ((U \to V) \to (U \to Z))$
  - $A_3 : (U \to V) \to (\neg V \to \neg U)$ (***modus tollens***)
  - $A_4 : (\forall x)U(x) \to U(t)$, where t is a term. (***universal instantiation***)
  - $A_5 : (U \to V(y)) \to (U \to (\forall x)V(x))$, where x is not free in U or V, y is free in V and does not appear in U.
- $R_{\mathrm{Pr}} = \{mp, gen\}$ is the ***set of inference rules***:
  - ***modus ponens*** symbolized as: $U, U \to V \vdash_{mp} V$
  - ***universal generalization rule*** symbolized as: $U(x) \vdash_{gen} (\forall x)U(x)$, x is a free variable in U

OPEN VERSUS CLOSED FORMULAS

Definition:

1. In a predicate formula the variables which are within the scope of a quantifier are called ***bound variables***, all the others are called ***free variables***.
2. A formula is called a ***closed formula*** if all its variables are bound.
3. If a formula contains at least one free variable, the ***formula is open***.

Example:

1. The predicate *formula* $(\forall x)(\exists z)(P(x,z,a) \vee (\exists y)Q(x,f(y)))$ is *closed*
   - where: $x,y,z \in Var$, $a \in Const$, $f \in F_1, P \in \mathrm{P}_3, Q \in \mathrm{P}_2$.
   - *all variables (x,y,z) are bound.*

2. The predicate *formula* $(\forall x)P(x,y) \wedge Q(z,a)$ is *open*,
   - where: $a \in Const$, $x,y,z \in Var$, $P,Q \in \mathrm{P}_2$
   - the *variables* y and z are *free*,
   - x is a *bound variable* (within the scope of $\forall$)

DEDUCTION IN FIRST-ORDER LOGIC

Definition:

Let $U_1, U_2, \ldots, U_n, V$ be first-order formulas. $U_1, U_2, \ldots, U_n$ are the ***hypotheses* (*premises*)** and V is the ***conclusion***. ***V is deducible (inferable, derivable) from*** $U_1, U_2, \ldots, U_n$, notation: $U_1, U_2, \ldots, U_n \vdash V$, if there exists a sequence of formulas $(f_1, f_2, \ldots, f_m)$ such that $f_m = V$ and $\forall i \in \{1, \ldots, m\}$ we have a) or *b*) or *c*) or *d*).

a) $f_i \in A_{\mathrm{Pr}}$ (*axiom* of predicate logic);

b) $f_i \in \{U_1, U_2, \ldots, U_n\}$ (*hypothesis* formula);

c) $f_{i_1}, f_{i_2} \vdash_{mp} f_i$, $i_1 < i$ *and* $i_2 < i$ (formula f_i is inferred, using *modus ponens* rule, from two formulas that exist in the sequence);

d) $f_j \vdash_{gen} f_i$, $j < i$ (formula f_i is obtained using the *universal generalization* rule from a formula that exists already in the sequence).

The sequence $(f_1, f_2, \ldots, f_m)$ is called the ***deduction*** of V from $U_1, U_2, \ldots, U_n$.

Definition:

A formula $U \in F_{\mathrm{Pr}}$, such that $\varnothing \vdash U$ (notation: $\vdash U$) is called a ***theorem***.

Remark: The theorems are the formulas deducible from the axioms, using modus ponens and the generalization rule.

INFERENCE RULES IN FIRST-ORDER LOGIC

	Inference rule
universal instantiation	$(\forall x)U(x) \vdash_{univ\_inst} U(t)$, *t* is a *term* (*variable* or *constant* of the domain)
universal generalization	$U(x) \vdash_{univ\_gen} (\forall x)U(x)$, *x* is a *free variable* in *U*
existential instantiation	$(\exists x)U(x) \vdash_{exist\_inst} U(c)$, *c* is a *new constant* of the domain
existential generalization	$U(t) \vdash_{exist\_gen} (\exists x)U(x)$, *t* is a *variable* or a *constant* of the domain, the variable *x* must not appear free in *U*

EXAMPLE 1 – MODELING REASONING

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Hypotheses:

H_1. All hummingbirds are richly colored.

H_2. No large birds live on honey.

H_3. Birds that do not live on honey are dull in color.

H_4. *Piky* is a hummingbird.

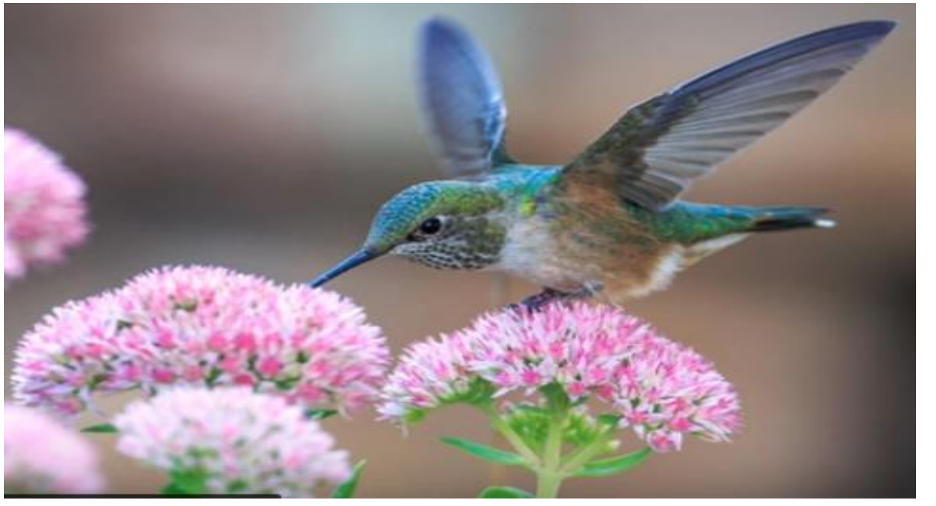
Conclusions:

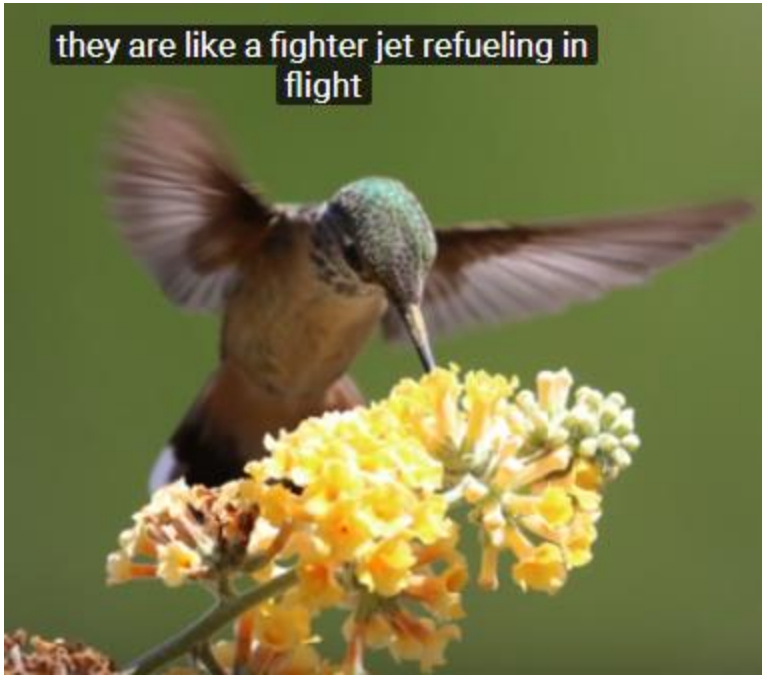
C_1. There is a bird which lives on honey.

C_2. All hummingbirds are small.

Check whether the following deductions hold or not.

$$H_1, H_2, H_3, H_4 \vdash C_1 \text{ and } H_1, H_2, H_3 \vdash C_2.$$





EXAMPLE 1 –MODELING REASONING (contd.)

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***Hypotheses*:**

H_1. All hummingbirds are richly colored.

H_2. No large birds live on honey.

H_3. Birds that do not live on honey are dull in color.

H_4. *Piky* is a hummingbird.

***Conclusions*:**

C_1. There is a bird which lives on honey.

C_2. All hummingbirds are small.

Check whether the following deductions hold or not.

$$H_1, H_2, H_3, H_4 \vdash C_1 \text{ and } H_1, H_2, H_3 \vdash C_2 .$$

First-order (predicate) formulas:

$$H_1 : (\forall x)(hb(x) \rightarrow rc(x))$$

$$H_2 : \neg(\exists x)(\neg sb(x) \wedge lh(x)) \equiv$$
$$\equiv (\forall x)(\neg sb(x) \rightarrow \neg lh(x))$$

$$H_3 : (\forall x)(\neg lh(x) \rightarrow \neg rc(x))$$

$$H_4 : hb(Piky)$$

$$C_1 : (\exists x) lh(x)$$

$$C_2 : (\forall x)(hb(x) \rightarrow sb(x))$$

D is the ***domain*** (the universe of birds)

Piky is a ***constant*** of the universe.

unary ***predicate symbols***: *hb, rc, sb, lh.*

$hb : D \rightarrow \{T, F\}$, $hb(x) = T$ if x is a *hummingbird*,

$rc : D \rightarrow \{T, F\}$, $rc(x) = T$ if x is *richly colored*,

$sb : D \rightarrow \{T, F\}$, $sb(x) = T$ if x is a *small bird*,

$lh : D \rightarrow \{T, F\}$, $lh(x) = T$ if x *lives on honey*.

EXAMPLE 1 – BUILDING DEDUCTIONS (LECTURE)

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Inference rules
$(\forall x)U(x) \vdash_{univ\_inst} U(t)$, t is a *term* (*variable* or *constant* of the domain)
$U(x) \vdash_{univ\_gen} (\forall x)U(x)$, x is a *free variable* in U
$(\exists x)U(x) \vdash_{exist\_inst} U(c)$, c is a *new constant* of the domain
$U(t) \vdash_{exist\_gen} (\exists x)U(x)$, t is a *variable* or a *constant* of the domain
$U, U \to V \vdash_{mp} V$ (*modus ponens*)
$U \to V, V \to Z \vdash_{\text{syllogism}} U \to Z$
$U \to V \vdash_{mt} \neg V \to \neg U$ (*modus tollens*) $\neg V, U \to V \vdash_{mt} \neg U$

First-order (predicate) formulas:

$H_1 : (\forall x)(hb(x) \to rc(x))$

$H_2 : \neg(\exists x)(\neg sb(x) \wedge lh(x)) \equiv$
$\equiv (\forall x)(\neg sb(x) \to \neg lh(x))$

$H_3 : (\forall x)(\neg lh(x) \to \neg rc(x))$

$H_4 : hb(Piky)$

$C_1 : (\exists x)lh(x)$

$C_2 : (\forall x)(hb(x) \to sb(x))$

Example 2

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Using the definition of deduction and the inference rules prove that:

$$(\forall x)(\forall y)(P(x) \vee Q(y)), \neg(\forall z)P(z) \vdash (\exists t)Q(t)$$

The sequence $(f_1, f_2, f_3, f_4, f_5, f_6, f_7)$ of predicate formulas is generated.

$f_1 : (\forall x)(\forall y)(P(x) \vee Q(y))$ – hypothesis

$f_2 : \neg(\forall z)P(z) \equiv (\exists z)\neg P(z)$ – hypothesis

We have a universal formula (f_1) and an existential formula (f_2). It is recommended to use first the *existential instantiation* inference rule that introduces a new constant, and then this constant will be used for *universal instantiations*.

$f_2 \vdash_{exist\_inst} f_3 = \neg P(c)$, c is a new constant;

$f_1 \vdash_{univ\_inst} f_4 = (\forall y)(P(c) \vee Q(y))$,

the outermost universally quantified variable x was instantiated using the constant c;

$f_4 \vdash_{univ\_inst} f_5 = P(c) \vee Q(c)$,

the universally quantified variable y was instantiated using the constant c;

$f_5 \equiv \neg P(c) \to Q(c)$

$f_3, f_5 \vdash_{mp} f_6 = Q(c)$, *modus* ponens was applied;

$f_6 \vdash_{exist\_gen} f_7 = (\exists t)Q(t)$,

existential generalization was applied, t is a variable introduced by this inference rule;

$(f_1, f_2, f_3, f_4, f_5, f_6, f_7)$ is the deduction (proof) of the conclusion f_7 from the hypotheses f_1 and f_2.

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EXAMPLE 3: SUCCESSION TO THE BRITISH THRONE

Hypotheses:

H1: If x is the king and y is his oldest son, then y can become the king.

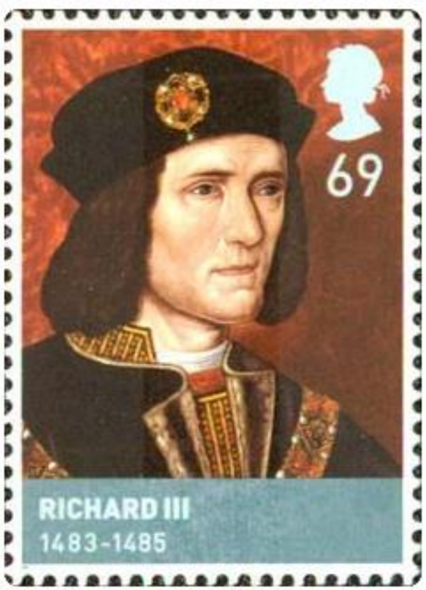
H2: If x is the king and y defeats x, then y will become the king

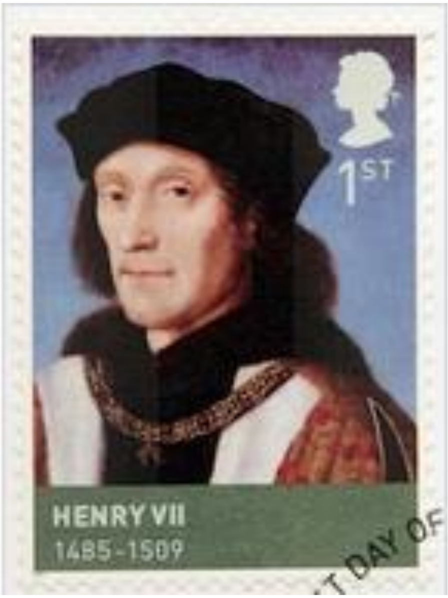
H3: *RichardIII* is the king.

H4: *HenryVII* defeated *RichardIII*.

H5: *HenryVIII* is *HenryVII*'s oldest son.

Conclusion C: Can *HenryVIII* become the king?





<u>EXAMPLE 3</u> (HOMEWORK)

SUCCESSION TO THE BRITISH THRONE

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Hypotheses:

H1: If x is the king and y is his oldest son, then y can become the king.

H2: If x is the king and y defeats x, then y will become the king

H3: *RichardIII* is the king.

H4: *HenryVII* defeated *RichardIII*.

H5: *HenryVIII* is *HenryVII*'s oldest son.

Conclusion C: Can *HenryVIII* become the king?

We transform the hypotheses and the conclusion into predicate formulas using:

variables: x,y,z,t **constants**: *RichardIII, HenryVII, HenryVIII*

predicate symbols: <u>unary</u>: *king*, <u>binary</u>: *oldest_son, defeats*

H1: $(\forall x)(\forall y)\ (king(x) \wedge oldest\_son(y,x) \rightarrow king(y))$

H2: $(\forall z)(\forall t)\ (king(z) \wedge defeats(t,z) \rightarrow king(t))$

H3: $king(RichardIII)$

H4: $defeats(HenryVII, RichardIII)$

H5: $oldest\_son(HenryVIII, HenryVII)$

C: $king(HenryVIII)$

Prove the **deduction** H1,H2,H3,H4,H5 $\vdash$ C.

Theorems

Theorem of deduction:

If $U_1,\ldots,U_{n-1},U_n \vdash V$ **, then** $U_1,\ldots,U_{n-1} \vdash U_n \to V$.

Reverse of the theorem of deduction:

If $U_1,\ldots,U_{n-1} \vdash U_n \to V$ **then** $U_1,\ldots,U_{n-1},U_n \vdash V$.

Refutation theorem:

If $U_1,\ldots,U_{n-1},U_n \cup \{\neg V\}$ is inconsistent **then** $U_1,\ldots,U_{n-1},U_n \vdash V$.

Note: The last theorem models "*reductio ad absurdum principle*" (proof by contradiction) and is used in proof methods such as:

- *resolution*,
- *semantic tableaux method*,

called ***refutation proof methods***.

Example 4: Using the theorem of deduction, prove that the formula

$$(\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\forall x)B(x)) \text{ is a theorem.}$$

Step1:

We apply the reverse of the theorem of deduction:

if $\vdash (\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\forall x)B(x))$

then $(\forall x)(A(x) \to B(x)) \vdash (\forall x)A(x) \to (\forall x)B(x)$

then $(\forall x)(A(x) \to B(x)), (\forall x)A(x) \vdash (\forall x)B(x)$ (*)

Step2: We prove

$(\forall x)(A(x) \to B(x)), (\forall x)A(x) \vdash (\forall x)B(x)$ (*)

using the definition of a deduction.

The sequence (f1, f2,…, f8) is generated:

f1: $(\forall x)(A(x) \to B(x))$ - hypothesis

f2: $(\forall x)(A(x) \to B(x)) \to (A(y) \to B(y))$

-- axiom A4, x is instantiated with y

f1, f2 $\vdash_{mp}$ f3 = $A(y) \to B(y)$

f4: $(\forall x)A(x)$ - hypothesis

f5: $(\forall x)A(x) \to A(y)$ --- axiom A4

f4, f5 $\vdash_{mp}$ f6 = $A(y)$

f3, f6 $\vdash_{mp}$ f7 = $B(y)$

f7 $\vdash_{gen}$ f8 = $(\forall x)B(x)$

Step3:

Using the deduction (*) proved at Step 2 we apply twice the theorem of deduction:

if $(\forall x)(A(x) \to B(x)), (\forall x)A(x) \vdash (\forall x)B(x)$ then

$(\forall x)(A(x) \to B(x)) \vdash (\forall x)A(x) \to (\forall x)B(x)$ then

$\vdash (\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\forall x)B(x))$

We have proved that the initial formula is a theorem.

- The semantics of predicate logic realize the connection between the *constant symbols*, the *function symbols*, the *predicate symbols* and the *real constants, functions, predicates from the modeled universe.*
- It is provided a meaning in terms of the modeled universe for each formula from the language.

Definition:

An ***interpretation*** of predicate formula is a pair $I=<D,m>$, where:

- D is a nonempty set called the domain of interpretation.
- m is a function that assigns:
  - a fixed value $m(c)\in D$ to the constant c.
  - a function $m(f):D^n \rightarrow D$ to each n-ary function symbol f;
  - a predicate $m(P):D^n \rightarrow \{T,F\}$ to each n-ary predicate symbol P.

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Notations: Let $I = <D, m>$ be an interpretation.

- $|I| = D$ is the domain of I, Var is the set of variables.
- $I|X|$ is $m(X)$ where X is a predicate symbol or a function symbol.
- $A_S(I)$ is the set of assignment functions for variables over the domain of I.

  $a \in As(I), a: Var \to |I|$.
- $[a]_x = \{a' \mid a' \in As(I) \text{ and } a'(y) = a(y), \text{ for every } y \neq x\}$.

Definition: Let I be an interpretation and $a \in As(I)$. The evaluation function v_a^I is defined:

- $v_a^I(x) = a(x), x \in Var$; $\qquad v_a^I(c) = I|c|, c \in Const$;
- $v_a^I(f(t_1, t_2, \ldots, t_n) = I|f|(v_a^I(t_1), v_a^I(t_2), \ldots, v_a^I(t_n)), f \in F_n, n > 0$;
- $v_a^I(P(t_1, t_2, \ldots, t_n) = I|P|(v_a^I(t_1), v_a^I(t_2), \ldots, v_a^I(t_n)), P \in \mathrm{P}_n, n > 0$;
- $v_a^I(\neg A) = \neg v_a^I(A)$; $\qquad v_a^I(A \wedge B) = v_a^I(A) \wedge v_a^I(B)$;
- $v_a^I(A \vee B) = v_a^I(A) \vee v_a^I(B)$; $\qquad v_a^I(A \to B) = v_a^I(A) \to v_a^I(B)$;
- $v_a^I((\exists x)A(x)) = T$ if and only if $v_{a'}^I(A(x)) = T$ for a function $a' \in [a]_x$
- $v_a^I((\forall x)A(x)) = T$ if and only if $v_{a'}^I(A(x)) = T$ for any function $a' \in [a]_x$

Note: The evaluation of a closed formula U depends only on the interpretation, notation: $v^I(U)$

Definitions (semantic concepts)

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- A ***formula*** A is ***satisfiable (consistent)*** if there is an interpretation I and an assignment function $a \in As(I)$ such that $v_a^I(A) = T$. Otherwise the formula is ***unsatisfiable (inconsistent)***.
- A formula A ***is true under the interpretation*** I if for any assignment function $a \in As(I)$, $v_a^I(A) = T$, notation: $\models_I A$, and I is called ***model*** of A.
- A formula A ***is false under the interpretation*** I if for any assignment function $a \in As(I)$, $v_a^I(A) = F$, and I is called ***anti-model*** of A.
- A ***formula*** A is ***valid (tautology)*** if A is true under all possible interpretations, notation: $\models A$.
- The ***formulas*** A and B are ***logically equivalent*** if $v_a^I(A) = v_a^I(B)$ for any interpretation I and any assignment function a, notation: $A \equiv B$.
- A set of formulas S ***logically implies*** the formula V if all the models of the set S are also models of the formula V. We say that V is a ***logical consequence*** of the set S, notation: $S \models V$.
- A ***set*** of formulas is ***consistent*** if the conjunction of all its formulas has at least one model.
  A ***set*** of formulas is ***inconsistent*** if the conjunction of all its formulas does not have a model.

Example 5

Build a **model** and an **anti-model** for the closed predicate formula:

$$U = (\forall x)(P(x) \vee Q(x)) \rightarrow (\forall x)P(x) \vee (\forall x)Q(x).$$

Let us consider the **interpretation** $I_1 = < D_1, m >$, where:

$D_1 = \mathbf{N}$ (the set of natural numbers)

$m(P) : \mathbf{N} \rightarrow \{T, F\}, m(P)(x) = "x \vdots 2"$,

$m(Q) : \mathbf{N} \rightarrow \{T, F\}, m(Q)(x) = "x \vdots 3"$.

$$v^{I_1}(U) = v^{I_1}((\forall x)(P(x) \vee Q(x))) \rightarrow v^{I_1}((\forall x)P(x) \vee (\forall x)Q(x))$$

$$= v^{I_1}((\forall x)(P(x) \vee Q(x))) \rightarrow v^{I_1}((\forall x)P(x)) \vee v^{I_1}((\forall x)Q(x))$$

$$= (\forall x)_{x \in \mathrm{N}}(x \vdots 2 \ \vee \ x \vdots 3) \rightarrow (\forall x)_{x \in \mathrm{N}}(x \vdots 2) \vee (\forall x)_{x \in \mathrm{N}}(x \vdots 3)$$

$$= F \rightarrow F \vee F = F \rightarrow F = T$$

$v^{I_1}(U) = T$, U is evaluated as true under the interpretation I_1 which is a **model** of U.

Example 5 (contd.)

$$U = (\forall x)(P(x) \vee Q(x)) \to (\forall x)P(x) \vee (\forall x)Q(x)$$

Let us consider the interpretation $I_2 = < D_2, m >$, where:

$D_2 = \{4, 9\}$ – the domain of interpretation;

$m(P): \{4, 9\} \to \{T, F\}, m(P)(x) = "x \vdots 2"$

$m(Q): \{4, 9\} \to \{T, F\}, m(Q)(x) = "x \vdots 3"$.

To evaluate the formula U under the interpretation I_2, with the finite domain $D_2 = \{4, 9\}$, the universally quantified subformulas are replaced by the conjunction of their instances for $x = 4$ and $x = 9$.

$$\begin{aligned}
v^{I_2}(U) &= v^{I_2}((\forall x)(P(x) \vee Q(x))) \to v^{I_2}((\forall x)P(x) \vee (\forall x)Q(x)) \\
&= v^{I_2}((\forall x)(P(x) \vee Q(x))) \to v^{I_2}((\forall x)P(x)) \vee v^{I_2}((\forall x)Q(x)) \\
&= (4 \vdots 2 \ \vee \ 4 \vdots 3) \wedge (9 \vdots 2 \ \vee \ 9 \vdots 3) \to (4 \vdots 2 \wedge 9 \vdots 2) \vee (4 \vdots 3 \wedge 9 \vdots 3) \\
&= (T \vee F) \wedge (F \vee T) \to (T \wedge F) \vee (F \wedge T) \\
&= T \wedge T \to F \vee F = T \to F = F
\end{aligned}$$

I_2 evaluates the formula U as false, I_2 is an anti-model of U.

LOGICAL EQUIVALENCES IN PREDICATE LOGIC

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- ***Expansion laws***

$(\forall x)A(x) \equiv (\forall x)A(x) \wedge A(t)$

– the universal quantifier is an infinitary conjunction

$(\exists x)A(x) \equiv (\exists x)A(x) \vee A(t)$

– the existential quantifier is a infinitary disjunction

- ***DeMorgan infinitary laws***

$\neg(\exists x)A(x) \equiv (\forall x)\neg A(x)$ and $\neg(\forall x)A(x) \equiv (\exists x)\neg A(x)$

- ***Quantifiers interchanging laws***

$(\exists x)(\exists y)A(x,y) \equiv (\exists y)(\exists x)A(x,y)$ and $(\forall x)(\forall y)A(x,y) \equiv (\forall y)(\forall x)A(x,y)$

Note: $(\exists x)(\forall y)B(x,y) \not\equiv (\forall y)(\exists x)B(x,y)$

Quantifiers of the same type commute, but quantifiers of different types do not commute.

Distributive laws

$(\exists x)(A(x)\vee B(x))\equiv(\exists x)A(x)\vee(\exists x)B(x)$ distributivity of '$\exists$' over '$\vee$'

$(\forall x)(A(x)\wedge B(x))\equiv(\forall x)A(x)\wedge(\forall x)B(x)$ distributivity of '$\forall$' over '$\wedge$'

Semi-distributivity of '$\exists$' over '$\wedge$':

$\models(\exists x)(A(x)\wedge B(x))\rightarrow(\exists x)A(x)\wedge(\exists x)B(x)$

$\not\models(\exists x)A(x)\wedge(\exists x)B(x)\rightarrow(\exists x)(A(x)\wedge B(x))$

Semi-distributivity of '$\forall$' over '$\vee$':

$\models(\forall x)A(x)\vee(\forall x)B(x)\rightarrow(\forall x)(A(x)\vee B(x))$

$\not\models(\forall x)(A(x)\vee B(x))\rightarrow(\forall x)A(x)\vee(\forall x)B(x))$

Semi-distributivity of '$\exists$' over '$\rightarrow$':

$\models((\exists x)A(x)\rightarrow(\exists x)B(x))\rightarrow(\exists x)(A(x)\rightarrow B(x)$

$\not\models(\exists x)(A(x)\rightarrow B(x))\rightarrow((\exists x)A(x)\rightarrow(\exists x)B(x))$

Semi-distributivity of '$\forall$' over '$\rightarrow$':

$\models(\forall x)(A(x)\rightarrow B(x))\rightarrow((\forall x)A(x)\rightarrow(\forall x)B(x))$

$\not\models((\forall x)A(x)\rightarrow(\forall x)B(x))\rightarrow(\forall x)(A(x)\rightarrow B(x))$

EXTRACTION OF QUANTIFIERS

$$A \vee (\exists x)B(x) \equiv (\exists x)(A \vee B(x)) \qquad A \vee (\forall x)B(x) \equiv (\forall x)(A \vee B(x))$$

$$A \wedge (\exists x)B(x) \equiv (\exists x)(A \wedge B(x)) \qquad A \wedge (\forall x)B(x) \equiv (\forall x)(A \wedge B(x))$$

where A does not contain x as a free variable.

$$(\exists x)A(x) \vee B \equiv (\exists x)(A(x) \vee B) \qquad (\forall x)A(x) \vee B \equiv (\forall x)(A(x) \vee B)$$

$$(\exists x)A(x) \wedge B \equiv (\exists x)(A(x) \wedge B) \qquad (\forall x)A(x) \wedge B \equiv (\forall x)(A(x) \wedge B)$$

where B does not contain x as a free variable.

Example 6

M.Lupea

Prove that the *universal* and *existential* quantifiers do not commute.

We have to prove that the logical equivalence:

$(\exists x)(\forall y)L(x,y) \equiv (\forall y)(\exists x)L(x,y)$ does not hold.

We choose the interpretation $I =< D, m >$, where:

- D is the set of all persons in the world
- $m(L): D \times D \to \{T, F\}, m(L)(x,y) = ''x\ loves\ y''$

Under the interpretation I, the formula $U_1 = (\exists x)(\forall y)L(x,y)$ has the meaning:

"*There exists a person who loves all persons.*"

The formula $U_2 = (\forall y)(\exists x)L(x,y)$ has the meaning:

"*All persons are loved by at least one person.*", under the same interpretation I.

These two natural language statements are not equivalent, so $U_1 \not\equiv U_2$,

but note that $U_1 \models U_2$.

PROPERTIES OF FIRST-ORDER LOGIC:

M.Lupea

SOUNDNESS, COMPLETENESS, SEMI-DECIDABILITY

Theorem of soundness and completeness states the equivalence between the "logical consequence" concept and the "syntactic consequence" concept.

Let $U_1,\ldots,U_{n-1},U_n,V$ be first-order formulas.

- ***completeness***: if $U_1,\ldots,U_{n-1},U_n \models V$ then $U_1,\ldots,U_{n-1},U_n \vdash V$.
- ***soundness***: if $U_1,\ldots,U_{n-1},U_n \vdash V$ then $U_1,\ldots,U_{n-1},U_n \models V$.

A particular case of this theorem is the following result:

"A formula is a tautology if and only if **it is a theorem in first-order logic."**

Theorem (Church 1936):

The problem of validity of a first-order formula is ***undecidable***, but is ***semi-decidable***.

If a procedure ***Proc*** is used to check the validity of a first-order formula ***U*** we have the following cases:

- **if** *U* is a valid formula, **then** *Proc* ends with the corresponding answer.
- **if** the formula *U* is not valid, **then** *Proc* ends with the corresponding answer **or** *Proc* may never stop.