Bledea Mihaela Alexandra

1.2.1.

a) $x'' + t^2 \cdot x = 0, \ x(0) = 0$

There are linearly independent solutions (x_1 and x_2) for the second order LHDE and the general solution is:

$$x = c_1 \cdot x_1 + c_2 \cdot x_2 \ , \ c_1, c_2 \in \mathbb{R}$$

a) $\begin{cases} x'' + t^2 \cdot x = 0 \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

By the existence and uniqueness theorem for the Initial Value Problem, the problem has a unique solution.

c) $x'' + t^2 \cdot x = 0$

$x(0) = 0$

$x'(0) = 0$

$x''(0) = 1$

Lets take $t_0 = 0 \Rightarrow \left. \begin{array}{l} x''(0) + 0 \cdot x(0) = 0 \\ x(0) = 0 \end{array} \right\} \Rightarrow x''(0) = 0 \Leftrightarrow 1 = 0$ (False) $\Rightarrow$

$\Rightarrow$ there is no solution for the equation

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1.2.5.

a) $x_p = a\cdot e^{t}$ $(a \in \mathbb{R})$

$\left.\begin{array}{l} x' - 2x = e^{t} \\ x_p' = a\cdot e^{t} \end{array}\right\} \Rightarrow a\cdot e^{t} - 2\cdot a\cdot e^{t} = e^{t} \Leftrightarrow -a\cdot e^{t} = e^{t} \Rightarrow a = -1 \Rightarrow$

$\Rightarrow$ A particular solution is $x_p = -e^{t}$

b) $x_p = b\cdot e^{-t}$

$\left.\begin{array}{l} x' - 2x = e^{-t} \\ x_p' = -b\cdot e^{-t} \end{array}\right\} \Rightarrow -b\cdot e^{-t} - 2\cdot b\cdot e^{-t} = e^{-t} \Leftrightarrow -3b\cdot e^{-t} = e^{-t} \Rightarrow$

$\Rightarrow b = -\frac{1}{3} \Rightarrow$ A particular solution is: $x_p = -\frac{1}{3}\cdot e^{-t}$

c) We denote $x' - 2x = \mathcal{L}(x)$ and $f = 5\cdot e^{t} - 3\cdot e^{-t}$

$f = 5\cdot f_1 - 3\cdot f_2$

a) $\Rightarrow \mathcal{L}(x) = f_1$ has $x_{p_1} = -e^{t}$

b) $\Rightarrow \mathcal{L}(x) = f_2$ has $x_{p_2} = -\frac{1}{3}e^{-t}$

$\Rightarrow$

$\Rightarrow$ From the Superposition Principle

$x_p = 5\cdot x_{p_1} - 3\cdot x_{p_2}$ is a particular solution of $\mathcal{L}(x) = f \Leftrightarrow$

$\Leftrightarrow x_p = -5\cdot e^{t} + e^{-t}$ is a particular solution of $\mathcal{L}(x) = f$

d) $x' - 2x = 5\cdot e^{t} - 3\cdot e^{-t} \,\Big|\cdot e^{-2t} \Leftrightarrow$

$x'\cdot e^{-2t} - 2x\cdot e^{-2t} = 5\cdot e^{-t} - 3\cdot e^{-3t} \Leftrightarrow$

$x'\cdot e^{-2t} + x\cdot(e^{-2t})' = 5\cdot e^{-t} - 3\cdot e^{-3t} \Leftrightarrow$

$(x\cdot e^{-2t})' = 5\cdot e^{-t} - 3\cdot e^{-3t} \,\Big|\int dt$

$x\cdot e^{-2t} = -5\cdot e^{-t} + 3e^{-3t} + C \,\Big|\cdot e^{2t}$

$x = -5\cdot e^{t} + 3\cdot e^{-t} + e^{2t}\cdot C$, $C \in \mathbb{R}$

$\hookrightarrow$ the general solution

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1.3.4.

Method 1: $x' - x = e^{t-1} \;|\cdot e^{-t}$

$x' \cdot e^{-t} - x \cdot e^{-t} = e^{-1}$

$(x \cdot e^{-t})' = \frac{1}{e} \;\Big| \int \Leftrightarrow \quad x \cdot e^{-t} = \int \frac{1}{e}\, dt \Leftrightarrow$

$x \cdot e^{-t} = \frac{t}{e} + c \;\Big|\cdot \frac{1}{e^{-t}} \Leftrightarrow \quad x = \frac{t}{e} \cdot e^{t} + c \cdot e^{t} \Rightarrow$

$x(t) = \frac{t}{e} \cdot e^{t} + c \cdot e^{t} \Rightarrow$ the general solution is:

$x(t) = t \cdot e^{t-1} + c \cdot e^{t}$

Method 2:

Step 1: Associated LHDE: $x' - x = 0$

Characteristic equation: $\lambda - 1 = 0 \Leftrightarrow \lambda = 1 \Rightarrow x_h = c \cdot e^{t}$

Step 2:

Let $f(t) = e^{t-1} \Rightarrow x_p$ is a particular solution of the form: $x_p = at\, e^{t-1}$

$x_p' = a \cdot t \cdot e^{t-1} + a \cdot e^{t-1}$

$x_p' = a \cdot e^{t-1}(t+1)$

$x_p' - x_p = e^{t-1} \Leftrightarrow a \cdot e^{t-1}(t+1) - a \cdot e^{t-1} \cdot t = e^{t-1} \Leftrightarrow$

$a \cdot e^{t-1} = e^{t-1} \Leftrightarrow a = 1 \Rightarrow$

$\Rightarrow x_p = t \cdot e^{t-1}$

$x = x_h + x_p \Rightarrow x = t \cdot e^{t-1} + c \cdot e^{t}$

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1.6.1. Method 1: Reduction to a second order diff. equation

j) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ -> it is an uncoupled system

$$\begin{cases} x_1' = 2x_1 \\ x_2' = x_1 + 2x_2 \end{cases}$$

$x_2'' = x_1' + 2x_2' = 2x_1 + 2x_2' \Leftrightarrow 2x_1 = x_2'' - 2x_2' \Leftrightarrow$

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2} x_2'' - x_2' \\ x_1 &= x_2' - 2x_2 \end{aligned}\right\} \Rightarrow x_2' - 2x_2 = \tfrac{1}{2} x_2'' - x_2' \,\Big|\cdot 2 \Leftrightarrow$$

$(x_1 = x_2' - a_{11} \cdot x_2)$

$x_2'' - 4x_2' + 4x_2 = 0$

↳ the characteristic equation: $r^2 - 4 \cdot r + r = 0 \Leftrightarrow$

$(r-2)^2 = 0 \Leftrightarrow r = 2$ root with multiplicity of $2 \Rightarrow$

$r = 2 \longmapsto e^{2t}, \; t \cdot e^{2t} \Rightarrow x_2 = c_1 \cdot e^{2t} + c_2 t \cdot e^{2t}$

$x_1 = x_2' - 2x_2 = \cancel{2c_1 \cdot e^{2t}} + c_2 \cdot e^{2t} + \cancel{2c_2 \cdot t \cdot e^{2t}} - \cancel{2c_1 e^{2t}} - \cancel{2c_2 \cdot t \cdot e^{2t}}$

$\Rightarrow x_1 = c_2 \cdot e^{2t}$

$$\begin{cases} x_1 = c_2 \cdot e^{2t} \\ x_2 = c_1 \cdot e^{2t} + c_2 \cdot t e^{2t} \end{cases}$$

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The first column of the e^{tA} is the solution of the

IVP $\begin{cases} X' = A \cdot X \\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$ (1)

The second column of the e^{tA} is the solution of the

IVP $\begin{cases} X' = A \cdot X \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$ (2)

$(1) \Rightarrow \begin{cases} x_1(0) = c_2 \\ x_2(0) = c_1 \end{cases} \Leftrightarrow \begin{cases} c_2 = 1 \\ c_1 = 0 \end{cases} \Rightarrow$

the solution of (1) is: $\begin{pmatrix} e^{2t} \\ t \cdot e^{2t} \end{pmatrix} = X_1$

$(2) \Rightarrow \begin{cases} x_1(0) = c_2 \\ x_2(0) = c_1 \end{cases} \Leftrightarrow \begin{cases} c_2 = 0 \\ c_1 = 1 \end{cases} \Rightarrow$

$\Rightarrow$ the solution of (2) is: $\begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} = X_2$

So, $e^{tA} = (X_1 X_2) = \begin{pmatrix} e^{2t} & 0 \\ t \cdot e^{2t} & e^{2t} \end{pmatrix} = U$

$\det U = (e^{2t})^2 \neq 0, \forall t \in \mathbb{R}$

$\Leftrightarrow$ U is a fundamental matrix solution

Method 2:

$\det(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)^2 = 0$

$u_1 = ?$ eigenvector corresponding to $\lambda_1 = 2 \Leftrightarrow u_1 \in \mathbb{R}$,

$u_1 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A u_1 = \lambda_1 \cdot u_1$

Bledea Mihaela Alexandra

Let $u_1 = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \Leftrightarrow \begin{cases} 2a = 2a \\ a + 2b = 2b \end{cases}$

$\Leftrightarrow \begin{cases} a \in \mathbb{R} \\ a + 2b = 2b \end{cases} \Rightarrow \begin{cases} a = 0 \\ b \in \mathbb{R} \end{cases}$

Let $b = 1 \Rightarrow u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_1 \Rightarrow u_2$ and u_1 are not linearly independent $\Rightarrow$

$\Rightarrow A$ is not diagonalizable.

c.

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b) $A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$

Method 1:

the characteristic equation:

$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} -\lambda & 4 \\ 5 & 1-\lambda \end{vmatrix} = 0 \iff$

$(1-\lambda)(-\lambda) - 20 = 0 \iff \lambda^2 - \lambda - 20 = 0$

$\Delta = 1 + 80$

$\Delta = 81 \Rightarrow \lambda_{1,2} = \frac{1 \pm 9}{2} \Rightarrow$

$\Rightarrow \lambda_1 = 5 \lor \lambda_2 = -4$

$u_1 = ?$ an eigenvector corresponding to $\lambda_1 = 5$

$u_1 \in \mathbb{R}^2,\ u_1 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$A \cdot u_1 = 5 \cdot u_1,\ u_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \end{pmatrix} \iff \begin{cases} 4b = 5a \\ 5a + b = 5b \end{cases} \iff \begin{cases} b = \frac{5}{4} a \\ 5a + \frac{5}{4} a = \frac{25}{4} a \end{cases} \iff$

$\begin{cases} b = \frac{5}{4} a \\ 20a + 5 \cdot a = 25 \cdot a \end{cases} \iff \begin{cases} b = \frac{5}{4} a \\ 25a = 25a \text{ (True, } \forall a \in \mathbb{R}) \end{cases}$

We take $a = 4 \Rightarrow b = 5 \Rightarrow u_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$u_2 = ?$ an eigenvector corresponding to $\lambda_2 = -4$

$u_2 \in \mathbb{R}^2,\ u_2 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$A \cdot u_2 = -4 \cdot u_2,\ u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4a \\ -4b \end{pmatrix} \iff \begin{cases} 4b = -4a \\ 5a + b = -4b \end{cases} \iff \begin{cases} b = -a \\ b = -a \end{cases} \Rightarrow$

we take $a = -1 \Rightarrow b = 1 \Rightarrow u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

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So A has the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -4$, both real, $\boxed{m=2}$ and we found two eigenvectors

$u_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Since:

$\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} = 4+5 = 9 \neq 0 \Rightarrow u_1$ and u_2 are linearly independent

$\Rightarrow$ Thus A is diagonalizable

We have that $e^{5t}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are linearly independent solutions of $X' = A \cdot X$

Its general solution is: $X = c_1 \cdot e^{5t}\begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_2 \cdot e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

$c_1, c_2 \in \mathbb{R}$

$X' = A \cdot X \Leftrightarrow \begin{cases} x = 4 \cdot c_1 \cdot e^{5t} - c_2 \cdot e^{-4t} \\ y = 5 \cdot c_1 \cdot e^{5t} + c_2 \cdot e^{-4t} \end{cases}, c_1, c_2 \in \mathbb{R}$

$X = \begin{pmatrix} x \\ y \end{pmatrix}$

In order to find e^{tA}, we have to identify D and P, since $e^{tA} = P \cdot D \cdot P^{-1}$, $D = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}$ is a fundamental matrix solution.

$D = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}, P = \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix}, \det P = -9 \Rightarrow P^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{5}{9} & -\frac{4}{9} \end{pmatrix}$

$e^{tA} = \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-4t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & -4 \end{pmatrix} \cdot \frac{1}{9}$

$e^{tA} = \left(\frac{1}{9}\right) \cdot \begin{pmatrix} 4e^{5t} & e^{-4t} \\ 5 \cdot e^{5t} & -e^{-4t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 5 & -4 \end{pmatrix}$

$e^{tA} = \left(\frac{1}{9}\right)\begin{pmatrix} 4 \cdot e^{5t} + 5 \cdot e^{-4t} & 4 \cdot e^{5t} - 4 \cdot e^{-t} \\ 5 \cdot e^{5t} - 5 \cdot e^{-4t} & 5 \cdot e^{5t} + 4 \cdot e^{-4t} \end{pmatrix}$

Method 2: Bledea Mihaela Alexandra

$x' = 4y$
$y' = 5x + y$

$y = \frac{1}{4}(x' - 0x)$

$x'' = 4y' = 20x + 4y = 20x + x'$

$x'' = 20x + x'$

$x'' - 20x - x' = 0 \Leftrightarrow x'' - x' - 20x = 0$

$\mapsto$ the characteristic equation: $\lambda^2 - \lambda - 20 = 0 \Leftrightarrow$

$\lambda_1 = 5 \vee \lambda = -4$

$\lambda_1 = 5 \mapsto e^{5t}$
$\lambda_2 = -4 \mapsto e^{-4t}$ $\Big\} \Rightarrow x = c_1 \cdot e^{5t} + c_2 \cdot e^{-4t}, \; c_1 c_2 \in \mathbb{R}$ (1)

$y = \frac{1}{4} \cdot x' - \frac{1}{4} \cdot 5 \cdot c_1 \cdot e^{5t} + \frac{1}{4} \cdot (-4) \cdot c_2 \cdot e^{-4t}$

$y = \frac{5}{4} \cdot c_1 \cdot e^{5t} - 1 \cdot c_2 \cdot e^{-4t}$ (2)

(1), (2) $\Rightarrow \begin{cases} x = c_1 \cdot e^{5t} + c_2 \cdot e^{-4t} \\ y = \frac{5}{4} \cdot c_1 \cdot e^{5t} - c_2 \cdot e^{-4t} \end{cases}, \; c_1 c_2 \in \mathbb{R}$

$\mapsto$ the general solution of the system $X' = A \cdot X$

The first column of the e^{tA} is the solution of the ivp $\begin{cases} X' = A \cdot X \\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$ (1)

The second column of e^{tA} is the solution of the ivp $\begin{cases} X' = A \cdot X \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$ (2)

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$$(1) \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ \frac{5}{4}c_1 - c_2 = 0 \end{cases} \Rightarrow \frac{9}{4} c_1 = 1 \Rightarrow c_1 = \frac{4}{9} \Rightarrow$$

$$\Rightarrow c_2 = 1 - \frac{4}{9} \Rightarrow c_2 = \frac{5}{9}$$

So, $X_1 = \begin{pmatrix} \frac{4}{9} e^{5t} + \frac{5}{9} \cdot e^{-4t} \\ \frac{4}{9} \cdot \frac{5}{4} \cdot e^{5t} - \frac{5}{9} \cdot e^{-4t} \end{pmatrix}$

$$(2) \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ \frac{5}{4} c_1 - c_2 = 1 \end{cases} \Rightarrow \frac{9}{4} c_1 = 1 \Leftrightarrow c_1 = \frac{4}{9} \Rightarrow c_2 = -\frac{4}{9}$$

So, $X_2 = \begin{pmatrix} \frac{4}{9} \cdot e^{5t} - \frac{4}{9} \cdot e^{-4t} \\ \frac{4}{9} \cdot \frac{5}{4} \cdot e^{5t} + \frac{4}{9} \cdot e^{-4t} \end{pmatrix}$

Thus, $e^{tA} = (X_1 \, X_2) = \begin{pmatrix} \frac{4}{9} e^{5t} + \frac{5}{9} \cdot e^{-4t} & \frac{4}{9} \cdot e^{5t} - \frac{4}{9} \cdot e^{-4t} \\ \frac{5}{9} \cdot e^{5t} - \frac{5}{9} \cdot e^{-4t} & \frac{5}{9} \cdot e^{5t} + \frac{5}{9} \cdot e^{-4t} \end{pmatrix}$

Method 1: Reduction to a second order diff. equation

m) $A=\begin{pmatrix}0&-2\\2&0\end{pmatrix}$ → it is an uncoupled system

$$\begin{cases} x' = -2y \\ y' = 2x \end{cases}$$

$x'' = -2y' = -4x \Rightarrow x''+4=0$

↳ the characteristic equation:

$\lambda^2+4=0 \Leftrightarrow \lambda^2=-4 \Leftrightarrow \lambda_{1,2}=\pm 2i$

$\lambda_{1,2}=\pm 2i \longmapsto \cos(2t),\ \sin(2t)$

$\Rightarrow x = c_1\cdot \cos(2t)+c_2\cdot \sin(2t)$ ①

$y=-\frac{1}{2}(x'-0\cdot x)=\frac{x'}{-2}$

$x'=-2c_1\cdot \sin(2t)+2c_2\cdot\cos(2t)$

$y=c_1\cdot \sin(2t)-c_2\cos(2t)$ ②

①, ②

↳ $\begin{cases} x=c_1\cdot\cos(2t)+c_2(\sin 2t) \\ y=c_1\cdot \sin(2t)-c_2\cos(2t) \end{cases}$

↳ the general solution of the system $X'=AX$

The first column of the e^{tA} is the solution of the

ivp $\begin{cases} X'=A\cdot X \\ X(0)=\begin{pmatrix}1\\0\end{pmatrix}\end{cases}$ (1)

The second column of the e^{tA} is the solution of the

ivp $\begin{cases} X'=A\cdot X \\ X(0)=\begin{pmatrix}0\\1\end{pmatrix}\end{cases}$ (2)

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(1) $\Rightarrow$

$$\begin{cases} c_1 = 1 \\ -c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

So $X_1 = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$

(2) $\Rightarrow$

$$\begin{cases} c_1 = 0 \\ -c_2 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = -1 \end{cases}$$

So $X_2 = \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix}$

Thus $e^{tA} = (X_1 X_2) = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} = U$

$\det U = \cos^2(2t) + \sin^2(2t) = 1 \neq 0, \forall t \in \mathbb{R} \Rightarrow U$ is a fundamental matrix solution

Method 2:

$$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 4 = 0 \Leftrightarrow$$

$\Leftrightarrow \lambda^2 = -4 \Leftrightarrow \lambda_{1,2} = \pm 2i \notin \mathbb{R} \Rightarrow A$ is not diagonalizable

1.7.5. Bradea Mihaela Alexandra

Euler's formula:

$$e^{\alpha + i\cdot\beta} = e^{\alpha}(\cos\beta + i\cdot\sin\beta)$$

$$e^{it} = e^{0+i\cdot t} = e^{0}(\cos t + i\sin t) = \cos t + i\sin t$$

$$e^{i\pi} = e^{0+i\pi} = e^{0}(\cos\pi + i\sin\pi) = -1 + i\cdot 0 = -1$$

$$e^{\frac{i\pi}{2}} = e^{0+\frac{i}{2}\pi} = e^{0}(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}) = 0 + i + 1 = i$$

$$e^{(-1+i)t} = e^{-t+i\cdot t} = e^{-t}(\cos t + i\sin t) = \frac{\cos t}{e^{t}} + \frac{\sin t}{e^{t}}\cdot i$$

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1.7.8.

$1+t(1+e^{-t})$ is a solution to a LHDE

$1+t(1+e^{-t}) = t\cdot e^{-t}+t+1 \Rightarrow$

$t\cdot e^{(-1)\cdot t}$, $t\cdot e^{0\cdot t}$, $1\cdot e^{0\cdot t}$ are solutions to the equation

$t\cdot e^{(-1)t}$ $e^{(-1)\cdot t}$

$\Rightarrow \lambda_{1,2}=-1,\ \lambda_{3,4}=0$ - double root

The characteristic equation $\lambda^2(\lambda+1)^2=0 \Leftrightarrow \lambda^2(\lambda^2+2\lambda+1)=0$

$\Rightarrow$ The differential equation: $x^{(4)}+2x'''+x''=0$

1.7.9. $k, \eta \in \mathbb{R}$ Bledea Mihaela Alexandra

$$x' = k(2\lambda - x), \quad x(0) = \eta$$

$x' + k \cdot x = k \cdot 2\lambda$ -> first order linear differential equation

The equation is in the first order linear ODE form:

$$y'(x) + p(x) \cdot y = g(x)$$
$$p(x) = k$$
$$g(x) = 2\lambda k$$

Find the integration factor $\mu(x)$, so that $\mu(x) \cdot p(x) = \mu'(x)$

$$\mu'(t) = p(t) \cdot \mu(t) \;\Big|\; : \mu(t)$$

$$\left.\begin{array}{l} \dfrac{\mu'(t)}{\mu(t)} = p(t) \iff \ln(\mu(t))' = p(t) \\ p(t) = k \end{array}\right\} \Rightarrow$$

$$\Rightarrow \ln(\mu(t))' = k \;\Big|\; \int$$

$$\ln(\mu(t)) = \int k \, dt \iff \ln(\mu(t)) = k \cdot t + c_1$$

$$\Rightarrow \mu(t) = e^{k \cdot t + c_1} \iff \mu(t) = e^{k \cdot t} \cdot e^{c_1}$$

The constant c_1 can be dropped (it will be absorbed into c

$$x' + k \cdot x = 2\lambda k \;\Big|\; \cdot e^{kt}$$

$$x' \cdot e^{kt} + k \cdot x \cdot e^{kt} = 2\lambda \cdot k \cdot e^{kt}$$

$$(x \cdot e^{kt})' = 2\lambda \cdot k \cdot e^{k \cdot t} \;\Big|\; \int$$

$$x \cdot e^{k \cdot t} = \int 2\lambda \cdot k \cdot e^{k \cdot t} \, dt$$

$$x \cdot e^{k \cdot t} = 2\lambda \cdot e^{k \cdot t} + c$$

$$x = 2\lambda + \frac{c}{e^{k \cdot t}}$$

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$x(0) = \eta$

$x(0) = 2\lambda + \frac{c}{e^0} = \eta \iff C = \eta - 2\lambda$

$x = 2\lambda + \frac{\eta - 2\lambda}{e^{h \cdot t}}$

↳ the solution of the IVP

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1.7.10. $x'' - x = t \cdot e^{-2t}$ - second order linear, non-homogeneous differential equation with constant coefficients

a) $x_p(t) = (a \cdot t + b) e^{-2t}$

$x_p'(t) = (a \cdot t + b)' \cdot e^{-2t} + (a \cdot t + b) \cdot (e^{-2t})'$

$x_p'(t) = a \cdot e^{-2t} - 2(a \cdot t + b) \cdot (e^{-2t})$

$x_p''(t) = -2a \cdot e^{-2t} - (2a \cdot t + 2b)' \cdot e^{-2t} - (2 \cdot a t + 2b)(e^{-2t})'$

$x_p''(t) = -2a \cdot e^{-2t} - 2a \cdot e^{-2t} - (-2)(2 \cdot a \cdot t + 2b) e^{-2t}$

$x_p''(t) = -4 \cdot a \cdot e^{-2t} + 4 \cdot a \cdot e^{-2t} \cdot t + 4b \cdot e^{-2t}$

$x_p''(t) - x(t) = t \cdot e^{-2t}$

$-4 \cdot a \cdot e^{-2t} + 4 \cdot a \cdot e^{-2t} \cdot t + 4 \cdot b \cdot e^{-2t} - a \cdot t \cdot e^{-2t} - b \cdot e^{-2t} = t \cdot e^{-2t}$

$3 \cdot a \cdot e^{-2t} \cdot t + 3 \cdot b \cdot e^{-2t} - 4 \cdot a \cdot e^{-2t} = t \cdot e^{-2t}$

$e^{-2t}(-3a \cdot t + 3b - 4a) = t \cdot e^{-2t} \quad | : e^{-2t}$

$t = 3 \cdot a \cdot t + (3b - 4a) \Rightarrow \begin{cases} 3a = 1 \\ 3b = 4a \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{3} \\ b = \frac{4}{9} \end{cases} \Rightarrow$

$\Rightarrow$ A particular solution would be: $x_p(t) = (\frac{1}{3} \cdot t + \frac{4}{9}) \cdot e^{-2t}$

b) The general solution can be written as:

$x = x_h + x_p$

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We find x_h by solving $x'' - x = 0$

$\lambda^2 - 1 = 0 \Leftrightarrow \lambda^2 = 1 \Leftrightarrow \lambda = 1 \vee \lambda = -1$ $\Big\} \Rightarrow$

$\Rightarrow x_h = c_1 \cdot e^t + c_2 \cdot e^{-t}$

From a) we know that $x_p = e^{-2t}\left(\frac{1}{3}t + \frac{4}{9}\right)$ $\Big\} \Rightarrow$

$$\Rightarrow x = x_h + x_p \Leftrightarrow x = c_1 \cdot e^t + c_2 \cdot e^{-t} + e^{-2t}\left(\frac{1}{3}t + \frac{4}{9}\right)$$

c) $x(0) = 0$
$x'(0) = 0$

$$x(0) = 0 \Leftrightarrow c_1 \cdot e^0 - c_2 \cdot e^0 + e^0\left(\frac{1}{3} \cdot 0 + \frac{4}{9}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow c_1 + c_2 + \frac{4}{9} = 0 \Leftrightarrow c_1 + c_2 = -\frac{4}{9} \quad (1)$$

$$x' = c_1 \cdot e^t - c_2 \cdot e^{-t} - 2 \cdot e^{-2t}\left(\frac{1}{3}t + \frac{4}{9}\right) + e^{-2 \cdot t} \cdot \frac{1}{3}$$

$$x'(0) = 0 \Leftrightarrow c_1 \cdot e^0 - c_2 \cdot e^0 - 2 \cdot e^0\left(\frac{1}{3} \cdot 0 + \frac{4}{9}\right) + e^0 \cdot \frac{1}{3} = 0$$

$$\Leftrightarrow c_1 - c_2 - \frac{8}{9} + \frac{1}{3} = 0 \Leftrightarrow c_1 - c_2 = \frac{5}{9} \quad (2)$$

$$(1), (2) \Rightarrow \begin{cases} c_1 + c_2 = -\frac{4}{9} \\ c_1 - c_2 = \frac{5}{9} \end{cases} \Rightarrow 2c_1 = \frac{1}{9} \Rightarrow c_1 = \frac{1}{18}$$

$$c_2 = -\frac{4}{9} - \frac{1}{18} \Rightarrow c_2 = \frac{-9}{18} \Rightarrow c_2 = -\frac{1}{2}$$

$$x = \frac{1}{18} \cdot e^t - \frac{1}{2} \cdot e^{-t} + e^{-2 \cdot t}\left(\frac{1}{3} \cdot t + \frac{4}{9}\right)$$

Bledea Mihaela Alexandra

1.7.12. $\mathcal{L} : C^2(\mathbb{R}) \to C(\mathbb{R})$, $\mathcal{L}(x) = x'' - 2x' + x$, $\forall x \in C^2(\mathbb{R})$

a) $\mathcal{L}$ is a linear map $\Leftrightarrow \mathcal{L}(\alpha x + \beta y) = \alpha \cdot \mathcal{L}(x) + \beta \cdot \mathcal{L}(y)$,
$\forall x, y \in C^2(\mathcal{I})$, $\forall \alpha, \beta \in \mathbb{R}$

$\mathcal{L}(\alpha \cdot x + \beta \cdot y) = (\alpha \cdot x + \beta \cdot y)'' - 2(\alpha \cdot x + \beta \cdot y)' + (\alpha \cdot x + \beta \cdot y)$

$= (\alpha \cdot x)'' + (\beta \cdot y)'' - 2(\alpha \cdot x)' - 2(\beta \cdot y)' + \alpha \cdot x + \beta \cdot y$

$= \alpha \cdot x'' - 2\alpha \cdot x' + \alpha x + \beta y'' - 2\beta \cdot y' + \beta y$

$= \alpha \underbrace{(x'' - 2x' + x)}_{\mathcal{L}(x)} + \beta \underbrace{(y'' - y' + y)}_{\mathcal{L}(y)}$

$= \alpha \cdot \mathcal{L}(x) + \beta \cdot \mathcal{L}(y) \Rightarrow \mathcal{L}$ is a linear map

Ker $\mathcal{L}$ has dimension 2.

b) $x_p = a \cdot \cos 2t + b \cdot \sin 2t$, $a, b \in \mathbb{R}$

$x'' - 2x' + x = \cos 2t$

$x_p' = -2a \sin 2t + 2b \cos 2t$

$x_p'' = -4a \cos 2t - 4b \cdot \sin 2t$

~~$-4a \cos 2t - 4b \sin 2t + 4a \sin$~~

$-4a \cdot \cos 2t - 4b \cdot \sin 2t - 2(-2a \sin 2t + 2b \cos 2t) + a \cdot \cos 2t + b \cdot \sin 2t = \cos 2t$

$\Leftrightarrow$

$-4a \cos 2t - 4b \sin 2t + 4a \sin 2t - 4b \cos 2t + a \cdot \cos 2t + b \cdot \sin 2t = \cos 2t$

$\cos(2t) \cdot (-4a - 4b + a) + \sin(2t) \cdot (-4b + 4a + b) = \cos 2t$

$\Leftrightarrow \begin{cases} -3a - 4b = 1 \\ 4a - 3b = 0 \end{cases}$

$$\begin{matrix} -3a-4b=1 & \mid\cdot 4 \\ 4a-3b=0 & \mid\cdot 3 \end{matrix} \Leftrightarrow \begin{cases} -12a-16b=4 \\ 12a-9b=0 \end{cases} \Leftrightarrow -25b=4 \Leftrightarrow$$

$$\left.\begin{matrix} b=-\frac{4}{25} \\ 4a=3b \end{matrix}\right\} \Rightarrow a=\frac{-3}{25} \Rightarrow$$

$\Rightarrow$ A particular solution would be: $x_p = -\frac{4}{25}\sin 2t - \frac{3}{25}\cos t$

In order to find the general solution which is of the form $x = x_h + x_p$, we have to find x_h

x_h - solution of $x'' - 2x' + x = 0$

$\lambda^2 - 2\lambda + 1 = 0 \Leftrightarrow (\lambda - 1)^2 = 0 \Leftrightarrow \lambda = 1$ with multiplicity of 2 $\Rightarrow$

$\Rightarrow x_h = c_1 \cdot e^t + c_2 \cdot t \cdot e^t$

$x = x_h + x_p \Leftrightarrow x = c_1 \cdot e^t + c_2 t e^t - \frac{4}{25}\sin(2t) - \frac{3}{25}\cdot\cos(2t)$

c) $y_1(t) = e^{2t}$ and $y_2(t) = e^{-2t}$, $\forall t \in \mathbb{R}$

$\mathcal{L}(x) = 3\cdot y_1 + 5\cdot y_2$

$\mathcal{L}(x) = y_1 \Leftrightarrow x'' - 2\cdot x' + x = e^{2t}$

We assume that the particular solution has a form: $x_p = a\cdot e^{2t}$

$x_p' = 2\cdot a\cdot e^{2t}$

$x_p'' = 4\cdot a\cdot e^{2t}$

$\cancel{4\cdot a\cdot e^{2t}} - \cancel{2\cdot 2\cdot a\cdot e^{2t}} + a\cdot e^{2t} = e^{2t} \Leftrightarrow a\cdot e^{2t} = e^{2t} \Leftrightarrow a = 1 \Rightarrow$

$\Rightarrow x_p = e^{2t}$ for $\mathcal{L}(x) = y_1$

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We assume that the particular solution has a form: $x_p = a \cdot e^{-2t}$

$x_p' = -2 \cdot a \cdot e^{-2t}$

$x_p'' = 4a \cdot e^{-2t}$

$4 \cdot a \cdot e^{-2t} + 4 \cdot a \cdot e^{-2t} + a \cdot e^{-2t} = e^{-2t}$

$9 \cdot a \cdot e^{-2t} = e^{-2t} \Leftrightarrow a = \frac{1}{9} \Rightarrow$

$\Rightarrow x_p = \frac{1}{9} e^{-2t}$ for $f(x) = f_2$

$f(x) = 3 \cdot f_1 + 5 \cdot f_2 \Rightarrow f_1 = 3$ and $f_2 = 5$

From the Superposition Principle $\Rightarrow x_p = 3 \cdot e^{2t} + 5 \cdot \frac{1}{9} \cdot e^{-2t}$

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1.7.19.

$$x'' + 4\cdot x = \cos(2\cdot t)$$

a) $x_p = t(a\cdot\cos(2\cdot t) + b\cdot\sin(2\cdot t))$, $a, b \in \mathbb{R}$

$$x_p' = a\cdot\cos(2\cdot t) + b\cdot\sin(2\cdot t) + t(-2a\cdot\sin(2t) + 2b\cdot\cos(2t))$$

$$x_p'' = -2a\sin(2t) + 2b\cos(2t) + (-2a\cdot\sin(2t) + 2b\cdot\cos(2t)$$
$$+ t(-4a\cos(2t) - 4b\sin(2t))$$

$x_p'' + 4\cdot x_p = \cos(2t) \Leftrightarrow$

$4b\cos(2t) - 4\cdot a\cdot\sin(2t) - \cancel{4\cdot a\cdot t\cos(2t)} - \cancel{4\cdot b\cdot t\cdot\sin(2t)} + \cancel{4\cdot a\cdot t\cdot\cos(2t)}$
$+ \cancel{4\cdot b\cdot t\cdot\sin(2t)} = \cos(2t) \Leftrightarrow$

$4b\cdot\cos(2t) - 4\cdot a\cdot\sin(2t) = \cos 2t \Leftrightarrow$

$$\begin{cases} 4b = 1 \\ a = 0 \end{cases} \Leftrightarrow b = \frac{1}{4} \wedge a = 0 \Rightarrow x_p = t\cdot\frac{1}{4}\cdot\sin(2t)$$

b) In order to find the general solution we have to find x_h which is the solution of $x'' + 4x = 0$

$r^2 + 4 = 0 \Leftrightarrow r^2 = -4 \Leftrightarrow \lambda_1 = -2i \vee \lambda_2 = 2i$

For two complex roots $r_1 \neq r_2$, where $r_1 = a + i\beta$, $r_2 = a - i\beta$

the general solution takes the form: $y = e^{a\cdot t}(c_1\cdot\cos(\beta\cdot t) + c_2\cdot\sin(\beta\cdot t)) \Rightarrow$

$\Rightarrow x_h = c_1\cdot\cos(2\cdot t) + c_2\cdot\sin(2t)$

$$x = x_p + x_h \Rightarrow x = c_1\cdot\cos(2\cdot t) + c_2\cdot\sin(2\cdot t) + \frac{t\cdot\sin(2t)}{4}$$

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c) $x'' + 4 \cdot x = \cos(2t)$ -> it's an undamped motion with an external force.

The general equation is: $x'' + \frac{k}{m} \cdot x = A \cdot \cos(w \cdot t)$

$x_h = c_1 \cdot \cos(w_0 t) + c_2 \cdot \sin(w_0 t)$, in our case from b) we have $x_h = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t) \Rightarrow w = w_0$. Then the general solution is $x = c_1 \cdot \cos(w_0 t) + c_2 \cdot \sin(w_0 t) + \frac{1}{2w_0} t \cdot \sin(w_0 t)$ in our case from b)

We have $x = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t) + \frac{1}{4} t \cdot \sin(2t)$

This function is unbounded. In this case oscillations occur with an amplitude that increases to ∞. This phenomen is called resonance.

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1.7.24.

$m x'' + 25 \cdot x = 12 \cos(36 \pi t)$

The DE will exhibit resonance when $\omega_0 = \omega$, where:

$$\left.\begin{array}{l} \omega = 36\pi \\ \omega_0 = \sqrt{\frac{k}{m}} \end{array} \Rightarrow \sqrt{\frac{k}{m}} = 36\pi \Leftrightarrow \begin{array}{l} \frac{k}{m} = 36^2 \pi^2 \\ k = 25 \end{array}\right\} \Rightarrow$$

$$\Rightarrow m = \frac{25}{36^2 \pi^2}$$

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1.7.25

$$\ddot{\theta} + \dot{\theta} + \theta = 0$$

The characteristic equation:

$$\lambda^2+\lambda+1=0 \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \quad (1)$$

$$e^{\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)t} = e^{-\frac{1}{2}t} \cdot e^{\pm i\frac{\sqrt{3}}{2}t} = e^{-\frac{1}{2}t}\left(\cos\left(\pm\frac{\sqrt{3}}{2}t\right) + i\sin\left(\pm\frac{\sqrt{3}}{2}t\right)\right)$$

$$= e^{-\frac{1}{2}t}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) \pm i\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$= e^{-\frac{1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) \pm i e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$\Rightarrow \theta(t) = c_1 e^{-\frac{1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \cdot e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$-1 \leq \cos\left(\frac{\sqrt{3}}{2}t\right) \leq 1 \;\Big|\cdot c_1 \cdot e^{-\frac{1}{2}t} \Leftrightarrow$$

$$-c_1 e^{-\frac{1}{2}t} \leq \cos\left(\frac{\sqrt{3}}{2}t\right) \cdot c_1 \cdot e^{-\frac{1}{2}t} \leq c_1 \cdot e^{-\frac{1}{2}t} \quad (1)$$

$$-1 \leq \sin\left(\frac{\sqrt{3}}{2}t\right) \leq 1 \;\Big|\cdot c_2 \cdot e^{-\frac{1}{2}t} \Leftrightarrow$$

$$-c_2 \cdot e^{-\frac{1}{2}t} \leq c_2 \cdot e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right) \leq c_2 \cdot e^{-\frac{1}{2}t} \quad (2)$$

$$(1)+(2) \Rightarrow \underbrace{e^{-\frac{1}{2}t}(-c_1-c_2)}_{\xrightarrow{t\to\infty} 0} \leq \underbrace{\theta(t)}_{\to 0} \leq \underbrace{e^{-\frac{1}{2}t}(c_1+c_2)}_{\xrightarrow{t\to\infty} 0} \Bigg\} \Rightarrow$$

$$\xRightarrow{\text{Squeeze Theorem}} \lim_{t\to\infty} \theta(t) = 0$$

Bledea Mihaela Alexandra

1.7.2.9

$$t^2 \cdot x'' + 2 \cdot t \cdot x' - 2x = 0, \quad t \in (0, \infty)$$

a) $x(t) = t^\lambda, \lambda \in \mathbb{R}$

$x'(t) = \lambda \cdot t^{\lambda-1}$

$x''(t) = \lambda(\lambda-1) \cdot t^{\lambda-2}$

$\lambda(\lambda-1) \cdot t^{\lambda-2} \cdot t^2 + 2 \cdot t \cdot \lambda \cdot t^{\lambda-1} - 2 \cdot t^\lambda = 0$

$t^\lambda(\lambda^2 - \lambda + 2\lambda - 2) = 0 \Leftrightarrow$

$t^\lambda(\lambda^2 + \lambda - 2) = 0 \Leftrightarrow t^\lambda = 0$ (not possible, $\forall t \in (0, \infty), \forall \lambda \in \mathbb{R}$)

$\vee$

$\lambda^2 + \lambda - 2 = 0 \Leftrightarrow \lambda^2 - \lambda + 2\lambda - 2 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda(\lambda-1) + 2(\lambda-1) = 0 \Leftrightarrow (\lambda-1)(\lambda+2) = 0 \Leftrightarrow \lambda = 1 \vee \lambda = -2 \Rightarrow$

$\Rightarrow x(t) = t \vee x(t) = t^{-2}$ - solutions

b) $t^2 \cdot x'' + 2 \cdot t \cdot x' - 2 \cdot x = 0 \to$ second order linear homogeneous differential equation without constant coefficients

$\cancel{t^2 x'' + 2 \cdot t \cdot x' - 2x = 0}$

$\cancel{t^2}$ from subpoint a) $\Rightarrow$ the general solution of the differential equation is $x = c_1 \cdot t + c_2 \cdot t^{-2}, c_1, c_2 \in \mathbb{R}$

c) $t^2 x'' + 2 \cdot t x' - 2x = 0, \; x(1) = 0, \; x'(1) = 1$

$x(1) = 0 \Leftrightarrow c_1 \cdot + c_2 \quad = 0 \Leftrightarrow \quad c_1 + c_2 = 0$

$x'(1) = 0 \Leftrightarrow c_1 \cdot \quad + c_2 \cdot (-2) \cdot \quad = 1 \Leftrightarrow \quad c_1 - 2c_2 = 1 \Leftrightarrow$

$$\begin{cases} c_1 - 2c_2 = 1 \\ c_1 + c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 - 2c_2 = 1 \\ c_1 = -c_2 \end{cases} \Leftrightarrow \begin{cases} -3c_2 = 1 \mid :(-3) \\ c_1 = -c_2 \end{cases} \Leftrightarrow$$

$\Leftrightarrow c_2 = -\frac{1}{3} \Leftrightarrow c_1 = \frac{1}{3} \Rightarrow$

$\Rightarrow x = \frac{1}{3} \cdot t - \frac{1}{3} t^{-2}$

1.7.34.

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$$f(x) = x'' + 25 \cdot x$$

(b) $f(x) = 0$, $x(0) = 0$, $x'(0) = 1$

$f(x) = 0 \Leftrightarrow x'' + 25 \cdot x = 0 \Leftrightarrow r^2 + 25 = 0 \Leftrightarrow$

$r^2 = -25 \Leftrightarrow r_1 = 5i \vee r_2 = -5i \Rightarrow$

$\Rightarrow x = c_1 \cdot \cos(5t) + c_2 \cdot \sin(5t)$

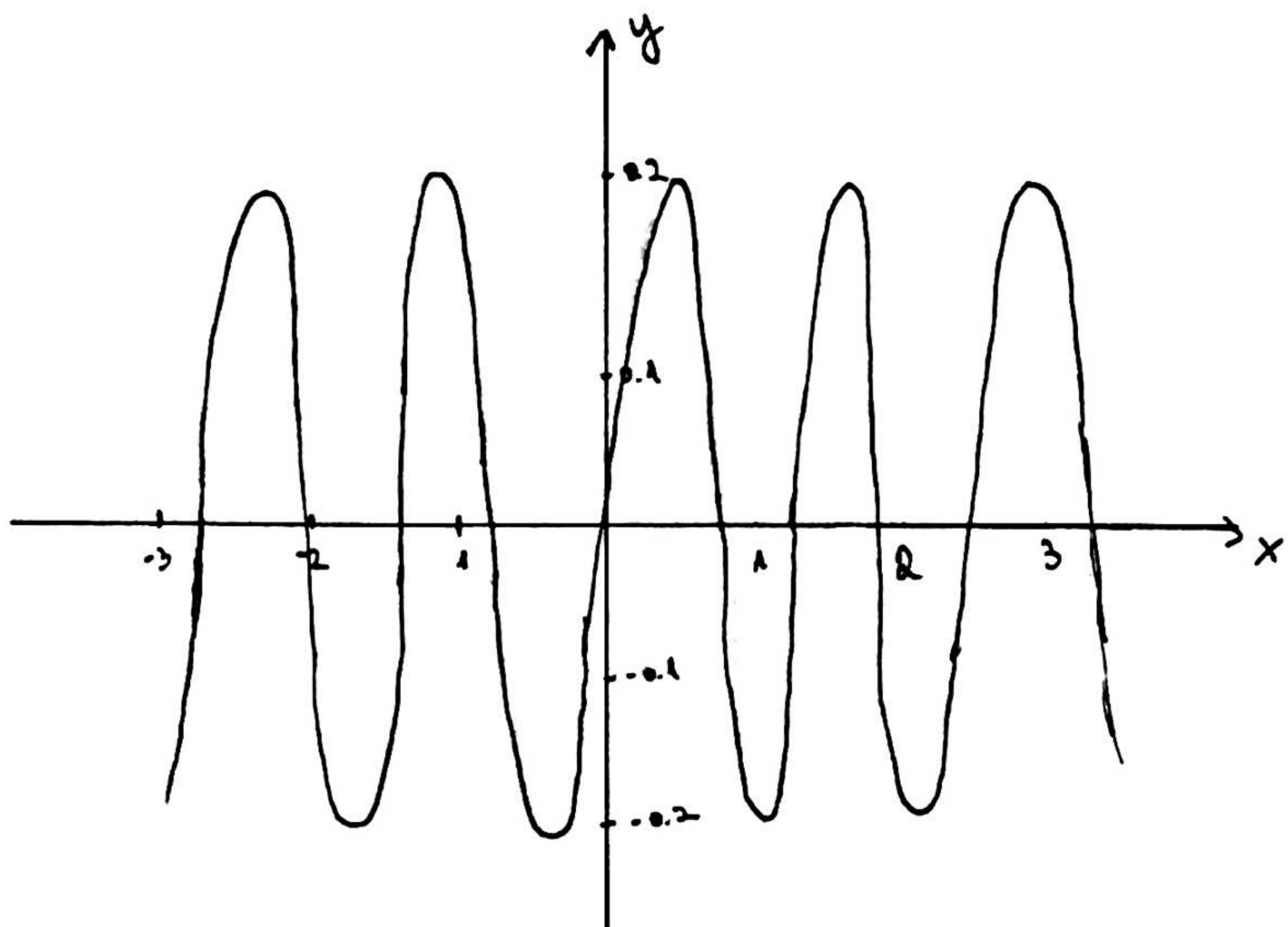
$x(0) = 0 \Leftrightarrow c_1 \cdot \underbrace{\cos 0}_{=1} + c_2 \cdot \underbrace{\sin(0)}_{=0} = 0 \Leftrightarrow c_1 = 0$

$x = c_2 \cdot \sin(5t)$

$x' = 5 c_2 \cdot \cos(5t)$

$x'(0) = 1 \Leftrightarrow 5 \cdot c_2 \cdot \underbrace{\cos 0}_{=1} = 1 \Leftrightarrow c_2 = \frac{1}{5} \Rightarrow$

$\Rightarrow x = \frac{1}{5} \sin(5t)$



$x(t)$ - is a periodic function with main period $T = 2\pi$

-> bounded between $-\frac{1}{5}$ and $\frac{1}{5}$

-> oscillatory around the value 0

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ii) $\varphi_1(t) = t \cdot \cos(5t)$, $\forall t \in \mathbb{R}$
$\varphi_2(t) = t \cdot \sin(5t)$,

$\mathcal{L}(5) = 5'' + 25 \cdot 5 = 125$

$\mathcal{L}(\varphi_1) = (t \cdot \cos(5t))'' + 25 \cdot t(\cos(5t))$

$(t \cdot \cos(5t))' = \cos(5 \cdot t) - 5 \cdot t \cdot \sin(5 \cdot t)$

$(t \cdot \cos(5t))'' = -5 \cdot \sin(5t) - 5 \cdot \sin(5 \cdot t) - 25 t \cdot \cos(5t)$

$\Rightarrow \mathcal{L}(\varphi_1) = -10 \cdot \sin(5t) - \cancel{25 t \cdot \cos(5t)} + \cancel{25 \cdot t(\cos(5t))}$

$\mathcal{L}(\varphi_1) = -10 \sin(5t)$

$\mathcal{L}(\varphi_2) = (t \cdot \sin(5t))'' + 25 \cdot t \cdot \sin(5t)$

$(t \cdot \sin(5 \cdot t))' = \sin(5 \cdot t) + 5t \cdot \cos(5t)$

$(t \cdot \sin(5t))'' = 5\cos(5t) + 5 \cdot \cos(5t) - 25 \cdot t \cdot \sin(5t)$

$\Rightarrow \mathcal{L}(\varphi_2) = 10 \cos(5t) - \cancel{25 \cdot t \cdot \sin(5t)} + \cancel{25 \cdot t \cdot \sin(5t)}$

$\mathcal{L}(\varphi_2) = 10 \cos(5 \cdot t)$

iii) $\mathcal{L}(x) = 5 \Leftrightarrow x'' + 25 \cdot x = 5$
We assume $x = a$ a solution $a \in \mathbb{R}$
$a'' = 0$
$0 + 25 \cdot a = 5 \Leftrightarrow a = \frac{1}{5} \Rightarrow x = \frac{1}{5}$ - is a constant solution

iv) $\mathcal{L}(x) = 25 - 25 \cdot \sin(5t) \Leftrightarrow x'' + 25 \cdot x = 25 - 25 \cdot \sin(5 \cdot t)$

$x = x_h + x_p$
x_h - solution of $\mathcal{L}(x) = 0$ (We know it from subpoint i)
$x_h = c_1 \cdot \cos(5t) + c_2 \cdot \sin(5t)$

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iv) $\mathcal{L}(x) = 25 - 25\cdot\sin(5t)$

$$\mathcal{L}(x) = 5(\underbrace{5}_{f_1}\underbrace{-5\cdot\sin(5t)}_{f_2})$$

$$\mathcal{L}(x) = 5(f_1 + f_2)$$

$\mathcal{L}(x) = f_1 \Leftrightarrow \mathcal{L}(x) = 5 \Rightarrow$ from iii) $\Rightarrow x_p = \frac{1}{5}$ (1)

$\mathcal{L}(x) = f_2 \Leftrightarrow \mathcal{L}(x) = -5\cdot\sin(5t)$

$$x'' + 25x = -5\cdot\sin(5t)$$

We assume that the solution has the following form:

$$x_p = a\cdot t\cdot\sin(5t) + b\cdot t\cdot\cos(5t)$$

$$x_p' = a\cdot\sin(5t) + 5\cdot a\cdot t\cdot\cos(5t) + b\cdot\cos(5t) + (-5)\cdot b\cdot t\cdot\sin(5t)$$

$$x_p'' = 5a\cdot\cos(5t) - 5b\cdot\sin(5t) + 5\cdot a\cdot\cos(5t) - 25\,a\cdot t\cdot\sin(5t) - 5b\cdot\sin(5t) - 25\cdot b\cdot t\cdot\cos(5t)$$

$$x_p'' = 10\,a\cdot\cos(5t) - 10\,b\cdot\sin(5t) - 25\,a\cdot t\cdot\sin(5t) - 25\,b\cdot t\cdot\cos(5t)$$

$x_p'' + 25\cdot x_p = -5\sin(5t) \Leftrightarrow$

$$10a\cdot\cos(5t) - 10b\cdot\sin(5t) - 25\cdot\cancel{a\cdot t\cdot\sin(5t)} - 25\cancel{b\cdot t\cdot\cos(5t)} + 25\cdot\cancel{a\cdot t\cdot\sin(5t)} + 25\cancel{b\cdot t\cdot\cos(5t)} = -5\sin(5t) \Leftrightarrow$$

$$10\cdot a\cdot\cos(5t) - 10\,b\cdot\sin(5t) = -5\cdot\sin(5t) \Leftrightarrow \begin{cases}10\cdot a = 0\\ -10\cdot b = -5\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}a = 0\\ b = \frac{1}{2}\end{cases} \Rightarrow x_p = \frac{1}{2}\cdot t\cdot\cos(5t) \ (2)$$

The SuperPosition Principle

(1), (2) $\Rightarrow x_p = 5\cdot\frac{1}{5} + 5\cdot\frac{1}{2}\cdot t\cdot\cos(5t)$ for $\mathcal{L}(x) = 5(f_1 + f_2)$

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$$\Rightarrow x_p = 1 + \frac{5 \cdot t \cdot \cos(5t)}{2}$$

x_h - solution of $\mathcal{L}(x) = 0$

From 1) $\Rightarrow \mathcal{L}(x) = 0 \Rightarrow x_h = c_1 \cdot \cos(5t) + c_2 \cdot \sin(5t)$

$$x = x_h + x_p \Rightarrow x = c_1 \cdot \cos(5t) + c_2 \cdot \sin(5t) + 1 + \frac{5 \cdot t \cdot \cos(5t)}{2}$$

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1.7.35.

$x' + \frac{1}{t^2} x = 0$, $t \in (-\infty, 0)$

a) $x = e^{\frac{1}{t}}$ solution?

$x' = \left(\frac{1}{t}\right)' \cdot e^{\frac{1}{t}} \Rightarrow x' = -\frac{1}{t^2} \cdot e^{\frac{1}{t}}$

$x' + \frac{1}{t^2} x = -\frac{1}{t^2} \cdot e^{\frac{1}{t}} + \frac{1}{t^2} \cdot e^{\frac{1}{t}} = 0 \Leftrightarrow 0 = 0$ (True) $\Rightarrow$

$\Rightarrow e^{\frac{1}{t}}$ - solution

b) $A(t) = -\int_{t_0}^{t} a(s) \cdot ds$, $a(t) = \frac{1}{t^2}$

$A(t) = -\int_{-\infty}^{t} a(s) \cdot ds = -\int_{-\infty}^{t} \frac{1}{s^2} ds = -\frac{1}{s}\Big|_{-\infty}^{t}$

$= \lim_{s \to -\infty} -\frac{1}{s} + \frac{1}{t} = \frac{1}{t}$

<u>the general solution</u>: $x(t) = c \cdot e^{A(t)} = c \cdot e^{\frac{1}{t}}$, $c \in \mathbb{R}$

ivp: $\begin{cases} x' + \frac{1}{t^2} x = 0 \\ x(-1) = 1 \end{cases}$

$x(-1) = 1 \Leftrightarrow c \cdot e^{-1} = 1 \Leftrightarrow c \cdot \frac{1}{e} = 1 \Leftrightarrow c = e \Rightarrow$

$\Rightarrow$ The solution of the ivp: $x(t) = e \cdot e^{\frac{1}{t}} \Rightarrow x(t) = e^{\frac{1}{t}+1}$

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c) $x' + \frac{1}{t^2} \cdot x = 1 + \frac{1}{t}$, $x \in (-\infty, 0)$

$x = x_h + x_p$

$x_h = c \cdot e^{\frac{1}{t}}$

$x' + \frac{1}{t^2} \cdot x = 1 + \frac{1}{t}$

We assume that x_p has the form: $x_p = t^{\lambda}$, $\lambda \in \mathbb{R}$

$x_p' = (t^{\lambda})' = \lambda \cdot t^{\lambda - 1}$

$\lambda \cdot t^{\lambda - 1} + \frac{1}{\cancel{t^2}_{1}} \cdot \cancel{t^{\lambda}}^{t^{\lambda-2}} = 1 + \frac{1}{t} \iff$

$\lambda \cdot t^{\lambda - 1} + t^{\lambda - 2} = 1 + \frac{1}{t}$

$t^{\lambda - 1} (\lambda + \frac{1}{t}) = (1 + \frac{1}{t}) \iff \lambda = 1 \Rightarrow x_p = t$

$x = x_p + x_h \Rightarrow x = c \cdot e^{\frac{1}{t}} + t$, $c \in \mathbb{R}$