Shortest Paths and Matrix Multiplication

Assumption : negative edge weights may be present, but no negative weight cycles.

(1) Structure of a Shortest Path :

- Consider a shortest path p_{ij}^{m} from v_i to v_j such that $|p_{ij}^{m}| \le m$
  - ► i.e., path p_{ij}^{m} has at most m edges.
- no negative-weight cycle $\Rightarrow$ all shortest paths are simple
  $\Rightarrow$ m is finite $\Rightarrow m \le n - 1$
- $i = j \Rightarrow |p_{ii}| = 0$ & $\omega(p_{ii}) = 0$
- $i \ne j \Rightarrow$ decompose path p_{ij}^{m} into p_{ik}^{m-1} & $v_k \to v_j$, where $|p_{ik}^{m-1}| \le m - 1$
  - ► p_{ik}^{m-1} should be a shortest path from v_i to v_k by optimal substructure property.
  - ► Therefore, $\delta(v_i, v_j) = \delta(v_i, v_k) + \omega_{kj}$

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(2) A Recursive Solution to All Pairs Shortest Paths Problem :

- d_{ij}^{m} = minimum weight of any path from v_i to v_j that contains at most "m" edges.

- $m = 0$: There exist a shortest path from v_i to v_j with no edges $\leftrightarrow i = j$.

  ► $$d_{ij}^{0} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \ne j \end{cases}$$

- $m \ge 1$: $d_{ij}^{m} = \min\{ d_{ij}^{m-1} , \min_{1\le k\le n \wedge k\ne j} \{ d_{ik}^{m-1} + \omega_{kj} \}\}$

  $= \min_{1\le k\le n} \{d_{ik}^{m-1} + \omega_{kj} \}$ for all $v_k \in V$,

  since $\omega_{jj} = 0$ for all $v_j \in V$.

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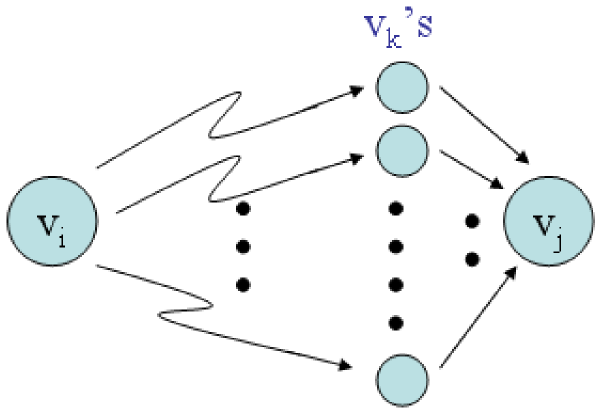
- to consider all possible shortest paths with $\le m$ edges from v_i to v_j
  ▶ consider shortest path with $\le m-1$ edges, from v_i to v_k, where $v_k \in R_{v_i}$ and $(v_k, v_j) \in E$



- note : $\delta(v_i, v_j) = d_{ij}^{n-1} = d_{ij}^{n} = d_{ij}^{n+1}$, since $m \le n-1 = |V| - 1$

Shortest Paths and Matrix Multiplication

(3) Computing the shortest-path weights bottom-up :

- given $W = D^1$, compute a series of matrices $D^2, D^3, \ldots, D^{n-1}$, where $D^m = (d_{ij}^m)$ for $m = 1, 2, \ldots, n-1$
  ▶ final matrix D^{n-1} contains actual shortest path weights, i.e., $d_{ij}^{n-1} = \delta(v_i, v_j)$

- SLOW-APSP(W)

```
    D^1 ← W
    for m ← 2 to n-1 do
        D^m ← EXTEND( D^(m-1) , W )
    return D^(n-1)
```

Shortest Paths and Matrix Multiplication

EXTEND (D , **W**)

▶ $D = (d_{ij})$ is an n x n matrix

```
for i ← 1 to n do
    for j ← 1 to n do
        d_ij ← ∞
        for k ← 1 to n do
            d_ij ← min{d_ij , d_ik + ω_kj}
return D
```

MATRIX-MULT (**A** , **B**)

▶ $C = (c_{ij})$ is an n x n result matrix

```
for i ← 1 to n do
    for j ← 1 to n do
        c_ij ← 0
        for k ← 1 to n do
            c_ij ← c_ij + a_ik x b_kj
return C
```

Shortest Paths and Matrix Multiplication

- relation to matrix multiplication $C = A \times B$: $\mathbf{c}_{ij} = \sum_{1 \le k \le n} \mathbf{a}_{ik} \times \mathbf{b}_{kj}$,

  ▶ $D^{m-1} \leftrightarrow A$ & $\mathbf{W} \leftrightarrow B$ & $D^m \leftrightarrow C$

  "min" ↔ "+" & "+" ↔ "x" & "∞" ↔ "0"

- Thus, we compute the sequence of matrix products

$$D^1 = D^0 \times W = W \ ; \text{ note } D^0 = \text{identity matrix,}$$

$$D^2 = D^1 \times W = W^2 \qquad \text{i.e., } d_{ij}^{\,0} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$$

$$D^3 = D^2 \times W = W^3$$

$$\vdots$$

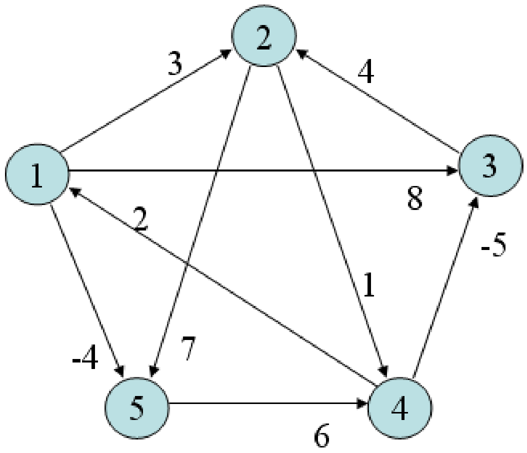
$$D^{n-1} = D^{n-2} \times W = W^{n-1}$$

- running time : $\Theta(n^4) = \Theta(V^4)$

  ▶ each matrix product : $\Theta(n^3)$

  ▶ number of matrix products : n-1

Example:



	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^1 = D^0W$

	1	2	3	4	5
1	0	3	8	2	-4
2	3	0	-4	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	8	∞	1	6	0

$D^2 = D^1W$

	1	2	3	4	5
1	0	3	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	11
4	2	-1	-5	0	-2
5	8	5	1	6	0

$D^3 = D^2W$

	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

$D^4 = D^3W$