sparse, and many algorithms have been developed for its fast inversion [85].

2.5 EEG Inverse Problem

The EEG inverse source imaging problem concerns the estimation of the brain activity given the potentials along the scalp and the mapping operator which describes the propagation of the electric activity of brain to the EEG measurement electrodes.

2.5.1 Review on the Source Reconstruction Methods

The spatial resolution and quality of the EEG scalp recordings are often poor due to the noise and inhomogeneities of the skull conductivity which makes the analysis challenging. The current flows towards the scalp from the intracranial sources and follows tangential trajectory within the skull which results in a significant voltage drop at the electrodes. To acquire more informative understanding of the brain activity, many source imaging techniques have been developed over the past 20 years for the accurate localization and visualization of the neural current sources (inside the brain) from the EEG measurements.

These source reconstruction methods can be divided into two main categories based on the assumptions about the sources and the utilized mathematical tools. The first methods concern the estimation of dipoles which fit best with the observations solving the problem in the least squares (Dipole fit model) [129, 151, 11]. These methods attempt to explain the measurement with a small number of dipole sources. The single or few dipole assumption has proven to be useful in cases where the underlying brain activity is concentrated in a relatively small volume (most of the brain is electrically silent). Additionally, the dipole term of a multi-pole source expansion is the principle factor affecting the potential measurements. The main disadvantages of the approach are that the number of dipole source needs to be predefined and that the approach is non linear. There are also studies in which the number of sources is predefined using temporal information [106]. However, in case the number of sources is incorrect, the approach fails to localize the activity. The approach often works well for up to 2 dipoles, especially for sources which are spatially separated.

When there are simultaneously several active regions then the actual source configuration must be modelled more accurately and the dipole fit model cannot be applied. A different approach called Distributed Source Modelling (DSM) [29] has been proposed based on the assumption that a large number of sources is distributed in an relatively extended area. Methods using the DSM attempt to compute a distribution of dipole moments at every point in a specified reconstruction space, e.g gray matter. These methods are sometimes called tomographic reconstruction techniques. The source locations are fixed while the amplitude and orientation are unknown. These methods have the advantage that the computational model is linear, there is no need for prior estimation of the number of sources and it can be used for extended source configuration. The solution space consists of all the possible source locations in this area and therefore a significant problem is that the computational model (linear system) is severely under-determined. The distributed source EEG problem is an ill-posed inverse problem and relatively strong assumptions are needed for regularization. Additionally, noisy and incomplete data impose the need for additional constraints related also to the physiological and anatomical information, for example recovering only radial cortical sources [24]

In the following sections, commonly used time-invariant source reconstruction methods based on the dipole fit modelling and the DSM employing different priors are reviewed. There are also other modelling methods which are not included in this overview. For example the spatial filtering method [171, 143] in which a set of spatial filters is designed in such a way that the filter passes signals originating from a specified location within the brain while eliminates signals from other locations and the Multiple Signal Classification (MUSIC) and (RAP)MUSIC [106, 90]. MUSIC separates the measured data into signal and noise subspaces and the best orthogonal projection operator of the signal onto the data-noise space is estimated. Subsequently, the orthogonal projection operator can be used to guide a recursive parametric dipole fitting algorithm. For an extended review of the inverse modelling methods, we suggest [66, 102, 38, 141]. Additionally, when spatio-temporal methods the interested reader is referred to [103, 131, 106, 90] where the temporal information in the signal is used. Furthermore, in spatio-temporal approaches, different dipole models [159] are used, for example the moving dipole model where all the dipole parameters change over time, the rotation dipole model in which the source location is constant during the EEG measurement acquisition [150, 105] or the fixed dipole mode, widely used in evoked response studies with both the orientation and the location constant.

2.5.2 Dipole Fit Model

In the dipole fit modelling, the neural activity in the brain is considered to be restricted to a relatively small volume allowing it to be well approximated by an equivalent single source or a small number of sources. The method tries to estimate a small number of dipoles which fit best with the observed potential measurements.

The sources are mathematically assumed to be dipoles with unknown positions and moments (magnitude and orientation) [67, 16]. In this approach the model equation (2.5) becomes

$$v = \sum_{i=1}^{n} k(\mathrm{x}_i)\mathrm{d}_i, \tag{2.14}$$

where n is the number of the candidate dipole locations and d_i is the i^{th} dipole moment at location x_i. The associated minimization problem is

$$\min_{\mathrm{x}_i,\mathrm{d}_i} \|v - \sum_{i=1}^{n} k(\mathrm{x}_i)\mathrm{d}_i\|_2^2. \tag{2.15}$$

The minimization problem is non convex and non linear. Direct search, non linear fitting methods or global optimization techniques can be used for the solution (e.g. the Levenberg-Marquardt algorithm). For several dipoles (i.e. $n > 2$) the problem (2.15) due to non convexity does not have a global minimum. Additionally, when the number of unknown parameters exceeds the observations the problem becomes under determined. In these cases, the DSM approach is used or temporal information is integrated in the model.

The simplest problem uses the assumption that there is only a single dipole source characterized by three parameters corresponding to magnitude, position and orientation (six degrees of freedom). These parameters are adjusted in such way that the resulting electrostatic potential matches the best with the given data [67]. The dipole moments are linear parameters and the three location parameters are non linear. This problem can be solved recursively by a sequence of linear and non linear solutions [129]. Specifically, if we consider a fixed source location then the problem becomes linear and over determined and it can be easily solved using linear least square method [45]. The least square estimate gives the optimal dipole for a given location. The non linear part of the problem is to find the best of all optimal dipoles by changing the location parameters.

The single dipole assumption can be used to estimate a distinctive spatial source. For multiple sources with overlapping fields the dipole fit approach breaks down.

2.5.3 Distributed Source Model

In the DSM approach, the electrical activity is considered in the entire brain (or in an extended region). The domain is divided into elements (voxels) and a dipole moment is placed at the centre of the element. The mesh with the dipole locations is the source space. If these locations are denoted with $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_N$, then (2.5) becomes

$$v = \sum_{i=1}^{N} k_i \mathrm{d}_i = Kd, \tag{2.16}$$

where $v \in \mathbb{R}^M$, $K \in \mathbb{R}^{M \times 3N}$ is the lead field matrix and $d \in \mathbb{R}^{3N}$ is the dipole distribution. With the distributed source model the problem becomes linear, however in most cases the linear system is severely underdetermined ($M \ll N$). To solve the inverse problem, different prior assumptions and constraints have been proposed such as the (weighted) minimum norm [41, 63] and smoothness properties [134] to ensure consistency. Also, anatomical and physiological priors have been proposed [86, 74] which reflect the nature and the properties of the brain activity. Additionally, for better performance, the number of unknowns can be reduced by restricting the region of interest, e.g. by considering sources only in the cortical surface. In the following paragraphs we review the most representative methods using the distributed source model (2.16).

The first distributed source model is the minimum norm estimate (MNE) introduced in [41]. The solution is based on the estimation of the dipole distribution with the lowest overall power which also satisfies the potential measurements. The problem can be expressed as

$$\min_d \|d\|_2^2 \text{ subject to } v = Kd, \tag{2.17}$$

where $\|.\|_2$ is the L_2 norm [101]. The solution is given by

$$\hat{d}_{MNE} = K^T(KK^T)^{-1}d. \tag{2.18}$$

The MNE was first introduced for the MEG problem however it can be similarly applied for the EEG inverse problem. Usually, to improve the result a particular brain area is selected.

In practice, the MNE is the solution of the Tikhonov regularization L_2 norm [42]

$$\hat{d}_{MNE} := \min_d \|Kd - v\|_2^2 + \alpha\|d\|_2^2, \tag{2.19}$$

where α is a regularization parameter. However, the MNE gives a solution with a maximum close to the boundaries. This is a consequence of the intrinsic depth bias of this lead field matrix which results in shifting the dipole sources near the surface [5, 174, 32]. For the compensation of the lead field matrix depth effect, the use of a weighted minimum norm estimate (WMNE) was suggested in [63, 31]. This problem was formulated as

$$\min_d \|Wd\|_2^2 \text{ subject to } v = Kd, \tag{2.20}$$

where the diagonal matrix W has elements $w_{ii} = \|K_i\|^{-1}$ with K_i denoting the column of the lead field matrix. The corresponding solution is $\hat{d}_{WMNE} = WW^TK^T(KWW^TK^T)^{-1}v$.

Another commonly used approach is the Low Resolution Electromagnetic Tomography (LORETA) [134] . LORETA is a generalized minimum norm estimate in which the distributed (smooth) electric activity is computed under the assumption that the neighboring neurons are simultaneously and synchronously active. The discrete Laplace operator B is used as a smoothness regularization and the problem is formulated as

$$\min_d \|BWd\|_2^2 \text{ subject to } v = Kd, \tag{2.21}$$

where W is a depth compensation diagonal matrix with $w_{ii} = \|K_i\|_2$. Matrix B is constructed to be a full rank, symmetric matrix employing a vanishing boundary condition. Consequently, with the LORETA approach, the superficial sources are forced to be close to zero, which is a drawback for the case of superficial brain activity.
The solution is $\hat{d}_{LORETA} = (WB^TBW)^{-1}K^T[K(WB^TBW)^{-1}K^2]\dagger v$ where $\dagger$ denotes the Moore Pensore pseudo-inverse [101].

The standardized low resolution tomography (sLORETA) [133] is an approach for the unbiased localization of a single underlying dipole source detection assuming no noise. The location of the source is designated by the maximum value of the standardized power given by $P_i = \hat{d}_i^T(R_{ii})^{-1}\hat{d}_i$ for $i = 1, \ldots, 3N$ where R_{ii} are the diagonal elements of the resolution matrix defined as $R = K^T(KK^T)^\dagger K$.

The previous described methods are linear and therefore the dipole distribution can be solved directly. However, the solution usually has many small dipoles and only a few dipoles with larger magnitude, and the reconstructed images have low spatial resolution. In the best cases, the L_2 norm regularization results in a blurred version of the actual dipole distribution [25] while sometimes imagining artefacts are introduced [98].

For better source resolution, the first proposed approach was the FOcal Underdetermined System Solver (FOCUSS) where focal sources were estimated using a recursive WMNE [37, 36]. In every iteration the weighting matrix is updated using the previous step values until most of the elements of the solution become nearly zero. The algorithm terminates when the rank of the weighting matrix drops below the number of the observations. The final solution of FOCUSS depends highly on the initial source distribution and is sensitive to noise. In [89], the authors discussed the integration of FOCUSS with LORETA in a similar fashion as in the WMNE.

For the detection detection of single dipoles, methods employing the L_1 norm [98, 99] or mixed norms [167, 43, 44, 28] have also been suggested. All these approaches are non linear and non linear optimization techniques need to be used for the solution. The first sparse inverse method is the minimum current estimate (MCE) that was introduced in [98] and is formulated as

$$\min_{d} \|d\|_1 \text{ subject to } v = Kd. \tag{2.22}$$

In [98], feasible solutions were obtained by selecting as many dipole moments as the the number of the observations. The L_1 norm imposes sparsity on the individual components of the dipole moments, and therefore the solution is axes-parallel. In [167], a two step algorithm was suggested such as to avoid the orientation bias i.e the orientation was solved using MNE and the amplitude via L_1 norm minimization.

Mixed norms, e.g. the dipole strength norm were suggested in [131, 43, 28] in order to eliminate the axes-parallel dipole distribution. In [43], the Focal Vector Field Reconstruction approach was based on the estimation of

$$\min_{d} \sum_{i=1}^{N} \|\mathrm{w}_i \mathrm{d}_i\|_2 + \alpha \sum_{i=1}^{N} \|\mathrm{w}_i \mathrm{t}_i\|_2, \text{ subject to } v = Kd, \tag{2.23}$$

where d_i is the dipole moment and t_i the Laplace operator evaluated in the components of d. The weights $\mathrm{w}_i \in \mathbb{R}^{3\times 3}$ are taken from the resolution matrix of the sLoreta [133]. With this approach imaging artefacts were reduced and sparsity is ensured, however the complexity of the algorithm and the tuning of the extra regularization parameters does not make the approach preferable.

2.5.4 EEG Inverse Problem using Bayesian Framework

The need to use prior information for the reconstruction of dipole sources and the availability of only a limited number of noise-corrupted potential measurements makes the use of probabilistic approaches appropriate. In this section, the basis of the probabilistic theory [71] and the Gaussian assumptions for the solution of EEG inverse problem are discussed. Under certain assumptions, the results of this section coincide with the results of the deterministic approaches