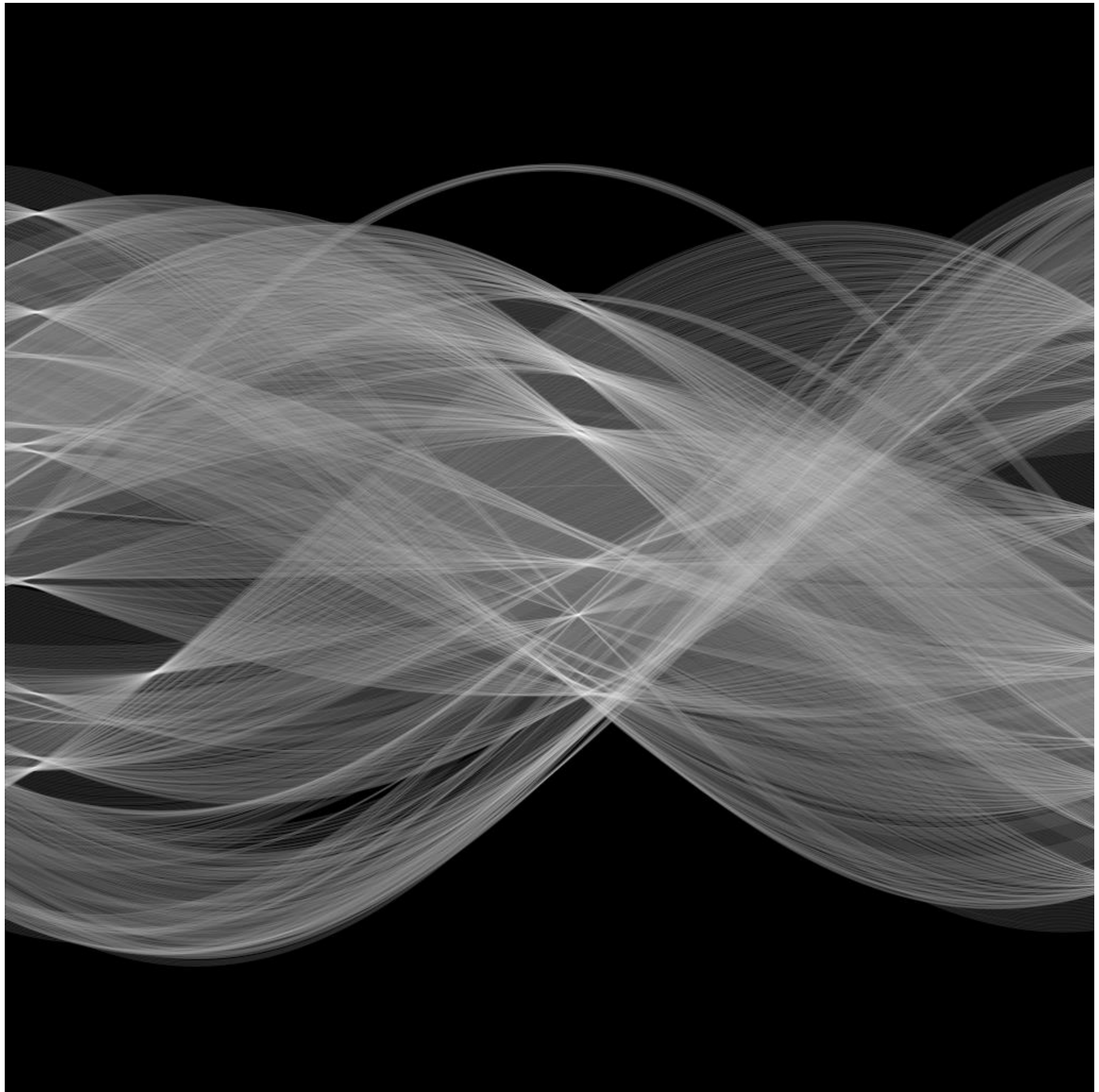
Curve fitting



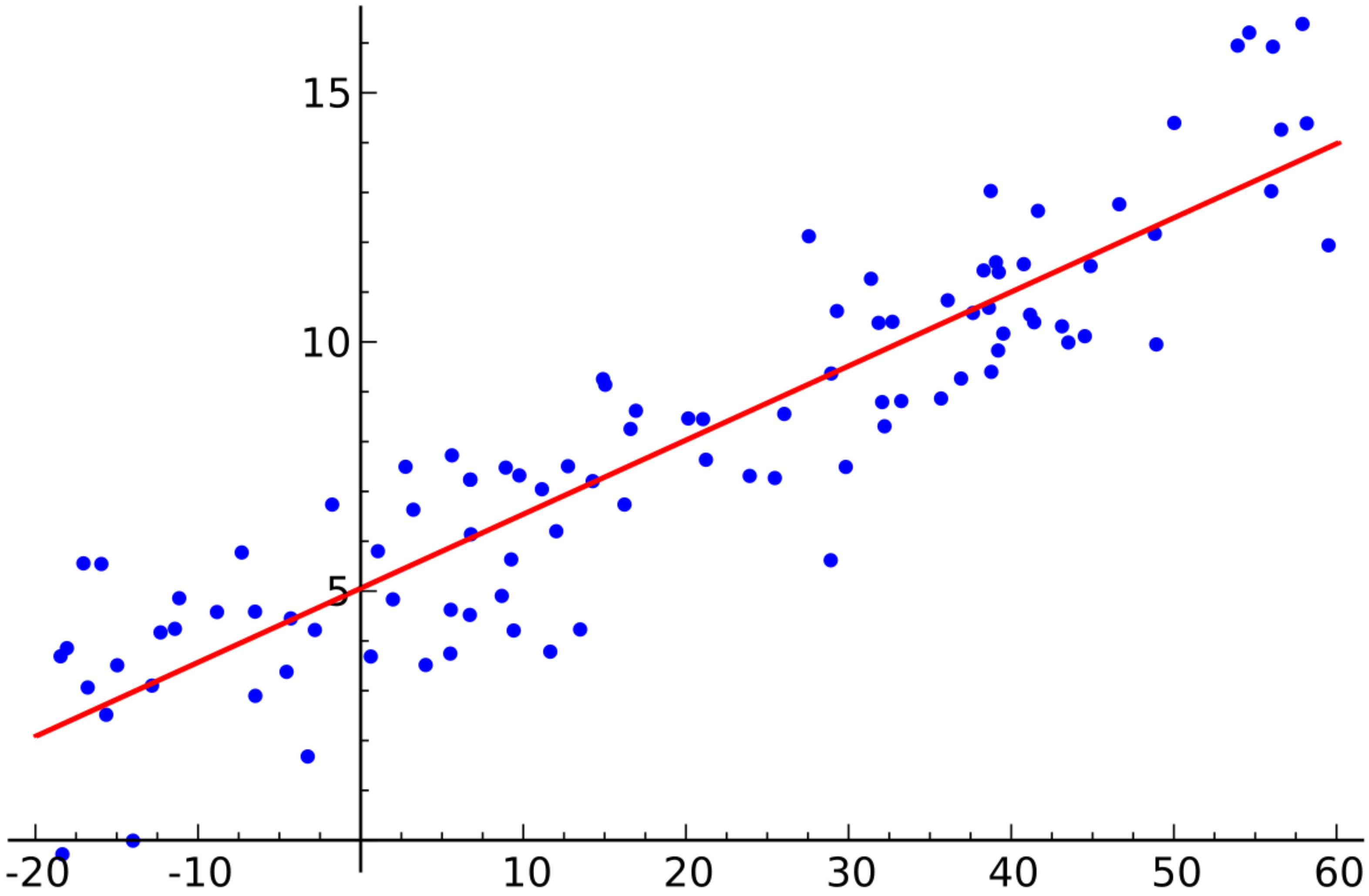
References

- http://szeliski.org/Book/
- http://www.cs.cornell.edu/courses/cs5670/2019sp/lectures/lectures.html
- http://www.cs.cmu.edu/~16385/

TOC

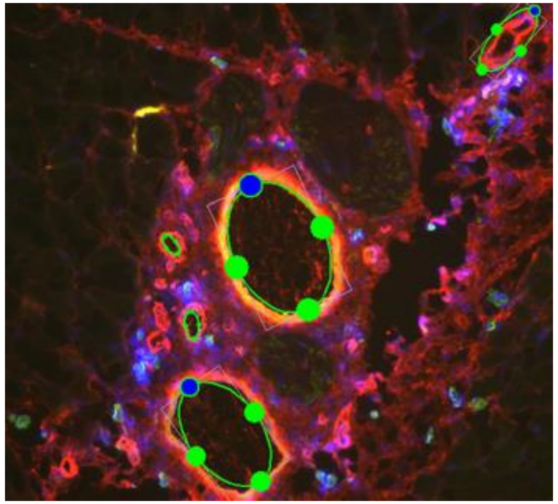
Some motivation

- What you think about when you here “line/curve fitting”

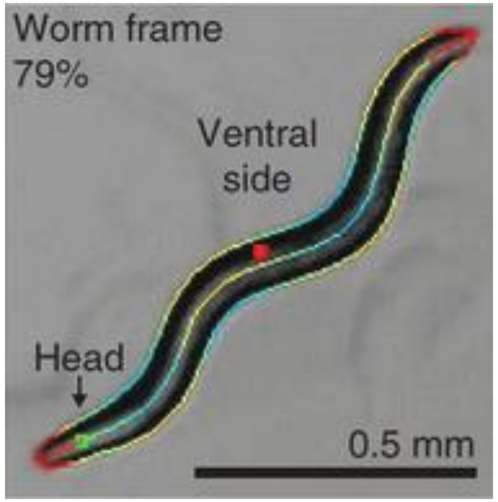


Some motivation

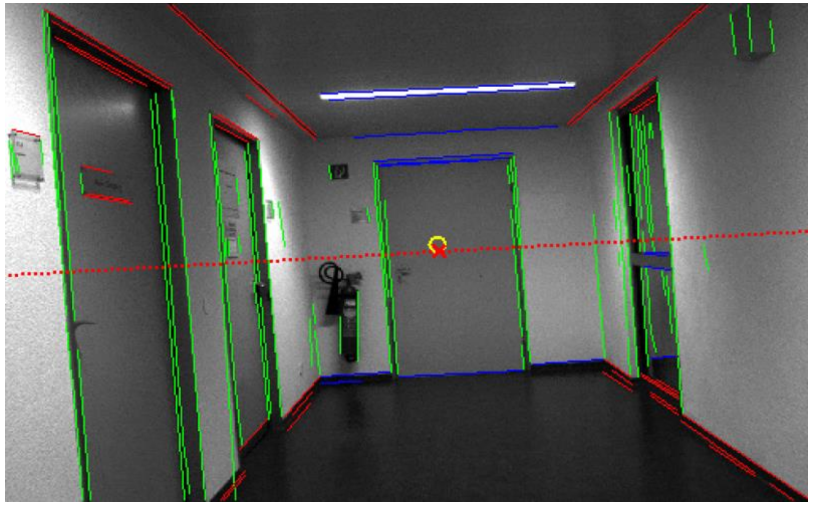
Autonomous Vehicles
(lane line detection)



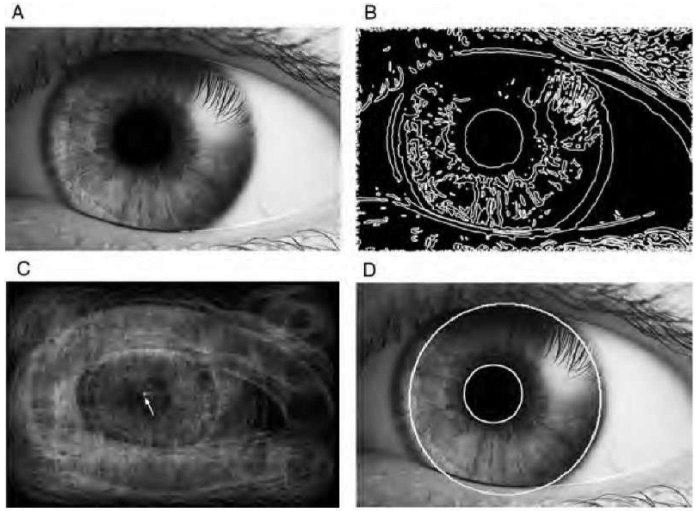
Bio-medical engineering
(blood vessel counting)



Biology
(earthworm contours)



Robotics
(scene understanding)



Psychology/ Human computer interaction
(eye tracking)

What is curve fitting

- **Curve fitting** is the process of constructing a curve, or mathematical function, that has the best fit to a series of data points. [Wikipedia]
- Input: dataset (e.g.: $\{(x_i, y_i)\}_{i=1,..,N}$ in 2D).
- Output: best representing function (e.g.: $f(x) = y$).

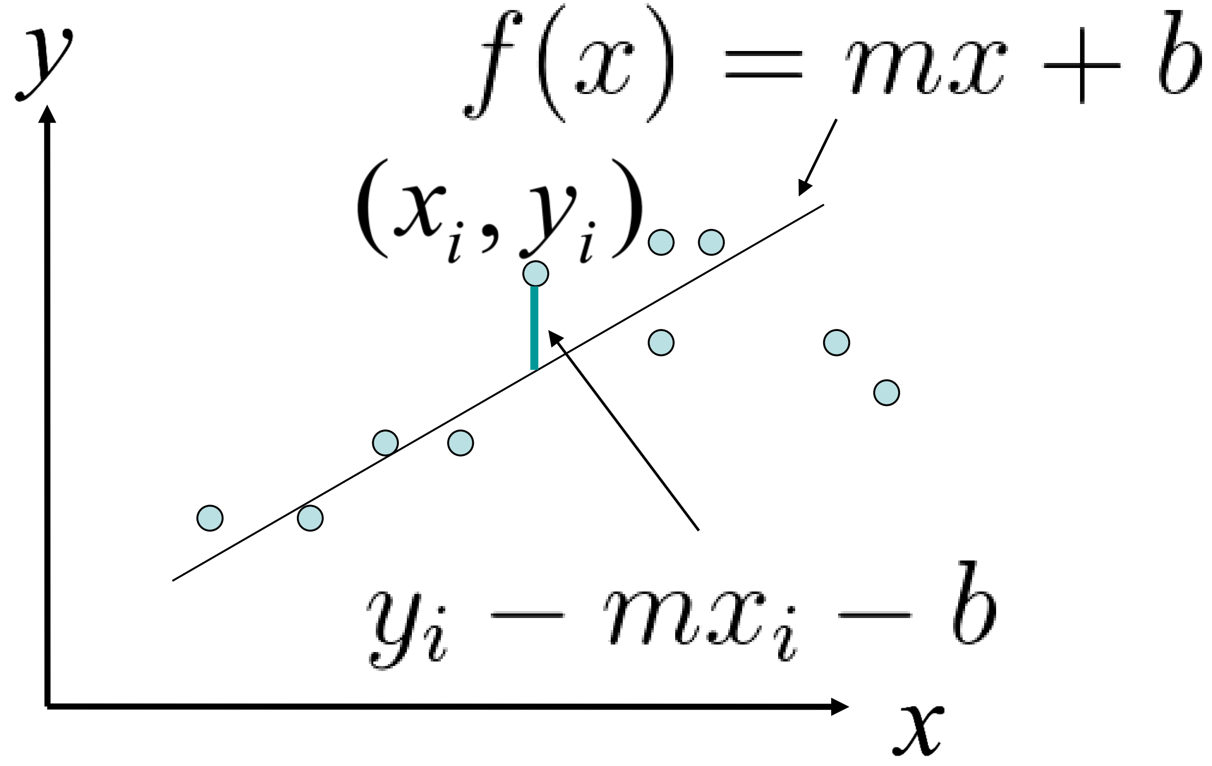
- This problem sometimes also called **Regression.**

TOC

- **Linear least squares**
- Total least squares
- Least squares
- RANSAC
- Hough transform
  - $(\boldsymbol{m}, \boldsymbol{b})$ parameter space
  - $(\boldsymbol{\rho}, \boldsymbol{\theta})$ parameter space

Linear least squares (LS): step by step example

- Given: $\{(x_i, y_i)\}_{i=1,\ldots,N}$
  find best line representation: $f(x) = mx + b$
- The best representation (m, b) are the ones that minimizes the total error e.



Side note: error / loss

- **Error** (also known as **loss**) is a method of evaluating how well specific algorithm models the given data.
  - If predictions deviates too much from actual results, error would be high.
- A popular error used a lot in CV and statistics is called **MSE** (mean square error, also known as **L2 loss** or **quadratic loss**).

$$\boxed{e_{MSE} = \frac{1}{N}\sum_{i=1}^{N}(y_i - f(x_i))^2}$$

- We need to find the set of variables that minimizes the error - **MMSE** (minimum mean square error).

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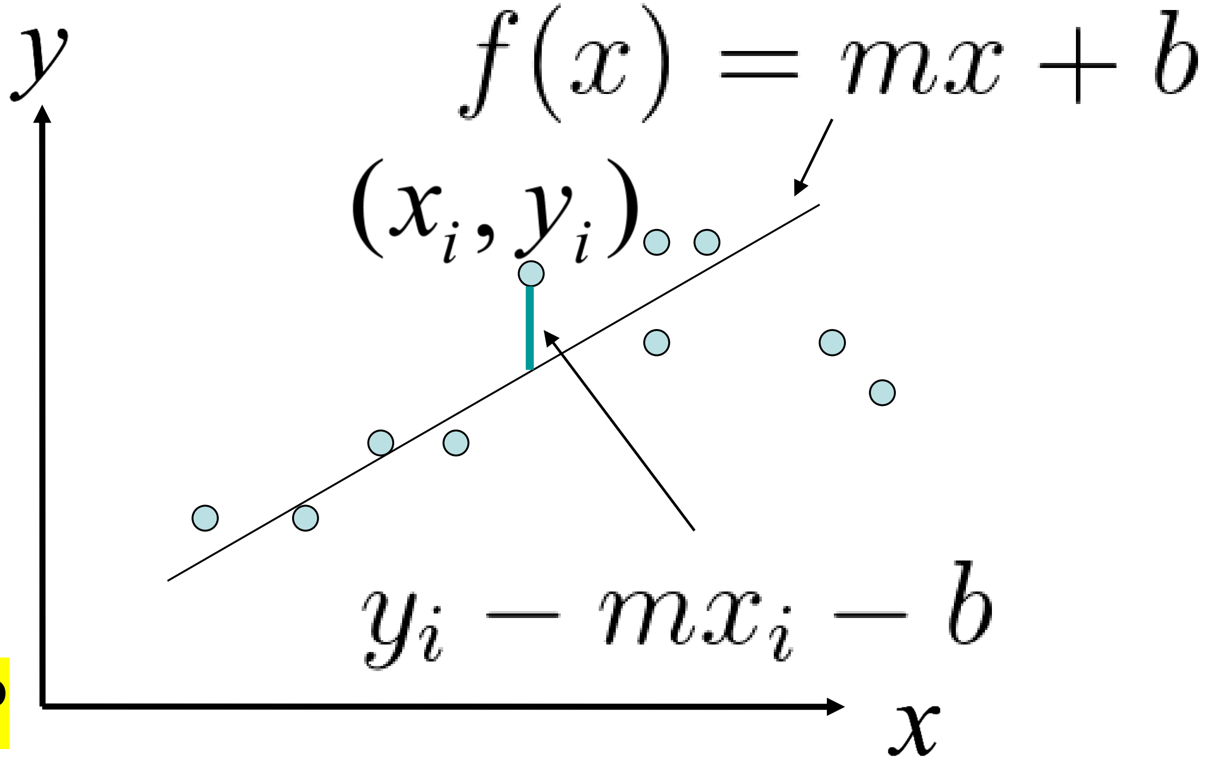
$$\downarrow$$

$$e = \frac{1}{N}\sum_{i=1}^{N}(y_i - mx_i - b)^2$$



- How do we find this set of variables?

Linear least squares (LS): step by step example

- Find derivatives of e with respect to both variables m, b s.t. (such that) we'll reach the minimum error (partial derivative of both variables equals zero…):

$$e = \frac{1}{N}\sum^{N}(y_i - mx_i - b)^2$$

Linear least squares (LS): step by step example

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  - Define: $\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i,\ \overline{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$

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$$0 = \frac{\partial e}{\partial n}$$

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$$0 = \frac{\partial e}{\partial n} = \frac{-2}{N}\sum_{i=1}^{N}(y_i - mx_i - b)$$

Linear least squares (LS): step by step example

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$$0 = \frac{\partial e}{\partial m} \mapsto \ldots \mapsto m = \frac{\sum_{i=1}^{N}(y_i - \overline{y})(x_i - \overline{x})}{\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

- Full derivation here.

Linear LS in matrix form

$$e = \frac{1}{N}\sum_{i=1}^{N}(y_i - mx_i - b)^2$$

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$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \quad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \quad \beta = \begin{bmatrix} m \\ b \end{bmatrix}$$

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$$e = \frac{1}{N}||y - X\beta||^2 = \frac{1}{N}(y - X\beta)^T(y - X\beta) =$$

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$$\frac{1}{N}(y^T y - \beta^T X^T y - y^T X\beta + \beta^T X^T X\beta) \overset{\beta^T X^T y = scalar = scalar^T = y^T X\beta}{=}$$

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$$\frac{1}{N}(y^Ty - \beta^TX^Ty - y^TX\beta + \beta^TX^TX\beta) \overset{\beta^TX^Ty=scalar=scalar^T=y^TX\beta}{=}$$

$$\frac{1}{N}(y^Ty - 2\beta^TX^Ty + \beta^TX^TX\beta)$$

$$0 = \frac{\partial e}{\partial \beta} = 2X^TX\beta - 2X^Ty$$

Linear LS in matrix form

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$$X^T X\beta = X^T y$$

$$\boxed{\beta = (X^T X)^{-1} X^T y}$$

Side note: pseudoinverse

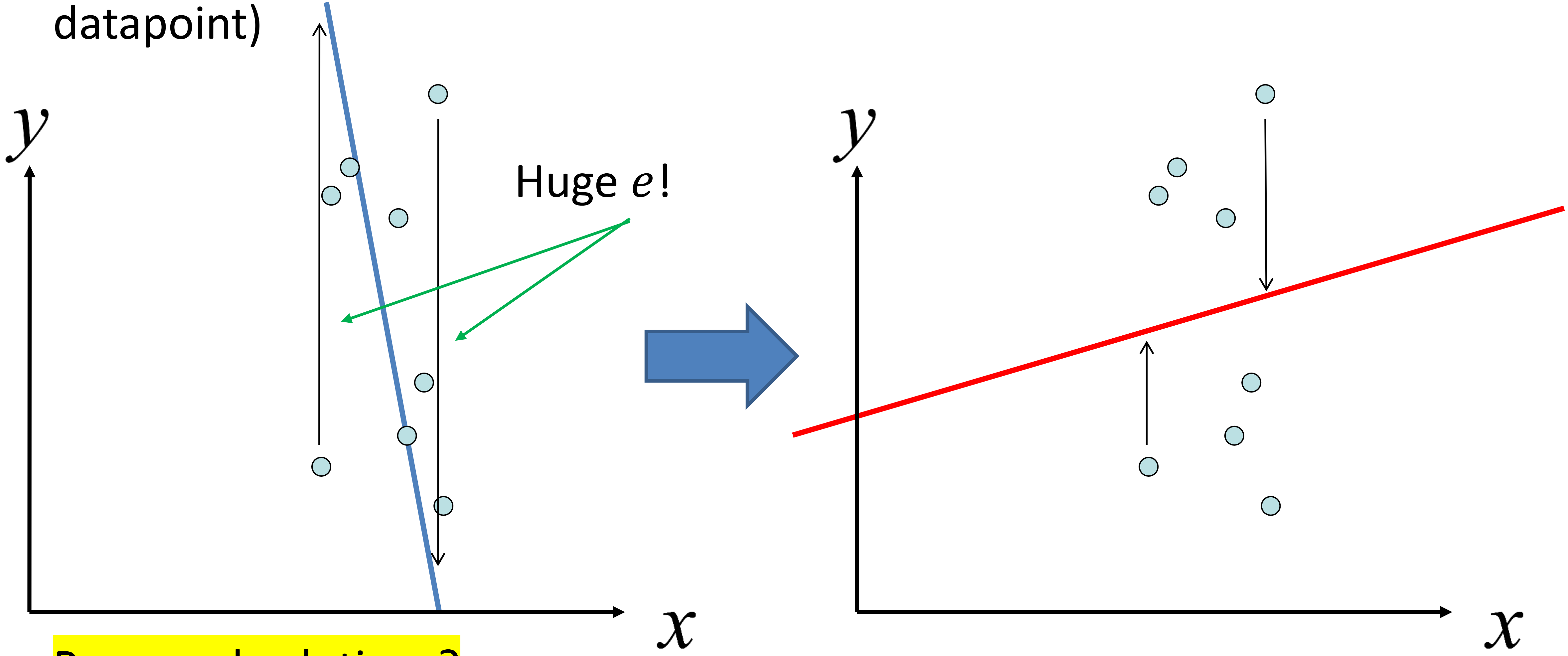
$$\boxed{\beta = \underbrace{(X^T X)^{-1} X^T} y}$$

- This is a known result which is called **the pseudoinverse matrix** of X. It is also known as Moore–Penrose inverse.
- If $X^T X$ is not invertible, the solution will be: $\beta = (X^T X + \epsilon I)^{-1} X^T y$
- The solution above is **a solution for any linear regression problem that is over-determined** (==more equations than unknowns).

- least_squares.ipynb

Problem 1: Linear LS with vertical data

- Near vertical data is hard to fit since the error is computed perpendicular to x axis + the bigger the error, the bigger the error squared! (more weight for this datapoint)



- Proposed solutions?

Problem 1: Linear LS with vertical data

- One possible solution is making all errors weigh the same (and not squared). This will make the far points have the same impact on error as closer points. One such error **MAE** (mean absolute error, also known as **L1 loss**) instead of MSE.

$$e_{MAE} = \frac{1}{N}\sum_{i=1}^{N} ||y_i - f(x_i)||$$

  – The derivation of MAE is out of the class scope, but some details can be found here.

- Another possible solution is computing the error distance of each point in a different way- one that takes into account the y data as well. One such algorithm is **linear total least squares.**

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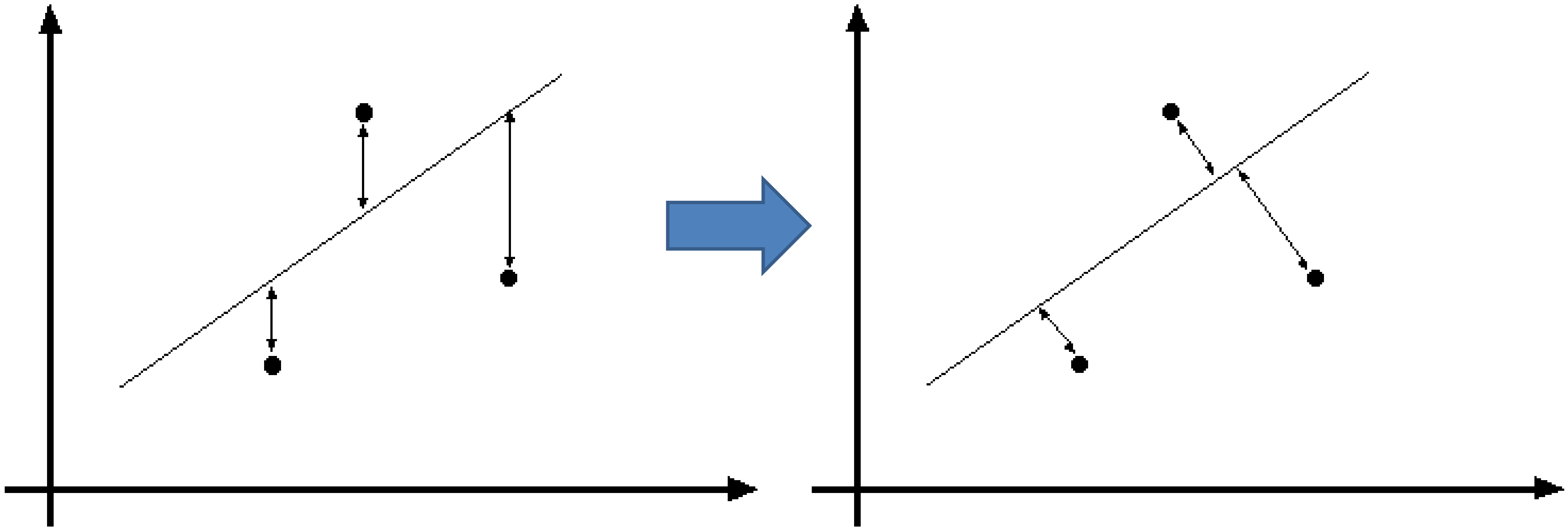
Linear total least squares



Linear TLS

- Another representation of a line: $ax + by + c = 0$
- Distance between a point (x_i, y_i) and line: $$d_i = \frac{|ax_i + by_i + c|}{\sqrt{a^2 + b^2}}$$

Linear TLS

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- The line equation above has 2 degrees-of-freedom (DOFs), so we can decide that, for later purpose, $a^2 + b^2 = 1$.

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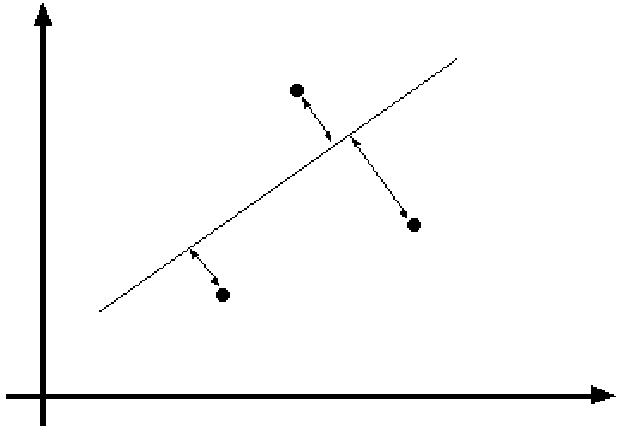
- The line equation above has 2 degrees-of-freedom (DOFs), so we can decide that, for later purpose, $a^2 + b^2 = 1$.
- The error to be minimized:

$$e_{MSE} = \frac{1}{N} \sum_{i=1}^{N} d_i^2 = \frac{1}{N} \sum_{i=1}^{N} (ax_i + by_i + c)^2$$



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$$0 = \frac{\partial e}{\partial c} = \frac{1}{N}\sum_{i=1}^{N} 2(ax_i + by_i + c)$$

Linear TLS

$$e_{MSE} = \frac{1}{N}\sum_{i=1}^{N} d_i^2 = \frac{1}{N}\sum_{i=1}^{N}(ax_i + by_i + c)^2$$

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$$c = -\frac{a}{N}\sum_{i=1}^{N} x_i - \frac{b}{N}\sum_{i=1}^{N} y_i = -a\overline{x} - b\overline{y}$$

Linear TLS

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$$e = \frac{1}{N}\sum_{i=1}^{N}(ax_i + by_i - a\overline{x} - b\overline{y})^2$$

Linear TLS

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Linear TLS

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Linear TLS

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$$\frac{1}{N}\|X\beta\|^2 = \frac{1}{N}\beta^T X^T X \beta$$

Linear TLS

$$e_{MSE} = \frac{1}{N}\sum_{i=1}^{N} d_i^2 = \frac{1}{N}\sum_{i=1}^{N}(ax_i + by_i + c)^2$$

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$$\frac{1}{N}\left\|X\beta\right\|^2 = \frac{1}{N}\beta^T X^T X\beta$$

$$\begin{cases} minimize & \frac{1}{N}\beta^T X^T X\beta \\ s.t. & a^2 + b^2 = 1 \end{cases}$$

Linear TLS

$$e_{MSE} = \frac{1}{N}\sum_{i=1}^{N} d_i^2 = \frac{1}{N}\sum_{i=1}^{N}(ax_i + by_i + c)^2$$

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$$\frac{1}{N}\left\|X\beta\right\|^2 = \frac{1}{N}\beta^T X^T X\beta$$

$$\begin{cases} minimize & \frac{1}{N}\beta^T X^T X\beta \\ s.t. & a^2+b^2=1 \end{cases} = \boxed{\begin{cases} minimize & \beta^T X^T X\beta \\ s.t. & \beta^T\beta = 1 \end{cases}}$$

Linear TLS -the minimization problem

- The minimization problem is:

$$\begin{cases} minimize & \beta^T X^T X \beta \\ s.t. & \beta^T \beta = 1 \end{cases}$$

- Recall eigendecomposition: $Av = \lambda v \mapsto v^T A v = \lambda$
  - Also recall that each eigenvector v is normalized ($||v|| = v^T v = 1$).
- $\boxed{\text{The solution to the minimization problem above is the eigenvector corresponding to smallest eigenvalue of } X^T X.}$

- **Watch out:** trying to minimize the problem above without the constraint $\beta^T \beta = 1$ will result with the trivial solution of $\beta = 0$.

- Another similar way to solve this problem is using SVD, but it’s out of scope…
- You can check the derivation out in: http://cmp.felk.cvut.cz/cmp/courses/XE33PVR/WS20072008/Lectures/Supporting/constrained_lsq.pdf
- Read more about SVD decomposition here: https://en.wikipedia.org/wiki/Singular_value_decomposition

TOC

LS

- What about curve fitting?
- Recall the matrix linear LS result: $\beta = (X^T X)^{-1} X^T y$

- The solution above is a solution for any **linear regression** problem that is over-determined (==more equations than unknowns).
  - Linear regression $\neq$ linear LS
  - Linear regression means only that the unknowns are linearly dependent in the data.
- For example- data set of a parabola:

  $\{(x_i, y_i)\}\ s.t.\ ax_i^2 + bx_i + c = y_i$

- How the matrices X, y, β will look like?

LS

- data set of a parabola:

  $\{(x_i, y_i)\}\ s.t.\ ax_i^2 + bx_i + c = y_i$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \quad X = \begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_N^2 & x_N & 1 \end{bmatrix} \quad \beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

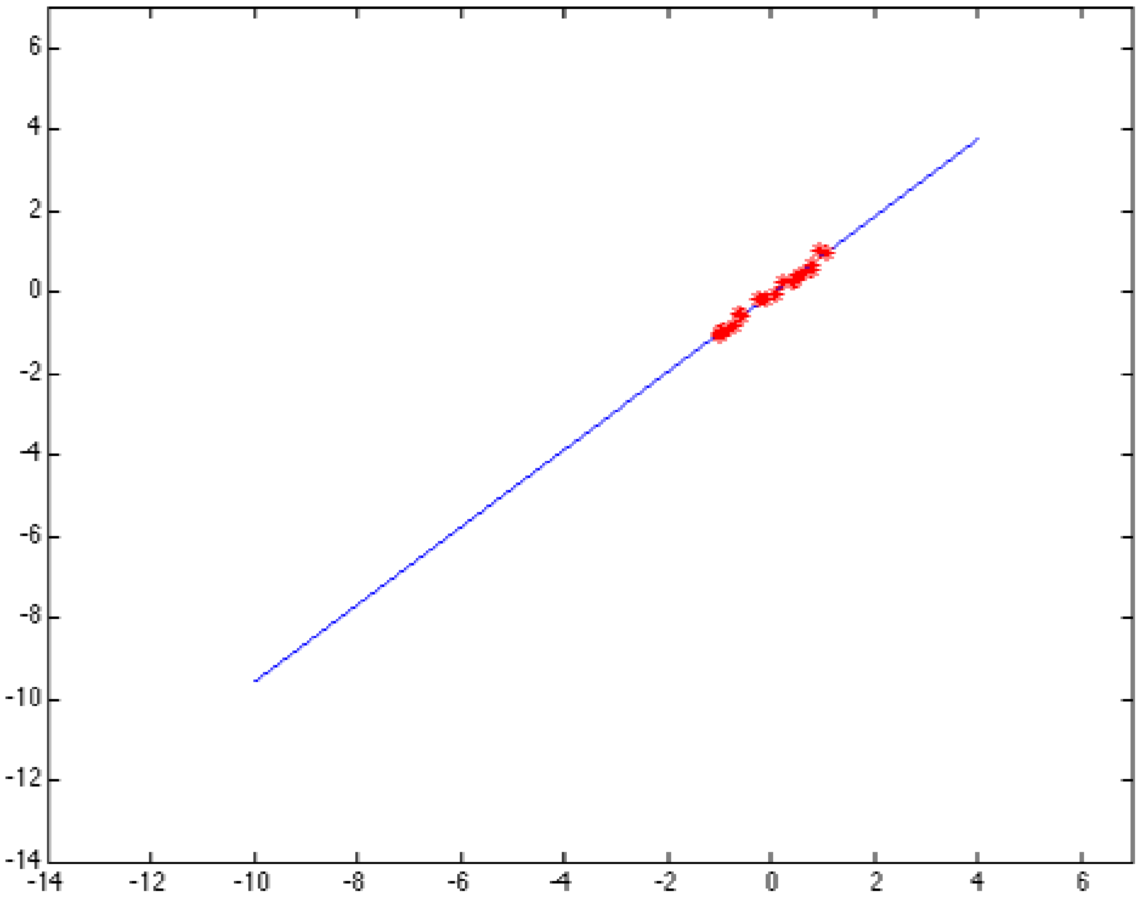
- The solution of this LS problem is the same (in our Derivation we didn't use the assumption that the data is linear…):

$$\beta = (X^T X)^{-1} X^T y$$

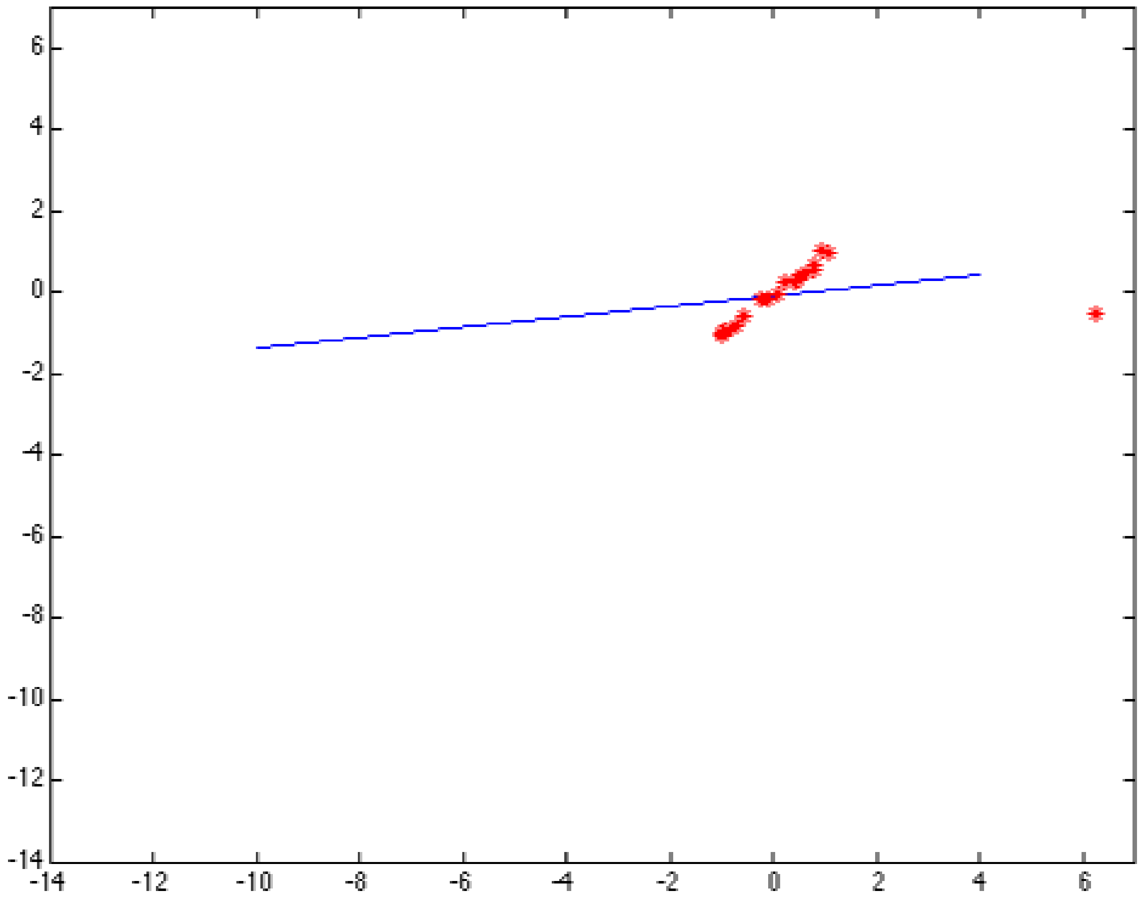
- Non-linear TLS also exist, but this can't be solved as before (the linearity assumption was used). This topic is out of scope- proof and examples here.

Problem 2: LS with outliers datapoints

- **Outlier**: a data point that differs significantly from other observations. [Wikipedia]



Least-squares error fit



Squared error heavily penalizes outliers

Problem 2: LS with outliers datapoints

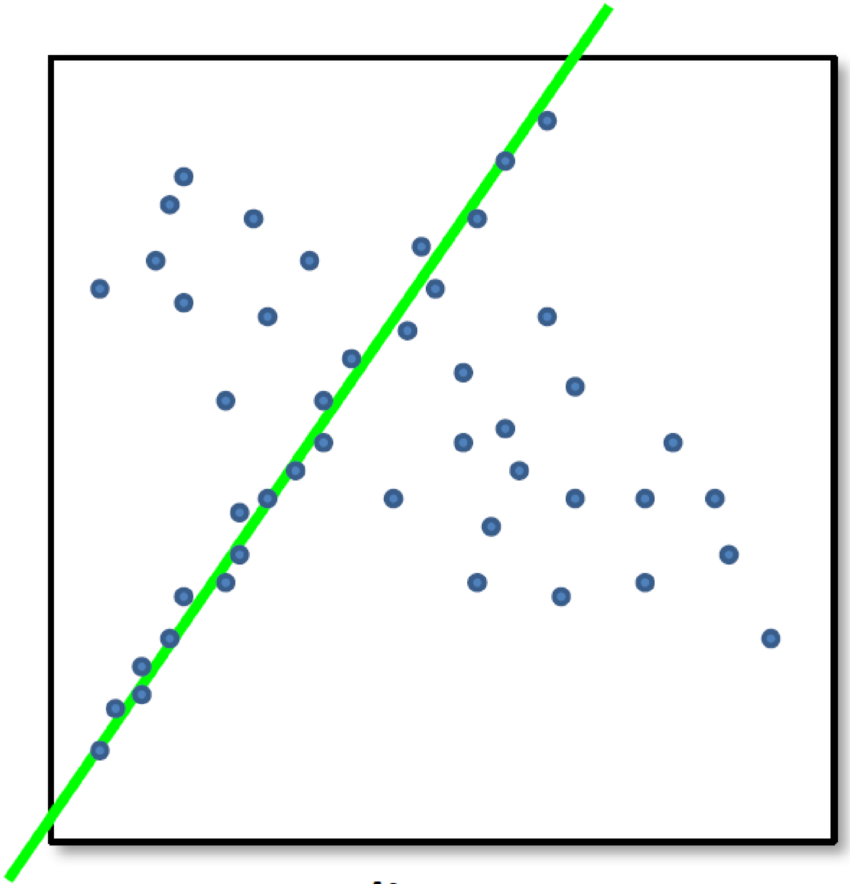
- One possible solution: MAE (again)- for less penalization on outliers.
- A better solution: removing the outliers! Possible algorithm to use: **RANSAC**

TOC

- Linear least squares
- Total least squares
- Least squares
- **RANSAC**
- Hough transform
  - $(\boldsymbol{m}, \boldsymbol{b})$ parameter space
  - $(\boldsymbol{\rho}, \boldsymbol{\theta})$ parameter space

RANSAC

- **Random sample consensus** (**RANSAC**) is an iterative method to estimate parameters of a mathematical model from a set of observed data that contains outliers. [Wikipedia]



Inliers: 20

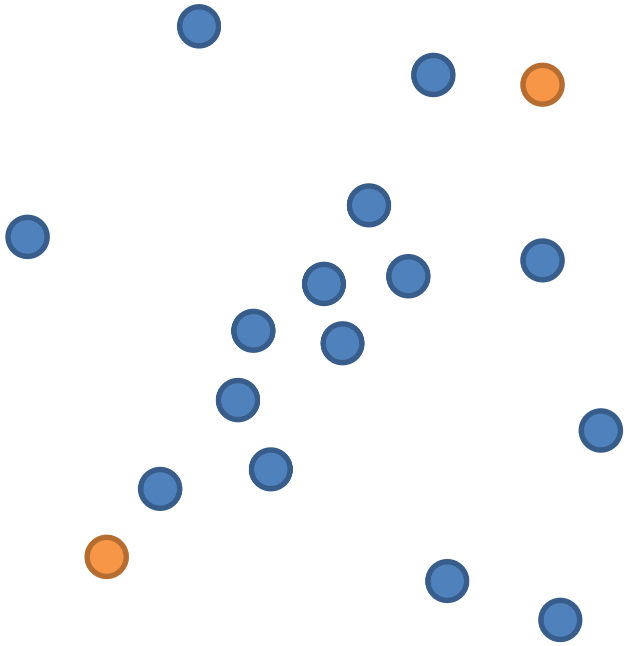
RANSAC algorithm

```
While searching for best model fit:
      Select a random subset of the original data.
      Fit a model to the data subset.
      find inliers that fit the given model.
      if number_inliers is bigger than the old best model
      number_inliers:
            Save as best model
```

RANSAC algorithm - step by step - 1

`Select` `a random subset of the original data.`

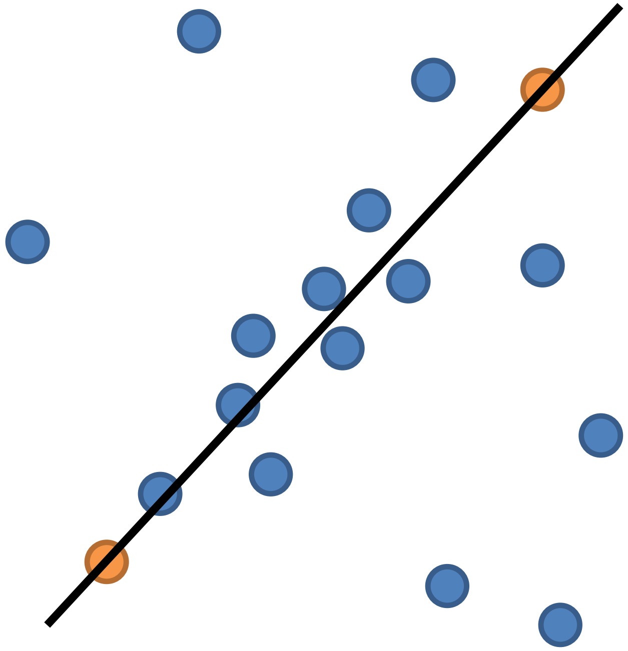
- The number of samples in the subset is the smallest needed to determine the model parameters (== number of unknow variables).
- Examples:
  - For line fit- only 2 datapoints.
  - For parabola fit- 3 datapoints.



RANSAC algorithm - step by step - 2

`Fit` `a model to the data subset.`

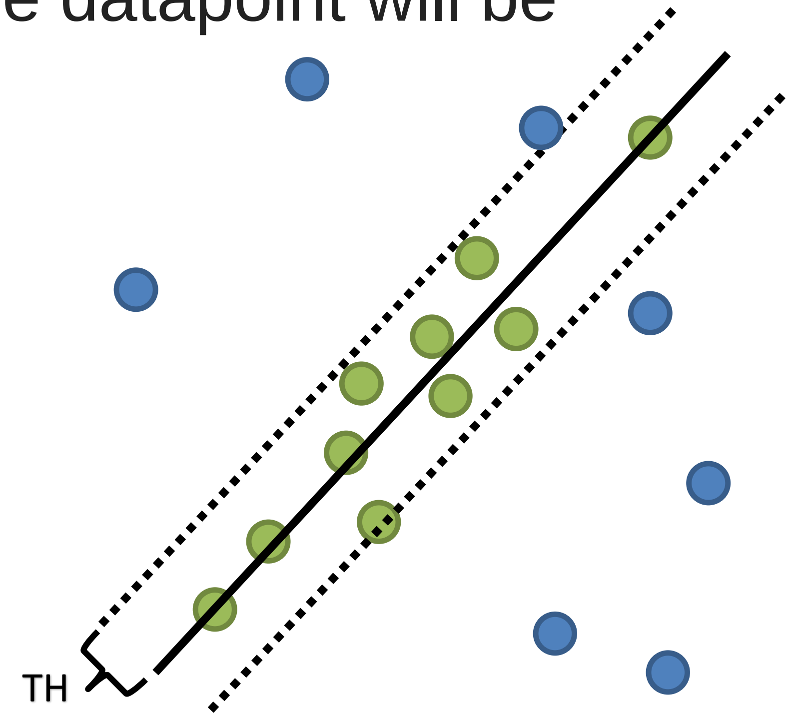
- Can be done using a chosen algorithm (for example LS).



RANSAC algorithm - step by step - 3

`find inliers that fit the given model.`

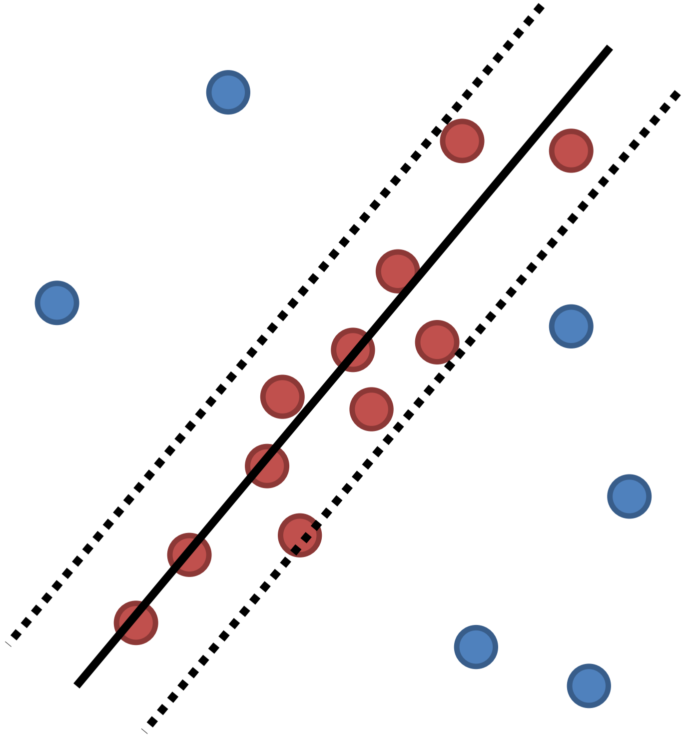
- Test all dataset against the fitted model. Points that fit the estimated model well, according to some chosen loss function, are considered as part of the consensus set. This points are called **inliers**.
- A possible loss function is MSE of the distances (same as TLS).
  - Choose a threshold for the error: below this TH the datapoint will be consider as an inlier.



RANSAC algorithm - step by step - 4

```
if number_inliers is bigger than the old best model number_inliers:
    Save as best model
```

- Trivial…
- An improvement to the final step can be iteratively re-fitting the model with all inliers to find a better describing model.



RANSAC algorithm - step by step - 0

`While` `searching for best model fit:`

- How do we know when to stop? How much we need to iterate before getting the best model?
- Let’s look at a possible statistical model for this question.

RANSAC convergence

- Denote

$$\omega = \frac{\#\ inliers}{\#\ total\ datapoints}:$$

  - getting ω ratio will be considered as reaching the best model.
  - This ratio is usually user specified according to dataset properties (or educated guess).

k: number of iterations (will be calculated).

p: percentage of achieving best model in chosen k iterations (also user specified).

n: number of initially sampled subset datapoints.

RANSAC convergence

- ω^n is the probability that all n points are inliers.
- $1 - \omega^n$ is the probability that at least one of the n points is an outlier.
- $1 - p = (1 - \omega^n)^k$ is the probability that in k iterations we will always have at least one outlier (meaning we didn’t get best match).
- $\boxed{k = \frac{\log(1-p)}{\log(1-\omega^n)}}$

TOC

- Linear least squares
- Total least squares
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  - $(\boldsymbol{m}, \boldsymbol{b})$ **parameter space**
  - $(\boldsymbol{\rho}, \boldsymbol{\theta})$ parameter space

$(\boldsymbol{m}, \boldsymbol{b})$ Parameter space

- Line function: $y = mx + b$
  - Usually we are given (m, b) constants, and the variables are (x, y).
- In a regression problem we are given (x, y), and the unknowns we wish to find are the best fit for (m, b).
  - **Let's look at $(\boldsymbol{m}, \boldsymbol{b})$ as our variables.**

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

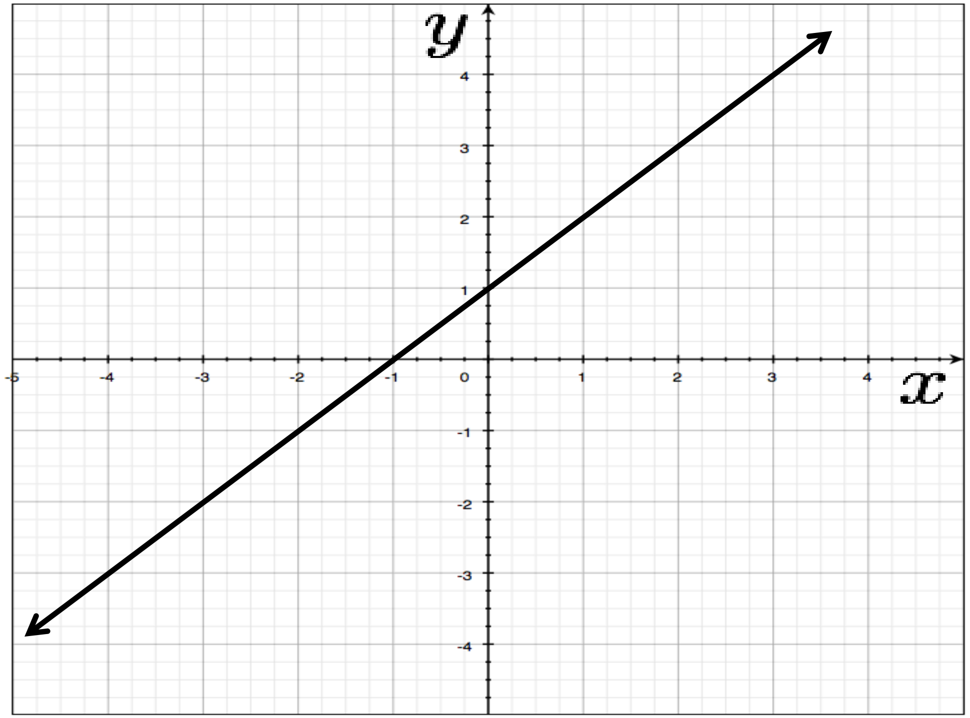


Image space

Let's look at (m,b) as our variables

Plots are from: http://www.cs.cmu.edu/~16385/

(m, b) Parameter space

variables

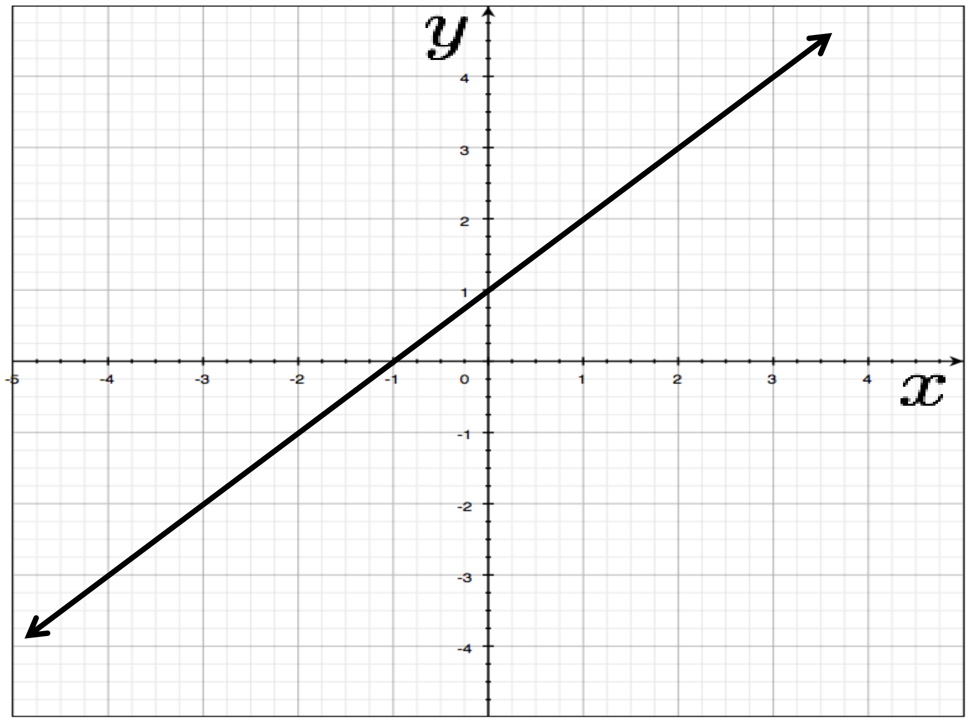
$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters



a line becomes a point

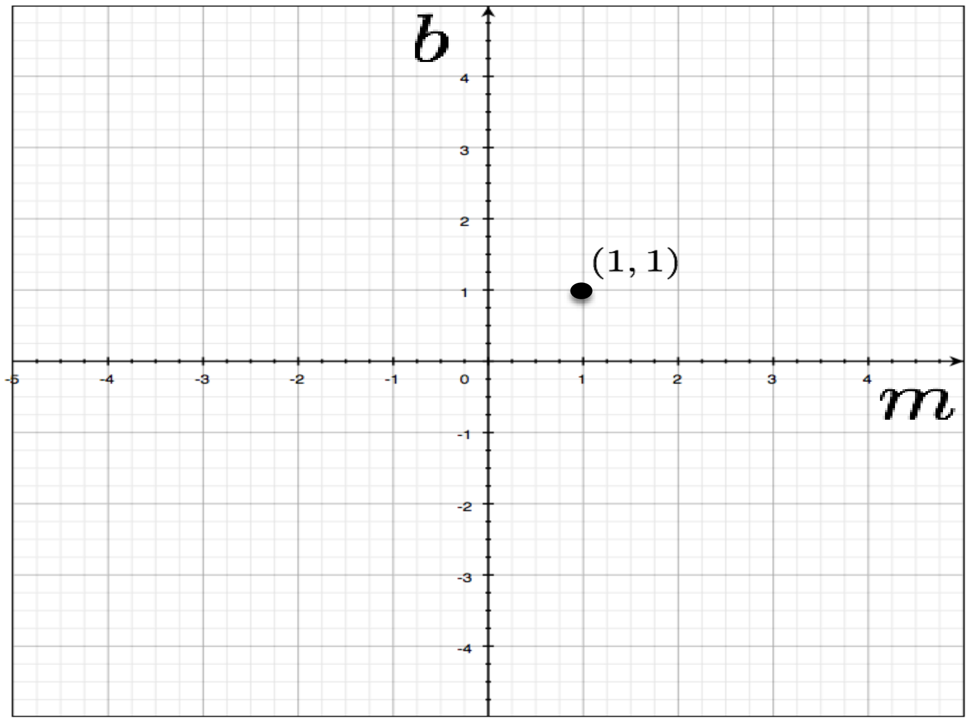


Image space

Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

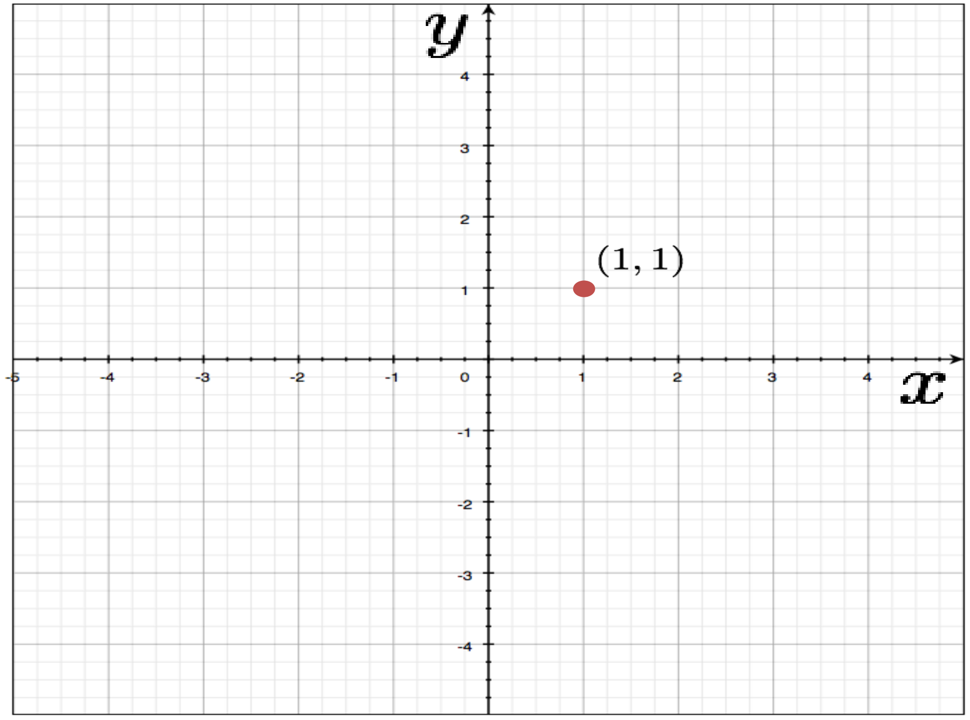


Image space

a point becomes a ?

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

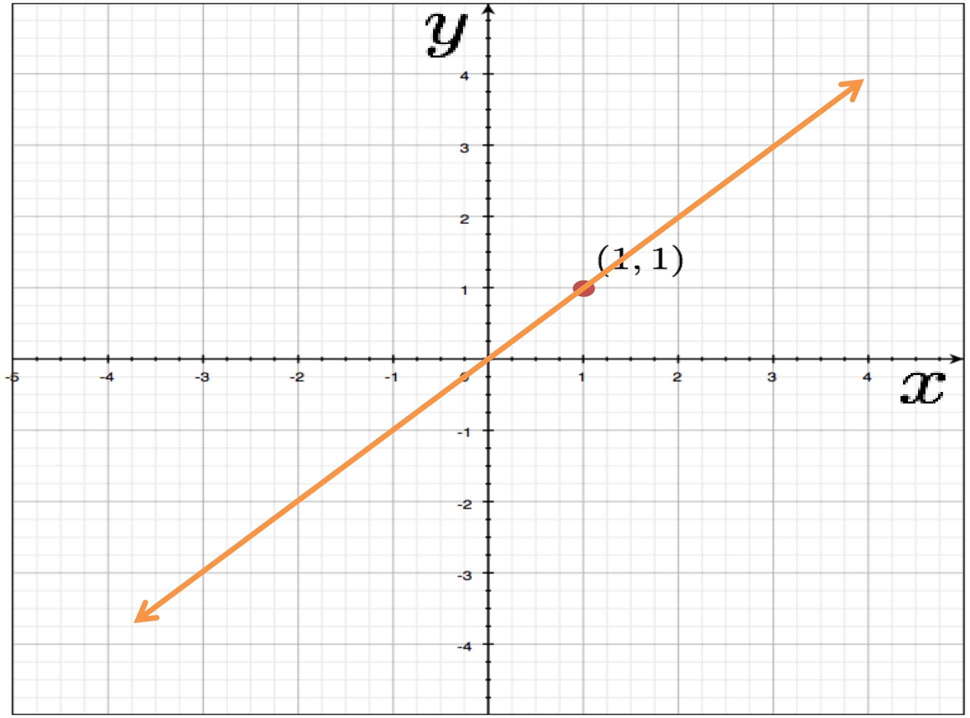
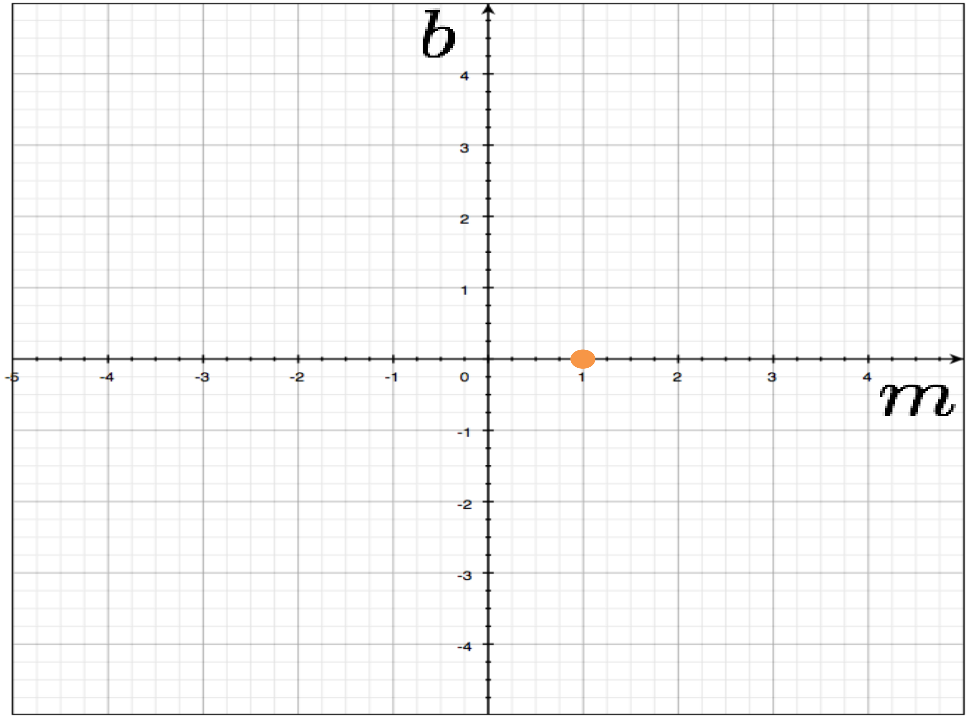


Image space

a point becomes a ?



Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

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parameters

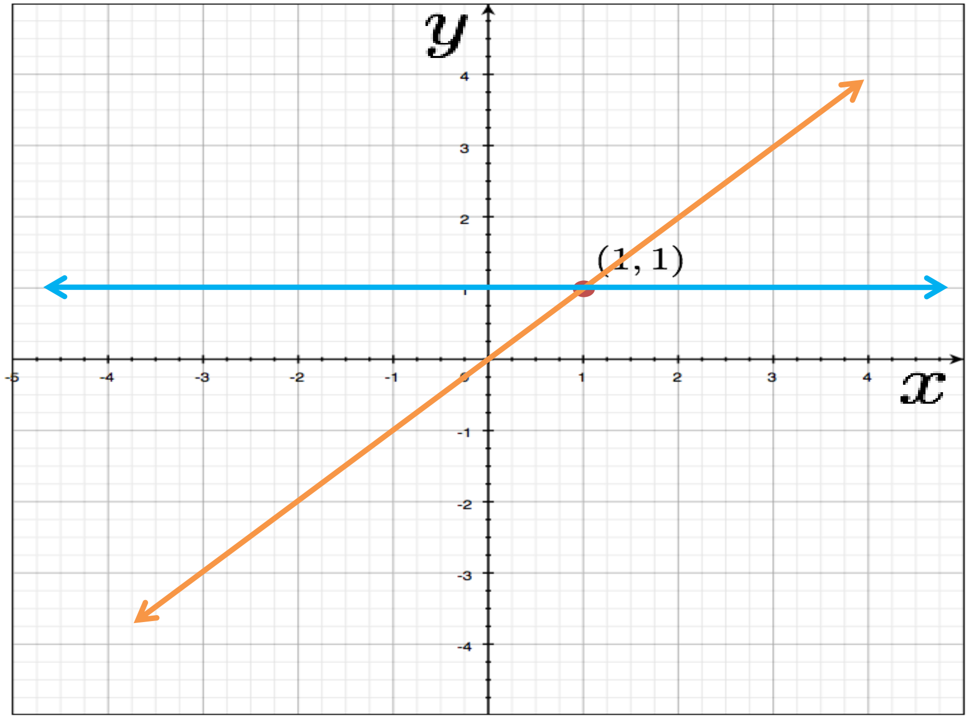
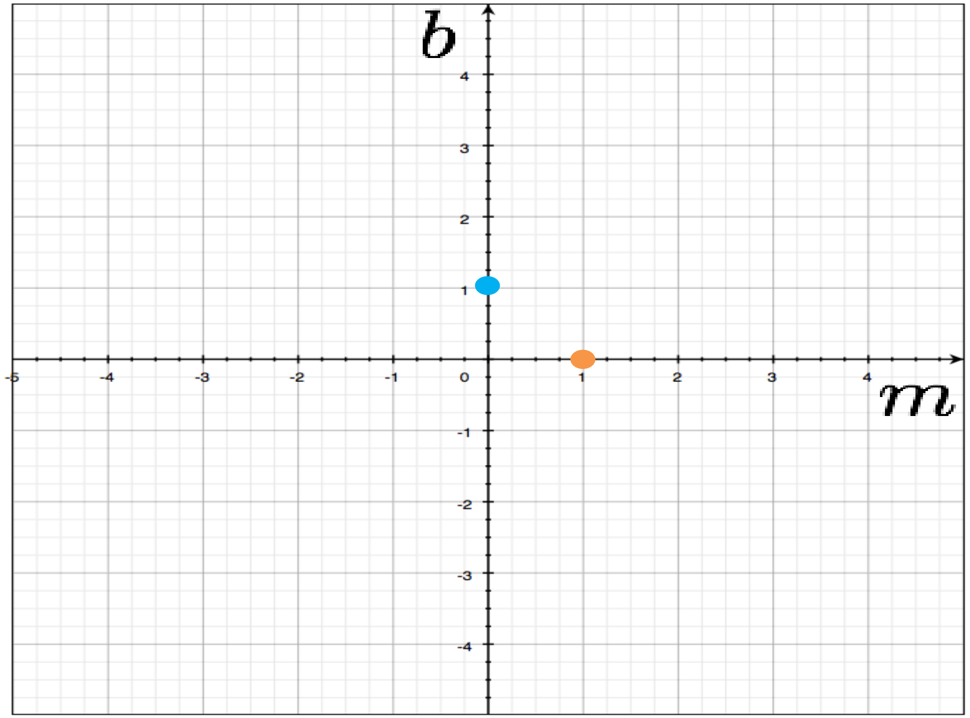


Image space

a point becomes a ?



Parameter space

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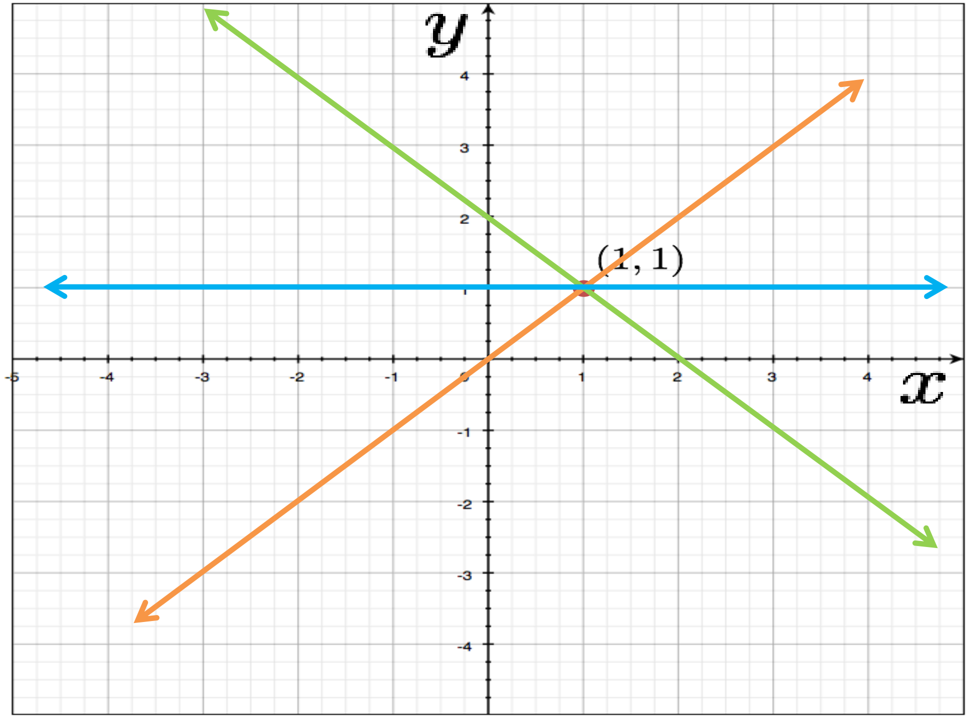
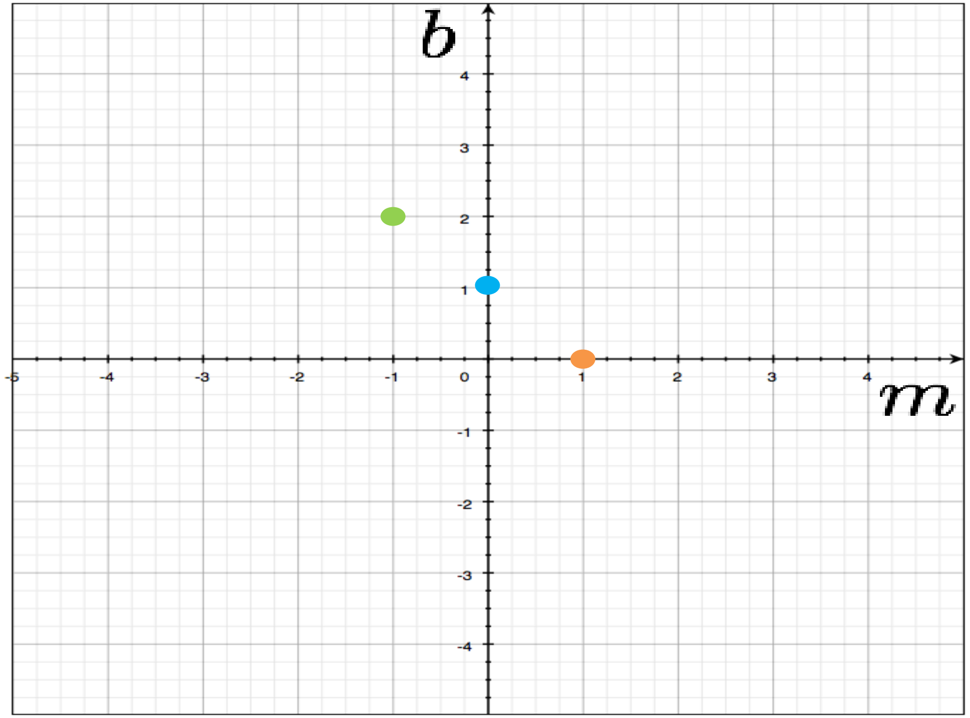


Image space

a point becomes a ?



Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

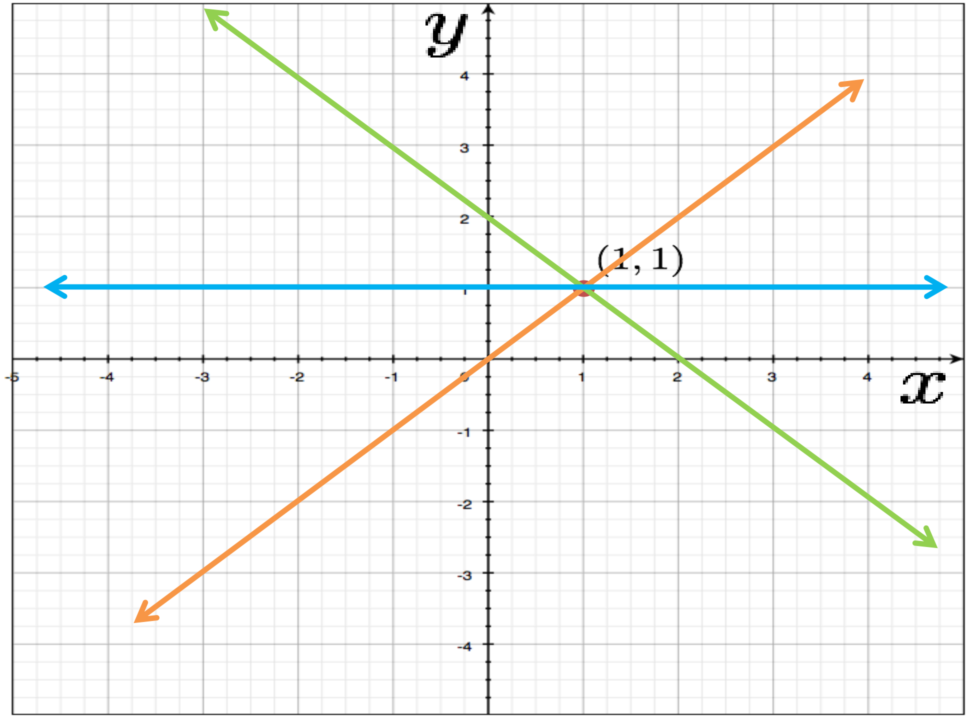
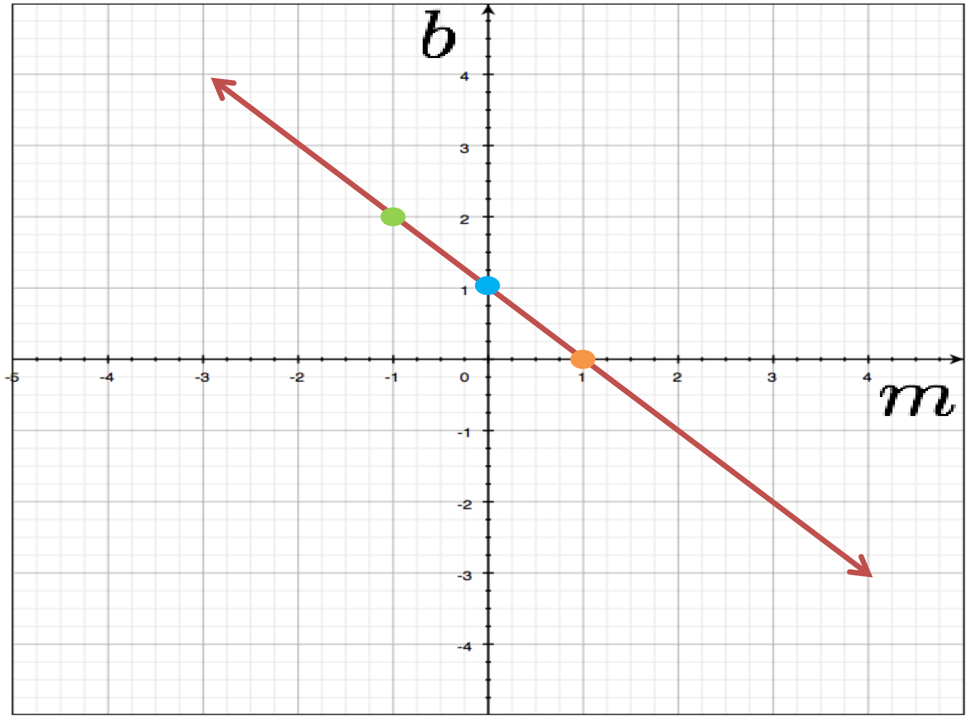


Image space

a point becomes a **line**



Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

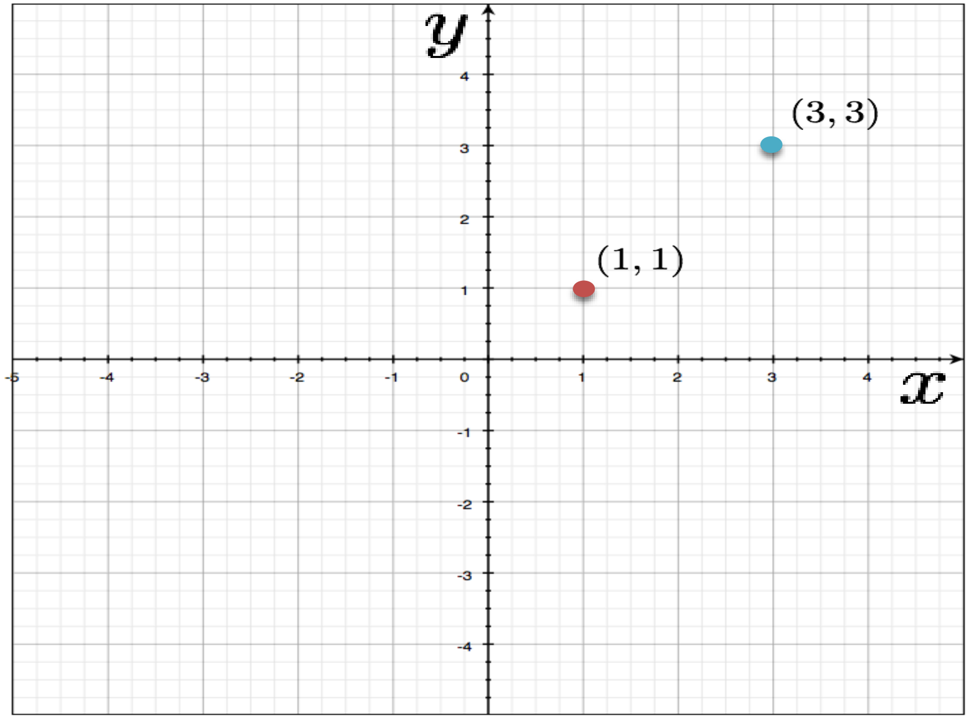
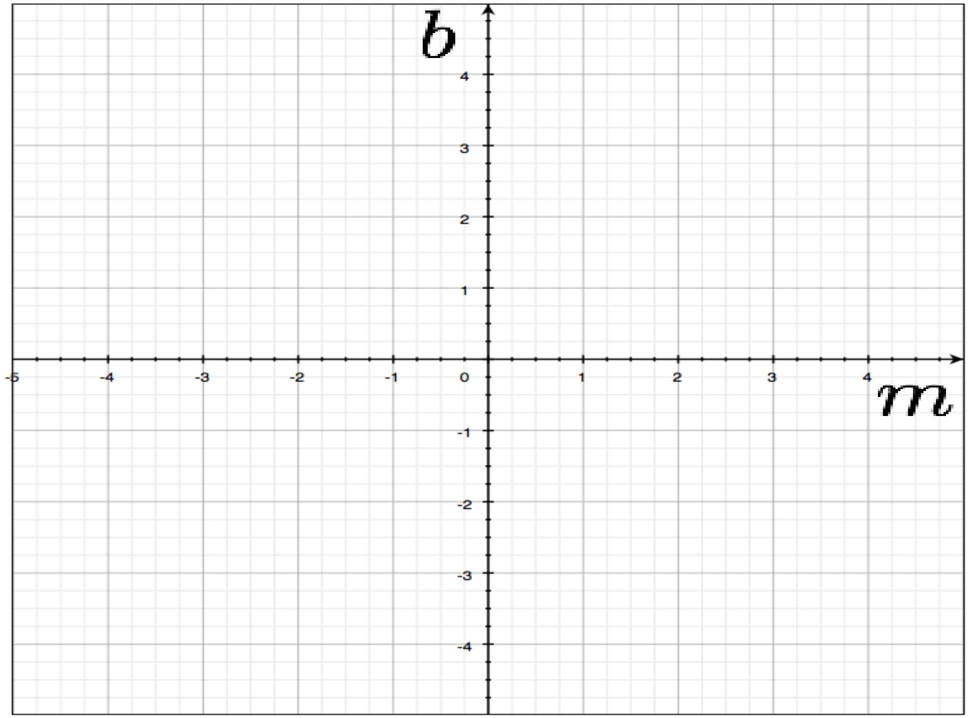


Image space

two points become ?



Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

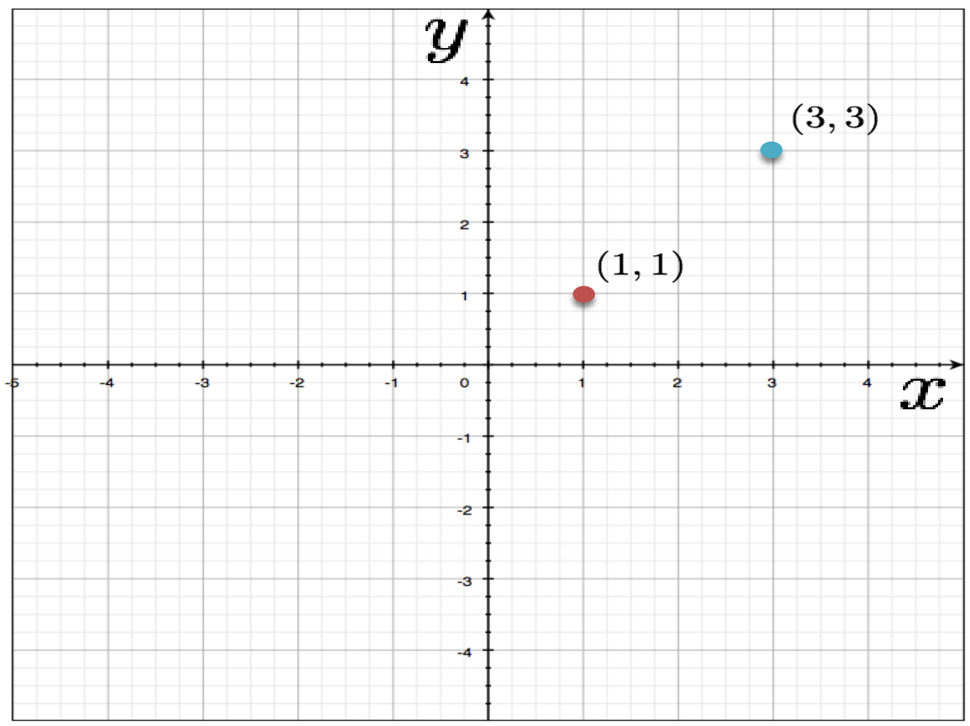
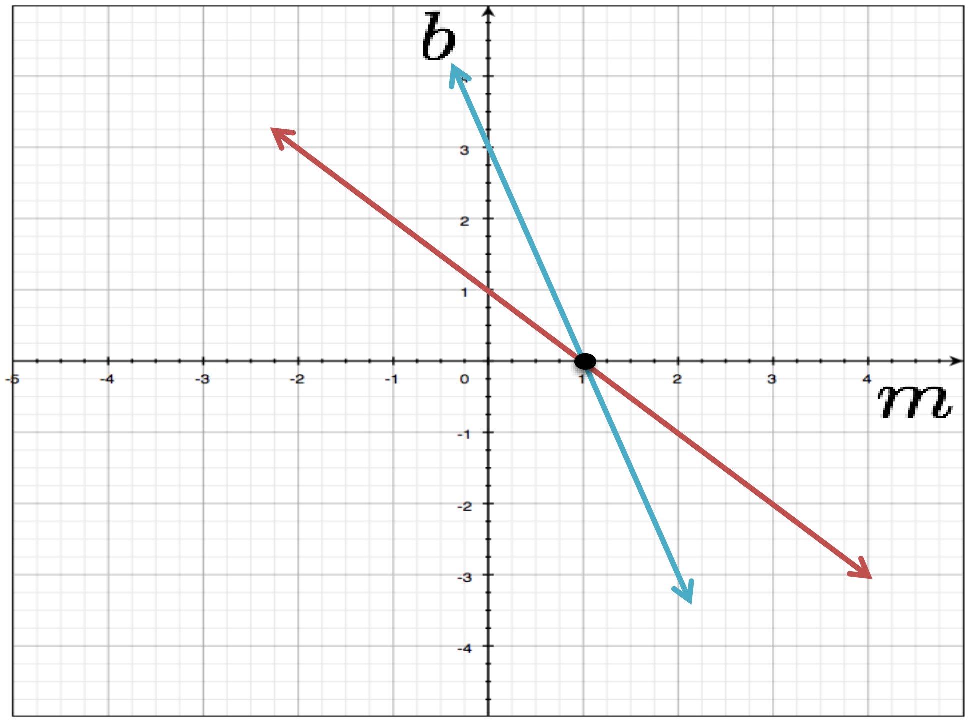


Image space

two points
become
Two lines



Parameter space

(m, b) Parameter space

variables

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variables

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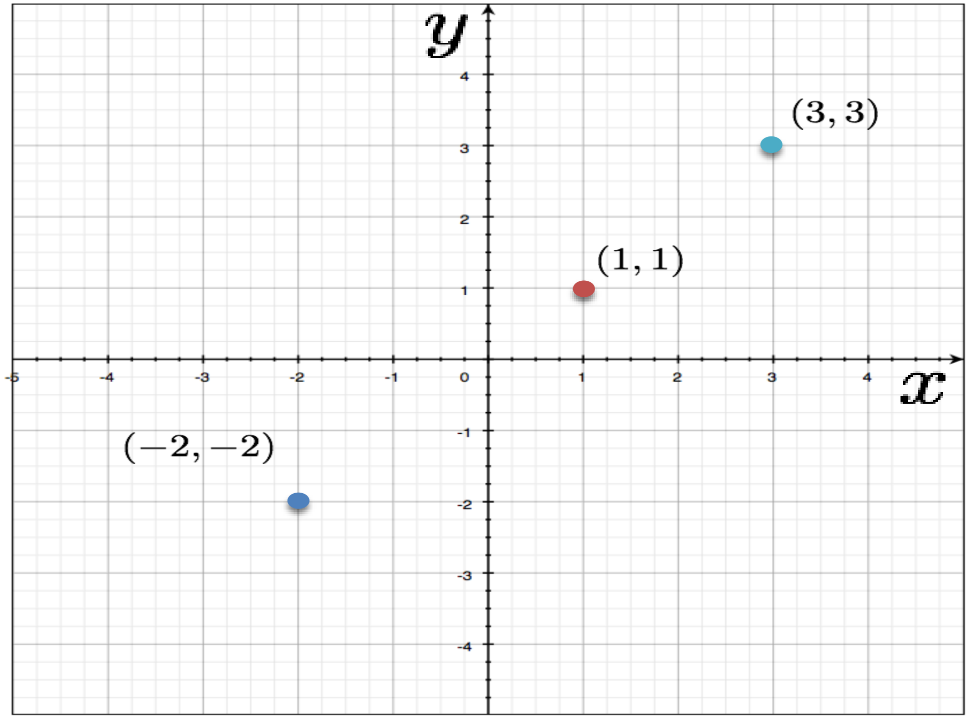
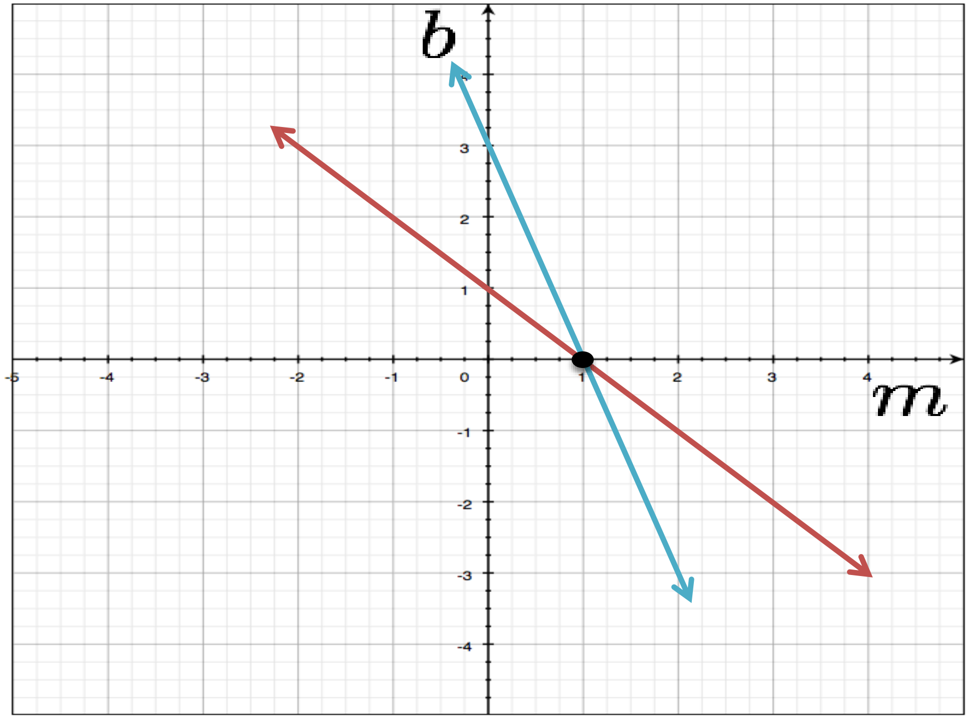


Image space

three points become ?



Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

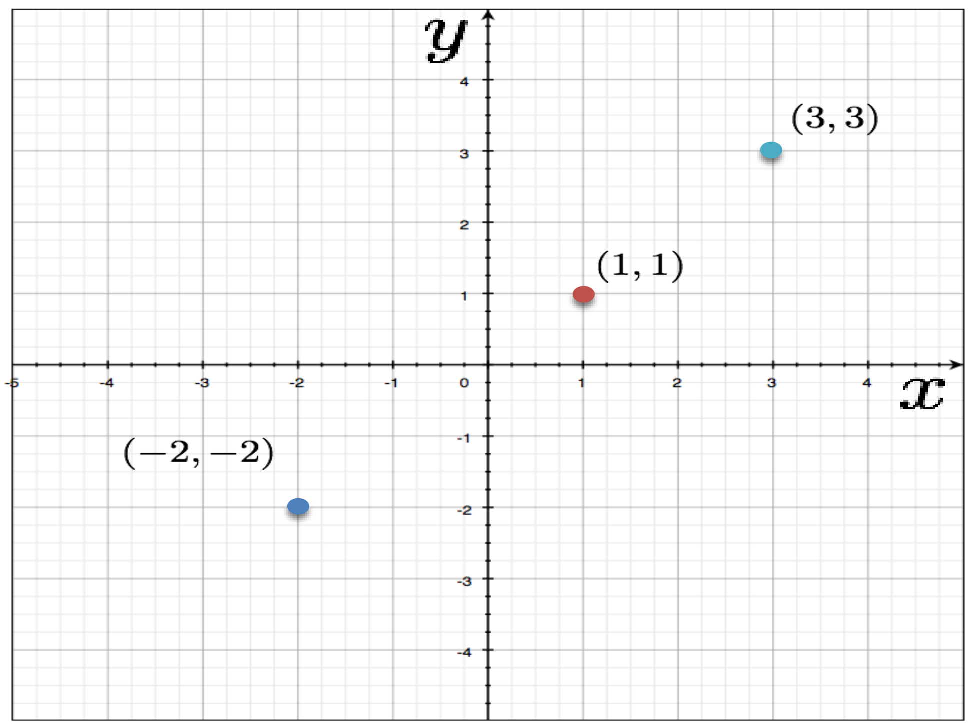
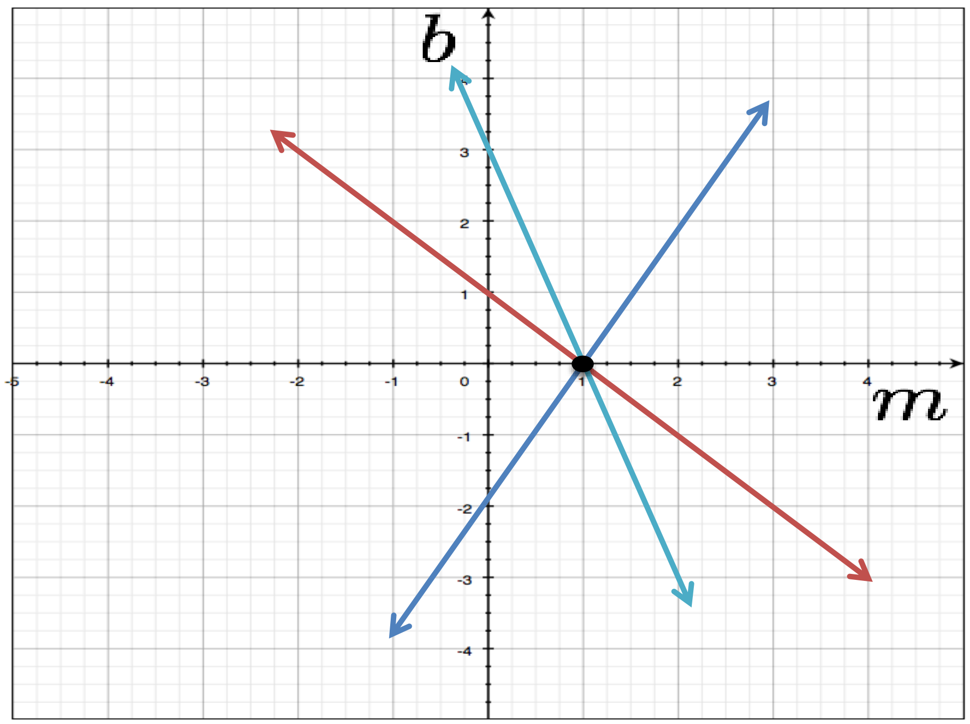


Image space

three points become Three lines



variables

$$y - mx = b$$

parameters

Parameter space

(m, b) Parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

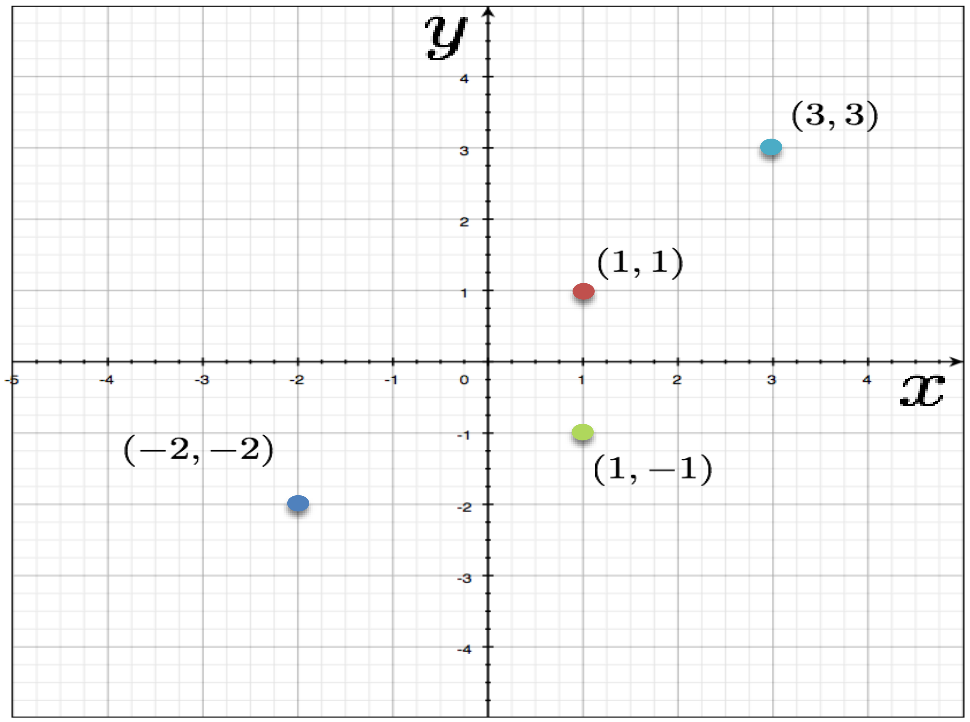
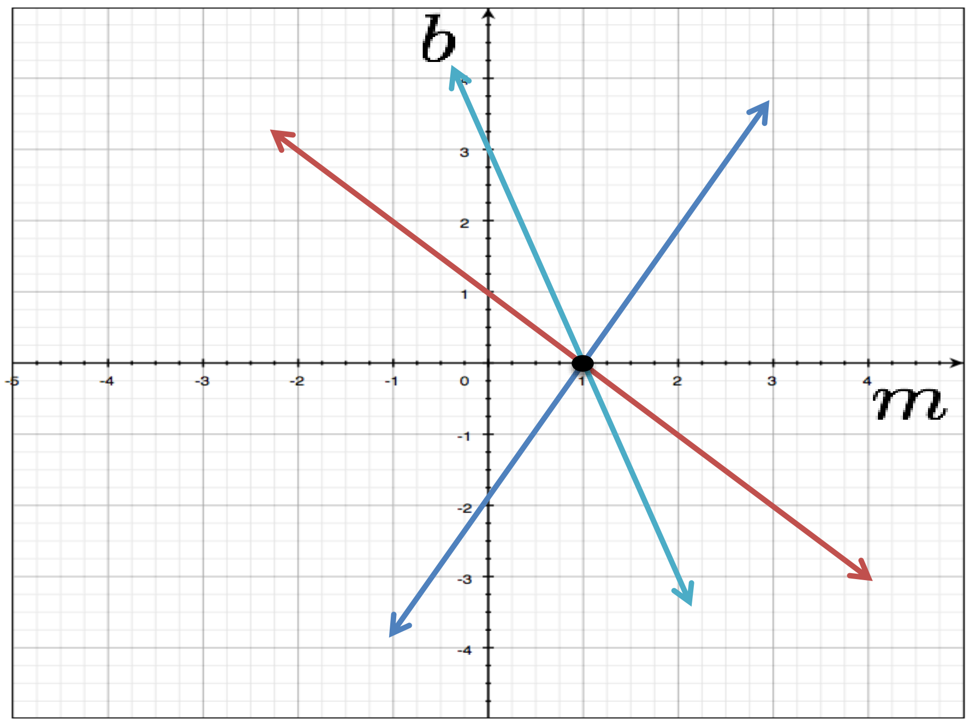


Image space

four points become ?



Parameter space

(m, b) Parameter space

variables

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parameters

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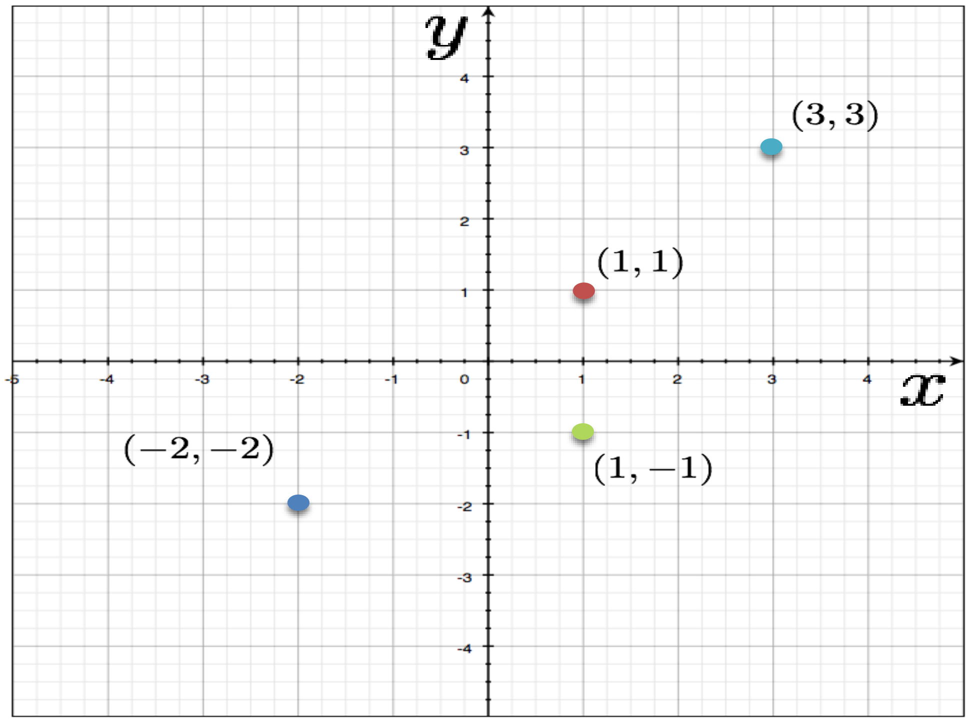
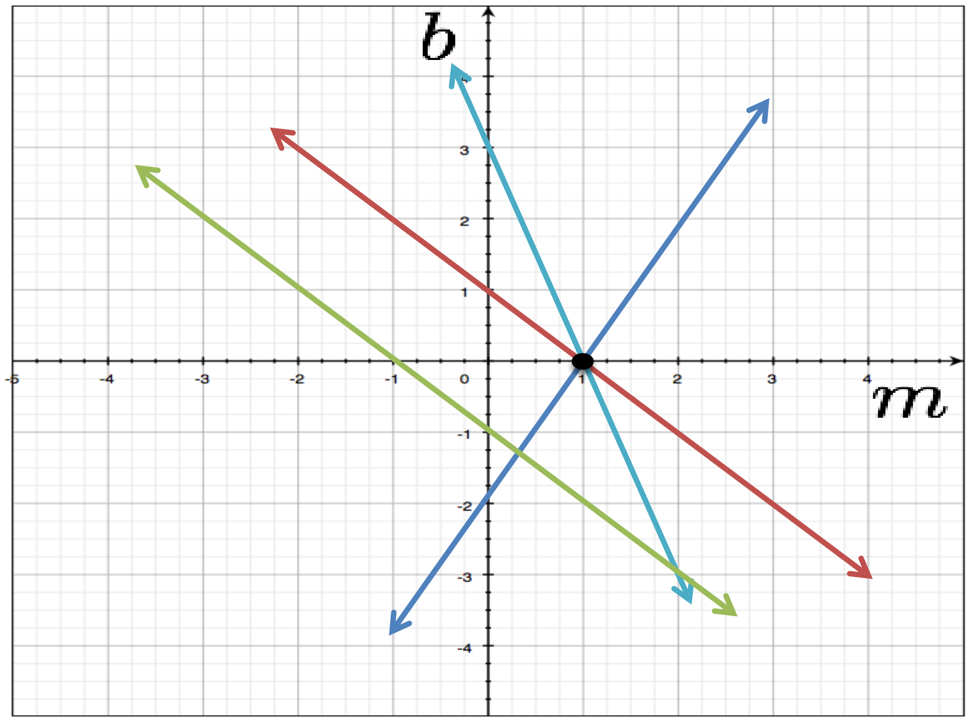


Image space

four points
become
Four lines



Parameter space

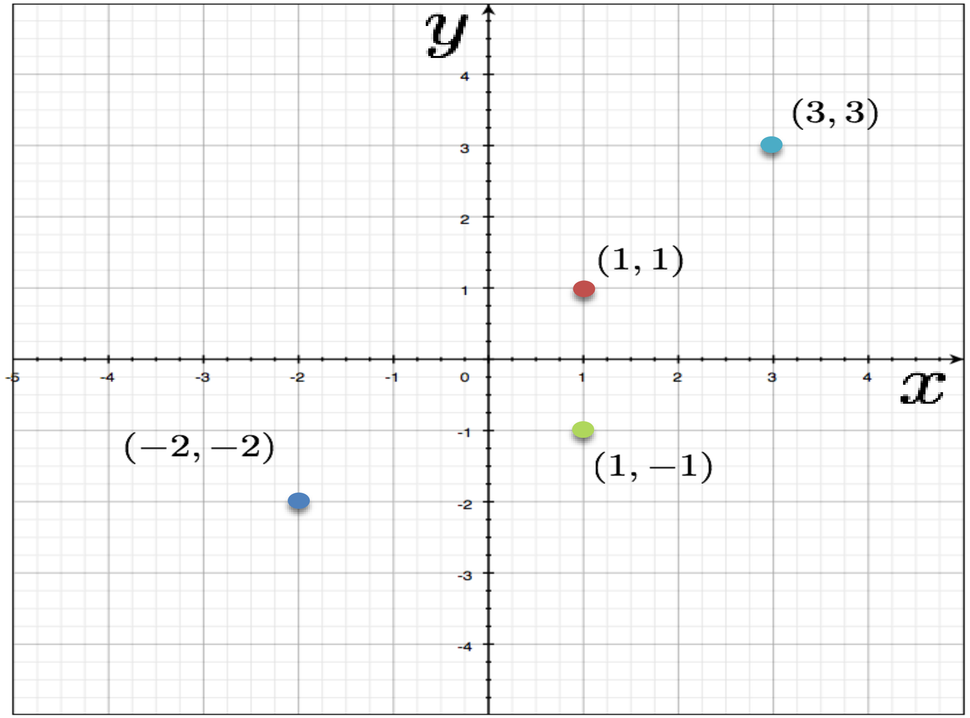
Parameter space- what for?

- How can we find lines in dataset using the parameter space?

Parameter space- what for?

- How can we find lines in dataset using the parameter space?

1. Quantize the output parameter space to user defined bins- we will call this table the **accumulator table.**



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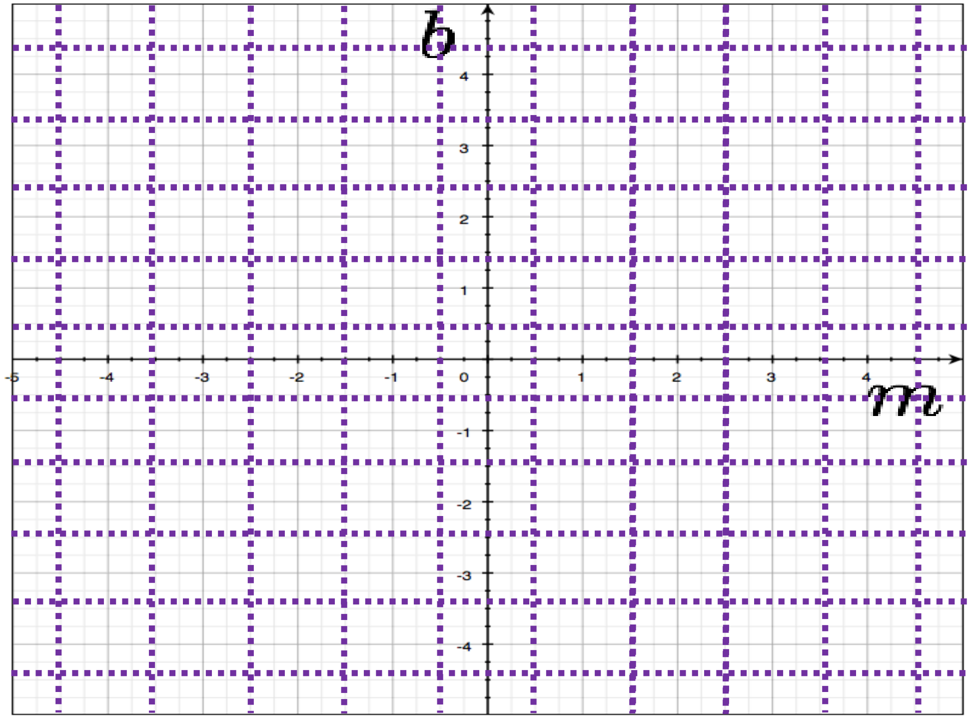
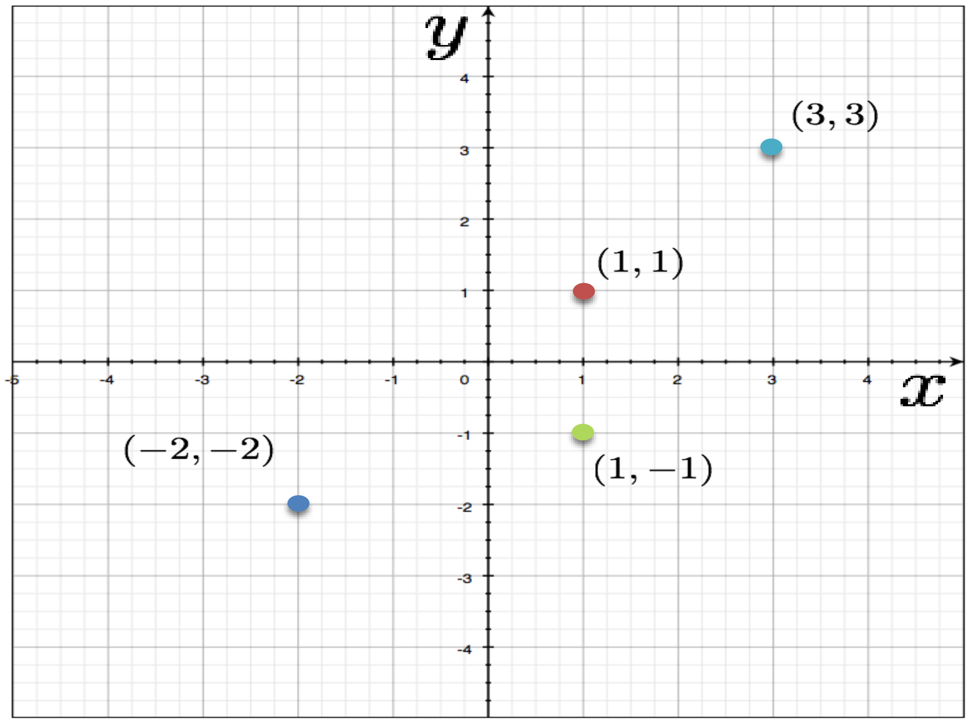


Parameter space

Parameter space- what for?

- How can we find lines in dataset using the parameter space?

2. For each point in image space- find corresponding line in parameter space and increment +1 the intersecting bins.



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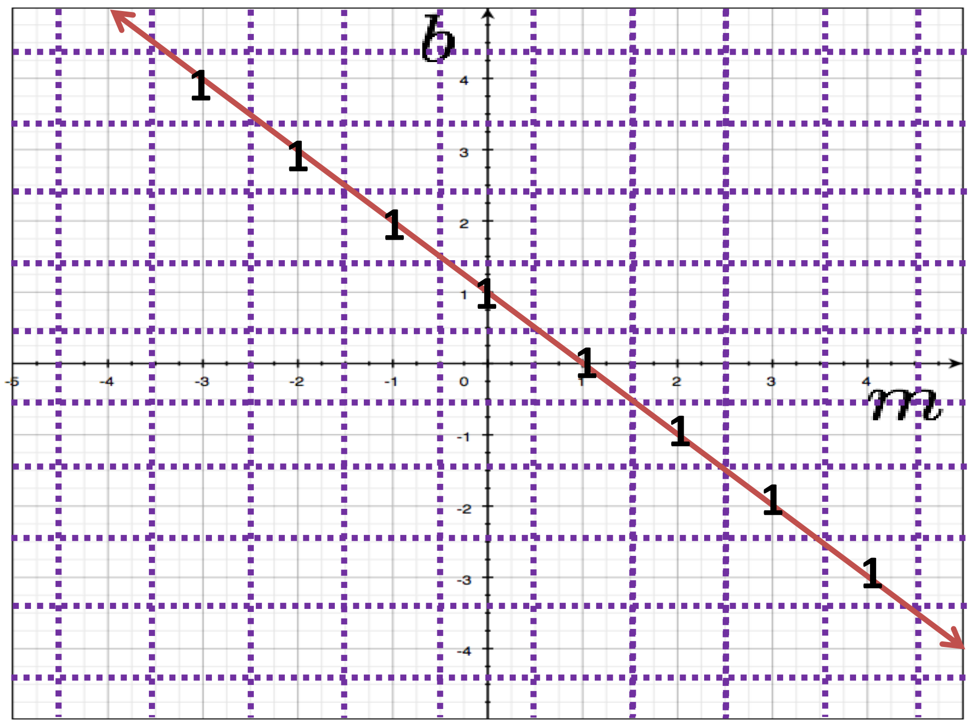

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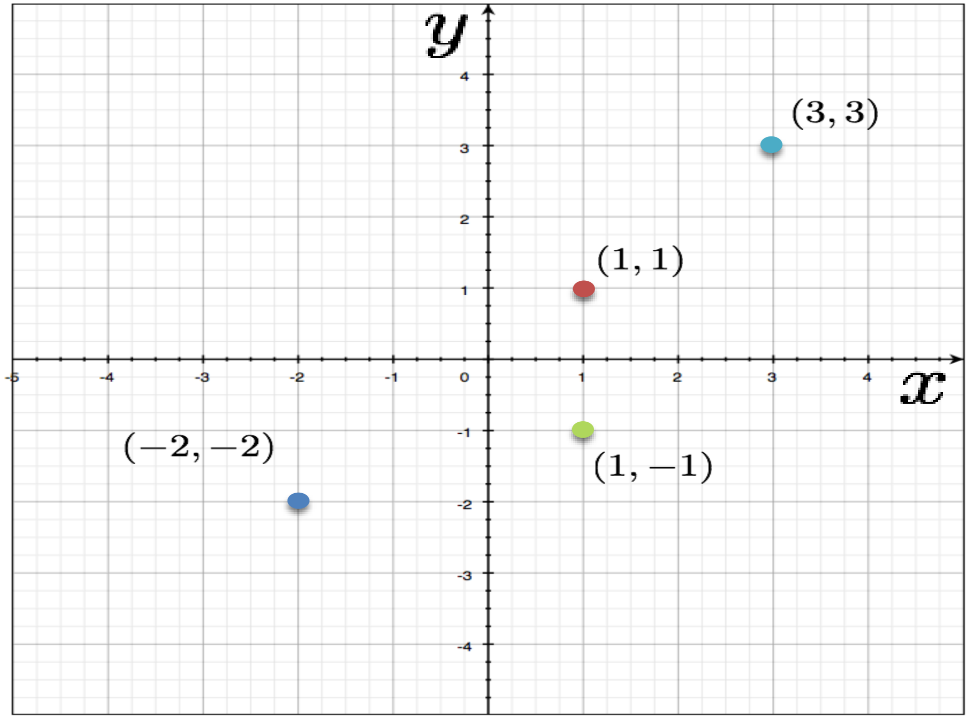
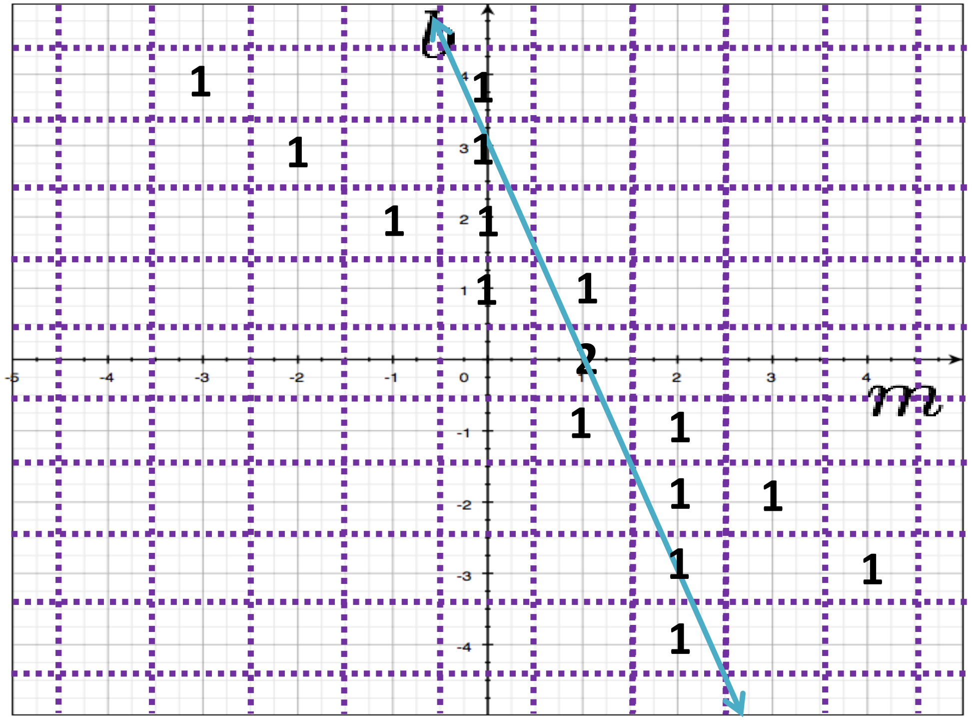


Image space



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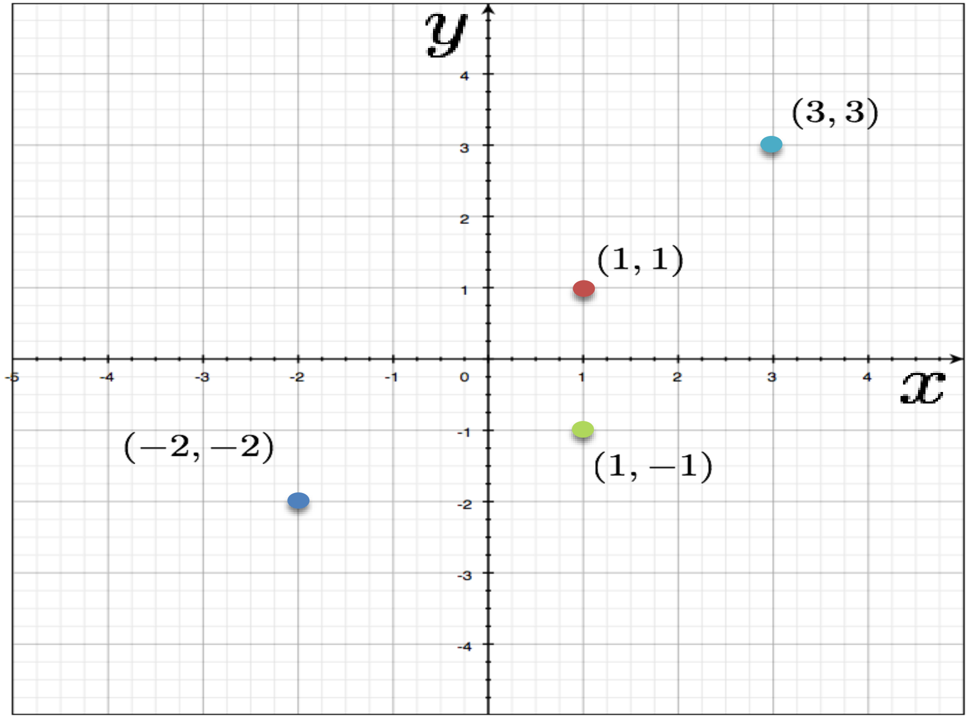

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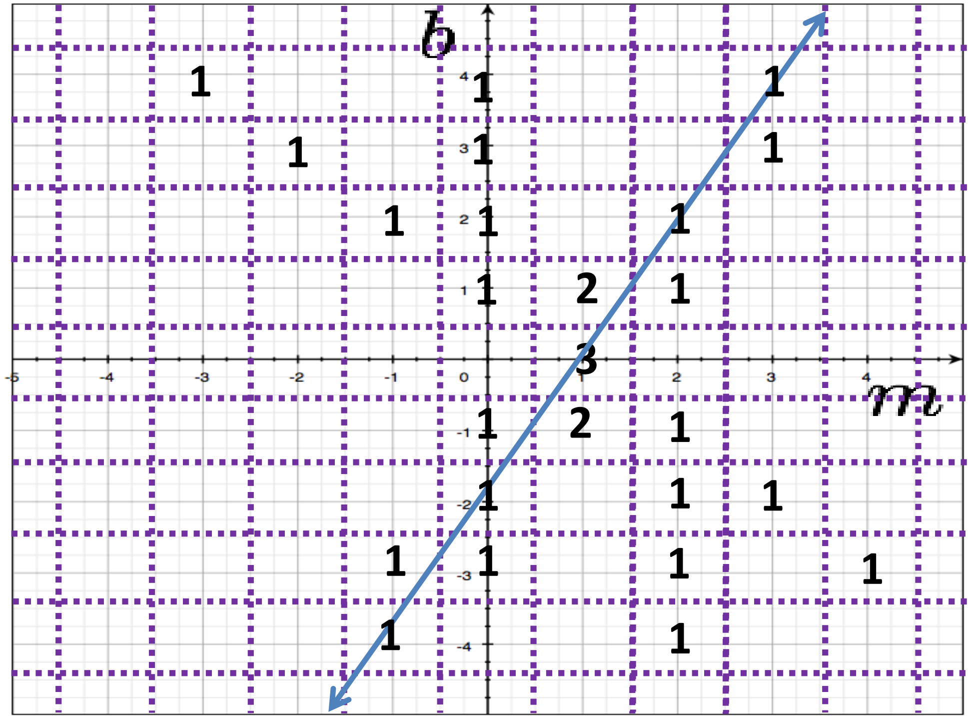

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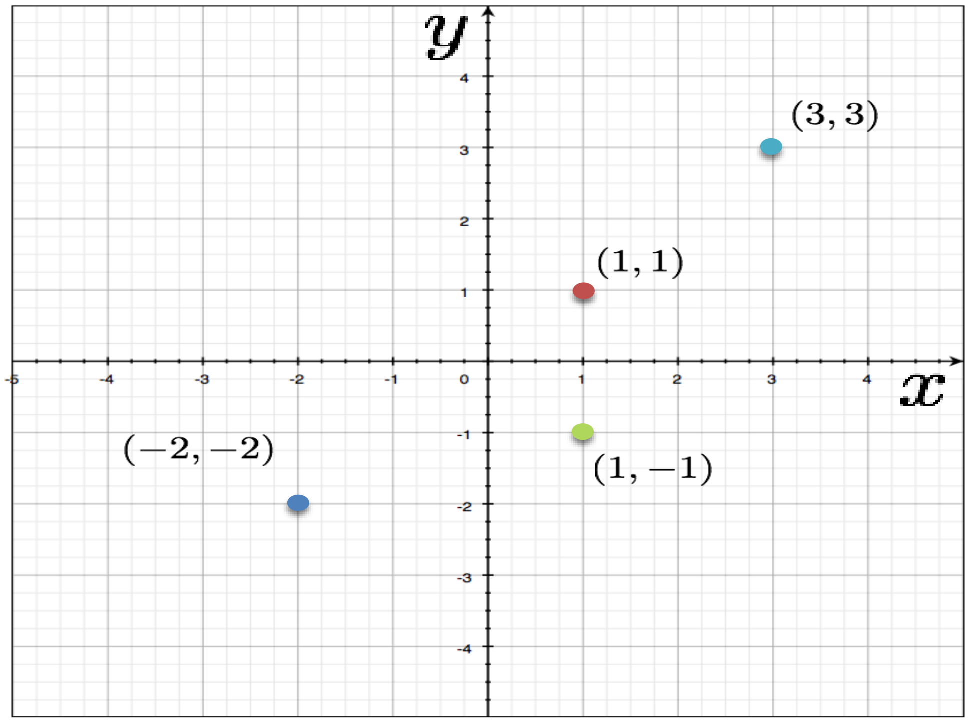
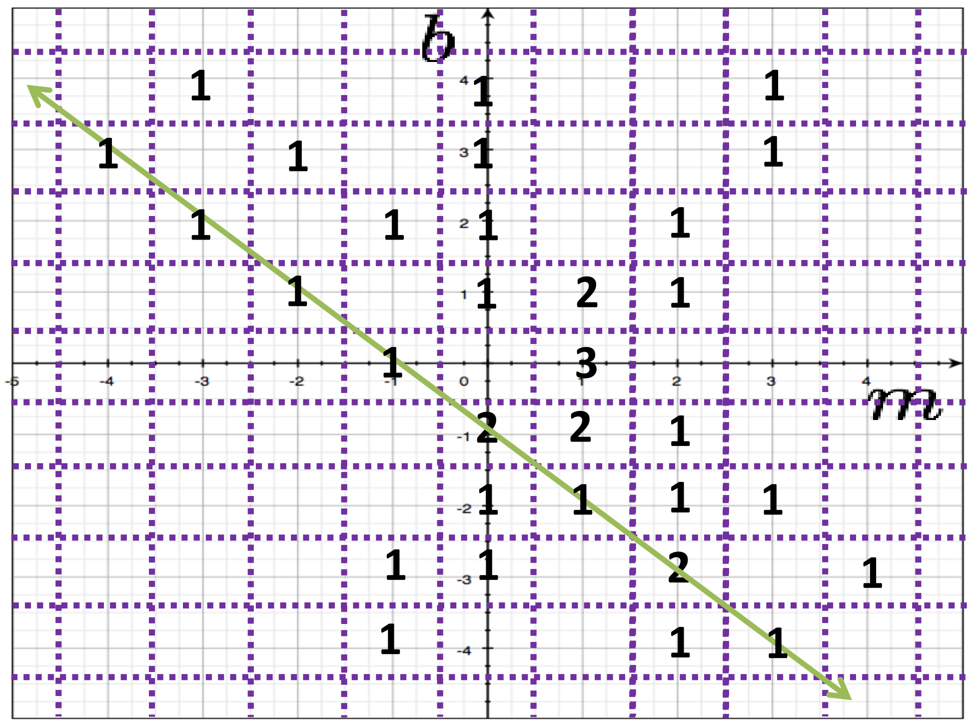


Image space



Parameter space

Parameter space- what for?

- How can we find lines in dataset using the parameter space?

3. Threshold the accumulator table result by some TH and get the corresponding line parameters.

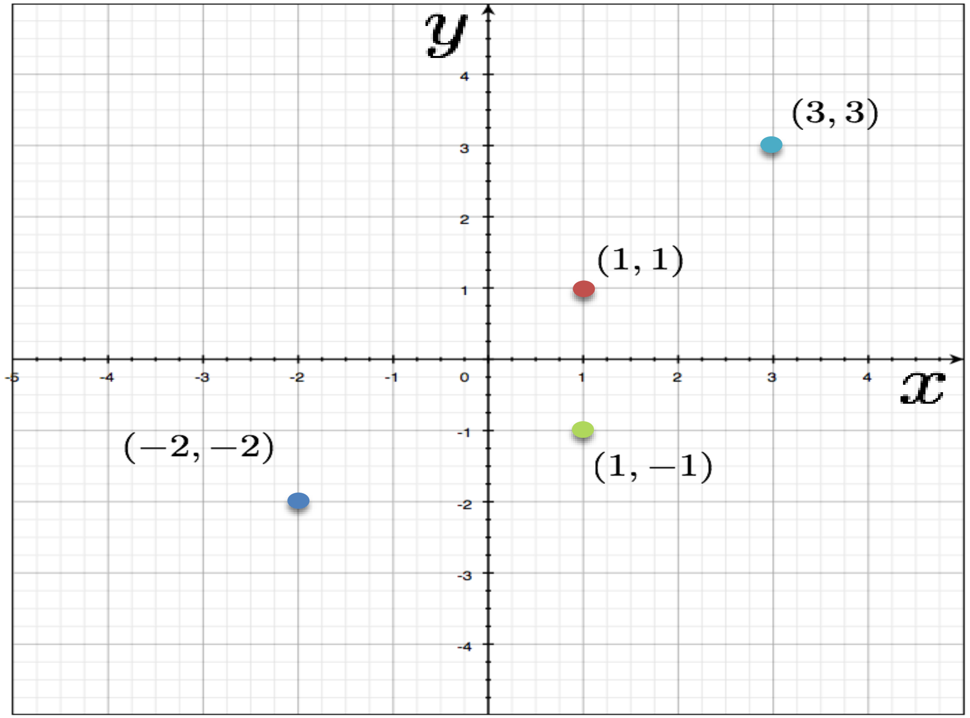
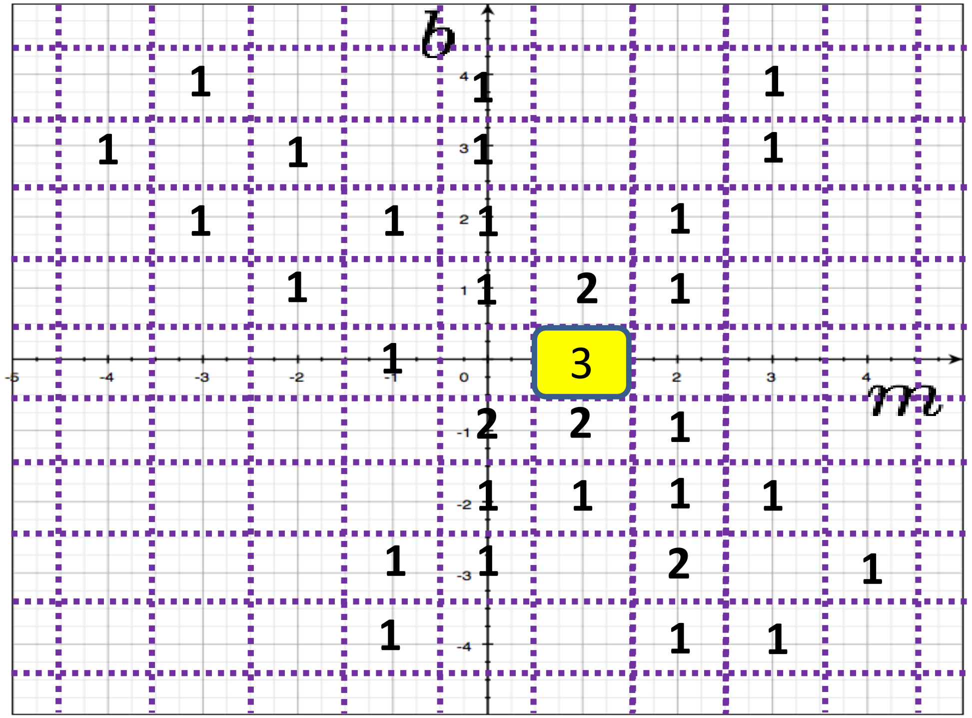


Image space



Parameter space

Hough transform

```
Build accumulator table.
For each point in image space:
   find corresponding line in parameter space and
   increment +1 the intersecting bins.
Threshold the accumulator table result by some TH and get
the corresponding line parameters.
```

Hough transform- pros & cons

- Pros:
  - Can detect multiple lines in image space.
  - can be extended to detect different parameterized curves (e.g.: circles, ellipsoids), and even un-parameterized curves (**generalized hough transform** [out of scope]- similar to template matching that will be covered later in course).
- Cons:
  - For the shown (m, b) parameter space, can't detect vertical lines. **Why?**
  - Susceptive to noise. **Why?**
  - Computationally costly.

Problem 1: vertical line detection

- Vertical (or near vertical) lines have a big slope: $m \to \infty$. This causes the accumulator table to be very big in m direction.
- A solution is to give a **different parameterization to lines**:

$$y = mx + b$$

Problem 1: vertical line detection

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$$\xrightarrow{x=0} y = b$$

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$$\xrightarrow{y=0} x = -\frac{b}{m}$$

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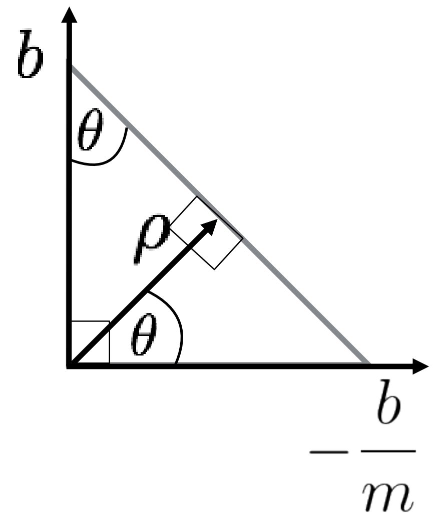

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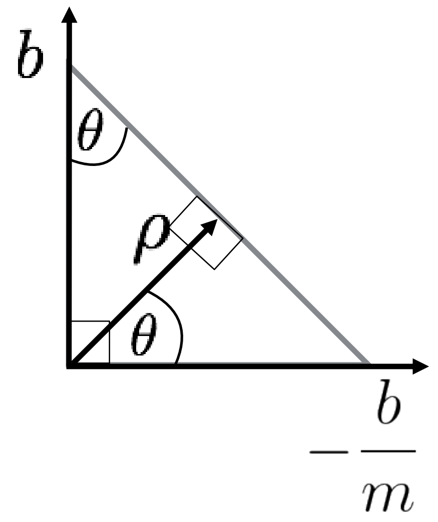
$$\xrightarrow{x=0} y = b$$

$$\xrightarrow{y=0} x = -\frac{b}{m}$$

$$\frac{\frac{-b}{m}}{b} = -\frac{1}{m} = tg\theta$$



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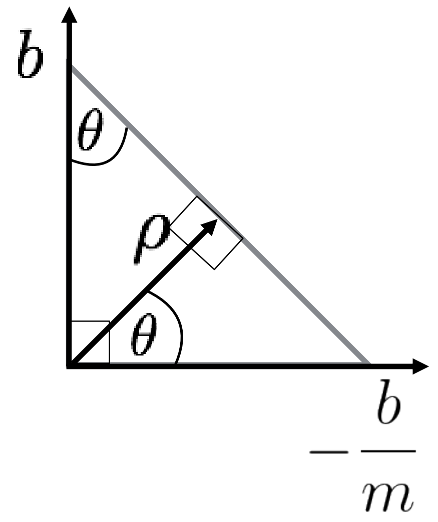
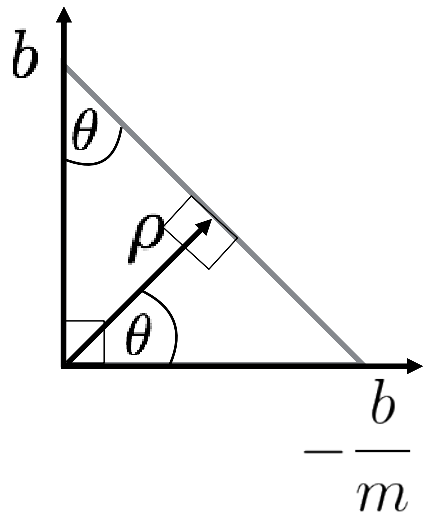
- A solution is to give a **different parameterization to lines**:

$$y = mx + b$$
$$\xrightarrow{x=0} y = b$$
$$\xrightarrow{y=0} x = -\frac{b}{m}$$
$$\frac{\frac{-b}{m}}{b} = -\frac{1}{m} = tg\theta \to m = -\frac{1}{tg\theta} = -\frac{cos\theta}{sin\theta}$$



Problem 1: vertical line detection

- Vertical (or near vertical) lines have a big slope: $m \to \infty$. This causes the accumulator table to be very big in m direction.
- A solution is to give a **different parameterization to lines**:



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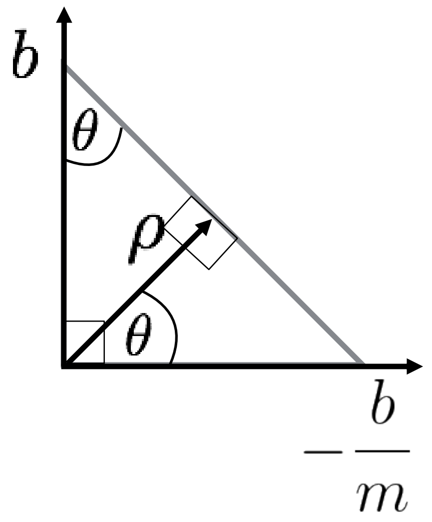

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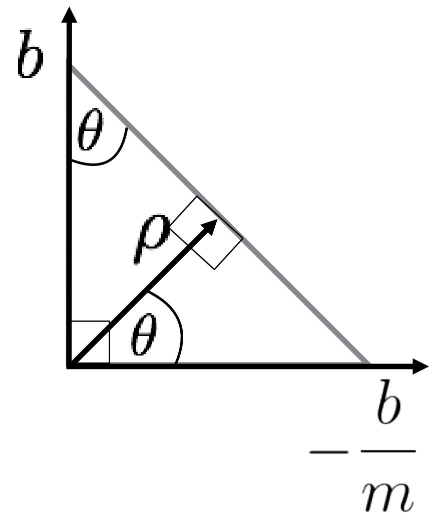


TOC

- Linear least squares
- Total least squares
- Least squares
- RANSAC
- Hough transform
  - $(\boldsymbol{m}, \boldsymbol{b})$ parameter space
  - $(\boldsymbol{\rho}, \boldsymbol{\theta})$ **parameter space**

(ρ, θ) parameter space

variables

$$y = mx + b$$

parameters

parameters

$$x\cos\theta + y\sin\theta = \rho$$

variables

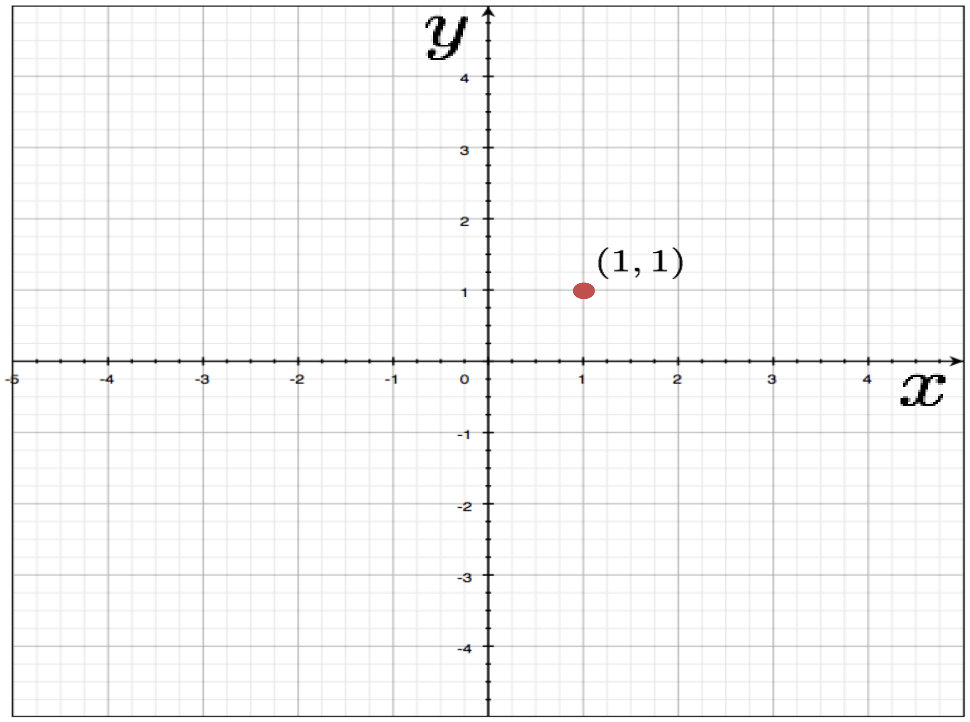
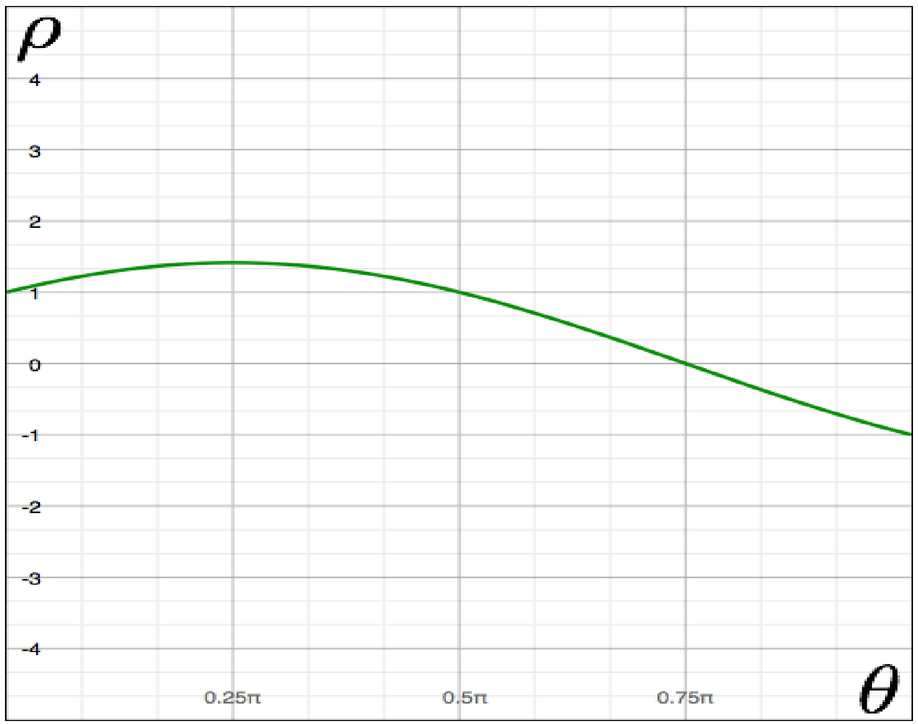


Image space

a point becomes a wave



Parameter space

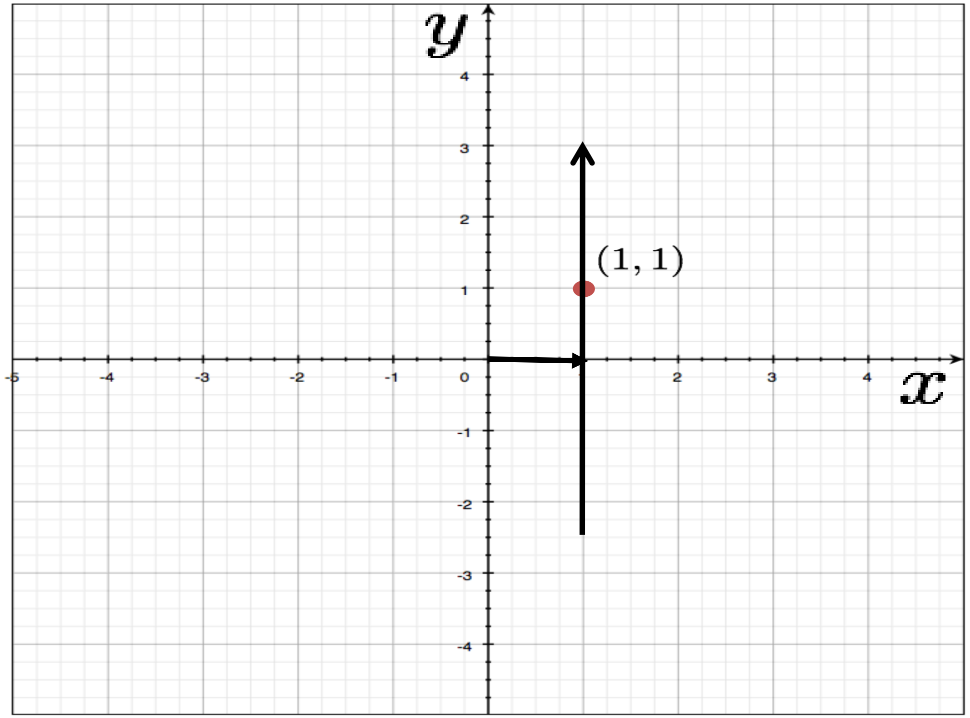
(ρ, θ) parameter space

variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$



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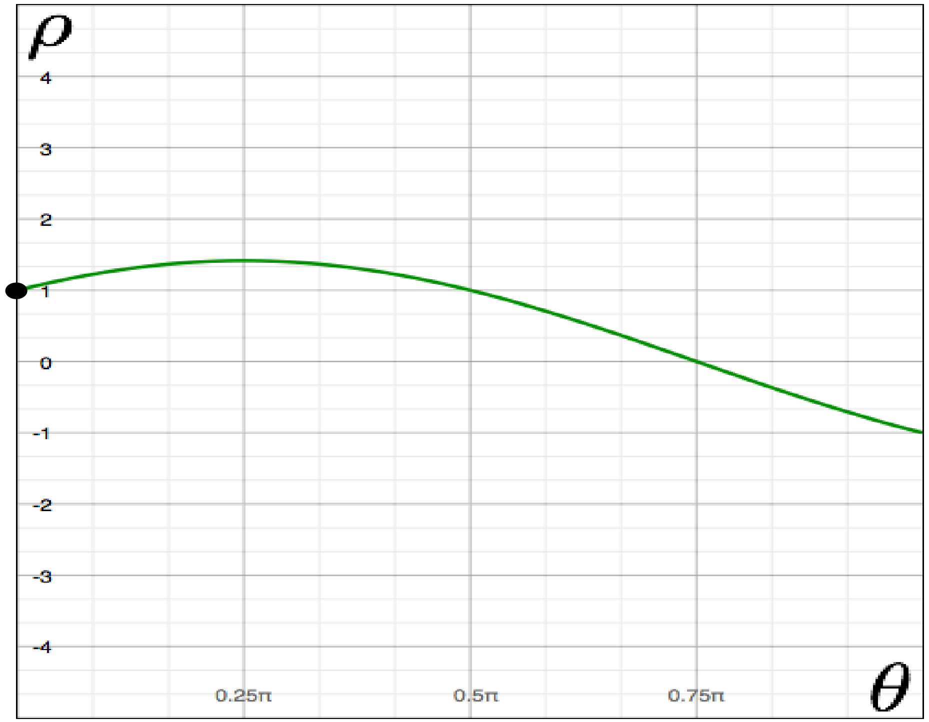


Parameter space

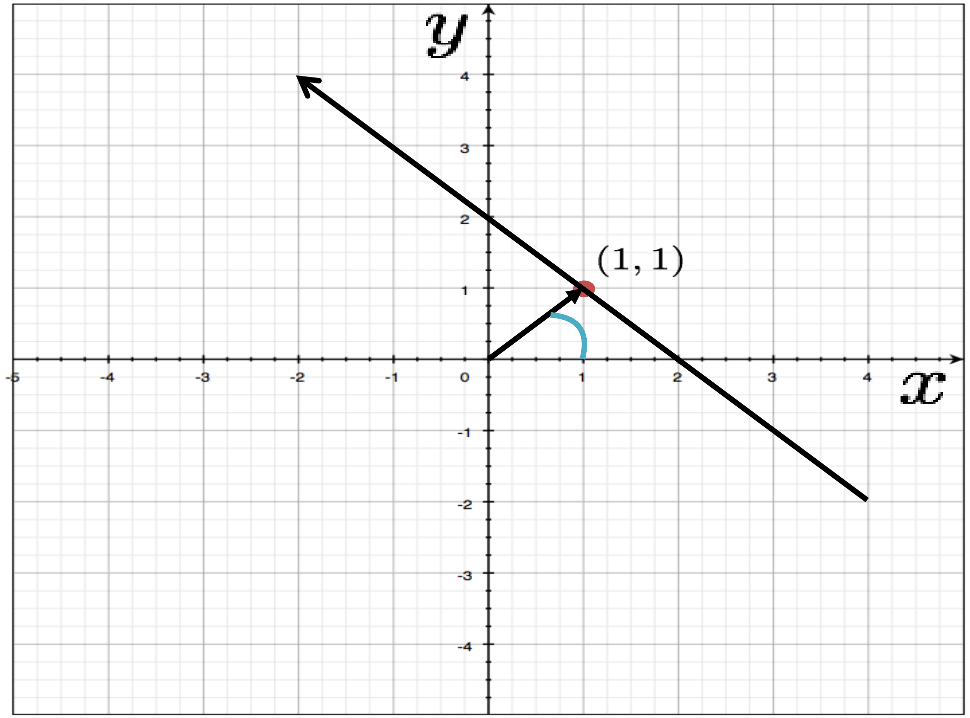
$(\boldsymbol{\rho}, \boldsymbol{\theta})$ parameter space

variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$



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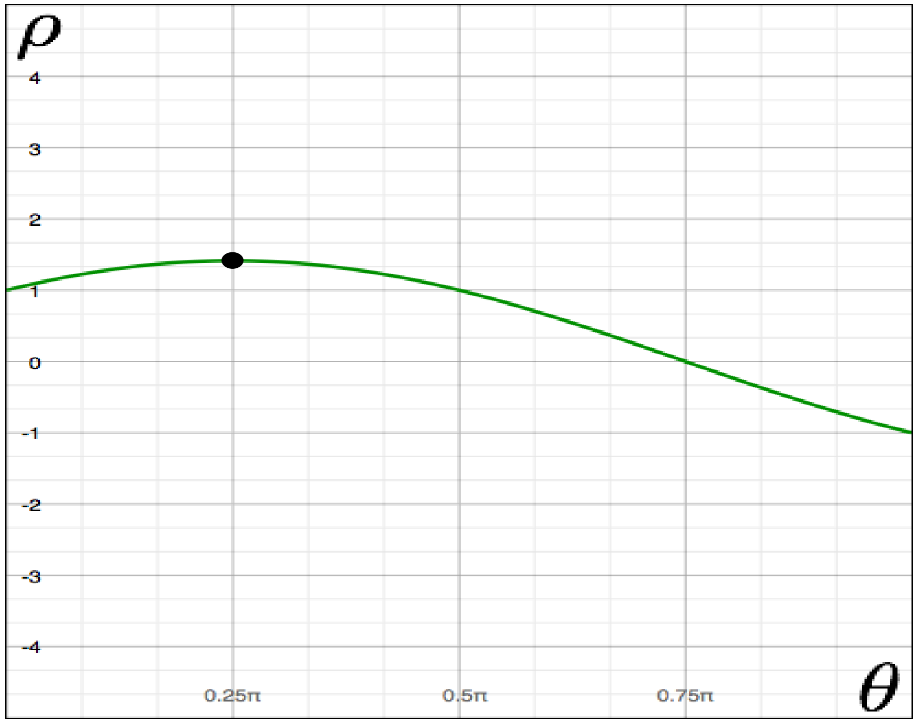

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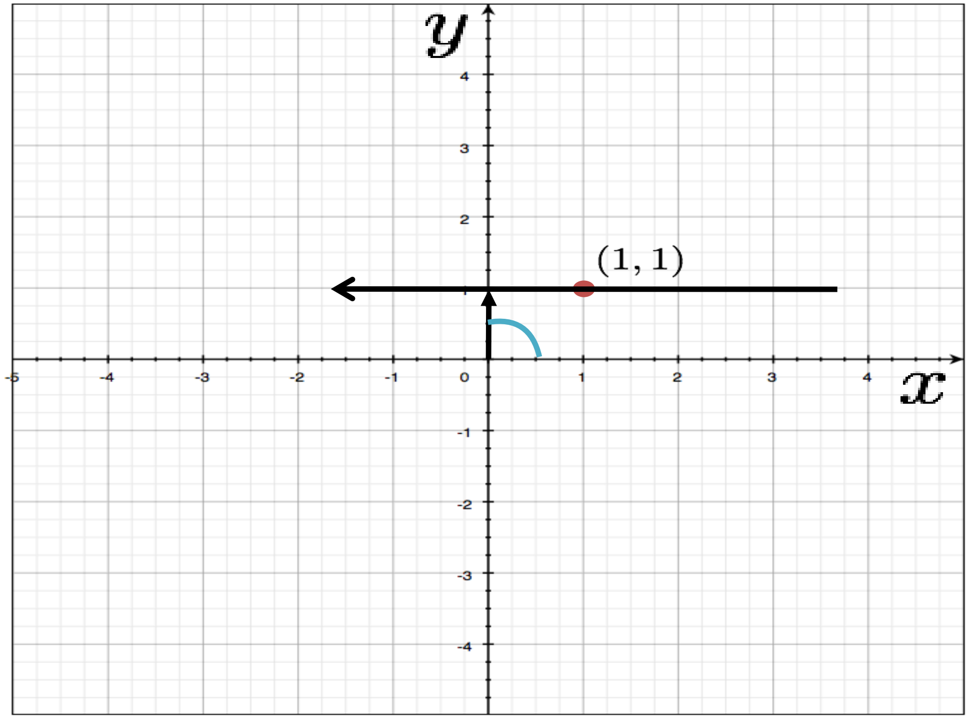

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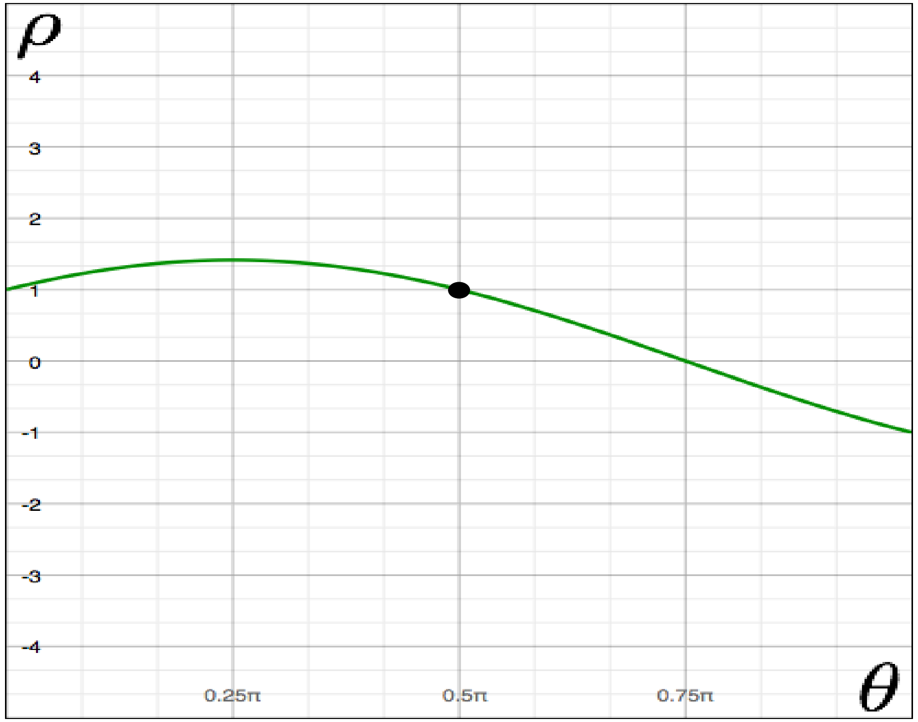

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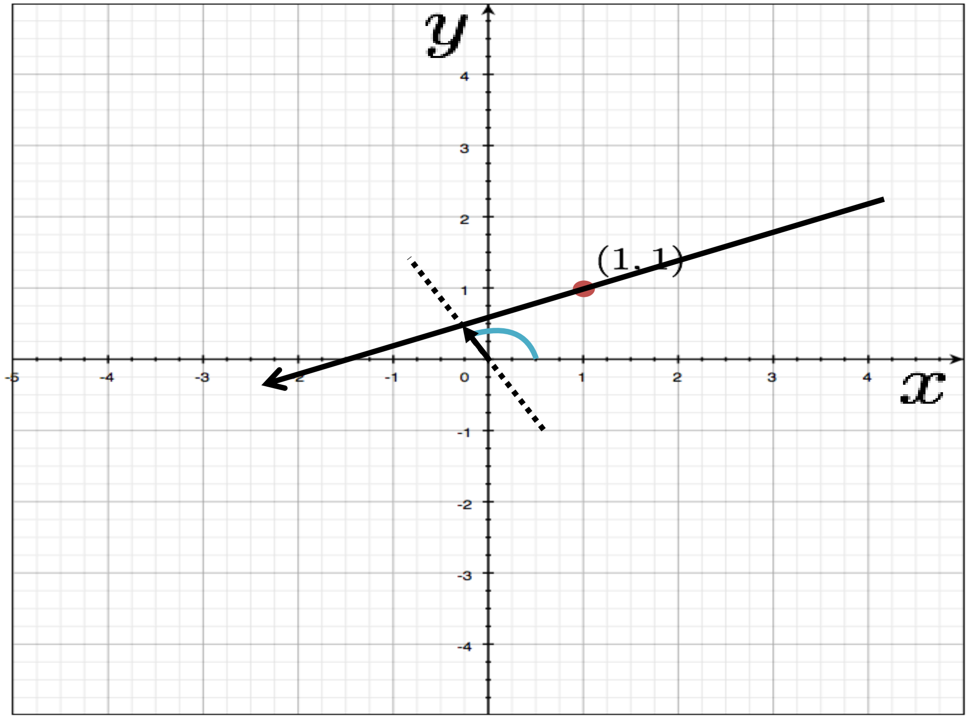

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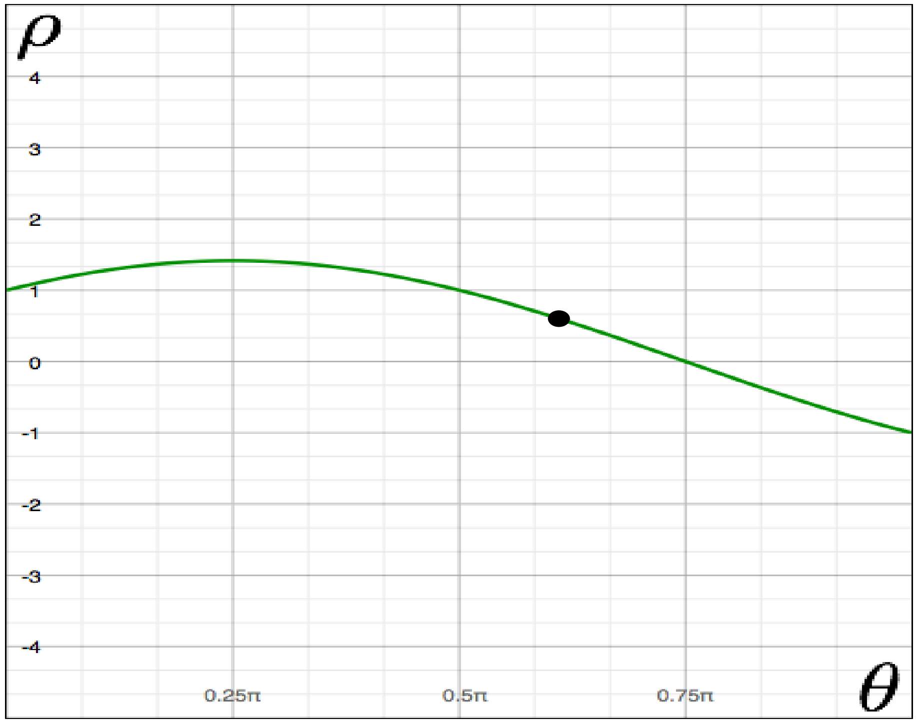

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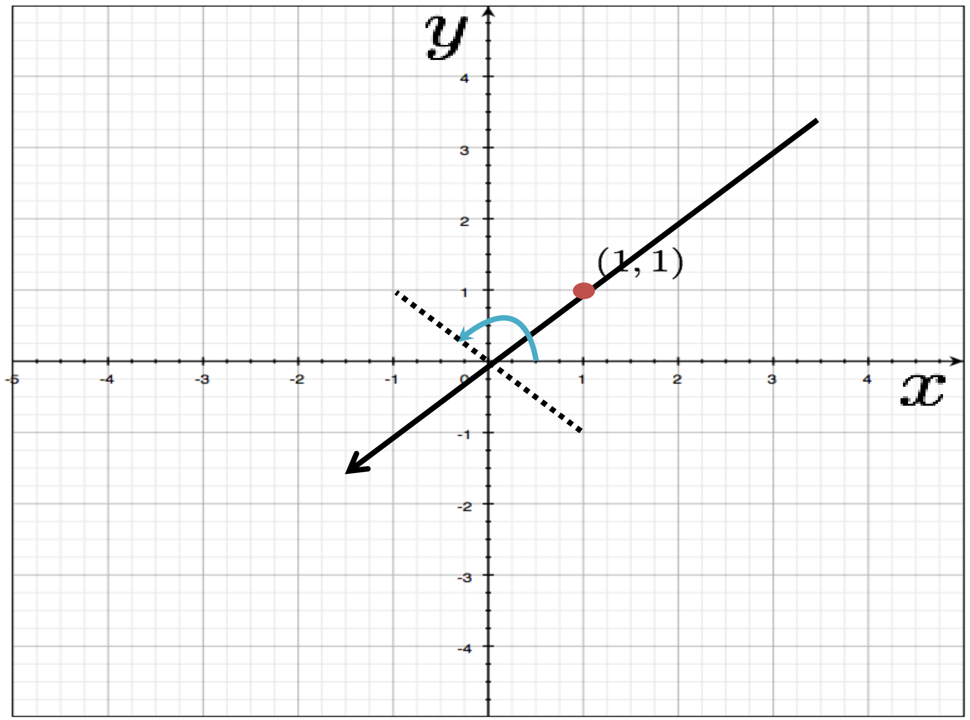

Image space

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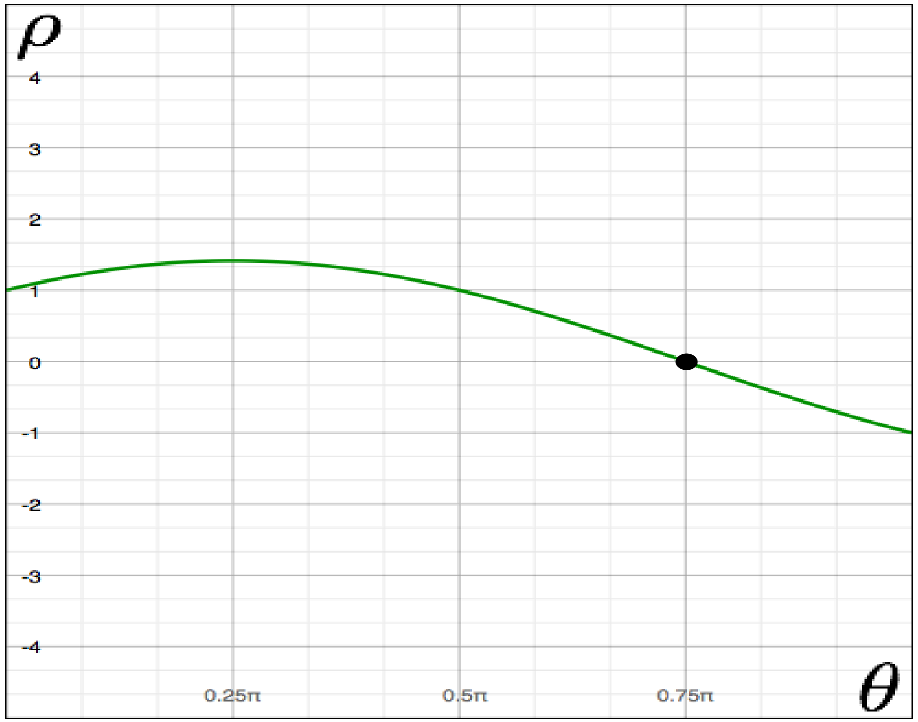

Parameter space

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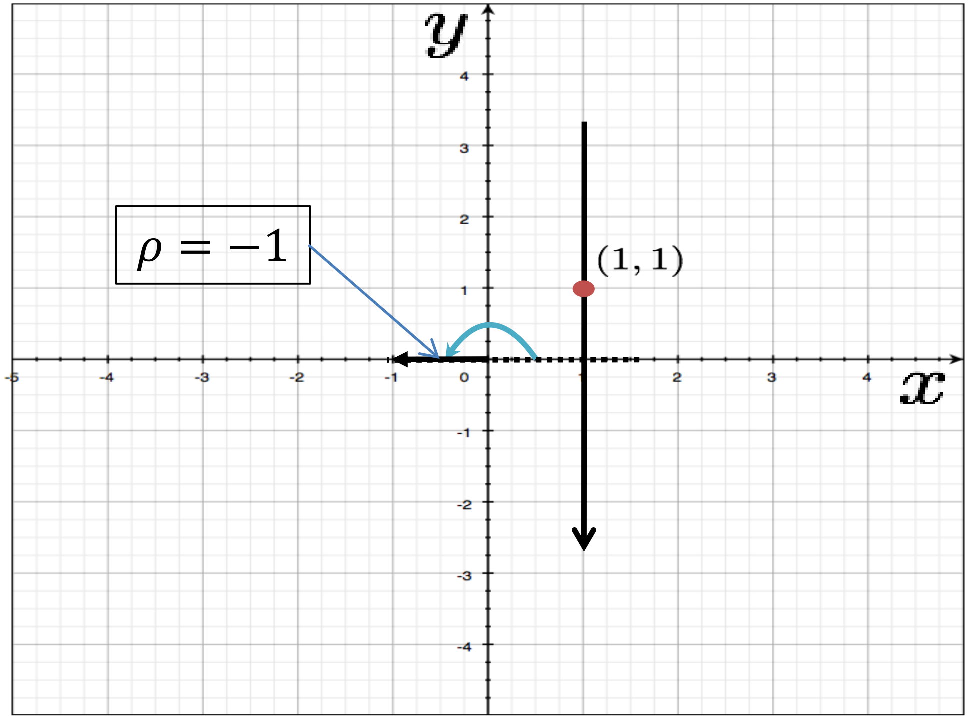

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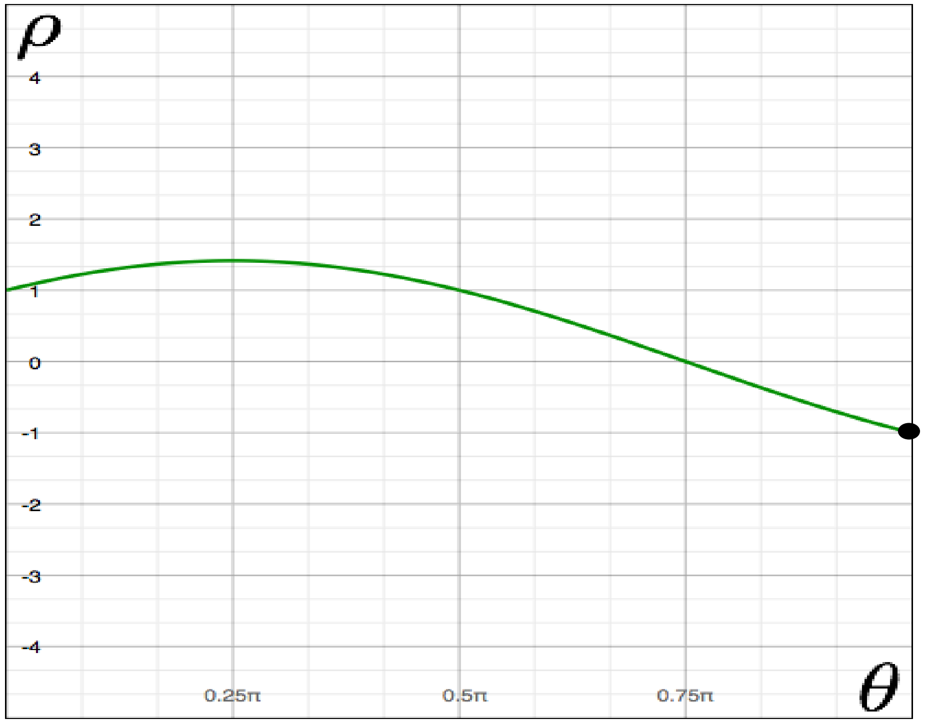


Image space

Parameter space

Notes on $(\boldsymbol{\rho}, \boldsymbol{\theta})$ parameter space

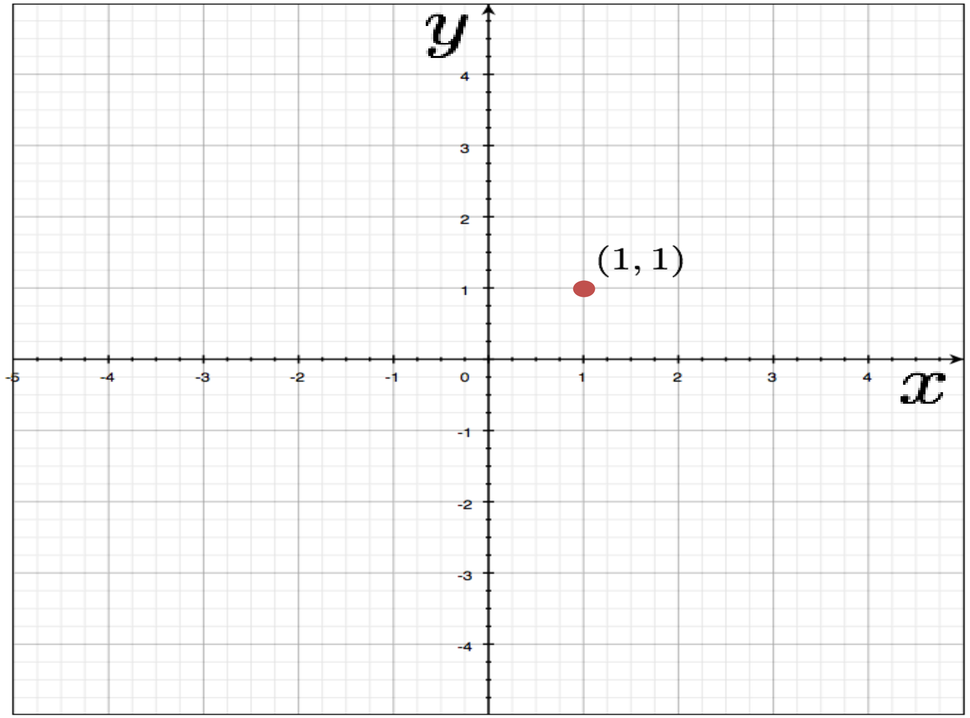
- We only care about $0 \leq \theta < \pi$, otherwise it's symmetric.
- As we so earlier, for some $\theta \rightarrow sign(\rho) = -1$. this is acceptable since the derivation earlier was right only for the first quadrant.

(ρ, θ) parameter space

variables

$$y = mx + b$$

parameters



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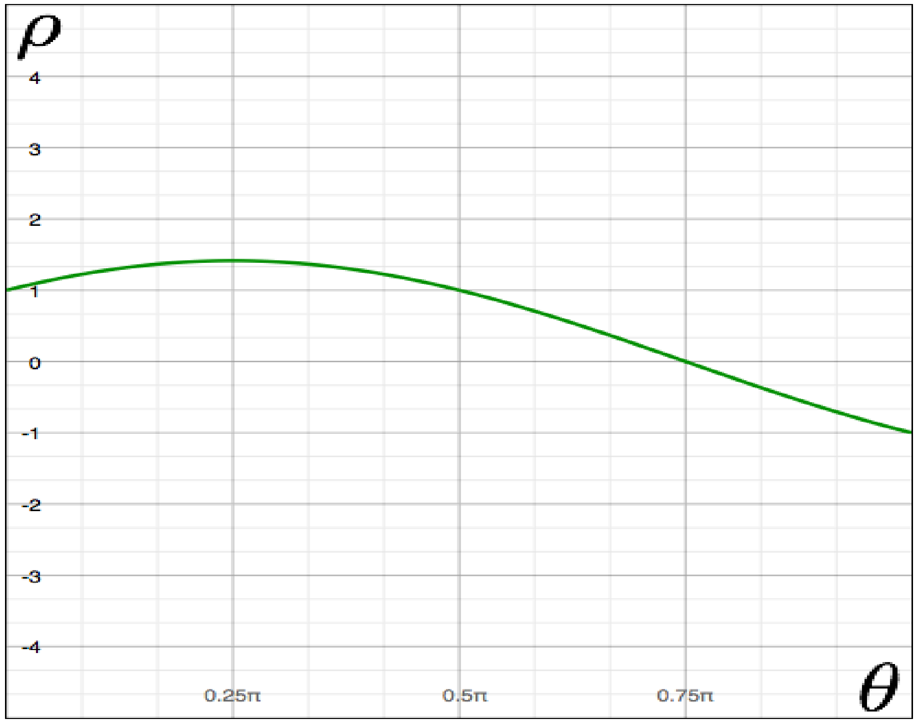

Parameter space

(ρ, θ) parameter space

variables

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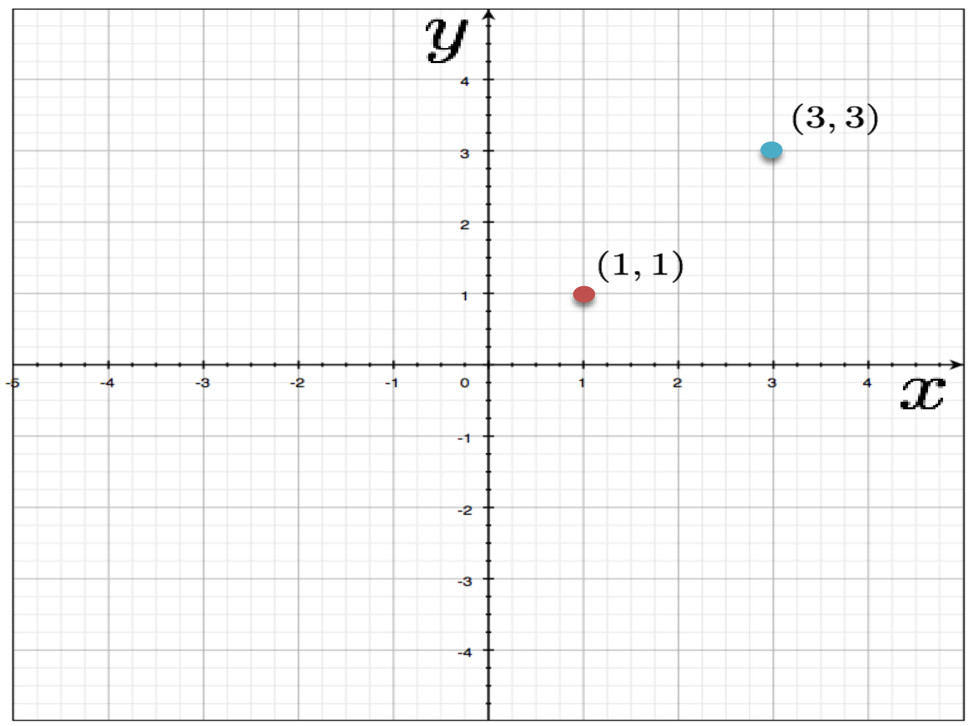

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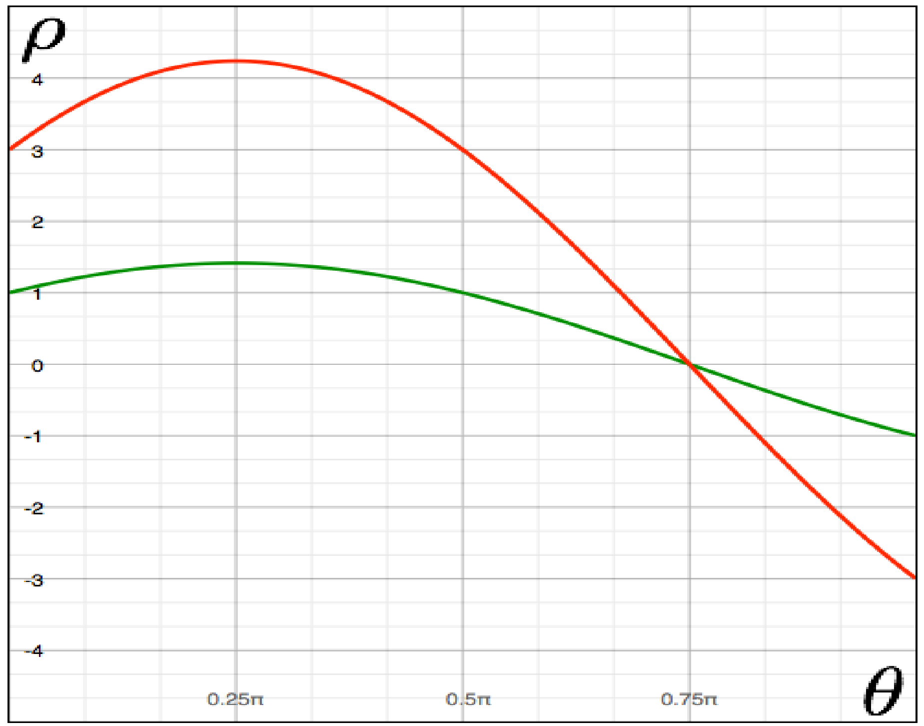

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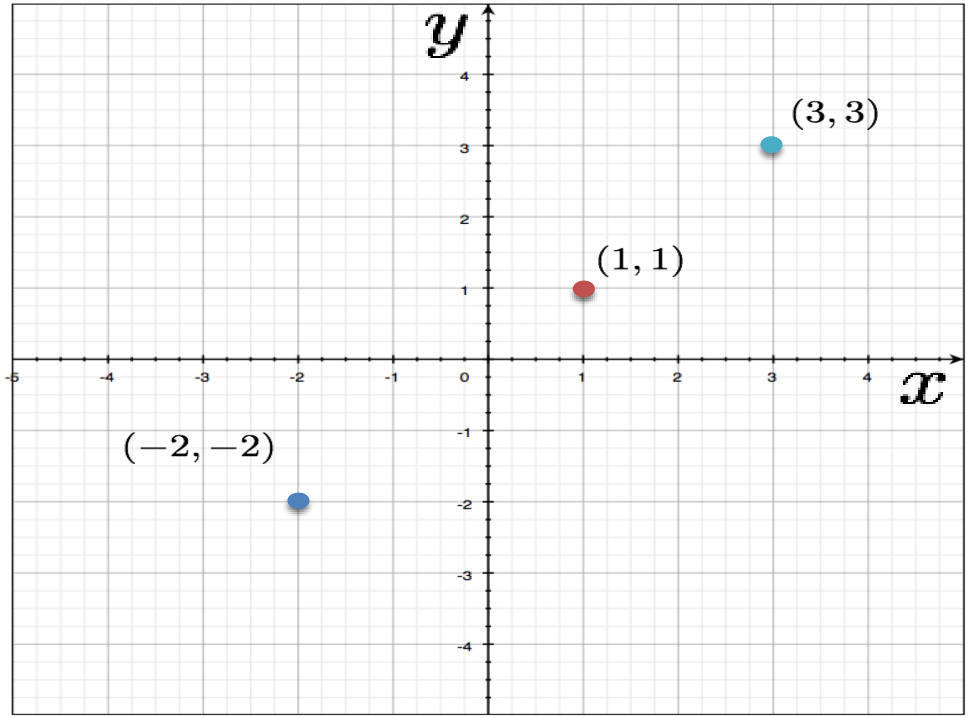

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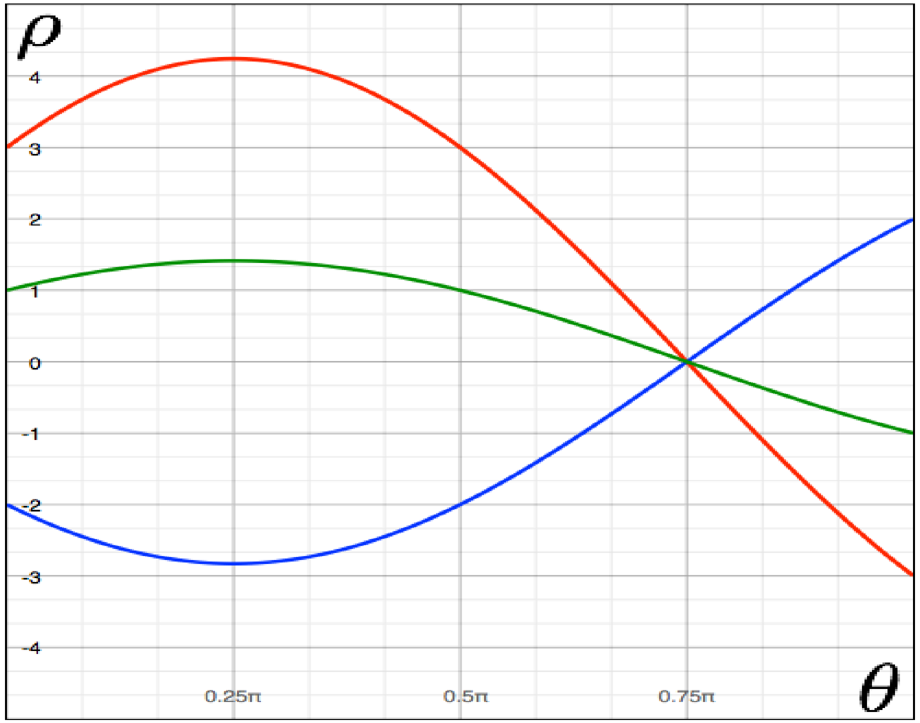

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(ρ, θ) parameter space

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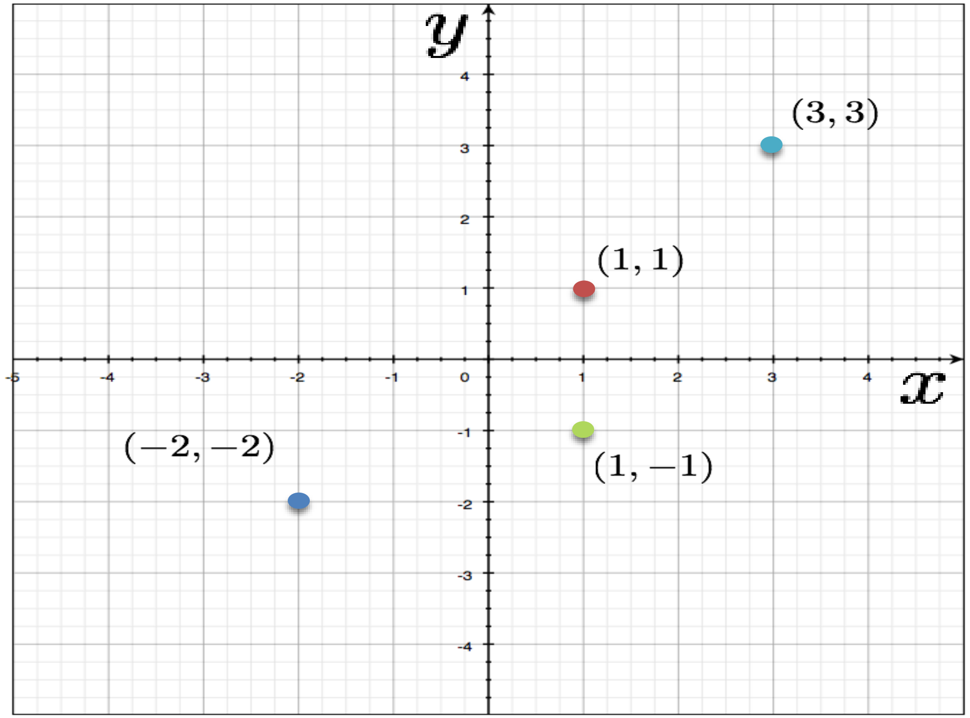

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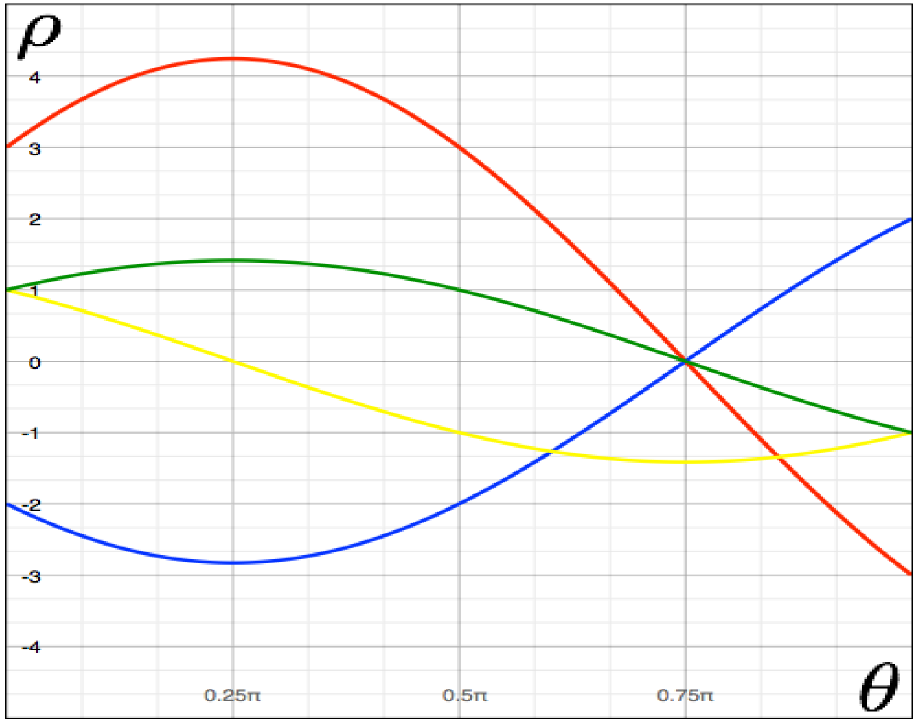

Parameter space

Hough transform algorithm stays the same!

```
Build accumulator table.
For each point in image space:
   find corresponding line in parameter space and
   increment +1 the intersecting bins.
Threshold the accumulator table result by some TH and get
the corresponding line parameters.
```

Hough transform- pros & cons

- Pros:
  - Can detect multiple lines in image space.
  - can be extended to detect different parameterized curves (e.g.: circles, ellipsoids), and even un-parameterized curves (**generalized hough transform** [out of scope]- similar to template matching that will be covered later in course).
- Cons:
  - For the shown (m, b) parameter space, can't detect vertical lines. **<u>Why?</u>**
  - Susceptive to noise. **<u>Why?</u>**
  - Computationally costly.

Hough transform noise

- In case that the discovered edge is noisy, the binning process in the accumulation matrix can be problematic.
- This is a known problem of Hough transform… some ways to get better results is to try:
  - Different bin size (different step size for R & θ).
  - Smooth the accumulation matrix before thresholding.