COURSE 3

2.2. Lagrange interpolation (continuation)

Let $[a, b] \subset \mathbb{R}$, $x_i \in [a, b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a, b] \to \mathbb{R}$.

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 1 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{1}$$

Corollary 2 *If* $f \in C^{m+1}[a,b]$ *then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ *denotes the uniform norm, and* $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$*.*

Example 3 *Which is the limit of the error for computing* $\sqrt{115}$ *using Lagrange interpolation formula for the nodes* $x_0 = 100$, $x_1 = 121$ *and* $x_2 = 144$? *Find the approximative value of* $\sqrt{115}$.

Example 4 *If we know that* $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, *find* $\lg 76$. *Study the approximation error.*

The Aitken's algorithm

Let $[a,b] \subset \mathbb{R}$, $x_i \in [a,b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a,b] \to \mathbb{R}$.

Usually, for a practical approximation problem, for a given function $f : [a,b] \to \mathbb{R}$ we have to find the approximation of $f(\alpha)$, $\alpha \in [a,b]$ with an error not greater than a given $\varepsilon > 0$.

If we have enough information about f and its derivatives, we use the inequality $|(R_m f)(x)| \leq \varepsilon$ to find m such that $(L_m f)(\alpha)$ approximates $f(\alpha)$ with the given precision.

We may use the condition $\frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty \leq \varepsilon$, but it should be known $\left\| f^{(m+1)} \right\|_\infty$ or a majorant of it.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm.** This consists in generating the table:

$$
\begin{array}{c|c|c|cccc}
x_0 & f_{00} & & & & & \\
x_1 & f_{10} & f_{11} & & & & \\
x_2 & f_{20} & f_{21} & f_{22} & & & \\
x_3 & f_{30} & f_{31} & f_{32} & f_{33} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} & \cdots & f_{mm}
\end{array}
$$

where

$$f_{i0} = f(x_i), \quad i = 0, 1, ..., m,$$

and

$$f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \qquad i = 0, 1, ..., m; j = 0, ..., i - 1.$$

For example,

$$
\begin{aligned}
f_{11} &= \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} \\
&= \frac{1}{x_1 - x_0} [f_{00}(x_1 - x) - f_{10}(x_0 - x)] \\
&= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x),
\end{aligned}
$$

so f_{11} is the value in x of Lagrange polynomial for the nodes x_0, x_1. We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$ being Lagrange polynomial for the nodes $x_0, x_1, ..., x_i$.

So $f_{11}, f_{22}, ..., f_{ii}, ..., f_{mm}$ is a sequence of approximations of $f(x)$.

If the interpolation procedure is convergent then the sequence is also convergent, i.e., $\lim\limits_{m \to \infty} f_{mm} = f(x)$. By Cauchy convergence criterion it follows

$$\lim_{i \to \infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left| f_{ii} - f_{i-1,i-1} \right| \le \varepsilon, \quad \text{for a given precision } \varepsilon > 0.$$

Recommendation is to sort the nodes $x_0, x_1, ..., x_m$ with respect to the distance to x, such that

$$|x_i - x| \le \left| x_j - x \right| \text{ if } i < j, \quad i, \quad j = 1, ..., m.$$

Example 5 *Approximate* $\sqrt{115}$ *with precision* $\varepsilon = 10^{-3}$, *using Aitken's algorithm.*

Newton interpolation polynomial

A useful representation for Lagrange interpolation polynomial is

$$(L_m f)(x) := (N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})(D^i f)(x_0) \tag{2}$$

$$= f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f],$$

which is called **Newton interpolation polynomial;** where $(D^i f)(x_0)$(or denoted $[x_0, ..., x_i; f]$) is the i-th order divided difference of the function f at x_0, given by the table

	f	$\mathcal{D}f$	$\mathcal{D}^2 f$	...	$\mathcal{D}^{\mathbf{m-1}} f$	$\mathcal{D}^m f$
x_0	f_0	$\mathcal{D}f_0$	$\mathcal{D}^2 f_0$	...	$\mathcal{D}^{m-1} f_0$	$\mathcal{D}^m f_0$
x_1	f_1	$\mathcal{D}f_1$	$\mathcal{D}^2 f_1$		$\mathcal{D}^{m-1} f_1$	
x_2	f_2	$\mathcal{D}f_2$	$\mathcal{D}^2 f_2$			
...	...	...				
x_{m-2}	f_{m-2}	$\mathcal{D}f_{m-2}$	$\mathcal{D}^2 f_{m-2}$			
x_{m-1}	f_{m-1}	$\mathcal{D}f_{m-1}$				
x_m	f_m					

Newton interpolation formula is

$$f = N_m f + R_m f,$$

where $R_m f$ denotes the remainder.

Assume that we add the point $(x, f(x))$ at the top of the table of divided differences:

	f	Df	...	$D^{m+1}f$
x	$f(x)$	$(Df)(x) = [x, x_0; f]$		$[x, x_0, ..., x_m; f]$
x_0	$f(x_0)$	$(Df)(x_0) = [x_0, x_1; f]$	...	
x_1	$f(x_1)$	$(Df)(x_1) = [x_1, x_2; f]$		
...	...	...		
x_{m-1}	$f(x_{m-1})$	$(Df)(x_{m-1}) = [x_{m-1}, x_m; f]$		
x_m	$f(x_m)$			

For obtaining the interpolation polynomial we consider

$$[x, x_0; f] = \frac{f(x_0) - f(x)}{x_0 - x} \Longrightarrow f(x) = f(x_0) + (x - x_0)[x, x_0; f] \quad (3)$$

$$[x, x_0, x_1; f] = \frac{[x_0, x_1; f] - [x, x_0; f]}{x_1 - x} \quad (4)$$

$$\Longrightarrow [x, x_0; f] = [x_0, x_1; f] + (x - x_1)[x, x_0, x_1; f].$$

Inserting (4) in (3) we get

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1; f] + (x - x_0)(x - x_1)[x, x_0, x_1; f].$$

If we continue eliminating the divided differences involving x in the same way, we get **Newton interpolation formula**

$$f(x) = (N_m f)(x) + (R_m f)(x),$$

with

$$(N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f]$$

and the remainder (the error) given by

$$(R_m f)(x) = (x - x_0)...(x - x_m)[x, x_0, ..., x_m; f]. \qquad (5)$$

Remark 6 *The remainder for Lagrange interpolation formula is also given by*

$$(R_m f)(x) = \frac{(x - x_0)...(x - x_m)}{(m+1)!} f^{(m+1)}(\xi),$$

with ξ between $x, x_0, ..., x_m$, so, by (5), it follows that **the divided differences are approximations of the derivatives**

$$[x, x_0, ..., x_m; f] = \frac{f^{(m+1)}(\xi)}{(m+1)!}.$$

Remark 7 *We notice that*

$$(N_i f)(x) = (N_{i-1} f)(x) + (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f]$$

so the Newton polynomials of degree $2, 3, ...,$ *can be iteratively generated, similarly to Aitken's algorithm.*

Example 8 *Find* $L_2 f$ *for* $f(x) = \sin \pi x$, *and* $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$, *in both forms.*

Sol. *a) We have* $u(x) = x(x - \frac{1}{6})(x - \frac{1}{2})$; $u_0(x) = (x - \frac{1}{6})(x - \frac{1}{2})$; $u_1(x) = x(x - \frac{1}{2})$; $u_2(x) = x(x - \frac{1}{6})$

$$\begin{aligned}(L_2 f)(x) &= \sum_{i=0}^{2} l_i(x) f(x_i) = \sum_{i=0}^{2} \frac{u_i(x)}{u_i(x_i)} f(x_i) \\ &= \frac{(x - \frac{1}{6})(x - \frac{1}{2})}{(-\frac{1}{6})(-\frac{1}{2})} 0 + \frac{x(x - \frac{1}{2})}{\frac{1}{6}(-\frac{1}{3})} \frac{1}{2} + \frac{x(x - \frac{1}{6})}{\frac{1}{2} \cdot \frac{1}{3}} 1 \\ &= -3x^2 + \frac{7}{2} x.\end{aligned}$$

b)

$$
\begin{aligned}
(N_2 f)(x) &= f(0) + \sum_{i=1}^{2} (x - x_0)...(x - x_{i-1})(D^i f)(x_0) \\
&= f(0) + (x - x_0)(Df)(x_0) + (x - x_0)(x - x_1)(D^2 f)(x_0) \\
&= x(Df)(x_0) + x(x - \frac{1}{6})(D^2 f)(x_0)
\end{aligned}
$$

The table of divided differences:

x	f	Df	$D^2 f$
0	0	3	-3
$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	
$\frac{1}{2}$	1		

so

$$(N_2 f)(x) = 3x - 3x(x - \frac{1}{6}) = -3x^2 + \frac{7}{2}x.$$