

## COURSE 5

### 2.4. Birkhoff interpolation

Let  $x_k \in [a, b]$ ,  $k = 0, 1, \dots, m$ ,  $x_i \neq x_j$  for  $i \neq j$ ,  $r_k \in \mathbb{N}$  and  $I_k \subset \{0, 1, \dots, r_k\}$ ,  $k = 0, 1, \dots, m$ ,  $f : [a, b] \rightarrow \mathbb{R}$  s.t.  $\exists f^{(j)}(x_k)$ ,  $k = 0, \dots, m$ ,  $j \in I_k$ , and denote  $n = |I_0| + \dots + |I_m| - 1$ , where  $|I_k|$  is the cardinal of the set  $I_k$ .

**The Birkhoff interpolation problem** (BIP) consists in determining the polynomial  $P$  of the smallest degree such that

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \dots, m; \quad j \in I_k.$$

**Remark 1** If  $I_k = \{0, 1, \dots, r_k\}$ ,  $k = 0, \dots, m$ , then (BIP) reduces to a (HIP). Birkhoff interpolation is also called lacunary Hermite interpolation.

In order to check if (BIP) has an unique solution, we consider the polynomial  $P(x) = a_n x^n + \dots + a_0$  and the  $(n+1) \times (n+1)$  linear system

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \dots, m; \quad j \in I_k, \quad (1)$$

having as unknowns the coefficients of the polynomial. If the determinant of the system (1) is nonzero then (BIP) has an unique solution.

**Definition 2** *A solution of (BIP), if exists, is called **Birkhoff interpolation polynomial**, denoted by  $B_n f$ .*

Birkhoff interpolation polynomial is given by

$$(B_n f)(x) = \sum_{k=0}^m \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k), \quad (2)$$

where  $b_{kj}(x)$  denote the Birkhoff fundamental interpolation polynomials. They fulfill relations:

$$\begin{aligned} b_{kj}^{(p)}(x_\nu) &= 0, \quad \nu \neq k, \quad p \in I_\nu \\ b_{kj}^{(p)}(x_k) &= \delta_{jp}, \quad p \in I_k, \quad \text{for } j \in I_k \text{ and } \nu, k = 0, 1, \dots, m, \end{aligned} \quad (3)$$

with  $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

**Remark 3** Because of the gaps of the interpolation conditions, it is hard to find an explicit expression for  $b_{kj}$ ,  $k = 0, \dots, m$ ;  $j \in I_k$ . They are found using relations (3).

Birkhoff interpolation formula is

$$f = B_n f + R_n f,$$

where  $R_n f$  denotes the remainder term.

**Example 4** Let  $f \in C^2[0, 1]$ , the nodes  $x_0 = 0$ ,  $x_1 = 1$  and we suppose that we know  $f(0) = 1$  and  $f'(1) = \frac{1}{2}$ . Find the corresponding interpolation formula.

**Sol.** We have  $m = 1$ ,  $I_0 = \{0\}$ ,  $I_1 = \{1\}$ , so  $n = 1 + 1 - 1 = 1$ .

We check if there exists a solution of the problem.

Consider  $P(x) = a_1x + a_0 \in \mathbb{P}_1$  and the system

$$\begin{cases} P(0) = f(0) \\ P'(1) = f'(1) \end{cases} \iff \begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases}.$$

The determinant of the system is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

so the problem has an unique solution.

The Birkhoff polynomial is

$$(B_1f)(x) = b_{00}(x)f(0) + b_{11}(x)f'(1) \in \mathbb{P}_1.$$

We have  $b_{00}(x) = ax + b \in \mathbb{P}_1$  and

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_1) = 0 \end{cases} \iff \begin{cases} b_{00}(0) = 1 \\ b'_{00}(1) = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \end{cases},$$

whence

$$b_{00}(x) = 1.$$

For  $b_{11}(x) = cx + d \in \mathbb{P}_1$  we have

$$\begin{cases} b_{11}(x_0) = 0 \\ b'_{11}(x_1) = 1 \end{cases} \iff \begin{cases} b_{11}(0) = 0 \\ b'_{11}(1) = 1 \end{cases} \iff \begin{cases} d = 0 \\ c = 1 \end{cases}$$

whence

$$b_{11}(x) = x.$$

So,

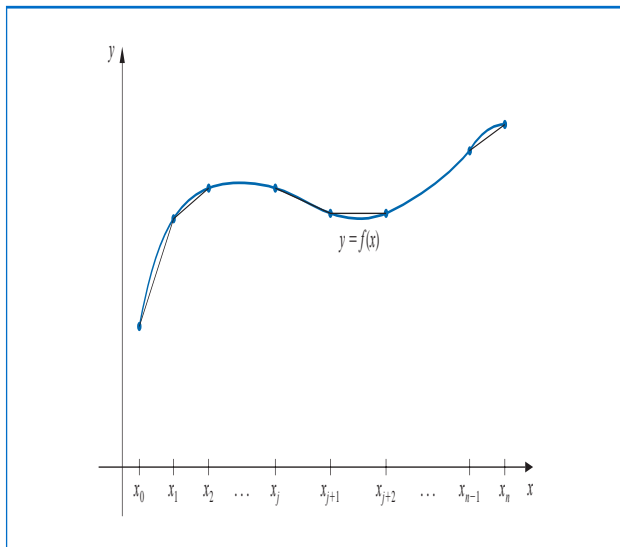
$$(B_1 f)(x) = f(0) + x f'(1) = 1 + \frac{1}{2}x.$$

**Example 5** Considering  $f'(0) = 1$ ,  $f(1) = 2$  and  $f'(2) = 1$ . Find the approximative value of  $f(\frac{1}{2})$ .

## 2.5. Cubic spline interpolation

Lagrange, Hermite, Birkhoff interpolants of large degrees could oscillate widely; a minor fluctuation over a small portion of the interval can induce large fluctuations over the entire interval.

An alternative: to divide the interval into a collection of subintervals and construct a (generally) different approximating polynomial on each subinterval. This is called **piecewise-polynomial approximation**.



Let  $f : [a, b] \rightarrow \mathbb{R}$  be the approximating function. Examples of piecewise-polynomial interpolation:

- piecewise-linear interpolation: consists of joining a set of data points  $\{(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))\}$  by a series of straight lines

Disadvantage: there is likely no differentiability at the endpoints of the subintervals, (the interpolating function is not “smooth”). Often, from physical conditions, that smoothness is required.

- Hermite interpolation when values of  $f$  and  $f'$  are known at the points  $x_0 < x_1 < \dots < x_n$ ;

Disadvantage: we need to know  $f'$  and this is frequently unavailable.

- spline interpolation: piecewise polynomials that require no specific derivative information, except perhaps at the endpoints of the interval.

**Definition 6** *The piecewise-polynomial approximation that uses cubic spline polynomials between each successive pair of nodes is called **cubic spline interpolation**.*

(The word “spline” was used to refer to a long flexible strip, generally of metal, that could be used to draw continuous smooth curves by forcing the strip to pass through specified points and tracing along the curve.)

**Definition 7** *Let  $f : [a, b] \rightarrow \mathbb{R}$  and the nodes  $a = x_0 < x_1 < \dots < x_n = b$ , a **cubic spline interpolant**  $S$  for  $f$  is the function that satisfies the following conditions:*



**(a)**  $S(x)$  is a cubic polynomial, denoted  $S_j(x)$  on the subinterval  $[x_j, x_{j+1}]$ ,  
 $\forall j = 0, 1, \dots, n-1$ , i.e.,

$$S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1] \\ S_1(x), & x \in [x_1, x_2] \\ \dots \\ S_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases}$$

**(b)**  $S_j(x_j) = f(x_j)$  and  $S_j(x_{j+1}) = f(x_{j+1})$ ,  $\forall j = 0, 1, \dots, n-1$ ;

**(c)**  $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$ ,  $\forall j = 0, 1, \dots, n-2$ ;

**(d)**  $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ ,  $\forall j = 0, 1, \dots, n-2$ ;

**(e)**  $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$ ,  $\forall j = 0, 1, \dots, n-2$ ;

**(f)** One of the following boundary conditions is satisfied:

- (i)  $S''(x_0) = S''(x_n) = 0$  ( $\iff S''_0(x_0) = S''_{n-1}(x_n) = 0$  *natural (or free) boundary*) **natural spline**;
- (ii)  $S'(x_0) = f'(x_0)$  and  $S'(x_n) = f'(x_n)$  ( $\iff S'_0(x_0) = f'(x_0)$  and  $S'_{n-1}(x_n) = f'(x_n)$  *clamped boundary*) **clamped spline**;
- (iii)  $S_1(x) = S_2(x)$  and  $S_{n-2} = S_{n-1}$  (**de Boor spline**).

**Remark 8** A cubic spline function defined on an interval divided into  $n$  subintervals will require determining  $4n$  constants.

We have the following expression of a cubic spline:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad \forall j = 0, 1, \dots, n-1. \quad (4)$$

**Theorem 9** If  $f$  is defined at  $a = x_0 < x_1 < \dots < x_n = b$ , then  $f$  has a unique natural spline interpolant  $S$  on the nodes  $x_0, x_1, \dots, x_n$ ; that satisfies the natural boundary conditions  $S''(a) = 0$  and  $S''(b) = 0$ .

**Theorem 10** If  $f$  is defined at  $a = x_0 < x_1 < \dots < x_n = b$  and differentiable at  $a$  and  $b$ , then  $f$  has a unique clamped spline interpolant  $S$  on the nodes  $x_0, x_1, \dots, x_n$ ; that satisfies the clamped boundary conditions  $S'(a) = f'(a)$  and  $S'(b) = f'(b)$ .

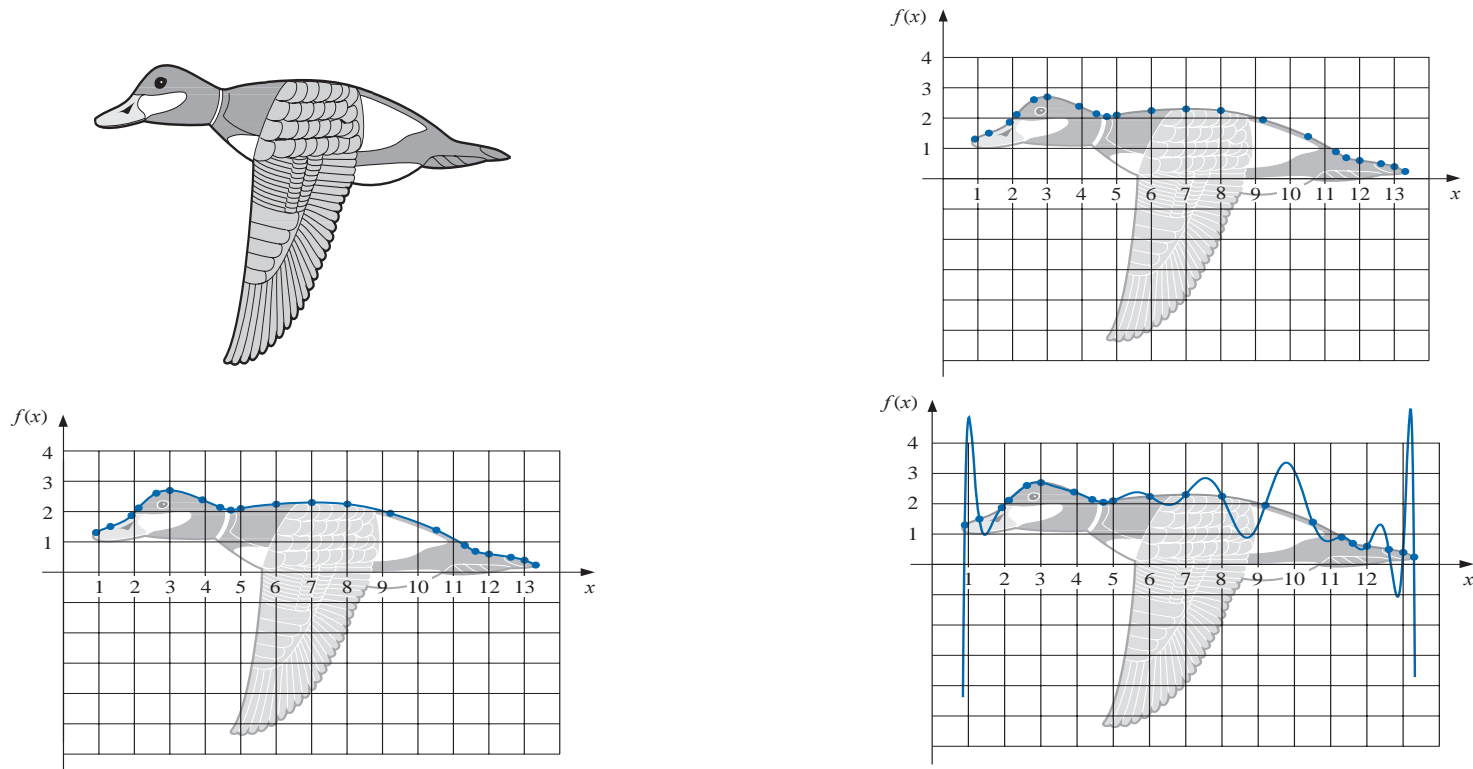
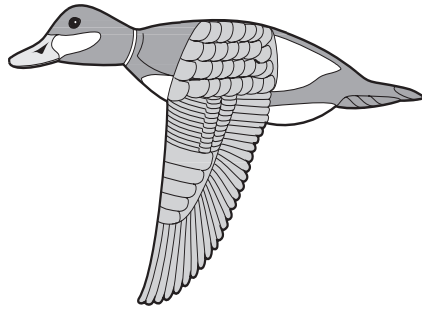
**Theorem 11** Let  $f \in C^4[a, b]$  with  $\max_{a \leq x \leq b} |f^{(4)}(x)| = M$ . If  $S$  is the unique clamped cubic spline interpolant to  $f$  with respect to the nodes  $a = x_0 < x_1 < \dots < x_n = b$ , then for all  $x$  in  $[a, b]$ ,

$$|f(x) - S(x)| \leq \frac{5M}{384} \max_{0 \leq j \leq n-1} (x_{j+1} - x_j)^4.$$

**Remark 12** A fourth-order error-bound result also holds in the case of natural boundary conditions, but it is more difficult to express.

**Remark 13** The natural boundary conditions will generally give less accurate results than the clamped conditions near the ends of the interval  $[x_0, x_n]$  unless the function  $f$  happens to nearly satisfy  $f''(x_0) = f''(x_n) = 0$ .

**Illustration.** To approximate the top profile of a duck, we have chosen 21 points along the curve through which we want the approximating curves to pass.



- 1) *The duck in flight.*
- 2) *The points.*
- 3) *The natural cubic spline.*
- 4) *The Lagrange interpolation polynomial.*

**Example 14** Construct a natural cubic spline that passes through the points  $(1, 2)$ ,  $(2, 3)$  and  $(3, 5)$ .

**Sol.** (Sketch of the solution) We follow Definition 7:

Here  $S(x)$  consists of two cubic splines,  $S_j(x)$  on the subinterval  $[x_j, x_{j+1}]$ ,  $\forall j = 0, 1$ , i.e.,

$$S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1] \\ S_1(x), & x \in [x_1, x_2] \end{cases}$$

given by (4),

$$S_0(x) = a_0 + b_0(x - 1) + c_0(x - 1)^2 + d_0(x - 1)^3,$$

$$S_1(x) = a_1 + b_1(x - 2) + c_1(x - 2)^2 + d_1(x - 2)^3.$$

There are 8 constants  $(a_i, b_i, c_i, d_i, i = 0, 1)$  to be determined, which requires 8 conditions, that come from (b),(c),(d),(e),(i).

**Example 15** Construct a clamped spline  $S$  that passes through the points  $(1, 2)$ ,  $(2, 3)$  and  $(3, 5)$  and that has  $S'(1) = 2$  and  $S'(3) = 1$ .