COURSE 4

2.3. Hermite interpolation

Example 1 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

Let $x_k \in [a, b]$, $k = 0, 1, ..., m$ be such that $x_i \neq x_j$, for $i \neq j$ and let $r_k \in \mathbb{N}$, $k = 0, 1, ..., m$. Consider $f : [a, b] \to \mathbb{R}$ such that there exist $f^{(j)}(x_k)$, $k = 0, 1, ..., m$; $j = 0, 1, ..., r_k$ and $n = m + r_0 + ... + r_m$.

The Hermite interpolation problem (HIP) consists in determining the polynomial P of the smallest degree for which

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m; \; j = 0, ..., r_k.$$

Definition 2 *A solution of (HIP) is called* **Hermite interpolation polynomial**, *denoted by* $H_n f$.

Hermite interpolation polynomial, $H_n f$, satisfies the interpolation conditions:

$$(H_n f)^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m; \; j = 0, ..., r_k.$$

Hermite interpolation polynomial is given by

$$(H_n f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \in \mathbb{P}_n, \tag{1}$$

where $h_{kj}(x)$ denote **the Hermite fundamental interpolation polynomials.** They fulfill the relations:

$$h_{kj}^{(p)}(x_\nu) = 0, \; \nu \neq k, \quad p = 0, 1, ..., r_\nu$$

$$h_{kj}^{(p)}(x_k) = \delta_{jp}, \; p = 0, 1, ..., r_k, \quad \text{for } j = 0, 1, ..., r_k \text{ and } \nu, k = 0, 1, ..., m,$$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

We denote by

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k+1} \qquad \text{and } u_k(x) = \frac{u(x)}{(x - x_k)^{r_k+1}}.$$

We have

$$h_{kj}(x) = \frac{(x - x_k)^j}{j!} u_k(x) \sum_{\upsilon=0}^{r_k-j} \frac{(x - x_k)^\nu}{\nu!} \left[\frac{1}{u_k(x)}\right]_{x=x_k}^{(\nu)}. \tag{2}$$

Example 3 *Find the Hermite interpolation polynomial for a function f for which we know $f(0) = 1, f'(0) = 2$ and $f(1) = -3$ (equivalent with $x_0 = 0$ multiple node of order 2 or double node, $x_1 = 1$ simple node).*

Sol. We have $x_0 = 0, x_1 = 1, m = 1, r_0 = 1, r_1 = 0, n = m + r_0 + r_1 = 2$

$$\begin{aligned}(H_2 f)(x) &= \sum_{k=0}^{1} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k)\\ &= h_{00}(x) f(0) + h_{01}(x) f'(0) + h_{10}(x) f(1).\end{aligned}$$

We have h_{00}, h_{01}, h_{10}. These fulfills relations:

$h_{kj}^{(p)}(x_\nu) = 0,\ \nu \neq k, \quad p = 0, 1, ..., r_\nu$
$h_{kj}^{(p)}(x_k) = \delta_{jp},\ p = 0, 1, ..., r_k, \quad$ for $j = 0, 1, ..., r_k$ and $\nu, k = 0, 1, ..., m$.

We have $h_{00}(x) = a_1 x^2 + b_1 x + c_1 \in \mathbb{P}_2$, with $a_1, b_1, c_1 \in \mathbb{R}$, and the system

$$\begin{cases} h_{00}(x_0) = 1 \\ h'_{00}(x_0) = 0 \\ h_{00}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{00}(0) = 1 \\ h'_{00}(0) = 0 \\ h_{00}(1) = 0 \end{cases}$$

that becomes

$$\begin{cases} c_1 = 1 \\ b_1 = 0 \\ a_1 + b_1 + c_1 = 0. \end{cases}$$

Solution is: $a_1 = -1, b_1 = 0, c_1 = 1$ so $h_{00}(x) = -x^2 + 1$.

We have $h_{01}(x) = a_2 x^2 + b_2 x + c_2 \in \mathbb{P}_2$, with $a_2, b_2, c_2 \in \mathbb{R}$. The system is

$$\begin{cases} h_{01}(x_0) = 0 \\ h'_{01}(x_0) = 1 \\ h_{01}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{01}(0) = 0 \\ h'_{01}(0) = 1 \\ h_{01}(1) = 0 \end{cases}$$

and we get $h_{01}(x) = -x^2 + x$.

We have $h_{10}(x) = a_3x^2 + b_3x + c_3 \in \mathbb{P}_2$, with $a_3, b_3, c_3 \in \mathbb{R}$. The system is

$$\begin{cases} h_{10}(x_0) = 0 \\ h'_{10}(x_0) = 0 \\ h_{10}(x_1) = 1 \end{cases} \Leftrightarrow \begin{cases} h_{10}(0) = 0 \\ h'_{10}(0) = 0 \\ h_{10}(1) = 1 \end{cases}$$

and we get $h_{10}(x) = x^2$.

The Hermite polynomial is

$$(H_2f)(x) = -x^2 + 1 - 2x^2 + 2x - 3x^2 = -6x^2 + 2x + 1.$$

The Hermite interpolation formula is

$$f = H_nf + R_nf,$$

where $R_n f$ denotes the remainder term (the error).

Theorem 4 *If* $f \in C^n[\alpha, \beta]$ *and* $f^{(n)}$ *is derivable on* (α, β), *with* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$, *then there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi). \tag{3}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_n f)(z) \\ u(x) & (R_n f)(x) \end{vmatrix}.$$

$F \in C^n[\alpha, \beta]$ and there exists $F^{(n+1)}$ on (α, β).

We have

$$F(x) = 0, \quad F^{(j)}(x_k) = 0, \qquad k = 0, ..., m; \;\; j = 0, ..., r_k;$$

because

$$u(x)=\prod_{k=0}^{m}(x-x_k)^{r_k+1}\Rightarrow u^{(j)}(x_k)=0,\ \ j=0,...,r_k$$

and

$$(R_mf)^{(j)}(x_k)=f^{(j)}(x_k)-(H_nf)^{(j)}(x_k)=f^{(j)}(x_k)-f^{(j)}(x_k)=0.$$

So, F and its derivatives have $n+2$ distinct zeros in (α,β). Applying successively Rolle's theorem it follows that F' has at least $n+1$ zeros in $(\alpha,\beta)\Rightarrow ...\Rightarrow F^{(n+1)}$ has at least one zero $\xi\in(\alpha,\beta)$, $F^{(n+1)}(\xi)=0$.

We have

$$F^{(n+1)}(z)=\begin{vmatrix} u^{(n+1)}(z) & (R_nf)^{(n+1)}(z) \\ u(x) & (R_nf)(x) \end{vmatrix},$$

with $u(z)=\prod\limits_{k=0}^{m}(z-z_k)^{r_k+1}\in\mathbb{P}_{n+1}\Rightarrow u^{(n+1)}(z)=(n+1)!$, and $(R_nf)^{(n+1)}(z)=f^{(n+1)}(z)-(H_nf)^{(n+1)}(z)=f^{(n+1)}(z)$ (as, $H_nf\in$

$\mathbb{P}_n$). We get

$$F^{(n+1)}(\xi) = \begin{vmatrix} (n+1)! & f^{(n+1)}(\xi) \\ u(x) & (R_n f)(x) \end{vmatrix} = 0,$$

whence it follows (3). ■

Corollary 5 *If* $f \in C^{n+1}[a,b]$ *then*

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ *denotes the uniform norm* ($\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$).

Remark 6 *In case of* $m = 0$, *i.e.,* $n = r_0$, *(HIP) becomes* **Taylor interpolation problem**. *Taylor interpolation polynomial is*

$$(T_n f)(x) = \sum_{j=0}^{n} \frac{(x - x_0)^j}{j!} f^{(j)}(x_0).$$

Example 7 *Find the Hermite interpolation formula for the function* $f(x) = xe^x$ *for which we know* $f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$,

$f(1) = 2.7183$, *(equivalent with* $x_0 = -1$ *simple,* $x_1 = 0$ *multiple of order 2 and* $x_2 = 1$ *simple). Which is the limit of the error for approximating* $f(\frac{1}{2})$?

Hermite interpolation with double nodes

Example 8 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

Consider $f : [a, b] \to \mathbb{R}$, $x_0, x_1, ..., x_m \in [a, b]$

and the values $f(x_0), f(x_1), ..., f(x_m), f'(x_0), f'(x_1), ..., f'(x_m)$.

The Hermite interpolation polynomial with double nodes, H_{2m+1}, satisfies the interpolation properties:

$$H_{2m+1}(x_i) = f(x_i), \;\; i = \overline{0, m},$$
$$H'_{2m+1}(x_i) = f'(x_i), \;\; i = \overline{0, m}.$$

It is a polynomial of $n = 2m + 1$ degree.

For computation: use Lagrange polynomial written in Newton form, with divided differences table having each node x_i written twice.

Consider $z_0 = x_0$, $z_1 = x_0$, $z_2 = x_1$, $z_3 = x_1$, $\ldots$, $z_{2m} = x_m$, $z_{2m+1} = x_m$.

Form divided differences table: each node appear twice, in the first column write the values of f for each node twice; in the second column, at the odd positions put the values of the derivatives of f; the other elements are computed using the rule from divided differences.

We obtain the following table:

z_0	$f(z_0)$	$(\mathcal{D}^1 f)(z_0) = f'(x_0)$	$(\mathcal{D}^2 f)(z_0)$		$(\mathcal{D}^{2m} f)(z_0)$	$(\mathcal{D}^{2m+1} f)(z_0)$
z_1	$f(z_1)$	$(\mathcal{D}^1 f)(z_1)$	$\vdots$		$(\mathcal{D}^{2m} f)(z_1)$	
z_2	$f(z_2)$	$(\mathcal{D}^1 f)(z_2) = f'(x_1)$				
z_3	$f(z_3)$	$\vdots$				
$\vdots$	$\vdots$	$(\mathcal{D}^1 f)(z_{2m-1})$	$(\mathcal{D}^2 f)(z_{2m-1})$	$\ddots$		
z_{2m}	$f(z_{2m})$	$(\mathcal{D}^1 f)(z_{2m}) = f'(x_m)$		$\ldots$		
z_{2m+1}	$f(z_{2m+1})$			$\ldots$		

Newton interpolation polynomial for the nodes $x_0, ..., x_n$ is

$$(N_n f)(x) = f(x_0) + \sum_{i=1}^{n} (x - x_0)...(x - x_{i-1})(\mathcal{D}^i f)(x_0),$$

and similarly, Hermite interpolation polynomial is

$$(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0)...(x - z_{i-1})(\mathcal{D}^i f)(z_0),$$

where $(\mathcal{D}^i f)(z_0)$, $i = 1, ..., 2m + 1$ are the elements from the first line and columns $2, ..., 2m + 1$.

Example 9 *Consider the double nodes* $x_0 = -1$ *and* $x_1 = 1$, *and* $f(-1) = -3, f'(-1) = 10, f(1) = 1, f'(1) = 2$. *Find the Hermite interpolation polynomial, that approximates the function* f, *in both forms, using the classical formula and using divided differences.*

Sol. We present here the method with divided differences. We have $m = 1, r_0 = r_1 = 1 \Rightarrow n = 3$

$z_0 = -1$	$f(-1) = -3$	$f'(-1) = 10$	$\frac{\frac{f(1)-f(-1)}{2} - f'(-1)}{z_2 - z_0} = -4$	$\frac{0-(-4)}{z_3 - z_0} = 2$
$z_1 = -1$	$f(-1) = -3$	$\frac{f(1)-f(-1)}{z_2 - z_1} = 2$	$\frac{f'(1) - \frac{f(1)-f(-1)}{2}}{z_3 - z_1} = 0$	
$z_2 = 1$	$f(1) = 1$	$f'(1) = 2$		
$z_3 = 1$	$f(1) = 1$			

The Hermite interpolation polynomial is

$$
\begin{aligned}
(H_3 f)(x) &= f(z_0) + \sum_{i=1}^{3} (x - z_0)\dots(x - z_{i-1})(\mathcal{D}^i f)(z_0) \\
&= f(z_0) + (x - z_0)(\mathcal{D}^1 f)(z_0) + (x - z_0)(x - z_1)(\mathcal{D}^2 f)(z_0) \\
&+ (x - z_0)(x - z_1)(x - z_2)(\mathcal{D}^3 f)(z_0)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(H_3 f)(x) =& f(-1) + (x+1)f'(-1) + (x+1)^2 \frac{f(1) - f(-1) - 2f'(-1)}{4} \\
& + (x+1)^2 (x-1) \frac{2f'(1) - f(1) + f(-1)}{4} \\
=& -3 + 10(x+1) - 4(x+1)^2 + 2(x+1)^2(x-1) \\
=& 2x^3 - 2x^2 + 1.
\end{aligned}
$$

Example 10 *Considering the the following data*

x	0	2	3
$f(x)$	0	10	12
$f\prime(x)$	5	3	7

find the corresponding Hermite interpolation polynomial.