Extracting the Sphaleron Rate from Euclidean Simulations

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1. Sphaleron rate
   - Non-conservation of baryon and lepton number
   - Topological structure of SU(N)
   - Topological charge and sphaleron rate definition
2. Analytic sphaleron rate in 1D quantum mechanics
   - Free particle winding around a circle
3. Numerical result of sphaleron rate
   - Monte Carlo method
   - Free particle winding around the circle
   - Sine Gordon model
   - $(1+1)$-dimensional abelian Higgs model without fermions
4. Numerical analytic continuation
   - Inverse theory and integral equation
   - Backus-Gilbert method
   - Test on a Breit-Wigner spectral function
5. Conclusion

Non-conservation of baryon and lepton number

Classically : $\partial_\mu j^\mu = 0$

Quantization of gauge theories $\longrightarrow$ anomalous symmetries

Consequence : Non-conservation of baryonic and leponic currents

$$\boxed{\partial_\mu j^\mu_B = \frac{g^2 n_f}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F^a_{\alpha\beta}F^a_{\mu\nu}}$$

$$\boxed{\partial_\mu j^\mu_L = \frac{g^2 n_f}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F^a_{\alpha\beta}F^a_{\mu\nu}}$$

$n_f \equiv$ the number of families of quarks and leptons.

B and **L** are not conserved, but **B**-**L** is conserved

Topological structure of SU(N), $N \geq 2$

RHS of the anomalous equation : $\mathrm{Tr}\mathrm{F}_{\mu\nu}\widetilde{\mathrm{F}}^{\mu\nu} = \partial_\mu \mathrm{K}^\mu$

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\mathrm{F}_{\nu\rho}\mathrm{A}_\sigma + \tfrac{2\mathrm{g}}{3}\mathrm{A}_\nu\mathrm{A}_\rho\mathrm{A}_\sigma\right)$$

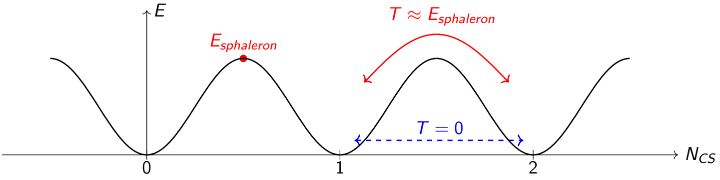
In a vacuum configuration $F_{\mu\nu} = 0$: $A_\mu = -\frac{i}{g}G^{-1}(\partial_\mu G)$

Chern-Simons number definition :

$$\begin{aligned}\mathrm{N}_{\mathrm{CS}} &= \frac{1}{16\pi^2}\int d^3xK^0 \\ &= \frac{1}{24\pi^2}\int d^3x\epsilon^{ijk}\mathrm{Tr}(\mathrm{G}^{-1}\partial_\mathrm{i}\mathrm{G}\mathrm{G}^{-1}\partial_\mathrm{j}\mathrm{G}\mathrm{G}^{-1}\partial_\mathrm{k}\mathrm{G})\end{aligned}$$

Topological structure of SU(N), $N \geq 2$

The Chern-Simons number $\longleftrightarrow$ vacuum.



$\boxed{\text{The homoptopy group of SU(N) : } \pi_3(\mathrm{SU(N)}) = \mathcal{Z}}$

Change of topological sector :

1. At zero temperature by tunneling $\longrightarrow$ Instanton solution ✗
2. At finite temperature $\longrightarrow$ Sphaleron solution ✓

Topological charge and sphaleron rate definition

Topological charge or winding number defnition :

$$\boxed{Q(t) = N_{CS}(t) - N_{CS}(0) = \frac{g^2 n_f}{32\pi^2} \int_0^t dt' \int d^3x F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a}$$

Winding of gauge fields $\longrightarrow$ Change of **B** and **L** by an amount of n_f

$$\longrightarrow \Delta(B+L) = 2n_f.$$

Homogeneity of spacetime : $\boxed{\langle Q^2(t) \rangle = \Gamma V t}$

The sphaleron rate Γ is the rate at which we go from one vacua to an other.

Sphaleron rate and Kubo formula

Computation of $\langle Q^2(t)\rangle$ for $t \longrightarrow \infty$:

$$\Gamma V t = \int d^4x G^{>}_{F\tilde{F}}(x,0),$$

$$G^{>}_{F\tilde{F}}(x,y) = \langle \frac{g^2 n_f}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a(x) \frac{g^2 n_f}{32\pi^2} F^b_{\alpha\beta}\tilde{F}^{\alpha\beta}_b(y)\rangle$$

This formula looks like the Kubo formula in response theory.

$$\boxed{\Gamma = \lim_{V,t\to\infty} \frac{\langle Q^2(t)\rangle}{Vt}}$$

The sphaleron rate is a response coefficient that can be computed using two-point function.

Objective 1

Compute the sphaleron rate for a theory in 1D quantum mechanics that have the same homotopy group than SU(N) : the free particle winding around a circle.

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Topological charge and sphaleron rate

Free particle : $\mathcal{L} = \frac{1}{2} m\dot{x}^2$

One turn $\longrightarrow$ change of topological sector by one units $\longrightarrow \pi(S_1) = \mathcal{Z}$.

Topological charge : $\boxed{Q = \frac{1}{R}\int_0^\beta \dot{x}(\tau) d\tau}$

The position in the circle is $x \in [0, R[$.

Equation of motion : $\ddot{x} = 0$

Boundary conditions : $x(0) = x(\beta) + QR$

Solution : $x(t) = -\frac{QR}{\beta}\tau$

This model have a non-trivial solution at finite energy
$\longrightarrow$ Analogue of instanton solution in QFT

Two-point function of velocities

Strategy to find the sphaleron rate :

1. Find $\langle Q^2(\tau)\rangle$.
2. Do the analytical continuation from Euclidean time to Minkovski time.

$$Z[\beta, J] \equiv \exp\left[-\int_0^\beta d\tau\left(\frac{1}{2}m\dot{x}^2(\tau) - J(\tau)\dot{x}(\tau)\right)\right]$$

$$\langle \dot{x}(\tau_1)\dot{x}(\tau_2)\rangle = \frac{1}{Z[\beta,0]}\frac{\delta}{\delta J(\tau_1)}\frac{\delta}{\delta J(\tau_2)}\,Z[\beta,J]|_{J=0}$$

Classical equation of motion : $\ddot{x}_{cl}(\tau) = \frac{\dot{J}(\tau)}{m}$

The two point function of velocities :

$$\boxed{\langle \dot{x}(\tau_1)\dot{x}(\tau_2)\rangle = \sum_{Q\in\mathbb{Z}}\left[\frac{1}{m}\delta(\tau_1-\tau_2) - \frac{1}{m\beta} + \frac{Q^2R^2}{\beta^2}\right]\exp\left[-\frac{mQ^2R^2}{2\beta}\right]}$$

Sphaleron rate

Using the expression of topological charge :

$$\boxed{\langle Q^2(\tau)\rangle = \frac{1}{R^2}\sum_{Q\in\mathbb{Z}}\left[\frac{1}{m}\tau - \left(\frac{1}{m\beta} - \frac{Q^2R^2}{\beta^2}\right)\tau^2\right]\exp\left[-\frac{mQ^2R^2}{2\beta}\right]}$$

Analytic continuation ($\tau \longrightarrow -it$) :

$$\boxed{\langle Q^2(t)\rangle = \frac{1}{R^2}\sum_{Q\in\mathbb{Z}}\left[-\frac{1}{m}it + \left(\frac{1}{m\beta} - \frac{Q^2R^2}{\beta^2}\right)t^2\right]\exp\left[-\frac{mQ^2R^2}{2\beta}\right]}$$

$\langle Q^2(t)\rangle$ is exponentially suppressed for another sectors than $Q = 0$.

Sphaleron solutions dominates at topological sector **Q=0**.

Topological susceptibility and analytic continuation

Euclidean time

Topological suscptiblity :

$$\langle Q^2 \rangle = \langle Q_E^2(\beta) \rangle = \sum_{Q \in \mathbb{Z}} Q^2 \exp\left[-\frac{mQ^2R^2}{2\beta}\right]$$

$$\langle Q_E^2(\beta) \rangle = 0, \text{ for } Q = 0$$

$$\boxed{\langle Q_E^2(\beta) \rangle \sim \exp\left[-\frac{mQ^2R^2T}{2}\right]}$$

Instanton

Minkowski time

$$\langle Q_M^2(t) \rangle \neq 0, \text{ for } Q = 0$$

$$\boxed{\langle Q_M^2(t) \rangle \sim \frac{T}{mR^2}}$$

Sphaleron

Objective 2

The computation of the sphaleron rate in QFT can not be done analytically. To do this we will require to numerical methods, here we will use Monte Carlo simulation.

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Monte Carlo method

Path integral average : $\langle \Gamma[x] \rangle := \frac{\int \mathcal{D}x(\tau)\Gamma[x]e^{-S_E[x]}}{\int \mathcal{D}x(\tau)e^{-S_E[x]}}$

The path integral average $\longrightarrow$ weighted average of $\Gamma[x]$ over all paths with weight/probability $e^{-S_E[x]}$.

Action in the lattice : $S_E[x] = \sum_{j=0}^{N-1} \left(\frac{m}{2a}(x_{j+1} - x_j)^2 + aV(x_j) \right)$

Idea : Generate paths that already have the right probability
$\longrightarrow$ **Metropolis algorithm**.

$$x^{(\alpha)} = \left\{ x_0^{(\alpha)}, x_1^{(\alpha)}, ..., x_{N-1}^{(\alpha)} \right\} \qquad P[x^{(\alpha)}] \propto e^{-S_E[x^{(\alpha)}]}$$

Monte Carlo method

The path integral average : $\langle \Gamma[x] \rangle \approx \bar{\Gamma} = \frac{1}{N_{\text{cf}}} \sum_{\alpha=1}^{N_{\text{cf}}} \Gamma[x^{(\alpha)}]$

Two-point function $G(t) := \langle x(\tau)x(0) \rangle$ in the lattice :

$$G(t) = \frac{1}{N} \sum_{i=0}^{N-1} \langle x(\tau_i + \tau) x(\tau_i) \rangle \xrightarrow[\text{lattice}]{} \boxed{\mathrm{G}_n^{(\alpha)} = \frac{1}{N} \sum_{i=0}^{N-1} \langle x_{(i+n) mod N}^{(\alpha)} x_i^{(\alpha)} \rangle}$$

$$G = \{G_0, ..., G_{N-1}\}, \text{ with } \mathrm{G_n} = \frac{1}{\mathrm{N_{cf}}} \sum_{\alpha=1}^{\mathrm{N_{cf}}} \mathrm{G_n^{(\alpha)}}$$

Simulation on the circle

Derivatives $\longrightarrow$ Distance between two points. On the circle with R=1 [1] :

$$S = \frac{1}{2}\sum_j \frac{[(x_{j+1}-x_j)mod\frac{1}{2}]^2}{a} + a\sum_j V(x_j)$$

$$(x-y)mod\frac{1}{2} = \begin{cases} x-y, & \text{if } |x-y| \leq \frac{1}{2} \\ x-y-1, & \text{if } x-y > \frac{1}{2} \\ x-y+1 & \text{if } x-y < -\frac{1}{2} \end{cases}$$

[1] Caludio Bonati and Massimo D'Elia, 2018, Phys. Rev. E.

Free particle winding around the circle

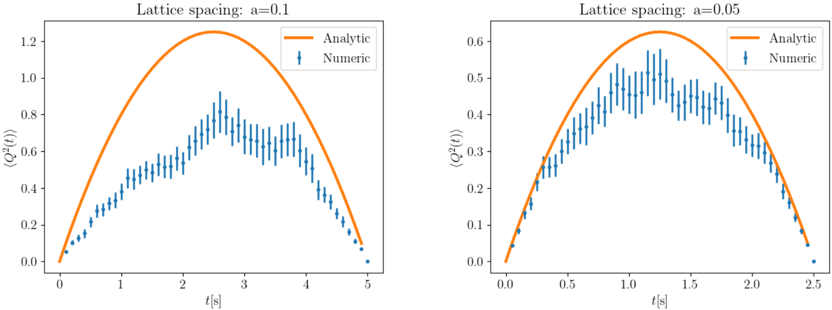

Figure – Monte Carlo parameter : $N_{cor} = 500$ and $N_{cf} = 500$
$N = 50$ points, and $Q = 0$

Free particle winding around the circle

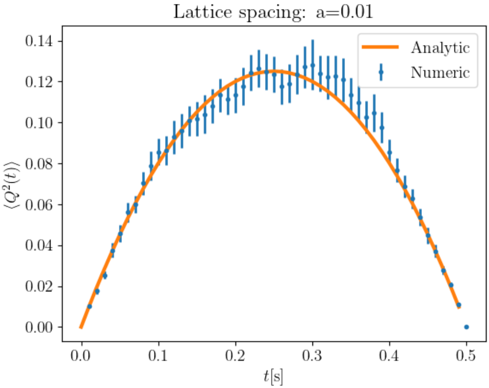

Figure – Monte Carlo parameter : $N_{cor} = 500$ and $N_{cf} = 500$
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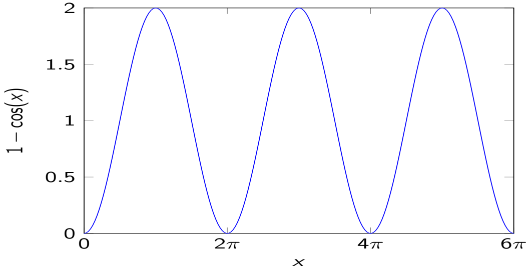
Sine Gordon model

Action : $S(x,\dot{x}) = \int_0^t dt' \left(\frac{m}{2}\dot{x}^2(t') - mA^2(1-\cos(x(t')))\right)$



Sine Gordon potential $\longrightarrow$ The periodicity of the potential must corresponds to the periodicity of the circle.

$$V(x) = mA^2(1 - cos(2\pi x))$$

Sine Gordon model

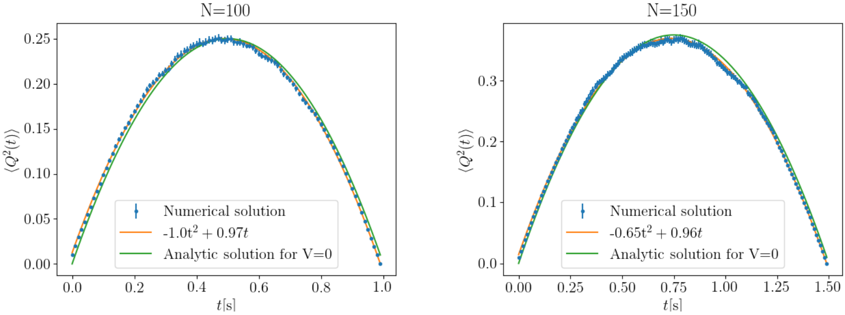


Figure – Sphaleron rate for $\beta = 1.0,\ 1.5$
Monte Carlo parameter are $N_{cor} = 500$ and $N_{cf} = 20000$.
High of the barrier is $A = 0.5$ and $Q = 0$

Sine Gordon model

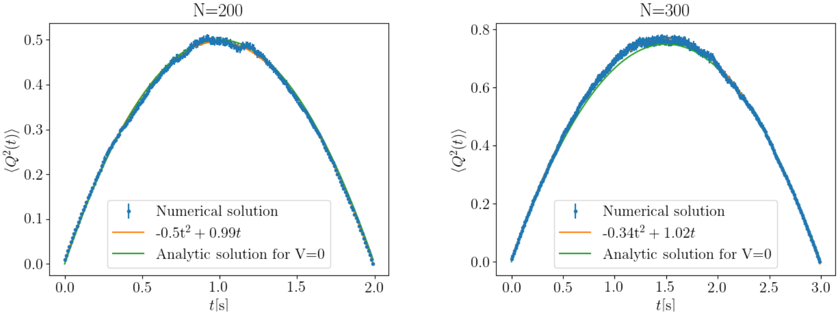


Figure – Sphaleron rate for $\beta = 2.0,\ 3.0$
Monte Carlo parameter are $N_{cor} = 500$ and $N_{cf} = 20000$.
High of the barrier is $A = 0.5$ and $Q = 0$

From numerical result

- It possible to extract the coefficient in front of t^2.
- For Sine Gordon model it is difficult to distinguish this result from the free particle one.

$(1+1)$-dimensional abelian Higgs model without fermions

Lagrangian : $\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D_\mu\phi) - \frac{1}{4}\lambda(\phi^2 - v^2)^2}$

Motivation :

1. This system have a non-trivial topological structure as SU(N).
   - Ground state : $A_\mu = 0$, $\phi = v$
   - Kink $\longrightarrow$ static, finite energy ($|\phi| = v$ for $|x| \longrightarrow \infty$) solution.
   - $\exists$ Mapping between spatial infinity and the classical vacuum (circle of radius v). In temporal gauge $A_0 = 0$

$$\phi_v(x) = ve^{i\alpha(x)}, \quad A_1 = \frac{1}{g}\partial_1\alpha(x)$$

$$\boxed{\mathrm{N}_{CS} = \frac{g}{2\pi}\int A_1 dx}$$

2. Compare with the simulation in classical effective field theory of the sphaleron rate [1].

[1] D.Yu. Gregorev, V.A. Rubakov and M.E. Shaposhnikov, 1989, Physics Letters B.

Objective 3

To find the sphaleron rate in quantum field theory it is necessary to do the analytic continuation. This can be done using Backus-Gilbert method

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Correlator and spectral function

Relation between the euclidean correlator $G_E(\tau)$ and the spectral function $\rho(\omega) \longrightarrow$ Integral equation :

$$\boxed{G_E(\tau) = \int_0^{+\infty} \frac{d\omega'}{\pi}\rho(\omega')\frac{\cosh\left(\omega'\left(\tau - \frac{\beta}{2}\right)\right)}{\sinh\left(\frac{\beta\omega'}{2}\right)}}$$

Relation between $\rho(\omega)$ and the retarded correlator $G_R(\omega)$:

$$G_R(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i}\frac{\rho(\omega')}{\omega' - \omega - i\epsilon}$$

Invert integral equation is a highly non-trivial problem that have its own theory, called **inverse theory**.

Inverse Theory

Equation to solve : $\boxed{G_i = \int u(x)k_i(x)dx + n_i}$

Idea : Discretize the space such that neither $u(x)$ nor k_i varies too much in an interval, then $\hat{u}(x)$ would be a good estimator for $u(x)$ around each discrete points.

Two functionals :

- $\mathcal{A}$: Quantify how the estimator $\hat{u}(x)$ is far from the true solution $u(x)$.
- $\mathcal{B}$: Evalutes the smoothness of the solution.

Best solution for $\hat{u}(x)$: $\boxed{\text{Minimizing} \quad \mathcal{A} + \lambda\mathcal{B}}$

$\lambda \longrightarrow$ the best balance between $\mathcal{A}$ and $\mathcal{B}$.

Inverse theory algorithm $\longrightarrow$ Different choice of $\mathcal{A}$ and $\mathcal{B}$.

Backus-Gilbert method

For our integral equation : $\boxed{\hat{\rho}(\omega) = \frac{G^E(\tau_i)\left[W_{ij}(\omega)+\lambda S_{ij}\right]^{-1} R_j}{R_k[W_{kl}(\omega)+\lambda S_{kl}]^{-1} R_l}}$

Relevant quantities :

$$K(\tau,\omega) = \frac{\cosh\left[\omega\left(\tau-\frac{\beta}{2})\right)\right]}{\sinh\left(\frac{\beta\omega}{2}\right)}$$

$$G_E(\tau_i) = \int_0^\infty d\omega K(\tau_i,\omega)\rho(\omega), \quad \text{for } \tau \in [0,\beta]$$

$$W_{ij}(\omega') = \int_0^\infty d\omega(\omega-\omega')^2 K(\tau_i,\omega)K(\tau_j,\omega)$$

$$R_i = \int_0^\infty d\omega K(\tau_i,\omega)$$

$$S_{ij} = Cov(G_E(\tau_i), G_E(\tau_j))$$

Test on a Breit-Wigner spectral function

Breit-Wigner spectral function : $\rho(\omega) = \frac{1}{\pi}\frac{2\omega\gamma}{(\omega^2-\gamma^2-M^2)^2+4\omega^2\gamma^2}$

Idea :

1. Compute the Euclidean correlator using the integral equation.
2. Reconstruct the spectral function $\hat{\rho}(\omega)$ using Backus-Gilbert algorithm.

Theoretical model $\longrightarrow S_{ij} = 0$.
$K(\tau,\omega) \longrightarrow$ Ill defined at $\tau = \infty \longrightarrow$ Remove this point.

W is extremely ill-conditioned $\longrightarrow$ Singular Value Decomposition (SVD) and Tikhonov regularisation to compute the inverse.

$$W^{-1} = VDU^{\dagger}, \quad D = \mathrm{diag}(\sigma_1^{-1}, ..., \sigma_{\mathrm{N}-1}^{-1})$$
$$D_{ij} \longrightarrow \tilde{D}_{ij} = \delta_{ij}\frac{\sigma_i}{\sigma_i^2+\lambda^2}$$

Numerical results

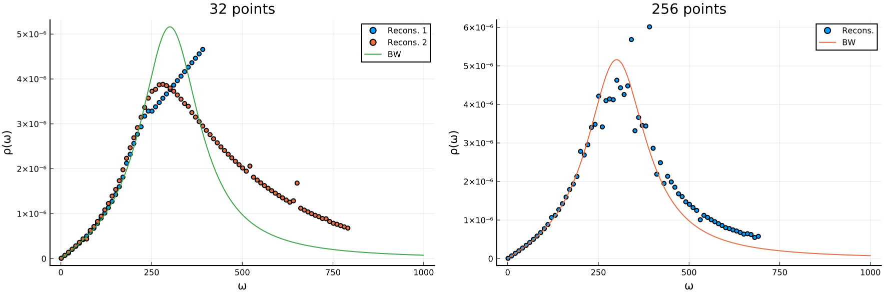

Figure – Backus-Gilbert reconstruction of the Breit-Wigner formula using $N = 32$ and $N=256$ points. The blue curves are realised without exponential damping

Numerical result

- For a small number of points the reconstruction breaks down after some ω, but we can modify the kernel to have a better reconstruction.
- For huge number of points N the simulation takes a lot of time, but gives a good reconstruction. In QFT we need N^4 points for the correlator.
- It requires high precision for the integral W and R.
- We can play with the Tikhonov regularisation parameter and with the precision of the integral to improve the smoothness of the solution.

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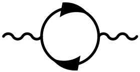
Conclusion

- Sphaleron rate for particle winding around a circle $\Longleftrightarrow$ Numerical simulation.
- Sine Gordon model $\longrightarrow$ Difficult to see the difference with the free particle.
- Sphaleron rate in QFT $\longrightarrow$ Method to do the analytical continuation.
- Backus-Gilbert method is unstable, but works relatively good for a large number of point or by playing with the kernel.
- The next step of this project $\longrightarrow$ computation of the sphaleron rate in (1+1)-dimensional abelian Higgs model without fermions using Backus-Gilbert method.

Anomalous symmetry

Two-dimensional QED : $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\not{D})\psi$

Vacuum polarisation diagram : 

Dimensional regularisation : $i\Pi^{\mu\nu}(q) = i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{e^2}{\pi}$

In a background EM field : $\langle j^\mu(q)\rangle = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{e}{\pi}A_\nu(q)$

$$\longrightarrow q_\mu\langle j^\mu(q)\rangle = 0$$

Chiral current : $\langle j^{\mu 5}(q)\rangle = -\epsilon^{\mu\nu}\langle j^\mu(q)\rangle = \epsilon^{\mu\nu}\left(A_\nu(q) - \frac{q_\nu q^\alpha}{q^2}A_\alpha(q)\right)\frac{e}{\pi}$

$$q_\mu\langle j^{\mu 5}(q)\rangle = \frac{e}{\pi}\epsilon^{\mu\nu}q_\mu A\nu(q) \longrightarrow \partial_\mu j^{\mu 5} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$

Total derivative calculation

$$
\begin{aligned}
\int d^4x \partial_\mu K^\mu &= \int_{\partial S} d\sigma_\mu K^\mu \\
&\sim \int_{\partial S} d\sigma_\mu Tr(A_\nu A_\rho A_\sigma) \\
&\sim \int d^4x \partial_\mu Tr(\epsilon^{\mu\nu\rho\sigma} A_\nu A_\rho A_\sigma) \\
&\sim \int d^4x \partial_0 Tr(\epsilon^{ijk} A_i A_j A_k) \\
&\sim Tr\left(\int d^3x \epsilon^{ijk} A_i A_j A_k\right) \Big|_{-\infty}^{+\infty} \\
&= \int d^3x Tr\left(\epsilon^{ijk}(F_{ij}A_k + \frac{2g}{3} A_i A_j A_k)\right) \Big|_{-\infty}^{+\infty} \\
&= \int d^3x K^0 \\
&= 2\pi(N_{CS}(\infty) - N_{CS}(-\infty))
\end{aligned}
$$

Sine Gordon potential

Equation of motion : $\ddot{x} + A^2 \sin x = 0$

Solution :

$$
\begin{aligned}
y(t) &= sgn(\dot{x}_0) kA(t - t_0) + \mathrm{sn}^{-1}(\mathrm{k}_0|\xi) \\
x(t) &= 2 \arcsin(\mathrm{sn}(\mathrm{y}(\mathrm{t})|\xi))\mathrm{sgn}(\mathrm{cn}(\mathrm{y}(\mathrm{t})|\xi)) \\
\dot{x}(t) &= sgn(\dot{x}_0)\sqrt{E_0}\mathrm{dn}(\mathrm{y}(\mathrm{t})|\xi) \\
\xi &= \frac{1}{k} = \sqrt{\frac{E_p}{E_0}} = \sqrt{\frac{4A^2}{E_0}}
\end{aligned}
$$

Sine Gordon potential

Instanton solutions $\ll$ Sphaleron solutions.

Probability of instanton solution : $P_I \sim e^{-8mA}$
Probability of sphaleron solution : $P_S \sim e^{-\beta E} \sim e^{-\beta T} \sim e^{-1}$

The two-point function of velocities :

$$\langle \dot{x}(\tau_1)\dot{x}(\tau_2)\rangle = \frac{1}{Z}\int_0^{2\pi} dx_0 \int_{\infty}^{-\infty} d\dot{x}_0 \dot{x}(\tau_1, x_0, \dot{x}_0)\dot{x}(\tau_2, x_0, \dot{x}_0) e^{-\beta H(x_0,\dot{x}_0)}$$

$$Z = \int_0^{2\pi} dx_0 \int_{\infty}^{-\infty} d\dot{x}_0 e^{-\beta H(x_0,\dot{x}_0)}$$

Sine Gordon potential

The integral can be compute numerically.

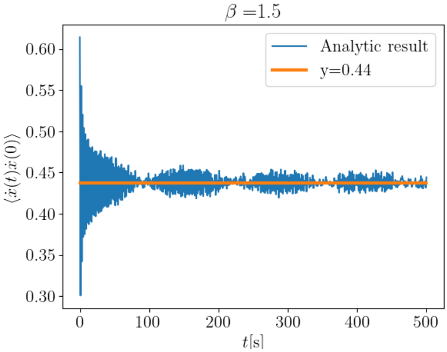


Figure – Two-point function of velocities for $T = 1.5$ MeV, $m = 1$ MeV, and $A = 0.5$

Sphaleron rate :

$$\boxed{\langle Q^2(t)\rangle \sim \mathrm{v_{const}\,t^2}}$$

Backus-Gilbert

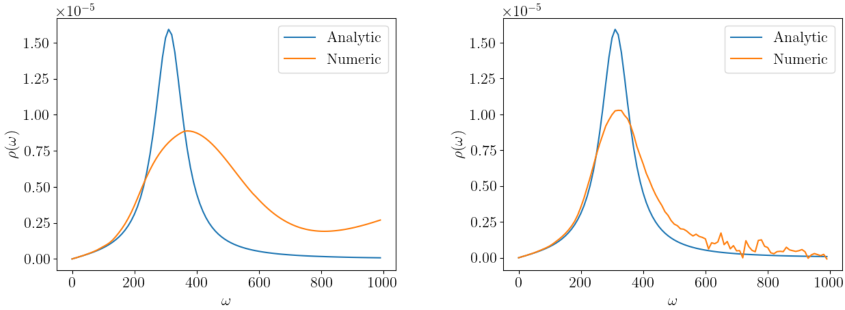


Figure – Backus-Gilbert reconstruction of the Breit-Wigner formula using N=32 and N=256 points, for the upper bound of integration $\omega = 1000$ and with the point $\tau = 0$.