Reinforcement Learning and Optimal Control

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Overview

- Fixed point of a mapping
  - Nonexpansive
  - Hölder-continuous
    - Lipschitz-continuous
  - Contractive
  - Contraction mapping theorem
    - Cauchy sequence
  - Linear maps
    - Matrix of a linear map
  - Neumann lemma (intro)

Fixed point of an operator

Most if not all algorithms studied in this course are of the form:

$$x^{(k+1)} = Tx^{(k)} \quad .$$

where $T : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping (operator).

We write $T : D \subset \mathbb{R}^n \to \mathbb{R}^m$ to denote a mapping T with domain D in $\mathbb{R}^n$ and range (image) in $\mathbb{R}^m$.

If there is a $x^\star$ such that $x^\star = Tx^\star$, we call call this point $x^\star$ a **fixed point** of T (there can be multiple fixed points). Fixed points are also solutions to the nonlinear equation of the form $x - Tx = 0$.

Hölder-continuous mapping

Definition (Hölder-continuous). A mapping $T : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be Hölder-continuous on $D_0 \subset D$ if there exists a $\gamma \geq 0$ and $p \in (0, 1]$ such that:

$$\|Tx - Ty\| \leq \gamma \|x - y\|^p, \quad \forall x, y \in D_0 \quad .$$

If the above holds for $p = 1$, then T is said to be **Lipschitz-continuous** on D_0.

Nonexpansive mapping

Definition (Nonexpansive). A mapping $T : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be **nonexpansive** on $D_0 \subset D$ if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D_0 \quad .$$

Furthermore, we say that a mapping is **strictly** nonexpansive on D_0 if:

$$\|Tx - Ty\| < \|x - y\|, \forall x, y \in D_0, x \neq y \quad .$$

At most one fixed point

Strictly nonexpansive mappings can have at most one fixed point.

To see this, let's pretend that $x^\star, y^\star \in D_0$ are two distinct fixed points of a strictly nonexpansive mapping $T : D \subset \mathbb{R}^n \to \mathbb{R}^n$.

By definition of strictly nonexpansive:

$$\|Tx^\star - Ty^\star\| < \|x^\star - y^\star\| \ .$$

By definition of fixed point:

$$\|x^\star - y^\star\| = \|Tx^\star - Ty^\star\| \ .$$

Leading to the contradiction:

$$\|x^\star - y^\star\| = \|Tx^\star - Ty^\star\| < \|x^\star - y^\star\| \ .$$

Existence in strictly nonexpansive mappings

Establishing that a mapping is strictly nonexpansive is
<u>not sufficient to conclude that there exists</u> a fixed point.

An example of this (to show) is the one-dimensional strictly nonexpansive mapping T defined as:

$$Tx \triangleq \begin{cases} x + \exp(-x/2) & x \geq 0 \\ \exp(x/2) & x \leq 0 \end{cases} .$$

Contractive mapping

Definition (Contractive). A mapping $T : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is **contractive** on $D_0 \subset D$ if there exists a $\gamma < 1$ such that:

$$\|Tx - Ty\| \leq \gamma \|x - y\|, \quad \forall x, y \in D_0$$

We call a mapping with this property a **contractive mapping**.

Contraction is a norm-dependent concept. We may establish that a mapping is contractive under a given norm of $\mathbb{R}^n$, but later find that it doesn't hold under a different norm.

Connection with other notion

Given the above definitions, we can also say that contractive mappings are also:

1. Strictly nonexpansive
   1.2 which implies that they can have at most one one fixed point
2. Hölder-continuous
   2.2 more precisely: Lipschitz-continuous

Contractive mappings are special in the sense that we can use the **contraction-mapping theorem** to establish the existence of a unique fixed-point. Furthermore, the proof of this theorem is constructive and gives us a procedure to find this fixed point.

Contraction-mapping theorem

Theorem (Contraction-mapping theorem, Banach 1922). Given a contractive mapping $T : D \subset \mathbb{R}^n \to \mathbb{R}^n$ on the closed set $D_0 \subset D$ and for which $TD_0 \subset D_0$, then T has a unique fixed point in D_0.

Proof Consider the sequence of iterates generated by:

$$x^{(k)} = Tx^{(k-1)}, \quad k = 1, 2, \ldots, \quad \text{given } x_0 \in D_0 \quad .$$

By assumption $TD_0 \subset D_0$, which implies that all $\{x^{(k)}\}$ remain in D_0. Using the contraction property:

$$\|x^{(k+1)} - x^{(k)}\| = \|Tx^k - Tx^{k-1}\| \leq \gamma \|x^{(k)} - x^{(k-1)}\| \quad .$$

Proof (continued)

Proof (continuation of the contraction-mapping proof) Writing the difference between the $k+p$-th iterate and the k-th one as a telescoping series:

$$\|x^{(k+p)} - x^{(k)}\| = \| \sum_{i=1}^{p} x^{(k+i)} - x^{(k+i-1)} \| \quad .$$

Using the subadditivity/triangle inequality:

$$\begin{aligned} \|x^{(k+p)} - x^{(k)}\| &\leq \sum_{i=1}^{p} \|x^{(k+i)} - x^{(k+i-1)}\| \\ &= \sum_{i=1}^{p} \|T^{k+i}x^{(0)} - T^{k+i-1}x^{(0)}\| \\ &\leq \sum_{i=1}^{p} \gamma^{k+i-1}\|x^{(1)} - x^{(0)}\| \quad . \end{aligned}$$

Proof (continued)

Lemma (Sum of the first $n+1$ terms of a geometric series) Let $\gamma \in \mathbb{R}, \gamma \neq 1$, and $\alpha \in \mathbb{R}$:

$$\sum_{i=0}^{n} \alpha\gamma^k = \alpha \frac{1-\gamma^{n+1}}{1-\gamma} \quad .$$

Continuing the above proof:

$$\|x^{(k+p)} - x^{(k)})\| \leq \sum_{i=1}^{p} \gamma^{k+i-1} \|x^{(1)} - x^{(0)}\| \quad = \gamma^k \frac{\left(1-\gamma^{(p+1)}\right)}{(1-\gamma)} \|x^{(1)} - x^{(0)}\|$$

$$\leq \frac{\gamma^k}{(1-\gamma)} \|x^{(1)} - x^{(0)}\| \quad .$$

Proof (continued)

Definition (Cauchy sequence) A sequence $\{x^{(k)}\}$, is called a **Cauchy sequence** if given any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that for all $m, n > N(\epsilon)$, $\|x^{(m)} - x^{(n)}\| < \epsilon$.

Practically speaking, this means that we can make $\|x^{(k+p)} - x^{(k)}\|$ arbitrarily small for a sufficiently large p.

$$\|x^{(k+p)} - x^{(k)})\| \leq \frac{\gamma^k}{(1-\gamma)}\|x^{(1)} - x^{(0)}\| \quad .$$

Using the fact that $\|x^{(k+p)} - x^{(k)}\|$ is a Cauchy sequence, we conclude that $\{x^{(k)}\}$ has a limit $x^\star$ in D_0 and that $\lim_{k\to\infty} Tx^{(k)} = Tx^\star$ and $x^\star$ is a fixed point of T.

Linear algebra

Norm

Definition (Norm) A **norm** $\|\cdot\|$ is a mapping from $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}$ with the following properties:

1. $\|x\| \geq 0, \forall x \in \mathbb{R}^n$ and $\|x\| = 0$ only if $x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|, \quad \forall x \in \mathbb{R}^n, \alpha \in \mathbb{R}$
3. $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}^n$

Linear maps

Definition (Linear map). A linear map T is a function $T : \mathcal{V} \rightarrow \mathcal{W}$ over a pair of vector spaces $\mathcal{V}$ and $\mathcal{W}$ satisfying two properties:

- Additivity: $T(u + v) = T(u) + T(v), \forall u, v \in \mathcal{V}$
- Homogeneity: $T(\alpha v) = \alpha T(v), \forall \alpha \in \mathcal{V}, \alpha \in \mathbb{F}$ where $\mathcal{F}$ is a field (like $\mathbb{R}$)

Examples:

- The identity map $I : \mathcal{V} \rightarrow \mathcal{V} : v \mapsto v$
- A definite integral can be seen as a linear map from the space of real-valued functions to the reals, that is: $T : (\mathbb{R}^n \rightarrow \mathbb{R}) \rightarrow \mathbb{R} : f \mapsto \int_a^b f(x)dx$
- The derivative (more on that below)

Vector space of linear maps

Definition (Vector space of linear maps) The set of all linear maps from a pair of vector spaces $\mathcal{V}$ and $\mathcal{W}$ is itself a vector space where the operations of addition and scalar multiplication are defined as:

- Addition: $(T + S)(v) = T(v) + S(v)$
- Scalar multiplication: $(\alpha T)(v) = \alpha(T(v)), \forall \alpha \in \mathcal{F}, v \in \mathcal{V}$

Furthermore, if $T : \mathcal{V} \rightarrow \mathcal{W}$ and $S : \mathcal{U} \rightarrow \mathcal{V}$, then the **product** or **composition** of T with S is the linear map $TS : \mathcal{U} \rightarrow \mathcal{W}$ defined as $(TS)(u) = T(S(u)) = (T \circ S)(u), u \in \mathcal{U}$.

Matrix of a linear map

An important property of linear maps is that they can be uniquely identified given their values on a basis. Let $T : \mathcal{V} \to \mathcal{W}$:

- If $\{v_1, \ldots, v_n\}$ is a basis of $\mathcal{V}$, we can represent any $v \in \mathcal{V}$ by a unique set of coefficients $\{c_1, ..., c_n\}$ called *coordinates* such that $v = \sum_{j=1}^{n} c_j v_j$
- If we have a linear map $T : \mathcal{V} \to \mathcal{W}$, then $T(v) = \sum_{j=1}^{n} c_j T(v_j)$.
- Since $T(v_j) \in \mathcal{W}$, then $T(v_j) = \sum_{j=1}^{m} a_{i,j} w_j$ where $\{w_1, \ldots, w_m\}$ is a basis of $\mathcal{W}$
- Therefore: $T(v) = \sum_{j=1}^{n} c_j T(v_j) = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{i,j} w_i$.
- The a's are the entries of the matrix A whose columns are the $T(v_j)$ in the basis of $\mathcal{W}$.

Neumann Lemma

Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, if $\sigma(A) < 1$, then $(I - A)^{-1}$ exists and:

$$(I - A)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} A^i \quad .$$

(To continue next class)