Assortative Matching and Wages: The Role of Selection*

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Abstract

We develop a random search model of the labor market with two-sided heterogeneity and match-specific productivity shocks. Our model has two main predictions: i) there is positively assortative matching and ii) the average log wage that a worker receives is increasing in the worker's and firm's productivity and is submodular. Sorting and wages are driven by selection. All workers are equally likely to meet all firms, but low (high) productivity workers have a higher average surplus from meeting low (high) productivity firms. The high surplus meetings result in matches more frequently, generating positive assortative matching. Since only meetings that result in matches are observed in administrative wage data, such data contain only a selected subset of meetings, driving the result that average log wages are increasing and submodular. We show that our findings are consistent with results in the empirical wage literature.

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1 Introduction

Why do high-wage workers tend to work at high-wage firms, even though low-wage workers seem to gain at least as much from such matches? Bonhomme, Lamadon and Manresa (2019) write that these patterns are "difficult to reconcile with models where sorting is driven by complementaries in production, as in Becker (1973) [p. 701]." We address this puzzle by developing a theory of frictional labor markets that highlights the crucial role of selective job acceptance decisions for observed wages.

We develop a random search framework where workers and firms are heterogeneous in their productivity. Output in a match depends on the worker's productivity, the firm's productivity, and a match-specific shock. The decision to form a match depends on the realization of this shock relative to the outside options of both the worker and the firm, with wages determined by Nash bargaining. This setup allows us to analyze how selection in the matching process affects observed wage patterns and sorting outcomes.

We prove that our model can generate outcomes that align with the empirical literature. We find conditions which ensure that (i) there is positively assortative matching between worker and firm productivities; (ii) average log wages increase with both worker and firm productivity, while possibly being submodular—meaning low-productivity workers' wages are more sensitive to firm type than high-productivity workers' wages; and (iii) high-productivity workers are high-wage workers and high-productivity firms are high-wage firms.

Our theoretical results arise from a novel mechanism, the selective nature of realized matches. While all workers are equally likely to meet all firms, the likelihood of a match forming depends on the interplay between worker and firm types. Low-productivity workers rarely receive acceptable offers from high-productivity firms, but when they do, these offers are exceptionally good. Similarly, high-productivity workers seldom accept offers from low-productivity firms unless the match-specific productivity is unusually high. This selection process creates a subset of realized matches and wages that differs systematically from the set of all meetings, driving both the observed sorting patterns and the behavior of average log wages.

Our theoretical findings are consistent with a growing body of empirical research on log wages and worker-firm sorting. Studies by Card, Heining and Kline (2013); Card, Cardoso and Kline (2016); Card, Cardoso, Heining and Kline (2018); Bonhomme, Lamadon and Manresa (2019); Kline, Saggio and Sølvsten (2020); Bonhomme, Holzheu, Lamadon, Manresa, Mogstad and Setzler (2023), among others, have found that average log wages increase with both worker and firm type and are approximately additively separable. These papers have also found that there is strong correlation between matched worker and firm types. Existing theoretical explanations of these empirical results appeal to unmeasured job attributes. For example, Bonhomme, Lamadon and Manresa (2019) conclude that "This may indicate that workers or firms care about other job attributes (Hwang, Mortensen and Reed, 1998), or that workers of a similar type enjoy working together as in the peer effects literature [p. 720]." We show that our sparsely-parameterized model is able to match the behavior of both wages and sorting. In particular, our model can match the variance decomposition of wages into the part attributable to worker heterogeneity, to firm heterogeneity, to worker-firm sorting, and to a residual.

Our final contribution is to distinguish between the surplus derived from meetings and the surplus derived from matches. The surplus from matches can readily be measured using administrative wage data. Using a generalized version of our model, we prove that the surplus from meetings is identified using administrative wage data and the assumption of random search. We also derive an expression for the surplus from meetings. This analysis reveals that the surplus from meetings is supermodular, with low-type workers actually preferring to meet intermediate-type firms rather than high-type firms. Importantly, we show that this identification result and estimator hinge on the random search assumption. This finding underscores the importance of carefully considering search frictions when interpreting wage data and highlights a novel aspect of our model's implications for labor market dynamics.

Our paper contributes to several strands of literature. First, we build on empirical work using matched employer-employee data to study wage determination and sorting (Abowd, Kramarz and Margolis, 1999; Andrews, Gill, Schank and Upward, 2008; Card, Heining and Kline, 2013; Card, Cardoso, Heining and Kline, 2018; Bonhomme, Lamadon and Manresa, 2019; Bonhomme, Holzheu, Lamadon, Manresa, Mogstad and Setzler, 2023; Kline, Saggio and Sølvsten, 2020). While these papers document important empirical regularities, we provide a theoretical framework that can explain these patterns through a novel mechanism.

Our theory cautions against the prevailing interpretation of these empirical regularities. For example, Card, Heining and Kline (2013) regress wages on worker and establishment fixed effects, with a focus on measuring "establishment-specific wage premiums" ψ_j. They "interpret the establishment effect ψ_j as a proportional pay premium (or discount) that is paid by establishment j to all employees (i.e., all those with $\mathcal{J}(i,t) = j$). Such a premium could represent rent-sharing, an efficiency wage premium, or strategic wage posting behavior [p. 987]." We argue that selection may shape the establishment-specific wage premium.

Second, we contribute to the theoretical literature on conditions for assortative matching in labor markets with random search (Shimer and Smith, 2000; Hagedorn, Law and Manovskii, 2017; Lopes de Melo, 2018; Bagger and Lentz, 2019). Our model extends this literature by incorporating match-specific shocks, as in Goussé, Jacquemet and Robin (2017)'s model of the marriage market, and highlighting the role of selection in job acceptance decisions. Our finding that there is positively assortative matching if the production function is strictly increasing and weakly log-supermodular recalls similar results in competitive search models (Shi, 2001, 2005; Shimer, 2005; Eeckhout and Kircher, 2010) and models with non-transferable utility (Smith, 2006; Bonneton and Sandmann, 2023). The details of the mechanisms are quite different, and in particular selection does not play a role in those papers. Those earlier papers also do not try to fit the empirical findings on how wages depend on worker and firm types. Our model bridges this gap by providing a framework that not only explains sorting patterns but also aligns with observed wage patterns in administrative data sets.

Third, the existing literature recognizes that endogenous mobility can create a selection bias in estimates of the AKM wage equation. Following Card, Heining and Kline (2013), much of the literature has focused on selection working through job-to-job selection transitions, as in Burdett and Mortensen (1998) and Lopes de Melo (2018). To stress that our model is not about selection from on-the-job search, we develop our main theoretical model in an environment without job-to-job mobility. In Appendix E, we show numerically that our results carry over to an environment with a quantitatively realistic amount of on-the-job search. We show why existing tests for selection cannot detect the type we highlight here. Moreover, building on an insight from Flinn and Heckman (1982), we explain why it is impossible to construct such a test without auxiliary assumptions, such as random search.

By highlighting the role of selection in job acceptance, our model offers a new interpretation of empirical wage patterns without requiring assumptions about unobserved job amenities or worker preferences for similar peers. This perspective has important implications for understanding labor market dynamics and interpreting wage data. The rest of the paper is structured as follows: Section 2 presents our model; Section 3 analyzes two special cases where we can prove analytical results; Section 4 discusses the role of selection in a quantitative version of the model, including a comparison of the surplus from meetings with the surplus from matches; and Section 5 concludes.

2 Model

We formulate a search model with two-sided heterogeneity (Shimer and Smith, 2000) and match-specific heterogeneity (Goussé, Jacquemet and Robin, 2017). The model is formulated in continuous time and we focus on steady states and so drop time arguments in what follows.

2.1 Assumptions

There is measure M of risk-neutral workers and measure N of risk-neutral firms. Everyone discounts the future at rate $r > 0$. There are X worker types indexed by $x = 1, \ldots, X$. The population measure of type-x workers is $m_x > 0$, with $\sum_{x=1}^{X} m_x = M$. There are Y firm types indexed by $y = 1, \ldots, Y$, with population measure $n_y > 0$ and $\sum_{y=1}^{Y} n_y = N$. Workers can be either unemployed or matched to one firm; likewise, firms can be either vacant or matched to one worker. Thus, in this model, a firm and a job are treated as identical.

Search is random and only unmatched firms and workers can search.[1] Let u_x be the population measure of unemployed type-x workers, so that $\frac{u_x}{m_x}$ is the unemployment rate for type x. Similarly, let v_y be the population measure of type-y vacancies, with $\frac{v_y}{n_y}$ is the vacancy rate of y.

All unemployed workers contact vacant type-y firms according to a Poisson process with arrival rate ρv_y for $y = 1, \ldots, Y$, where $\rho > 0$. Likewise, all vacant firms contact unemployed type-x workers at rate ρu_x for $x = 1, \ldots, X$. When a worker and firm meet, they draw a match-specific productivity shock z from a distribution function with density $s(z)$ and survival function $S(z)$. Draws are independent across matches for every worker and firm. By definition, the survival function $S : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing. For expositional simplicity, we also assume that S is continuous and strictly positive for all $z > 0$. Finally, we assume $\int_0^\infty zs(z)dz$ is finite, which is necessary for existence of an equilibrium.

After an unemployed worker of type x meets a vacant firm of type y and draws a match-specific productivity shock z, they decide whether to match. If they match, they stop searching and produce flow output $zf_{x,y}$. A matched worker and firm split the surplus according to Nash bargaining, with worker's bargaining power equal to $\gamma \in (0, 1)$. We assume $f_{x,y} > 0$ for all x and y. Matches end at rate $\delta > 0$, leaving the worker unemployed and the job vacant.

2.2 Value Functions

We start by formulating the value functions of workers and firms. For a type-x unemployed worker, let the value be V_x^u:

$$rV_x^u = \rho \sum_{y=1}^{Y} v_y \int_0^\infty \max\left\{V_{x,y}^e(z, W_{x,y}(z)) - V_x^u, 0\right\} s(z)dz. \tag{1}$$

[1] In Appendix E, we extend the model to allow for search by employed workers and show numerically that our main conclusions carry over to this more realistic environment.

At rate ρv_y, the worker meets a vacant type y firm. They then draw match-specific productivity z from a distribution with density $s(z)$. After that, they decide whether to match. If they do, the worker's value jumps to $V^e_{x,y}(z, W_{x,y}(z))$, the value of a type x worker matched to a type y firm in a match with productivity z and earning the equilibrium wage $W_{x,y}(z)$. Nash bargaining implies that both parties agree on whether to match, so the worker matches whenever the value of being in the match, $V^e_{x,y}(z, W_{x,y}(z))$, exceeds the value of being unmatched, V^u_x.

Once the worker is in the match, we have the corresponding Bellman equation

$$rV^e_{x,y}(z, W) = W + \delta(V^u_x - V^e_{x,y}(z, W)). \tag{2}$$

This equation describes a type x worker at a type y firm with match-specific shock z earning an arbitrary wage W. The worker earns the wage until the match ends exogenously. This implies that the worker will accept the match ($V^e_{x,y}(z, W) \geq V^u_x$) if and only if $W \geq rV^u_x \equiv \bar{w}_x$, the worker's reservation wage.

The Bellman equations for firms are symmetric:

$$rV^v_y = \rho \sum_{x=1}^{X} \int_0^\infty u_x \max\left\{V^f_{y,x}(z, W_{x,y}(z)) - V^v_y, 0\right\} s(z)dz \tag{3}$$

$$rV^f_{y,x}(z, W) = zf_{x,y} - W + \delta(V^v_y - V^f_{y,x}(z, W)). \tag{4}$$

Notably, a type y firm earns flow profit $zf_{x,y} - W$ when employing a type x worker in a match with productivity z and paying a wage W. The firm will accept the match ($V^f_{y,x}(z, W) \geq V^v_y$) if and only if $zf_{x,y} - W \geq rV^v_y \equiv \bar{\pi}_y$, the firm's reservation profit.

We define the match surplus as

$$V^s_{x,y}(z) \equiv V^e_{x,y}(z, W) + V^f_{x,y}(z, W) - V^u_x - V^v_y = \frac{\max\{zf_{x,y} - \bar{w}_x - \bar{\pi}_y, 0\}}{r + \delta}, \tag{5}$$

where the second equation follows from equations (2) and (4) and the definitions of the reservation wage $\bar{w}_x$ and reservation profit $\bar{\pi}_y$. Equivalently, define the reservation productivity level $\bar{z}_{x,y}$:

$$\bar{z}_{x,y} \equiv \frac{\bar{w}_x + \bar{\pi}_y}{f_{x,y}}. \tag{6}$$

Then

$$V^s_{x,y}(z) = \frac{f_{x,y} \max\{z - \bar{z}_{x,y}, 0\}}{r + \delta}. \tag{7}$$

The match surplus is positive, so there exists a W which is acceptable both to the worker ($V^e_{x,y}(z, W) \geq V^u_x$) and firm ($V^f_{y,x}(z, W) \geq V^v_y$), if and only if productivity exceeds the reservation level $\bar{z}_{x,y}$.

Finally, when the match surplus is positive, we use Nash bargaining to pin down the equilibrium wage $W_{x,y}(z)$:

$$W_{x,y}(z) = \arg\max_{W} (V^e_{x,y}(z, W) - V^u_x)^\gamma (V^f_{y,x}(z, W) - V^v_y)^{1-\gamma}. \tag{8}$$

Using equations (2) and (4), as well as the definition of the reservation productivity level in equation (6), it is straightforward to show that this implies

$$W_{x,y}(z) = \bar{w}_x + \gamma f_{x,y}(z - \bar{z}_{x,y}). \tag{9}$$

Putting this together with the fact that a worker and firm match whenever $z \geq \bar{z}_{x,y}$, we get that type x workers match with type y firms at rate $\rho u_x v_y S(\bar{z}_{x,y})$, with matches formed whenever $W_{x,y}(z) \geq \bar{w}_x$, a type-$x$ worker's reservation wage. Symmetrically, a type-y firm agrees to match with a type-x worker whenever $z f_{x,y} - W_{x,y}(z) \geq \bar{\pi}_y$, a type-$y$ firm's reservation profit.

We can now combine these equations to get a simpler expression for a worker's reservation wage and a firm's reservation profit. Eliminate the wage from equation (2) using equation (9) to get $V^e_{x,y}(z, W_{x,y}(z)) - V^u_x = \frac{\gamma f_{x,y}(z - \bar{z}_{x,y})}{r+\delta}$. Substitute that into equation (1) to get

$$\bar{w}_x = \frac{\gamma\rho}{r+\delta} \sum_{y=1}^{Y} v_y f_{x,y} \int_{\bar{z}_{x,y}}^{\infty} (z - \bar{z}_{x,y}) s(z) dz. \tag{10}$$

Analogous steps lead to the equation for a firm's reservation profit:

$$\bar{\pi}_y = \frac{(1-\gamma)\rho}{r+\delta} \sum_{x=1}^{X} u_x f_{x,y} \int_{\bar{z}_{x,y}}^{\infty} (z - \bar{z}_{x,y}) s(z) dz. \tag{11}$$

2.3 State Variables and Equilibrium

To close the model, we need to find the steady state values of u_x and v_y. To do this, we first define the steady state measure of (x, y) matches, $\phi_{x,y}$. This satisfies

$$\delta \phi_{x,y} = \rho u_x v_y S(\bar{z}_{x,y}). \tag{12}$$

The left hand side is the rate that these matches end, while the right hand side is the rate that unmatched type x agents (measure u_x) meet unmatched type y agents (ρv_y) in a match with an acceptable z (share $S(\bar{z}_{x,y})$).

By adding partner types, we can then recover the unemployment and vacancy measures:

$$u_x = m_x - \sum_{y=1}^{Y} \phi_{x,y} \tag{13}$$

$$v_y = n_y - \sum_{x=1}^{X} \phi_{x,y}. \tag{14}$$

A steady state equilibrium is given by $(\bar{w}, \bar{\pi}, \bar{z}, \phi, u, v)$ satisfying equations (6), (10), (11), (12), (13), and (14). We can prove

Proposition 1 *An equilibrium exists. In any equilibrium, the reservation wage $\bar{w}_x$ and reservation profit $\bar{\pi}_y$ are strictly positive for all x and y.*

In general, a model like this may have multiple equilibria (Burdett and Coles, 1997). All of our claims apply to any steady state equilibrium.

2.4 Monotonicity

We first prove a useful preliminary result, that the reservation wage and reservation profit are increasing if the production function is increasing:

Lemma 1 *Assume $f_{x,y}$ is strictly increasing in x and y. Then the reservation wage $\bar{w}_x$ and reservation profit $\bar{\pi}_y$ are strictly increasing.*

The proof is in Appendix A.

We note that since $\bar{w}_x$ and $\bar{\pi}_y$ are strictly positive, equation (6) implies $\bar{z}_{x,y}$ is strictly positive as well. And since there are a finite number of types of workers and firms, $\underline{z} \equiv \min_{x,y} \bar{z}_{x,y}$ is strictly positive as well. It follows from the definition of equilibrium that in any steady state equilibrium, the behavior of $S(z)$ at $z < \underline{z}$ does not affect the equilibrium $(\bar{w}, \bar{\pi}, \bar{z}, u, v)$. We build on this observation in Section 3.1 below.

3 Special Cases

This section characterizes the equilibrium for two special cases of the match quality distribution, a Pareto and an exponential. We provide three results for each case. First, we find conditions which ensure assortative matching, in the sense that the measure of matches of a higher type worker is shifted towards higher type firms, and similarly for the measure of matches of a higher firm type. And second, we prove that more productive workers earn higher wages at any firm type and conversely more productive firms pay higher wages to any worker type. Finally, we show that the two conditions together imply that more productive workers earn higher wages on average, and similarly more productive firms pay higher wages on average.

The conditions for assortative matching extend known results for an economy without idiosyncratic shocks (Shimer and Smith, 2000) into a more realistic environment where matching between a worker and firm is probabilistic. We note that conditions for positively assortative matching and the proof of positively assortative matching are much simpler here than in that earlier work.

3.1 Pareto-Distributed Match Quality

We first characterize sorting and wage behavior under the assumption that z has a Pareto distribution with minimum value $z_0 > 0$ and tail parameter $\theta > 1$, so $S(z) = (z/z_0)^{-\theta}$ for $z \geq z_0$ and $S(z) = 1$ otherwise. We also parameterize the contact rate as $\rho = \bar{\rho} z_0^{-\theta}$, so the rate that an unemployed worker contacts a vacant type-y firm and has productivity at least z is $\bar{\rho} z^{-\theta} v_y$ for $z > z_0$. Notably, this is independent of z_0. We focus throughout on cases where there is an interior solution for the threshold $\bar{z}_{x,y}$ for all pairs (x, y), $z_0 < \underline{z} = \min_{x,y} \bar{z}_{x,y}$, as will be the case if z_0 is sufficiently small.[2]

We first find conditions for positively assortative matching:

Proposition 2 *Assume $S(z) = (z/z_0)^{-\theta}$ with $z_0 > 0$ and $\theta > 1$. Also assume $f_{x,y}$ is strictly increasing and weakly log-supermodular. Then the measure of matches $\phi_{x,y}$ is strictly log-supermodular.*

All the proofs in this section are in Appendix A. Weak log-supermodularity of f is equivalent to $f_{x_1,y_1} f_{x_2,y_2} \geq f_{x_1,y_2} f_{x_2,y_1}$ for all $x_1 < x_2$ and $y_1 < y_2$, and strict log-supermodularity of ϕ implies $\phi_{x_1,y_1} \phi_{x_2,y_2} > \phi_{x_1,y_2} \phi_{x_2,y_1}$. A corollary of this finding is that higher type workers and firms have a better distribution of match partners in the sense of first order stochastic dominance and that the correlation between the types of matched workers and firms is strictly positive (see Shimer, 2005, pp. 1013–1014).

Next, define the *average log wage* in an (x, y) match:

$$w^*_{x,y} \equiv \frac{\int_{\bar{z}_{x,y}}^{\infty} \log(W_{x,y}(z)) s(z) dz}{S(\bar{z}_{x,y})}. \tag{15}$$

In Appendix B, we discuss the relationship between the average log wage and the more familiar log-linear wage equation proposed by Abowd, Kramarz and Margolis (1999). In

[2] For given $\bar{\rho}$, a change in z_0 does not affect the equilibrium allocation as long as $z_0 \leq \underline{z}$. Thus focusing

short, that paper proposed that the log wage $w_{i,t}$ of worker i at firm j in period t can be expressed as

$$w_{i,t} = \alpha_i + \psi_j + \varepsilon_{i,t},$$

where $\varepsilon_{i,t}$ is a random variable with mean zero for all i, j, and t. Under these conditions, $w^*_{x_i,y_j} = \alpha_i + \psi_j$, where x_i is the type of worker i and y_j is the type of firm j. Our formulation allows for non-separabilities in w^*, which the following result proves are present:

Proposition 3 *Assume $S(z) = (z/z_0)^{-\theta}$ with $z_0 > 0$ and $\theta > 1$. Then*

$$w^*_{x,y} \equiv \int_0^\infty \log\big(\bar{w}_x + \gamma(\bar{w}_x + \bar{\pi}_y)q\big)\theta(1+q)^{-\theta-1}dq. \tag{16}$$

Moreover,

1. *for all x_1 and x_2 with $\bar{w}_{x_1} < \bar{w}_{x_2}$, $w^*_{x_1,y} < w^*_{x_2,y}$ for all y;*
2. *for all y_1 and y_2 with $\bar{\pi}_{y_1} < \bar{\pi}_{y_2}$, $w^*_{x,y_1} < w^*_{x,y_2}$ for all x;*
3. *for all x_1, x_2, y_1, and y_2 with $\bar{w}_{x_1} < \bar{w}_{x_2}$ and $\bar{\pi}_{y_1} < \bar{\pi}_{y_2}$, $w^*_{x_1,y_2} + w^*_{x_2,y_1} > w^*_{x_1,y_1} + w^*_{x_2,y_2}$.*

In particular, the average log wage is strictly increasing and strictly submodular for any strictly increasing $f_{x,y}$.

Our proof in the appendix allows for any transformation of the wage, not just the log. We focus in the text on the average log wage because this is the focus of the empirical literature.

There are many pieces to unpack from Proposition 3. First, the average log wage depends only on four numbers: the worker's reservation wage $\bar{w}_x$, the firm's reservation profits $\bar{\pi}_y$, the worker's bargaining power γ, and the Pareto tail parameter θ. The production technology $f_{x,y}$ does not explicitly enter this expression. We view this as both good news and bad news for empirical research. The bad news is that wage data are not useful for learning about the production technology f. The good news is that the model makes strong and testable predictions for how the average wages behaves across different types of matches. We discuss these predictions in Section 4.

Second, workers who have a higher reservation wage earn more at any type of employer, and similarly firms that have a higher reservation profit level pay more to any type of worker. To understand the power of this result, it is useful to consider a non-monotonic production function f. That is, suppose that there are worker types x_1 and x_2 and firm

on small values of z_0 does not change the arrival rate of "good" matches. We could sidestep any discussion of z_0 by assuming that meetings with match quality at least equal to z occur at rate $\bar{\rho}z^{-\theta}$ times the relevant vacancy or unemployment rate for any z, as in Oberfield (2018) and Buera and Oberfield (2020).

types y_1 and y_2 such that $f_{x_1,y_1} > f_{x_1,y_2}$ and $f_{x_2,y_1} < f_{x_2,y_2}$. One might conjecture that with Nash bargaining, this non-monotonicity would imply a similar ordering of average log wages, $w^*_{x_1,y_1} > w^*_{x_1,y_2}$ and $w^*_{x_2,y_1} < w^*_{x_2,y_2}$. Proposition 3 establishes that this cannot happen: $\bar{\pi}_{y_2} \gtreqless \bar{\pi}_{y_1} \Rightarrow w^*_{x,y_2} \gtreqless w^*_{x,y_1}$ for all x.

Proposition 3 also implies that all workers have the same ranking of average log wages across firm types y, determined by the reservation profit $\bar{\pi}_y$. This result stands in contrast to one in Shimer and Smith (2000), a similar model except that z has a degenerate distribution. In that case, an unproductive worker's wage may be maximized at low-type firm, while a more productive worker's wage is maximized at a high-type firm. See Lopes de Melo (2018) for an elaboration of this observation.

Finally, we prove that the average log wage is submodular in the worker's reservation wage $\bar{w}_x$ and the firm's reservation profits $\bar{\pi}_y$. That means that the average increase in the log wage of a given worker who moves from a low $\bar{\pi}_y$ firm to a high $\bar{\pi}_y$ firm is decreasing in the worker's reservation wage $\bar{w}_x$. Again, this places strong and testable restrictions for how wages vary across different types of matches.

The combination of positively assortative matching (Proposition 2) and submodular average log wages (Proposition 3) may seem surprising. After all, if low productivity workers gain proportionately more from moving to high productivity firms, why do they work there less frequently? According to our model, the answer is selection. We use wage data as measured in a typical administrative data set, the wage paid by a firm to its employee. Such data sets do not have information about meetings that do not result in matches, i.e. about wage offers that are below the worker's reservation wage. Low wage workers rarely get acceptable wage offers from high productivity firms, and so rarely match there; but when such wage offers do materialize, our model predicts that the average log wage is higher than at a low productivity firm. We return to the role of selection in Section 4.

3.2 Exponential-Distributed Match Quality

Next we characterize sorting and wage behavior under the assumption that z has an exponential distribution with parameter θ, so $S(z) = e^{-\theta z}$ for all $z \geq 0$.

We first find conditions for positively assortative matching.

Proposition 4 *Assume $S(z) = e^{-\theta z}$ with $\theta > 0$. Also assume $f_{x,y}$ is strictly increasing and $-1/f_{x,y}$ is weakly supermodular. Then the measure of matches $\phi_{x,y}$ is strictly log-supermodular.*

For a strictly increasing function f, the assumption that $-1/f_{x,y}$ is weakly supermodular implies that $f_{x,y}$ is strictly log supermodular. Thus the conditions for positively assortative matching we give here are stronger than the conditions in Proposition 2. Numerical results in Section 4 suggest that it may be possible to relax this assumption.

Next, we give the following characterization of the average log wage:

Proposition 5 *Assume $S(z) = e^{-\theta z}$ with $\theta > 0$. Then*

$$w^*_{x,y} = \theta \int_0^\infty \log(\bar{w}_x + \gamma q f_{x,y}) e^{-\theta q} dq. \tag{17}$$

Assume that for all $x_1 < x_2$ and y, $f_{x_1,y} < f_{x_2,y}$. Then $w^*_{x_1,y} < w^*_{x_2,y}$. Additionally, take any x, y_1, and y_2. Then $w^*_{x,y_1} \gtreqless w^*_{x,y_2}$ if and only if $f_{x,y_1} \gtreqless f_{x,y_2}$.

Again, our proof in the appendix allows for any transformation of the wage, not just the log.

Some of these results parallel the results for a Pareto distribution. In particular, if f is strictly increasing in both x and y, then so is the average log wage. Thus the model predicts that more productive workers are paid higher wages at any firm, and that more productive firms pay higher wages to any worker.

On the other hand, other results suggest more flexibility with the exponential distribution. For example, the average log wage depends on the entire production function f, not just on the reservation wage and profit. Additionally, monotonicity of the average log wage in y depends on monotonicity of the production function. If the production function is not monotonic, different workers may find that different firms pay them the highest wage. Still, the basic point remains with an exponential distribution of match quality: even when there is positively assortative matching, the average log wage may be increasing in both worker and firm type.

A natural question is whether monotonicity of the production function f generally guarantees monotonicity of the average log wage for other distributions of the idiosyncratic shock z. In general, the answer is no. First, if the distribution of idiosyncratic productivity shocks is degenerate, wages are generally a hump-shaped function of a firm's type for a given worker (Shimer and Smith, 2000; Lopes de Melo, 2018). Second, we find that if z has a normal distribution, possibly truncated at zero, then the average log wage for some workers may be decreasing in firm type even if the production function is strictly increasing. Still, it is easy to construct other examples where both the production function and wage are monotone in both the worker and firm types, and for this reason we believe that the special cases we highlight here are useful for understanding more general properties of the model.

3.3 Productivity and Wages

Define the average log wage paid to a type x worker and the average log wage paid by a type y firm:

$$\lambda_x \equiv \frac{\sum_{y=1}^{Y} w^*_{x,y}\phi_{x,y}}{\sum_{y=1}^{Y}\phi_{x,y}}, \tag{18}$$

$$\mu_y \equiv \frac{\sum_{x=1}^{X} w^*_{x,y}\phi_{x,y}}{\sum_{x=1}^{X}\phi_{x,y}} \tag{19}$$

The earlier results in this section give conditions for λ_x and μ_y to be strictly increasing:

Corollary 1

1. *Assume $S(z) = (z/z_0)^{-\theta}$ with $z_0 > 0$ and $\theta > 1$. Also assume $f_{x,y}$ is strictly increasing and weakly log-supermodular. Then λ_x and μ_y are strictly increasing.*

2. *Assume $S(z) = e^{-\theta z}$ with $\theta > 0$. Also assume $f_{x,y}$ is strictly increasing and $-1/f_{x,y}$ is weakly supermodular. Then λ_x and μ_y are strictly increasing.*

The proof follows immediately from monotonicity of $w^*_{x,y}$ in both arguments (Propositions 3 and 5) and first order stochastic dominance of the match density ϕ (Propositions 2 and 4).[3]

This result implies that if f is strictly increasing and satisfies the appropriate supermodularity condition, there is a one-to-one mapping between a worker's type x their average log wage λ_x, and a one-to-one mapping between a firm's productivity π_y and its average log wage μ_y. Using the monotonicity of λ and μ, Propositions 2 and 4 then imply that the correlation between λ_x and μ_y among matched workers and firms is strictly positive when f is strictly increasing and satisifies the relevant supermodularity condition.

In Borovičková and Shimer (2020), we develop unbiased estimates of λ for each worker and μ for each firm. We also show how to obtain consistent estimates of the variance of λ_x across employed workers, the variance of μ_y across filled jobs, and the covariance between λ_x and μ_y across matched workers and firms using a short panel with many workers and firms. Using administrative data from Austria, we verify that in fact the correlation between λ and μ is strictly positive.

[3] A natural question is whether the same reduced-form assumptions, monotonicity of $w^*_{x,y}$ and log-supermodularity of $\phi_{x,y}$, imply monotonicity of the Abowd, Kramarz and Margolis (1999) fixed effects $\bar{\alpha}_x$ and $\bar{\psi}_y$, defined in Appendix B and equations (35) and (36). It turns out that if $X \geq 2$ and $Y \geq 3$, we can construct counterexamples where $\bar{\psi}_y$ is not monotonic, and similarly for $\bar{\alpha}_x$ if $X \geq 3$ and $Y \geq 2$.

4 The Role of Selection

4.1 Numerical Example

We start this section by comparing the numerical predictions of our model with data from Bonhomme, Lamadon and Manresa (2019). Using administrative data from Sweden, Bonhomme, Lamadon and Manresa (2019) estimate the average log wage $w^*_{x,y}$ in an (x, y) match, as well as the share of such matches $\phi_{x,y}$. Their main findings are depicted in Figure 2 of their paper, which we summarize here. First, there is strong sorting between workers and firms, with low-type firms mostly employing low-type workers and high-type firms mostly employing high-type workers. Second, average log earnings are increasing in the worker type and firm type. Finally, the average log earnings of the lowest worker type is the most responsive to the firm type, consistent with submodular average log wages.[4]

We put numbers into our model to compare it to these findings. We focus here on Pareto-distributed match quality. We present results with an exponentially-distributed match quality in Appendix D, and also show results from an extension of the model to allow for on-the-job search in Appendix E.4. We set the number of types at $X = Y = 10$ and assume equal numbers of each type, so $m_x = \frac{1}{X}$ and $n_y = \frac{1}{Y}$. The production function is CES with elasticity of substitution ξ,

$$f_{x,y} = \left(\tfrac{1}{2} p_x^{\frac{\xi-1}{\xi}} + \tfrac{1}{2} q_y^{\frac{\xi-1}{\xi}} \right)^{\frac{\xi}{\xi-1}},$$

where $p_1 = q_1 = 1$, $p_x = (1 + \Delta_p)p_{x-1}$ for $x = 2, \ldots X$, and $q_y = (1 + \Delta_q)q_{y-1}$ for $y = 2, \ldots Y$. We let $S(z) = (z/z_0)^{-\theta}$ and set the meeting rate to $\rho = z_0^{-\theta}$.[5] We choose $\gamma = \frac{2}{3}$, $r = 0.05$, and $\delta = 0.5$, which implies expected duration of a match of 2 years. We choose the remaining parameters, Δ_p, Δ_q, θ, and ξ to match the variance decomposition in Bonhomme, Lamadon and Manresa (2019). Specifically we target the variance of worker type, firm types, the covariance and the variance of the residual, so we have four parameters to match four

[4]Other papers present evidence consistent with submodular average log wages. Card, Heining and Kline (2013) regress log wages on worker and establishment (firm) fixed effects, as we discuss in Section B. They then sort workers and firms into deciles based on their estimated person and establishment effects and compute the average residual for each of the one hundred combinations of deciles. Figure VI in their paper shows that these residuals are on average positive when high types match with low types and negative when low types match with low types or high types match wit high types. This is consistent with submodular average log wages, so low worker-types increase their average log wage by more than the typical worker when they increase the firm type. On the other hand, Card, Cardoso and Kline (2016) show analogous calculations using Portuguese data in Figures B5 and B6 of their online appendix. They find that log wage residuals are positive for low-type workers working low-type firms and negative for low-type workers in high-type firms, which is consistent with average log wages being supermodular.

[5]We show in Appendix C that changing ρ does not affect equilibrium allocations and has only a level effect on mean log wages.

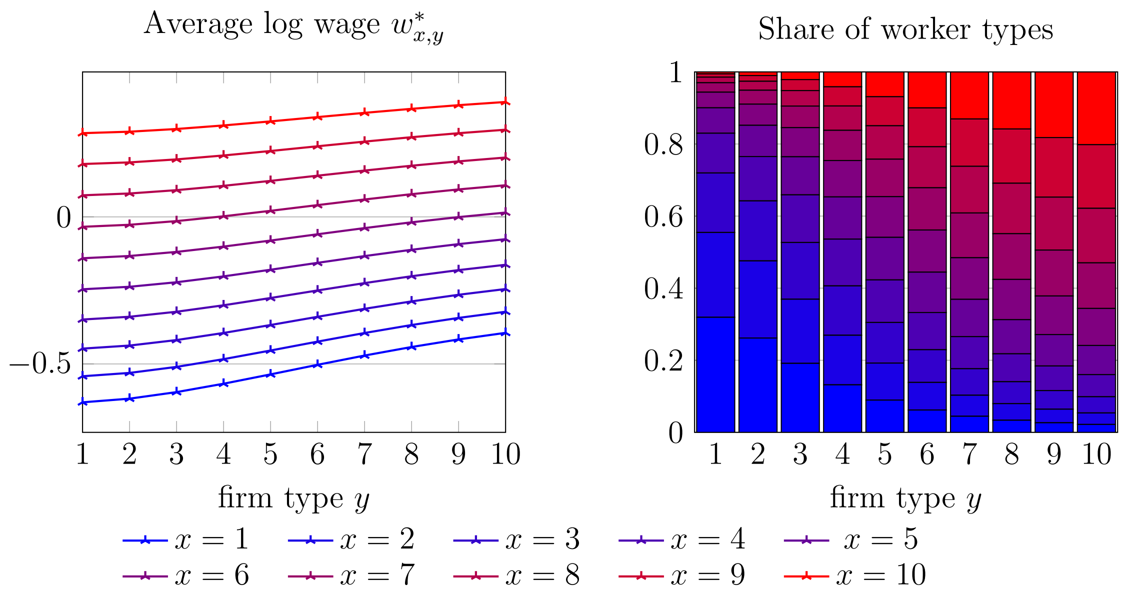

Figure 1: Average log wages and distribution of worker types conditional on firm type. The left panel shows the average log wage $w^*_{x,y}$ paid by different firms for different worker types. Each line represents one worker type. The right panel shows the distribution of worker types x in firms with different firm types y.

moments.[6] In these calculations, we assume we have infinitely much data, so we know each worker's type x and each firm's type y and so all distributions are deterministic functions of model parameters. Thus we abstract from important econometric considerations that arise in real-world data sets where we observe each worker only for a few years. The parameter values which allow us to match the target moments are $\Delta_p = 0.077$, $\Delta_q = 0.854$, $\xi = 0.713$ and $\theta = 7.146$. This implies that the most productive worker is on average 62 percent more productive than the least productive worker, and the most productive firm is on average 402 percent more productive than the least productive one [7]. The unemployment rate, which is primarily dictated by parameter θ, is 16.6 percent.

Our Figure 1 is a direct analogue of Figure 2 in Bonhomme, Lamadon and Manresa (2019). The results are qualitatively very similar, though naturally results from our model are smoother than those from real-world data. The left panel of Figure 1 shows the average log wage as a function of firm type y. Each line corresponds to a different worker type x. As we proved in Proposition 3, the average log wage is increasing in x and y and submodular.

[6] Since these targets are a non-linear function of the model parameters, there is no guarantee that we can match these moments. If we reduced workers' bargaining power to $\gamma = \frac{1}{2}$, we would not be able to do so.

[7] This parametrization implies that worker heterogeneity, as measured by $f_{X,y}/f_{1,y} - 1$, lies between 0.37 and 0.82, depending on the value of y. Firm heterogeneity, as measured by $f_{x,Y}/f_{x,1} - 1$, lies between 3.35 and 4.78, depending on value of x.

Still, the lines are nearly parallel, which implies that the average log wage is almost additively separable in worker and firm types.[8]

The right panel of Figure 1 shows the distribution of worker types in the different firm types, $\phi_{x,y}/\sum_{x'=1}^{X}\phi_{x',y}$. It is clear from the figure that high-type firms employ relatively more high-type workers than do low-type firms, as we proved in Proposition 2. That is, there is positive assortative matching.

4.2 The Value of Meetings and the Value of Matches

Bonhomme, Lamadon and Manresa (2019) view the combination of positive assortative matching and submodular average low wages as puzzling, writing "the results in this section are difficult to reconcile with models based on revealed preferences for wages only [p. 720]." Low-type workers have the most to gain from working in high-type firms yet they are predominantly employed by low-type firms. They propose several factors besides the average log wage that may drive sorting behavior. For example, they recognize that workers may care about other job attributes which differ across employers, such as amenities. For these to drive sorting patterns, they propose that high-wage workers value relatively more the amenities offered by high-wage firms, so these unobserved amenities drive sorting patterns. Alternatively, the propose that workers like to work with similar peers, and again this drives the empirical sorting patterns. Thus in their view, wage data give a misleading picture of the value that workers get from matching with different types of employers.

Positive assortative matching and submodular average low wages arise in our model as well. According to the numbers behind Figure 1, 32 percent of the lowest firm decile's workers are drawn from the lowest worker decile, compared to less than 2 percent for the highest firm decile. For the highest worker decile, these numbers are nearly flipped. They represent 21 percent of employment at the highest firm decile and less than 1 percent at the lowest firm decile. This is positively assortative matching. At the same time, the lowest worker decile has an average log wage of -0.63 in the lowest firm decile and -0.39 in the highest firm decile. The corresponding gain for the highest worker decile is slightly smaller, going from 0.28 to 0.39, so the increase in the average log wage is smaller, consistent with submodular average log wages.

Why is there positively assortative matching when average log wages are submodular? According to our model, wages describe the value that workers derive from matching with

[8] Another prominent feature of Figure 1 is that the average log wage is convex in the firm type y. This feature depends on the units used to measure the firm type. For example, if we measure the average log wage in an (x, y) match as a function of the firm's reservation profit, $\bar{\pi}_y$, or as a function of the average log wage that the firm pays, μ_y, the resulting curves would be concave.

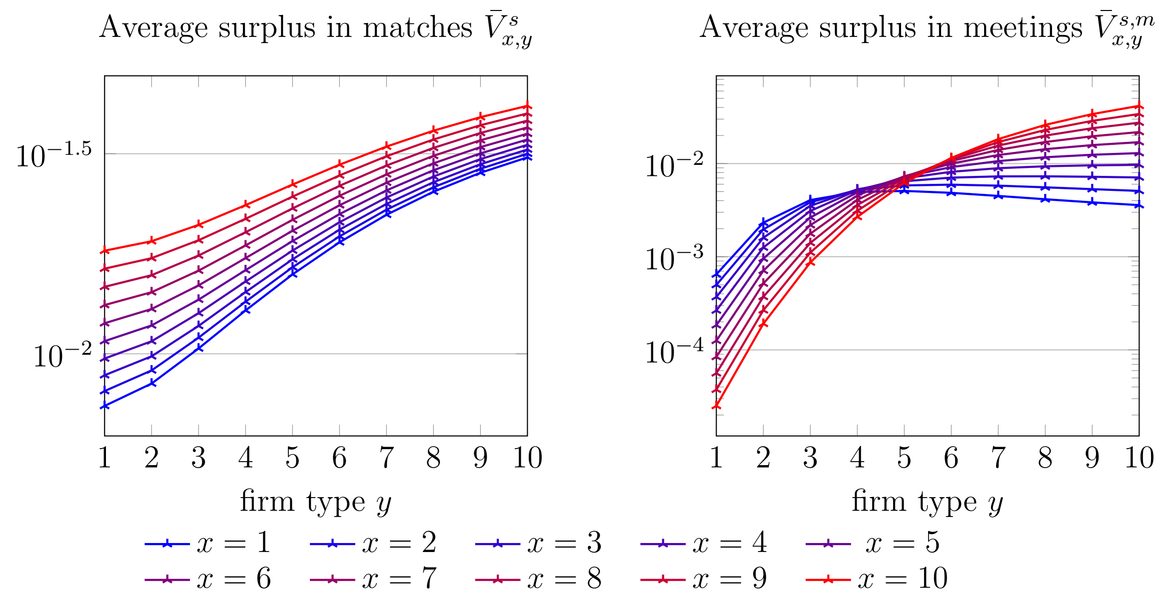

Figure 2: Average surplus in meetings when $z_0 = \underline{z}$, and average surplus in matches, conditional on firm type. Each line represents one worker type. Vertical scales are logarithmic.

different types of employers. However, they do not capture the value that workers obtain from merely *meeting* different types of employers. The value of a meeting is captured by the match surplus $V^s_{x,y}(z)$ in equation (5). For illustrative purposes, we set the lower bound on the idiosyncratic shock distribution, z_0, equal to the minimum acceptance threshold across (x, y) pairs: $z_0 = \underline{z} \equiv \min_{x,y} \bar{z}_{x,y}$. We then define the average surplus in meetings as

$$\bar{V}^{s,m}_{x,y} \equiv \int_{z_0}^{\infty} V^s_{x,y}(z) s(z) dz. \tag{20}$$

We also define the average surplus conditional on matching,

$$\bar{V}^s_{x,y} \equiv \frac{1}{S(\bar{z}_{x,y})} \int_{\bar{z}_{x,y}}^{\infty} V^s_{x,y}(z) s(z) dz. \tag{21}$$

In Figure 2, we plot the average surplus in an (x, y) match, $\bar{V}^s_{x,y}$, in the left panel, and the average surplus in an (x, y) meeting, $\bar{V}^{s,m}_{x,y}$, in the right panel.

The left panel reflects the results in Figure 1. The average surplus in matches is increasing in firm type for each worker type and is log submodular. Looking at this panel, one might be tempted to conclude that low-type workers benefit relatively more from meeting a high-type firm than do high-type workers. The right panel of the same figure, where we show average surplus in meetings, contradicts this conclusion. We observe that low-type workers have only small gains from meeting a higher-type firm, while high-type workers gain a lot more from meeting a high-type firm. Indeed, the lowest worker type has the highest average surplus in a meeting with a type-5 firm and gets 30 percent less surplus from meeting a type-10 firm.

The difference between the two figures is selection: while the left panel shows the average surplus in meetings that result in matches, the right panel shows the average surplus in all meetings, keeping the support of the z distribution the same for every pair. In other words, the right panel is not affected by selection of worker types into firms based on the realized value of z. Based on the right hand panel of Figure 2, it should no longer be surprising that low-type workers rarely take high-type jobs, as illustrated in the right panel of Figure 1. Average surplus in meetings, not in observed matches, drive sorting.

4.3 Identification of Meeting Surplus with Random Search

In this section, we prove that it is possible to identify the extent of log-supermodularity of the average surplus in meetings using the structure of our model and the type of information in standard administrative data sets. More precisely, we extend our model in several directions, relaxing the assumption that wages are determined by Nash bargaining and allowing for a more general matching technology. Importantly, we maintain the assumption that search is random.

In our more general model, an unemployed type x worker meets a type y firm at rate $\rho^u_x \rho^v_y$, a multiplicatively separable function of the worker and firm types. This allows for the possibility that some workers and firms are better at searching, but maintains the assumption that search is random rather than directed. Upon meeting, the worker samples from a wage distribution with survival function $\tilde{S}_{x,y}(w)$ and density $\tilde{s}_{x,y}(w)$, and chooses whether to accept or reject the job. This may be the reduced form coming from the Nash wage equation (9), the production technology f, and the distribution of match-specific shocks, but we do not need this interpretation for our results in this section. Finally, the job ends at rate δ_x and the worker discounts the future at rate r.

Under these assumptions, we have the value functions for type x workers both when they are unemployed and when they are employed at a wage w:

$$rV^u_x = \rho^u_x \sum_y \rho^v_y \int \max\{V^e_x(w) - V^u_x, 0\} \tilde{s}_{x,y}(w) dw, \tag{22}$$

$$rV^e_x(w) = w + \delta_x (V^u_x - V^e_x(w)). \tag{23}$$

Note that the value of an employed worker depends on the wage but not the firm type. Moreover, from equation (23), we get that the worker accepts any job with $w > \bar{w}_x = rV^u_x$, the worker's reservation wage. Additionally, we use the fact that in steady state, the measure of (x, y) matches solves

$$\delta_x \phi_{x,y} = \rho_x^u \rho_y^v u_x \tilde{S}_{x,y}(\bar{w}_x), \tag{24}$$

with $u_x = m_x - \sum_y \phi_{x,y}$. Finally, define the average wage (not log wage) in an (x, y) match as

$$W^*_{x,y} \equiv \frac{\int_{\bar{w}_x}^{\infty} w \tilde{s}_{x,y}(w) dw}{\tilde{S}_{x,y}(\bar{w}_x)}. \tag{25}$$

Combining equations (22)–(25), we obtain

$$\bar{w}_x = \frac{\delta_x}{r u_x + \delta_x m_x} \sum_y W^*_{x,y} \phi_{x,y}. \tag{26}$$

Note that if $r = 0$, this is simply the average wage multiplied by the employment rate, while with discounting it is a bit lower.[9] Using the methodology in Bonhomme, Lamadon and Manresa (2019), the average wage $W^*_{x,y}$ and the share of type-x workers matched to type-y firms, $\phi_{x,y}/m_x$, are identified. Moreover, a slight extension to the methodology allows us to recover how many type-x workers separate each period, $\delta_x m_x$, as well as the type-specific unemployment rate u_x. Thus the reservation wage is identified if we use other information to recover the discount rate r

The average surplus from an (x, y) meeting, defined in equation (20), is

$$\bar{V}^{s,m}_{x,y} = \int \max\{V^e_x(w) - V^u_x, 0\} \tilde{s}_{x,y}(w) dw. \tag{27}$$

We simplify this using equations (23), (24), and (25) to obtain

$$\bar{V}^{s,m}_{x,y} = \frac{(W^*_{x,y} - \bar{w}_x)\delta_x \phi_{x,y}}{(r + \delta_x)\rho_x^u \rho_y^v u_x}. \tag{28}$$

Note that the average surplus from meetings, $\bar{V}^{s,m}_{x,y}$, is log-supermodular if and only if $(W^*_{x,y} - \bar{w}_x)\phi_{x,y}$ is log-supermodular. Each element of this expression is identified using the methodology in Bonhomme, Lamadon and Manresa (2019), and so we can identify the extent of log-supermodularity in the gains meetings without observing meetings that do not result in matches.

[9] In practice, the distinction is small because r and u_x are each an order of magnitude smaller than δ_x and m_x, respective.

4.4 Testing for Selection with a General Search Technology

We just showed that the average surplus from meetings is identified using the structure of our model and standard administrative data. Here we ask whether we can distinguish the average surplus from meetings from the average surplus from matches without using the structure of our model. The answer is no.

We show here that if we relax the random search assumption, it is no longer possible to test for selection. The insight follows from Flinn and Heckman (1982). Return to our main (and more restrictive) model, where wages vary within an (x, y) match due to the idiosyncratic productivity shock z. By comparison, consider an alternative model which generates the same administrative data on wages in accepted matches. In the alternative model, a type-x worker meets a type-y firm at rate $\rho S(\bar{z}_{x,y})v_y$ (rather than ρv_y). In that event, they draw a match-specific productivity shock with survival function $S(z)/S(\bar{z}_{x,y})$ for $z \geq \bar{z}_{x,y}$ (rather than $S(z)$ for $z \geq z_0$), and the wage is given by $W_{x,y}(z)$ defined in equation (9). We call this the "exogenous sorting model."

By construction, all meetings result in matches in the exogenous sorting model and so there is no selection and no distinction between the average surplus from an (x, y) meeting and the average surplus from an (x, y) match. Nevertheless, all observable outcomes in standard administrative data sets—the joint distribution of wages and match partners—are identical to our model of selection and endogenous sorting. We conclude that it is impossible to test whether there is selection once we relax the random search assumption. This limit to identification naturally carries over to the more general model in Section 4.3.

Testing for selection is closely related to testing for exogenous mobility in the literature that follows Abowd, Kramarz and Margolis (1999). That literature proposes regressing the log wage $W_{i,t}$ of worker i at time t on fixed effects for the worker, the employer $\mathcal{J}_{i,t}$, and an error term:

$$\log W_{i,t} = \alpha_i + \psi_{\mathcal{J}_{i,t}} + \varepsilon_{i,t}. \tag{29}$$

This literature interprets α_i to be the average effect that worker i would have on their wage at any firm, not just the firms that employ them. It seems natural to interpret this as the value of the worker's human capital. Similarly, ψ_j is the average effect that firm j would have on the wage of any worker, not just the workers that it employs. Card, Heining and Kline (2013) state that "such a premium could represent rent-sharing, an efficiency wage premium, or strategic wage posting behavior [p. 987]." In our model, firms matter for wages because of rent-sharing.

To estimate equation (29), one makes the usual assumption that the mean of the error term $\varepsilon_{i,t}$ is independent of the regressors α_i and $\psi_{\mathcal{J}_{i,t}}$. Card, Heining and Kline (2013), Abowd, McKinney and Schmutte (2019), and others call this assumption "exogenous mobility." As is well-known, the exogenous mobility assumption cannot be tested in the data. Instead, the literature evaluates the plausibility of the exogenous mobility assumption by considering data generating processes (models) that would violate exogenous mobility and then testing auxiliary implications of those models.

We believe our model's data generating process is useful for evaluating the exogenous mobility assumption. First we ask whether the exogenous mobility assumption is satisfied. According to our model, the wage satisfies equation (9) when the match-specific productivity shock exceeds the threshold $\bar{z}_{x,y}$. While our model does not make predictions about the wage in less productive meetings, a reasonable bound is that the (latent) wage is less than the worker's reservation wage $\bar{w}_x$ when the match is not formed; for example, it might still satisfy equation (9) in unacceptable matches. This means that, if our model were the data generating process and one estimated the AKM wage equation using an administrative data set, one would be dropping all of the lowest wage observations for each worker-firm pair. The mean error term $\varepsilon_{i,t}$ among matched worker-firm pairs thus varies in a way that depends on how likely the worker and firm are to match when they meet, violating the exogenous mobility assumption.

We now turn to practical tests of the exogenous mobility assumption, i.e. of other aspects of models that violate exogenous mobility. Card, Heining and Kline (2013) propose two such tests. In the first, they look at whether the wage gains of workers who move from one establishment to another are equal in magnitude and opposite in sign to the wage gains of workers who move in the opposite direction. This would not be the case in many models of on-the-job search, such as Burdett and Mortensen (1998), where workers always experience wage increases when switching jobs. Card, Heining and Kline (2013) and subsequent research by these authors find remarkable support for the "equal in magnitude and opposite in sign" prediction. We show the same pattern for Austrian data in Appendix F.

This finding is consistent with both our model of endogenous sorting and the alternative model of exogenous sorting, and so cannot distinguish between them. Take a cross-section of workers and select those working at a type-y_1 firm. The share of type-x workers in this pool is $\phi_{x,y_1}/\sum_{x'=1}^{X}\phi_{x',y_1}$. Some of those workers will have their next job at a type-y_2 firm. The probability of this event for a type x worker is $\phi_{x,y_2}/\sum_{y'=1}^{Y}\phi_{x,y'}$. We weight the original pool with this probability to find the share of type-x workers among those who move from y_1 to y_2:

$$\frac{\phi_{x,y_1}\frac{\phi_{x,y_2}}{\sum_{y'=1}^{Y}\phi_{x,y'}}}{\sum_{x'=1}^{X}\phi_{x',y_1}\frac{\phi_{x',y_2}}{\sum_{y'=1}^{Y}\phi_{x',y'}}}=\frac{\frac{u_x S(\bar{z}_{x,y_1})S(\bar{z}_{x,y_2})}{\sum_{y'=1}^{Y}v_{y'}S(\bar{z}_{x,y'})}}{\sum_{x'=1}^{X}\frac{u_{x'}S(\bar{z}_{x',y_1})S(\bar{z}_{x',y_2})}{\sum_{y'=1}^{Y}v_{y'}S(\bar{z}_{x',y'})}},$$

where the equation eliminates ϕ using equation (12) and simplifies. This means that average increase in the log wage of a worker whom we observe moving from y_1 to y_2 is

$$w^*_{y_1 \to y_2} = \frac{\sum_{x=1}^{X} (w^*_{x,y_2} - w^*_{x,y_1}) \frac{u_x S(\bar{z}_{x,y_1}) S(\bar{z}_{x,y_2})}{\sum_{y'=1}^{Y} v_{y'} S(\bar{z}_{x,y'})}}{\sum_{x=1}^{X} \frac{u_x S(\bar{z}_{x,y_1}) S(\bar{z}_{x,y_2})}{\sum_{y'=1}^{Y} v_{y'} S(\bar{z}_{x,y'})}} = -w^*_{y_2 \to y_1}$$

That is, the gains of a worker observed moving going from a type-y_1 to a type-y_2 establishment are the same as the wage losses for a worker going in the other direction. This is because the distribution of wages depends only on the worker's type and the firm's type, and that any worker is as likely to go from y_1 to y_2 as to go from y_2 to y_1.

In their second test, Card, Heining and Kline (2013) look at the variance of the wage residual in equation (29). They write that a large "idiosyncratic match component of wages changes the interpretation of the estimated establishment effects, since different workers have different wage premiums at any given establishment, depending on the value of their match component (Card, Heining and Kline, 2013, p. 989)." In their empirical work, they find that the variance of the match-specific wage residual is between 0.060 and 0.075, and argue that this is small enough to constitute evidence against endogenous mobility. According to our model, the wage residual in equation (29) depends on the variance of the idiosyncratic shock z as well as any nonlinearities in the average log wage. Importantly, it is the same in both our model with selection and the alternative model with exogenous sorting, and so this test cannot distinguish between the two models. Moreover, we calibrated our model to match this wage residal. In Table 1 in Appendix B, we show that our model can simultaneously generate a realistic relationship between the average log wage and worker and firm types, a realistic amount of sorting between workers and firms, and a realistically wage residual.

Finally, Abowd, McKinney and Schmutte (2019) look at whether the wage residual $\hat{\varepsilon}_{i,t}$ in a worker's current match forecasts a future employer's fixed effect $\hat{\psi}_{\mathcal{J}_{i,t'}}$ for $t' > t$, conditional on the worker's and current employer's fixed effects $\hat{\alpha}_i$ and $\hat{\psi}_{\mathcal{J}_{i,t}}$. In both our model of selection and the alternative model with exogenous mobility, the worker's type x determines the distribution over future employers' fixed effects $\bar{\psi}_y$, and so the current wage residual does not help to predict future employers. When $T \to \infty$, the estimated worker fixed effects $\hat{\alpha}_i$ converge to $\bar{\alpha}_{x_i}$ and the estimated firm fixed effects $\hat{\psi}_{\mathcal{J}_{i,t}}$ converge to $\bar{\psi}_{y_{\mathcal{J}_{i,t}}}$, which means that the Abowd, McKinney and Schmutte (2019) test would not reject exogenous mobility. The same paper also looks at the question from the employer's perspective, forecasting future employee fixed effects. Again, under either model, the estimated firm fixed effects predict future employees and so this test would not reject exogenous mobility when $T \to \infty$.[10]

[10] If T is finite, then OLS gives noisy measures of worker and firm fixed effects. These noisy measures are

5 Conclusion

We have developed a random search model of the labor market with ex ante heterogeneous workers and firms and ex post match-specific productivity shocks. When the distribution of match-specific shocks is Pareto or exponential, we obtain simple proofs of three results. First, we show that there is positively assortative matching when the production function is sufficiently complementary. Second, we show that on average, more productive workers earn higher wages at any type of firm and more productive firms pay higher wages to any type of worker. And finally, we show that more productive workers earn more on average and more productive firms pay more on average, so high productivity workers and firms are high wage workers and firms.

We also argue that our model is consistent with a variety of facts in the empirical labor economics literature built around the Abowd, Kramarz and Margolis (1999) wage equation and the non-additive specification in Bonhomme, Lamadon and Manresa (2019). In particular, we are able to replicate the decomposition of the cross-sectional variance in log wages into the component due to ex ante worker heterogeneity, the component due to ex ante firm heterogeneity, the component due to the sorting of workers and firms, and a residual due to the match-specific productivity shocks.

Additionally, our model predicts that average log wages may be submodular—low-wage workers gain proportionately more than high-wage workers when they move from low-wage to high-wage firms—while simultaneously low-wage workers may be disproportionately employed at low-wage firms. This is an empirically plausible pattern. In our model, this reflects a selection mechanism. Low-wage workers rarely accept high-wage jobs, but when they do, productivity is so high that their wage is also very high.

Our model has implications for whether the empirical literature stemming from the Abowd, Kramarz and Margolis (1999) wage equation can tell us something about how wages are determined. To continue the previous paragraph, take the empirical finding that average log wages are submodular. This result, viewed through the lens of some models where administrative wage data are not a selected sub-sample of all wage offers, implies that low-wage workers would particularly benefit from meeting high-wage firms. That they don't often match with such firms is then a puzzle, one that Bonhomme, Lamadon and Manresa (2019) resolve through an assumption that low-wage workers enjoy substantially higher amenities when employed by low-wage firms. Our model offers an alternative mechanism, that observed

minimum variance among linear unbiased estimators, but the error term $\hat{\varepsilon}_{i,t}$ may still contain information, e.g. higher moments, that is useful for predicting types. As a result, with finite T, data generated from our model or from the alternative model may fail the Abowd, McKinney and Schmutte (2019) test for exogenous mobility.

wages are a selected subset of all (latent) wage offers, and that selection shapes the average log wage. A corollary of this is that a regression of log wages on worker and firm fixed effects need not shed light on firms' role in wage determination because of sample selection issues.

In a similar vein, recent papers by Engbom, Moser and Sauermann (2023) and Lachowska, Mas, Saggio and Woodbury (2023) estimate the AKM regression with time-varying firm types, $\psi_{j,t}$, and interpret changes in $\psi_{j,t}$ as changes in firm pay policies. Our model implies that time variation in firm fixed effects can capture other things, such as changes in selection due to time-varying variation in the distribution of match-specific shocks. Hence time-variation in estimated firm fixed effects again may not have a structural interpretation as a change in how firms set wages.

Our analysis also cautions against attempts to learn about the production function using only administrative data on average log wages. Bonhomme, Lamadon and Manresa (2019) write "As a first way to quantify the economic magnitude of complementarities, we next assess the explanatory power of worker types and firm classes when those enter the regression interactively as opposed to additively. The R^2 coefficient in the linear regression is 74.8%, while in the regression that includes all interactions between worker type indicators and firm class indicators, the R^2 is 75.8%. Hence, while the left panel of Figure 2 suggests the presence of some complementarity between firms and lower-type workers, those complementarities explain only a small part of the variance of log-earnings [pp. 718–719]." Proposition 3 shows that when the distribution of idiosyncratic shocks is Pareto, the production function does not directly enter the expression for the average log wage $w^*_{x,y}$, and thus this conclusion may be unwarranted in our model.

We close by mentioning one assumption that is important for our selection results: firms have an opportunity costs of filling a vacancy. In our model, this is extreme because a firm can only hire one worker. At the opposite extreme, if firms had a constant returns to scale production technology using only labor, as is the case in Card, Cardoso, Heining and Kline (2018), then the opportunity cost of hiring would be zero. This is effectively equivalent to an environment where firms have no bargaining power, $\gamma = 1$, and so firms' reservation profit is driven to zero. In this case, it is easy to verify that when the idiosyncratic shock has a Pareto distribution, workers have the same wage distribution at every type of firm, so the model does not generate a firm premium. We believe that, because of diminishing returns to scale and because a firm cannot costlessly fire an old worker if a better match comes along, hiring a worker incurs an opportunity cost. To the extent that the opportunity cost of hiring differs systematically across firms, the results we present in this paper are relevant.

One other assumption is unimportant. We assume that the matching technology is quadratic (Diamond and Maskin, 1979), so the total number of matches is a homogeneous of degree two in unemployment $\{u_x\}_{x=1}^X$ and vacancies $\{v_Y\}_{y=1}^Y$. It is straightforward to extend most of our results to a linear matching technology (homogeneous of degree one) by assuming that the contact rate ρ is a homogeneous of degree minus-one function of aggregate unemployment $\sum_{x=1}^X u_x$ and aggregate vacancies $\sum_{y=1}^Y v_y$. In particular, all characterizations of equilibrium, including Propositions 2–5, as well as our discussions of the gains from meetings and matches and of selection, carry over to this environment. Only the proof of existence of equilibrium (Proposition 1) would need to be extended to allow for a more general random matching technology.

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Appendix

A Proofs

Proof of Proposition 1. Consider the following $2(X+Y)$ functions of $(\bar{w}_x, u_x)_{x=1}^X$ and $(\bar{\pi}_y, v_y)_{y=1}^Y$:

$$
\begin{aligned}
T_{1,x}(\bar{w}, \bar{\pi}, u, v) &= \frac{\gamma\rho}{r+\delta} \sum_{y=1}^{Y} v_y f_{x,y} \int_{\bar{z}_{x,y}}^{\infty} (z - \bar{z}_{x,y}) s(z) dz, \\
T_{2,y}(\bar{w}, \bar{\pi}, u, v) &= \frac{(1-\gamma\rho)}{r+\delta} \sum_{x=1}^{X} u_x f_{x,y} \int_{\bar{z}_{x,y}}^{\infty} (z - \bar{z}_{x,y}) s(z) dz, \\
T_{3,x}(\bar{w}, \bar{\pi}, u, v) &= m_x - \sum_{y=1}^{Y} \frac{\rho u_x v_y S(\bar{z}_{x,y})}{\delta}, \\
T_{4,y}(\bar{w}, \bar{\pi}, u, v) &= n_y - \sum_{x=1}^{X} \frac{\rho u_x v_y S(\bar{z}_{x,y})}{\delta},
\end{aligned}
$$

where $\bar{z}_{x,y} = (\bar{w}_x + \bar{v}_y)/f_{x,y}$, as in equation (6). We refer to this mapping collectively as $T = (T_1, T_2, T_3, T_4)$. It is immediate that any vector $(\bar{w}, \bar{\pi}, u, v)$, together with $\bar{z}$ solving equation (6) and ϕ solving equation (12), is an equilibrium if and only if it is a fixed point of T, i.e. $T(\bar{w}, \bar{\pi}, u, v) = (\bar{w}, \bar{\pi}, u, v)$.

Now define the mapping $\tilde{T}$:

$$
\begin{aligned}
\tilde{T}_{1,x}(\bar{w}, \bar{\pi}, u, v) &= \min\left\{T_{1,x}(\bar{w}, \bar{\pi}, u, v), \bar{\bar{w}}_x\right\}, \\
\tilde{T}_{2,y}(\bar{w}, \bar{\pi}, u, v) &= \min\left\{T_{2,y}(\bar{w}, \bar{\pi}, u, v), \bar{\bar{\pi}}_y\right\}, \\
\tilde{T}_{3,x}(\bar{w}, \bar{\pi}, u, v) &= \max\left\{T_{3,x}(\bar{w}, \bar{\pi}, u, v), 0\right\}, \\
\tilde{T}_{4,y}(\bar{w}, \bar{\pi}, u, v) &= \max\left\{T_{4,y}(\bar{w}, \bar{\pi}, u, v), 0\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\bar{w}}_x &= \frac{\gamma\rho}{r+\delta} \sum_{y=1}^{Y} n_y f_{x,y} \int_{\bar{\bar{w}}_x/f_{x,y}}^{\infty} (z - \bar{\bar{w}}_x/f_{x,y}) s(z) dz, \\
\bar{\bar{\pi}}_y &= \frac{(1-\gamma)\rho}{r+\delta} \sum_{x=1}^{X} m_x f_{x,y} \int_{\bar{\bar{\pi}}_y/f_{x,y}}^{\infty} (z - \bar{\bar{\pi}}_y/f_{x,y}) s(z) dz.
\end{aligned}
$$

The mapping $\tilde{T}$ is continuous. Moreover, it maps points satisfying $\bar{w}_x \in [0, \bar{\bar{w}}_x]$, $\bar{\pi}_y \in [0, \bar{\bar{\pi}}_y]$,

$u_x \in [0, m_x]$, and $v_y \in [0, n_y]$ for all $x \in \{1, \ldots, X\}$ and $y \in \{1, \ldots, Y\}$ into itself. Therefore $\tilde{T}$ has a fixed point by Brouwer's fixed point theorem.

We prove that at any $(\bar{w}, \bar{\pi}, u, v)$ which is a fixed point of $\tilde{T}$, $T(\bar{w}, \bar{\pi}, u, v) = \tilde{T}(\bar{w}, \bar{\pi}, u, v)$, and thus $(\bar{w}, \bar{\pi}, u, v)$ is also a fixed point of T. In the first step we prove that $u_x > 0$ for all x. This is because if $u_x = 0$, $\tilde{T}_{3,x}(\bar{w}, \bar{\pi}, u, v) = m_x$, contradicting $(\bar{w}, \bar{\pi}, u, v)$ being a fixed point. This implies $\tilde{T}_{3,x}(\bar{w}, \bar{\pi}, u, v) = T_{3,x}(\bar{w}, \bar{\pi}, u, v)$ for all x at any fixed point. Similarly $v_y > 0$ at any fixed point of $\tilde{T}$ and so $\tilde{T}_{4,y}(\bar{w}, \bar{\pi}, u, v) = T_{4,y}(\bar{w}, \bar{\pi}, u, v)$ for all y at any fixed point.

Next, any fixed point has $\bar{w}_x > 0$ for all x: $\tilde{T}_{1,x}(\bar{w}, \bar{\pi}, u, v)$ is continuous and decreasing in $\bar{w}_x$ and is strictly positive at $\bar{w}_x = 0$ since $v_y > 0$ for all y. Similarly any fixed point has $\bar{\pi}_y > 0$ for all y.

Finally, in any fixed point of $\tilde{T}$, any solution to $\tilde{T}_{1,x}(\bar{w}, \bar{\pi}, u, v) = \bar{w}_x$ has $\bar{w}_x < \bar{\bar{w}}_x$ because $\bar{\pi}_y > 0$. This implies that the fixed point of $\tilde{T}$ also solves $\tilde{T}_{1,x}(\bar{w}, \bar{\pi}, u, v) = T_{1,x}(\bar{w}, \bar{\pi}, u, v)$ for all x. The same logic implies $\tilde{T}_{2,y}(\bar{w}, \bar{\pi}, u, v) = T_{2,y}(\bar{w}, \bar{\pi}, u, v)$ for all y in any fixed point.

In summary, we have proved that there exists a vector $(\bar{w}, \bar{\pi}, u, v)$ with $\tilde{T}(\bar{w}, \bar{\pi}, u, v) = (\bar{w}, \bar{\pi}, u, v)$, that at any such vector $T(\bar{w}, \bar{\pi}, u, v) = \tilde{T}(\bar{w}, \bar{\pi}, u, v)$ and so $(\bar{w}, \bar{\pi}, u, v)$ is a fixed point of T, and that any fixed point of T is an equilibrium. This proves an equilibrium exists.

Along the way we also proved that $\bar{w}_x > 0$ for all x and $\bar{\pi}_y > 0$ for all y at any fixed point of $\tilde{T}$, and hence in any equilibrium. ■

Proof of Lemma 1. We prove that $\bar{w}_x$ is strictly increasing. The proof that $\bar{\pi}_y$ is strictly increasing is analogous.

To find a contradiction, suppose there exists an $x_1 < x_2$ with $\bar{w}_{x_1} \geq \bar{w}_{x_2}$. Since f is monotonic, $f_{x_2,y} > f_{x_1,y}$ for all y. Then using equation (6), we have $\bar{z}_{x_1,y} > \bar{z}_{x_2,y}$ for all y. Additionally, observe that $\int_{\bar{z}}^{\infty} (z - \bar{z}) s(z) dz$ is strictly positive for all $\bar{z}$, since $S(z) > 0$ for all z. Additionally, the integral is decreasing in $\bar{z}$, as can be confirmed directly. This means that $\int_{\bar{z}_{x_2,y}}^{\infty} (z - \bar{z}_{x_2,y}) s(z) dz > \int_{\bar{z}_{x_1,y}}^{\infty} (z - \bar{z}_{x_1,y}) s(z) dz > 0$.

Putting this together, if there exists an $x_1 < x_2$ with $\bar{w}_{x_1} \geq \bar{w}_{x_2}$,

$$\begin{aligned} \bar{w}_{x_2} &= \frac{\gamma\rho}{r+\delta} \sum_{y=1}^{Y} v_y f_{x_2,y} \int_{\bar{z}_{x_2,y}}^{\infty} (z - \bar{z}_{x_2,y}) s(z) dz \\ &> \frac{\gamma\rho}{r+\delta} \sum_{y=1}^{Y} v_y f_{x_1,y} \int_{\bar{z}_{x_1,y}}^{\infty} (z - \bar{z}_{x_1,y}) s(z) dz = \bar{w}_{x_1}, \end{aligned}$$

where the two equations use the value function (10) and the inequality uses $f_{x_2,y} > f_{x_1,y}$

and $\int_{\bar{z}_{x_2,y}}^{\infty}(z-\bar{z}_{x_2,y})s(z)dz > \int_{\bar{z}_{x_1,y}}^{\infty}(z-\bar{z}_{x_1,y})s(z)dz > 0$, together with strict positivity of the remaining terms. But this is a contradiction, proving $\bar{w}_{x_1} < \bar{w}_{x_2}$. ∎

Proof of Proposition 2. We first prove that $\bar{z}_{x,y}$ is strictly log-submodular. Substituting equation (6) for $\bar{z}$, we must prove that for $x_1 < x_2$ and $y_1 < y_2$,

$$\left(\frac{\bar{w}_{x_1}+\bar{\pi}_{y_1}}{f_{x_1,y_1}}\right)\left(\frac{\bar{w}_{x_2}+\bar{\pi}_{y_2}}{f_{x_2,y_2}}\right) < \left(\frac{\bar{w}_{x_1}+\bar{\pi}_{y_2}}{f_{x_1,y_2}}\right)\left(\frac{\bar{w}_{x_2}+\bar{\pi}_{y_1}}{f_{x_2,y_1}}\right).$$

Weak log supermodularity of f implies $f_{x_1,y_1}f_{x_2,y_2} \geq f_{x_1,y_2}f_{x_2,y_1}$. And we see that the product of the numerators on the left hand side is smaller than the product of the numerators on the right hand side if and only if

$$\bar{w}_{x_1}\bar{\pi}_{y_2} + \bar{w}_{x_2}\bar{\pi}_{y_1} < \bar{w}_{x_1}\bar{\pi}_{y_1} + \bar{w}_{x_2}\bar{\pi}_{y_2} \Leftrightarrow (\bar{w}_{x_2}-\bar{w}_{x_1})(\bar{\pi}_{y_2}-\bar{\pi}_{y_1}) > 0.$$

This is immediate because $\bar{w}$ and $\bar{\pi}$ are strictly increasing (Lemma 1).

We now use the assumption that $S(z) = (z/z_0)^{-\theta}$. Then since $\log \bar{z}_{x,y}$ is strictly submodular, $\log S(\bar{z}_{x,y}) = \theta \log z_0 - \theta \log \bar{z}_{x,y}$ is strictly supermodular. Finally, equation (12) implies that $\log \phi_{x,y}$ inherits the strict supermodularity of $\log S(\bar{z}_{x,y})$. ∎

Proof of Proposition 3. We prove a more general version of this proposition. For any strictly increasing function $G : \mathbb{R}_+ \to \mathbb{R}$, define

$$w_{x,y}^G = \frac{\int_{\bar{z}_{x,y}}^{\infty} G(W_{x,y}(z))s(z)dz}{S(\bar{z}_{x,y})}. \tag{30}$$

From equations (6) and (9), we have

$$W_{x,y}(z) = \bar{w}_x + \gamma(\bar{w}_x + \bar{\pi}_y)\left(\frac{z}{\bar{z}_{x,y}} - 1\right).$$

Using equation (30) and the functional form of the Pareto distribution, we obtain

$$w_{x,y}^G = \frac{\int_{\bar{z}_{x,y}}^{\infty} G\Big(\bar{w}_x + \gamma(\bar{w}_x+\bar{\pi}_y)\big(z/\bar{z}_{x,y}-1\big)\Big)\theta z^{-\theta-1}dz}{\bar{z}_{x,y}^{-\theta}}.$$

Now let $q = z/\bar{z}_{x,y} - 1$ and perform a change in the variable of integration to obtain

$$w_{x,y}^G \equiv \int_0^{\infty} G\big(\bar{w}_x + \gamma(\bar{w}_x+\bar{\pi}_y)q\big)\theta(1+q)^{-\theta-1}dq.$$

Setting $G(W) = \log(W)$ for all W gives us equation (16). This equation implies that $w^G_{x,y}$ is simply a weighted average of $G\big(\bar{w}_x + \gamma(\bar{w}_x + \bar{\pi}_y)q\big)$, with the same weights for all (x, y). This means that x and y only affect $w^G_{x,y}$ through $\bar{w}_x$ and $\bar{\pi}_y$.

Now since G is strictly increasing, it is straightforward to verify that increasing either $\bar{w}_x$ or $\bar{\pi}_y$ raises the integrand in equation (16) at all $q > 0$, and hence raises $w^G_{x,y}$. This establishes the first two enumerated points.

We next prove the third point if G is strictly concave. If G is strictly convex, we prove $w^G_{x_1,y_2} + w^G_{x_2,y_1} < w^G_{x_1,y_1} + w^G_{x_2,y_2}$. If G is linear, this is an equality. Take x_1 and x_2 with $\bar{w}_{x_1} < \bar{w}_{x_2}$; and y_1 and y_2 with $\bar{\pi}_{y_1} < \bar{\pi}_{y_2}$. Let

$$\lambda \equiv \frac{(1 + \gamma q)(\bar{w}_{x_2} - \bar{w}_{x_1})}{(1 + \gamma q)(\bar{w}_{x_2} - \bar{w}_{x_1}) + \gamma q(\bar{\pi}_{y_2} - \bar{\pi}_{y_1})}.$$

The assumptions on x_1, x_2, y_1, and y_2 ensure that $\lambda \in (0, 1)$ for all $q > 0$. Then verify algebraically that

$$\begin{aligned}
&\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_2})q = \lambda(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_1})q) + (1 - \lambda)(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_2})q), \text{ and}\\
&\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_1})q = (1 - \lambda)(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_1})q) + \lambda(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_2})q).
\end{aligned}$$

Thus G concave (convex) implies

$$\begin{aligned}
&G(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_2})q) > (<)\lambda G(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_1})q) + (1 - \lambda)G(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_2})q), \text{ and}\\
&G(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_1})q) > (<)(1 - \lambda)G(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_1})q) + \lambda G(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_2})q).
\end{aligned}$$

Summing these gives

$$\begin{aligned}
G\big(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_2})q\big) + G\big(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_1})q\big)&\\
> (<)G\big(\bar{w}_{x_1} + \gamma(\bar{w}_{x_1} + \bar{\pi}_{y_1})q\big) + G\big(\bar{w}_{x_2} + \gamma(\bar{w}_{x_2} + \bar{\pi}_{y_2})q\big)&
\end{aligned}$$

when G is concave (convex). Integrating over the density $\theta(1 + q)^{-\theta-1}$ for $q > 0$ delivers the third bullet point.

Finally, we note that if $f_{x,y}$ is strictly increasing in x and y, Lemma 1 implies $\bar{w}_x$ and $\bar{\pi}_y$ are strictly increasing. Then the three numbered conditions in the statement of the proposition imply that $w^G_{x,y}$ is strictly increasing and strictly submodular (supermodular) in

x and y when G is strictly concave (convex). ■

Proof of Proposition 4. Take any $x_1 < x_2$ and $y_1 < y_2$. Then

$$\left(\frac{1}{f_{x_2,y_2}} - \frac{1}{f_{x_2,y_1}}\right)(\bar{w}_{x_2} - \bar{w}_{x_1}) < 0,$$
$$\left(\frac{1}{f_{x_2,y_2}} - \frac{1}{f_{x_1,y_2}}\right)(\bar{\pi}_{y_2} - \bar{\pi}_{y_1}) < 0,$$

since in both cases the first term is negative (since f is positive and strictly increasing) and the second term is positive by Lemma 1. Additionally, $1/f$ submodular implies

$$\frac{1}{f_{x_1,y_1}} + \frac{1}{f_{x_2,y_2}} \leq \frac{1}{f_{x_1,y_2}} + \frac{1}{f_{x_2,y_1}}.$$

Multiply each term in the last inequality by $\bar{w}_{x_1} + \bar{\pi}_{y_1}$ and add to the preceding inequalities to get

$$\frac{\bar{w}_{x_1} + \bar{\pi}_{y_1}}{f_{x_1,y_1}} + \frac{\bar{w}_{x_2} + \bar{\pi}_{y_2}}{f_{x_2,y_2}} < \frac{\bar{w}_{x_1} + \bar{\pi}_{y_2}}{f_{x_1,y_2}} + \frac{\bar{w}_{x_2} + \bar{\pi}_{y_1}}{f_{x_2,y_1}},$$

From equation (6), this implies that $\bar{z}_{x,y}$ is strictly submodular.

Next, in steady state with an exponential distribution, we have from equation (12) that

$$\log \phi_{x,y} = \log(\rho/\delta) + \log u_x + \log v_y - \theta \bar{z}_{x,y}.$$

Since $\bar{z}_{x,y}$ is strictly submodular and the other terms are amodular, this proves that $\log \phi_{x,y}$ is strictly supermodular, i.e. that ϕ is strictly log-supermodular. ■

Proof of Proposition 5. We again work with the more general version of the proposition, using w^G defined in equation (30) for an arbitrary increasing function G. From the wage equation (9), and the exponential distribution, we have

$$w^G_{x,y} = \theta \int_{\bar{z}_{x,y}}^{\infty} G(\bar{w}_x + \gamma(z f_{x,y} - \bar{w}_x - \bar{\pi}_y)) e^{-\theta(z - \bar{z}_{x,y})} dz.$$

Let $q = z - \bar{z}_{x,y}$ to get

$$w^G_{x,y} = \theta \int_0^{\infty} G(\bar{w}_x + \gamma((q + \bar{z}_{x,y}) f_{x,y} - \bar{w}_x - \bar{\pi}_y)) e^{-\theta q} dq.$$

From equation (6), we can reduce this to

$$w_{x,y}^{G} = \theta \int_{0}^{\infty} G(\bar{w}_x + \gamma q f_{x,y}) e^{-\theta q} dq,$$

which is equivalent to equation (17) when $G(W) = \log W$ for all W. If f is strictly increasing, so is $\bar{w}_x$ (Lemma 1). And since G is strictly increasing, then the integrand in equation (17) is strictly increasing in both x and y for all q. Thus w^G is strictly increasing. ■

B Log-Linear Wage Equation

Most of the empirical literature does not aim to estimate the average log wage, $w^*_{x,y}$. Instead, following Abowd, Kramarz and Margolis (1999), authors impose a log-linear wage structure to estimate worker and firm fixed effects. In this section we analyze what that procedure recovers if our model is data-generating process, and in particular the relationship between a log-linear wage equation and the average log wage $w^*_{x,y}$.

B.1 Econometric Framework

Consider a panel data set containing the wage $W_{i,t}$ of worker $i \in \{1, \ldots, I\}$ at time $t \in \{1, 2, \ldots, T\}$ as well as the employer identifier $\mathcal{J}_{i,t} \in \{1, \ldots, J\}$. Since the worker may not always be employed, we let $\mathcal{T}_i \subseteq \{1, 2, \ldots, T\}$ denote the periods when the worker earns a wage. We assume, in line with the literature, that we observe neither a wage nor a wage offer for i when they are not employed, at $t \in \mathcal{T}_i^c \equiv \{1, 2, \ldots, T\} \backslash \mathcal{T}_i$. For notational simplicity and following the literature, we impose that each worker only works for one firm at each point in time, for example by focusing on their main job in each period that they are employed.

Following Abowd, Kramarz and Margolis (1999), we could regress the log wage on a full set of worker and firm fixed effects and an error term: For all $i \in \{1, \ldots, I\}$ and $t \in \mathcal{T}_i$,

$$\log W_{i,t} = \alpha_i + \psi_{\mathcal{J}_{i,t}} + \varepsilon_{i,t}. \tag{31}$$

We are interested in the coefficient estimates $\hat{\alpha}_i$ and $\hat{\psi}_j$ when estimating equation (31) using ordinary least squares (OLS). Regardless of the economic model and the data set, OLS is a statistical procedure which minimizes the sum of squared errors:

$$\{\hat{\alpha}_i, \hat{\psi}_j\} = \arg \min_{\{\alpha_i, \psi_j\}} \sum_{i=1}^{I} \sum_{t \in \mathcal{T}_i} (\log W_{i,t} - \alpha_i - \psi_{\mathcal{J}_{i,t}})^2. \tag{32}$$

The first order condition for α_i from equation (32) is

$$\sum_{t\in\mathcal{T}_i}(\log W_{i,t} - \hat{\alpha}_i - \hat{\psi}_{\mathcal{J}_{i,t}}) = 0. \tag{33}$$

Symmetrically, define $\mathcal{I}_{j,t}$ as the set of workers whom j employs at t, so $j = \mathcal{J}_{i,t}$ if and only if $i \in \mathcal{I}_{j,t}$.[11] Then the first order condition for ψ_j is

$$\sum_{t=1}^{T}\sum_{i\in\mathcal{I}_{j,t}}(\log W_{i,t} - \hat{\alpha}_i - \hat{\psi}_j) = 0. \tag{34}$$

Under the assumption that all workers and firms are connected though the matching graph,[12] Abowd, Creecy and Kramarz (2002) establish that equations (33) and (34) pin down $\hat{\alpha}$ and $\hat{\psi}$ up to an additive constant. That is, we can increase all the worker fixed effects by k and decrease all the firm fixed effects by k without changing the fit of equation (32).

B.2 Estimates in Model-Generated Data

Next consider estimating equation (31) using an ideal data set generated by our model. We assume that there is a large number of workers I and a large number of firms J. Each worker i has an unobserved type x_i, and similarly each firm j has an unobserved type y_j. We assume i and j behave according to the decision rules in our model. That is, when i is unemployed, they meet a type-y vacant job in a match with productivity at least z at rate $S(z)v_y$, and they accept the job and earn a wage $W_{x_i,y}(z)$ if and only if $z \geq \bar{z}_{x_i,y}$. Symmetrically, when j has a vacant job, it meets a type-x unemployed worker in a match with productivity at least z at rate $S(z)u_x$, and it hires the worker and earns profits $zf_{x,y_j} - W_{x,y_j}(z)$ if and only if $z \geq \bar{z}_{x,y_j}$.

We are interested in an environment where there is a large but finite number of workers and jobs and where we observe each worker and job for a very long time, $T \to \infty$.[13] In this case, worker i with type x_i will spend a fraction u_{x_i}/m_{x_i} of their time unemployed and fraction $\phi_{x_i,y}/m_{x_i}$ of their time matched to a type-y firm. In such matches, the density of match productivity will be $s(z)/S(\bar{z}_{x_i,y})$ for $z \geq \bar{z}_{x_i,y}$ and the wage will be $W_{x_i,y}(z)$. Similarly,

[11] In our model, $\mathcal{I}_{j,t}$ has either zero or one element, depending on whether the job is filled or vacant. In real world data, firms can employ multiple workers and so $\mathcal{I}_{j,t}$ typically has multiple elements.

[12] Formally, we require that any worker i_0 can be linked to any firm j through a finite sequence of steps $t_0, t_1, \ldots, t_n$: $j_t = \mathcal{J}_{i_{t-1},t}$ for t odd and $i_t \in \mathcal{I}_{j_{t-1},t}$ for t even, with $j = \mathcal{J}_{t_n}$.

[13] In the real world, T is finite, so the OLS estimates $\hat{\alpha}_i$ and $\hat{\psi}_j$ are unbiased but noisy estimates of $\bar{\alpha}_{x_i}$ and $\bar{\psi}_{y_j}$, respectively. This creates econometric issues which we sidestep in this paper through our idealized "large T" assumption.

the relative likelihood of firm j with type y_j matching with a type-x worker is proportional to ϕ_{x,y_j}. Again, in such matches, the density of match productivity will be $s(z)/S(\bar{z}_{x,y_j})$ for $z \geq \bar{z}_{x,y_j}$ and the wage will be $W_{x,y_j}(z)$. Since there is no uncertainty about these long-run distributions, $\hat{\alpha}_i$ and $\hat{\psi}_j$ have well-behaved limits in the limit as $T \to \infty$. These limits depend only on the worker's and firm's type, since the distribution of partners and wages only depend on types. We let $\bar{\alpha}_{x_i}$ denote the limiting value of $\hat{\alpha}_i$ and $\bar{\psi}_{y_j}$ denote the limiting value of $\hat{\psi}_j$ when $T \to \infty$.[14] We are interested in characterizing and interpreting those values.

Using equations (33) and (34) and the model structure, we obtain

$$\bar{\alpha}_x = \frac{\sum_{y=1}^{Y}(w^*_{x,y} - \bar{\psi}_y)\phi_{x,y}}{\sum_{y=1}^{Y}\phi_{x,y}} = \lambda_x - \frac{\sum_{y=1}^{Y}\bar{\psi}_y\phi_{x,y}}{\sum_{y=1}^{Y}\phi_{x,y}}, \tag{35}$$

$$\bar{\psi}_y = \frac{\sum_{x=1}^{X}(w^*_{x,y} - \bar{\alpha}_x)\phi_{x,y}}{\sum_{x=1}^{X}\phi_{x,y}} = \mu_y - \frac{\sum_{x=1}^{X}\bar{\alpha}_x\phi_{x,y}}{\sum_{x=1}^{X}\phi_{x,y}}, \tag{36}$$

where $w^*_{x,y}$ is the average log wage in an (x, y) match, defined in equation (15), $\phi_{x,y}$ is the measure of (x, y) matches, defined in equation (12), and λ_x and μ_y are defined in equations (18) and (19). Our model ensures that all workers and firms are connected when $T \to \infty$, since $S(\bar{z}_{x,y}) > 0$ for all (x, y). That means that we can solve equations (35) and (36) for $(\bar{\alpha}_x, \bar{\psi}_y)$ up to the additive constant k mentioned before.

If the average log wage were an additively separable function of the worker's type and the firm's type, $w^*_{x,y} = a_x + b_y$, then equations (35) and (36) would imply $\bar{\alpha}_x = a_x + k$ and $\bar{\psi}_y = b_y - k$, where k is the irrelevant additive constant discussed before. In this case, $\bar{\psi}_{y_2} - \bar{\psi}_{y_1}$ is the average difference in the log wage that any worker earns at a type-y_2 firm compared to a type-y_1 firm when following the equilibrium decision rules. Our model implies that $w^*_{x,y}$ is increasing in x and y but is submodular rather than linear.

Finally, we follow Abowd, Kramarz and Margolis (1999), Andrews, Gill, Schank and Upward (2008), Card, Heining and Kline (2013), and others in focusing on the cross-sectional variance of $\bar{\alpha}$ and $\bar{\psi}$ as well as the covariance between α and ψ in matched pairs. Define

$$\mathbb{E}(g_{x,y}) \equiv \frac{\sum_{x=1}^{X}\sum_{y=1}^{Y} g_{x,y}\phi_{x,y}}{\sum_{x=1}^{X}\sum_{y=1}^{Y}\phi_{x,y}}$$

[14] In practice, the literature estimating equation (31) looks only at one job per worker per year. In Abowd, Kramarz and Margolis (1999), this is the job where the individual works the most days. In Card, Cardoso and Kline (2016), this is the job where the worker has the most hours during a reference week. In Bonhomme, Lamadon and Manresa (2019), a worker is only included in the sample if they are employed by a single firm in all twelve months. Since the duration distribution of all jobs is the same in our model, none of these selection criterion affect the asymptotic values $\bar{\alpha}_x$ and $\bar{\psi}_y$.

	$Var(w)$	$Var(\alpha)$	$Var(\beta)$	$2Cov(\alpha,\beta)$	$Var(\varepsilon)$	$Corr(\alpha,\beta)$
BLM	0.124	0.074	0.003	0.015	0.031	0.491
model	0.124	0.074	0.003	0.015	0.031	0.491
		$\frac{Var(\alpha)}{Var(w)}$	$\frac{Var(\beta)}{Var(w)}$	$\frac{2Cov(\alpha,\beta)}{Var(w)}$	$\frac{Var(\varepsilon)}{Var(w)}$	
model		0.603	0.026	0.122	0.252	

Table 1: Decomposition of the variance of log wages into four components: variance of worker types, firm types, twice its covariance and the variance of the error term. The first row shows the decomposition reported in Bonhomme, Lamadon and Manresa (2019), the second row is our model.

for any g. We let $\sigma^2_{\bar{\alpha}} \equiv \mathbb{E}\big((\bar{\alpha}_x - \mathbb{E}(\bar{\alpha}_x))^2\big)$ and $\sigma^2_{\bar{\psi}} \equiv \mathbb{E}\big((\bar{\psi}_y - \mathbb{E}(\bar{\psi}_y)))^2\big)$ denote the variance of α_{x_i} and ψ_{y_j} across employed workers i and filled jobs j. Also let $\text{cov}_{\bar{\alpha},\bar{\psi}} \equiv \mathbb{E}\big((\bar{\alpha}_x - \mathbb{E}(\bar{\alpha}_x))(\bar{\psi}_y - \mathbb{E}(\bar{\psi}_y))\big)$ denote the covariance between $\bar{\alpha}_{x_i}$ and $\bar{\psi}_{y_j}$ across matched pairs (i,j) with $j = J_{i,t}$. The unidentified additive constant k does not affect any of these moments.

If we estimate $\hat{\alpha}_i$ and $\hat{\psi}_j$ using OLS, the well-known issue of limited mobility (Andrews, Gill, Schank and Upward, 2008) biases estimates of the variances and covariances when T is finite. Throughout this paper, we assume an idealized environment, either one where $T \to \infty$, or alternatively a statistical procedure that gives unbiased estimates of the variance and covariance, as proposed by Andrews, Gill, Schank and Upward (2008), Bonhomme, Lamadon and Manresa (2019), or Kline, Saggio and Sølvsten (2020).

Using the parametrization described in Section 4, we conduct the variance decomposition and report the results in Table 1. By construction, we exactly match the empirical moments.

C Normalizations for Pareto and Exponential Case

Equilibrium is described by equations (6) (10), (11), (12), (13), (14), which we repeat here for convenience.

$$\bar{z}_{x,y} = \frac{\bar{w}_x + \bar{\pi}_y}{f_{x,y}}$$
$$\bar{w}_x = \frac{\gamma\rho}{r+\delta} \sum_{y=1}^{Y} v_y f_{x,y} \int_{\bar{z}_{x,y}}^{\infty} (z - \bar{z}_{x,y}) s(z) dz$$
$$\bar{\pi}_y = \frac{(1-\gamma)\rho}{r+\delta} \sum_{x=1}^{X} u_x f_{x,y} \int_{\bar{z}_{x,y}}^{\infty} (z - \bar{z}_{x,y}) s(z) dz$$
$$\delta\phi_{x,y} = \rho u_x v_y S(\bar{z}_{x,y})$$
$$u_x = m_x - \sum_{y=1}^{Y} \phi_{x,y}$$
$$v_y = n_y - \sum_{x=1}^{X} \phi_{x,y}.$$

In the Pareto case, $S(z) = \left(\frac{z}{z_0}\right)^{\theta}$ and $s(z) = -\theta\left(\frac{z^{-\theta-1}}{z_0^{-\theta}}\right)$. Guess and verify that changing ρ to $\lambda\rho$ changes thresholds to $\lambda^{1/\theta}\bar{z}_{x,y}$ and reservation wages and profits to $\lambda^{1/\theta}\bar{w}_x$, $\lambda^{1/\theta}\bar{\pi}_y$ but does not affect $\phi_{x,y}, u_x, v_y$. It is immediate to verify this guess for the last three equations. To verify the equation for the reservation values $\bar{w}_x$ and $\bar{\pi}_y$, use that

$$\int_a^{\infty} (z-a)s(z)dz = \left(\frac{b}{a}\right)^{1-\theta} \int_b^{\infty} (z-b)s(z)dz.$$

Finally, equation (16) implies that changing ρ has a level effect on mean log wages which increase by $\frac{1}{\theta}\log(\lambda)$ for all x, y. We therefore normalize $\rho = 1$ in our numerical evaluation.

In the exponential case, $S(z) = e^{-\theta z}$, changing θ to $\lambda\theta$ changes thresholds to $\frac{\bar{z}_{x,y}}{\lambda}$ and reservation wages and profits to $\frac{\bar{w}_x}{\lambda}$, $\frac{\bar{\pi}_y}{\lambda}$ but does not change $\phi_{x,y}, u_x, v_y$. Using that $s(z) = \theta e^{-\theta z}$, it can be easily verified that the equilibrium equations hold. Finally, equation (17) implies that change in θ has a level effect on mean log wage which increase by $-\log(\lambda)$. Hence in the exponential case, we can normalize $\theta = 1$.

	$Var(w)$	$Var(\alpha)$	$Var(\beta)$	$2Cov(\alpha,\beta)$	$Var(\varepsilon)$	$Corr(\alpha,\beta)$
BLM	0.124	0.074	0.003	0.015	0.031	0.491
model	0.124	0.078	0.003	0.014	0.029	0.442

Table 2: Decomposition of variance of log wages in the model with exponentially distributed match quality shocks.

D Exponentially-Distributed Match Quality

We present results for the model with exponentially distributed match quality. We proceed as in the case of Pareto distribution. We set the number of types at $X = Y = 10$ and assume a uniform type distribution, $m_x = \frac{1}{X}$ and $n_y = \frac{1}{Y}$. The production function is CES with elasticity of substitution ξ, $f_{x,y} = (0.5p_x^{\frac{\xi-1}{\xi}} + 0.5q_y^{\frac{\xi-1}{\xi}})^{\frac{\xi}{\xi-1}}$ where $p_1 = q_1 = 1$, $p_x = (1+\Delta_p)p_{x-1}$ for $x = 2, \ldots X$, and $q_y = (1+\Delta_q)q_{y-1}$ for $y = 2, \ldots Y$. We let $S(z) = e^{-z}$ because we normalize $\theta = 1$. We choose $\gamma = \frac{2}{3}$, and $r = 0.05$, $\delta = 0.5$, which implies expected duration of a match of 2 years. We choose the remaining parameters, $\Delta_p, \Delta_q, \rho, \xi$ to match the variance decomposition in Bonhomme, Lamadon and Manresa (2019). As before, we target the variance of worker type, firm types, the covariance and the variance of the residual, and we assume that we have infinitely much data, so we know each worker's type x and each firm's type y and so all distributions are deterministic functions of model parameters. The parameter values which allow us to match the target moments are $\Delta_p = 0.065$, $\Delta_q = 1.962$, $\xi = 0.694$ and $\rho = 2.9 \times 10^5$. Moments are summarized in Table 2.

This parametrization implies that the most productive worker is on average 59% more productive than the least productive worker, and the most productive firm is on average 443% more productive than the least productive one. In particular, $f_{10y}/f_{1y} - 1$ lies between 0.30 and 0.75, depending on the value of y. For firms, $f_{xy_{10}}/f_{xy_1} - 1$ lies between 3.68 and 5.27, depending on value of x. The unemployment rate is 14.0%.

Figure 3 shows the results. In the first row we show the average log wage in matches and the distribution of worker types conditional on firm type. The log wage is monotone in firm type for each worker type, so again, one is tempted to conclude that low-type workers gain a lot from meeting a very productive firms. The top right panel shows that there is strong sorting, so low productive workers are only rarely employed by firms with high productivity. The bottom row shows that the reason behind this pattern is selection. While the bottom left figure shows a large increase in the mean surplus in matches as one moves from low to high productivity firm, the right panel of the bottom row shows that this is not the case for the average surplus in meetings.

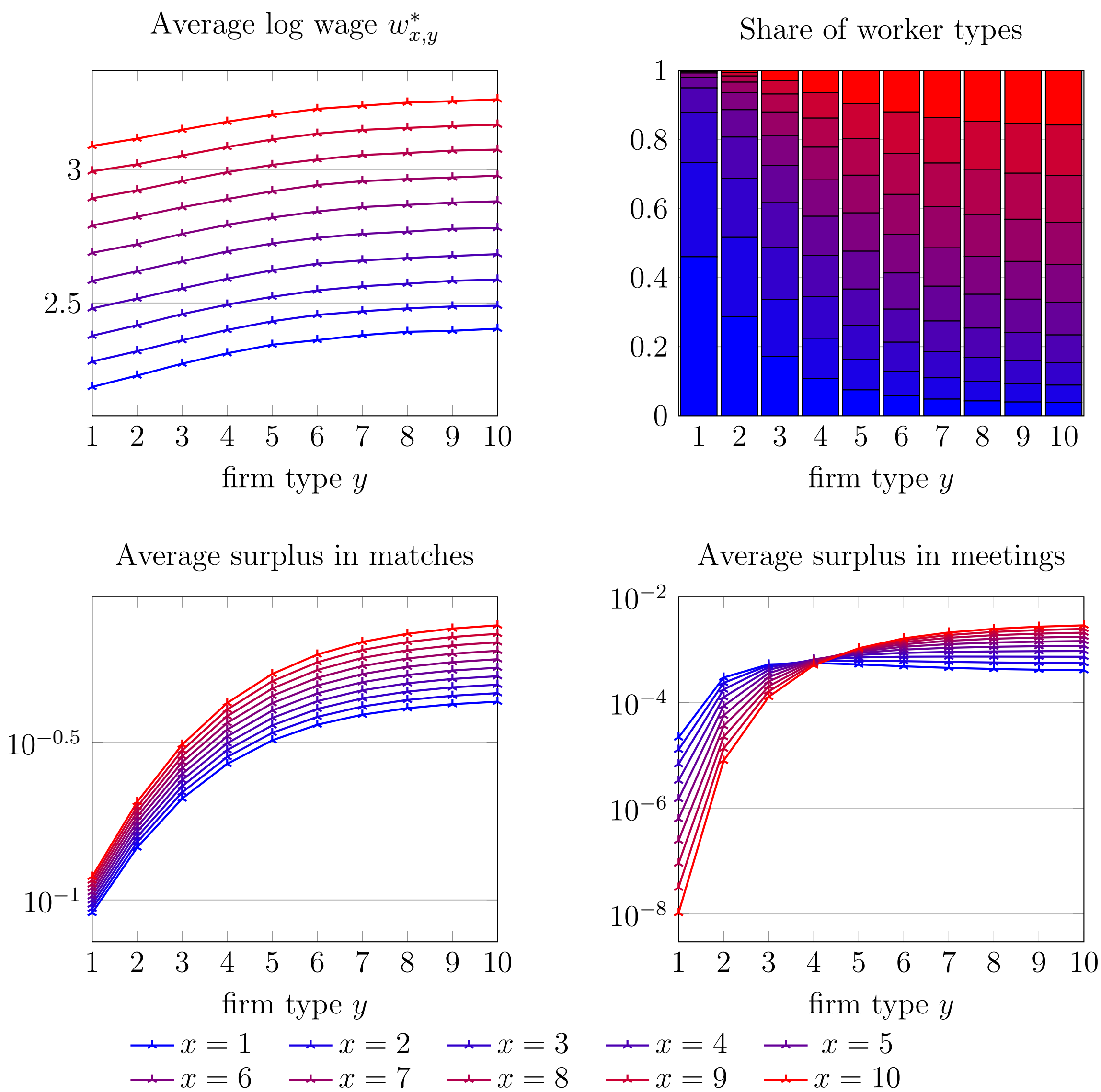

Figure 3: Results for model with exponentially distributed match quality shocks.

E On-the-Job Search

We extend the model to introduce on-the-job search. The goal is to illustrate that it does not change our message regarding selection.

We introduce on-the-job search to our baseline model. We assume that unemployed workers meet a vacancy at the rate ρ_0, employed workers at the rate ρ_1. Search is random. Once a workers and a firm meet, they draw a productivity shock z from a distribution with density $s(z)$ and survival function $S(z)$. We assume that the wage in an (x, y, z) match is $W_{x,y}(z)$, does not depend on whether a worker has been hired from employment or unemployment, and is such that the worker's fraction γ of the joint match surplus.

E.1 Bellman equations

We start by formulating the value functions of a worker with type x. The Bellman equation for the value of being unemployed is

$$rV_x^u = \rho_0 \sum_{y=1}^{Y} v_y \int_0^\infty \max\left\{V_{x,y}^e(z) - V_x^u, 0\right\} s(z)dz. \tag{37}$$

For an employed worker, the Bellman equation is

$$\begin{aligned} rV_{x,y}^e(z) &= W_{x,y}(z) + \delta(V_x^u - V_{x,y}^e(z)) \\ &\quad + \rho_1 \sum_{y'=1}^{Y} v_{y'} \int_0^\infty \max\left\{V_{x,y'}^e(z') - V_{x,y}^e(z), 0\right\} s(z')dz'. \end{aligned} \tag{38}$$

We next formulate Bellman equations for a firm. A vacant job can meet employed or unemployed worker, and the corresponding Bellman equation is

$$\begin{aligned} rV_y^v &= \rho_0 \sum_{x=1}^{X} \int_0^\infty u_x \max\left\{V_{y,x}^f(z)) - V_y^v, 0\right\} s(z)dz \\ &\quad + \rho_1 \sum_{x=1}^{X}\sum_{y'=1}^{Y} \int_0^\infty \phi_{x,y'}(z') \left(\int_0^\infty \max\left\{V_{y,x}^f(z) - V_y^v, 0\right\} \mathbb{1}_{(x,y,z)\succeq(x,y',z')} s(z)dz\right) dz', \end{aligned} \tag{39}$$

where $\phi_{x,y}(z)$ is the measure of (x, y, z) matches, and $\mathbb{1}_{(x,y,z)\succeq(x,y',z')} = 1$ if a worker prefers match (x, y, z) to (x, y', z'), and zero otherwise.

The value of a match to the firm is

$$rV^f_{y,x}(z) = zf_{x,y} - W_{x,y}(z) + \left(\delta + \rho_1 \sum_{y'=1}^{Y} v_{y'} \int_0^\infty \mathbb{1}_{(x,y',z')\succeq(x,y,z)} s(z')dz'\right)\left(V^v_y - V^f_{y,x}(z)\right). \quad (40)$$

As in our benchmark model, we define reservation wages and reservation profits as $\bar{w}_x = rV^u_x$ and $\bar{\pi}_y = rV^v_y$.

The match surplus is defined as $rV^s_{x,y}(z) = rV^e_{x,y}(z) + rV^f_{y,x}(z) - (\bar{w}_x + \bar{\pi}_y)$. Then we have

$$(r+\delta)V^s_{x,y}(z) = zf_{x,y} + + \rho_1 \sum_{y'=1}^{Y} v_{y'} \int_0^\infty \mathbb{1}_{(x,y',z')\succeq(x,y,z)} \left(V^e_{x,y'}(z') - V^u_x - V^s_{x,y}(z)\right) s(z')dz' - (\bar{w}_x + \bar{\pi}_y). \quad (41)$$

We assume that the wage is such that the worker receives fraction γ of the joint surplus, $V^e_{x,y}(z) - V^u_x = \gamma V^s_{x,y}(z)$. An unemployed worker accepts the offer when $rV^e_{x,y}(z) - \bar{w}_x = \gamma r V^s_{x,y}(x) \geq 0$. Similarly, a vacant job wants to match with an unemployed worker if $rV^f_{y,x}(z) - \bar{\pi}_y = (1-\gamma)rV^s_{x,y}(z) \geq 0$. Therefore, a match is created if and only if $V^s_{x,y}(z) \geq 0$, which defines a threshold rule $\bar{z}_{x,y} : V^s_{x,y}(\bar{z}_{x,y}) = 0$.

An employed worker (x,y,z) accepts a new job (x,y',z') when $V^e_{x,y'}(z') \geq V^e_{x,y}(z)$. Using the surplus sharing rule this is equivalent to $V^s_{x,y'}(z') \geq V^s_{x,y}(z)$. It thus follows that $\mathbb{1}_{(x,y',z')\succeq(x,y,z)} = 1$ if and only if $\mathbb{1}_{V^s_{x,y'}(z')\geq V^s_{x,y}(z)} = 1$. This defines a threshold rule (assuming that V^s is increasing in z) $\bar{z}^e_{x,y,z,y'}$ such that $V^s_{x,y}(z) = V^s_{x,y'}(\bar{z}^e_{x,y,z,y'})$ and $V^s_{x,y'}(z') \geq V^s_{x,y}(z)$ for any $z' \geq \bar{z}^e_{x,y,z,y'}$.

Using these results, we derive an equation for the surplus

$$(r+\delta)V^s_{x,y}(z) = zf_{x,y} - (\bar{w}_x + \bar{\pi}_y) + \rho_1 \sum_{y'=1}^{Y} v_{y'} \int_0^\infty \mathbb{1}_{V^s_{x,y'}(z')\geq V^s_{x,y}(z)} \left(\gamma V^s_{x,y'}(z') - V^s_{x,y}(z)\right) s(z')dz'. \quad (42)$$

E.2 Distributions

The steady-state measure of (x, y, z) matches with $V^s_{x,y}(z) \geq 0$ satisfies

$$\phi_{x,y}(z)\left(\delta + \rho_1 \sum_{y'} v_{y'} \int_0^\infty \mathbb{1}_{V^s_{x,y'}(z') \geq V^s_{x,y}(z)} s(z') dz'\right)$$
$$= v_y s(z)\left(\rho_0 u_x \mathbb{1}_{V^s_{x,y}(z)\geq 0} + \rho_1 \sum_{y'=1}^{Y} \int_0^\infty \mathbb{1}_{V^s_{x,y}(z) \geq V^s_{x,y'}(z')} \phi_{x,y'}(z') dz'\right). \quad (43)$$

The left hand side is the measure of matches that are destroyed through exogenous separation and through endogenous on-the-job search. The right hand side counts matches which are created through a vacancy meeting an unemployed worker (first term) or an employed worker who accepts this new offer.

Finally, the measure of unemployed and vacant jobs satisfy

$$u_x = m_x - \sum_{y=1}^{Y} \int_0^\infty \phi_{x,y}(z) dz \quad (44)$$

$$v_y = n_y - \sum_{x=1}^{X} \int_0^\infty \phi_{x,y}(z) dz. \quad (45)$$

E.3 Wage

Consider the value of being employed, equation (38), subtract V^u_x from both sides, and replace worker's surplus with the surplus sharing rule. We obtain

$$(r+\delta)(V^e_{x,y}(z) - V^u_x) = W_{x,y}(z) - \bar{w}_x + \rho_1 \gamma \sum_{y'=1}^{Y} v_{y'} \int_0^\infty \max\left\{V^s_{x,y'}(z') - V^s_{x,y}(z), 0\right\} s(z') dz'.$$

Rearrange the terms to find an expression for wage in terms of V^s and V^u:

$$W_{x,y}(z) = \bar{w}_x + \gamma(r+\delta)V^s_{x,y}(z) - \rho_1 \gamma \sum_{y'=1}^{Y} v_{y'} \int_0^\infty \max\left\{V^s_{x,y'}(z') - V^s_{x,y}(z), 0\right\} s(z') dz'. \quad (46)$$

E.4 Results

We use the same parameter values as in the model without on-the-job search and a Pareto distribution of match-specific productivity shocks (Section 4.1). We set the efficiency of on-the-job search to 30 percent of unemployed search, $\rho_1 = 0.3\rho_0$. This is in line with relative intensities reported in the literature. Using French administrative data, Postel-Vinay and Robin (2002) estimate this ratio to be between 0.31 and 0.48, depending on worker occupation and skill category. Jolivet, Postel-Vinay and Robin (2006) estimate this ratio to be 0.03 and 0.19 for different European countries. We deliberately choose the value closer to the upper bound of this range to show how on-the-job search affects our findingsP. Figure 4 shows the results. They are very similar to the results in Figure 1.

F Empirical Investigation

We analyze matched employer-employee data from Austria. We have two goals. First is to see whether the patterns, that the literature interprets as tests of exogenous mobility, hold in our data as well. The second is to understand the importance of on-the-job search.

F.1 Data Description

We use panel data set from the Austrian social security registry, the Arbeitsmarktdatenbank (AMDB, Labor Market Database), described in Zweimuller, Winter-Ebmer, Lalive, Kuhn, Wuellrich, Ruf and Buchi (2009). The AMDB covers the period from 1986 to 2018. For each worker, the data set contains information about every job they hold. More precisely, in every calendar year and for every worker-firm pair,[15] we observe annual earnings and days worked during the year. We see two sources of earnings, regular wage payments and bonus payments, which we combine together to compute annual earnings. Earnings are top-coded at the maximum social security contribution level, which rises over time.[16] We further observe registered unemployment, maternity and retirement spells. There is limited demographic information on firms and workers, including workers' birth year and sex, and region and industry for firms.

Following Card, Heining and Kline (2013), we focus on workers age 20–60. We do not observe an indicator for part-time jobs which might be a problem for studying women's wages since part-time jobs are prevalent among them. Between 1994 and 2007, on average 4.7 percent of employed men and 34.0 of employed women worked part-time.[17] We therefore focus on men. We drop marginal jobs (less than 10 hours per week) and data that include an apprenticeship.

[15] A firm is identified by its employer identification number (EIN). Some firms may have multiple EINs.

[16] For example, in 2018, the cap for monthly wage earnings is Eur 5,130 and the cap for annual bonus payment is Eur 10,260. The fraction of male worker-firm observations affected by top-coding fell from a peak of 15.2 percent in 1990 to 10.3 percent in 2018. Top-coding affects far fewer female worker-firm observations, varying from 1.7 to 4.8 percent during our sample period.

[17] These statistics come from the Statistical office of Austria, https://www.statistik.at.

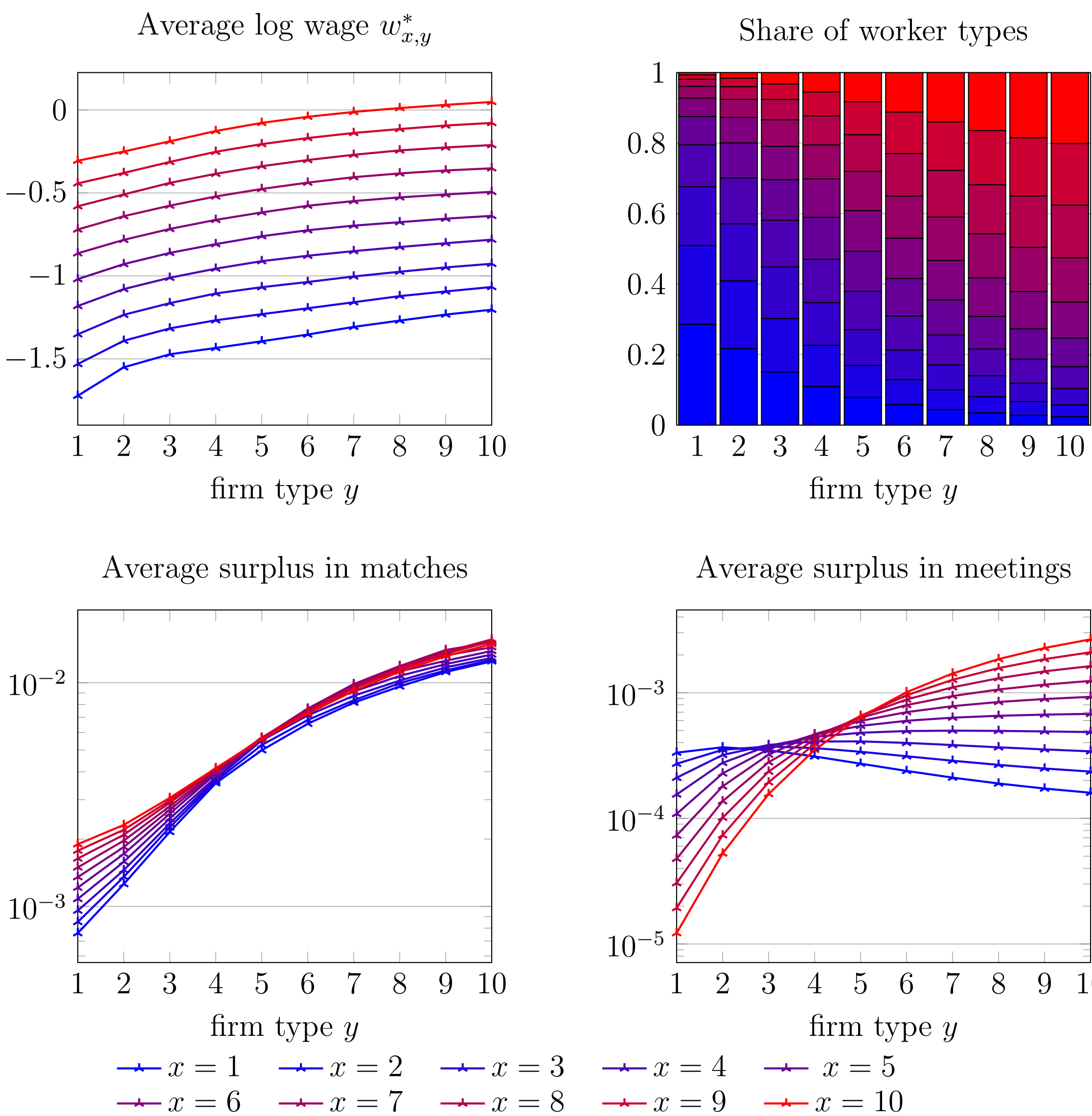

Figure 4: Results for the model with Pareto distributed match quality shocks and on-the-job search. We use the benchmark parametrization and the ratio of search intensity between employed and unemployed $\frac{\rho_1}{\rho_0} = 0.3$.

For each worker-firm-year, we construct a measure of the log real daily wage by taking the difference between log real annual earnings, which is the sum of wages and bonus, and log days worked. We define worker's main job in a calendar year as the one with the highest number of days worked and keep only this job. We then regress this on a full set of dummies for the calendar year and age. The first set of dummies captures the effects of aggregate nominal wage growth, while the second removes a standard age-earnings profile. Our analysis focuses on these wage residuals.

F.2 Patterns in the Data

We start by replicating Figure V in Card, Heining and Kline (2013) which shows average wage gains and loses of workers who move between different employers. We select workers who change their main employer between two calendar years. We further require that the origin and destination firm is the worker's main employer the year before and the year after the move, respectively. We classify firms into quartiles based on co-workers' average wages using the full sample of workers (that is, not only movers). We then study average wage among movers conditional on the quartile of the origin and destination firm.

Figure 5 shows the results which are qualitatively very similar to Card, Heining and Kline (2013). That is, moves from a firm in the lower quartile to a firm in the higher quartile are associated with a wage gain, moves in the opposite direction with a wage loss. Conditional on the quartile of the origin firm, the average wage before the move is increasing with the quartile of the destination firm. There is no pre-trend in wages of job movers in the sense that we do not see wages of workers who ended up doing a downward move going down before the move itself. The literature has interpreted this figure as evidence for exogenous mobility in the sense that there does not seem to be much movements across jobs associated with direct job-to-job movements as those should typically lead to a wage gain.

The wage gains and losses from the movement across quartiles appear to be symmetric. The left panel of Figure 6 shows the wage change associated with a downward move against the wage change associated with the symmetric upward move. That is, for every pair of quartiles (i, j) with $i < j$, we compute the average wage gain when moving from i to j, and the average wage loss associated with a move from j to i. Every mark in Figure 6 represents one such pair (i, j). If the gains and losses are symmetric, these marks would be aligned along the negative 45-degree line, which is the black dashed line in the figure.

The black squares represent all movers. We observe that they are close to the minus 45-degree line, with several squares below the line. This means that the magnitude of the wage loss from a downward move is higher than a wage gain from an upward move. This finding is similar to Figure 5 in Card, Cardoso, Heining and Kline (2018) for Portuguese data.

We next analyze this symmetry for four different age groups defined based on worker's age at the time of the move. We observe that the youngest workers, 20–29 years old, have mostly symmetric wage changes, and if anything, the wage gains from an upward move is higher than the magnitude of a wage loss from a downward move. Workers aged 30–39 are very similar to the full population. The older cohorts, however, lie further below the negative 45-degree line, implying that their wage gains from an upward move are much lower than wage losses from a downward move.

The right panel of Figure 6 shows the number of workers by age group making the upward and downward move for any (i, j) pair of quartiles. The black square represent all movers. They lie a little below the 45-degree line implying that there are more workers moving upward than downward. There are again differences across age groups. Youngest workers are more likely to move up than down, but as age increases, this tends to flip, and workers in the oldest age group make many more downward than upward moves.

To summarize, we observe that there is a life-cycle component in how workers move across different employers. Young workers make more upward than downward moves, and the average wage gains from an upward move are higher than the magnitude of the wage losses from a downward move. For older workers, the wage gain from upward move is lower than the magnitude of wage losses from a downward move, and upward moves are less frequent. This figure suggests that on-the-job search might be important for young workers, but becomes less important in later ages where we observe many moves associated with a wage loss.

We further investigate this hypothesis by analyzing the relationship between the non-employment duration and the wage gain. We regress the wage change after a move on the full set of dummy variables capturing the non-employment gap in months. We find that wage gain is hump-shaped in duration of employment gap, peaking at the duration of 3 months and becoming negative at duration of 6 months. This pattern suggests that we can think of movers with employment gap shorter than 3 months as representing job-to-job moves. About half of all movers have employment gap shorter than 3 months, and 25% of movers have duration between 4 and 6 months.

The relationship between wage change and duration of employment gap changes with age. For each age category and each non-employment duration, we compute the mean wage gain associated with the employer switching. These are depicted in Figure 7. We observe that the wage gains are decreasing in employment gap for every age category, but the decline depends on age. Workers younger than 29 experience large positive wage gains when the

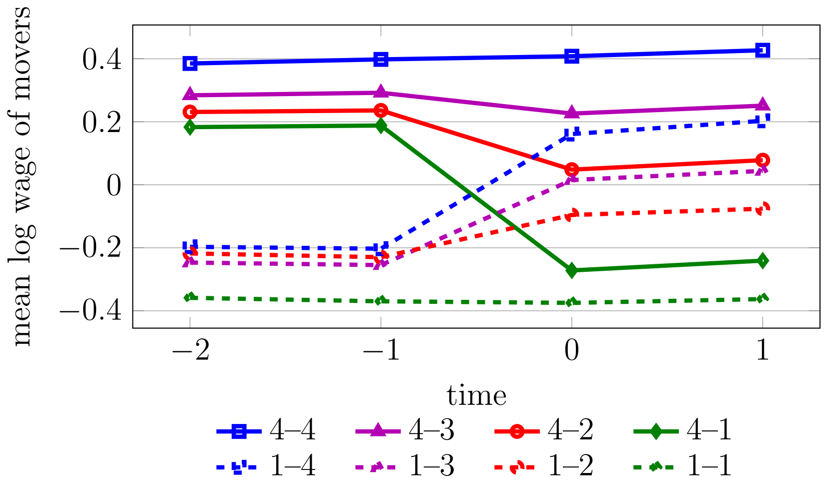

Figure 5: Mean log residual wages of job changers classified by quartile of mean log residual wage of co-workers at origin and destination firm. Log wage residuals are constructed by removing calendar year and age fixed effects.

employment gap is shorter than 6 months, and a small negative loss when the duration is longer than 7 months. As workers age, the gains at short employment gap become smaller and losses at longer employment gap become larger.

The data hence suggest that job-to-job transitions are an important source of wage growth for youngest workers, but not so much for workers older than 30. This provides empirical justification for abstracting from them in the theoretical part.

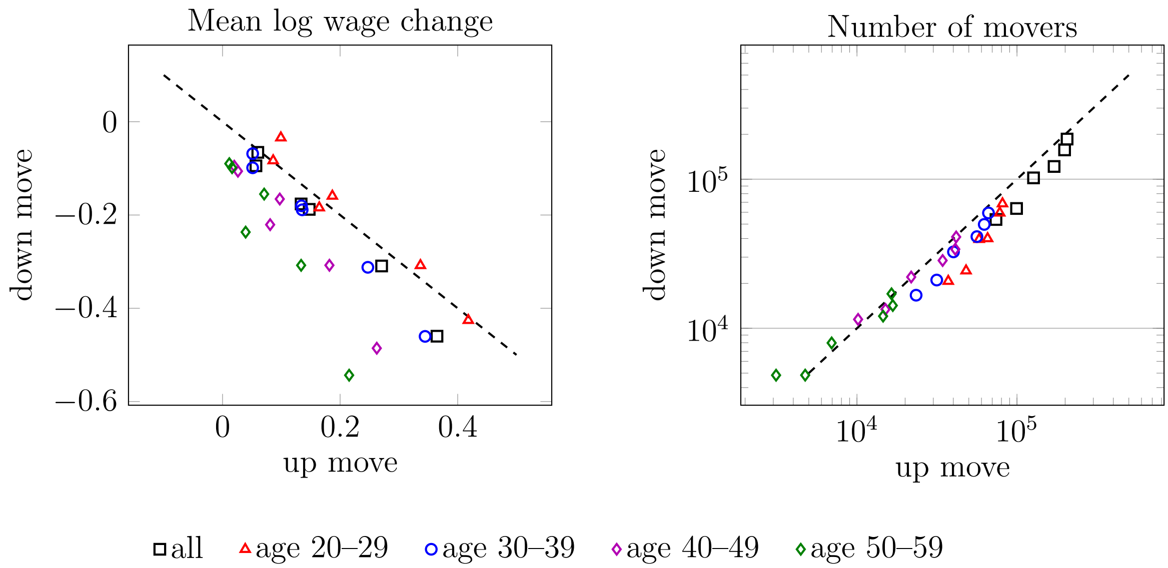

Figure 6: Test of symmetry of log residual wage changes and the number of movers across coworkers' residual log wage quartiles, separately for different age categories. Each dot represents a pair of quartile (i, j) with $i < j$. In the left panel, the horizonal line shows the mean wage growth associated with move from i to j, and the vertical axis the mean wage growth of movers from j to i. The dashed line represents symmetric changes for upward and downward moves. Different colors represent movers of different age group. The right panel is analogous, only showing the number of movers in each (i, j) pair of quartiles.

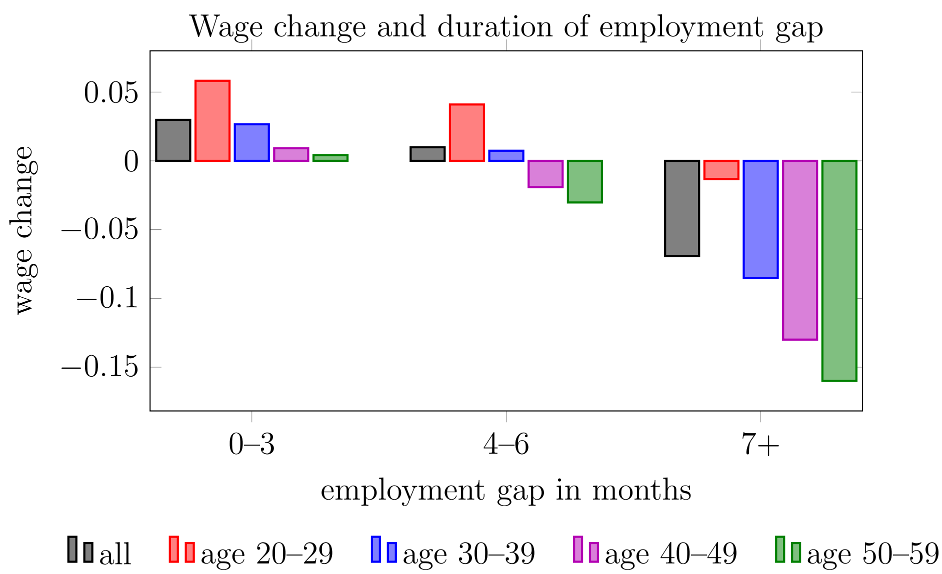

Figure 7: Mean wage growth of job movers for different categories of non-employment duration gap, measured in months.