ON THE KOLMOGOROV-SMIRNOV TEST FOR THE EXPONENTIAL DISTRIBUTION WITH MEAN UNKNOWN

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The standard tables used for the Kolmogorov-Smirnov test are valid when testing whether a set of observations are from a completely specified continuous distribution. If one or more parameters must be estimated from the sample then the tables are no longer valid.

A table is given in this note for use with the Kolmogorov-Smirnov statistic for testing whether a set of observations is from an exponential population when the mean is not specified but must be estimated from the sample. The table is obtained from a Monte Carlo calculation.

THE Kolmogorov-Smirnov statistic provides a means of testing whether a set of observations are from some completely specified continuous distribution, $F_0(X)$. When certain parameters of the distribution must be estimated from the sample, then the Kolmogorov-Smirnov test no longer applies—at least not using the commonly tabulated critical points.

In [2] it is shown that if the parameters estimated are parameters of scale or location, and the estimators satisfy certain general conditions, then when one applies the probability integral transformation, the joint distribution of the transformed variables will not depend on the true parameter values. The distribution will, in general, depend on the functional form of the distribution of the original variables. Thus one can construct tables for use with the Kolmogorov-Smirnov statistic for that particular distribution. In [3] a table is given for use with the Kolmogorov-Smirnov statistic when testing for normality with mean and variance unknown.

This note presents a table for use with the Kolmogorov-Smirnov statistic when testing that a set of observations is from an exponential population but the population mean is not specified. The procedure is: Given a sample of N observations, one determines $D=\max_X|F^*(X)-S_N(X)|$, where $S_N(X)$ is the sample cumulative distribution function and $F^*(X)$ is the cumulative exponential distribution function with $1/\lambda=\overline{X}$, the sample mean. If the value of D exceeds the critical value in the table, one rejects the hypothesis that the observations are from an exponential population.

Critical values for the D statistic were obtained by a Monte Carlo calculation. For all odd values of N between 3 and 19 as well as $N=20$, 25, 30, and 35, 5000 samples were drawn and the distribution of D was thus estimated. The calculations were performed at the George Washington University Computing Center on an IBM 360-40. Using a method provided by one of the referees, the exact critical values were determined for $N=3$. (These values differed by no more than .002 from the Monte Carlo results.)

When the values so obtained were compared with those in the standard table for the Kolmogorov-Smirnov test ([1], [5], [6]) it was found that for a fixed significance level the ratio of the standard values increases smoothly for small

TABLE 1. TABLE OF CRITICAL VALUES OF D

The values of D given in the table are critical values associated with selected values of N. Any value of D which is greater than or equal to the tabulated value is significant at the indicated level of significance. These values were obtained as a result of Monte Carlo calculations, using 5,000 samples for $N=3$ (2) 19, 20, 25, 30, interpolation for $N=4$ (2) 18, and extrapolating (see text) for N over 30.

Sample Size N	Level of Significance for $D=\text{Max}\,\lvert F^*(X)-S_N(X)\rvert$				
	.20	.15	.10	.05	.01
3	.451	.479	.511	.551	.600
4	.396	.422	.449	.487	.548
5	.359	.382	.406	.442	.504
6	.331	.351	.375	.408	.470
7	.309	.327	.350	.382	.442
8	.291	.308	.329	.360	.419
9	.277	.291	.311	.341	.399
10	.263	.277	.295	.325	.380
11	.251	.264	.283	.311	.365
12	.241	.254	.271	.298	.351
13	.232	.245	.261	.287	.338
14	.224	.237	.252	.277	.326
15	.217	.229	.244	.269	.315
16	.211	.222	.236	.261	.306
17	.204	.215	.229	.253	.297
18	.199	.210	.223	.246	.289
19	.193	.204	.218	.239	.283
20	.188	.199	.212	.234	.278
25	.170	.180	.191	.210	.247
30	.155	.164	.174	.192	.226
Over 30	$\frac{.86}{\sqrt{N}}$	$\frac{.91}{\sqrt{N}}$	$\frac{.96}{\sqrt{N}}$	$\frac{1.06}{\sqrt{N}}$	$\frac{1.25}{\sqrt{N}}$

N and is very nearly constant for any value of $N \geq 9$. This fact was used to fill in the rest of the table and to smooth the Monte Carlo results. The quantity tabulated for the "over 30" points was obtained by applying the above mentioned constant ratio to the standard critical values for $n=50$.

Comparing Table 1 with the standard table for the Kolmogorov-Smirnov test for [5], it is seen that the critical values in Table 1 for a .05 significance level are for each value of N about the same as the critical values for a .20 significance level using the standard tables. Thus the result of using the standard table when values of the mean and standard deviation are estimated from the sample would be to obtain an extremely conservative test in the sense that the actual significance level would be much lower than that given by the table.

Table 2 gives the results of some Monte Carlo calculations of the power of the Kolmogorov-Smirnov test using Table 1 for the critical values. The underlying distributions used, the log normal and chi-square with 1 d.f. were chosen because they were somewhat similar in appearance to the exponential.

It would appear that this specialized Kolmogorov-Smirnov test for expo-

TABLE 2

Probability of rejecting hypothesis of exponential distribution using Kolmogorov-Smirnov statistic with Table 1. The numbers are the result of Monte Carlo calculations with 1000 samples for each case.

Underlying Distribution	Critical Level	Sample Size		
		10	20	50
Log normal	.01	.023	.046	.085
	.05	.082	.113	.215
	.10	.153	.204	.431
Chi square, 1 d.f.	.01	.105	.265	.623
	.05	.230	.441	.816
	.10	.383	.569	.929

nentiality should have the same advantages over the chi-square test as does the usual Kolmogorov-Smirnov test when testing for a completely specified distribution. Clearly it provides a test which can be used with sample sizes which are too small for use of the chi-square test.

In [4] results are presented for the power of the chi-square goodness of fit test for the same two alternatives as in Table 2. In either case for sample sizes of 10, 20, and 50, the Kolmogorov-Smirnov test appears to be a more powerful test.

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