1 Probability and independence

a) Decomposition

We aim to validate

$$(X \perp Y, W \mid Z) \implies (X \perp Y \mid Z) \tag{1.1}$$

Proof. We suppose the statement $(X \perp Y,\ W \mid Z)$ is true. It follows from the definition of the conditional independence that $p(x, y, w \mid z) = p(x \mid z)p(y, w \mid z)$ for all $x \in \Omega_x$, $(y, w) \in \Omega_y \times \Omega_w$ and $z \in \Omega_z$. We then consider the marginalize $p(x, y, w \mid z)$:

$$\begin{aligned} p(x, y|z) &= \sum_{w\, \in\, \Omega_w} p(x, y, w \mid z) \\ &= \sum_{w\, \in\, \Omega_w} p(x \mid z)p(y, w \mid z) \\ &= p(x \mid z) \sum_{w\, \in\, \Omega_w} p(y, w \mid z) \\ &= p(x \mid z)p(y \mid z) \end{aligned}$$

from which we conclude that $(X \perp Y \mid Z)$ $\square$. By symmetry of the argument, we can show that $(X \perp W \mid Z)$ is true as well.

b)

We aim to validate

$$(X \perp Y \mid Z) \text{ and } (X, Y \perp W \mid Z) \implies (X \perp W \mid Z) \tag{1.2}$$

Proof. Suppose $(X, Y \perp W \mid Z)$ and $(X \perp Y \mid Z)$ are true. We know from the symmetry and decomposition properties of the conditional independence that $(X, Y \perp W \mid Z) \implies (W \perp X, Y \mid Z) \implies (X \perp W \mid Z)$. Therefore $(X \perp W \mid Z)$ is true $\square$.

c)

We aim to validate

$$(X \perp Y, W \mid Z) \text{ and } (Y \perp W \mid Z) \implies (X, W \perp Y \mid Z) \tag{1.3}$$

Proof. Suppose $(X \perp Y, W \mid Z)$ is true. Then it follows from the definition of conditional independence that

$$p(x, y, w \mid z) = p(x \mid z)p(y, w \mid z)$$

Then assume $(Y \perp W \mid Z)$ is true. The second factor can be factorized

$$p(x, y, w \mid z) = p(x \mid z)p(y \mid z)p(w \mid z)$$

From the decomposition property, we know $(X \perp W \mid Z)$ is true. Thus

$$p(x, y, w \mid z) = p(x, w \mid z)p(y \mid z)$$

From which we conclude $(X, W \perp Y \mid Z)$ is true $\square$.

d)

We aim to validate

$$(X \perp Y \mid Z) \text{ and } (X \perp Y \mid W) \implies (X \perp Y \mid Z, W) \tag{1.4}$$

Counter example. We consider the following R.V.

1. X: Person A arrive late for diner;
2. Y: Person B arrive late for diner;
3. W: They come from the same city;
4. Z: They work in the same city.

For this situation, we see that X and Y are conditionally independent when given either W or Z. If we know they are from the same city, then they might work in different cities and take different route home. Thus knowing person A was late doesn't inform us on the probability of person B to arrive late.

A similar argument can be made for $(X \perp Y \mid Z)$.

Thus the LHS of the proposition is true, yet the RHS is clearly false in our case. Assuming we were given that W, Z are true, this is akin to a geolocalisation of person A and B, so if we were given that person A would be late for diner, then we'd be able to make a good guess that person B would be late as well (they both would be impacted by the traffic jam or whatnot). Thus the proposition is false.

2 Bayesian inferance and MAP