# 1 DGM

We consider $G$, a DAG:

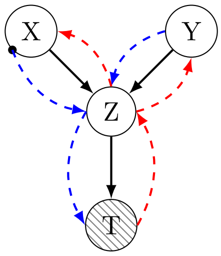



Figure 1: Graph $G$, where $T$ is observed. An active path (blue dashed line) can lead from $X$ to $Y$ since this is an undirected path. The arrow is there to indicate the motion of the Bayes Ball. The starting point of the algorithm is represented as a black dot.

To prove that $X \perp\!\!\!\perp Y \mid T$, we use the Bayes Ball algorithm (see algorithm 1 in the appendix). We conclude that $\boxed{X \not\!\perp\!\!\!\perp Y \mid T}$ because there exists an undirected active path (blue dashed line in figure 1) from $X$ to $Y$. This could also have been observed using the fact that the unobserved node $Z$ with two parents has a descendant that is observed.

A distribution $p \in \mathcal{L}(G)$ satisfy the factorization

$$\mathcal{L}(G) = \left\{ p \mid p(x_V) = \prod_{i=1}^{n} p(x_i \mid x_{\pi_i}) \right\}$$

where $\pi_i$ is the set of all parents of node $i$ and $V$ is the set of vertex in the graph. Therefore, $p \in \mathcal{L}(G)$ must satisfy

$$p(x_V) = p(x)p(y)p(z \mid x, y)p(t \mid z)$$

# 2 D-separation in DGM

We consider the graph $G$:

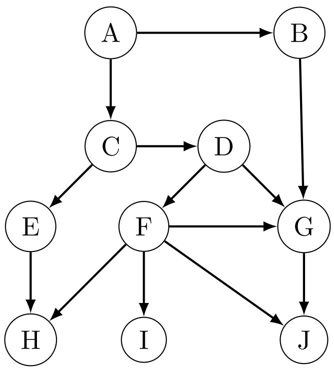



Figure 2: Complete graph $G$.

We are interested in the verification of several conditional independence relations. For each case, we will verify the relation using Bayes Ball algorithm (see algorithm 1). Here, we exploit the fact that unobserved node do not have the ability to bounce the ball when visited by a parent. Therefore, for each cases, we only consider a relevant sub-graph of $G$ built by plucking away all the unobserved nodes until an observed node is found or a node of interest is found.

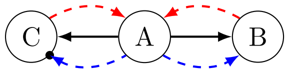
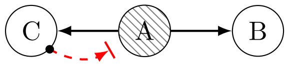
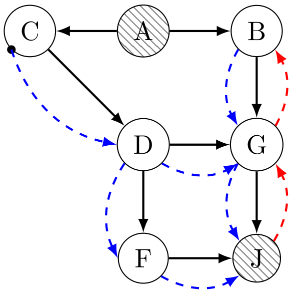
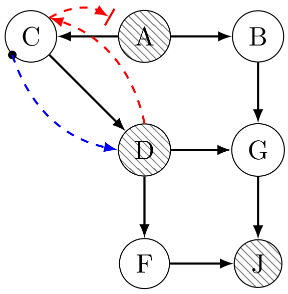
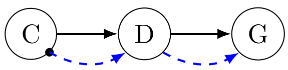
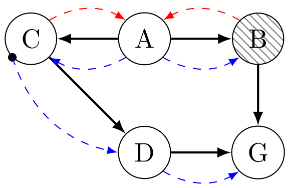
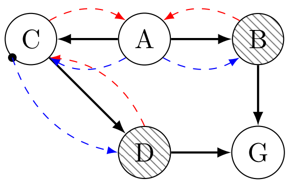
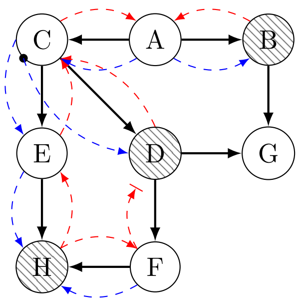
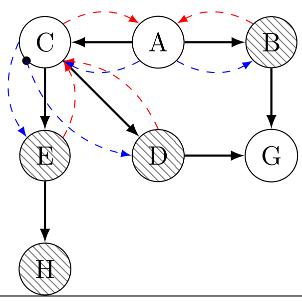
| Relevant sub-graph | Conditional independence | True/False |
|---|---|---|
|  | $C \perp\!\!\!\perp B \mid \emptyset$ | False |
|  | $C \perp\!\!\!\perp B \mid A$ | True |
|  | $C \perp\!\!\!\perp B \mid A, J$ | False |
|  | $C \perp\!\!\!\perp B \mid A, J, D$ | True |
|  | $\mathrm{C} \perp\!\!\!\perp \mathrm{G} \mid \emptyset$ | False |
|  | $C \perp\!\!\!\perp G \mid B$ | False |
|  | $C \perp\!\!\!\perp G \mid B, D$ | True |
|  | $C \perp\!\!\!\perp G \mid B, D, H$ | True |
|  | $C \perp\!\!\!\perp G \mid B, D, H, E$ | True |

Table 1: Conditional independence statements and active trail (blue dashed paths) shown in the relevant path column for questions (a) to (i).

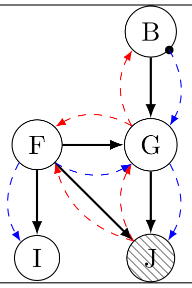
| Relevant sub-graph | Conditional independence | True/False |
|---|---|---|
|  | $B \perp\!\!\!\perp I \mid J$ | False |

Table 2: Conditional independence statement of question (j). In the relevant sub-graph we omitted the node $D$ because we wanted to show the shortest active trail.

# 3 Appendix

## a) Bayes Ball algorithm

Here we focus our attention to the Bayes Ball algorithm for probabilistic node only. In that case, conditional independence $X_J \perp\!\!\!\perp X_L \mid X_K$, with $J, K, L \subseteq V$ requires the notion of d-separation:

**Active path** An active path from $J$ to $L$ given $K$ is an undirected path between $\ell \in L$ and $j \in J$ such that every node $i$ with two parents in this chain is observed ($i \in K$) or has a descendant in $K$ ($i \in K \cap \text{descendant}(i)$)

**D-separation** $X_j$ is said to be conditionally independent to $X_L$ (or *d-separate* $X_L$) given $X_K$ if there is no active path from $J$ to $L$ given $K$.

With this description, we can devise a simple algorithm that will find all active path in the graph in linear time [1]. We will use the following convention

- $\dashrightarrow$ Dashed blue arrows indicate the ball drop from parent to child;
- $\dashrightarrow$ Dashed red arrow indicate the ball bounce back to the parent;
- Barred arrows mean the node will block the ball (neither bounce it back nor let it pass through). These can be used as an indicator and are not necessary for the algorithm.

With this convention, we describe an algorithm that will find all active path from a node $\ell \in L$ to a node $j \in J$ given $K$ if they exists:

**Algorithm 1:** Bayes Ball

**Result:** active paths in the graph $\dashrightarrow$
initialize a schedule with all $j \in J$ as though they were visited by a child;
**while** *schedule not empty* **do**
  pick a node $j$ and remove it from the schedule;
  **if** $j \notin K$ ***and*** $j$ *is visited from a child* ***and*** *there is no* $\dashrightarrow$ *linking* $j$ *to previous node* **then**
    draw $\dashrightarrow$ going into $j$;
    schedule its parents to be visited;
    schedule its children to be visited;
  **else if** $j$ *is visited from a parent* ***and*** *there is no* $\dashrightarrow$ *linking* $j$ *to previous node* **then**
    draw $\dashrightarrow$ going into $j$;
    **if** $j \in K$ **then**
      schedule its parents to be visited;
    **else**
      schedule its children to be visited;

# References

[1] Shachter, R. D. (2013). Bayes-Ball: The Rational Pastime (for Determining Irrelevance and Requisite Information in Belief Networks and Influence Diagrams). http://arxiv.org/abs/1301.7412