A Really Catchy Title for My Technical Report

Technical Report

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Abstract

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In this section, we ...

Lemma 1. *Consider graph $G = (V, E)$ with some vertex $v \in V$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a minimum clique cover of G of size $\theta(G)$. Choose $C_v \in \mathcal{C}$ such that $v \in C_v$. We can construct $\mathcal{C}'$ from $\mathcal{C}$ such that for any $C, \in (\mathcal{C}' \setminus \{C_v\}), C \cap \{v\} = \emptyset$ (i.e., only clique C_v covers v) and $|\mathcal{C}'| = |\mathcal{C}| = \theta(G)$.*

Proof. Let $X \in \mathcal{C}$ be a set of cliques such that for all $C \in X$, $v \in C$. Consider some clique $C_x \in (X \setminus \{C_v\})$. By the definition of a clique, $C'_x = C_x \setminus \{v\}$ denotes some clique that covers all $u \in C_x$ and $u \neq v$.

Let $X' = (X \setminus \{C\}) \cup \{C'\}$, and note that the single removal of C and single addition of C' ensures that $|X| = |X'|$. Since $v \in C_v$, X' covers the same vertices as X.

If we remove v from all $C \in (X \setminus \{C_v\})$, then v is only covered by C_v. Therefore, in $\mathcal{C}' = (\mathcal{C} \setminus \{X\}) \cup \{X'\}$, v is covered by a single clique. Notice that the single removal of C and single addition of C' ensures that $|\mathcal{C}| = |\mathcal{C}'| = \theta(G)$, $\mathcal{C}'$ is a minimum clique cover of G. □

Reduction 1. *If graph $G = (V, E)$ contains a vertex $v \in V$ such that the closed neighborhood of v, denoted $N[v]$, is a clique, then $N[v]$ is in some minimum (vertex) clique cover and can be removed from G to form reduced graph $G' = G[V \setminus N[v]]$, where $\theta(G) = 1 + \theta(G')$.*

Proof. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_{\theta(G)}\}$ be a given minimum clique cover for G. Since $\mathcal{C}$ is a clique cover, there exists some clique in $\mathcal{C}$ that covers v. Let $C_v \in \mathcal{C}$ denote some clique such that $v \in C_v$. Let clique $N[v]$ be denoted by X.

If $X \in \mathcal{C}$, then X is in a minimum clique cover, as desired. If $X \notin \mathcal{C}$, then we argue that we can swap X with C_v and maintain a minimum clique cover of G.

Observe that, by the definition of a clique, C_v consits only of v and vertices $u \in N(v)$. Therefore, $C_v \subset X$. Hence, all $v \in C_v$ are covered by X. Let $\mathcal{C}' = (\mathcal{C} \setminus \{C_v\}) \cup \{X\}$ and notice that the single removal of C_v and single addition of clique X ensures that $|\mathcal{C}'| = |\mathcal{C}| = \theta(G)$. Thus, we have a minimum clique cover $\mathcal{C}'$ such that $X \in \mathcal{C}'$.

If $X = C_v$, then all $v \in X$ reside in a single clique X, as desired. If $X \neq C_v$, then there exists some $u \in N(v)$ such that $u \notin C_v$. Since $\mathcal{C}$ is a clique cover, then there exists some clique in $\mathcal{C}$ that covers u. Then by Lemma 1, we can pick X to cover u such that in clique cover $\mathcal{C}''$, for any $C \in (\mathcal{C}'' \setminus \{X\})$, $u \notin C$ and $|\mathcal{C}''| = |\mathcal{C}'| = \theta(G)$.

Observe that all $v \in X$ reside in a single clique X. Therefore, we can remove X from G to form reduced graph $G' = G[V \setminus N[v]]$, where $\theta(G) = 1 + \theta(G')$, as desired. □

Lemma 2. *Consider a graph $G = (V, E)$. Let $v \in V$ denote a degree two vertex. Let u and w be the verticies in the neighborhood of v, denoted $N(v)$. Let v' be a new vertex denoting he contraction of $\{u, v, w\} \in V$ such that $G' = (V', E')$ with vertex set $V' = (V \setminus \{u, v, w\}) \cup \{v'\}$ and edge set $E' = (E \setminus \{\{x, y\} | x \in \{u, v, w\} \vee y \in \{u, v, w\}\}) \cup \{\{x, v'\} | x \in (N(u) \cup N(w))\}$. Let $\mathcal{C}' = \{C_1, C_2, \ldots, C_{\theta(G')}$ be a minimun clique cover of G'. Let $(C \cup \{v'\}) \in \mathcal{C}'$ denote a clique such that covers v'. Then $(C \cup \{v'\}) \in \mathcal{C}'(G)$ (i.e., the contraction of $\{u, v, w\}$ into v' is a valid reduction), if and only if $(C \cup \{u\}) \in \mathcal{C}(G) \veebar (C \cup \{w\}) \in \mathcal{C}(G)$. Then $\theta(G') = \theta(G') + 1$, and letting $\mathcal{C}(G'')$ be a minimum vertex clique cover of G'', there are two cases:*

(a) if $C \subseteq N(u)$, then $(\mathcal{C}(G') \setminus \{(C \cup \{v'\})\}) \cup \{(C \cup \{u\}), \{v, w\}\}$ is a minimum clique cover of G, and

(b) otherwise $C \subseteq N(w)$, then $(\mathcal{C}(G') \setminus \{(C \cup \{v'\})\}) \cup \{(C \cup \{w\}), \{u, v\}\}$ is a minimum clique cover of G.

Proof. We preform a proof in three parts. First we show that for any clique cover $\mathcal{C}'$ of G', there exists a clique cover $\mathcal{C}$ of G of size $|\mathcal{C}| = |\mathcal{C}' + 1|$. Then we show that $\theta(G) = \theta(G') + 1$. This then implies that if $|\mathcal{C}'| = \theta(G')$, then $|\mathcal{C}| = \theta(G)$. We then conclude by constructing a minimum clique cover of G from a minimum clique cover of G'.

Claim 1. *Let $\mathcal{C}'$ be some clique cover of G', we show there exists a clique cover $\mathcal{C}$ of G with size $|\mathcal{C}| = |\mathcal{C}'| + 1$.*

Let $(C \cup \{v'\}) \in \mathcal{C}'$ denote some clique that contains v'. Note that $N(v') = N(u) \cup N(w)$. Observe that by the definition of a clique, either $C \subseteq N(u)$, $C \subseteq N(w)$, or $C \subseteq (N(u) \cup N(w))$ (i.e., there exists some $x \in C$ such that $x \in (N(u) \cup N(w))$ and $x \notin (N(u) \cap N(w))$).

Case 1 ($C \subseteq N(u)$). Since C is a clique composed of elements from the neighborhood of u, then by the definition of a clique, $C \cup \{u\}$ is also a clique. Observe that since $w \in N(v)$, $\{v, w\}$ is a valid clique. Note that $\mathcal{C}'$ covers all $v \in (V \backslash \{u, v, w\})$. Then, let $\mathcal{C} = (\mathcal{C}' \backslash (C \cup \{v'\})) \cup \{(C \cup \{u\}), \{v, w\}\}$ is a clique cover of G with size $|\mathcal{C}| = |\mathcal{C}'| - 1 + 2 = |\mathcal{C}'| + 1$.

Case 2 ($C \subseteq N(w)$). Then, by a symmetric argument to case 1, $\mathcal{C} = (\mathcal{C}' \setminus (C \cup \{v'\})) \cup \{(C \cup \{w\}), \{u, v\}\}$ is a clique cover of G with size $|\mathcal{C}| = |\mathcal{C}'| + 1$.

Case 3 ($C \subseteq (N(u) \cup N(w))$) In this case, there exists some $x \in C$ such that $x \in N(u)$, $x \notin N(w)$ or $x \in N(w)$, $x \notin N(u)$. C is a valid clique in $\mathcal{C}'$ since $v' \in (N(u) \cap N(w))$, but violates the definition of a clique in G. Therefore, this reduction is valid if and only if $(C \cup \{u\}) \in \mathcal{C}(G) \veebar (C \cup \{w\}) \in \mathcal{C}(G)$.

Claim 2. $\theta(G) = \theta(G') + 1$.

We show that $\theta(G) = \theta(G') + 1$ by showing that $\theta(G) \leq \theta(G') + 1$ and $\theta(G) \geq \theta(G') + 1$.

Case 1 ($\theta(G) \leq \theta(G') + 1$). Let $\mathcal{C}(G')$ be a minimum clique cover of G'. Then by Claim 1, there is a clique cover $\mathcal{C}$ of G of size $|\mathcal{C}| = |\mathcal{C}(G')| + 1 = \theta(G') + 1$. Since any clique cover of G has size at least $\theta(G)$, we have that $\theta(G) \leq |\mathcal{C}| = \theta(G') + 1$.

Case 2 ($\theta(G) \geq \theta(G') + 1$). Let $\mathcal{C}(G)$ be a minimum clique cover of G. Then either $\{(C \cup \{u\}), \{v, w\}\} \in \mathcal{C}(G)$ or $\{(C \cup \{w\}), \{u, v\}\} \in \mathcal{C}(G)$.

- *Case 2a.* Suppose $\{(C \cup \{u\}), \{v, w\}\} \in \mathcal{C}(G)$. Let $X = \mathcal{C}(G) \setminus \{v, w\}$ and observe that $|\mathcal{C}(G)| = |X| + 1$. We note that $\mathcal{C}(G') = ((\mathcal{C}(G) \setminus \{v, w\}) \setminus (C \cup \{u\})) \cup (C \cup \{v'\}))$ is a clique cover of G' with size $|\mathcal{C}(G')| = |\mathcal{C}(G)| - 2 + 1 = |\mathcal{C}(G)| - 1 = |X|$. Therefore $\theta(G) = |\mathcal{C}(G)| = |X| + 1 = |\mathcal{C}(G')| + 1 \geq \theta(G') + 1$.

- *Case 2b.* Suppose $\{(C \cup \{w\}), \{u, v\}\} \in \mathcal{C}(G)$, then we can make a symmetrical argument to case 2a, $\theta(G) \geq \theta(G') + 1$.

We now show how to construct a minimum clique cover of G from a minimum clique cover of G'. Let $\mathcal{C}(G')$ be a minimum clique cover of G'. Applying Claim 1 with $G' = \mathcal{C}(G')$ gives a clique cover C of G of size $|C| = |C'| + 1 = \theta(C') + 1 = \theta(G)$ if and only if $(C \cup \{u\}) \in \mathcal{C}(G) \veebar (C \cup \{w\}) \in \mathcal{C}(G)$, where the last equality is by Claim 2 and thus C is a minimum clique cover of G. Then by the proof of Claim 1 we have that

(a) if $C \subseteq N(u)$, then $(\mathcal{C}(G') \setminus \{(C \cup \{v'\})\}) \cup \{(C \cup \{u\}), \{v, w\}\}$ is a minimum clique cover of G, and

(b) otherwise $C \subseteq N(w)$, then $(\mathcal{C}(G') \setminus \{(C \cup \{v'\})\}) \cup \{(C \cup \{w\}), \{u, v\}\}$ is a minimum clique cover of G.

$\square$

Reduction 2. *Consider a graph $G = (V, E)$. Let $P = \{v_1, v_2, v_\ell\}$ be a set of verticies in a degree two path in G. Let vertices $u, w \in V$, $u, w \notin P$ be endpoints of the path such that $(u, v_1), (v_\ell, w) \in E$. We can reduce G in the following cases:*

(1) If $|P|$ is odd and $u = w$, then we can remove $P \cup \{u\}$ from G to form $G' = G[V \setminus (P \cup \{u\})]$, where $\theta(G) = \theta(G') + \frac{|P|+1}{2}$.

Proof of Reduction 2 (1). Let v denote $v_t \in P$ where $t = \frac{|P|+1}{2}$. Note that since $v \in P$, v is degree two. Let $a, b \in V$ be the neighbors of v such that $(a, v), (v, b) \in E$. Then, by Lemma 2, we can contract $\{a, v, b\}$ into a single vertex v' such that $G' = (V', E')$ with vertex set $V' = (V \setminus \{a, v, b\}) \cup \{v'\}$ and edge set $E' = (E \setminus \{\{x, y\} | x \in \{a, v, b\} \vee y \in \{a, v, b\}\}) \cup \{\{x, v'\} | x \in (N(a) \cup N(b))\}$ and $\theta(G) = \theta(G') + 1$. Then $P' = \{v_1, v_2, \ldots, v_{\ell-2}\}$ with size $|P'| = |P| - 2$.

If we repeatedly apply Lemma 2 in this fashion, until v_t does not satisfy the conditions of Lemma 2, we are left with $G' = (V', E')$ with vertex set $V' = (V \setminus P) \cup \{v'\}$ and edge set $E' = (E \setminus \{\{x, y\} | x \in P \vee y \in P\}) \cup \{\{u, v'\}\}$ since when $|P| = 1$, v is degree one.

To reduce G such that $|P'| = 1$, we apply Lemma 2 $\frac{|P|-1}{2}$ times such that $\theta(G) = \theta(G') + \frac{|P|-1}{2}$. Since the last $v \in P'$ is degree one, by the proof of reduction 1, we can remove u, v from G such that $G'' = G'[V \setminus \{u, v\}]$ where $\theta(G') = \theta(G'') + 1$. Therefore we have $G'' = G[V \setminus (P \cup \{u\})]$ where $\theta(G) = \theta(G') + \frac{|P|-1}{2} + 1 = \theta(G') + \frac{|P|+1}{2}$. $\square$

5 Specific Contribution 2

In this section, we …

5.1 A Subsection Title

6 Experimental Results

In this section…

- item 1
- item 2
- item 3

7 Conclusion

7.1 Contributions

7.2 Future Work

Acknowledgments