A Really Catchy Title for My Technical Report

Technical Report

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Abstract

1 Introduction

Why this is important ...

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Akiba and Iwata [?]...

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4 Specific Contribution 1

In this section, we ...

Lemma 1. *Consider graph $G = (V, E)$ with some vertex $v \in V$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a minimum clique cover of G of size $\theta(G)$. Choose $C_v \in \mathcal{C}$ such that $v \in C_v$. We can construct $\mathcal{C}'$ from $\mathcal{C}$ such that for any $C, \in (\mathcal{C}' \setminus \{C_v\}), C \cap \{v\} = \emptyset$ (i.e., only clique C_v covers v) and $|\mathcal{C}'| = |\mathcal{C}| = \theta(G)$.*

Proof. Let $X \in \mathcal{C}$ be a set of cliques such that for all $C \in X$, $v \in C$. Consider some clique $C_x \in (X \setminus \{C_v\})$. By the definition of a clique, $C'_x = C_x \setminus \{v\}$ denotes some clique that covers all $u \in C_x$ and $u \neq v$.

Let $X' = (X \setminus \{C\}) \cup \{C'\}$, and note that the single removal of C and single addition of C' ensures that $|X| = |X'|$. Since $v \in C_v$, X' covers the same vertices as X.

If we remove v from all $C \in (X \setminus \{C_v\})$, then v is only covered by C_v. Therefore, in $\mathcal{C}' = (\mathcal{C} \setminus \{X\}) \cup \{X'\}$, v is covered by a single clique. Notice that the single removal of C and single addition of C' ensures that $|\mathcal{C}| = |\mathcal{C}'| = \theta(G)$, $\mathcal{C}'$ is a minimum clique cover of G. $\square$

Reduction 1. *If graph $G = (V, E)$ contains a vertex $v \in V$ such that the closed neighborhood of v, denoted $N[v]$, is a clique, then $N[v]$ is in some minimum (vertex) clique cover and can be removed from G to form reduced graph $G' = G[V \setminus N[v]]$, where $\theta(G) = 1 + \theta(G')$.*

Proof. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_{\theta(G)}\}$ be a given minimum clique cover for G. Since $\mathcal{C}$ is a clique cover, there exists some clique in $\mathcal{C}$ that covers v. Let $C_v \in \mathcal{C}$ denote some clique such that $v \in C_v$. Let clique $N[v]$ be denoted by X.

If $X \in \mathcal{C}$, then X is in a minimum clique cover, as desired. If $X \notin \mathcal{C}$, then we argue that we can swap X with C_v and maintain a minimum clique cover of G.

Observe that, by the definition of a clique, C_v consits only of v and vertices $u \in N(v)$. Therefore, $C_v \subset X$. Hence, all $v \in C_v$ are covered by X. Let $\mathcal{C}' = (\mathcal{C} \setminus \{C_v\}) \cup \{X\}$ and notice that the single removal of C_v and single addition of clique X ensures that $|\mathcal{C}'| = |\mathcal{C}| = \theta(G)$. Thus, we have a minimum clique cover $\mathcal{C}'$ such that $X \in \mathcal{C}'$.

If $X = C_v$, then all $v \in X$ reside in a single clique X, as desired. If $X \neq C_v$, then there exists some $u \in N(v)$ such that $u \notin C_v$. Since $\mathcal{C}$ is a clique cover, then there exists some clique in $\mathcal{C}$ that covers u. Then by Lemma 1, we can pick X to cover u such that in clique cover $\mathcal{C}''$, for any $C \in (\mathcal{C}'' \setminus \{X\})$, $u \notin C$ and $|\mathcal{C}''| = |\mathcal{C}'| = \theta(G)$.

Observe that all $v \in X$ reside in a single clique X. Therefore, we can remove X from G to form reduced graph $G' = G[V \setminus N[v]]$, where $\theta(G) = 1 + \theta(G')$, as desired. $\square$

Lemma 2. *Consider a graph $G = (V, E)$ with minimum degree two. Let $P = \{v_1, v_2, \ldots, v_\ell\}$ be the set of vertices in a maximal, degree two path in G such that $|P| \geq 5$ is odd. Let $v_a = v_{\ell-4}$, $v_b = v_{\ell-3}$, $v_c = v_{\ell-2}$, $v_d = v_{\ell-1}$, $v_e = v_\ell$ be the last five vertices in P. Then, we can contract $\{v_b, v_c, v_d\} \in P$ into a single vertex v', giving us a new graph $G' = (V', E')$ with vertex set $V' = (V \setminus \{v_b, v_c, v_d\}) \cup \{v'\}$ and edge set $E' = (E \setminus \{\{x, y\} | x \in \{v_b, v_c, v_d\} \vee y \in \{v_b, v_c, v_d\}\}) \cup \{\{x, v'\} | x \in \{v_a, v_e\}\}$. Then $\theta(G) = \theta(G') + 1$, and letting $\mathcal{C}(G')$ be a minimum vertex clique cover of G', we have two cases:*

- *(a) if $\{v_a, v'\} \in \mathcal{C}(G')$, then $(\mathcal{C}(G') \setminus \{\{v_a, v'\}\}) \cup \{\{v_a, v_b\}, \{v_c, v_d\}\}$ is a minimum clique cover of G, and*
- *(b) otherwise $\{v', v_e\} \in \mathcal{C}(G')$, and $(\mathcal{C}(G') \setminus \{\{v, v_e\}\}) \cup \{\{v_b, v_c\}, \{v_d, v_e\}\}$ is a minimum clique cover of G.*

Proof. We perform a proof in three parts. First, we show that for any clique cover $\mathcal{C}'$ of G', there is a clique cover $\mathcal{C}$ of G of size $|\mathcal{C}| = |\mathcal{C}'| + 1$. Then we show that $\theta(G) = \theta(G') + 1$. This then implies that if $|C'| = \theta(G)$ (i.e., C' is a minimum clique cover of G'), then $|I| = \theta(G)$ (i.e., C is a minimum clique cover of G). We then conclude by constructing a minimum clique cover of G from a minimum clique cover of G'.

Claim 1. *Let C' be a clique cover of of G', we show there exists a clique cover C of G with size $|C| = |C'| + 1$.*

There are two cases.

Case 1 ($\{v_a, v'\} \in C'$). Then v_a is in some clique $\{v_a, x\}$ where $x \in v'$. Since P is degree two, only $v_b \in v'$ is in the neighborhood of v_a. Thus $\{v_a, v_b\}, \{v_c, v_d\} \in C$. Therefore, $C = (C' \setminus \{\{v_a, v'\}\}) \cup \{\{v_a, v_b\}, \{v_c, v_d\}\}$ is a clique cover of G with size $|C| = |C'| - 1 + 2 = |C'| + 1$.

Case 2 ($\{v', v_e\} \in C'$). Since P is degree two, we can make a symmetric argument to case one, $|C| = |C'| + 1$.

Claim 2. $\theta(G) = \theta(G') + 1$.

We show that $\theta(G) = \theta(G') + 1$ by showing that $\theta(G) \leq \theta(G') + 1$ and $\theta(G) \geq \theta(G') + 1$.

Case 1 ($\theta(G) \leq \theta(G') + 1$). Let $C(G')$ be a minimum clique cover of G'. Then by Claim 1, there is a clique cover C of G of size $|C| = |C(G')| + 1 = \theta(G') + 1$. Since any clique cover of G has size at least $\theta(G)$, we have that $\theta(G) \leq |C| = \theta(G') + 1$.

Case 2 ($\theta(G) \geq \theta(G') + 1$). Let $C(G)$ be a minimum clique cover of G. Then either $\{v_a, v_b\} \in C(G)$ or $\{v_d, v_e\} \in C(G)$.

- *Case 2a.* Suppose $\{v_a, v_b\} \in C(G)$. Then, we have that $\{v_c, v_d\} \in C(G)$. Let $X = C(G) \setminus \{v_c, v_d\}$ and observe that $|C(G)| = |X|+1$. We note that $C(G') = ((C(G) \setminus \{v_c, v_d\}) \setminus \{v_a, v_b\}) \cup \{v_a, v'\})$ is a clique cover of G' with size $|C(G')| = |C(G)|-2+1 = |C(G)|-1 = |X|$. Therefore $\theta(G) = |C(G)| = |X| + 1 = |C(G')| + 1 \geq \theta(G') + 1$.

- *Case 2b.* Suppose $\{v_d, v_e\} \in C(G)$, then we can make a symmetrical argument to case 2a, $\theta(G) \geq \theta(G') + 1$.

We now show how to construct a minimum clique cover of G from a minimum clique cover of G'. Let $C(G')$ be a minimum clique cover of G'. Applying Claim 1 with $G' = C(G')$ gives a clique cover C of G of size $|C| = |C'| + 1 = \theta(C') + 1 = \theta(G)$, where the last equality is by Claim 2 and thus C is a minimum clique cover of G. Then by the proof of Claim 1 we have that

(a) if $\{v_a, v'\} \in C(G')$, then $(C(G') \setminus \{v_a, v'\}) \cup \{\{v_a, v_b\}, \{v_c, v_d\}\}$ is a minimum clique cover of G, and

(b) otherwise $\{v', v_e\} \in C(G')$, and $(C(G') \setminus \{v, v_e\}) \cup \{\{v_b, v_c\}, \{v_d, v_e\}\}$ is a minimum clique cover of G.

□

Lemma 3. *Consider a graph $G = (V, E)$ with minimum degree two. Let $P = \{v_1, v_2, \ldots, v_\ell\}$ be the set of vertices in a maximal, degree two path in G such that $|P| \geq 6$ is even. Let $v_a = v_{\ell-5}$, $v_b = v_{\ell-4}$, $v_c = v_{\ell-3}$, $v_d = v_{\ell-2}$, $v_e = v_{\ell-1}$, $v_f = v_\ell$ be the last six vertices in P. Then, we can contract $\{v_b, v_c, v_d, v_e\} \in P$ into two vertices v'_1 and v'_2, giving us a new graph $G' = (V', E')$ with* *vertex set* $V' = (V \setminus \{v_b, v_c, v_d, v_e\}) \cup \{v_1', v_2'\}$ *and edge set* $E' = (E \setminus \{\{x, y\} | x \in \{v_b, v_c, v_d, v_e\} \vee y \in \{v_b, v_c, v_d, v_e\}\}) \cup \{\{v_a, v_1'\}, \{v_1', v_2'\}, \{v_2', v_f\}\}$. *Then* $\theta(G) = \theta(G') + 1$, *and letting* $C(G')$ *be a minimum vertex clique cover of* G', *we have three cases:*

(a) *if* $\{v_1', v_2'\} \in C(G')$, *then* $(C(G') \setminus \{v_1', v_2'\}) \cup \{\{v_b, v_c\}, \{v_d, v_e\}\}$ *is a minimum clique cover of* G, *and*

(b) *otherwise* $\{v_a, v_1'\}, \{v_2', v_f\} \in C(G')$, *then* $(C(G') \setminus \{\{v_a, v_1'\}, \{v_2', v_f\}\}) \cup \{\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\}\}$ *is a minimum clique cover of* G.

Proof. We perform a proof in three parts. First, we show that for any clique cover C' of G', there is a clique cover C of G of size $|C| = |C'| + 1$. Then we show that $\theta(G) = \theta(G') + 1$. This then implies that if $|C'| = \theta(G)$ (i.e., C' is a minimum clique cover of G'), then $|I| = \theta(G)$ (i.e., C is a minimum clique cover of G). We then conclude by constructing a minimum clique cover of G from a minimum clique cover of G'.

Claim 1. *Let* C' *be a clique cover of of* G', *we show there exists a clique cover* C *of* G *with size* $|C| = |C'| + 1$.

There are two cases.

Case 1 ($\{v_1', v_2'\} \in C'$). Then $v_a, v_f \ni n\{v_1', v_2'\}$, then some minimum clique cover of $\{v_b, v_c, v_d, v_e\} \in C$. Since P is degree two, there is a single unique minimum clique cover of $\{v_b, v_c, v_d, v_e\}$. Thus $\{v_b, v_c\}, \{v_d, v_e\} \in C$. Therefore, $C = (C' \setminus \{\{v_1', v_2'\}\}) \cup \{\{v_b, v_c\}, \{v_d, v_e\}\}$ is a clique cover of G with size $|C| = |C'| - 1 + 2 = |C'| + 1$.

Case 2 ($\{v_a, v_1'\}, \{v_f, v_2'\} \in C'$). Then v_a is in some clique $\{v_a, x\}$ where $x \in v_1'$ and v_f is in some clique $\{y, v_f\}$ where $y \in v_2'$. Since P is degree two, only $v_b \in v_1'$ is in the neighborhood of v_a and only $v_e \in v_2'$ is in the neighborhood of v_f. Thus $\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\} \in C$. Therefore, $C = (C' \setminus \{\{v_a, v_1'\}, \{v_2', v_f\}\}) \cup \{\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\}\}$ is a clique cover of G with size $|C| = |C'| - 2 + 3 = |C'| + 1$.

Claim 2. $\theta(G) = \theta(G') + 1$.

We show that $\theta(G) = \theta(G') + 1$ by showing that $\theta(G) \leq \theta(G') + 1$ and $\theta(G) \geq \theta(G') + 1$.

Case 1 ($\theta(G) \leq \theta(G') + 1$). Let $C(G')$ be a minimum clique cover of G'. Then by Claim 1, there is a clique cover C of G of size $|C| = |C(G')| + 1 = \theta(G') + 1$. Since any clique cover of G has size at least $\theta(G)$, we have that $\theta(G) \leq |C| = \theta(G') + 1$.

Case 2 ($\theta(G) \geq \theta(G') + 1$). Let $C(G)$ be a minimum clique cover of G. Then either $\{v_b, v_c\}, \{d, e\} \in C(G)$ or $\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\} \in C(G)$.

- *Case 2a.* Suppose $\{v_b, v_c\}, \{d, e\} \in C(G)$. Let $X = C(G) \setminus \{\{v_c, v_d\}\}$ and observe that $|C(G)| = |X| + 1$. We note that $C(G') = (C(G) \setminus \{\{v_b, v_c\}, \{v_d, v_e\}\}) \cup \{v_1', v_2'\})$ is a clique cover of G' with size $|C(G')| = |C(G)| - 2 + 1 = |C(G)| - 1 = |X|$. Therefore $\theta(G) = |C(G)| = |X| + 1 = |C(G')| + 1 \geq \theta(G') + 1$.

- *Case 2b.* Suppose $\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\} \in C(G)$. Let $X = C(G) \setminus \{\{v_c, v_d\}, \{v_e, v_f\}\}$ and observe that $|C(G)| = |X| + 2$. We note that $C(G') = (C(G) \setminus \{\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\}\}) \cup$

$\{\{v_a, v_1'\}, \{v_2', v_f\}\}$ is a clique cover of G' with size $|C(G')| = |C(G)| - 3 + 2 = |C(G)| - 1 = |X|$. Therefore $\theta(G) = |C(G)| = |X| + 1 = |C(G')| + 1 \geq \theta(G') + 1$.

We now show how to construct a minimum clique cover of G from a minimum clique cover of G'. Let $C(G')$ be a minimum clique cover of G'. Applying Claim 1 with $G' = C(G')$ gives a clique cover C of G of size $|C| = |C'| + 1 = \theta(C') + 1 = \theta(G)$, where the last equality is by Claim 2 and thus C is a minimum clique cover of G. Then by the proof of Claim 1 we have that

(a) if $\{v_1', v_2'\} \in C(G')$, then $(C(G') \setminus \{v_1', v_2'\}) \cup \{\{v_b, v_c\}, \{v_d, v_e\}\}$ is a minimum clique cover of G, and

(b) otherwise $\{v_a, v_1'\}, \{v_2', v_f\} \in C(G')$, then $(C(G') \setminus \{\{v_a, v_1'\}, \{v_2', v_f\}\}) \cup \{\{v_a, v_b\}, \{v_c, v_d\}, \{v_e, v_f\}\}$ is a minimum clique cover of G.

□

5 Specific Contribution 2

In this section, we …

5.1 A Subsection Title

6 Experimental Results

In this section…

- item 1
- item 2
- item 3

7 Conclusion

7.1 Contributions

7.2 Future Work

Acknowledgments