

CSC477

Introduction to Mobile Robotics

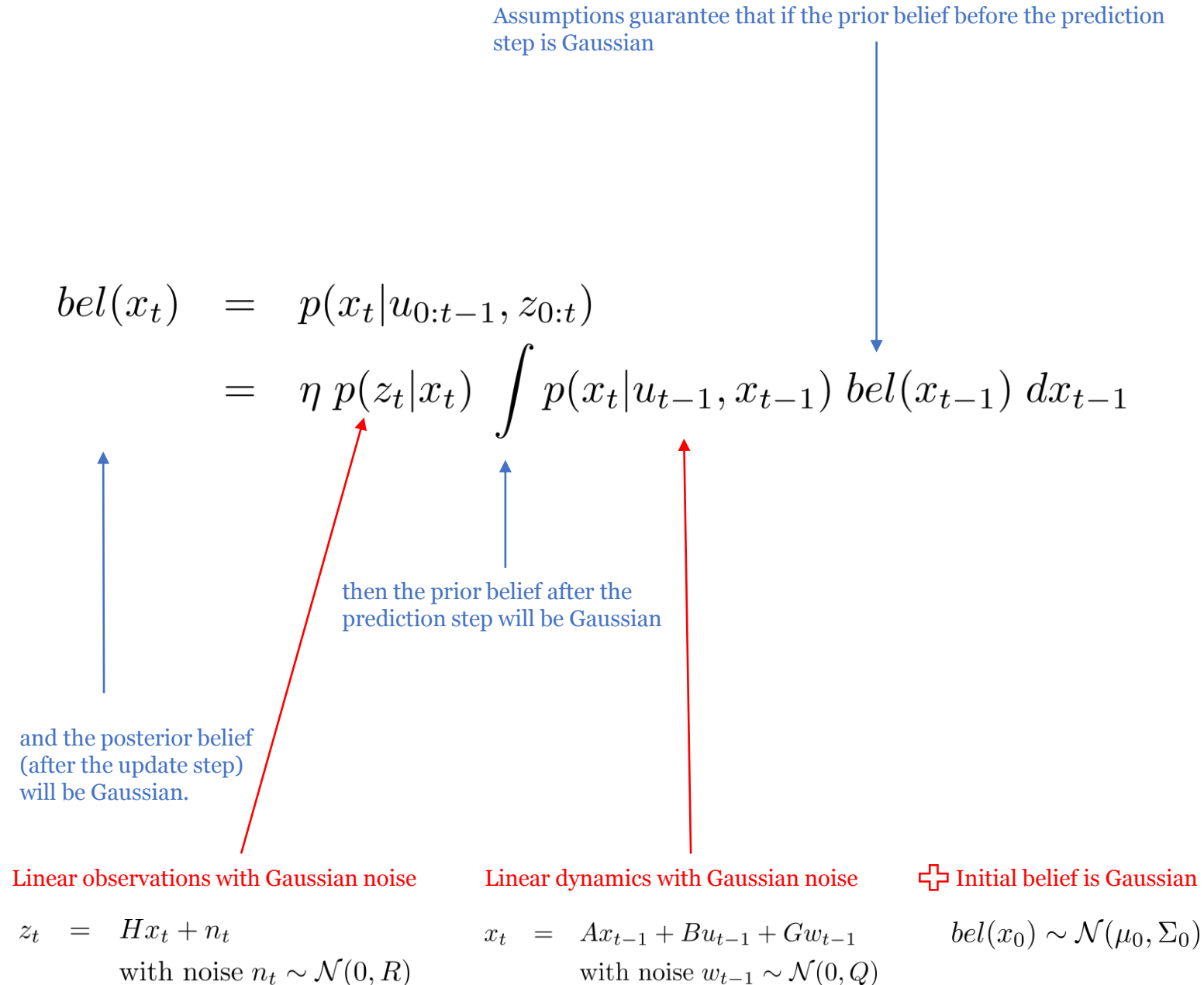
Florian Shkurti

Week #9: Extended Kalman Filter

Recommended reading

- Chapter 3.3 in Probabilistic Robotics for EKF
- Chapter 7.4 in Probabilistic Robotics for EKF-Localization
- Chapter 10.2 in Probabilistic Robotics for EKF-SLAM

Kalman Filter: an instance of Bayes' Filter



Kalman Filter: an instance of Bayes' Filter

Suppose you replace
the linear models with
nonlinear models.

Does the posterior $bel(x_t)$ remain
Gaussian?

$$\begin{aligned} bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\ &= \eta p(z_t | x_t) \int p(x_t | u_{t-1}, x_{t-1}) bel(x_{t-1}) dx_{t-1} \end{aligned}$$

Assumptions guarantee that if the prior belief before the prediction step is Gaussian

then the prior belief after the prediction step will be Gaussian

and the posterior belief
(after the update step)
will be Gaussian.

Linear observations with Gaussian noise

$$\begin{aligned} z_t &= h(x_t) + n_t \\ &\text{with noise } n_t \sim \mathcal{N}(0, R) \end{aligned}$$

Linear dynamics with Gaussian noise

$$\begin{aligned} x_t &= f(x_{t-1}, u_{t-1}) + Gw_{t-1} \\ &\text{with noise } w_{t-1} \sim \mathcal{N}(0, Q) \end{aligned}$$

⊕ Initial belief is Gaussian

$$bel(x_0) \sim \mathcal{N}(\mu_0, \Sigma_0)$$

Kalman Filter: an instance of Bayes' Filter

Suppose you replace
the linear models with
nonlinear models.

Does the posterior $bel(x_t)$ remain
Gaussian? **NO**

$$\begin{aligned} bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\ &= \eta p(z_t | x_t) \int p(x_t | u_{t-1}, x_{t-1}) bel(x_{t-1}) dx_{t-1} \end{aligned}$$

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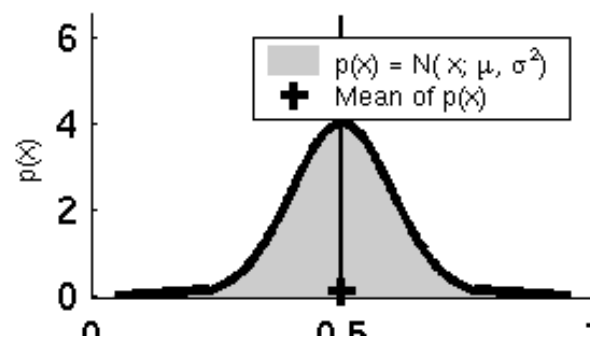
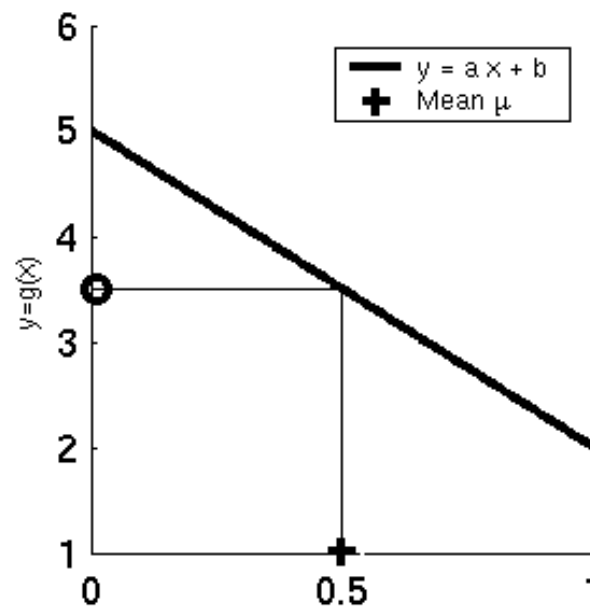
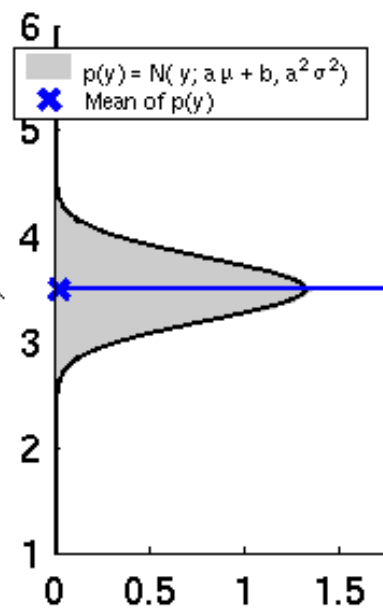
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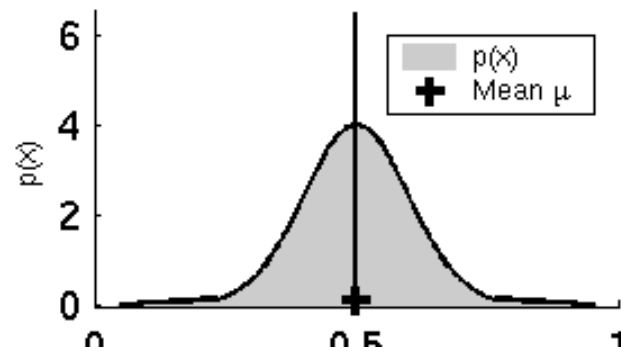
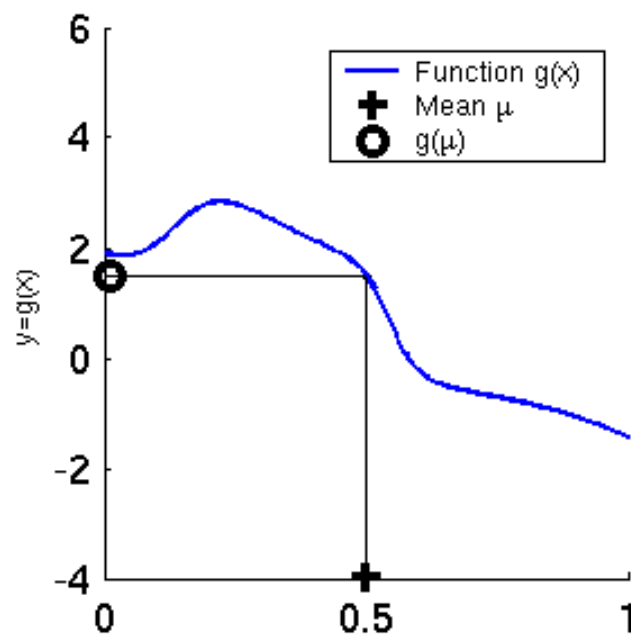
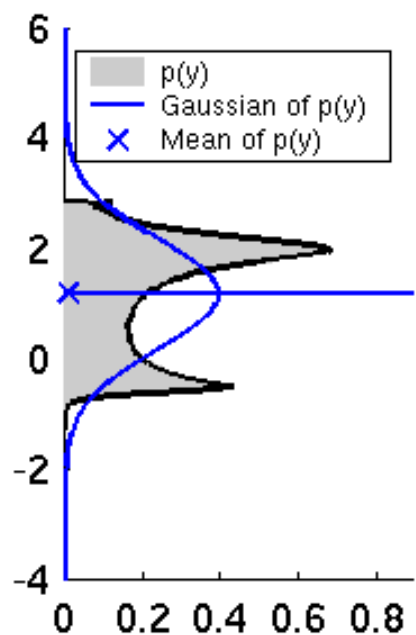
Linearity Assumption Revisited



If $y = ax + b$
and $x \sim \mathcal{N}(\mu, \sigma^2)$

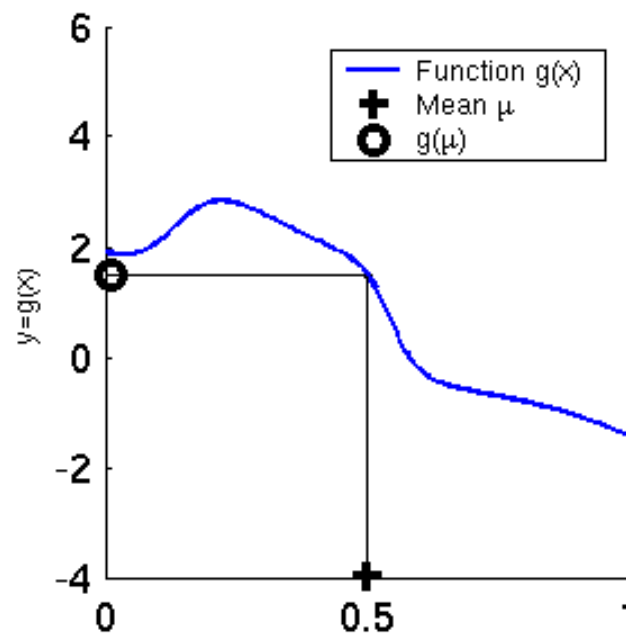
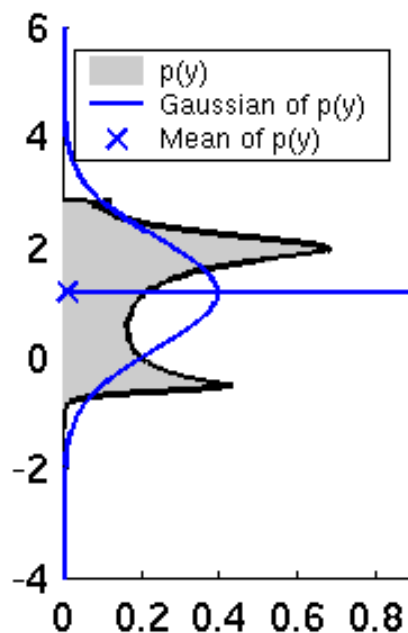
then $y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Nonlinear Function



If $y = g(x)$
and $x \sim \mathcal{N}(\mu, \sigma^2)$
then y is not necessarily
distributed as a Gaussian.

Nonlinear Function

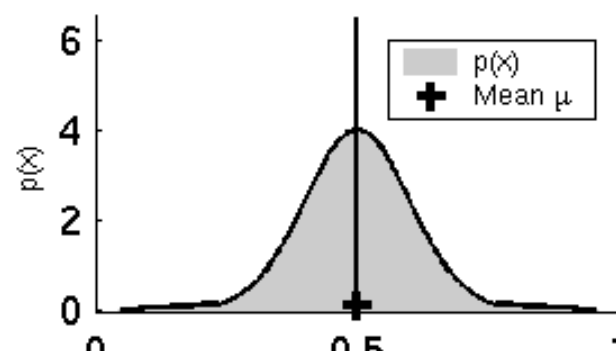


If $y = g(x)$
and $x \sim \mathcal{N}(\mu, \sigma^2)$
then y is not necessarily
distributed as a Gaussian.

How can we approximate $p(y)$ using a single Gaussian, without having a formula for $p(y)$?

IDEA #1: MONTE CARLO SAMPLING

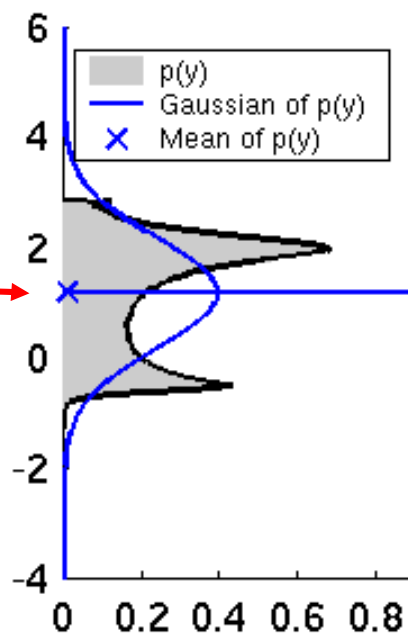
- Draw many (e.g. 10^6) samples $x_i \sim p(x)$
- Pass them through the nonlinear function $y_i = g(x_i)$
- Compute the empirical mean (m) and covariance (S) of the samples y_i
- Return $\text{Normal}(m, S)$



IDEA #2: LINEARIZE THE NONLINEAR FUNCTIONS f, h

- Then we are in the case $y = Gx + c$, so $p(y)$ is a Gaussian

Nonlinear Function



This is how we computed the Gaussian approximation of $p(y)$.

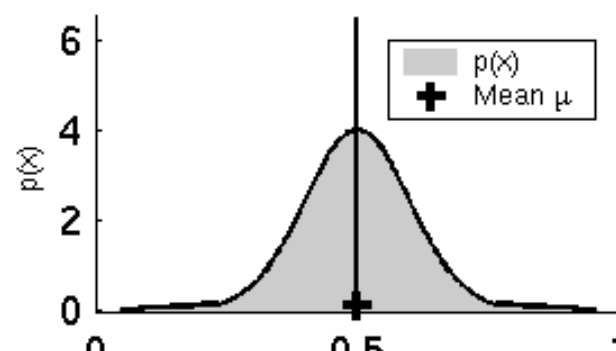
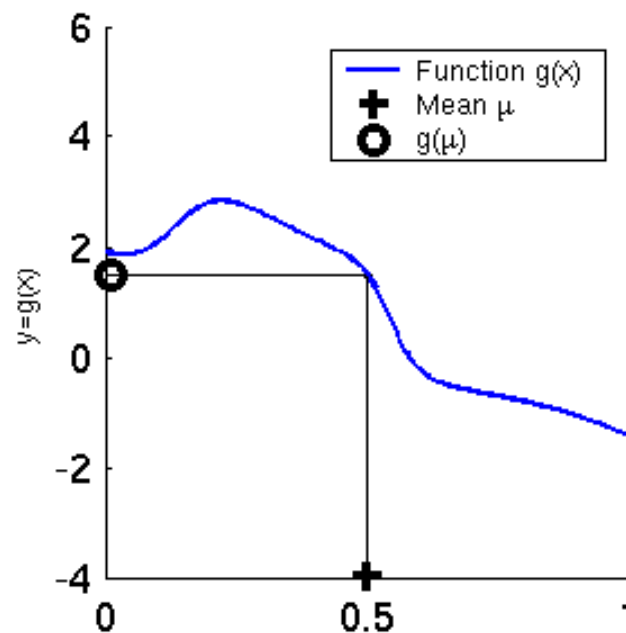
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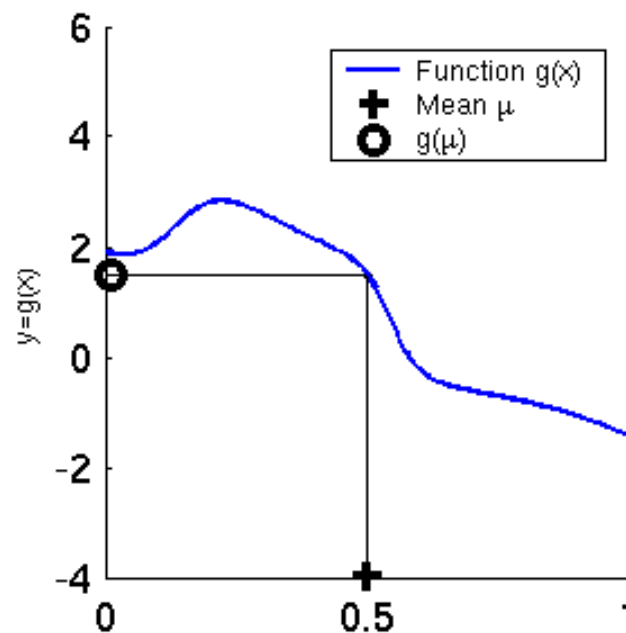
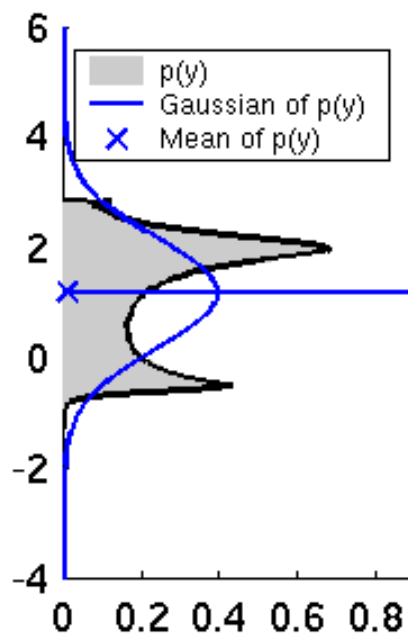
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Nonlinear Function



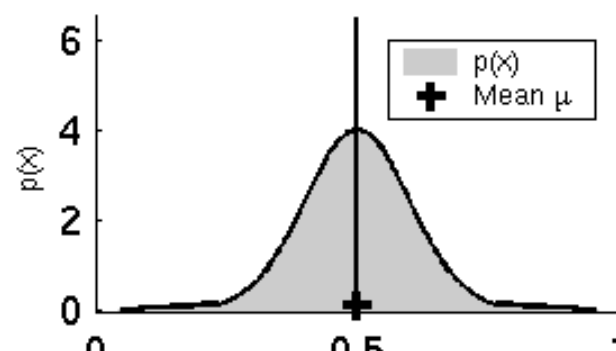
If $y = g(x)$
and $x \sim \mathcal{N}(\mu, \sigma^2)$
then y is not necessarily
distributed as a Gaussian.

Bad idea to use in Kalman Filter
because we want to approximate
 $p(y)$ efficiently.

How can we approximate $p(y)$ using a single Gaussian, without having
a formula for $p(y)$?

IDEA #1: MONTE CARLO SAMPLING

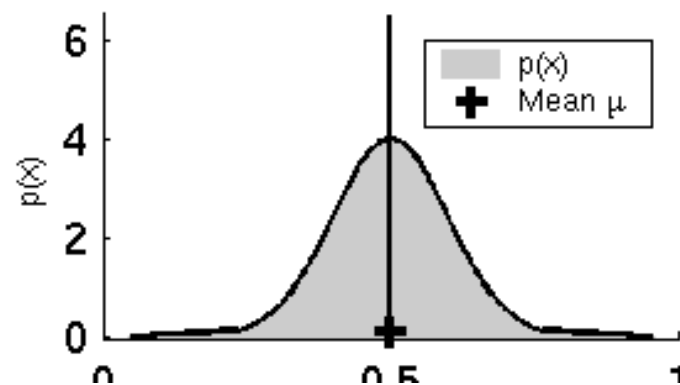
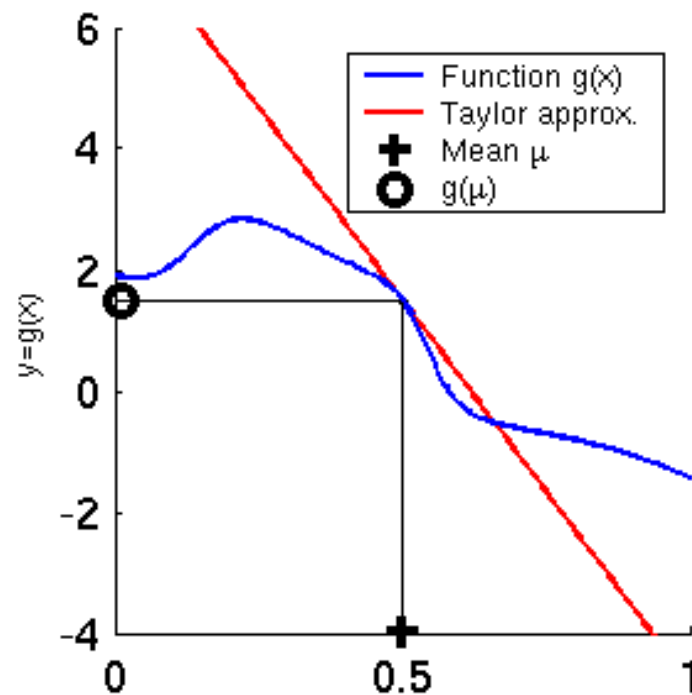
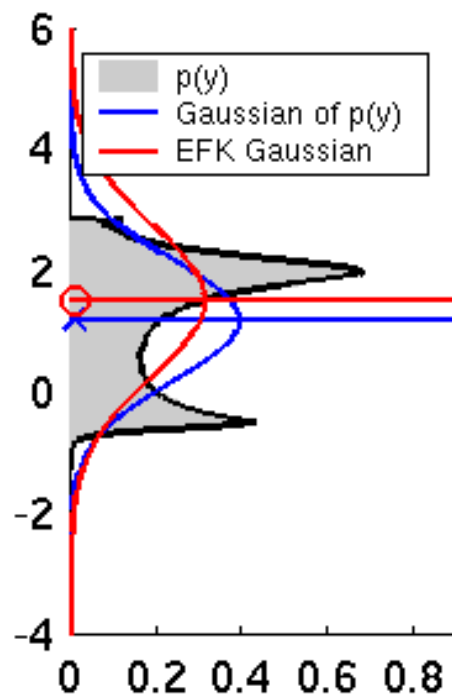
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IDEA #2: LINEARIZE THE NONLINEAR FUNCTIONS f, h

- Then we are in the case $y = Gx + c$, so $p(y)$ is a Gaussian

Linearization



Notice how the Linearization approximation differs from the Monte Carlo approximation (which is better, provided sufficiently many samples).

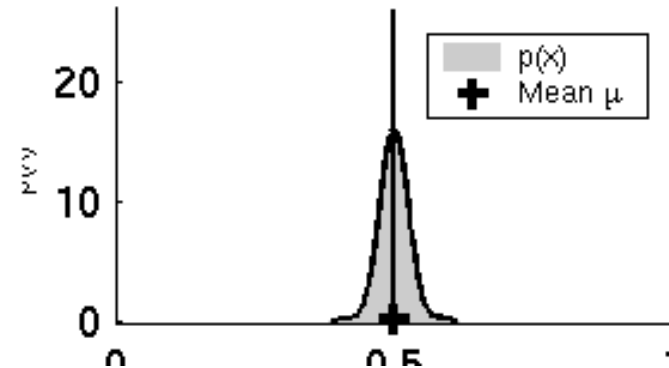
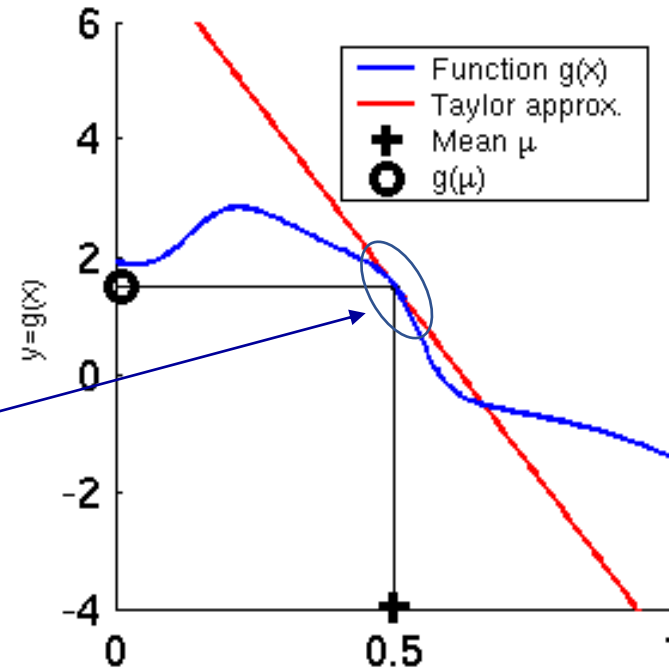
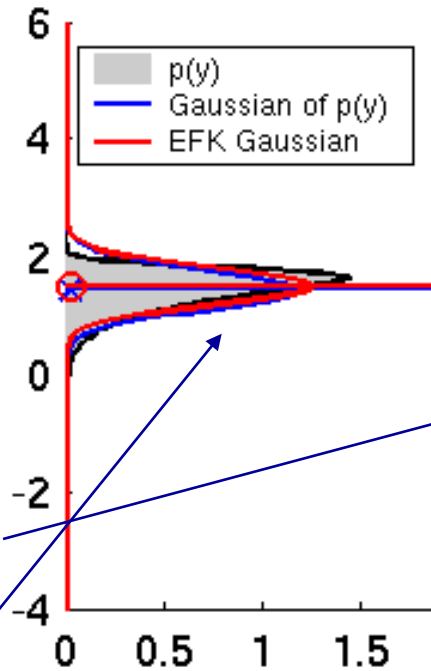
That said, the Linearization approximation can be computed efficiently, and can be integrated into the Kalman Filter estimator \rightarrow Extended Kalman Filter

Linearization with low approximation error

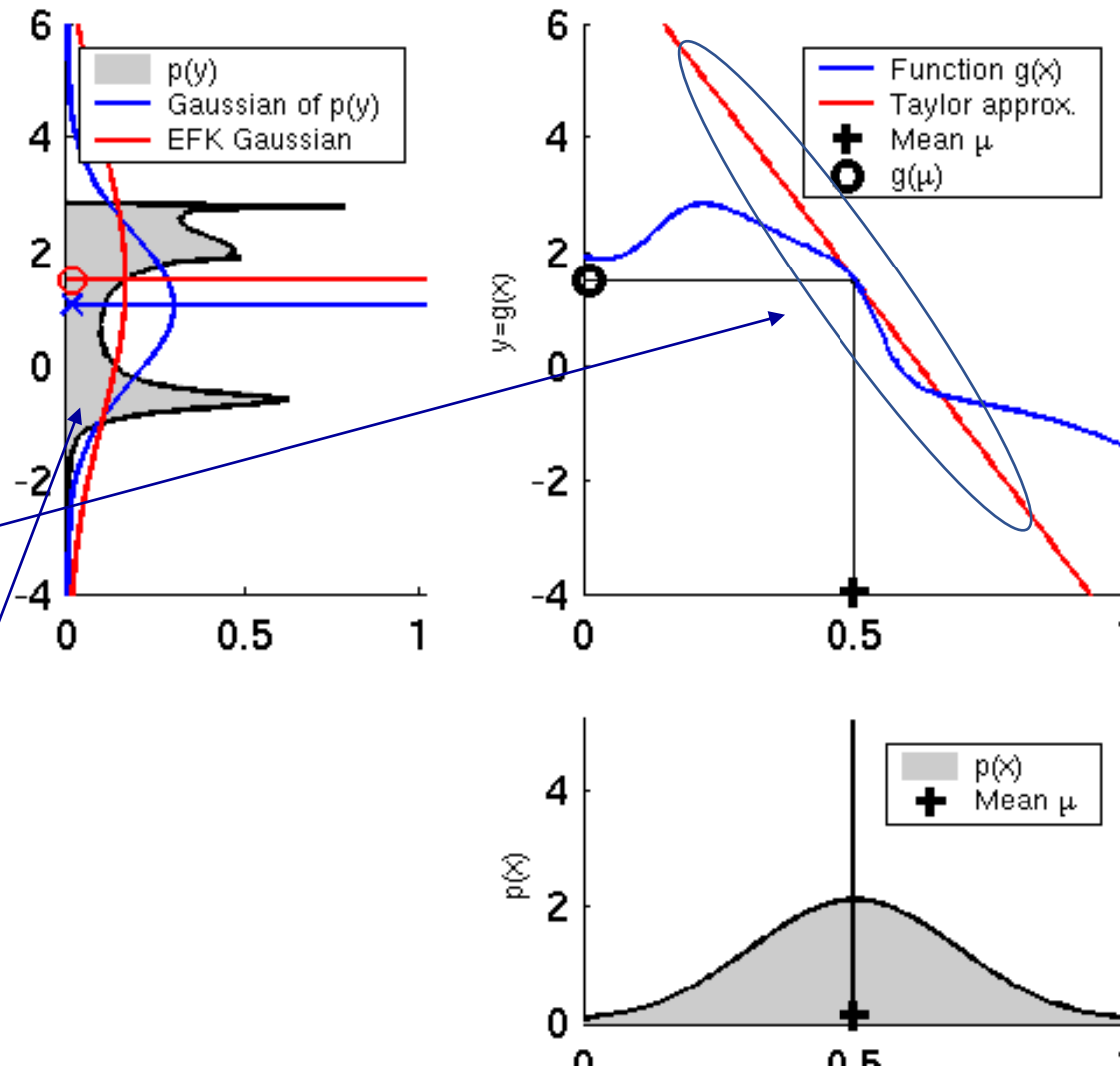
The quality of the linearization approximation depends on the uncertainty of $p(x)$ but also on the shape of the nonlinearity $g(x)$.

In this example $p(x)$ has small variance so most points will be concentrated around 0.5 and will pass through a very small region of $g(x)$, where $g(x)$ is close to linear.

In this case $p(y)$ is nearly Gaussian, and the Linearization approximation matches the Monte Carlo approximation.



Linearization with high approximation error



The quality of the linearization approximation depends on the uncertainty of $p(x)$ but also on the shape of the nonlinearity $g(x)$.

In this example $p(x)$ has high variance so points $g(x)$ will be spread out around $g(0.5)$, where $g(x)$ is not close to linear.

In this case $p(y)$ is multimodal, and the Linearization approximation matches the Monte Carlo approximation are both suboptimal approximations. Again Monte Carlo is better, provided sufficient samples.

How do we linearize?

- Using the first order Taylor expansion around the mean of the previous update step's state estimate:

$$\begin{aligned}x_{t+1} &= f(x_t, u_t) + w_t \\&\approx f(\mu_{t|t}, u_t) + \frac{\partial f}{\partial x}(\mu_{t|t}, u_t)(x_t - \mu_{t|t}) + w_t \\&= f(\mu_{t|t}, u_t) + F_t(x_t - \mu_{t|t}) + w_t \\&= F_t x_t + \boxed{f(\mu_{t|t}, u_t) - F_t \mu_{t|t}} + w_t \\&= F_t x_t + \bar{u}_t + w_t\end{aligned}$$

Constant term
with respect to
the state

Recall how to compute the Jacobian matrix. For example, if

$$f(x_1, x_2, u) = [x_1 + x_2^2, x_2 + 3u, x_1^4 - u^2] \in \mathbb{R}^3$$

then the Jacobian of f with respect to (x_1, x_2) at (μ_1, μ_2, u) is

$$\begin{aligned}\frac{\partial f}{\partial x_{1:2}}(\mu_1, \mu_2, u_1) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix}(\mu_1, \mu_2, u_1) \\&= \begin{bmatrix} 1 & 2\mu_2 \\ 0 & 1 \\ 4\mu_1^3 & 0 \end{bmatrix}\end{aligned}$$

How do we linearize?

- Using the first-order Taylor expansion around the mean of the previous prediction step's state estimate:

$$\begin{aligned} z_{t+1} &= h(x_{t+1}) + n_{t+1} \\ &\approx h(\mu_{t+1|t}) + \frac{\partial h}{\partial x}(\mu_{t+1|t})(x_t - \mu_{t+1|t}) + n_{t+1} \\ &= h(\mu_{t+1|t}) + H_{t+1}(x_{t+1} - \mu_{t+1|t}) + n_{t+1} \\ &= H_{t+1}x_{t+1} + \boxed{h(\mu_{t+1|t}) - H_{t+1}\mu_{t+1|t}} + n_{t+1} \\ &= H_{t+1}x_{t+1} + \bar{c}_{t+1} + n_{t+1} \end{aligned}$$

Constant term
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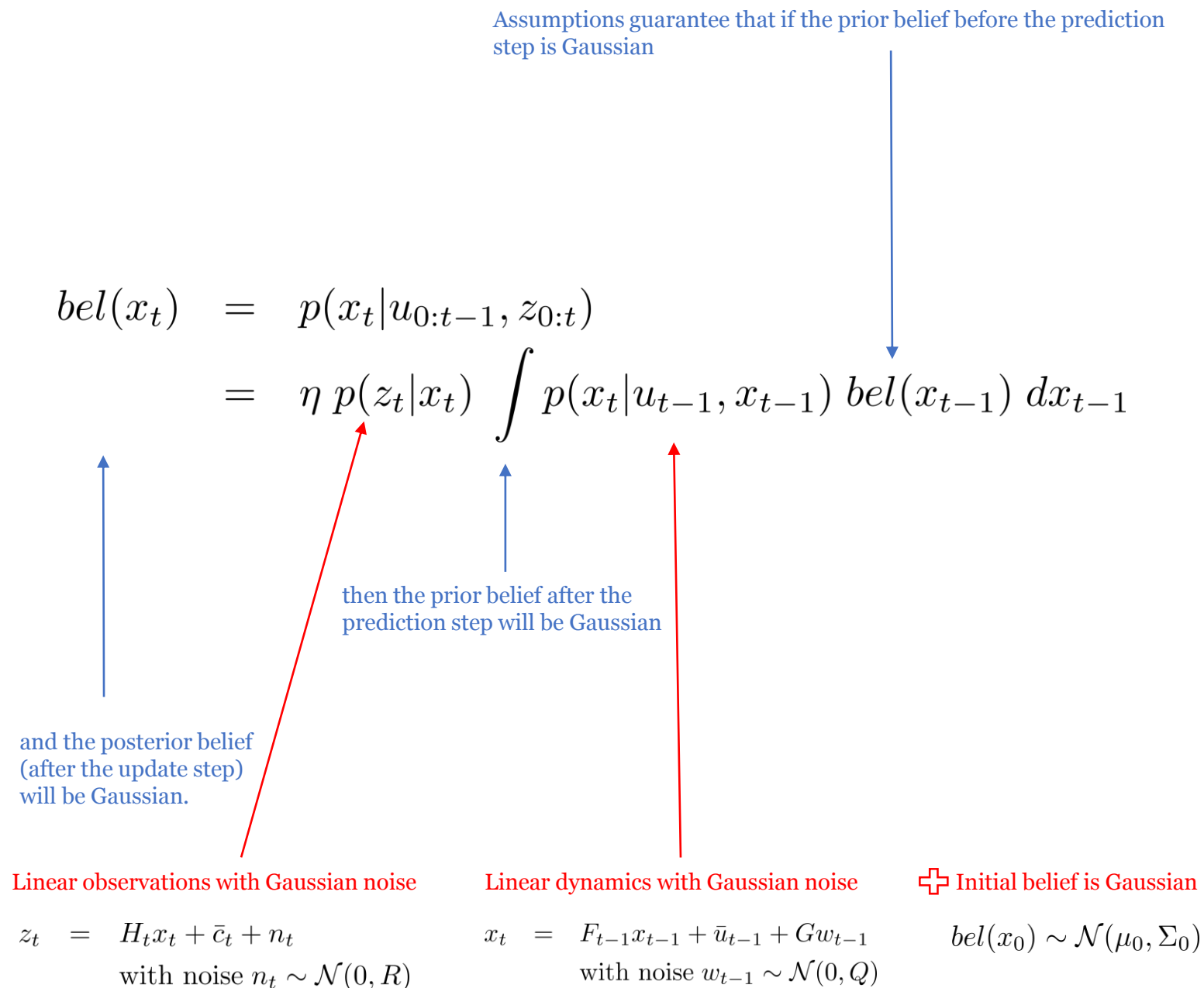
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Extended Kalman Filter: an instance of Bayes' Filter



Dynamics

$$x_{t+1} = f(x_t, u_t) + Gw_t$$
$$w_t \sim \mathcal{N}(0, Q)$$

Measurements

$$z_t = h(x_t) + n_t$$
$$n_t \sim \mathcal{N}(0, R)$$

EKF in N dimensions

Init

$$bel(x_0) \sim \mathcal{N}(\mu_{0|0}, \Sigma_{0|0})$$

Prediction Step

$$\mu_{t+1|t} = f(\mu_{t|t}, u_t)$$
$$\Sigma_{t+1|t} = F_t \Sigma_{t|t} F_t^T + G Q G^T$$

Update Step

Received measurement \bar{z}_{t+1} but expected to receive $\mu_{z_{t+1}} = h(\mu_{t+1|t})$

Prediction residual is a Gaussian random variable $\delta z \sim \mathcal{N}(\bar{z}_{t+1} - \mu_{z_{t+1}}, S_{t+1})$
where the covariance of the residual is $S_{t+1} = H_{t+1} \Sigma_{t+1|t} H_{t+1}^T + R$

Kalman Gain (optimal correction factor): $K_{t+1} = \Sigma_{t+1|t} H_{t+1}^T S_{t+1}^{-1}$

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}(\bar{z}_{t+1} - \mu_{z_{t+1}})$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1} H_{t+1} \Sigma_{t+1|t}$$

EKF Summary

- **Efficient**: Polynomial in measurement dimensionality k and state dimensionality n :

$$O(k^{2.376} + n^2)$$

- **Not optimal** (unlike the Kalman Filter for linear systems)
- Can **diverge** if nonlinearities are large
- Works surprisingly well even when all assumptions are violated

As in KF, inverting
the covariance of the
residual is $O(k^{2.376})$

Example #1: EKF-Localization

“Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities.”
[Cox '91]

- **Given**
 - Map of the environment.
 - Sequence of sensor measurements.
- **Wanted**
 - Estimate of the robot's position.
- **Problem classes**
 - Position tracking
 - Global localization
 - Kidnapped robot problem (recovery)

Landmark-based Localization

Landmarks, whose position $(l_x^{(i)}, l_y^{(i)})$ in the world is known.

Each robot measures its range and bearing from each landmark to localize itself.

State of a robot:

$$x_t = \begin{bmatrix} p_x(t) \\ p_y(t) \\ \theta(t) \end{bmatrix}$$



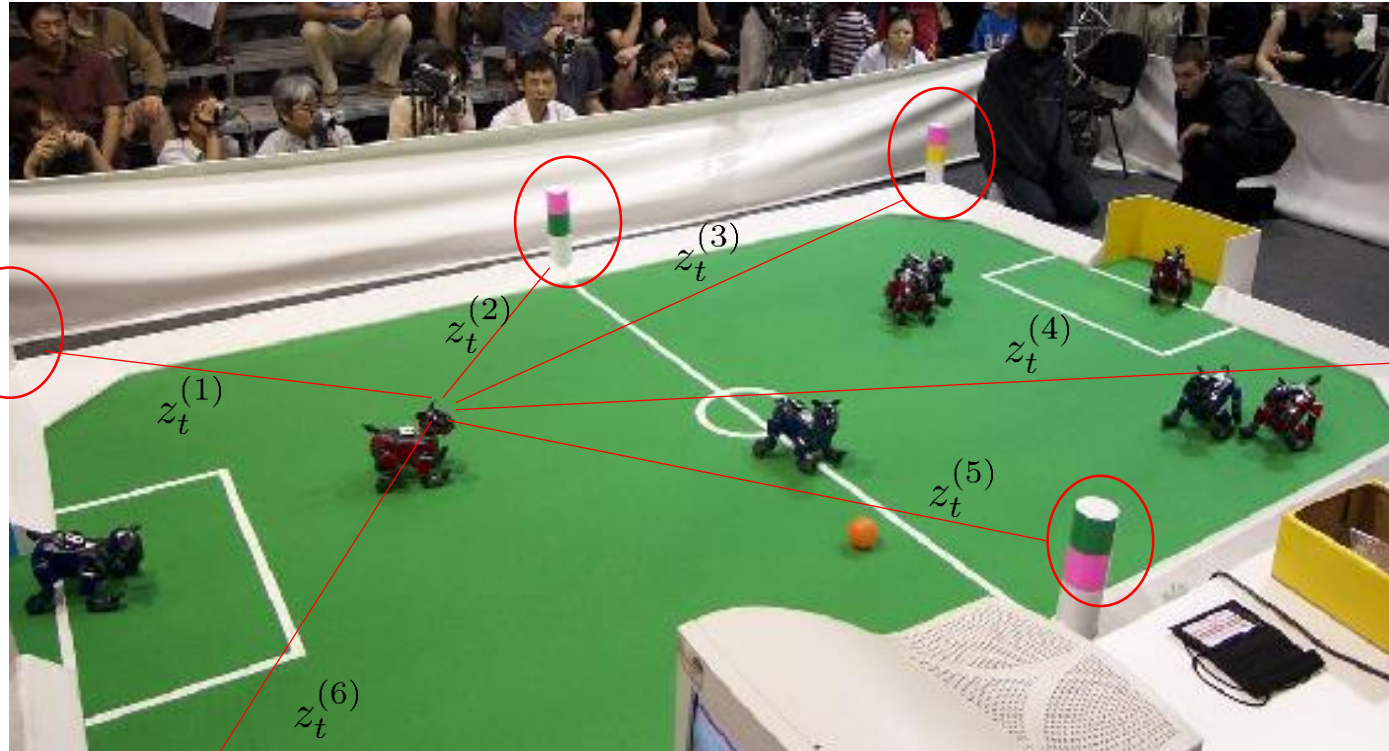
Landmark-based Localization

Measurement at time t ,

$$z_t = \begin{bmatrix} \dots \\ z_t^{(i)} \\ \dots \end{bmatrix}$$

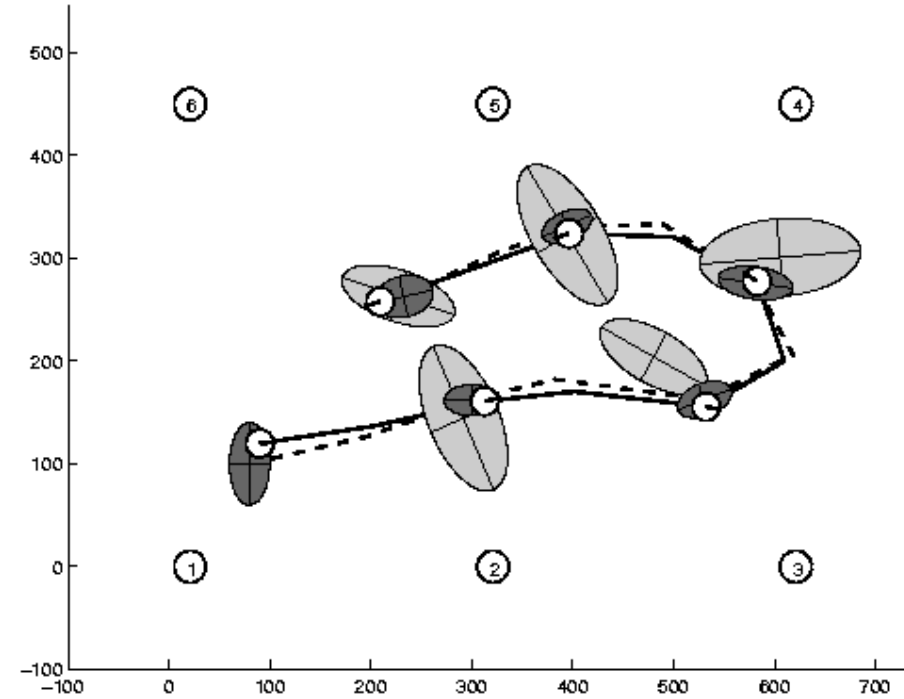
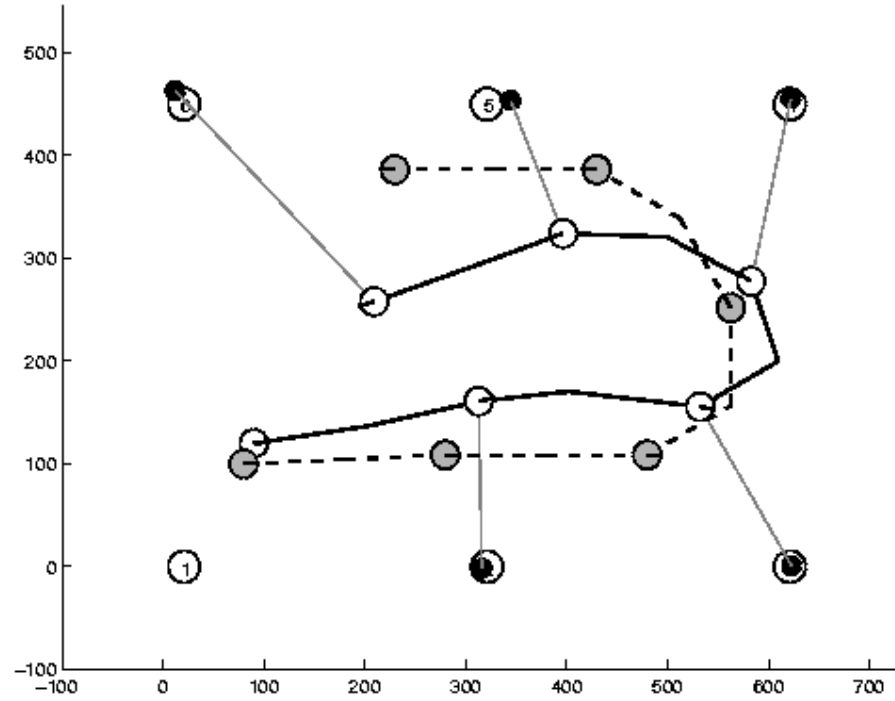
is a variable-sized vector, depending on the landmarks that are visible at time t .

Each measurement is a 2D vector, containing range and bearing from the robot to a landmark.

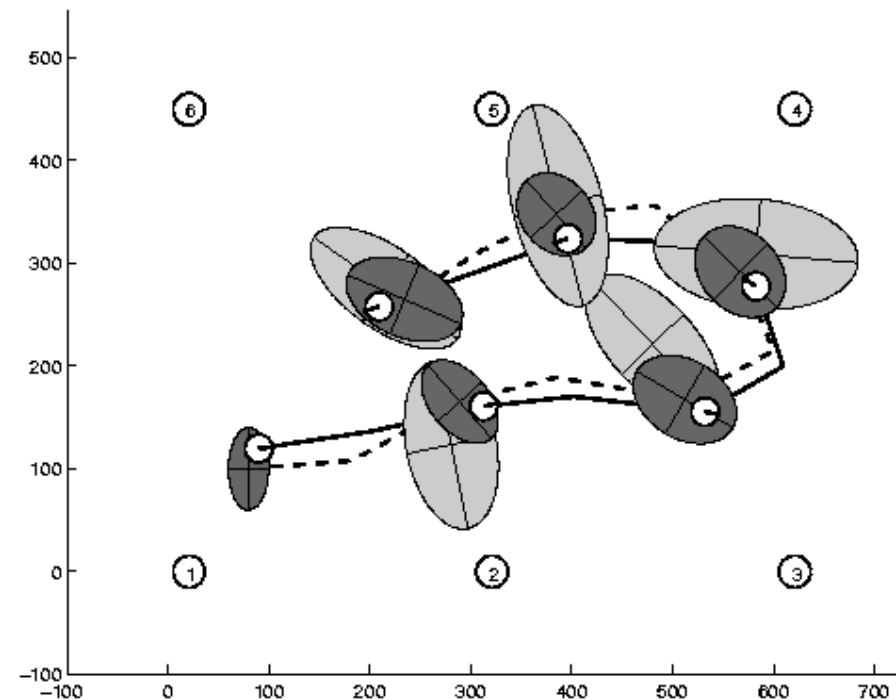
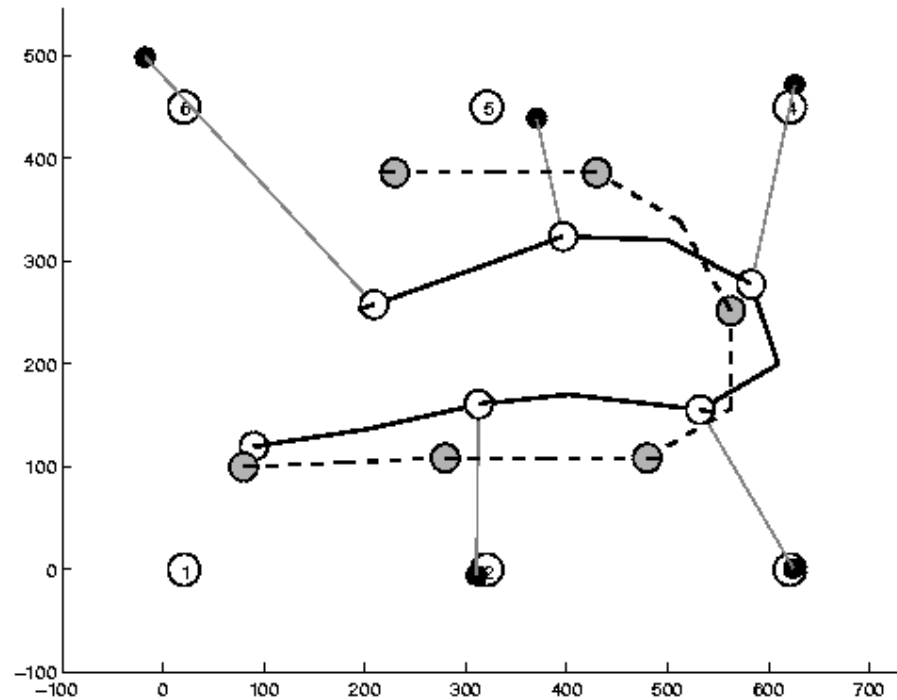


$$z_t^{(i)} = h_i(x_t) = \begin{bmatrix} \sqrt{(p_x(t) - l_x^{(i)})^2 + (p_y(t) - l_y^{(i)})^2} \\ \text{atan2}(p_y(t) - l_y^{(i)}, p_x(t) - l_x^{(i)}) - \theta(t) \end{bmatrix} + n_t$$

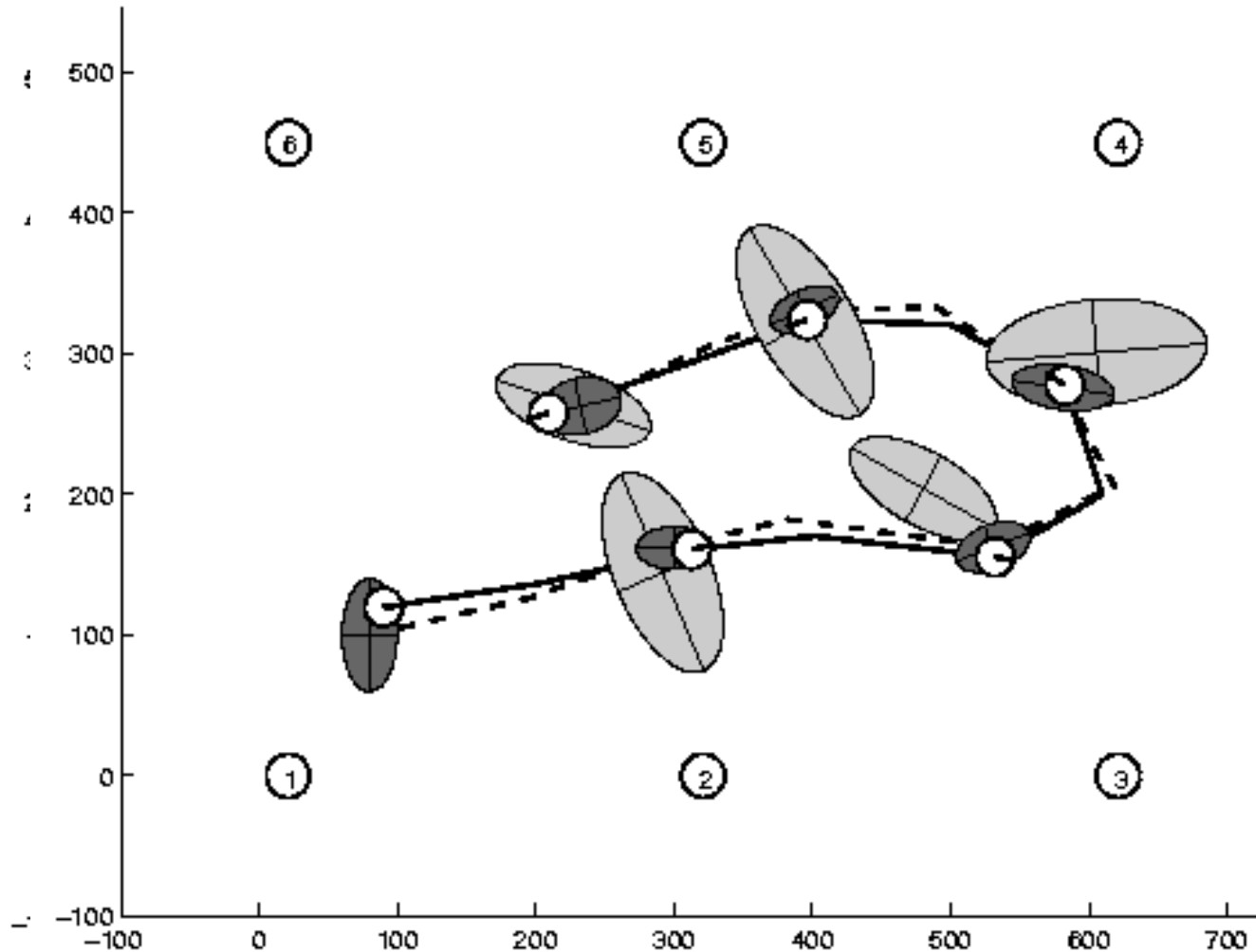
Estimation Sequence (1)



Estimation Sequence (2)



Comparison to true trajectory



Example #2: EKF-SLAM

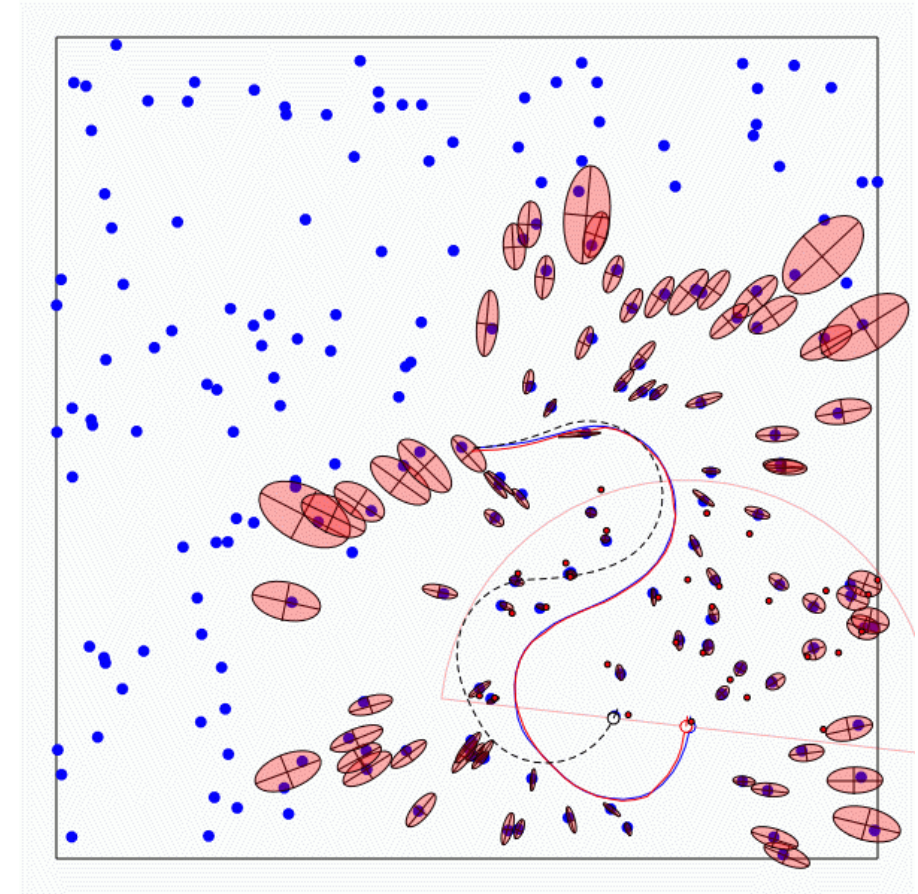
A robot is exploring an unknown, static environment.

Given:

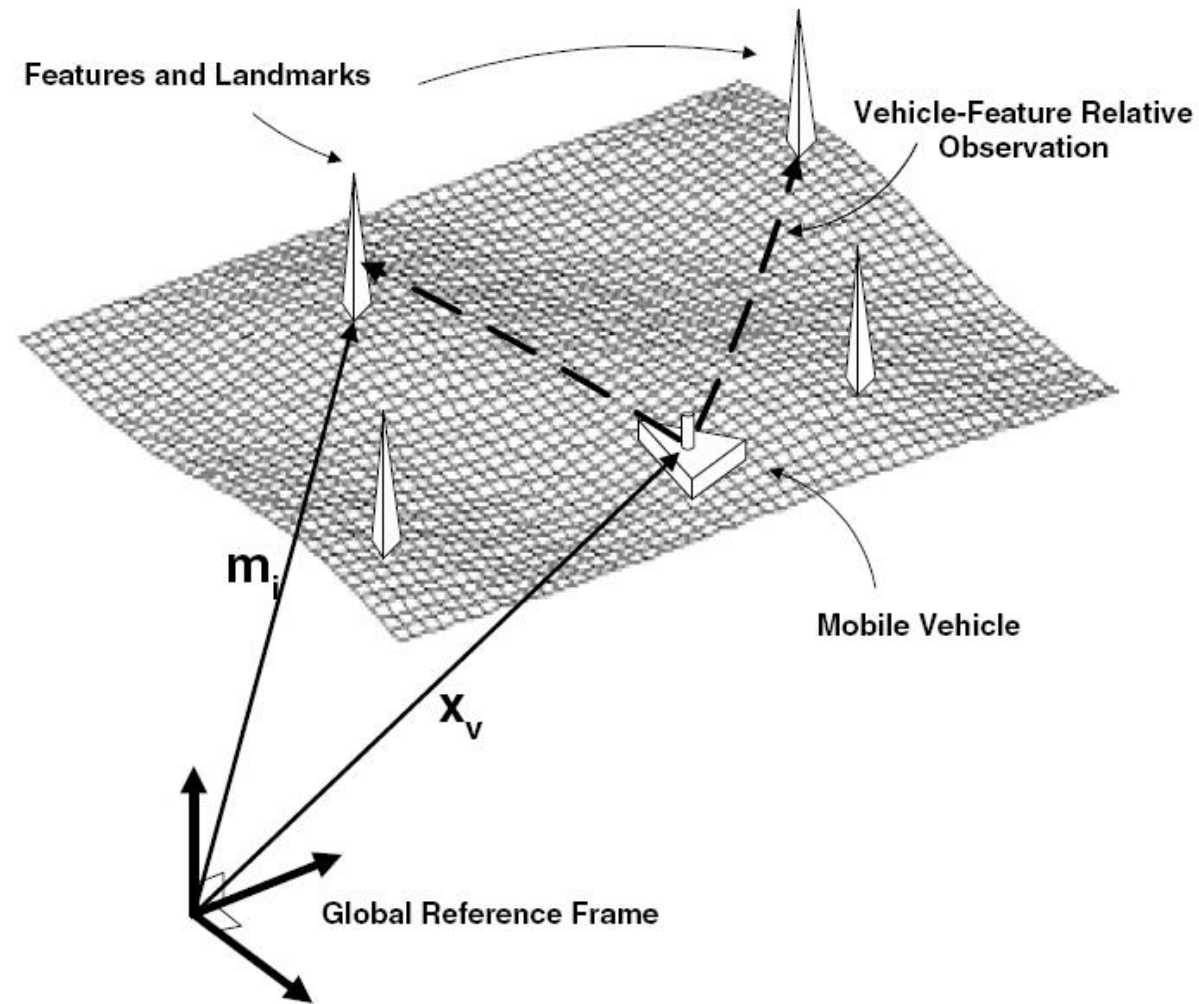
- The robot's controls
- Observations of nearby features

Estimate:

- Map of features
- Path of the robot

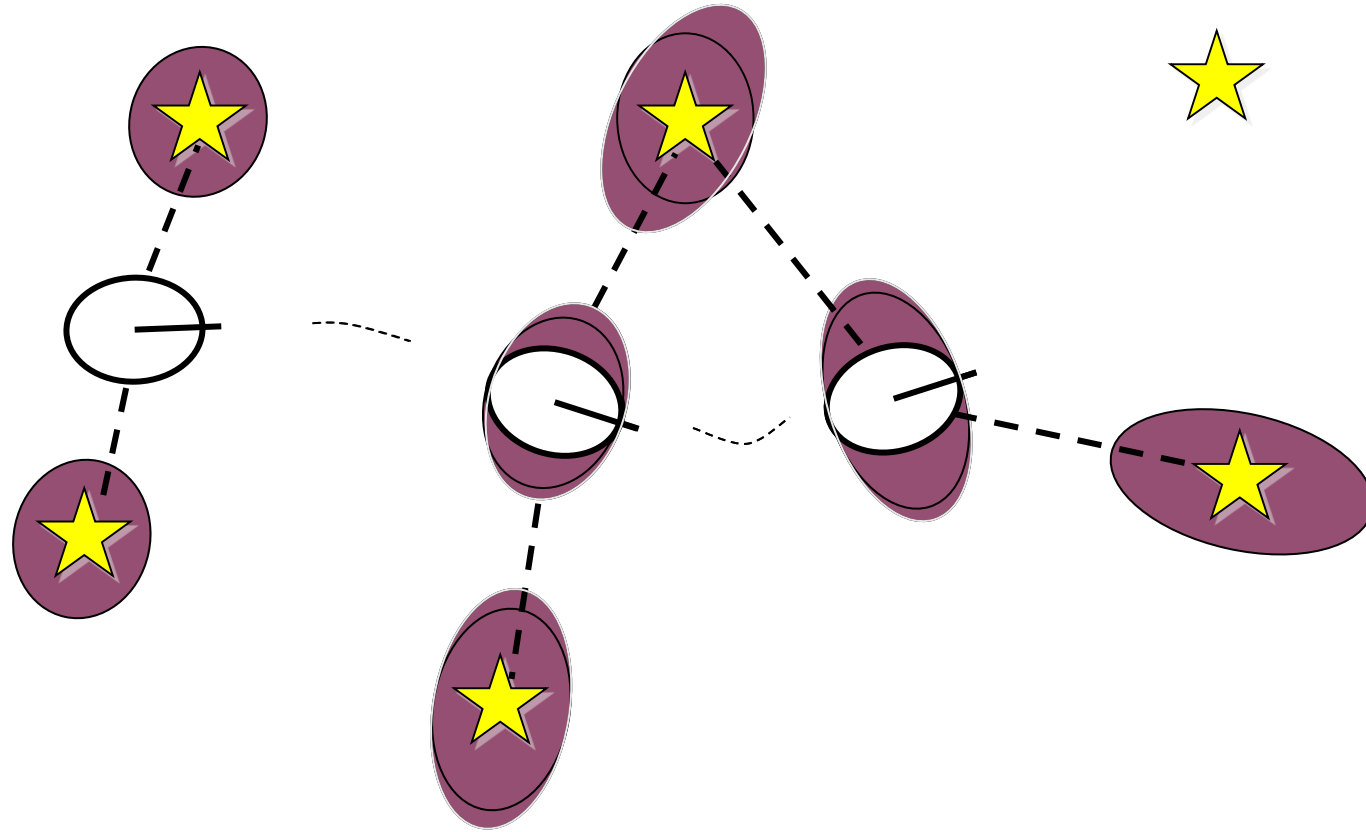


Structure of Landmark-based SLAM



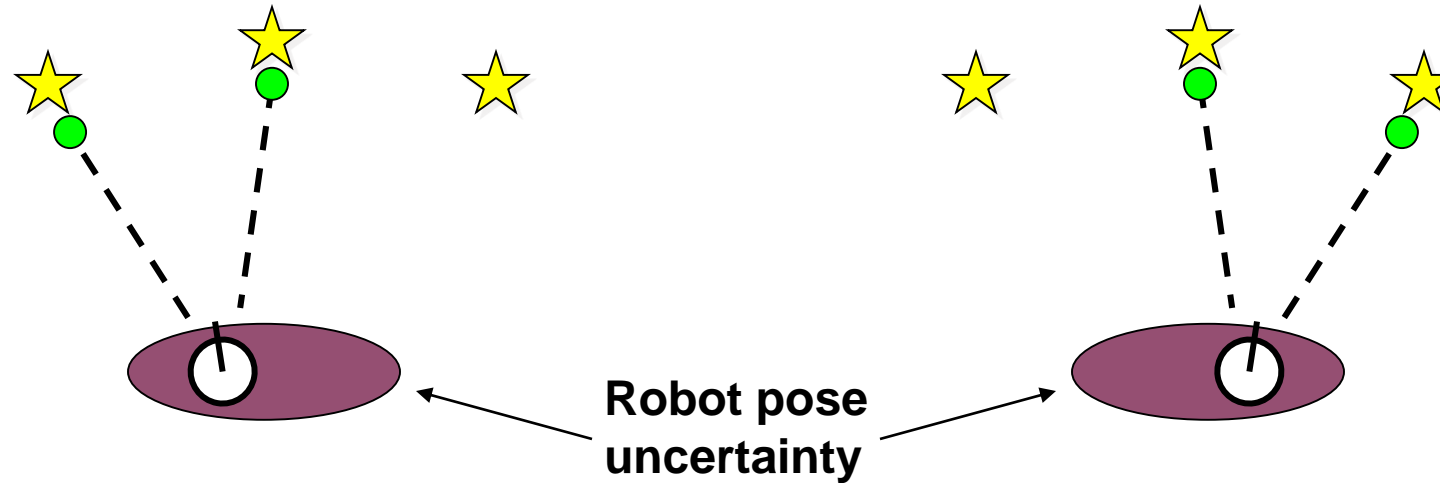
Why is SLAM a hard problem?

SLAM: robot path and map are both **unknown**



Robot path error correlates errors in the map

Why is SLAM a hard problem?



- In the real world, the mapping between observations and landmarks is unknown
- Picking wrong data associations can have catastrophic consequences
- Pose error correlates data associations

EKF-SLAM

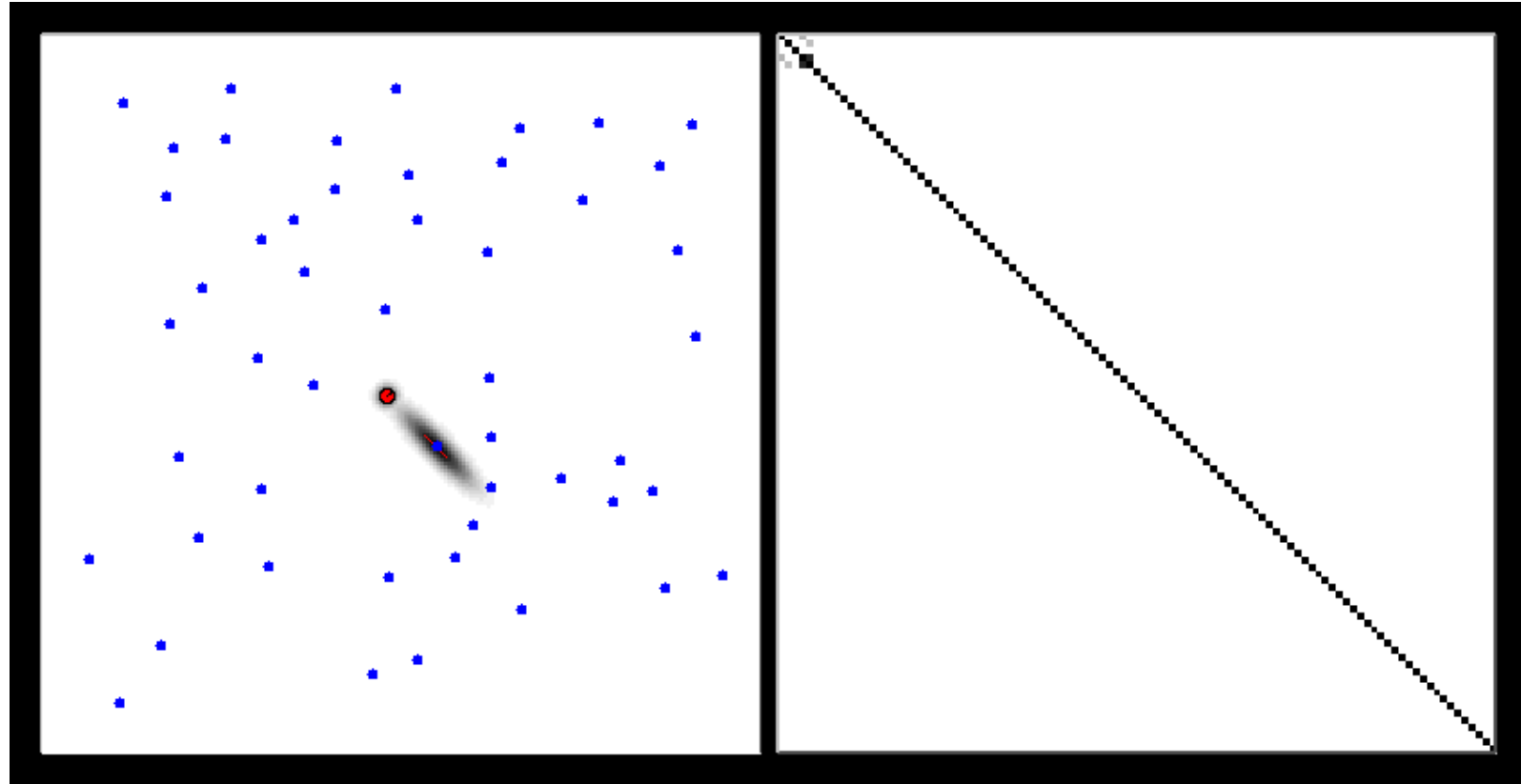
- Map with N landmarks: (3+2N)-dimensional Gaussian

$$Bel(x_t, m_t) = \left(\begin{array}{c} x \\ y \\ \theta \\ l_1 \\ l_2 \\ \vdots \\ l_N \end{array} \right), \left(\begin{array}{ccc|cccc} \sigma_x^2 & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xl_1} & \sigma_{xl_2} & \cdots & \sigma_{xl_N} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\theta} & \sigma_{yl_1} & \sigma_{yl_2} & \cdots & \sigma_{yl_N} \\ \sigma_{x\theta} & \sigma_{y\theta} & \sigma_\theta^2 & \sigma_{\theta l_1} & \sigma_{\theta l_2} & \cdots & \sigma_{\theta l_N} \\ \hline \sigma_{xl_1} & \sigma_{yl_1} & \sigma_{\theta l_1} & \sigma_{l_1}^2 & \sigma_{l_1 l_2} & \cdots & \sigma_{l_1 l_N} \\ \sigma_{xl_2} & \sigma_{yl_2} & \sigma_{\theta l_2} & \sigma_{l_1 l_2} & \sigma_{l_2}^2 & \cdots & \sigma_{l_2 l_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{xl_N} & \sigma_{yl_N} & \sigma_{\theta l_N} & \sigma_{l_1 l_N} & \sigma_{l_2 l_N} & \cdots & \sigma_{l_N}^2 \end{array} \right)$$

- Can handle hundreds of dimensions

Appendix

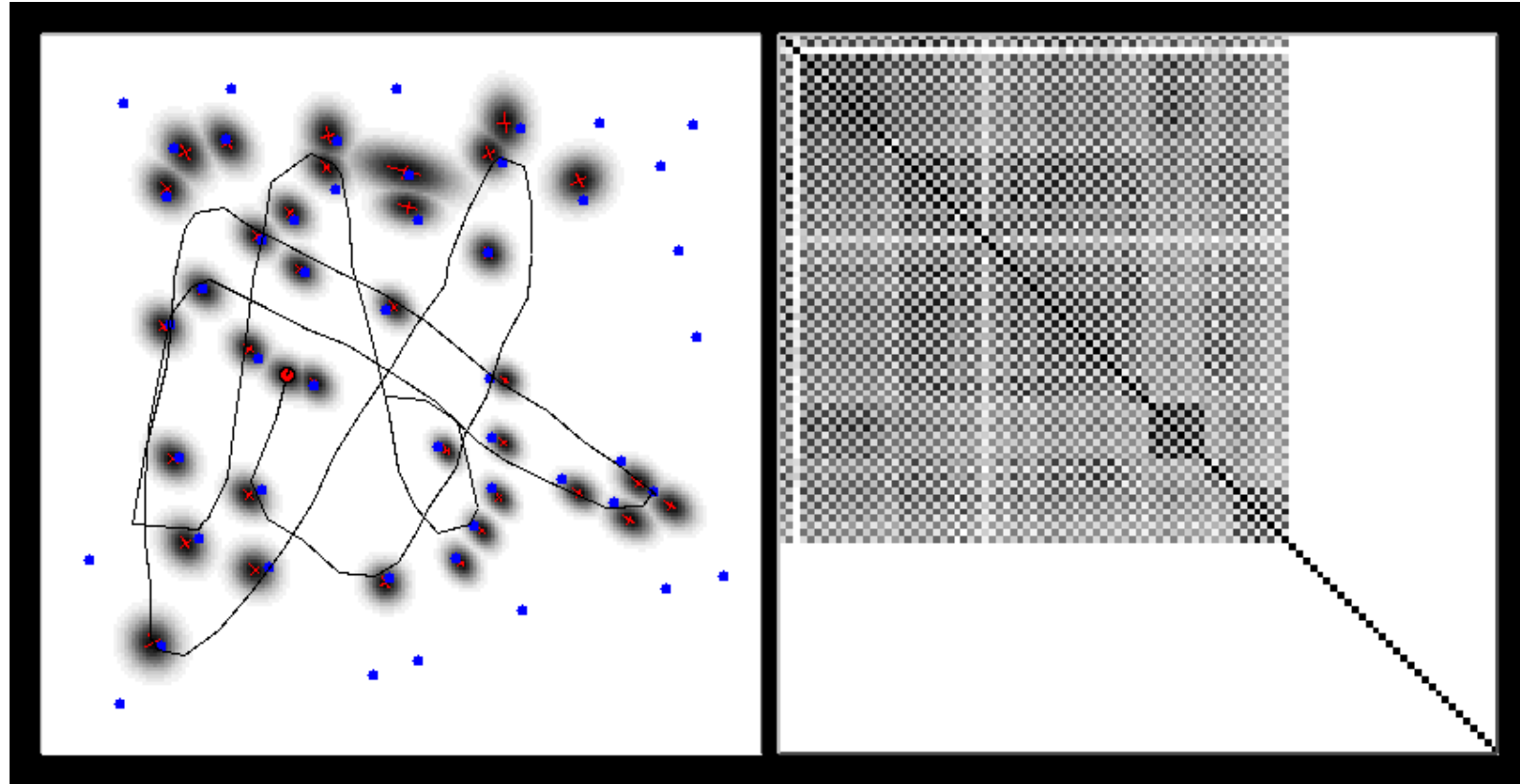
EKF-SLAM



Map

Covariance matrix

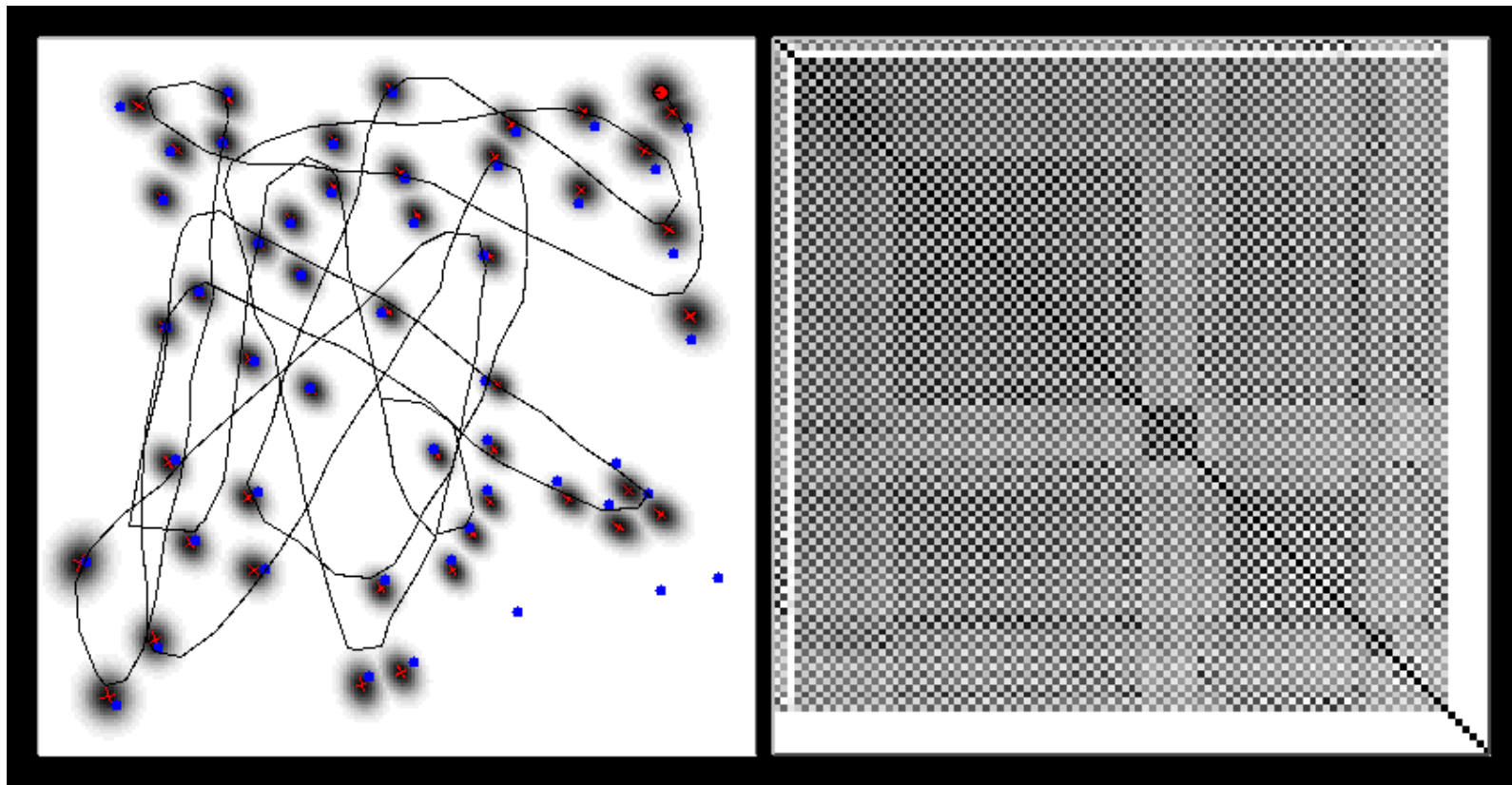
EKF-SLAM



Map

Covariance matrix

EKF-SLAM



Map

Covariance matrix

Properties of EKF-SLAM (Linear Case)

[Dissanayake et al., 2001]

Theorem:

The determinant of any sub-matrix of the map covariance matrix decreases monotonically as successive observations are made.

Theorem:

In the limit the landmark estimates become fully correlated