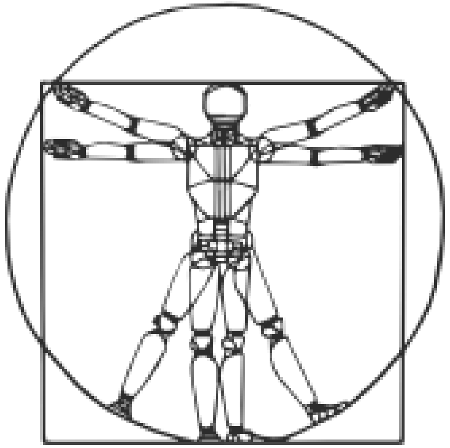

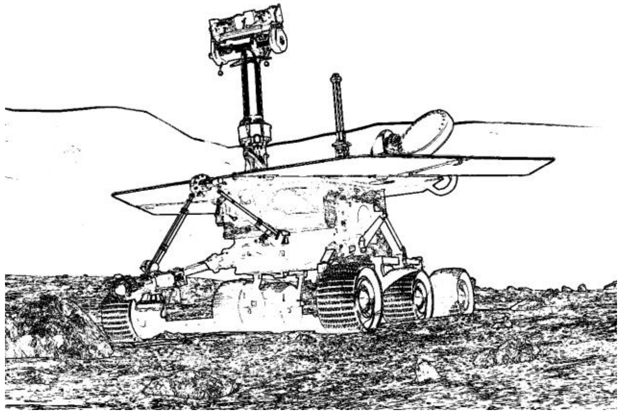

# COMP417
# Introduction to Robotics and Intelligent Systems

## Lecture 2: Kinematics and Dynamics


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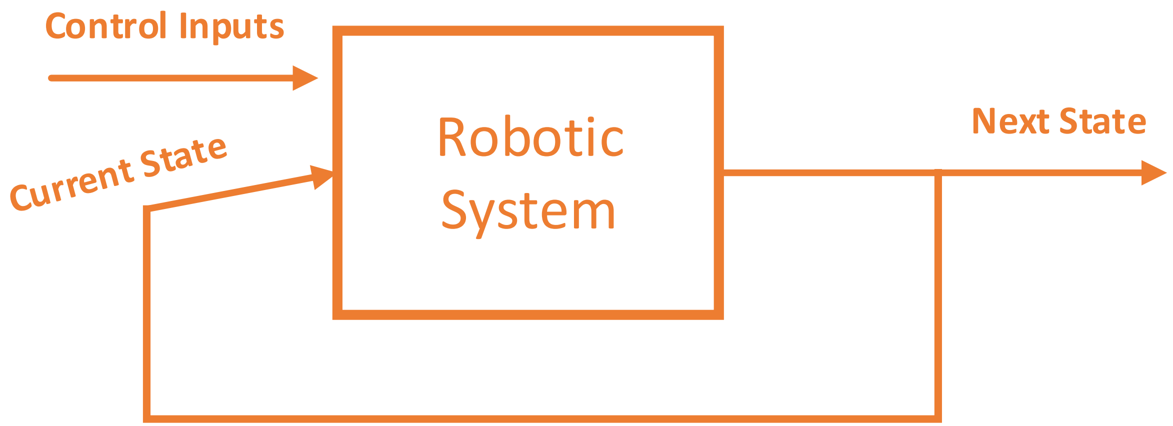
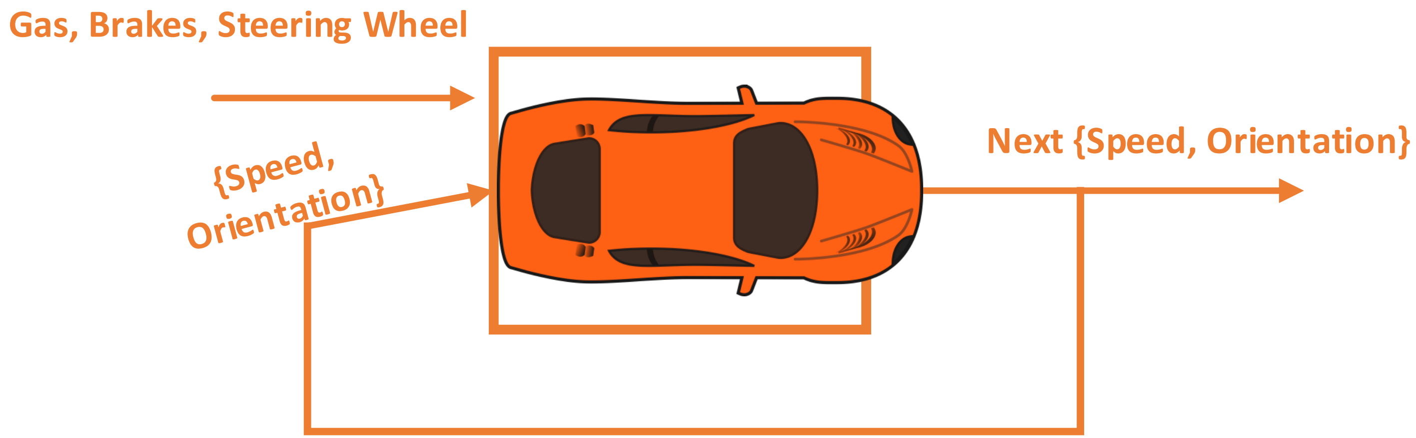
# Physical models of how systems move

Kinematics & Dynamics:
physical models of
robotic systems and sensors




Main question: what is the next state given the current state and controls?

# Today

- Idealized physical models of robotic vehicles

# Why idealized?

- “All models are wrong, but some are useful” – George Box (statistician)

- Model: a function that describes a physical phenomenon or a system, i.e. how a set of input variables cause a set of output variables.

- Models are useful if they can predict reality <u>up to some degree</u>.

- Mismatch between model prediction and reality = **error / noise**

# Noise

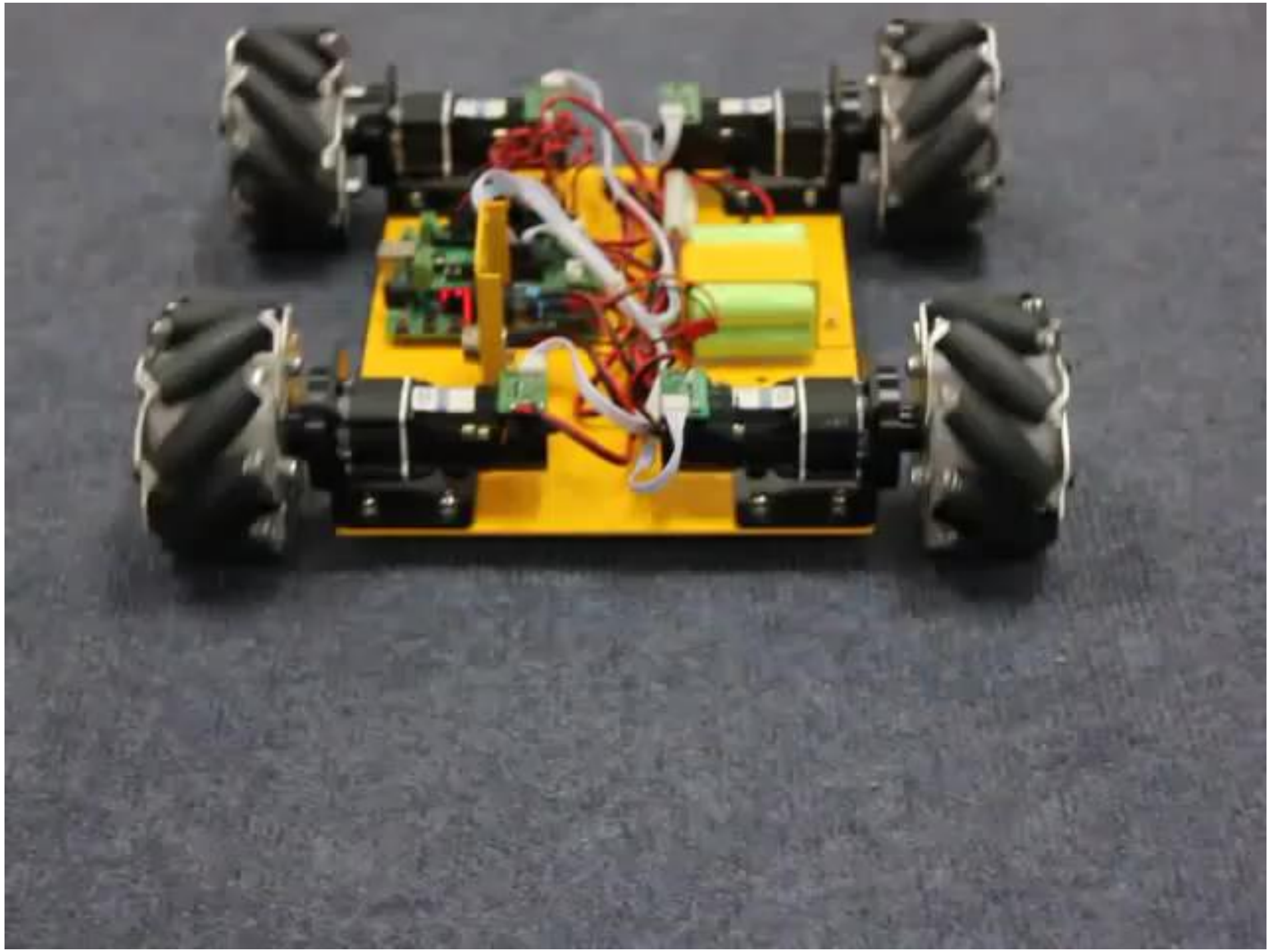
- Anything that we do not bother modelling with our model
- Example 1: “assume frictionless surface”
- Example 2: Taylor series expansion (only first few terms are dominant)
- With models, can be thought of as approximation error.

# Idealized physical models of robotic vehicles

- Omnidirectional motion
- Dubins car
- Differential drive steering
- Ackerman steering
- Unicycle
- Cartpole
- Quadcopter

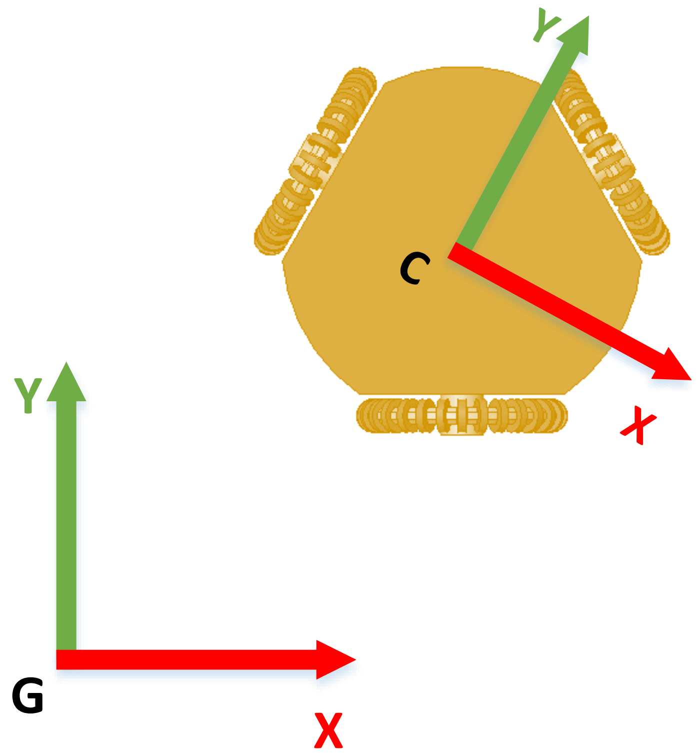
# Omnidirectional Robots

# Omnidirectional Robots



# The state of an omnidirectional robot

State := Configuration := $\mathbf{x}$ : = vector of physical quantities of interest about the system

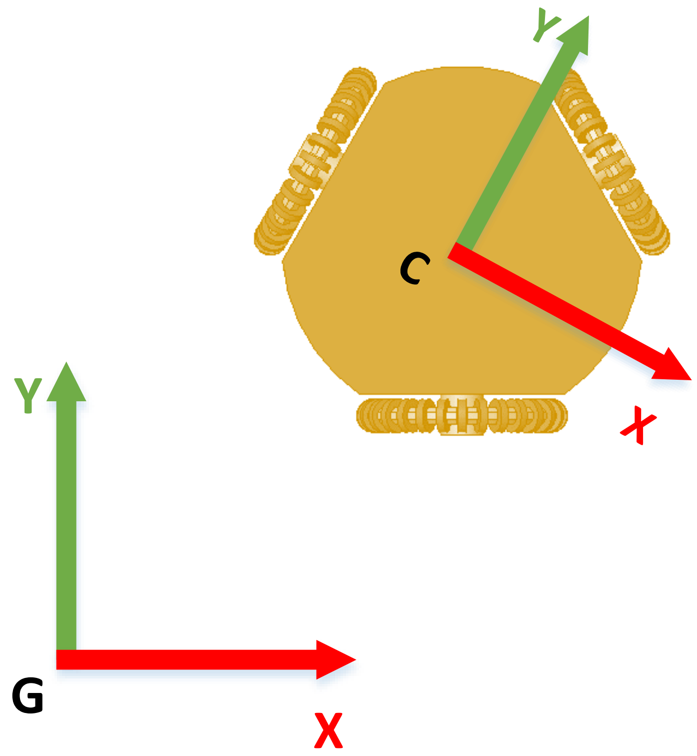



$$\mathbf{x} = [{}^{G}p_x, {}^{G}p_y, {}^{G}\theta]$$

State = [Position, Orientation]
Position of the robot's frame of reference C with respect to a fixed frame of reference G, expressed in coordinates of frame G. Angle is the orientation of frame C with respect to frame G.

# Control of an omnidirectional robot

Control := $\mathbf{u}$ := a vector of input commands that can modify the state of the system

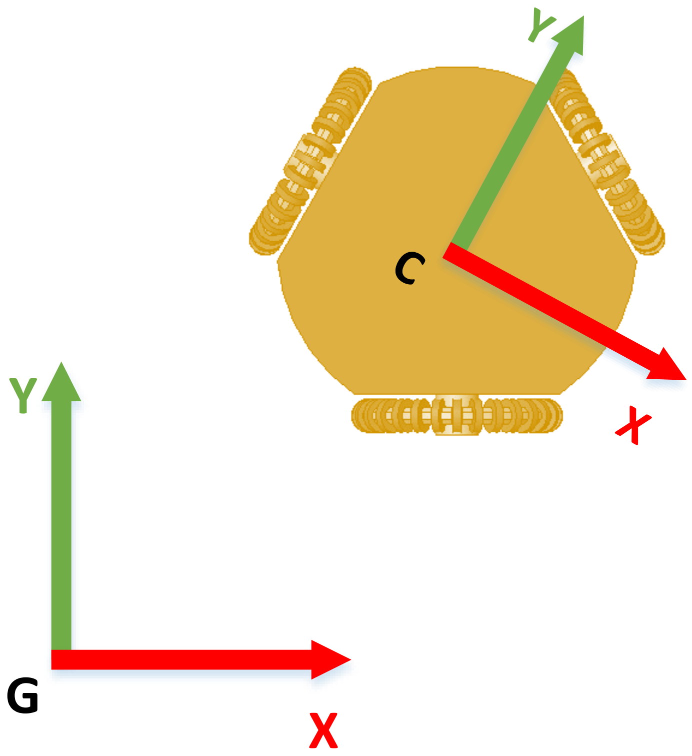



$$\mathbf{u} = [{}^{C}v_x, {}^{C}v_y, {}^{C}\omega_z]$$

Control = [Linear velocity, Angular velocity]
Linear and angular velocity of the robot's frame of reference C with respect to a fixed frame of reference G, expressed in coordinates of frame C.

# Dynamics of an omnidirectional robot

Dynamical System : = Dynamics := a function that describes the time evolution of the state in response to a control signal




Continuous case: $$\frac{\mathbf{dx}}{\mathbf{dt}} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

$$\begin{aligned} \dot{p}_x &= v_x \\ \dot{p}_y &= v_y \\ \dot{\theta} &= \omega_z \end{aligned}$$

Note: reference frames have been removed for readability.

# Inertial frames of reference

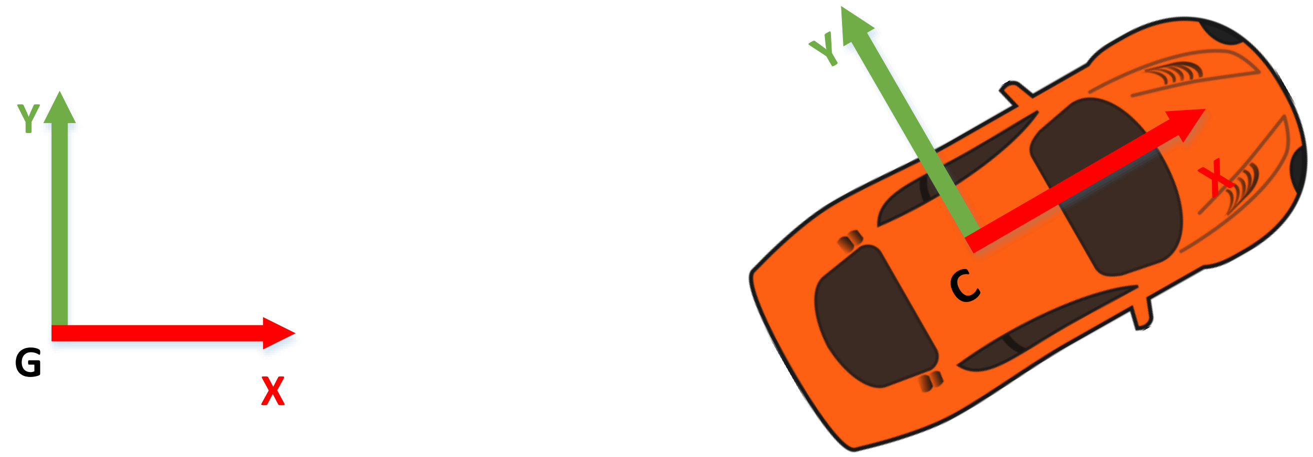
- G, the global frame of reference is fixed, i.e. with zero velocity in our previous example.

- But, in general it can move as long as it has zero acceleration. Such a frame is called an "inertial" frame of reference.

- Newton's laws hold for inertial reference frames only. For reference frames with non-constant velocity we need the theory of General Relativity.

- So, make sure that your global frame of reference is inertial, preferably fixed.

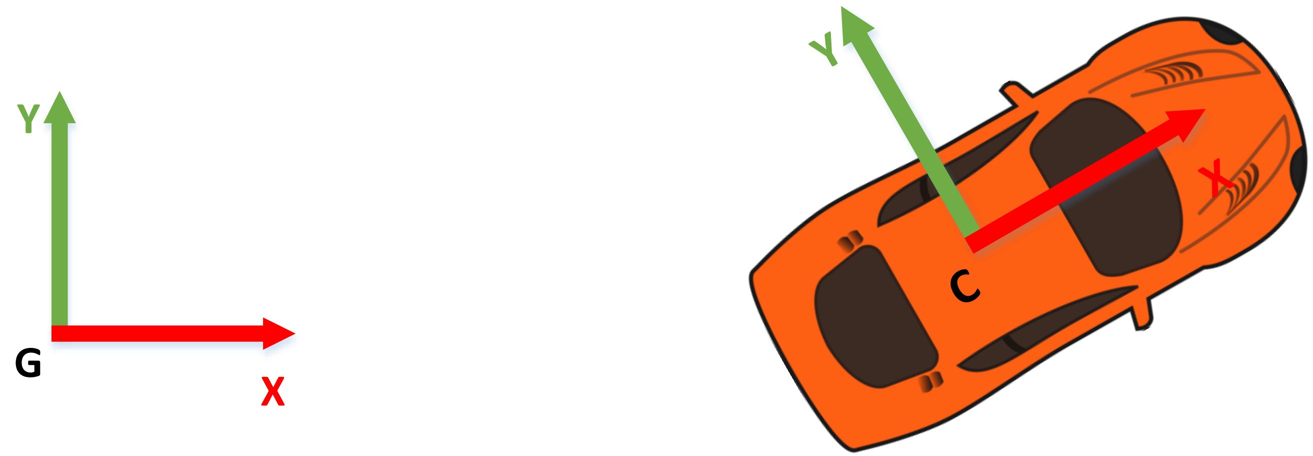
# The state of a simple car




State = [Position and orientation]
Position of the car's frame of reference C with respect to a fixed frame of reference G, expressed in frame G.
The angle is the orientation of frame C with respect to G.

$$\mathbf{x} = [{}^{G}p_x, {}^{G}p_y, {}^{G}\theta]$$

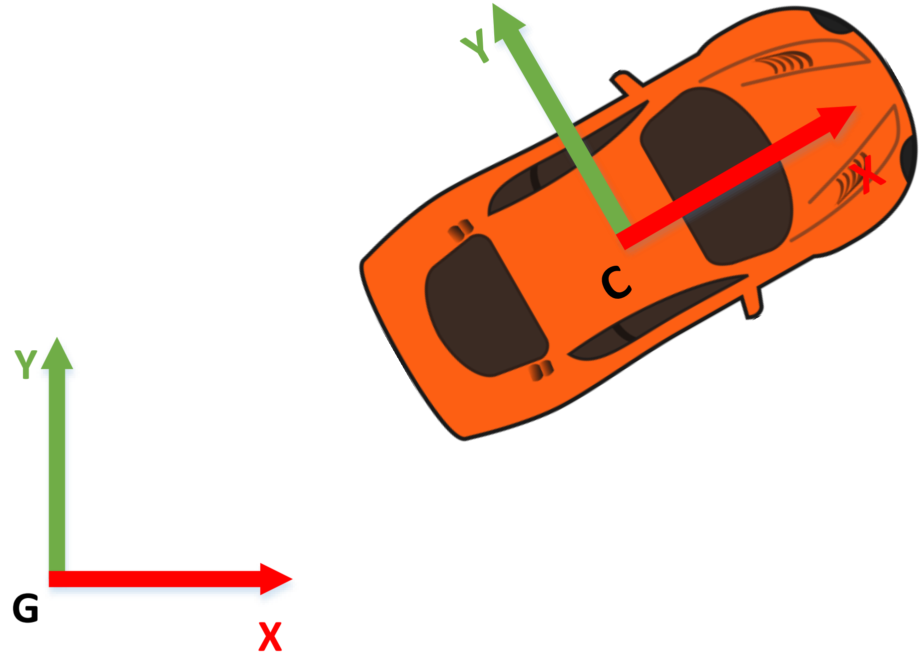
# The controls of a simple car




Controls = [Forward speed and angular velocity]
Linear velocity and angular velocity of the car's frame of reference C with respect to a fixed frame of reference G, expressed in coordinates of C.

$$\mathbf{u} = [{}^{C}v_x, {}^{C}\omega_z]$$

# The dynamical system of a simple car




$$\begin{aligned} \dot{p}_x &= v_x \cos(\theta) \\ \dot{p}_y &= v_x \sin(\theta) \\ \dot{\theta} &= \omega_z \end{aligned}$$

Note: reference frames have been removed for readability.

# Kinematics vs Dynamics

- Kinematics considers models of locomotion independently of external forces and control.

- For example, it describes how the speed of a car affects the state without considering what the required control commands required to generate those speeds are.

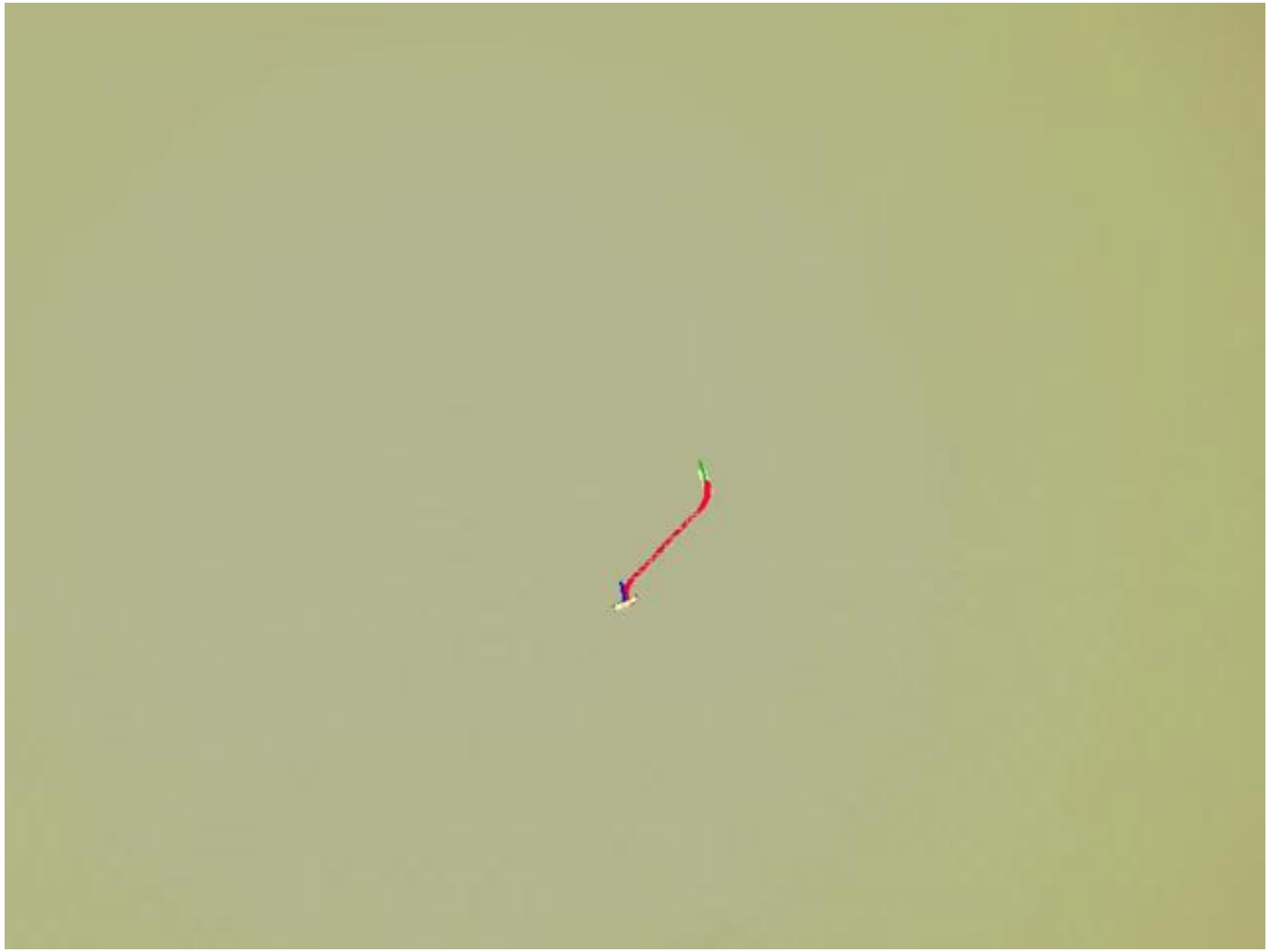
- Dynamics considers models of locomotion as functions of their control inputs and state.

# Special case of simple car: Dubins car

- Can only go forward
- Constant speed

$$ {}^{C}v_x = \text{const} > 0 $$

- You only control the angular velocity



# Special case of simple car: Dubins car

- Can only go forward
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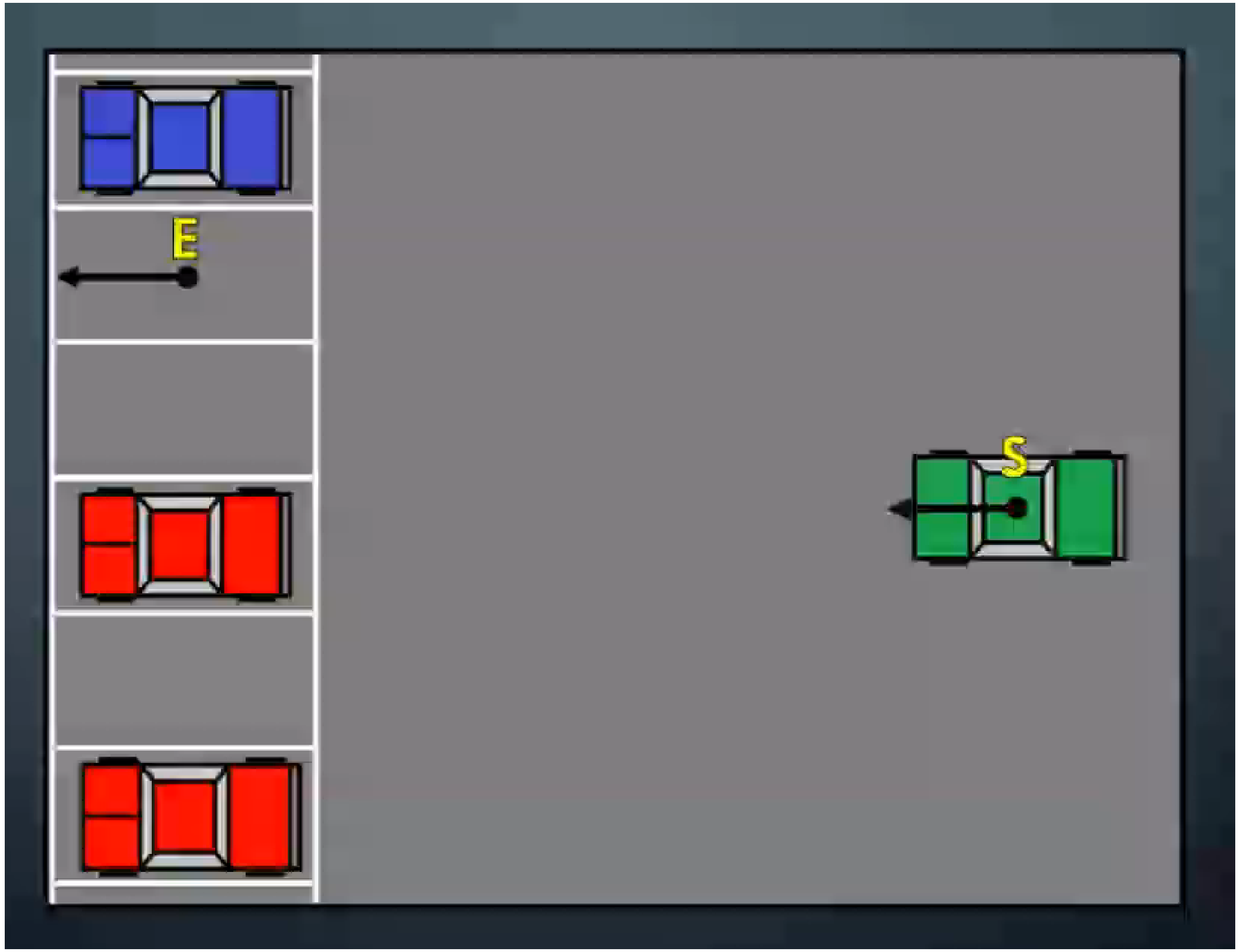
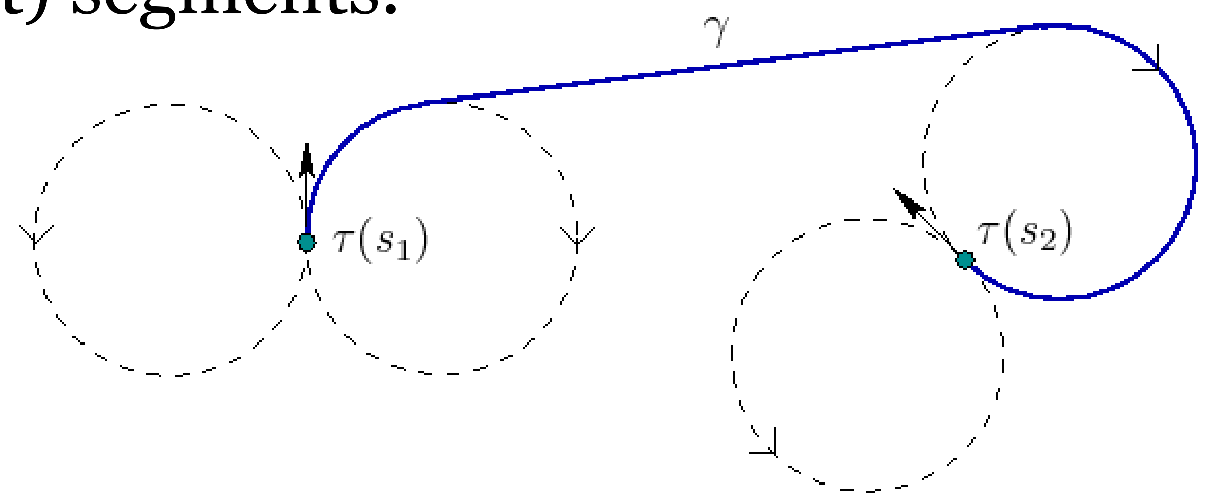


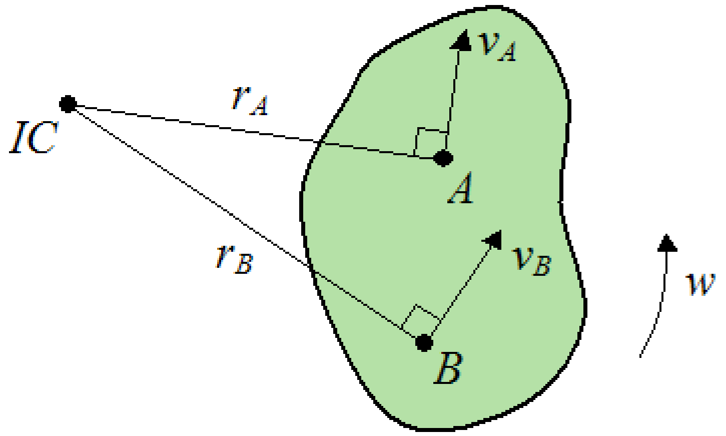
# Dubins car: motion primitives

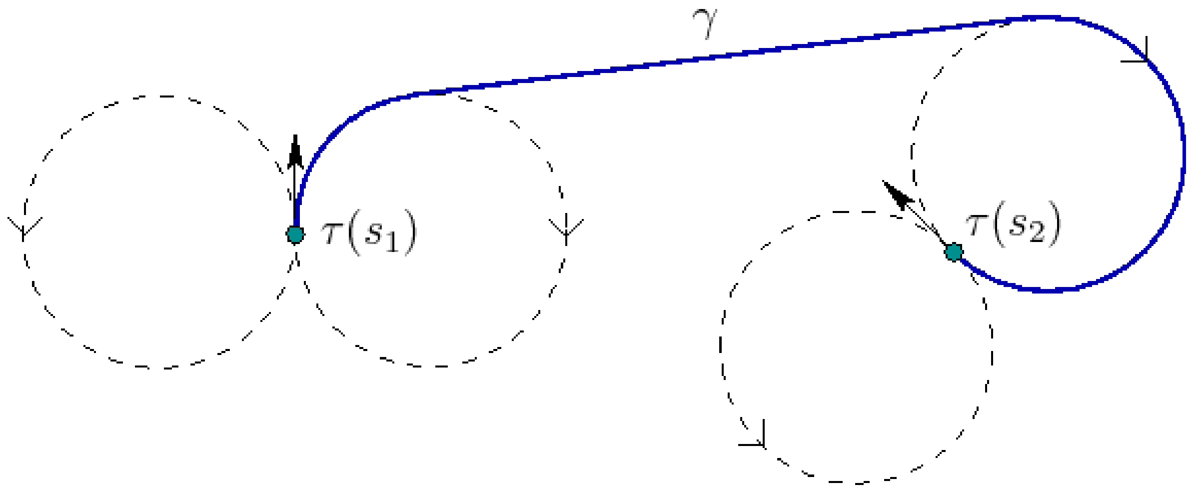
- The path of the car can be decomposed to L(eft), R(ight), S(traight) segments.




RSR path

# Instantaneous Center of Rotation

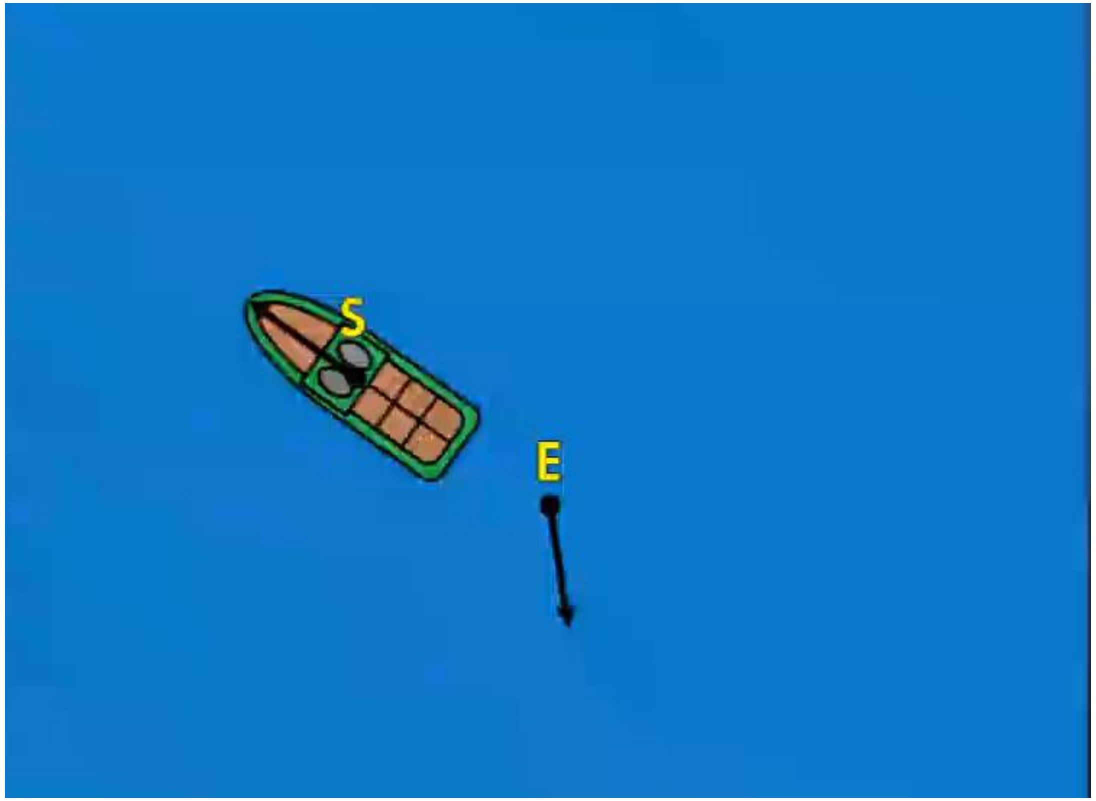



IC = Instantaneous Center of Rotation
The center of the circle circumscribed by the turning path.
Undefined for straight path segments.

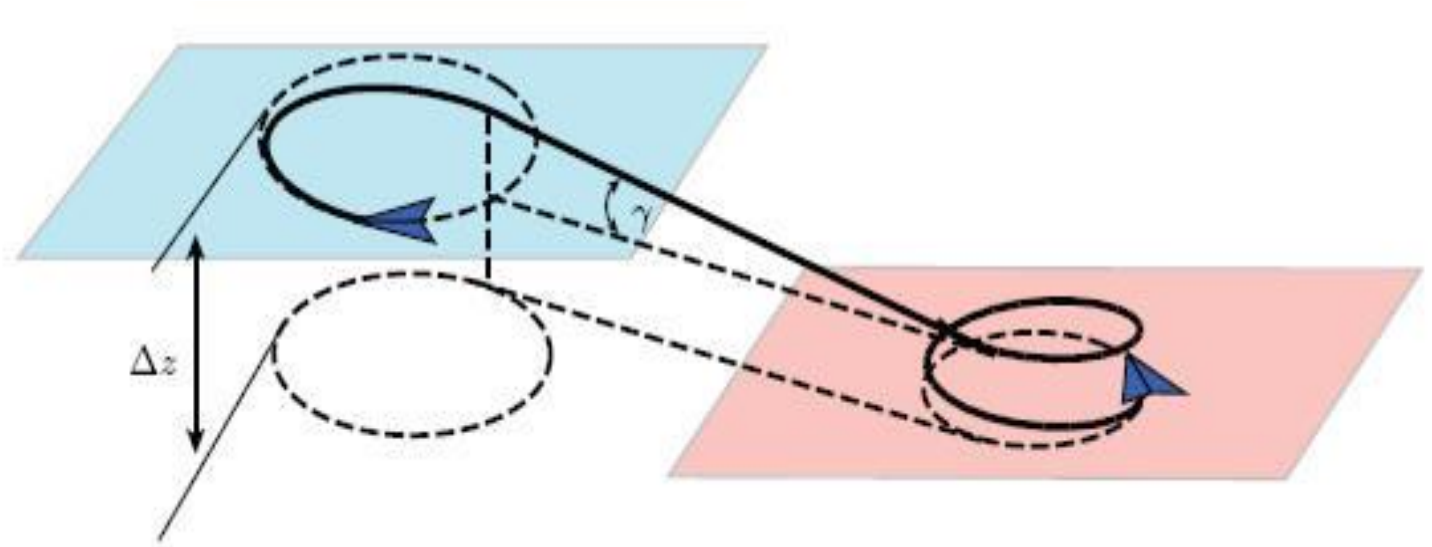
# Dubins car → Dubins boat

- Why do we care about a car that can only go forward?
- Because we can also model idealized airplanes and boats
- Dubins boat = Dubins car



# Dubins car $\longrightarrow$ Dubins airplane in 3D

- Pitch angle $\phi$ and forward velocity determine descent rate
- Yaw angle $\theta$ and forward velocity determine turning rate



(a) The 3D view of the path.

$$
\begin{aligned}
\dot{p}_x &= v_x \cos(\theta)\sin(\phi) \\
\dot{p}_y &= v_x \sin(\theta)\sin(\phi) \\
\dot{p}_z &= v_x \cos(\phi) \\
\dot{\theta} &= \omega_z \\
\dot{\phi} &= \omega_y
\end{aligned}
$$

$\theta$ is yaw

$\phi$ is pitch

# Holonomic constraints

- Equality constraints on the state of the system, but not on the higher-order derivatives:

$$\mathbf{f}(\mathbf{x}, t) = 0$$

- For example, if you want to constrain the state to lie on a circle:

$$||\mathbf{x}||^2 - 1 = 0$$

- Another example: train tracks are a holonomic constraint.

# Non-holonomic constraints

- Equality constraints that involve the derivatives of the state (e.g. velocity) in a way that it cannot be integrated out into holonomic constraints, i.e.

$$\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t) = 0$$

but not

$$\mathbf{f}(\mathbf{x}, t) = 0$$

# The Dubins car is non-holonomic

- Dubins car is constrained to move straight towards the direction it is currently heading. It cannot move sideways. It needs to "parallel park" to move laterally.

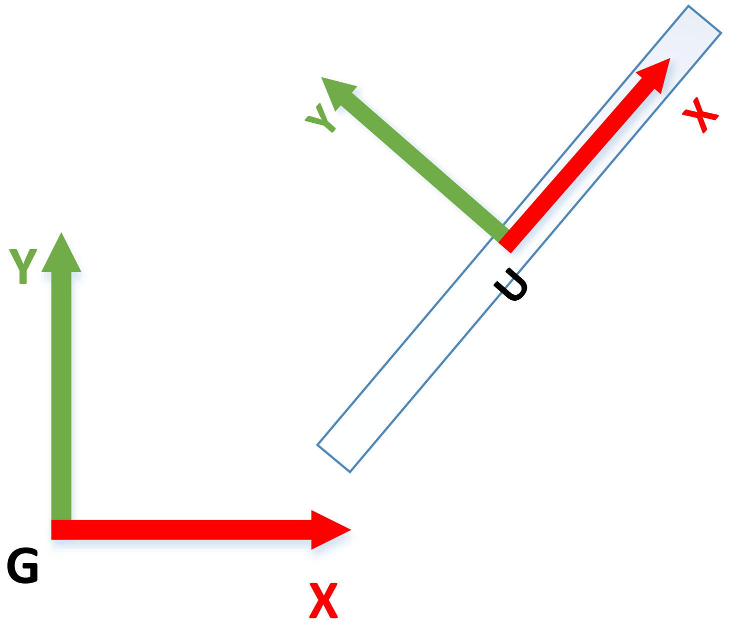
- In a small time interval dt the vehicle is going to move by $\delta p_x$ and $\delta p_y$ in the global frame of reference. Then from the dynamical system:

$$\begin{aligned} \delta p_x \sin(\theta) &= v_x \cos(\theta)\sin(\theta) \\ \delta p_y \cos(\theta) &= v_x \sin(\theta)\cos(\theta) \end{aligned} \Rightarrow \delta p_x \sin(\theta) = \delta p_y \cos(\theta) \Rightarrow v_x \sin(\theta) = v_y \cos(\theta)$$

Car is constrained to move along the line of current heading, i.e. non-holonomic

# The state of a unicycle

$$\mathbf{x} = [{}^G p_x, {}^G p_y, {}^G \theta]$$



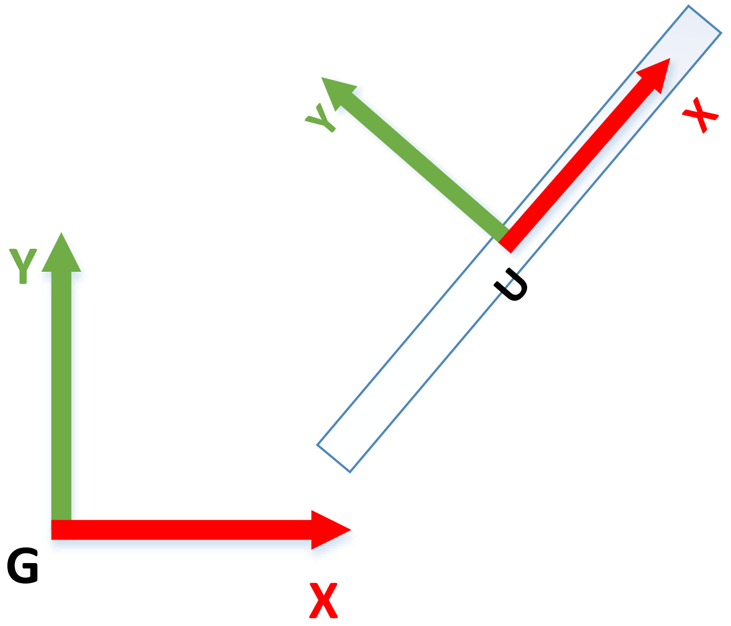

Top view of a unicycle

State = [Position, Orientation]
Position of the unicycle's frame of reference U with respect to a fixed frame of reference G, expressed in coordinates of frame G. Angle is the orientation of frame U with respect to frame G.

Q: Would you put the radius of the unicycle to be part of the state?

# The state of a unicycle

$$\mathbf{x} = [{}^{G}p_x, {}^{G}p_y, {}^{G}\theta]$$



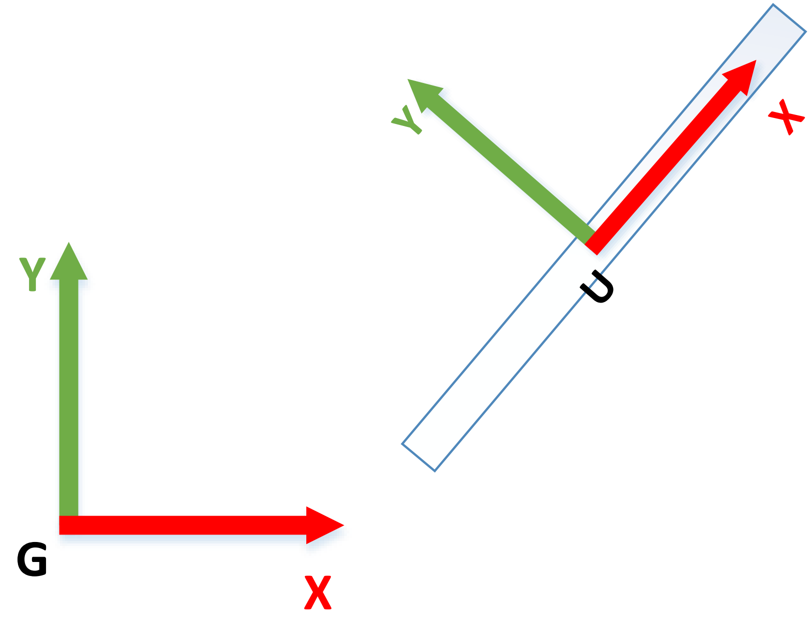
Top view of a unicycle

State = [Position, Orientation]
Position of the unicycle's frame of reference U with respect to a fixed frame of reference G, expressed in coordinates of frame G. Angle is the orientation of frame U with respect to frame G.

Q: Would you put the radius of the unicycle to be part of the state?
A: Most likely not, because it is a constant quantity that we can measure beforehand. But, if we couldn't measure it, we need to make it part of the state in order to estimate it.

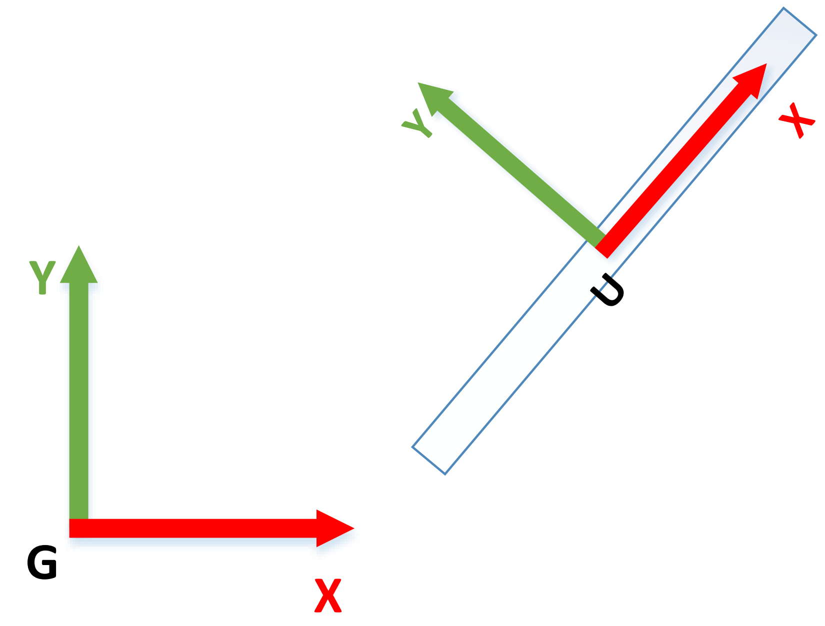
# Controls of a unicycle

$$\mathbf{u} = [{}^{U}\omega_z, {}^{U}\omega_y]$$




Controls = [Yaw rate,  and pedaling rate]
Yaw and pedaling rates describe the angular velocities of the respective axes of the unicycle's frame of reference U with respect to a fixed frame of reference G, expressed in coordinates of U.

# Dynamics of a unicycle

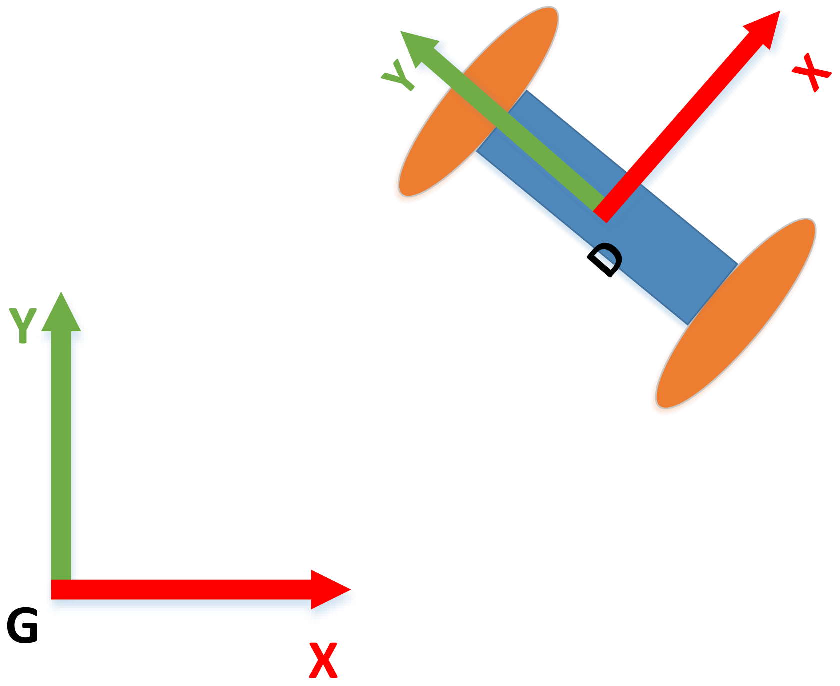



$$
\begin{aligned}
\dot{p}_x &= r\omega_y \cos(\theta) \\
\dot{p}_y &= r\omega_y \sin(\theta) \\
\dot{\theta} &= \omega_z
\end{aligned}
$$

r = the radius of the wheel

$r\omega_y$ is the forward velocity of the unicycle

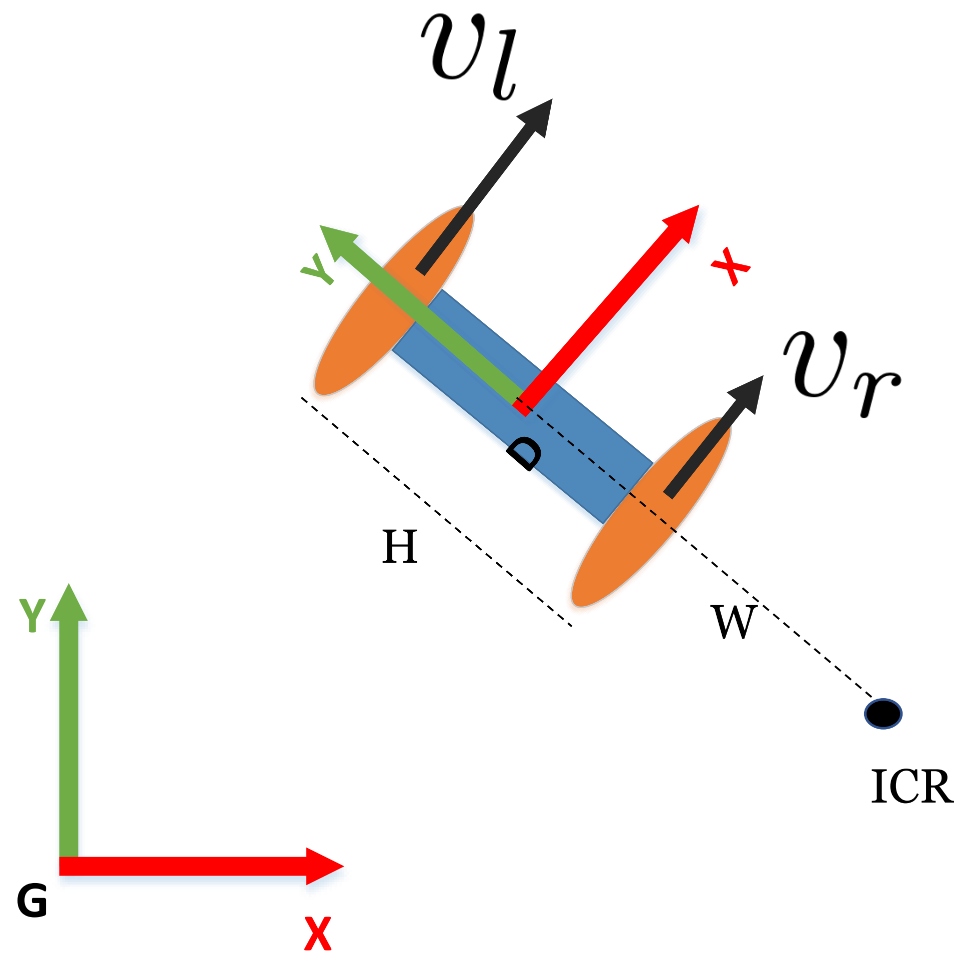
# The state of a differential drive vehicle




$$\mathbf{x} = [{}^{G}p_x, {}^{G}p_y, {}^{G}\theta]$$

State = [Position, Orientation]
Position of the vehicle's frame of reference D with respect to a fixed frame of reference G, expressed in coordinates of frame G. Angle is the orientation of frame D with respect to frame G.

# Controls of a differential drive vehicle




$$\mathbf{u} = [{}^D\omega_l, {}^D\omega_r]$$

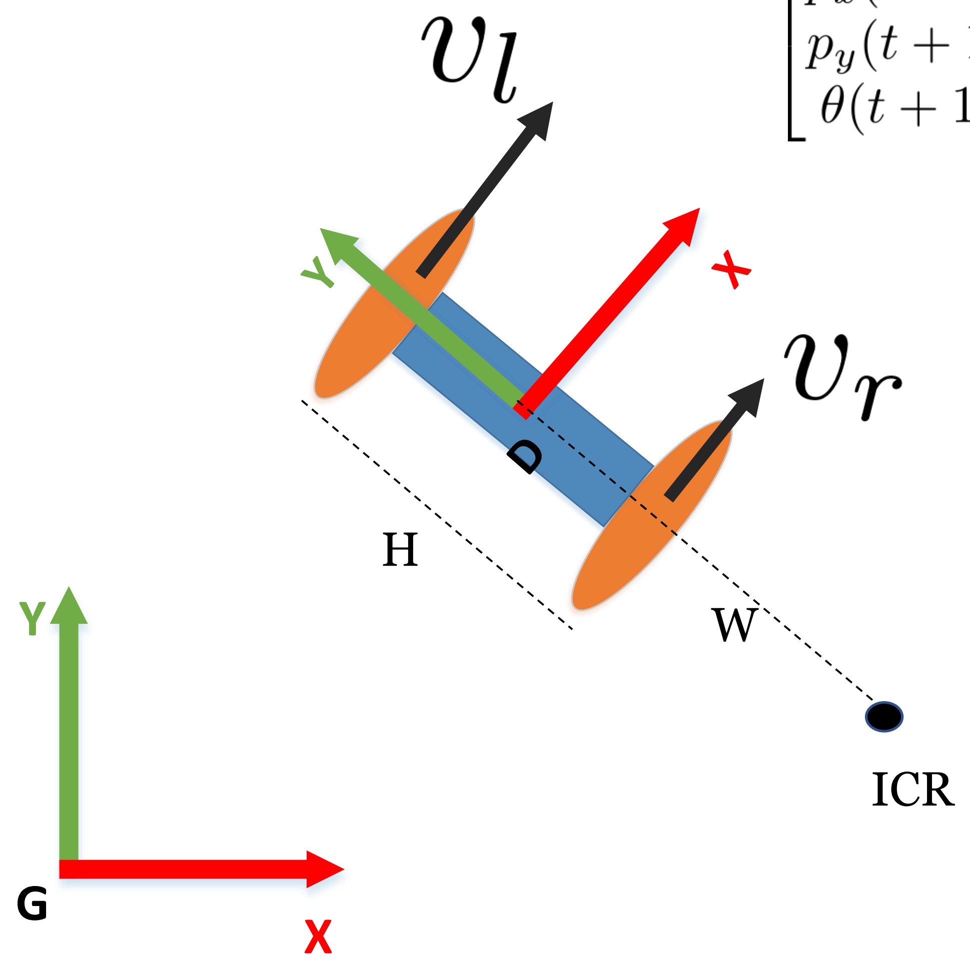
Controls = [Left wheel and right wheel turning rates]
Wheel turning rates determine the linear velocities of the respective wheels of the vehicle's frame of reference D with respect to a fixed frame of reference G, expressed in coordinates of D.

$$v_l = (W - H/2)\omega_l$$

$$v_r = (W + H/2)\omega_r$$

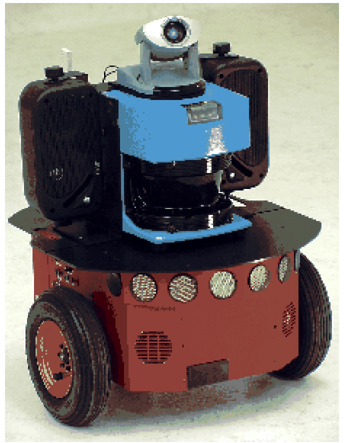
$$v_x = (v_l + v_r)/2$$

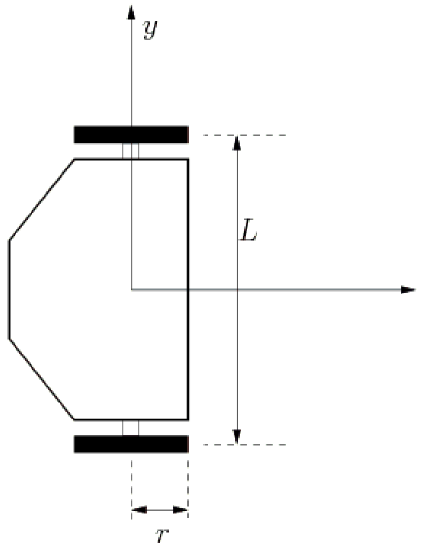
# Dynamics of a differential drive vehicle



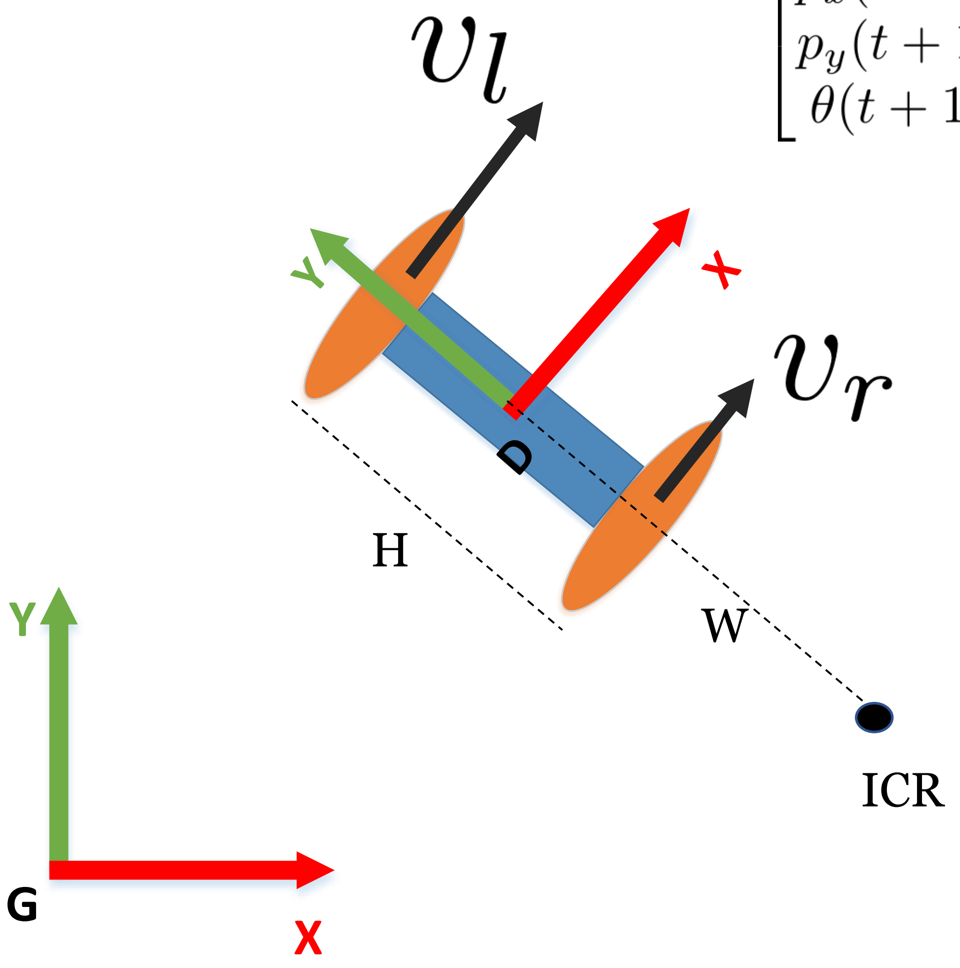

$$\begin{bmatrix} p_x(t+1) \\ p_y(t+1) \\ \theta(t+1) \end{bmatrix} = \begin{bmatrix} \cos(\omega\delta t) & -\sin(\omega\delta t) & 0 \\ \sin(\omega\delta t) & \cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x(t) - \mathrm{ICR}_x \\ p_y(t) - \mathrm{ICR}_y \\ \theta(t) \end{bmatrix} + \begin{bmatrix} \mathrm{ICR}_x \\ \mathrm{ICR}_y \\ \omega\delta t \end{bmatrix}$$

$$\mathrm{ICR} = [p_x - W\sin\theta, p_y + W\cos\theta]$$





# Dynamics of a differential drive vehicle



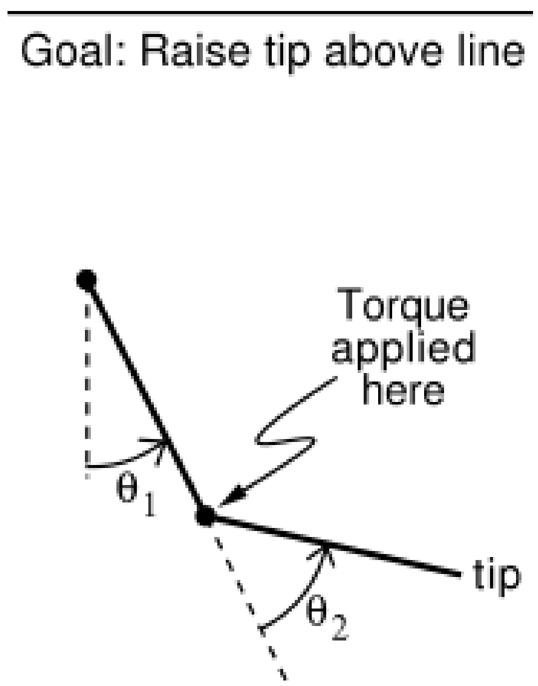

$$\begin{bmatrix} p_x(t+1) \\ p_y(t+1) \\ \theta(t+1) \end{bmatrix} = \begin{bmatrix} \cos(\omega\delta t) & -\sin(\omega\delta t) & 0 \\ \sin(\omega\delta t) & \cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x(t) - \text{ICR}_x \\ p_y(t) - \text{ICR}_y \\ \theta(t) \end{bmatrix} + \begin{bmatrix} \text{ICR}_x \\ \text{ICR}_y \\ \omega\delta t \end{bmatrix}$$

$$\text{ICR} = [p_x - W\sin\theta, p_y + W\cos\theta]$$

Special cases:

- moving straight $v_l = v_r$
- in-place rotation $v_l = -v_r$
- rotation about the left wheel $v_l = 0$

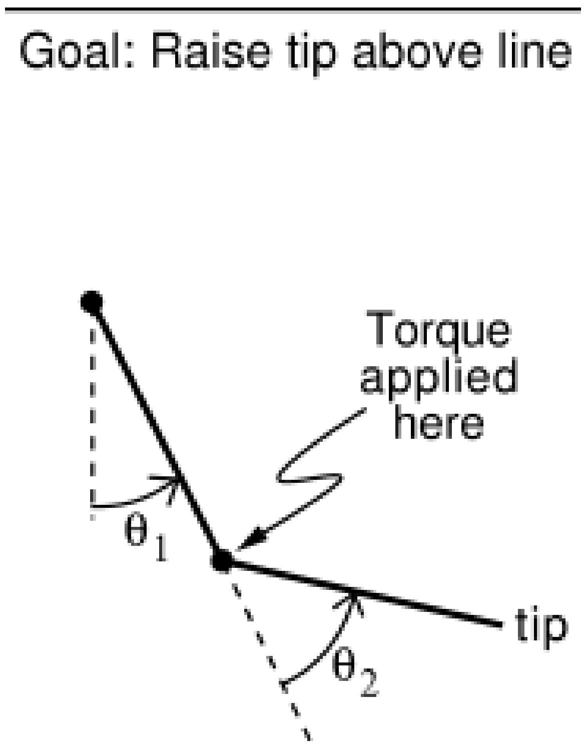
# The state of a double-link inverted pendulum (a.k.a. Acrobot)




$$\mathbf{x} = [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]$$

State = [angle of joint 1, joint 2, joint velocities]
Angle of joint 2 is expressed with respect to joint 1. Angle of joint 1 is expressed compared to down vector.

# Controls of a double-link inverted pendulum (a.k.a. Acrobot)




$$\mathbf{u} = [\tau_1]$$

Controls = [torque applied to joint 1]

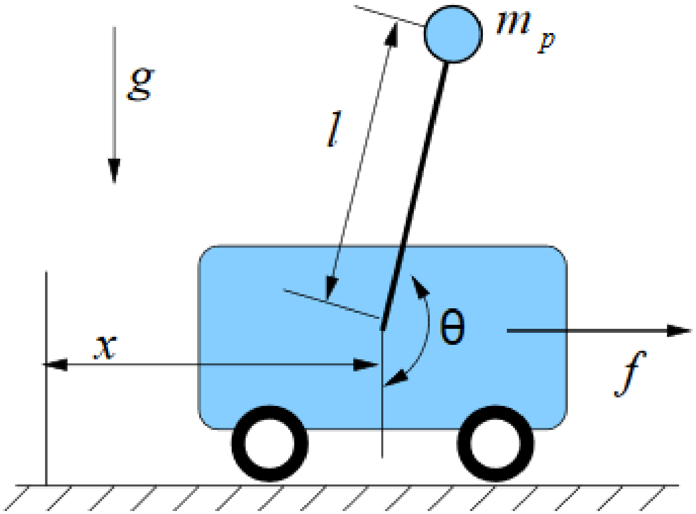
# Dynamics of a double-link inverted pendulum (a.k.a Acrobot)

$$
\begin{aligned}
\ddot{\theta}_1 &= -d_1^{-1}(d_2\ddot{\theta}_2 + \phi_1) \\
\ddot{\theta}_2 &= \left(m_2 l_{c2}^2 + I_2 - \frac{d_2^2}{d_1}\right)^{-1}\left(\tau + \frac{d_2}{d_1}\phi_1 - m_2 l_1 l_{c2}\dot{\theta}_1^2 \sin\theta_2 - \phi_2\right) \\
d_1 &= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}\cos\theta_2) + I_1 + I_2 \\
d_2 &= m_2(l_{c2}^2 + l_1 l_{c2}\cos\theta_2) + I_2 \\
\phi_1 &= -m_2 l_1 l_{c2}\dot{\theta}_2^2 \sin\theta_2 - 2 m_2 l_1 l_{c2}\dot{\theta}_2\dot{\theta}_1 \sin\theta_2 \\
&\quad + (m_1 l_{c1} + m_2 l_1) g \cos(\theta_1 - \pi/2) + \phi_2 \\
\phi_2 &= m_2 l_{c2} g \cos(\theta_1 + \theta_2 - \pi/2)
\end{aligned}
$$

Provided here just for reference and completeness. You are not expected to know this.

# Dynamics of a double-link inverted pendulum (a.k.a Acrobot)

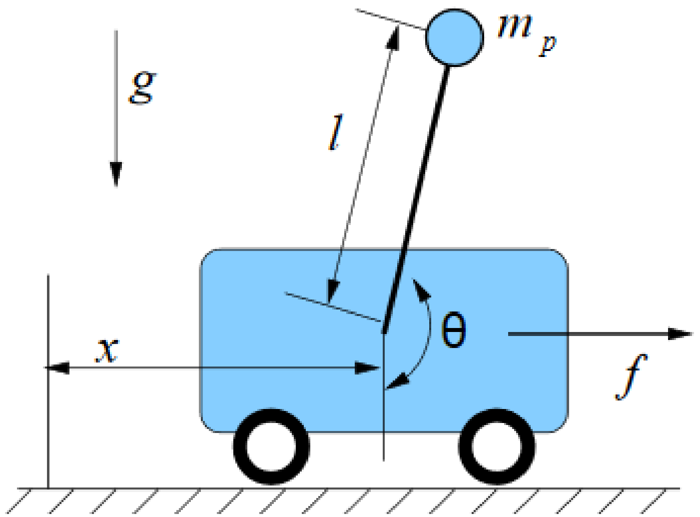
# The state of a single-link cartpole




$$\mathbf{x} = [{}^G p_x, {}^G \dot{p}_x, {}^G \theta, {}^G \dot{\theta}]$$

State = [Position and velocity of cart, orientation and angular velocity of pole]

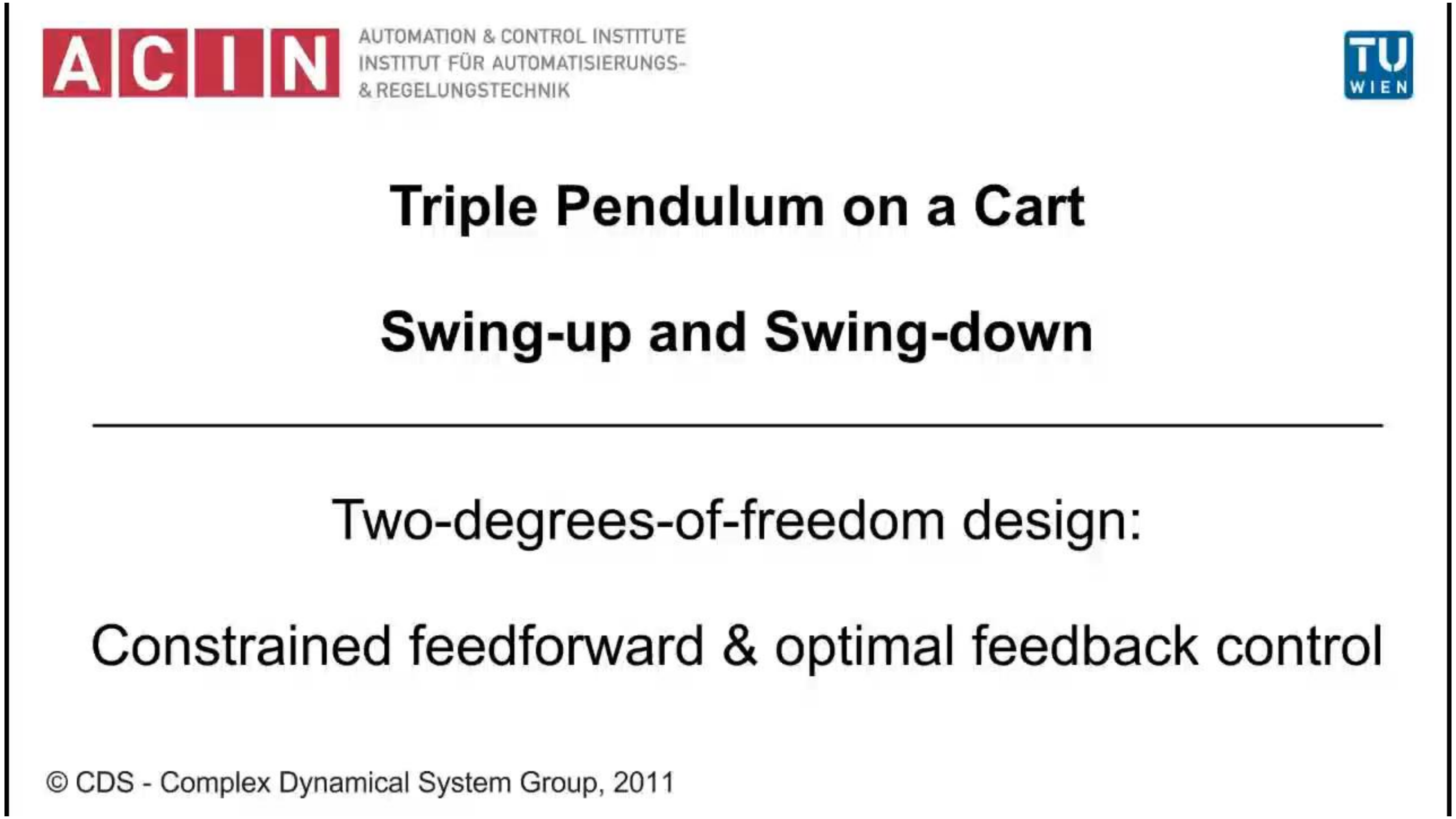
# Controls of a single-link cartpole



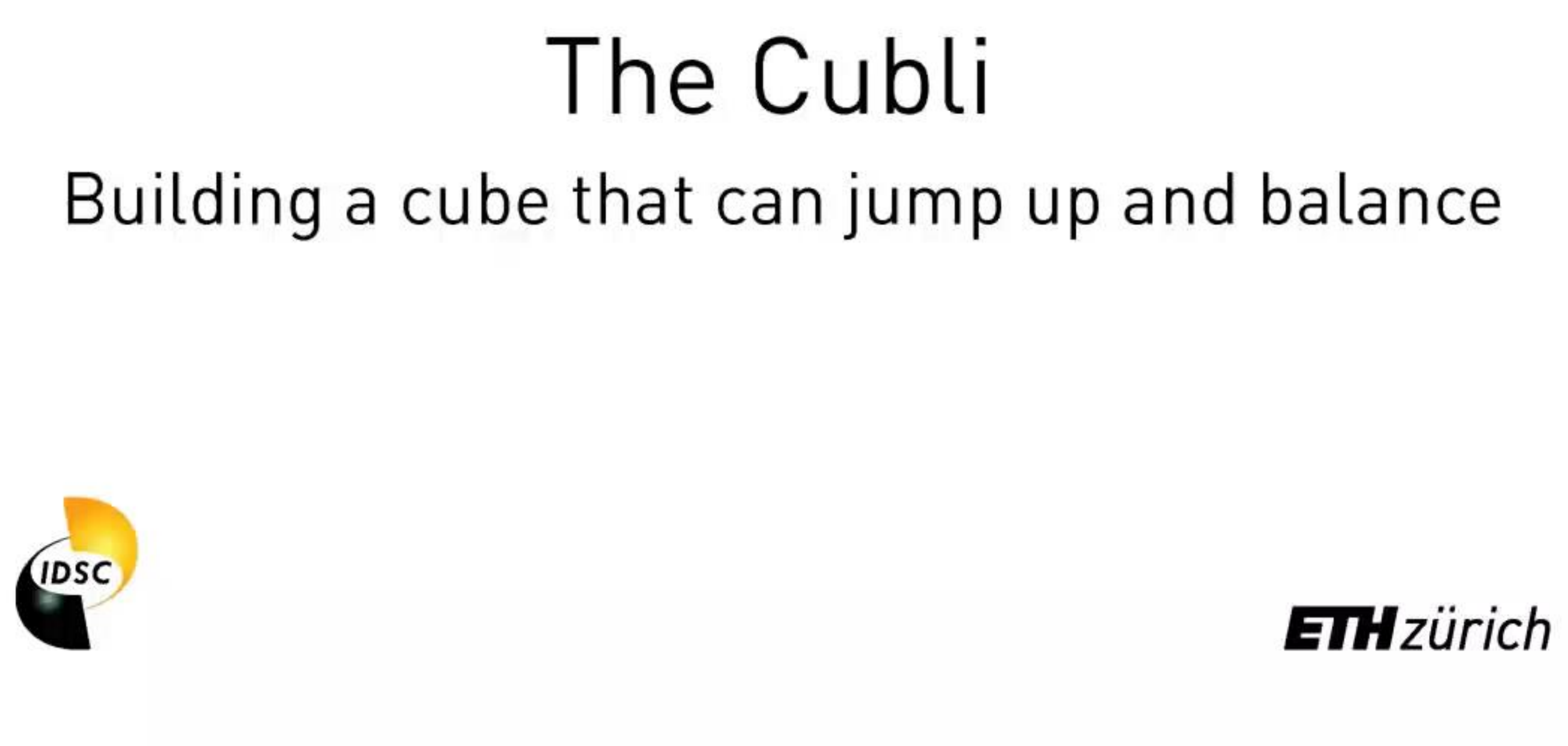

$$\mathbf{u} = [f]$$

Controls = [Horizontal force applied to cart]

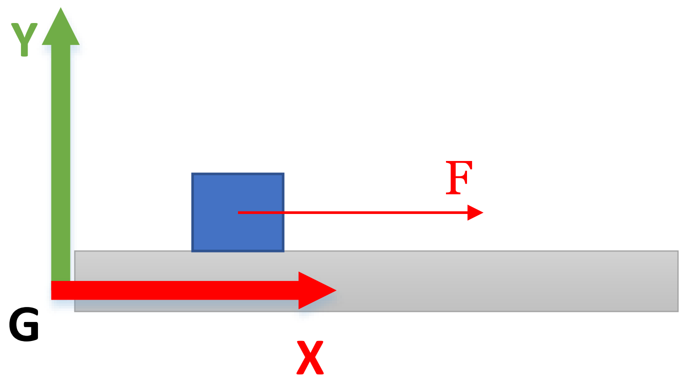
# Balancing a triple-link pendulum on a cart



# Extreme Balancing



# The state of a double integrator



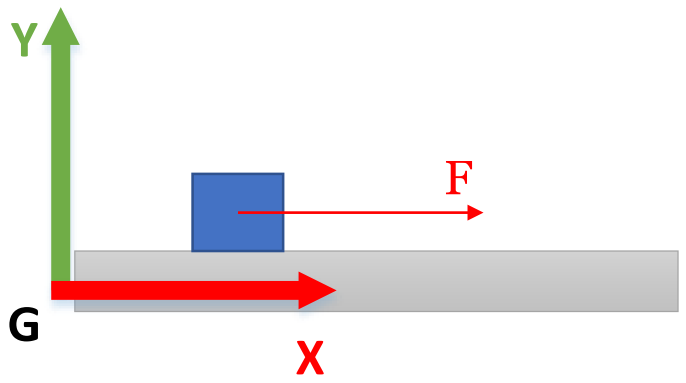

$$\mathbf{x} = \left[ {}^{G}p_x \right]$$

State = [Position along x-axis]

# Controls of a double integrator

$$\mathbf{u} = [{}^{G}u_x]$$

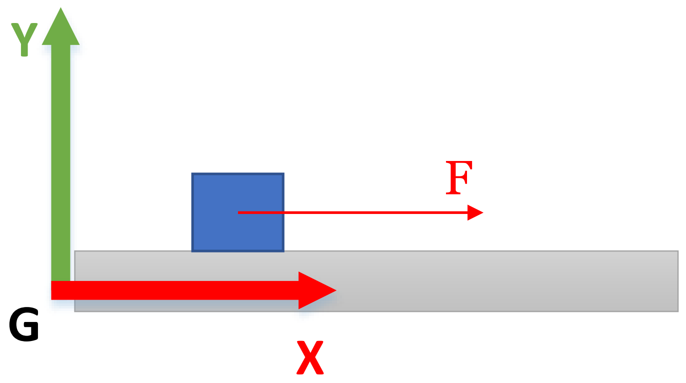
Controls = [Force along x-axis]



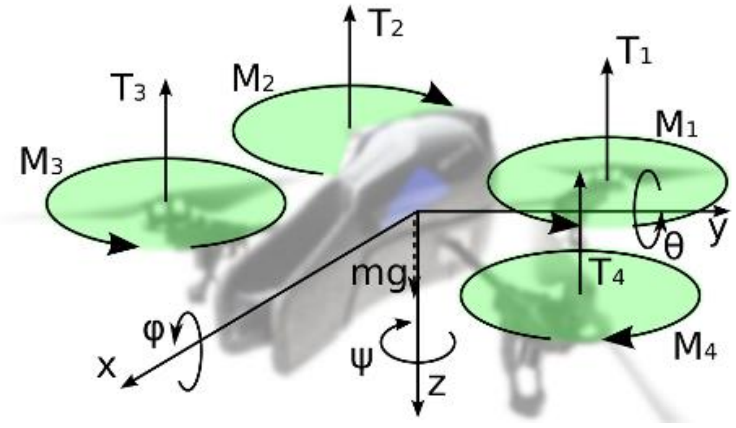
# Dynamics of a double integrator

$$\ddot{\mathbf{x}} = \mathbf{F}$$

This corresponds to applying force to a brick of mass 1 to move on frictionless ice. Where is the brick going to end up? Similar to curling.

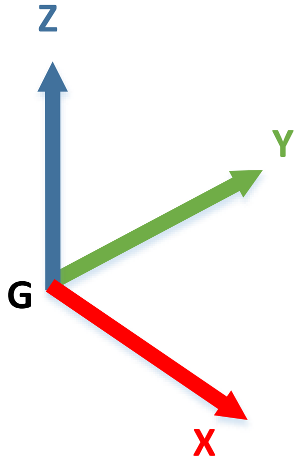


# The state of a quadrotor

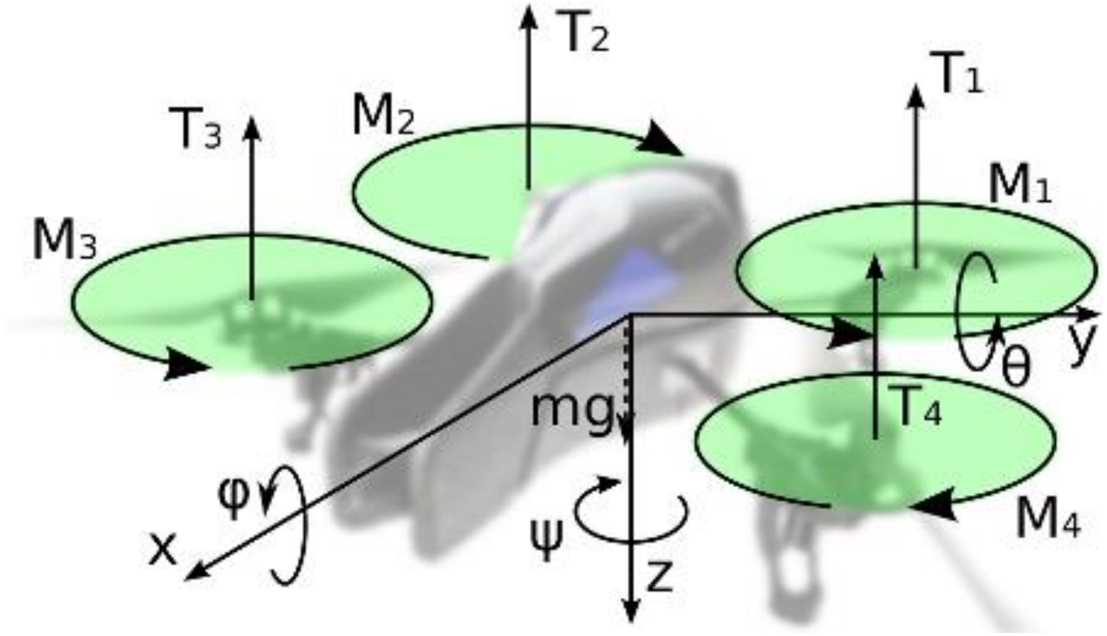



$$\mathbf{x} = [{}^G\phi, {}^G\theta, {}^G\psi, {}^G\dot{\phi}, {}^G\dot{\theta}, {}^G\dot{\psi}]$$

State = [Roll, pitch, yaw, and roll rate, pitch rate, roll rate]
Angles are with respect to the global frame.



# Controls of a quadrotor




$$\mathbf{u} = [T_1, T_2, T_3, T_4]$$

Controls = [Thrusts of four motors]

OR

$$\mathbf{u} = [M_1, M_2, M_3, M_4]$$

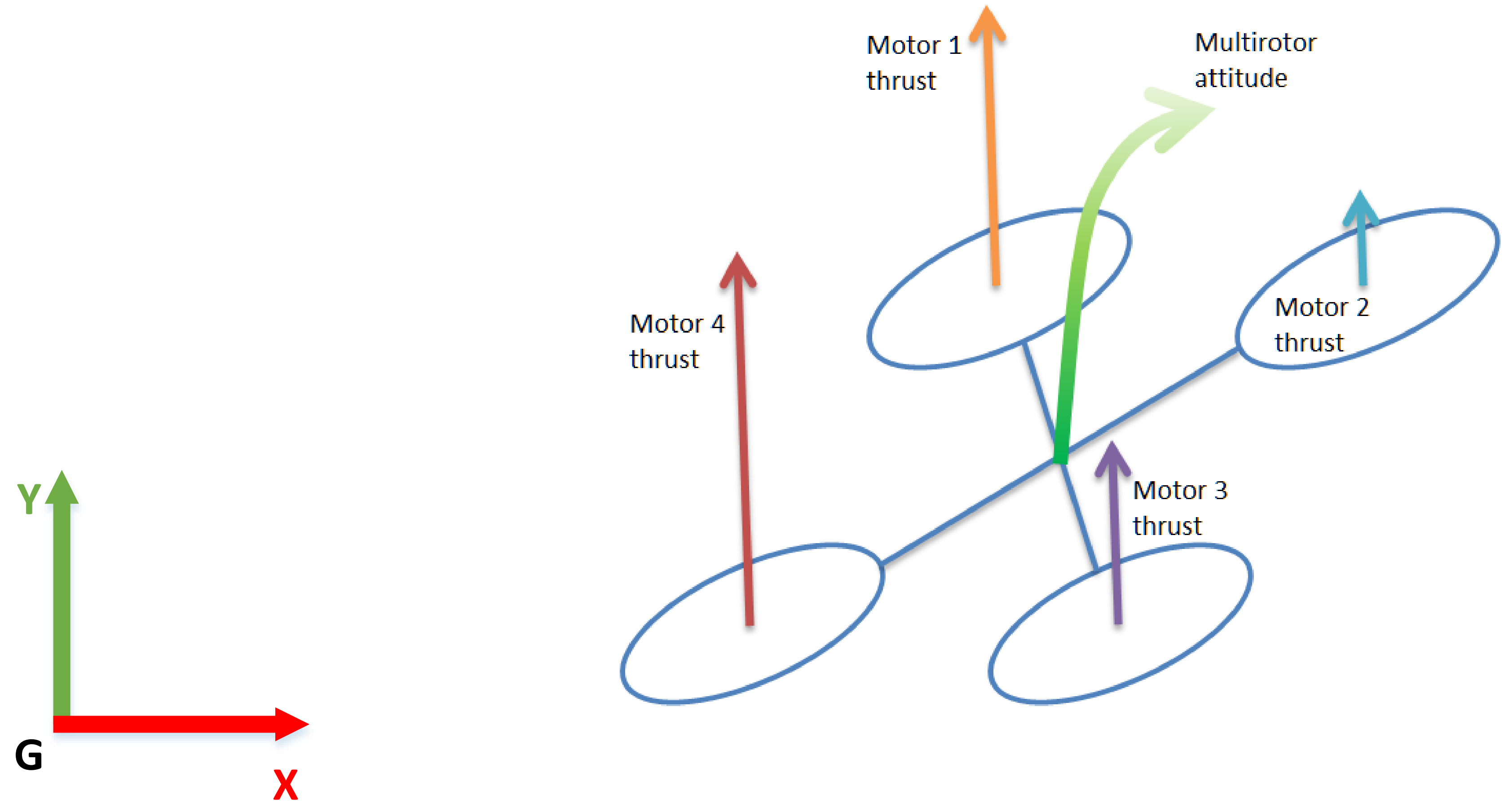
Controls = [Torques of four motors]

Notice how adjacent motors spin in opposite ways. Why?

# What if all four motors spin the same direction?

# Dynamics of a quadrotor

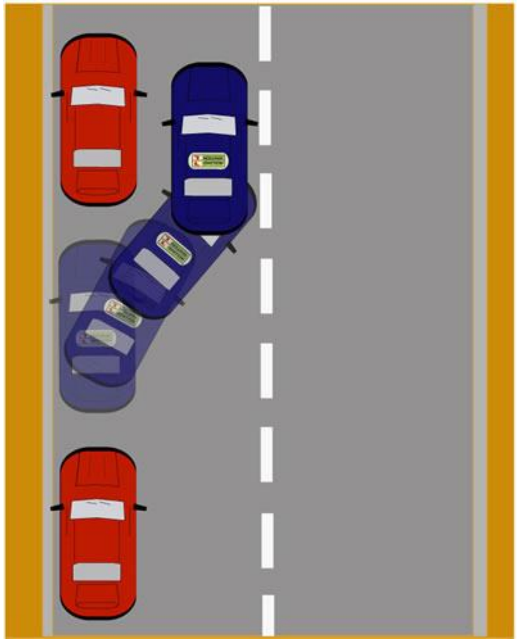
Multirotor(quadcopter)-thrust scheme



# Controllability

- A system is controllable if there exist control sequences that can bring the system from any state to any other state, in finite time.

- For example, even though cars are subject to non-holonomic constraints (can't move sideways directly), they are controllable, They can reach sideways states by parallel parking.



# Passive Dynamics

- Dynamics of systems that operate without drawing (a lot of) energy from a power supply.

- Interesting because biological locomotion systems are more efficient than current robotic systems.

# Passive Dynamics

- Dynamics of systems that operate without drawing (a lot of) energy from a power supply.
- Usually propelled by their own weight.
- Interesting because biological locomotion systems are more efficient than current robotic systems.