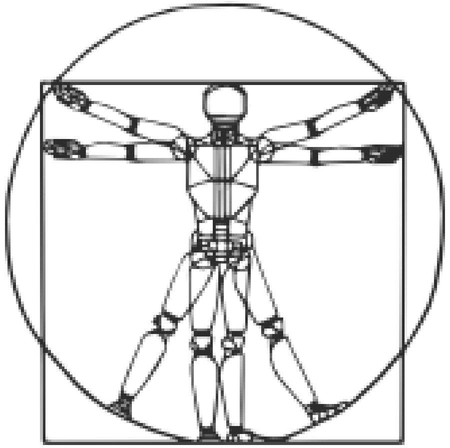

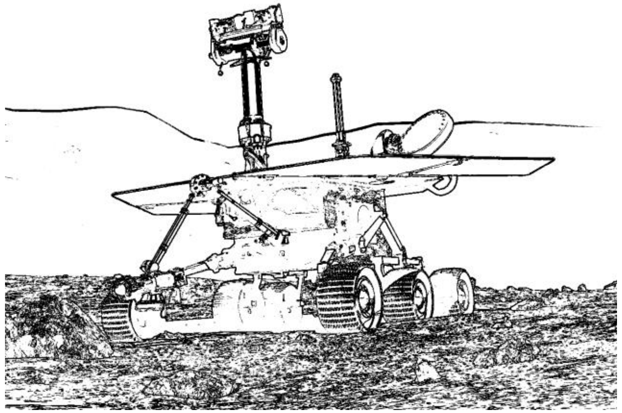

COMP417
Introduction to Robotics and Intelligent Systems

Lecture 13: GraphSLAM

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McGill

MRL Mobile Robotics Lab at McGill University

Goal

- Enable a robot to simultaneously build a map of its environment and estimate where it is in that map.

- This is called SLAM (Simultaneous Localization And Mapping)

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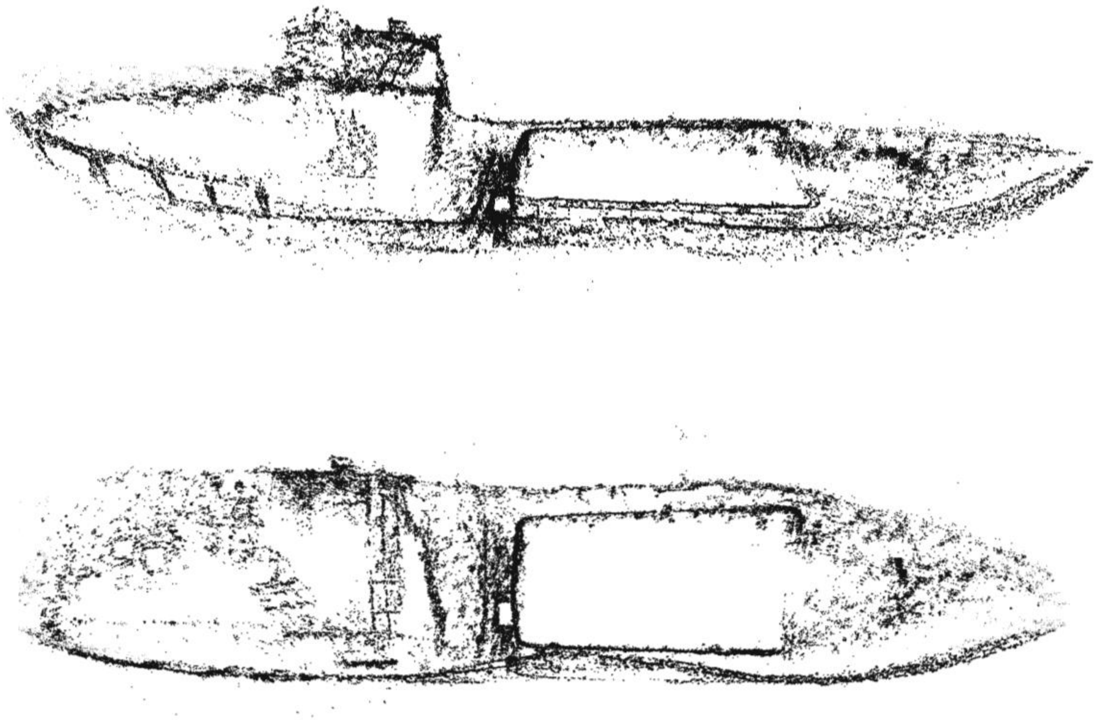
- Enable a robot to simultaneously build a map of its environment and estimate where it is in that map.

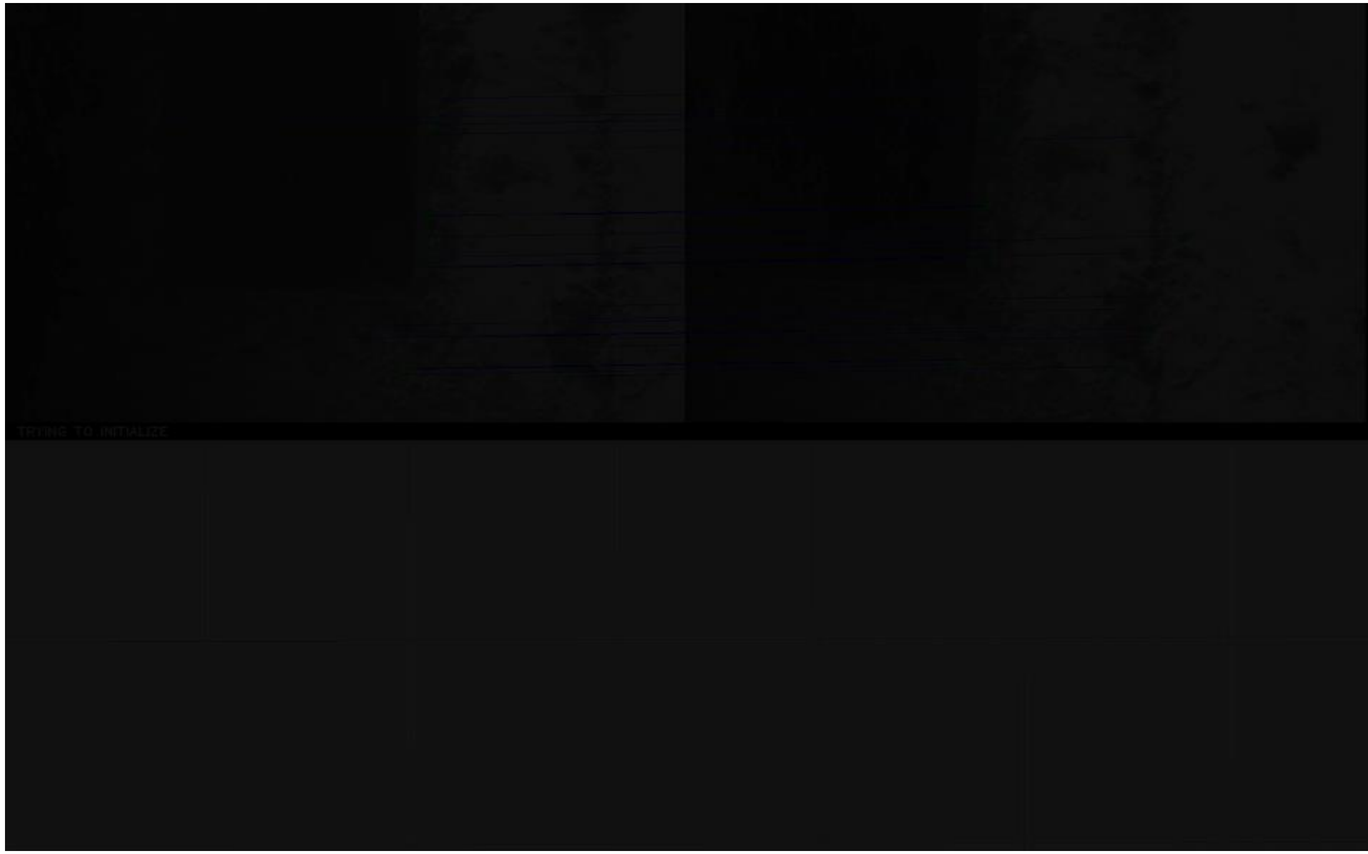
- This is called SLAM (Simultaneous Localization And Mapping)

- Today we are going to look at the batch version, i.e. collect all measurements and controls, and later form an estimate of the states and the map.

- We are going to solve SLAM using least squares

Examples of SLAM systems





MORESLAM system, McGill, 2016

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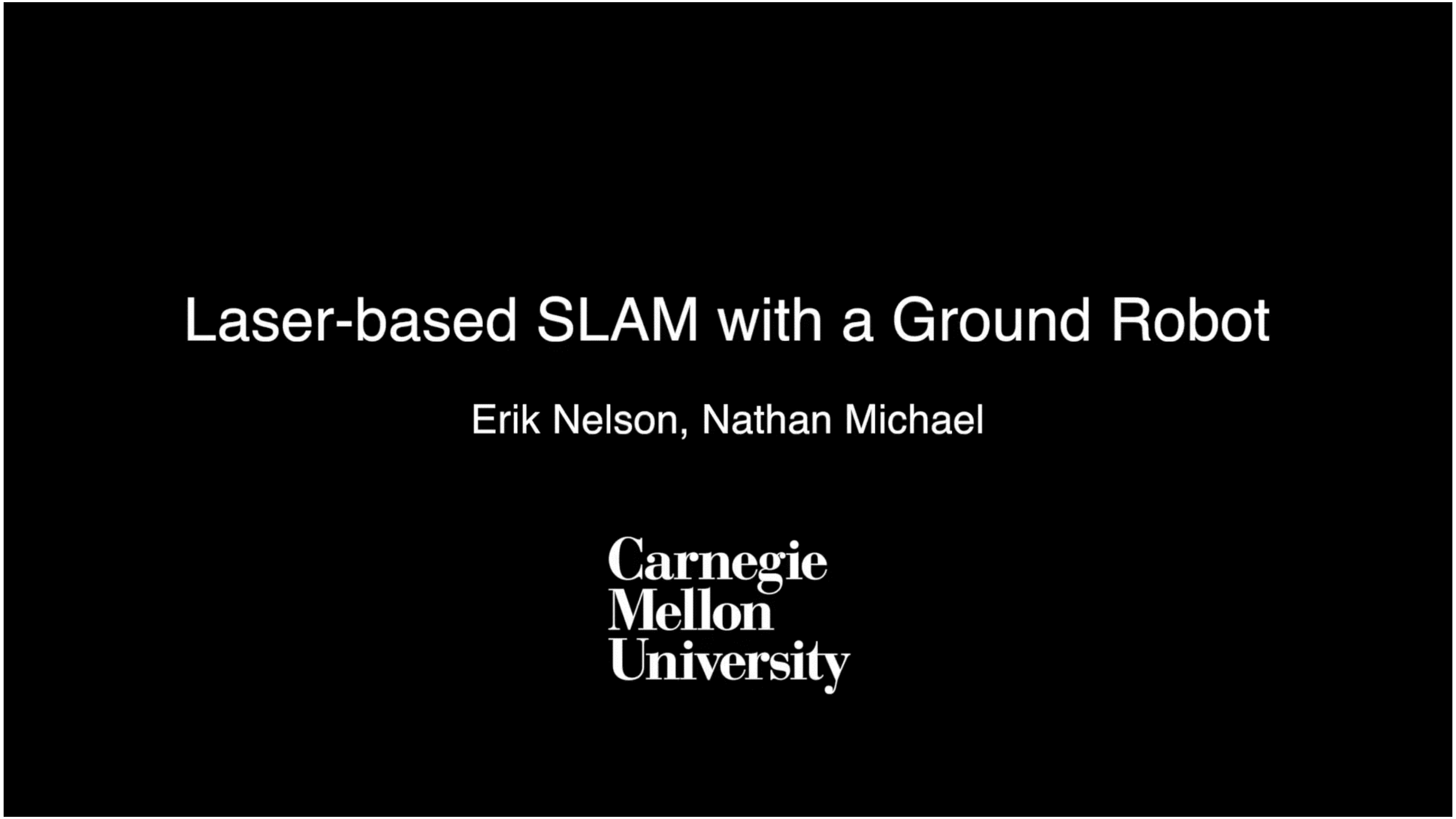
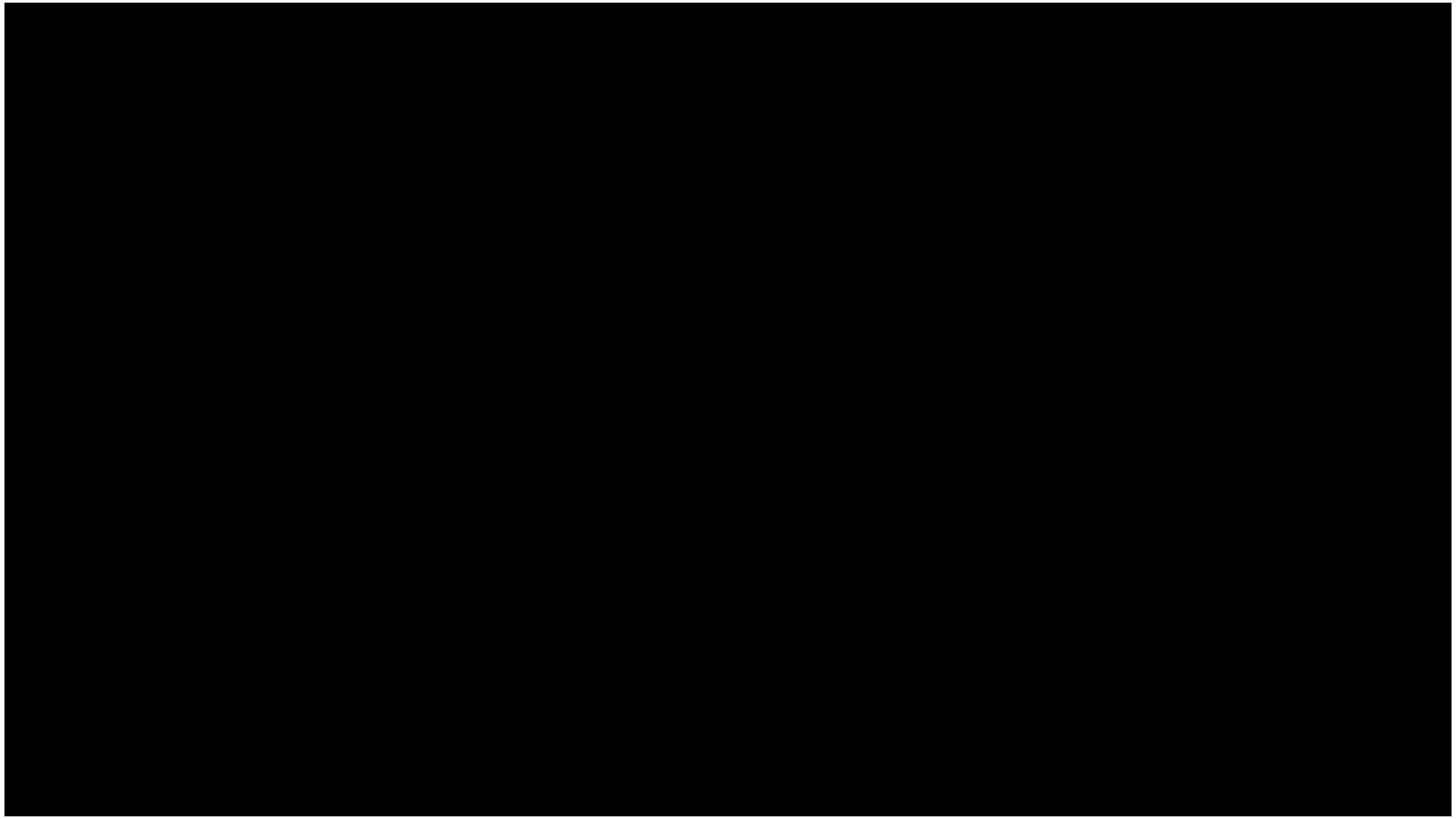
Examples of SLAM systems



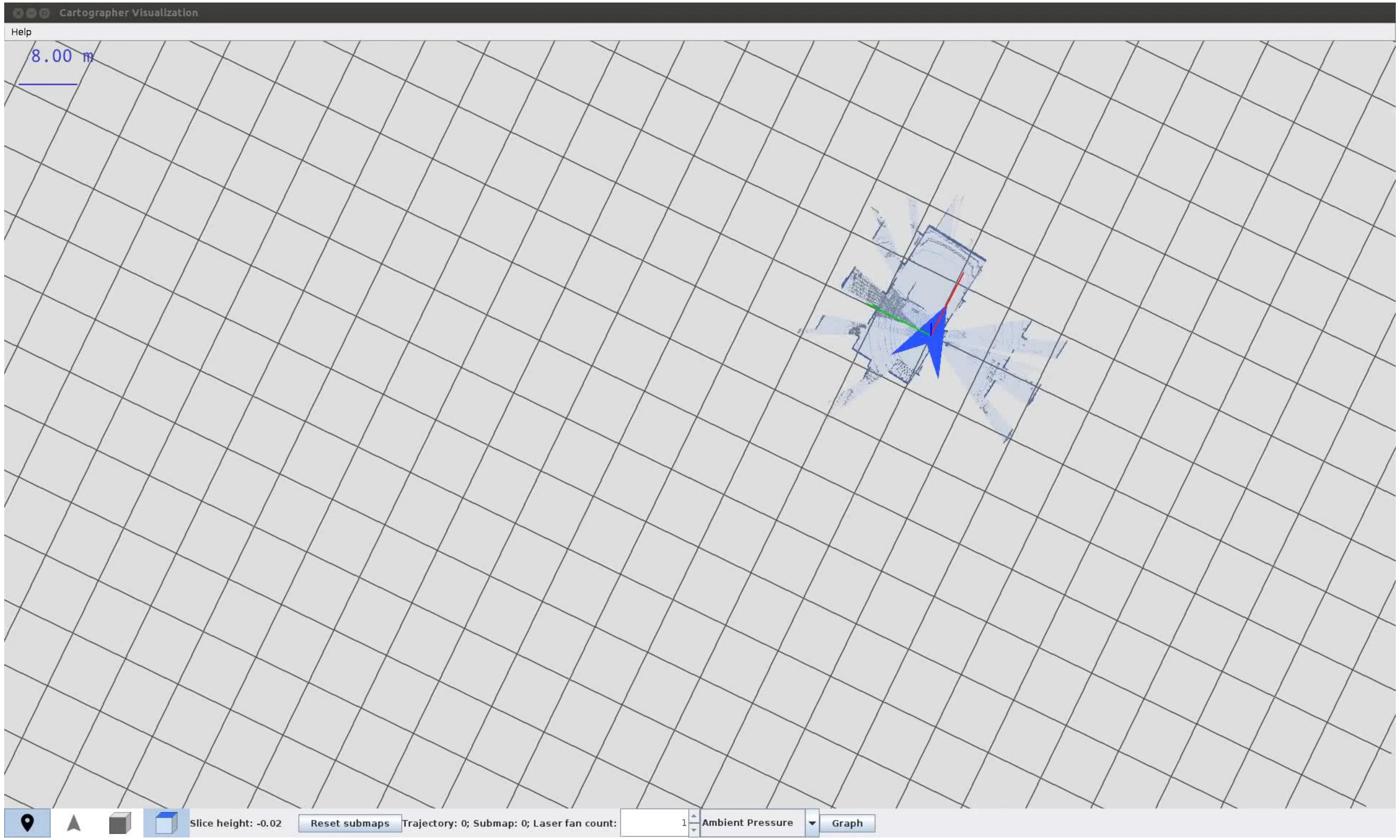
Examples of SLAM systems



Source Code: https://github.com/erik-nelson/blam

Examples of SLAM systems

Google Cartographer: 2D and 3D laser SLAM



Code: https://github.com/googlecartographer/cartographer

SLAM: possible problem definitions

- Smoothing/Batch/Full SLAM

Initial state

$$p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$$

Sequence of states

Map

Sequence of sensor measurements

Sequence of commands

SLAM: possible problem definitions

- Smoothing/Batch/Full SLAM

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Sequence of commands

- Filtering SLAM

$$p(\mathbf{x}_t, \mathbf{m}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1}, \mathbf{x}_0)$$

SLAM: possible problem definitions

In this lecture

- Smoothing/Batch/Full SLAM

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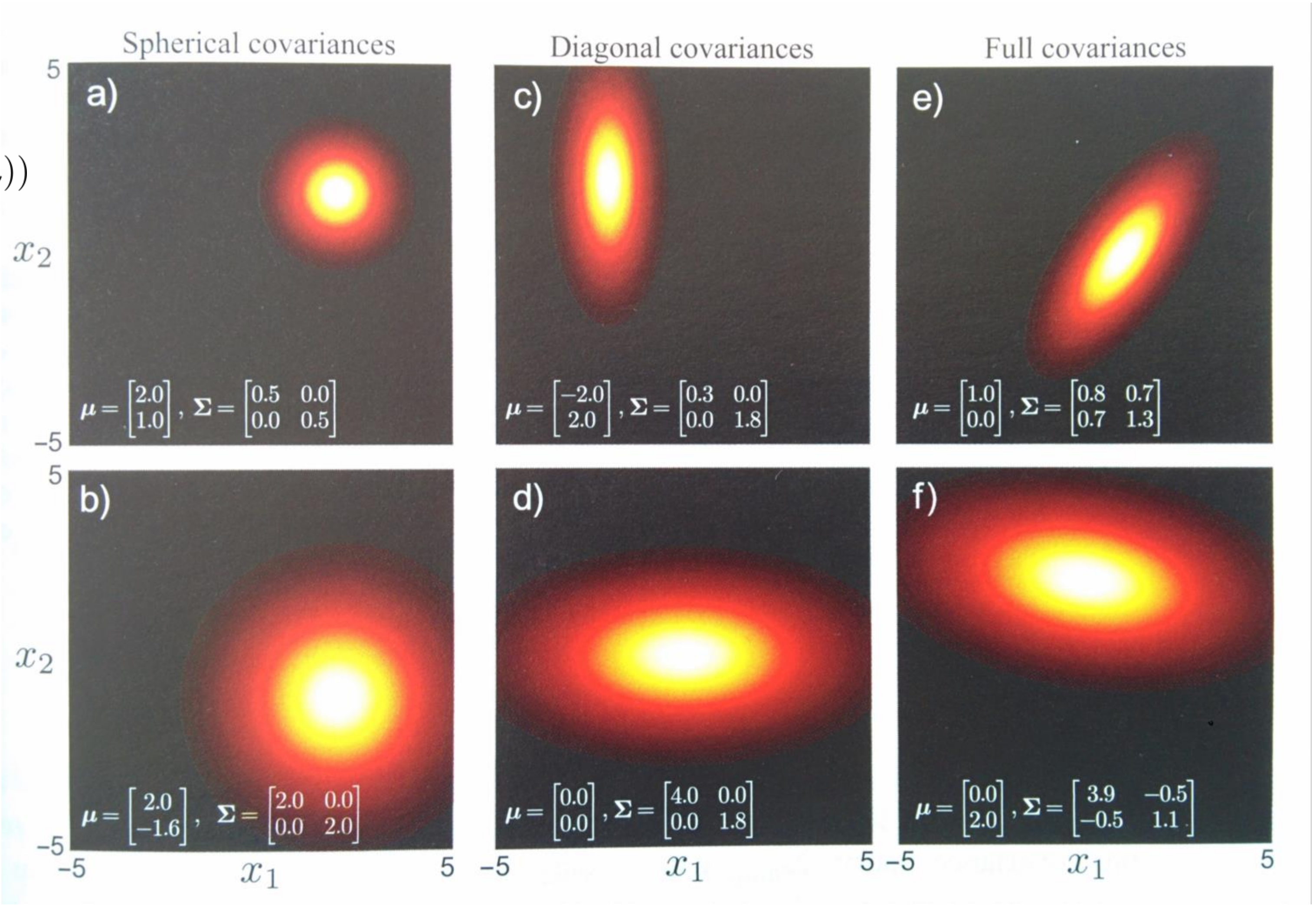
- Filtering SLAM

$$p(\mathbf{x}_t, \mathbf{m}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1}, \mathbf{x}_0)$$

Background: Multivariate Gaussian Distribution

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}\det(\boldsymbol{\Sigma})^{1/2}}\exp(-0.5(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}))$$



From “Computer Vision: Models, Learning, and Inference” Simon Prince

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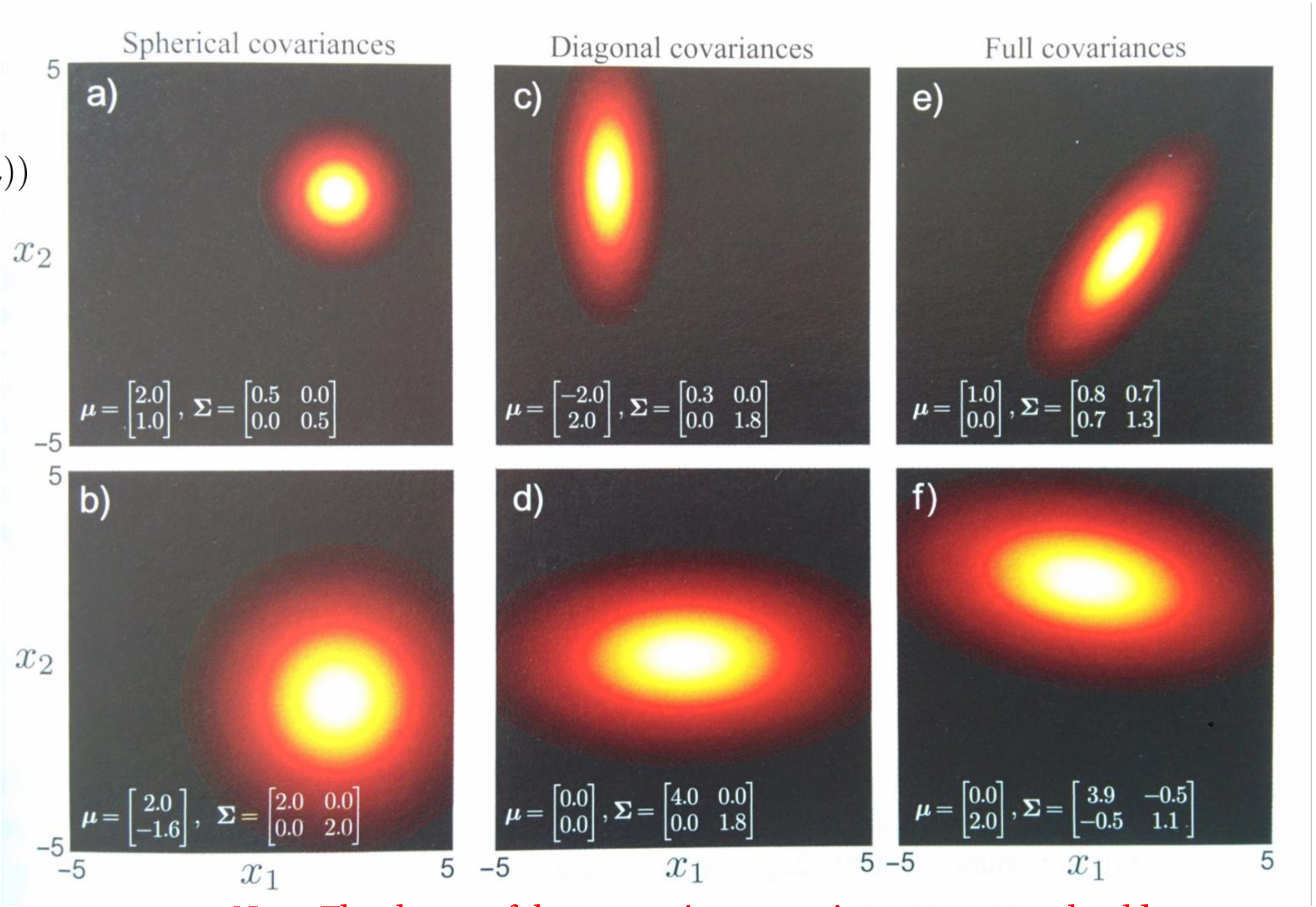
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}\det(\boldsymbol{\Sigma})^{1/2}}\exp(-0.5||\mathbf{x}-\boldsymbol{\mu}||^2_{\boldsymbol{\Sigma}})$$

Shortcut notation: $||\mathbf{x}||^2_{\boldsymbol{\Sigma}} = \mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}$



From "Computer Vision: Models, Learning, and Inference" Simon Prince

Note: The shapes of these covariances are important, you should know them well. In particular, when are x1 and x2 correlated?

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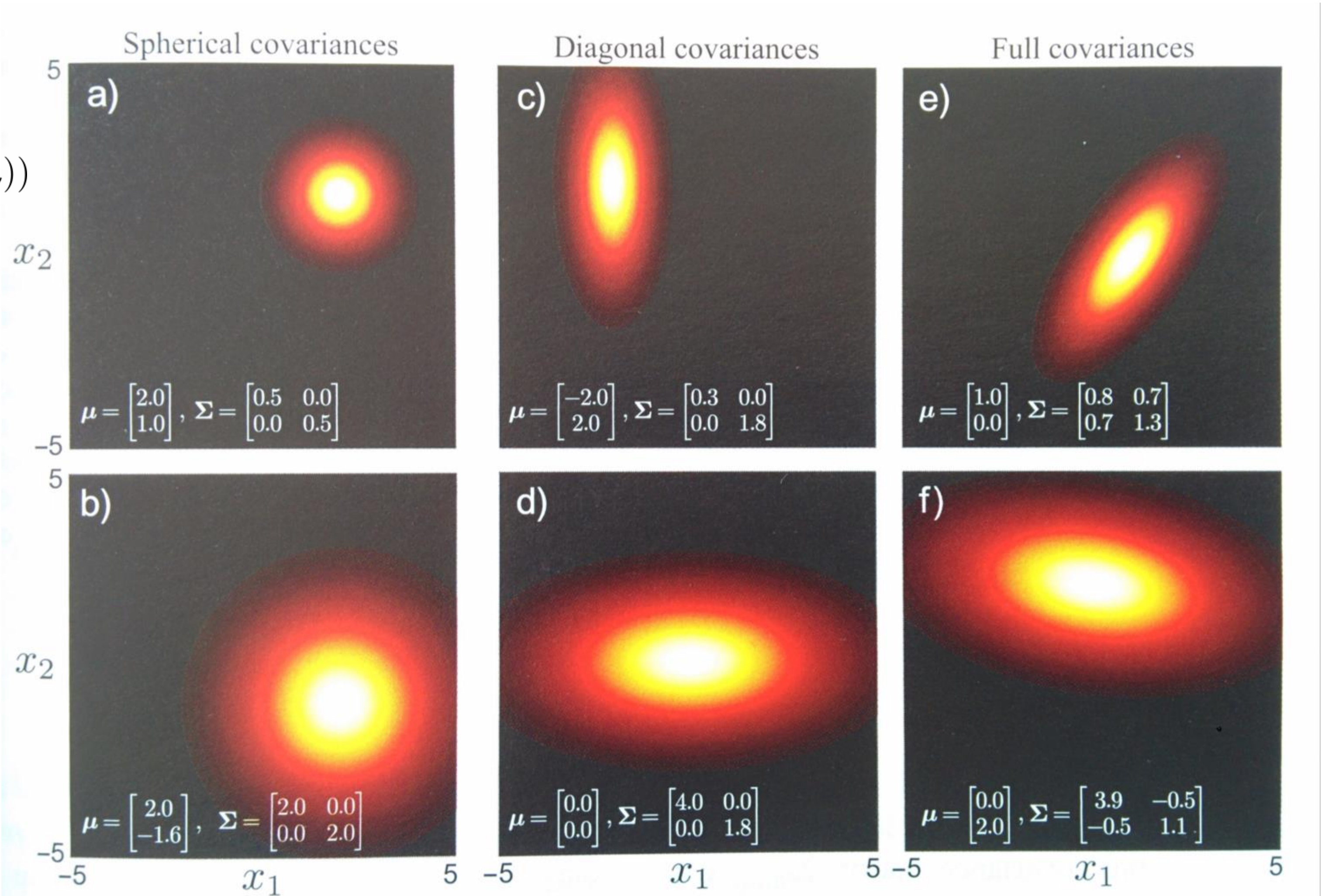


x1 and x2 are correlated when the shape of the ellipse is rotated,
i.e. when there are nonzero off-diagonal terms in the covariance matrix.
In this example, (e) and (f)

From "Computer Vision: Models, Learning, and Inference" Simon Prince

Confidence regions

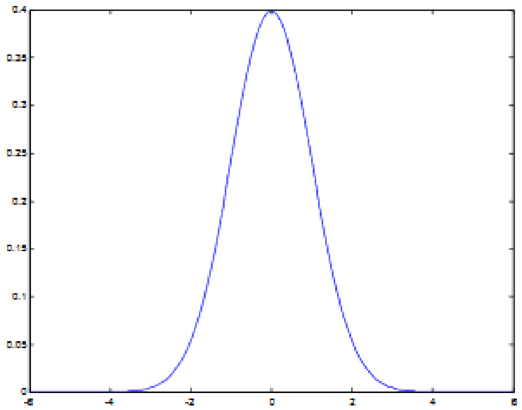
• To quantify confidence and uncertainty define a confidence region R about a point x (e.g. the mode) such that at a confidence level c $\leq$ 1

$$p(x \in R) = c$$

• we can then say (for example) there is a 99% probability that the true value is in R

• e.g. for a univariate normal distribution $N(\mu,\sigma^2)$

$$\begin{aligned} p(|x-\mu| < \sigma) &\approx 0.67 \\ p(|x-\mu| < 2\sigma) &\approx 0.95 \\ p(|x-\mu| < 3\sigma) &\approx 0.997 \end{aligned}$$



Expectation

- Expected value of a random variable X:

$$E_{p(X)}[X] = \int_x xp(X = x)dx$$

- E is linear: $E_{p(X)}[X + c] = E_{p(X)}[X] + c$

$$E_{p(X)}[AX + b] = AE_{p(X)}[X] + b$$

- If X,Y are independent then [Note: inverse does not hold]

$$E_{p(X,Y)}[XY] = E_{p(X)}[X]E_{p(Y)}[Y]$$

Covariance Matrix

- Measures linear dependence between random variables X, Y. Does **not** measure independence.

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y]$$

- Variance of X

$$\mathrm{Var}[X] = \mathrm{Cov}[X] = \mathrm{Cov}[X, X] = E[X^2] - E[X]^2$$

$$\mathrm{Cov}[AX + b] = A\mathrm{Cov}[X]A^T$$

$$\mathrm{Cov}[X + Y] = \mathrm{Cov}[X] + \mathrm{Cov}[Y] - 2\mathrm{Cov}[X, Y]$$

Covariance Matrix

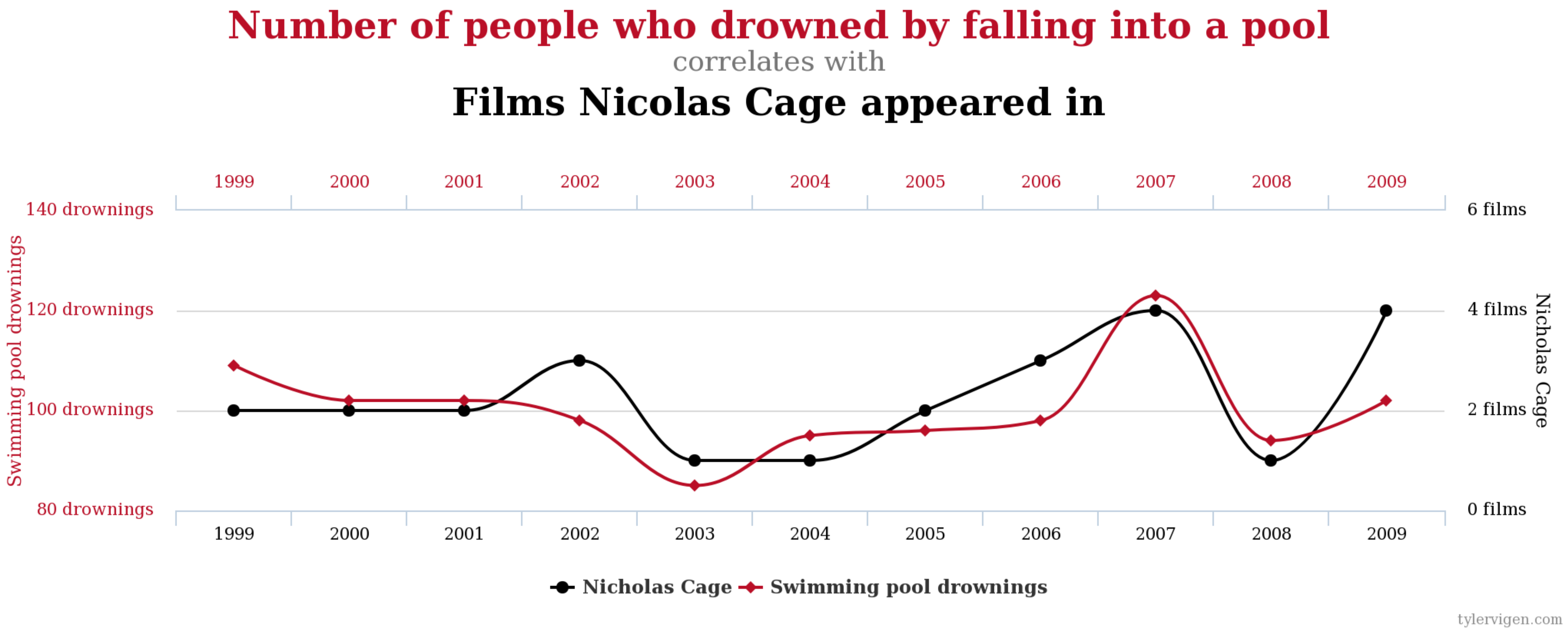
- Measures linear dependence between random variables X, Y. Does **not** measure independence.

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y]$$

- Entry (i,j) of the covariance matrix measures whether changes in variable X_i co-occur with changes in variable Y_j

- It does not measure whether one causes the other.

Correlation does not imply causation



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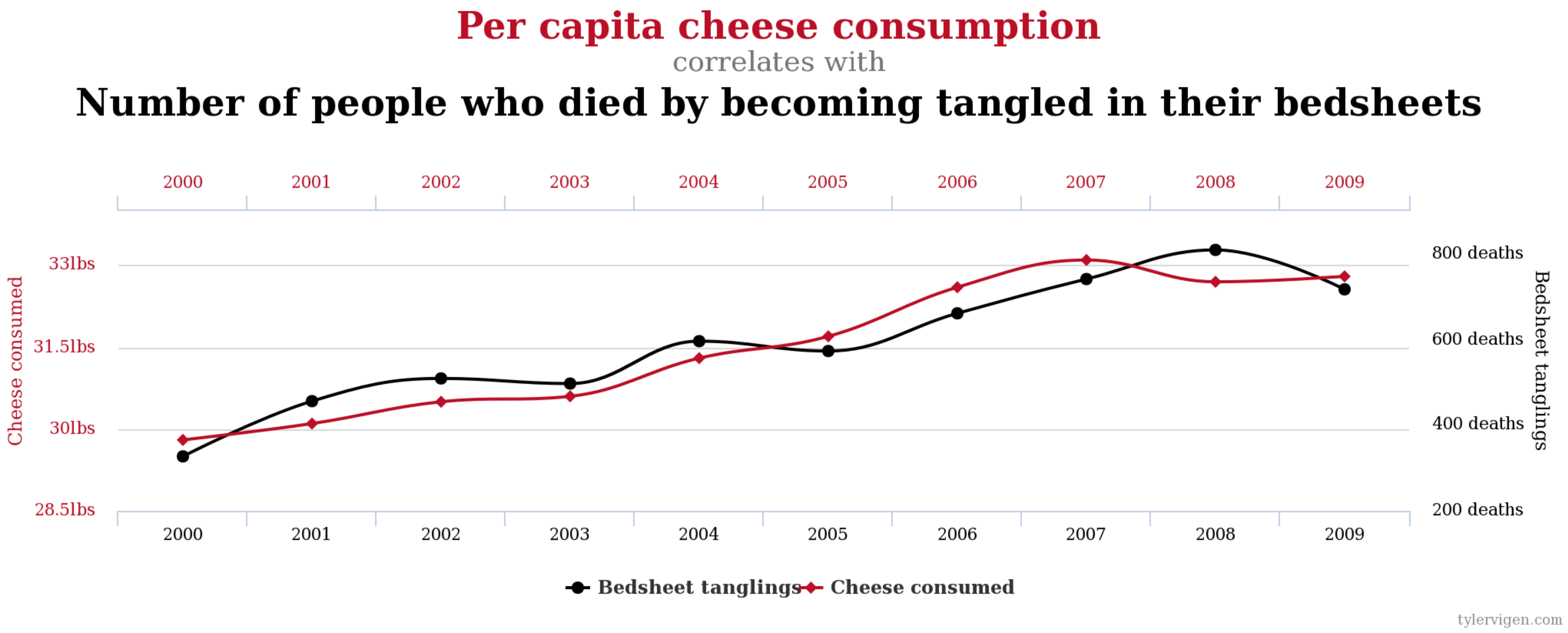

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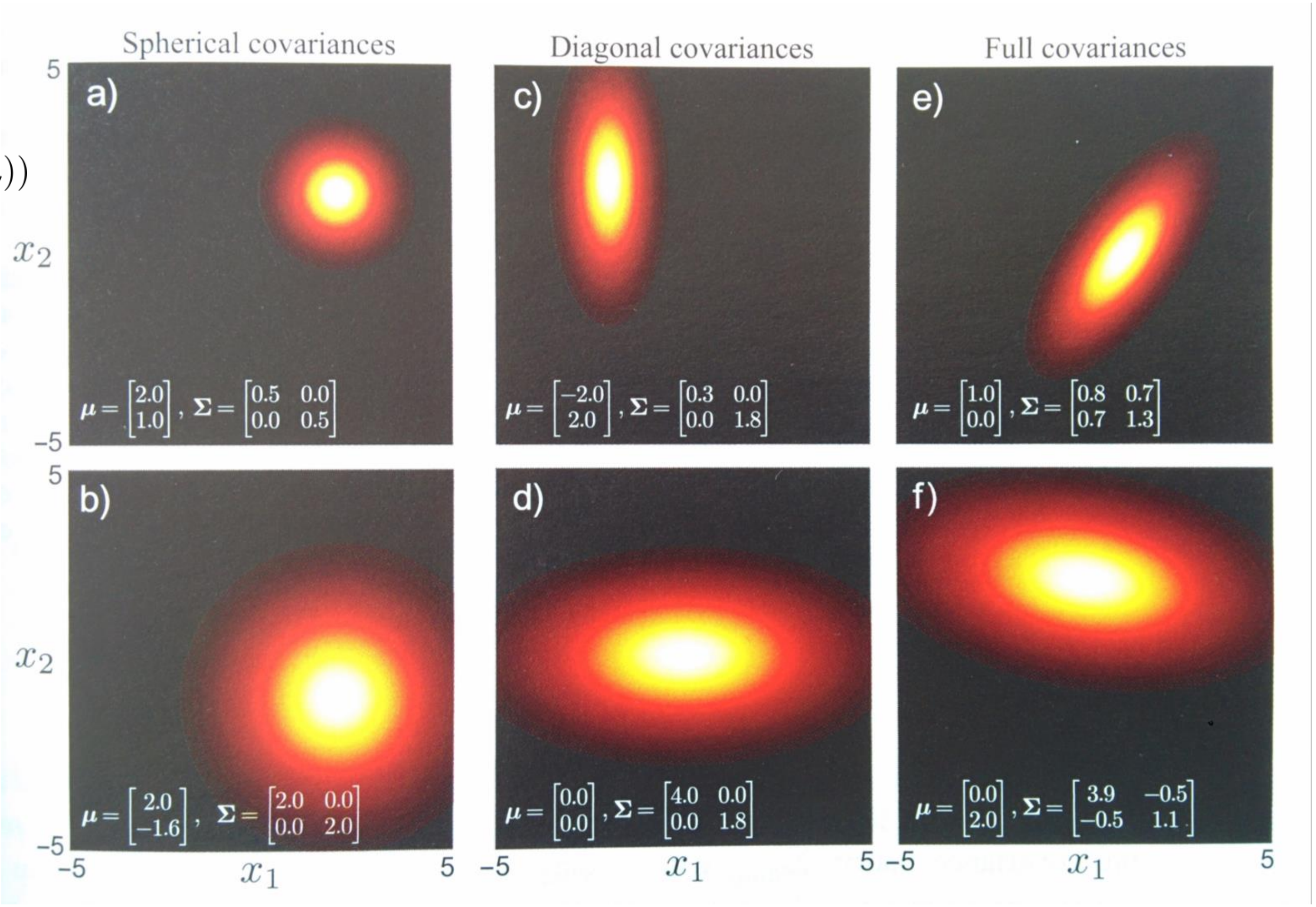
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}\det(\boldsymbol{\Sigma})^{1/2}}\exp(-0.5||\mathbf{x}-\boldsymbol{\mu}||^2_{\boldsymbol{\Sigma}})$$

For multivariate Gaussians:

$$E[\mathbf{x}] = \boldsymbol{\mu}$$

$$\mathrm{Cov}[\mathbf{x}] = \boldsymbol{\Sigma}$$



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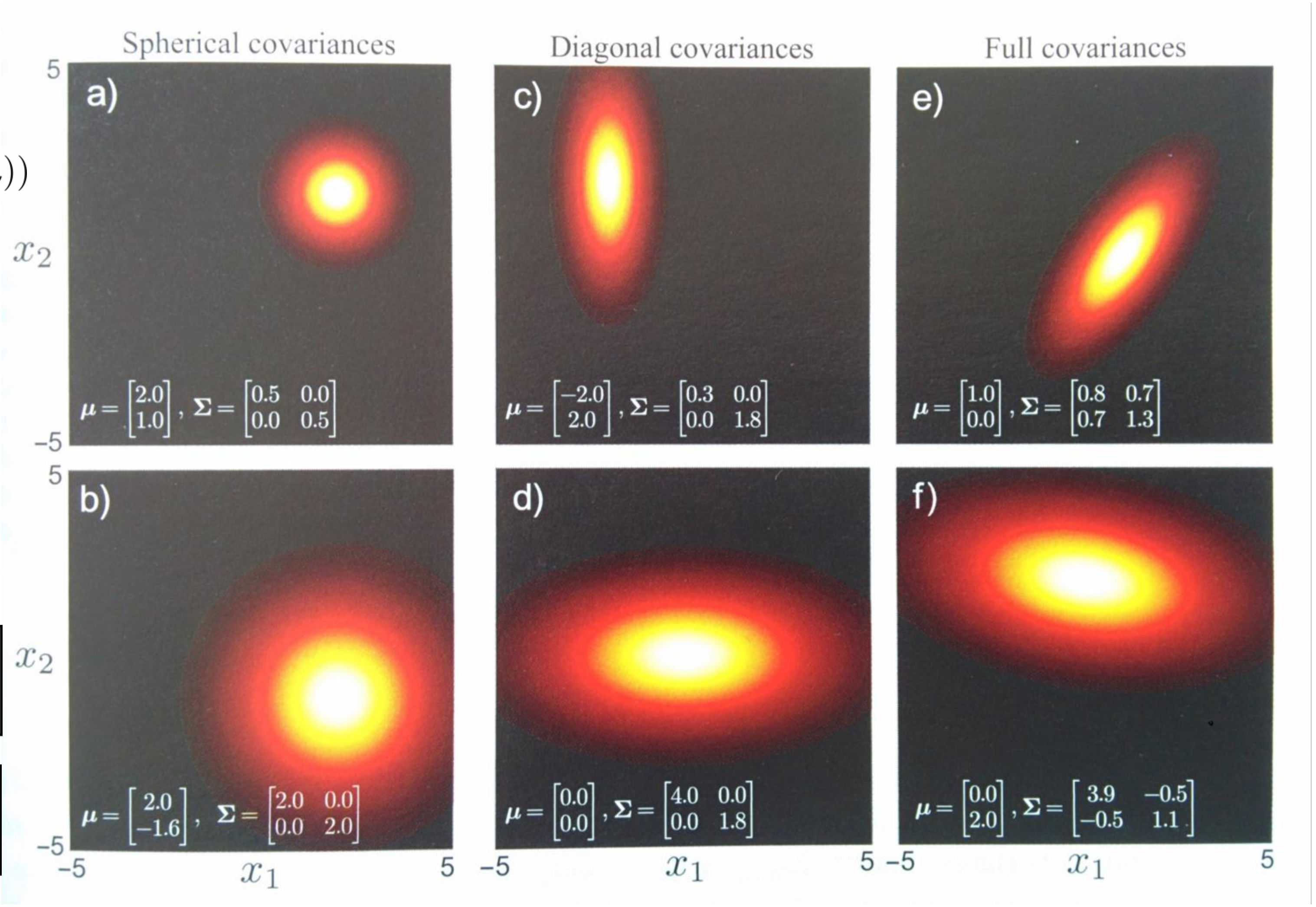
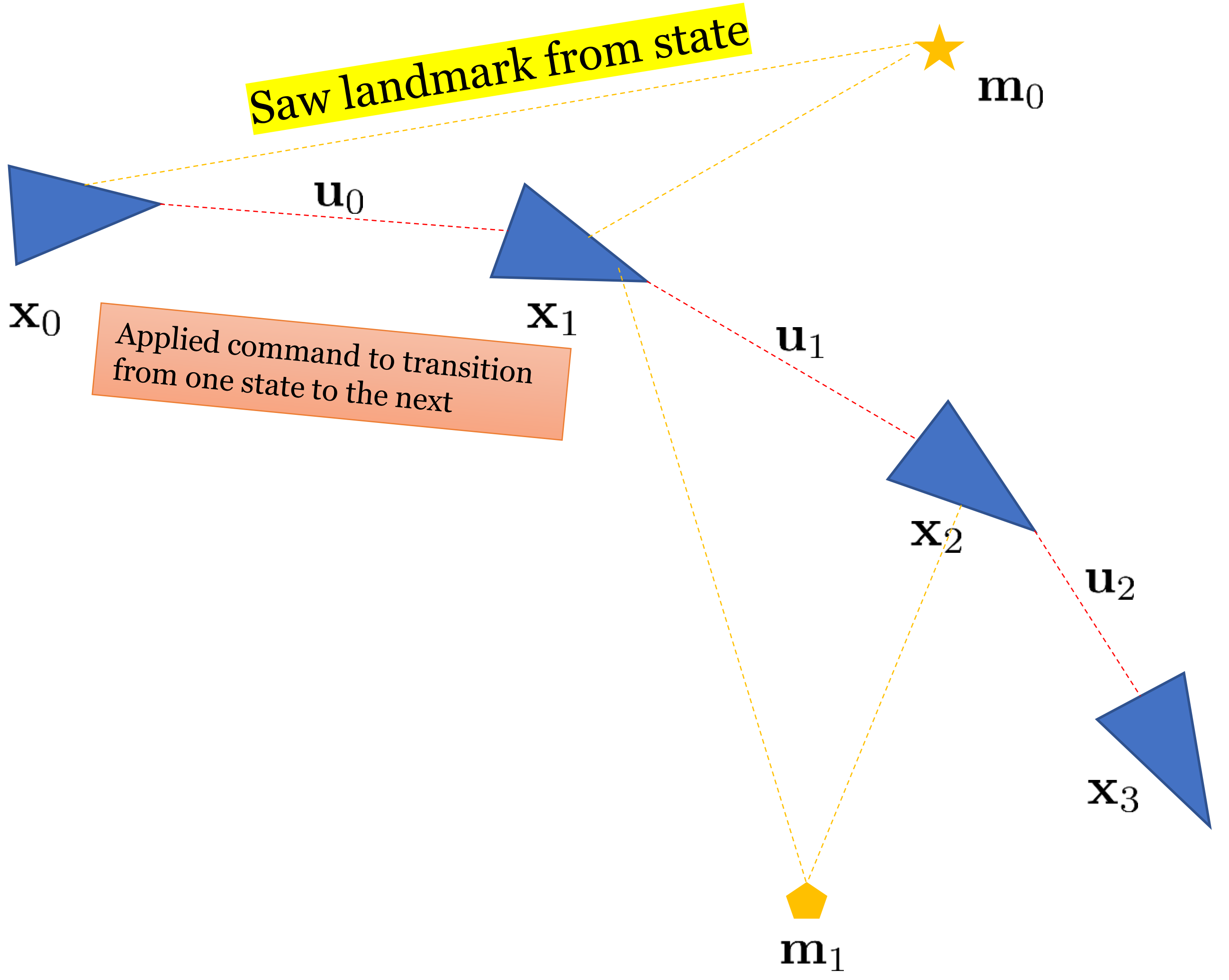
Since we have 2D examples here:

$$\mathrm{Cov}[\mathbf{x}] = \mathbf{\Sigma} = \begin{bmatrix} \mathrm{Cov}[x_1, x_1] & \mathrm{Cov}[x_1, x_2] \\ \mathrm{Cov}[x_2, x_1] & \mathrm{Cov}[x_2, x_2] \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Var}[x_1] & \mathrm{Cov}[x_1, x_2] \\ \mathrm{Cov}[x_2, x_1] & \mathrm{Var}[x_2] \end{bmatrix}$$



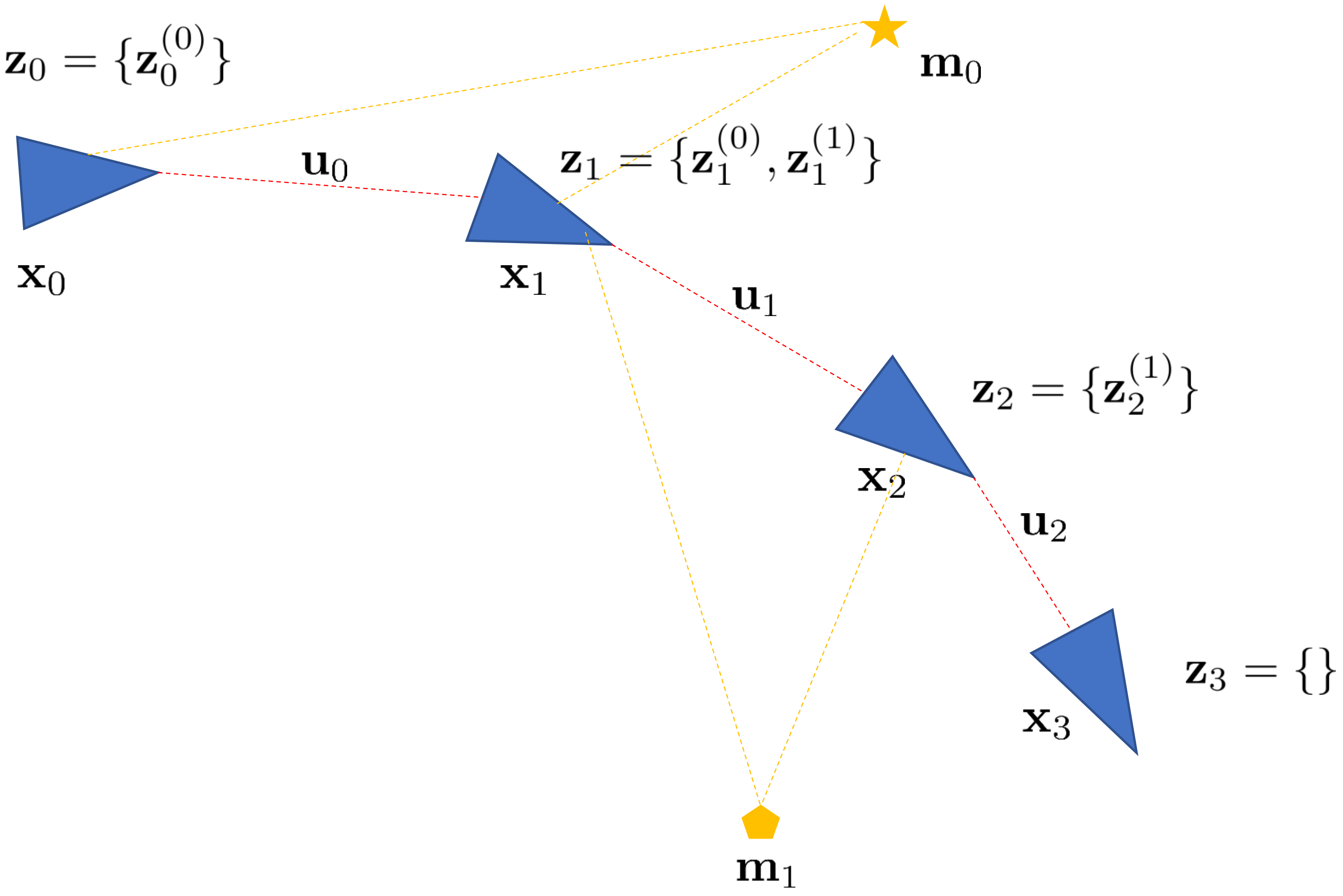
SLAM: graph representation



Map $\mathbf{m} = \{\mathbf{m}_0, \mathbf{m}_1\}$ consists of landmarks that are easily identifiable and cannot be mistaken for one another.

i.e. we are avoiding the data association problem here.

SLAM: graph representation



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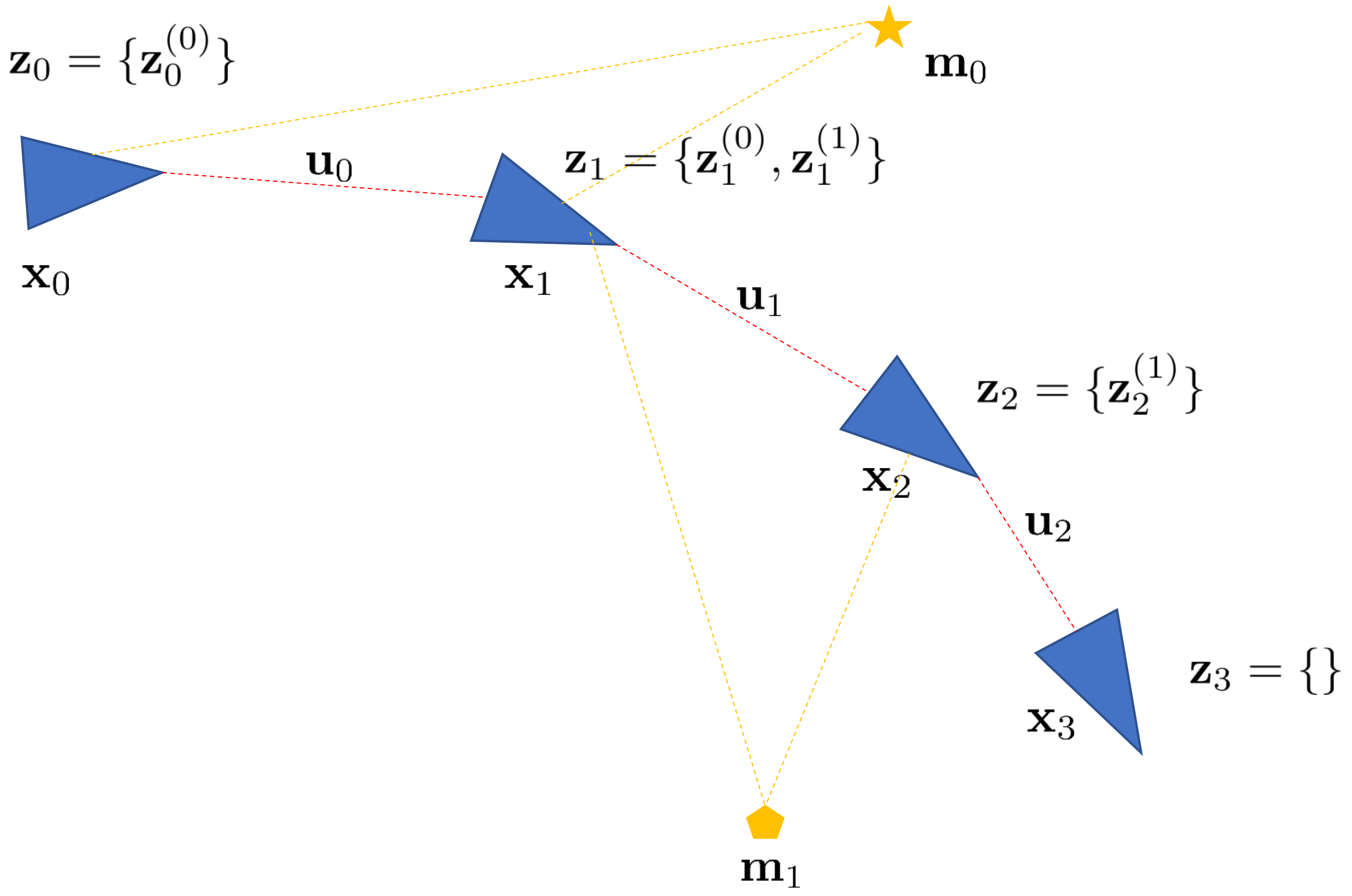
SLAM: graph representation



Notice that the graph is mostly sparse as long as not many states observe the same landmark.

That implies that there are many symbolic dependencies between random variables in

$$p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$$

that are not necessary and can be dropped.

GraphSLAM:
SLAM as a Maximum A Posteriori Estimate

Instead of computing the posterior $p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$ we are going to compute its max

See least squares lecture

$$\mathbf{x}^*_{1:T}, \mathbf{m}^* = \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}}\, p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$$

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\begin{aligned}
\mathbf{x}_{1:T}^*, \mathbf{m}^* &= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}}\; p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[\frac{p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1} | \mathbf{x}_0)}{p(\mathbf{z}_{0:T}, \mathbf{u}_{0:T-1} | \mathbf{x}_0)} \right]
\end{aligned}
$$

by definition of conditional distribution

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&= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[\frac{p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1} | \mathbf{x}_0)}{p(\mathbf{z}_{0:T}, \mathbf{u}_{0:T-1} | \mathbf{x}_0)} \right] \\
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\end{aligned}
$$

denominator does not depend on optimization variables

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&= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[\prod_{t=1}^{T} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) \prod_{t=0}^{T} \prod_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} p(\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k) \right]
\end{aligned}
$$

Observation of landmark k at time t

See Appendix 1 for the derivation of this step

Set of observations that were made at time t

Set of observations that were made at time t

GraphSLAM: SLAM as a Maximum A Posteriori Estimate

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\end{aligned}
$$

Probabilistic sensor measurement model

Probabilistic dynamics model

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&= \underset{\mathbf{x}_{1:T},\mathbf{m}}{\operatorname{argmax}} \left[\sum_{t=1}^{T} \log p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + \sum_{t=0}^{T} \sum_{\mathbf{z}_t^{(k)} \in\, \mathbf{z}_t} \log p(\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k) \right]
\end{aligned}
$$

GraphSLAM:
SLAM as a Maximum A Posteriori Estimate

Instead of computing the posterior $p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$ we are going to compute its max

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Main GraphSLAM assumptions:

1. Uncertainty in the dynamics model is Gaussian

$$\mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + \mathbf{w}_t$$

$$\mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$$

so

$$\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1} \sim \mathcal{N}(\mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}), \mathbf{R}_t)$$

2. Uncertainty in the sensor model is Gaussian

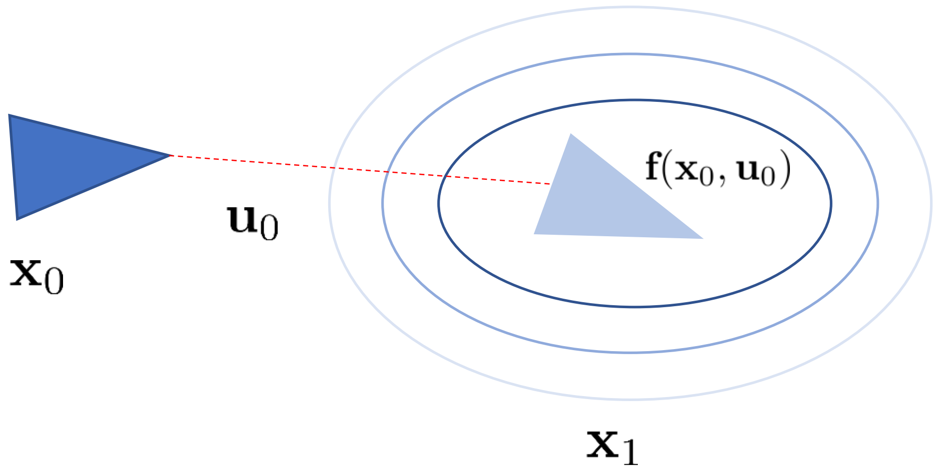
$$\mathbf{z}_t^{(k)} = \mathbf{h}(\mathbf{x}_t, \mathbf{m}_k) + \mathbf{v}_t$$

$$\mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t)$$

so

$$\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k \sim \mathcal{N}(\mathbf{h}(\mathbf{x}_t, \mathbf{m}_k), \mathbf{Q}_t)$$

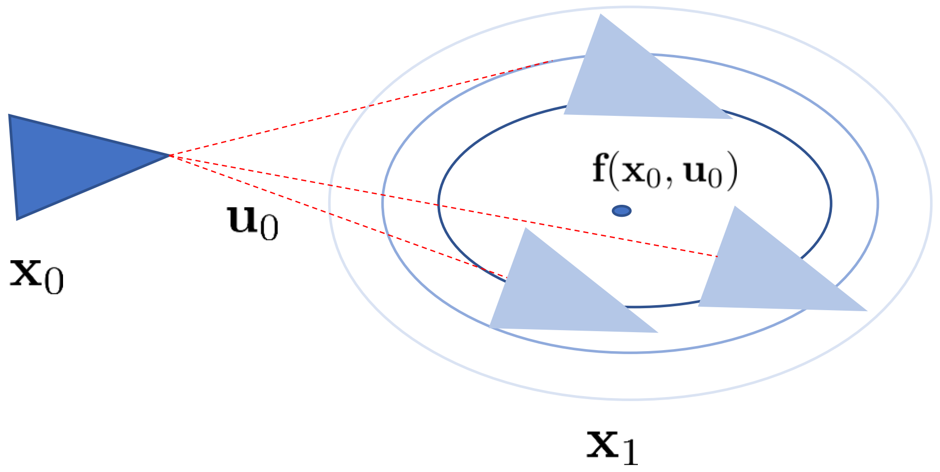
SLAM: noise/errors



$$\mathbf{x}_1 | \mathbf{x}_0, \mathbf{u}_0 \sim \mathcal{N}(\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0), \mathbf{R}_0)$$

Expected to end up at $\mathbf{x}_1 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$ from $\mathbf{x}_0$

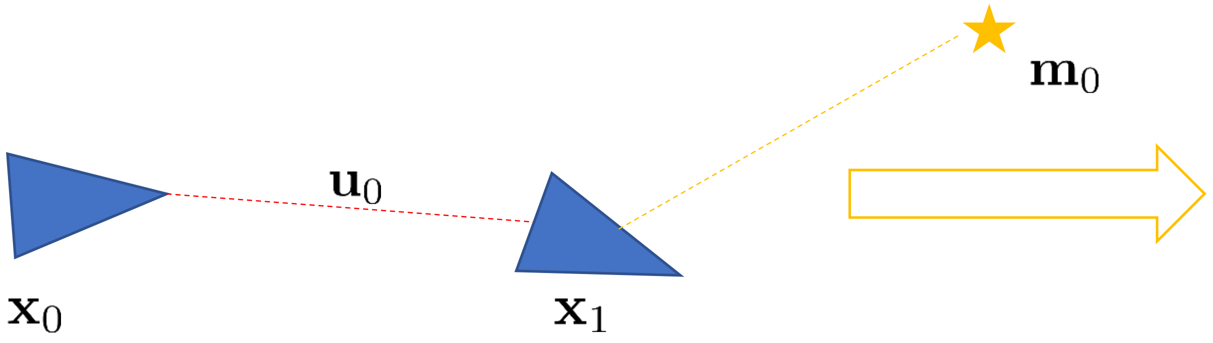
SLAM: noise/errors

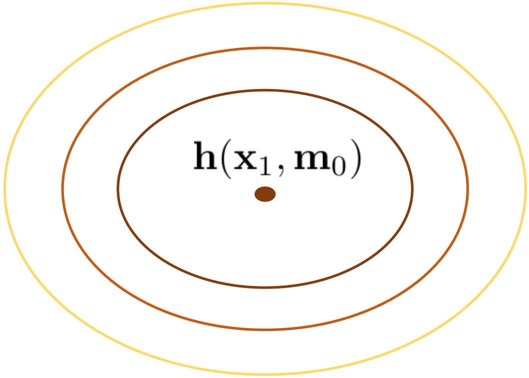


$$\mathbf{x}_1 | \mathbf{x}_0, \mathbf{u}_0 \sim \mathcal{N}(\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0), \mathbf{R}_0)$$

Expected to end up at $\mathbf{x}_1 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$ from $\mathbf{x}_0$ but we might end up around it, within the ellipse defined by the covariance matrix $\mathbf{R}_0$

SLAM: noise/errors





$$\mathbf{z}_1^{(0)} | \mathbf{x}_1, \mathbf{m}_0 \sim \mathcal{N}(\mathbf{h}(\mathbf{x}_1, \mathbf{m}_0), \mathbf{Q}_1)$$

Expected to get measurement $\mathbf{h}(\mathbf{x}_1, \mathbf{m}_0)$ at state $\mathbf{x}_1$ but it might be somewhere within the ellipse defined by the covariance matrix $\mathbf{Q}_1$

GraphSLAM: SLAM as a least squares problem

Instead of computing the posterior $p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$ we are going to compute its max

$$\begin{aligned}
\mathbf{x}^*_{1:T}, \mathbf{m}^* &= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[\sum_{t=1}^{T} \log p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + \sum_{t=0}^{T} \sum_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} \log p(\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k) \right] \\
&= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[-\sum_{t=1}^{T} ||\mathbf{x}_t - \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})||^2_{\mathbf{R}_t} - \sum_{t=0}^{T} \sum_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} ||\mathbf{z}_t^{(k)} - \mathbf{h}(\mathbf{x}_t, \mathbf{m}_k)||^2_{\mathbf{Q}_t} \right]
\end{aligned}$$

Notation:

$\mathbf{x}^T \mathbf{Q}^{-1} \mathbf{x} = ||\mathbf{x}||^2_{\mathbf{Q}}$

$\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1} \sim \mathcal{N}(\mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}), \mathbf{R}_t)$

$\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k \sim \mathcal{N}(\mathbf{h}(\mathbf{x}_t, \mathbf{m}_k), \mathbf{Q}_t)$

GraphSLAM:
SLAM as a least squares problem

Instead of computing the posterior $p(\mathbf{x}_{1:T}, \mathbf{m} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0)$ we are going to compute its max

$$
\begin{aligned}
\mathbf{x}_{1:T}^*, \mathbf{m}^* &= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[\sum_{t=1}^{T} \log p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + \sum_{t=0}^{T} \sum_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} \log p(\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k) \right] \\
&= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmax}} \left[-\sum_{t=1}^{T} ||\mathbf{x}_t - \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})||_{\mathbf{R}_t}^2 - \sum_{t=0}^{T} \sum_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} ||\mathbf{z}_t^{(k)} - \mathbf{h}(\mathbf{x}_t, \mathbf{m}_k)||_{\mathbf{Q}_t}^2 \right] \\
&= \underset{\mathbf{x}_{1:T}, \mathbf{m}}{\operatorname{argmin}} \left[\sum_{t=1}^{T} ||\mathbf{x}_t - \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})||_{\mathbf{R}_t}^2 + \sum_{t=0}^{T} \sum_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} ||\mathbf{z}_t^{(k)} - \mathbf{h}(\mathbf{x}_t, \mathbf{m}_k)||_{\mathbf{Q}_t}^2 \right]
\end{aligned}
$$

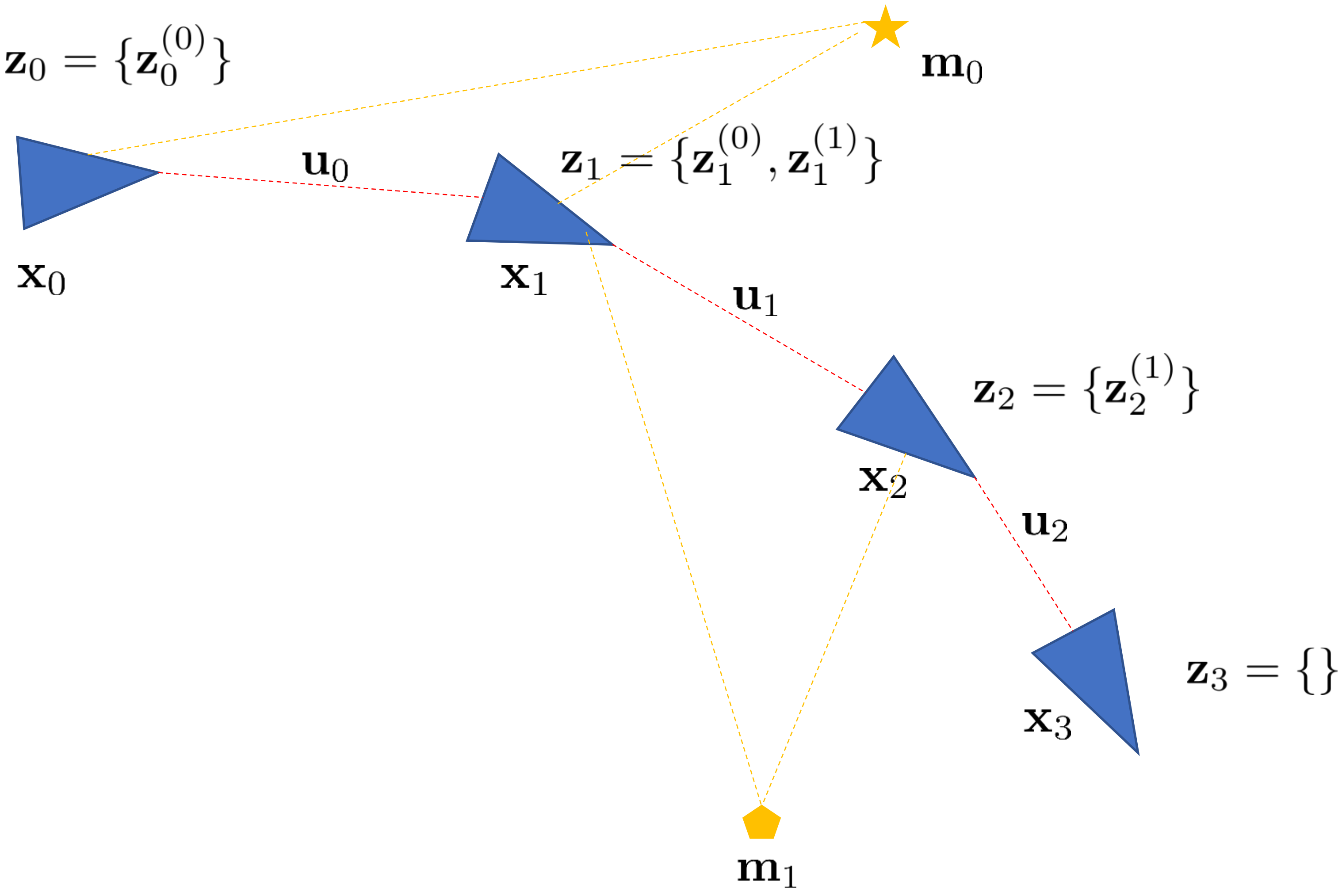
Notation:

$\mathbf{x}^T \mathbf{Q}^{-1} \mathbf{x} = ||\mathbf{x}||_{\mathbf{Q}}^2$

$\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1} \sim \mathcal{N}(\mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}), \mathbf{R}_t)$

$\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k \sim \mathcal{N}(\mathbf{h}(\mathbf{x}_t, \mathbf{m}_k), \mathbf{Q}_t)$

GraphSLAM: example



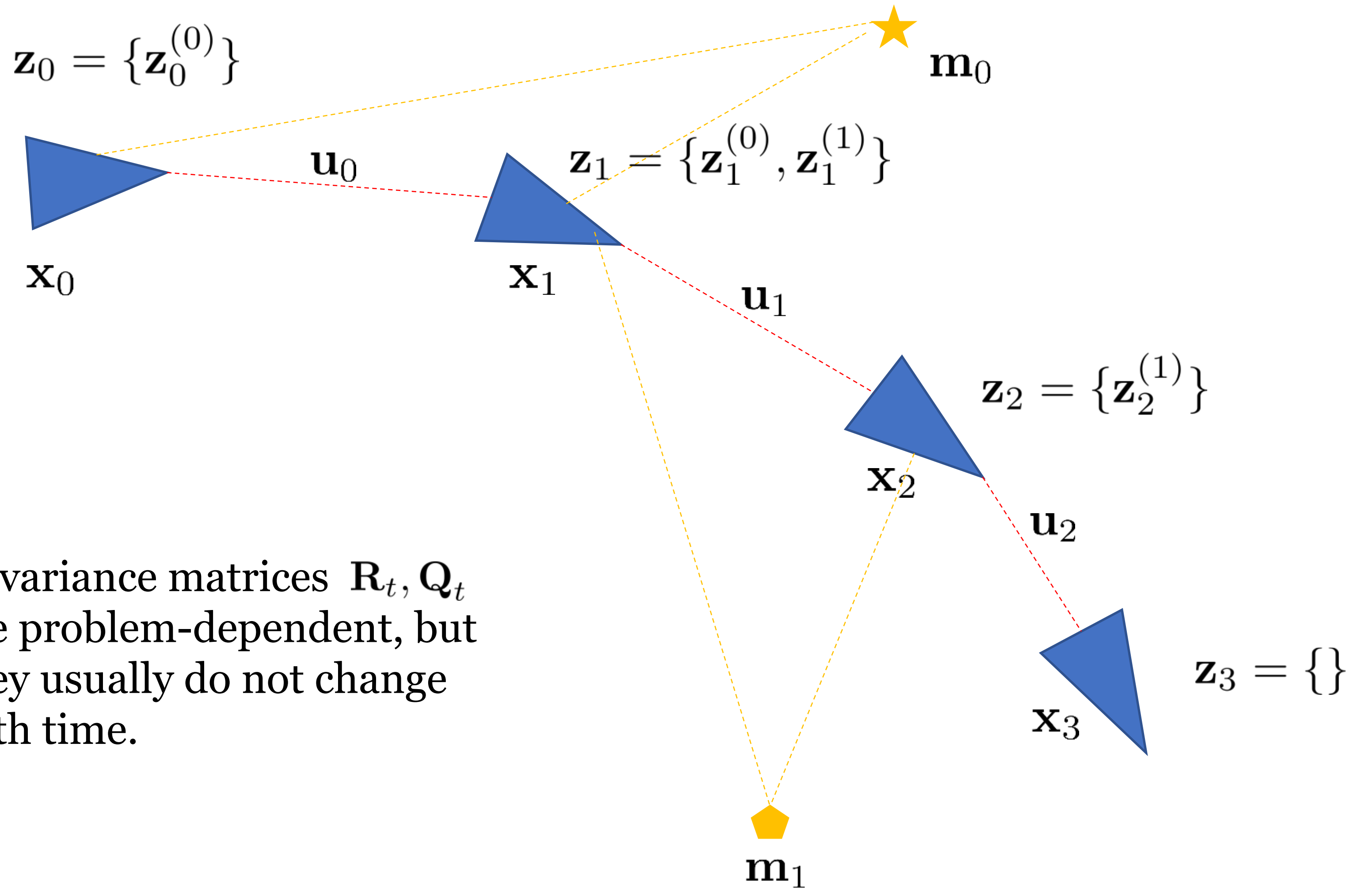
Need to minimize the sum of the following quadratic terms:

$$||\mathbf{x}_1 - \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)||^2_{\mathbf{R}_1}$$
$$||\mathbf{x}_2 - \mathbf{f}(\mathbf{x}_1, \mathbf{u}_1)||^2_{\mathbf{R}_2}$$
$$||\mathbf{x}_3 - \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2)||^2_{\mathbf{R}_3}$$
$$||\mathbf{z}_0^{(0)} - \mathbf{h}(\mathbf{x}_0, \mathbf{m}_0)||^2_{\mathbf{Q}_0}$$
$$||\mathbf{z}_1^{(0)} - \mathbf{h}(\mathbf{x}_1, \mathbf{m}_0)||^2_{\mathbf{Q}_1}$$
$$||\mathbf{z}_1^{(1)} - \mathbf{h}(\mathbf{x}_1, \mathbf{m}_1)||^2_{\mathbf{Q}_1}$$
$$||\mathbf{z}_2^{(1)} - \mathbf{h}(\mathbf{x}_2, \mathbf{m}_1)||^2_{\mathbf{Q}_2}$$

with respect to variables:
$\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \mathbf{m}_0 \quad \mathbf{m}_1$

initial state $\mathbf{x}_0$ is given

GraphSLAM: example



Covariance matrices $\mathbf{R}_t, \mathbf{Q}_t$ are problem-dependent, but they usually do not change with time.

Need to minimize the sum of the following quadratic terms:

$$||\mathbf{x}_1 - \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)||^2_{\mathbf{R}_1}$$
$$||\mathbf{x}_2 - \mathbf{f}(\mathbf{x}_1, \mathbf{u}_1)||^2_{\mathbf{R}_2}$$
$$||\mathbf{x}_3 - \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2)||^2_{\mathbf{R}_3}$$
$$||\mathbf{z}_0^{(0)} - \mathbf{h}(\mathbf{x}_0, \mathbf{m}_0)||^2_{\mathbf{Q}_0}$$
$$||\mathbf{z}_1^{(0)} - \mathbf{h}(\mathbf{x}_1, \mathbf{m}_0)||^2_{\mathbf{Q}_1}$$
$$||\mathbf{z}_1^{(1)} - \mathbf{h}(\mathbf{x}_1, \mathbf{m}_1)||^2_{\mathbf{Q}_1}$$
$$||\mathbf{z}_2^{(1)} - \mathbf{h}(\mathbf{x}_2, \mathbf{m}_1)||^2_{\mathbf{Q}_2}$$

with respect to variables:
$\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \mathbf{m}_0 \quad \mathbf{m}_1$

initial state $\mathbf{x}_0$ is given

Examples of dynamics and sensor models

$$\mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + \mathbf{w}_t$$

Can be any of the dynamical systems we saw in Lecture 2.

$$\mathbf{z}_t^{(k)} = \mathbf{h}(\mathbf{x}_t, \mathbf{m}_k) + \mathbf{v}_t$$

$\mathbf{z}_t^{(k)}$ can be any of the sensors we saw in Lecture 4:

- Laser scan $\{(r_i, \theta_i)\}_{i=1:K}$ where $\mathbf{m}_k$ is an occupancy grid
- Range and bearing (r, θ) to the landmark $\mathbf{m}_k = (x_k, y_k, z_k)$
- Bearing measurements from images
- Altitude/Depth
- Gyroscope
- Accelerometer

Appendix 1

Claim: $$p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \quad = \quad p(\mathbf{x}_0) \prod_{t=1}^{T} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) \prod_{t=0}^{T} \prod_{\mathbf{z}_t^{(k)} \in \, \mathbf{z}_t} p(\mathbf{z}_t^{(k)} | \mathbf{x}_t, \mathbf{m}_k)$$

Proof:

$$\begin{aligned}
p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \quad &= \quad p(\mathbf{z}_T | \mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T-1}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \; p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T-1}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \quad p(\mathbf{z}_T | \mathbf{x}_T, \mathbf{m}) \; p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T-1}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \quad p(\mathbf{z}_T | \mathbf{x}_T, \mathbf{m}) \; p(\mathbf{z}_{T-1} | \mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T-2}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \; p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T-2}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \quad p(\mathbf{z}_T | \mathbf{x}_T, \mathbf{m}) \; p(\mathbf{z}_{T-1} | \mathbf{x}_{T-1}, \mathbf{m}) \; p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{z}_{0:T-2}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&\dots \\
&= \quad \prod_{t=0}^{T} p(\mathbf{z}_t | \mathbf{x}_t, \mathbf{m}) \; p(\mathbf{x}_{1:T}, \mathbf{m}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \quad \prod_{t=0}^{T} p(\mathbf{z}_t | \mathbf{x}_t, \mathbf{m}) \; p(\mathbf{x}_T | \mathbf{x}_{1:T-1}, \mathbf{m}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \;\; p(\mathbf{x}_{1:T-1}, \mathbf{m}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \quad \prod_{t=0}^{T} p(\mathbf{z}_t | \mathbf{x}_t, \mathbf{m}) \; p(\mathbf{x}_T | \mathbf{x}_{T-1}, \mathbf{u}_{T-1}) \; p(\mathbf{x}_{1:T-1}, \mathbf{m}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&= \quad \prod_{t=0}^{T} p(\mathbf{z}_t | \mathbf{x}_t, \mathbf{m}) \; p(\mathbf{x}_T | \mathbf{x}_{T-1}, \mathbf{u}_{T-1}) \; p(\mathbf{x}_{T-1} | \mathbf{x}_{T-2}, \mathbf{m}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \; p(\mathbf{x}_{1:T-2}, \mathbf{m}, \mathbf{u}_{0:T-1}, \mathbf{x}_0) \\
&\dots \\
&= \quad \left[\prod_{t=0}^{T} p(\mathbf{z}_t | \mathbf{x}_t, \mathbf{m}) \right] \; p(\mathbf{x}_0) \; \prod_{t=1}^{T} p(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1})
\end{aligned}$$

Appendix 1

Claim: $p(\mathbf{z}_t|\mathbf{x}_t, \mathbf{m}) = \prod_{\mathbf{z}_t^{(k)} \in \mathbf{z}_t} p(\mathbf{z}_t^{(k)}|\mathbf{x}_t, \mathbf{m}_k)$ where $\mathbf{z}_t = \{\mathbf{z}_t^{(k)} \text{ iff landmark } \mathbf{m}_k \text{ was observed}\}$

$\mathbf{m} = \{\text{landmarks } \mathbf{m}_k\}$

Proof:

Suppose without loss of generality that $\mathbf{z}_t = \{\mathbf{z}_t^{(k)},\ k = 1...K\}$ and $\mathbf{m} = \{\mathbf{m}_k,\ k = 1...K\}$
i.e. that all landmarks were observed from the state at time t. Then:

$$
\begin{aligned}
p(\mathbf{z}_t^{(1)}, ..., \mathbf{z}_t^{(K)}|\mathbf{x}_t, \mathbf{m}) &= p(\mathbf{z}_t^{(1)}|\mathbf{z}_t^{(2)}, ..., \mathbf{z}_t^{(K)}, \mathbf{x}_t, \mathbf{m})\ p(\mathbf{z}_t^{(2)}, ..., \mathbf{z}_t^{(K)}|\mathbf{x}_t, \mathbf{m}) \\
&= p(\mathbf{z}_t^{(1)}|\mathbf{x}_t, \mathbf{m}_1)\ p(\mathbf{z}_t^{(2)}, ..., \mathbf{z}_t^{(K)}|\mathbf{x}_t, \mathbf{m}) \\
&= p(\mathbf{z}_t^{(1)}|\mathbf{x}_t, \mathbf{m}_1)\ p(\mathbf{z}_t^{(2)}|\mathbf{z}_t^{(3)}, ..., \mathbf{z}_t^{(K)}, \mathbf{x}_t, \mathbf{m})\ p(\mathbf{z}_t^{(3)}, ..., \mathbf{z}_t^{(K)}|\mathbf{x}_t, \mathbf{m}) \\
&= p(\mathbf{z}_t^{(1)}|\mathbf{x}_t, \mathbf{m}_1)\ p(\mathbf{z}_t^{(2)}|\mathbf{x}_t, \mathbf{m}_2)\ p(\mathbf{z}_t^{(3)}, ..., \mathbf{z}_t^{(K)}|\mathbf{x}_t, \mathbf{m}) \\
&\dots \\
&= \prod_{k=1}^{K} p(\mathbf{z}_t^{(k)}|\mathbf{x}_t, \mathbf{m}_k)
\end{aligned}
$$