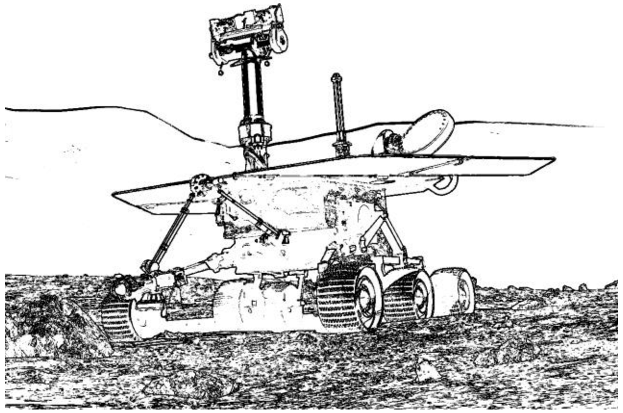

CSC477
Introduction to Mobile Robotics

Florian Shkurti

Week #8: Bayes’ Filters and Kalman Filter

Recommended reading

- Chapters 2 and 3.2 from Probabilistic Robotics
- Chapters 4.9 and 8.3 from Computational Principles of Mobile Robotics
- Lesson 2 in https://www.udacity.com/course/artificial-intelligence-for-robotics--cs373
- This illustrative blog post:

  http://www.bzarg.com/p/how-a-kalman-filter-works-in-pictures/

  Careful: the figure between equations (9) and (10) is wrong. The blue Gaussian should be taller and peakier than the other two Gaussians, the prior and the measurement models. This is not fixed as of March 15, 2017.

Filtering vs. Smoothing

- Smoothing/Batch Estimation

$$p(\mathbf{x}_{0:T} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1})$$

Sequence of states

Sequence of sensor measurements

Sequence of commands

- Filtering Estimation

$$p(\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1})$$

What's the difference?

- Smoothing/Batch Estimation

$$p(\mathbf{x}_{0:T} \mid \mathbf{z}_{0:T}, \mathbf{u}_{0:T-1})$$

All measurements and
controls are known
in advance

- Filtering Estimation

$$p(\mathbf{x}_t \mid \mathbf{z}_{0:t}, \mathbf{u}_{0:t-1})$$

Measurements and controls
are processed online as they come.
Future measurements are unknown.

Why do we use filtering?

- Online belief updates: filters provide a principled way to incorporate noisy information from sensor measurements, which can change our prior belief, in an online fashion.

- Sensor fusion: filters enable us to combine measurements from multiple different noisy sensors into one coherent state estimate. E.g. camera + laser, camera + IMU, multiple cameras, sonar and IMU, GPS and IMU etc.

Technically speaking, this is also true for smoothing estimators.

Bayes' Filter

- A generic class of filters that make use of Bayes' rule and assume the following:

  - **Markov Assumption For Dynamics**: the state x_t is conditionally independent of past states and controls, given the previous state x_{t-1} . In other words, the dynamics model is assumed to satisfy

$$p(x_t|x_{0:t-1}, u_{0:t-1}) = p(x_t|x_{t-1}, u_{t-1})$$

  - **Static World Assumption:** the current observation is conditionally independent of past observations and controls, given the current state

$$p(z_t|x_t, u_{0:t-1}, z_{0:t-1}) = p(z_t|x_t)$$

Bayes' Filter

- A generic class of filters that make use of Bayes' rule and assume the following:
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  - **Static World Assumption:** the current observation is conditionally independent of past observations and controls, given the current state

$$p(z_t|x_t, u_{0:t-1}, z_{0:t-1}) = p(z_t|x_t)$$

Note: the Markov assumption is more general than what we have presented here.

Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\
&= \eta \; p(z_t | x_t, u_{0:t-1}, z_{0:t-1}) \; p(x_t | u_{0:t-1}, z_{0:t-1})
\end{aligned}
$$

Normalizing factor that makes the integral/sum of the numerator in Bayes' Rule be 1.

Conditional Bayes' Rule

x_t z_t

$$p(A|B,C) = \frac{p(C|A,B)\, p(A|B)}{p(C|B)}$$

$u_{0:t-1}, z_{0:t-1}$ $1/\eta$

Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta \, p(z_t|x_t, u_{0:t-1}, z_{0:t-1}) \, p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta \, p(z_t|x_t) \, p(x_t|u_{0:t-1}, z_{0:t-1})
\end{aligned}
$$

Static World Assumption

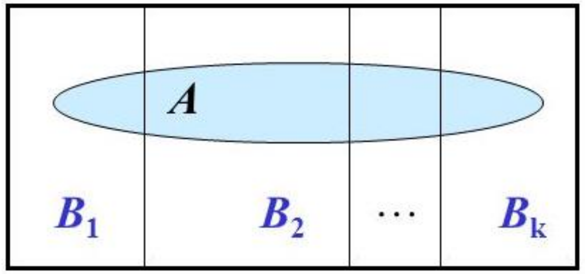
Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\
&= \eta \, p(z_t | x_t, u_{0:t-1}, z_{0:t-1}) \, p(x_t | u_{0:t-1}, z_{0:t-1}) \\
&= \eta \, p(z_t | x_t) \, p(x_t | u_{0:t-1}, z_{0:t-1}) \\
&= \eta \, p(z_t | x_t) \int p(x_t, x_{t-1} | u_{0:t-1}, z_{0:t-1}) \, dx_{t-1}
\end{aligned}
$$

Marginalization, or law of total probability

$$p(A) = \sum_{B_i} p(A, B_i)$$

where the sum enumerates all possibilities over the variable Bi. If we see Bi as a set, then the collection of Bi's must be pairwise disjoint. I.e. the collection of subsets Bi must be a partition of the sample space.



Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t, u_{0:t-1}, z_{0:t-1})\; p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t)\; p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t) \int p(x_t, x_{t-1}|u_{0:t-1}, z_{0:t-1})\; dx_{t-1}
\end{aligned}
$$

Marginalization, or law of total probability

$$p(A) = \sum_{B_i} p(A, B_i)$$

Here we are actually using the law of total probability for conditional distributions, so

$$p(A|C) = \sum_{B_i} p(A, B_i|C)$$

Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t, u_{0:t-1}, z_{0:t-1})\, p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t)\, p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t) \int p(x_t, x_{t-1}|u_{0:t-1}, z_{0:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{0:t-1}, z_{0:t-1}, x_{t-1})\, p(x_{t-1}|z_{0:t-1}, u_{0:t-1})\, dx_{t-1}
\end{aligned}
$$

Definition of conditional distribution

$$p(A, B|C) = p(A|B, C)p(B|C)$$

Bayes' Filter: Derivation

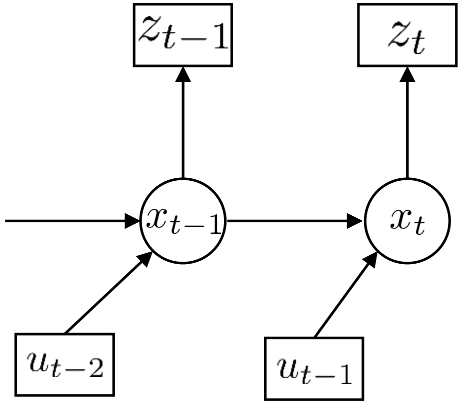
$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t, u_{0:t-1}, z_{0:t-1})\, p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t)\, p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t) \int p(x_t, x_{t-1}|u_{0:t-1}, z_{0:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{0:t-1}, z_{0:t-1}, x_{t-1})\, p(x_{t-1}|z_{0:t-1}, u_{0:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, p(x_{t-1}|z_{0:t-1}, u_{0:t-1})\, dx_{t-1}
\end{aligned}
$$

Markov assumption for dynamics

Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t, u_{0:t-1}, z_{0:t-1})\; p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t)\; p(x_t|u_{0:t-1}, z_{0:t-1}) \\
&= \eta\, p(z_t|x_t) \int p(x_t, x_{t-1}|u_{0:t-1}, z_{0:t-1})\; dx_{t-1} \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{0:t-1}, z_{0:t-1}, x_{t-1})\; p(x_{t-1}|z_{0:t-1}, u_{0:t-1})\; dx_{t-1} \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\; p(x_{t-1}|z_{0:t-1}, u_{0:t-1})\; dx_{t-1} \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\; p(x_{t-1}|z_{0:t-1}, u_{0:t-2})\; dx_{t-1}
\end{aligned}
$$

This is the belief at the previous time step!
This means we can perform filtering recursively.



Control at time t-1 only affects state at time t

Bayes' Filter: Derivation

$$\begin{aligned} bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\ &= \eta \, p(z_t | x_t) \int p(x_t | u_{t-1}, x_{t-1}) \, bel(x_{t-1}) \, dx_{t-1} \end{aligned}$$

Bayes' Filter: Derivation

$$
\begin{aligned}
bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\
&= \eta \, p(z_t | x_t) \underbrace{\int p(x_t | u_{t-1}, x_{t-1}) \, bel(x_{t-1}) \, dx_{t-1}}_{\substack{\text{Computes the probability density of reaching state} \\ x_t \text{ from any possible previous state } x_{t-1} \\ \text{via the command } u_{t-1}}}
\end{aligned}
$$

Bayes' Filter: Derivation

$$\begin{aligned} bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\ &= \underbrace{\eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}}_{\substack{\text{Computes the probability density of reaching state} \\ x_t \text{ from any possible previous state } x_{t-1} \\ \text{via the command } u_{t-1} \text{ and observing } z_t}} \end{aligned}$$

Bayes' Filter: Derivation

$$\begin{aligned} bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\ &= \underbrace{\eta\, p(z_t | x_t) \underbrace{\int p(x_t | u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}}_{\text{Belief after prediction step}}}_{\text{Belief after update step}} \end{aligned}$$

Kalman Filter: an instance of Bayes' Filter

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}
\end{aligned}
$$

Linear dynamics with Gaussian noise

$$
\begin{aligned}
x_t &= Ax_{t-1} + Bu_{t-1} + Gw_{t-1} \\
&\quad \text{with noise } w_{t-1} \sim \mathcal{N}(0, Q)
\end{aligned}
$$

Linear observations with Gaussian noise

$$
\begin{aligned}
z_t &= Hx_t + n_t \\
&\quad \text{with noise } n_t \sim \mathcal{N}(0, R)
\end{aligned}
$$

Initial belief is Gaussian

$$bel(x_0) \sim \mathcal{N}(\mu_0, \Sigma_0)$$

Kalman Filter: assumptions

- Two assumptions inherited from Bayes' Filter
- Linear dynamics and observation models
- Initial belief is Gaussian
- Noise variables and initial state

  $x_0, w_0, w_1, ..., n_0, n_1, ...$

  are jointly Gaussian and independent
- Noise variables w_t are independent and identically distributed $\mathcal{N}(0, Q)$
- Noise variables n_t are independent and identically distributed $\mathcal{N}(0, R)$

Kalman Filter: why so many assumptions?

- Two assumptions inherited from Bayes' Filter
- Linear dynamics and observation models
- Initial belief is Gaussian
- Noise variables and initial state

  $x_0, w_0, w_1, \ldots, n_0, n_1, \ldots$

  are jointly Gaussian and independent
- Noise variables w_t are independent and identically distributed $\mathcal{N}(0, Q)$
- Noise variables n_t are independent and identically distributed $\mathcal{N}(0, R)$

Without linearity there is no closed-form solution for the posterior belief in the Bayes' Filter. Recall that if X is Gaussian then Y=AX+b is also Gaussian. This is not true in general if Y=h(X).

Also, we will see later that applying Bayes' rule to a Gaussian prior and a Gaussian measurement likelihood results in a Gaussian posterior.

Kalman Filter:
why so many assumptions?

- Two assumptions inherited from Bayes' Filter
- Linear dynamics and observation models
- Initial belief is Gaussian
- Noise variables and initial state

  $x_0, w_0, w_1, ..., n_0, n_1, ...$

  are jointly Gaussian and independent
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- Noise variables n_t are independent and identically distributed $\mathcal{N}(0, R)$

This results in the belief remaining Gaussian after each propagation and update step. This means that we only have to worry about how the mean and the covariance of the belief evolve recursively with each prediction step and update step
→ COOL!

Kalman Filter: why so many assumptions?

- Two assumptions inherited from Bayes' Filter
- Linear dynamics and observation models
- Initial belief is Gaussian
- Noise variables and initial state

  $x_0, w_0, w_1, ..., n_0, n_1, ...$

  are jointly Gaussian and independent
- Noise variables w_t are independent and identically distributed $\mathcal{N}(0, Q)$
- Noise variables n_t are independent and identically distributed $\mathcal{N}(0, R)$

This makes the recursive updates of the mean and covariance much simpler.

Kalman Filter: an instance of Bayes' Filter

Assumptions guarantee that if the prior belief before the prediction step is Gaussian

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}
\end{aligned}
$$

then the prior belief after the prediction step will be Gaussian

and the posterior belief (after the update step) will be Gaussian.

Kalman Filter: an instance of Bayes' Filter

$$
\begin{aligned}
bel(x_t) &= p(x_t | u_{0:t-1}, z_{0:t}) \\
&= \eta \, p(z_t | x_t) \, p(x_t | u_{0:t-1}, z_{0:t-1}) \\
&= \eta \, p(z_t | x_t) \int p(x_t | u_{t-1}, x_{t-1}) \, bel(x_{t-1}) \, dx_{t-1} \\
&= \eta \, p(z_t | x_t) \, \overline{bel}(x_t)
\end{aligned}
$$

Belief after prediction step (to simplify notation)

So, under the Kalman Filter assumptions we get

$$bel(x_{t-1}) \sim \mathcal{N}(\mu_{t-1|t-1}, \Sigma_{t-1|t-1})$$

$$\overline{bel}(x_t) \sim \mathcal{N}(\mu_{t|t-1}, \Sigma_{t|t-1})$$

$$bel(x_t) \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$$

Notation: estimate at time t given history of observations and controls up to time t-1

Kalman Filter: an instance of Bayes' Filter

$$
\begin{aligned}
bel(x_t) &= p(x_t|u_{0:t-1}, z_{0:t}) \\
&= \eta\, p(z_t|x_t) \int p(x_t|u_{t-1}, x_{t-1})\, bel(x_{t-1})\, dx_{t-1} \\
&= \eta\, p(z_t|x_t)\, \overline{bel}(x_t)
\end{aligned}
$$

So, under the Kalman Filter assumptions we get

$$bel(x_{t-1}) \sim \mathcal{N}(\mu_{t-1|t-1}, \Sigma_{t-1|t-1})$$

$$\downarrow$$

$$\overline{bel}(x_t) \sim \mathcal{N}(\mu_{t|t-1}, \Sigma_{t|t-1})$$

$$\downarrow$$

$$bel(x_t) \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$$

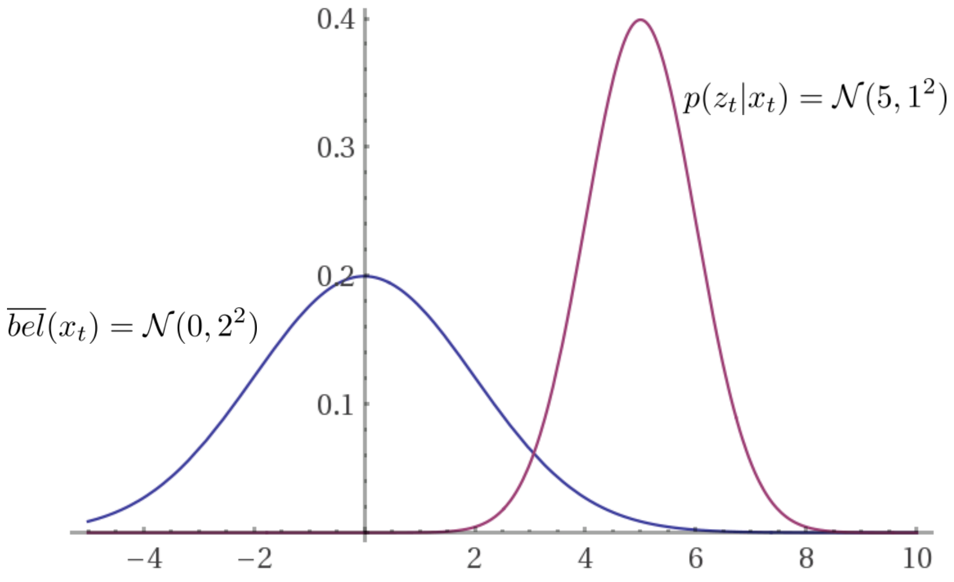
Two main questions:

1. How to get prediction mean and covariance from prior mean and covariance?
2. How to get posterior mean and covariance from prediction mean and covariance?

These questions were answered in the 1960s. The resulting algorithm was used in the Apollo missions to the moon, and in almost every system in which there is a noisy sensor involved → COOL!

Kalman Filter with 1D state

- Let's start with the update step recursion. Here's an example:



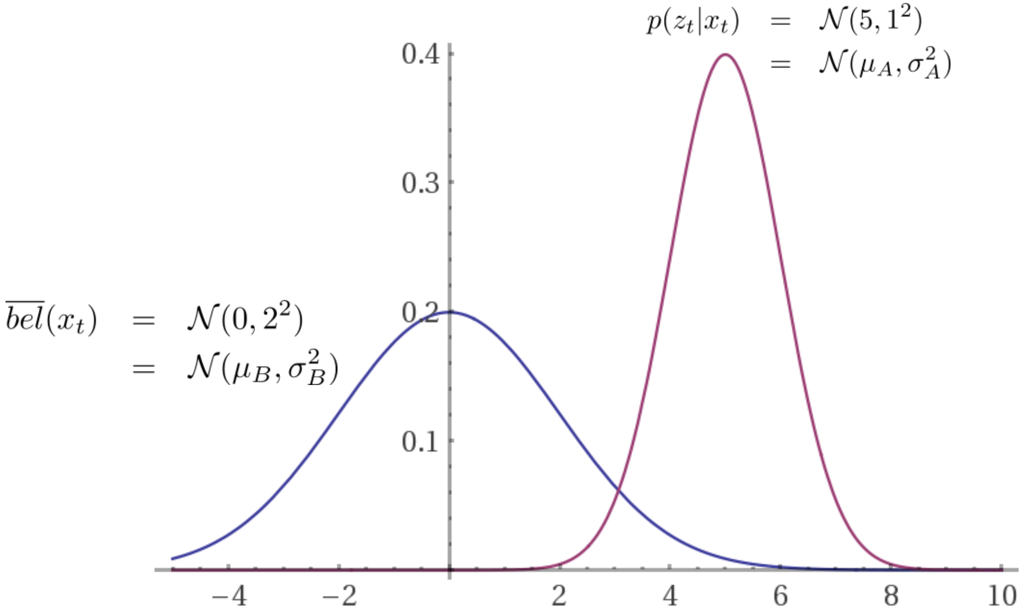
Suppose your measurement model is $z_t = x_t + n_t$ with $n_t \sim \mathcal{N}(0, 1^2)$

Suppose your belief after the prediction step is $\overline{bel}(x_t) = \mathcal{N}(0, 2^2)$

Suppose your first noisy measurement is $z_0 = 5$

Q: What is the mean and covariance of $bel(x_t)$?

Kalman Filter with 1D state: the update step



From Bayes' Filter we get $bel(x_t) = \eta\, p(z_t|x_t)\, \overline{bel}(x_t)$ so

$$\begin{aligned} p(z_t|x_t)\, \overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\ &= \ldots \\ &= \text{see Appendix 1 for proof} \\ &= \ldots \\ &= \mathcal{N}(\mu, \sigma^2)/\eta \end{aligned}$$

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B)$$

$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$$

Kalman Filter with 1D state: the update step

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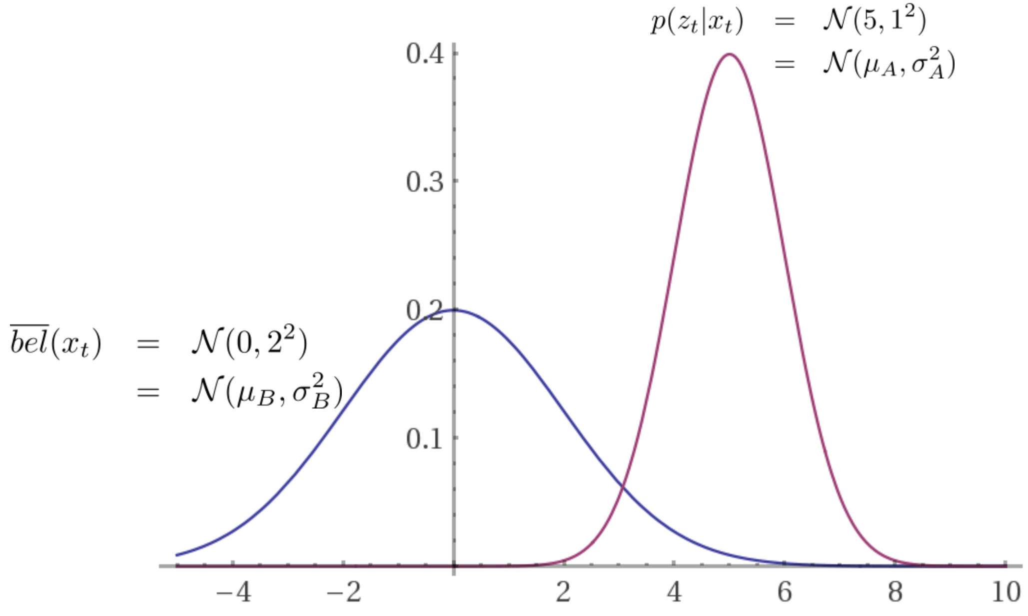


$$
\begin{aligned}
p(z_t|x_t)\,\overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\
&= \ldots \\
&= \text{see Appendix 1 for proof} \\
&= \ldots \\
&= \mathcal{N}(\mu, \sigma^2)/\eta
\end{aligned}
$$

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\boxed{(\mu_A - \mu_B)}$$

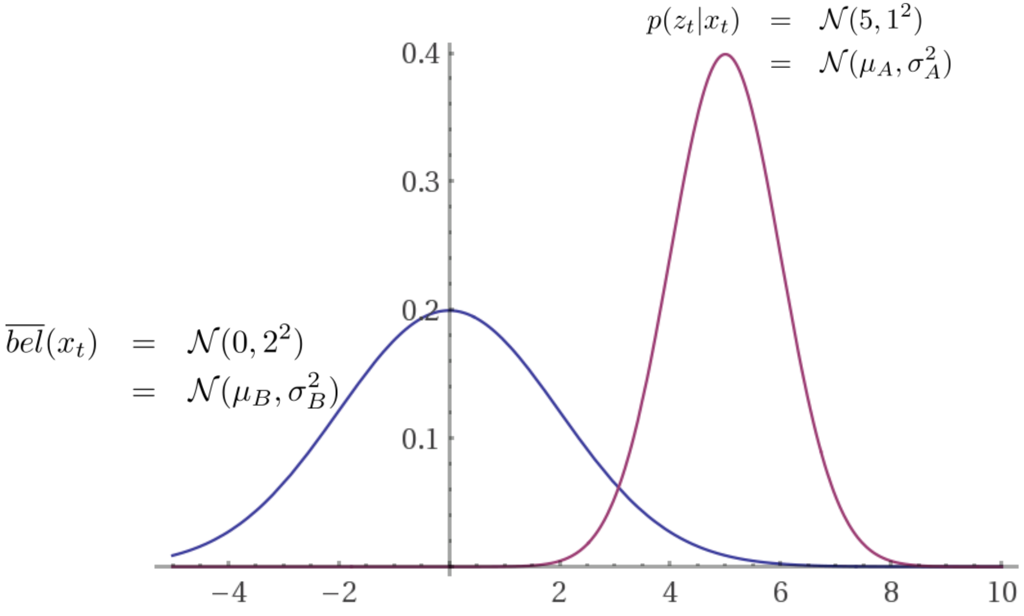
$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$$

Prediction residual/error between actual observation and expected observation.

You expected the measured mean to be 0, according to your prediction prior, but you actually observed 5.

The smaller this prediction error is the better your estimate will be, or the better it will agree with the measurements.

Kalman Filter with 1D state: the update step



From Bayes' Filter we get $bel(x_t) = \eta \; p(z_t|x_t) \; \overline{bel}(x_t)$ so

$$
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p(z_t|x_t) \; \overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\
&= \ldots \\
&= \text{see Appendix 1 for proof} \\
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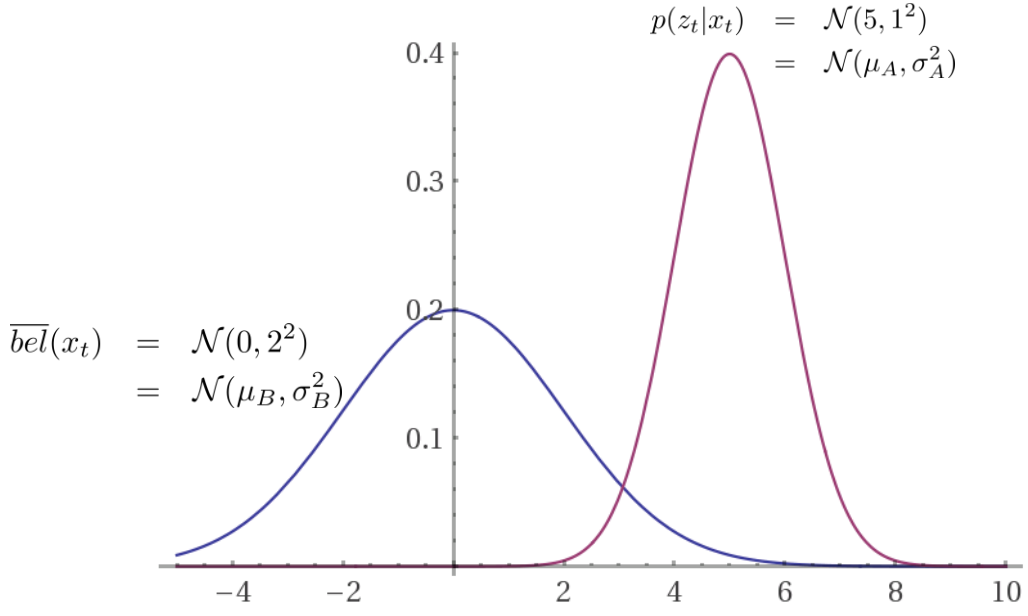
$$\mu = \mu_B + \boxed{\frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}}(\mu_A - \mu_B)$$

$$\sigma^2 = \sigma_B^2 - \boxed{\frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}}\sigma_B^2$$

Kalman Gain: specifies how much effect will the measurement have in the posterior, compared to the prediction prior. Which one do you trust more, your prior $\overline{bel}(x_t)$ or your measurement $p(z_t|x_t)$?

Kalman Filter with 1D state: the update step

From Bayes' Filter we get $bel(x_t) = \eta\, p(z_t|x_t)\, \overline{bel}(x_t)$ so



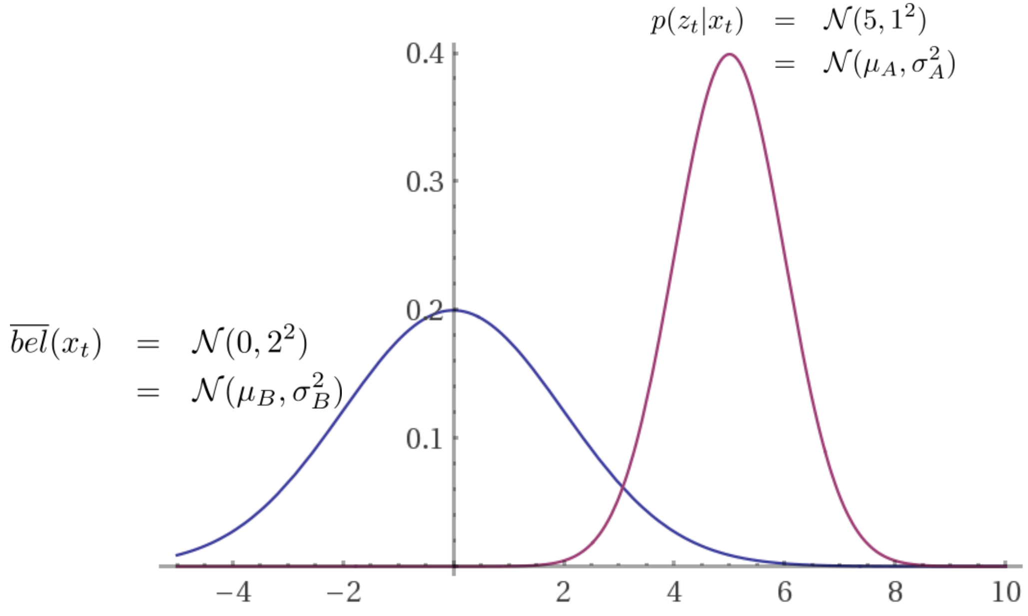
$$
\begin{aligned}
p(z_t|x_t)\,\overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\
&= \ldots \\
&= \text{see Appendix 1 for proof} \\
&= \ldots \\
&= \mathcal{N}(\mu, \sigma^2)/\eta
\end{aligned}
$$

$$\mu = \mu_B + \boxed{\frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}}(\mu_A - \mu_B)$$

The measurement is more confident (lower variance) than the prior, so the posterior mean is going to be closer to 5 than to 0.

$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$$

Kalman Filter with 1D state: the update step



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$$
\begin{aligned}
p(z_t|x_t) \, \overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\
&= \ldots \\
&= \text{see Appendix 1 for proof} \\
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&= \mathcal{N}(\mu, \sigma^2)/\eta
\end{aligned}
$$

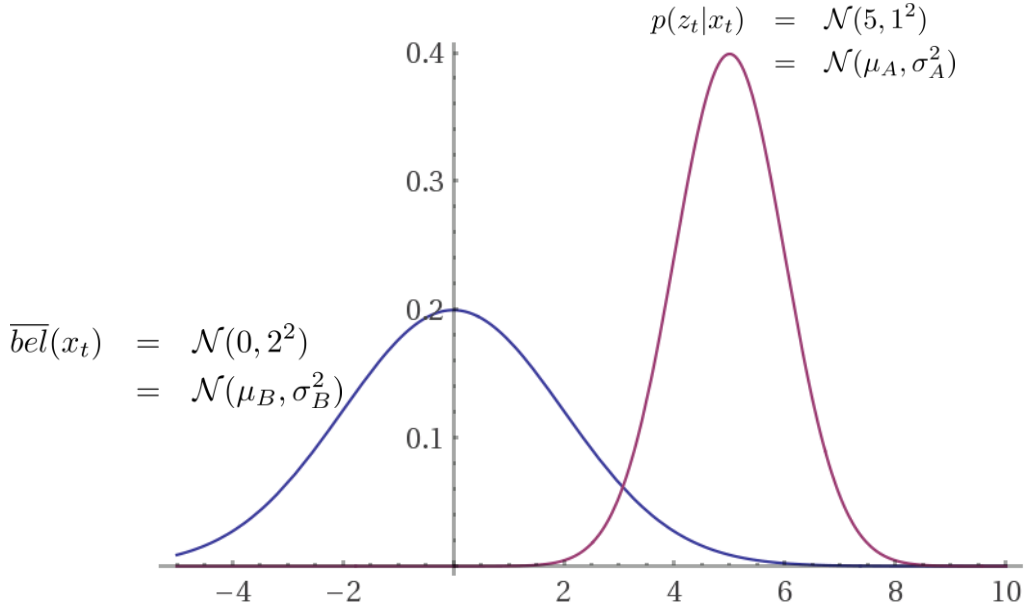
$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B)$$

$$\sigma^2 = \sigma_B^2 \boxed{-} \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$$

No matter what happens, the variance of the posterior is going to be reduced. I.e. new measurement increases confidence no matter how noisy it is.

Kalman Filter with 1D state: the update step

From Bayes' Filter we get $bel(x_t) = \eta\, p(z_t|x_t)\, \overline{bel}(x_t)$ so



$$
\begin{aligned}
p(z_t|x_t)\, \overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\
&= \ldots \\
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\end{aligned}
$$

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B)$$

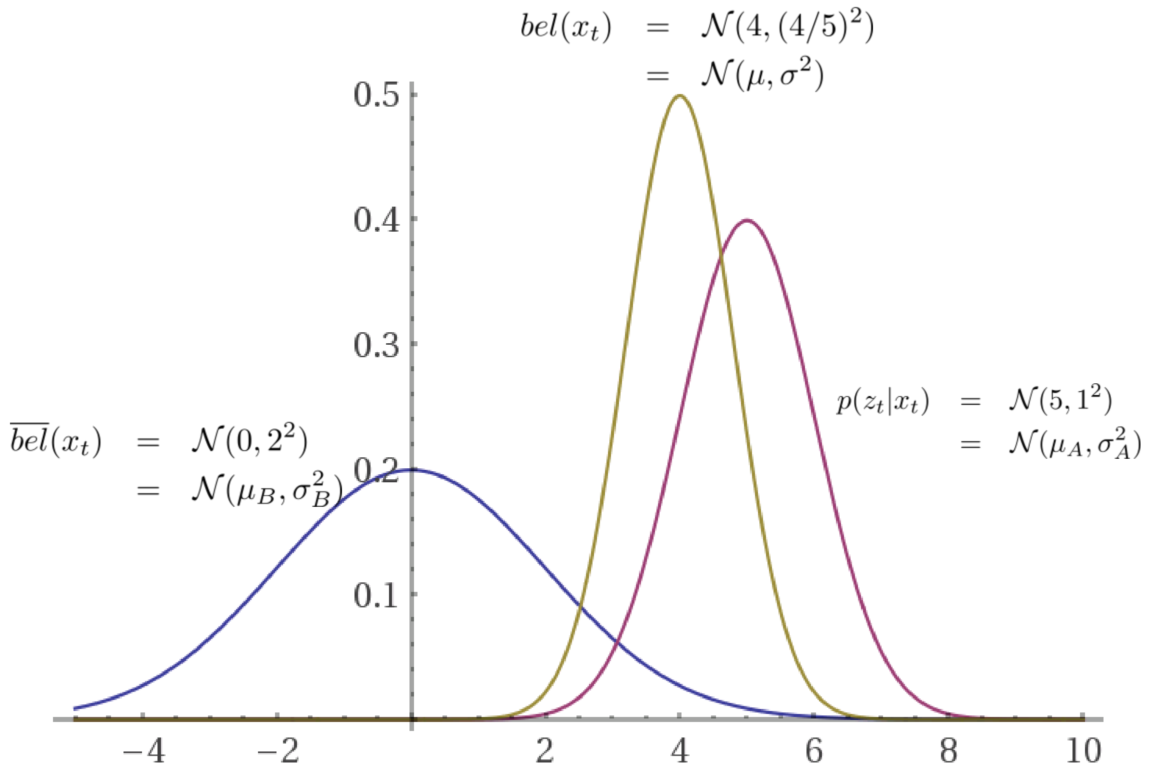
$$\boxed{\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2}$$

In fact you can write this as $\frac{1}{\sigma^2} = \frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}$

so $\sigma < \sigma_A$ and $\sigma < \sigma_B$

I.e. the posterior is more confident than both the prior and the measurement.

Kalman Filter with 1D state: the update step



From Bayes' Filter we get $bel(x_t) = \eta\, p(z_t|x_t)\, \overline{bel}(x_t)$ so

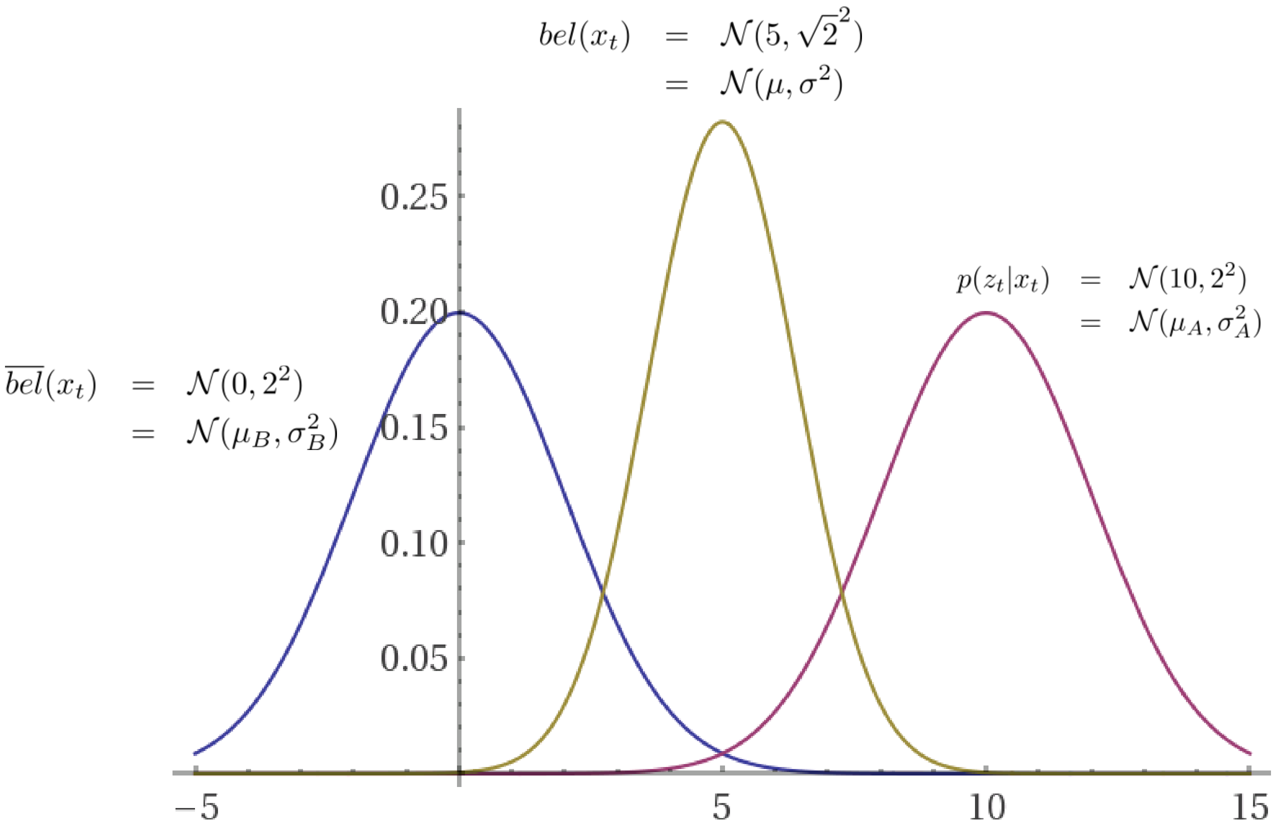
$$\begin{aligned} p(z_t|x_t)\, \overline{bel}(x_t) &= \mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \\ &= \ldots \\ &= \text{see Appendix 1 for proof} \\ &= \ldots \\ &= \mathcal{N}(\mu, \sigma^2)/\eta \end{aligned}$$

In this example:

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B) = 4$$

$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2 = 4/5$$

Kalman Filter with 1D state: the update step



Another example:

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B) = 5$$

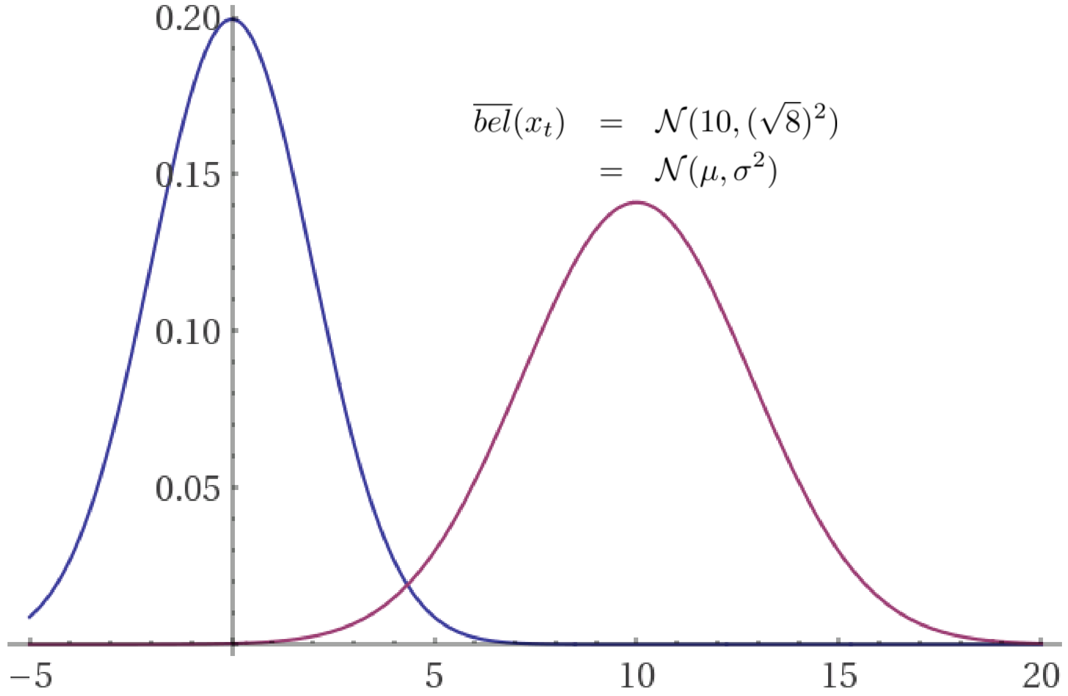
$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2 = \sigma_B^2/2 = 2$$

Kalman Filter with 1D state: the update step

Take-home message: new observations, no matter how noisy, always **reduce uncertainty** in the posterior. The mean of the posterior, on the other hand, only changes when there is a nonzero prediction residual.

Kalman Filter with 1D state: the propagation/prediction step

$$
\begin{aligned}
bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\
&= \mathcal{N}(\mu_C, \sigma_C^2)
\end{aligned}
$$



$$
\begin{aligned}
\overline{bel}(x_t) &= \mathcal{N}(10, (\sqrt{8})^2) \\
&= \mathcal{N}(\mu, \sigma^2)
\end{aligned}
$$

Suppose that the dynamics model is

$$x_t = x_{t-1} + u_{t-1} + w_{t-1} \quad \text{with} \quad w_{t-1} \sim \mathcal{N}(0, q^2)$$

and you applied the command $u_{t-1} = 10$. Then

$$
\begin{aligned}
\mu &= \mathbb{E}[x_t | z_{0:t-1}, u_{0:t-1}] \\
&= \mathbb{E}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathbb{E}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\
&= \mu_C + u_{t-1}
\end{aligned}
$$

Recall: this notation means expected value with respect to conditional expectation, i.e

$$\int x_t \, p(x_t | z_{0:t-1}, u_{0:t-1}) \, dx_t$$

$$= \int x_t \, \overline{bel}(x_t) \, dx_t$$

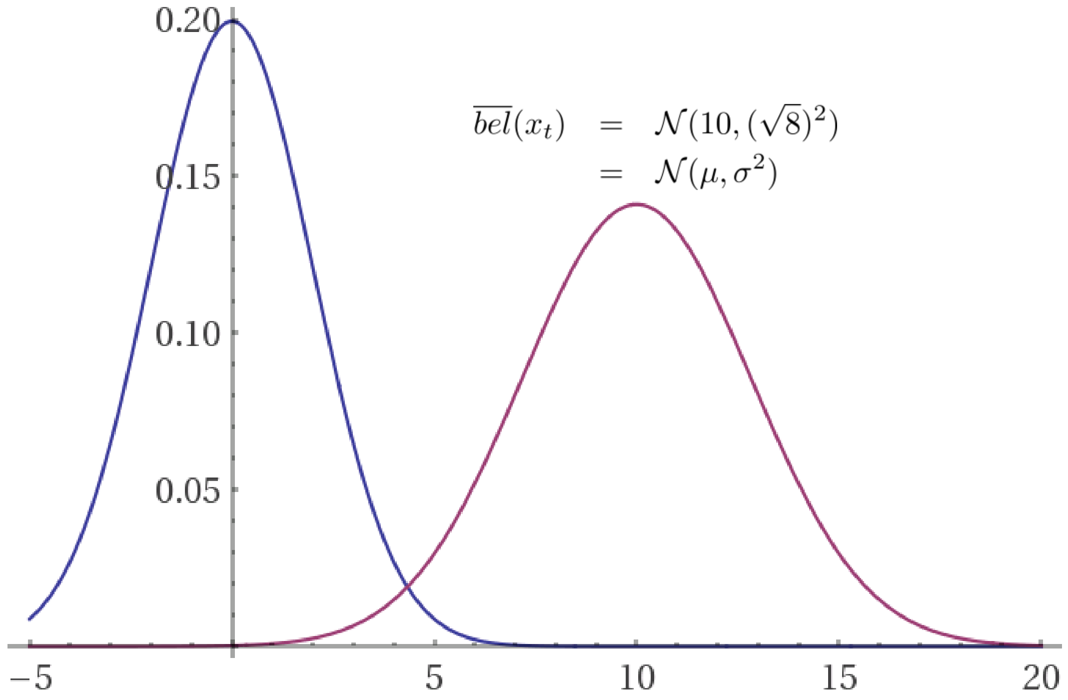
Control is a constant with respect to the distribution

$$\overline{bel}(x_t)$$

Dynamics noise is zero mean, and independent of observations and controls

Kalman Filter with 1D state: the propagation/prediction step

$$\begin{aligned} bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\ &= \mathcal{N}(\mu_C, \sigma_C^2) \end{aligned}$$



Suppose that the dynamics model is

$$x_t = x_{t-1} + u_{t-1} + w_{t-1} \quad \text{with} \quad w_{t-1} \sim \mathcal{N}(0, q^2)$$

and you applied the command $u_{t-1} = 10$. Then

$$\begin{aligned} \mu &= \mathbb{E}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \mathbb{E}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathbb{E}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\ &= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\ &= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\ &= \mu_C + u_{t-1} \end{aligned}$$

$$\begin{aligned} \sigma^2 &= \mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \end{aligned}$$

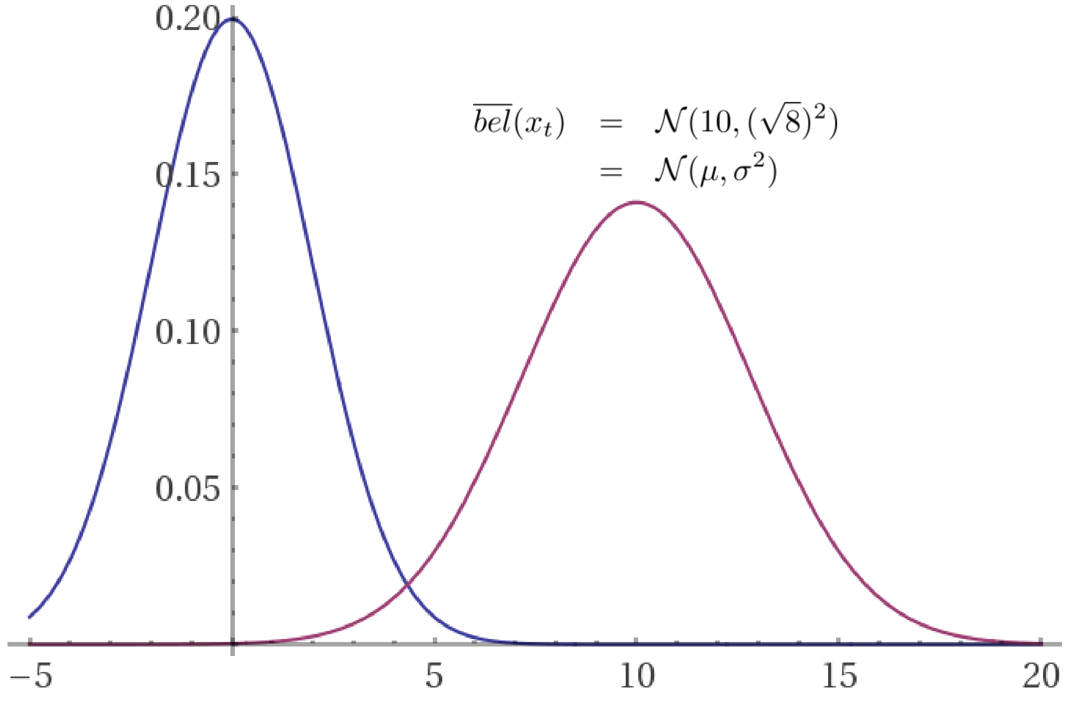
Recall: this notation means covariance with respect to conditional expectation, i.e

$$\mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] = \mathbb{E}[x_t^2 | z_{0:t-1}, u_{0:t-1}] - (\mathbb{E}[x_t | z_{0:t-1}, u_{0:t-1}])^2$$

Kalman Filter with 1D state: the propagation/prediction step

$$bel(x_{t-1}) = \mathcal{N}(0, 2^2) = \mathcal{N}(\mu_C, \sigma_C^2)$$

$$\overline{bel}(x_t) = \mathcal{N}(10, (\sqrt{8})^2) = \mathcal{N}(\mu, \sigma^2)$$



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and you applied the command $u_{t-1} = 10$. Then

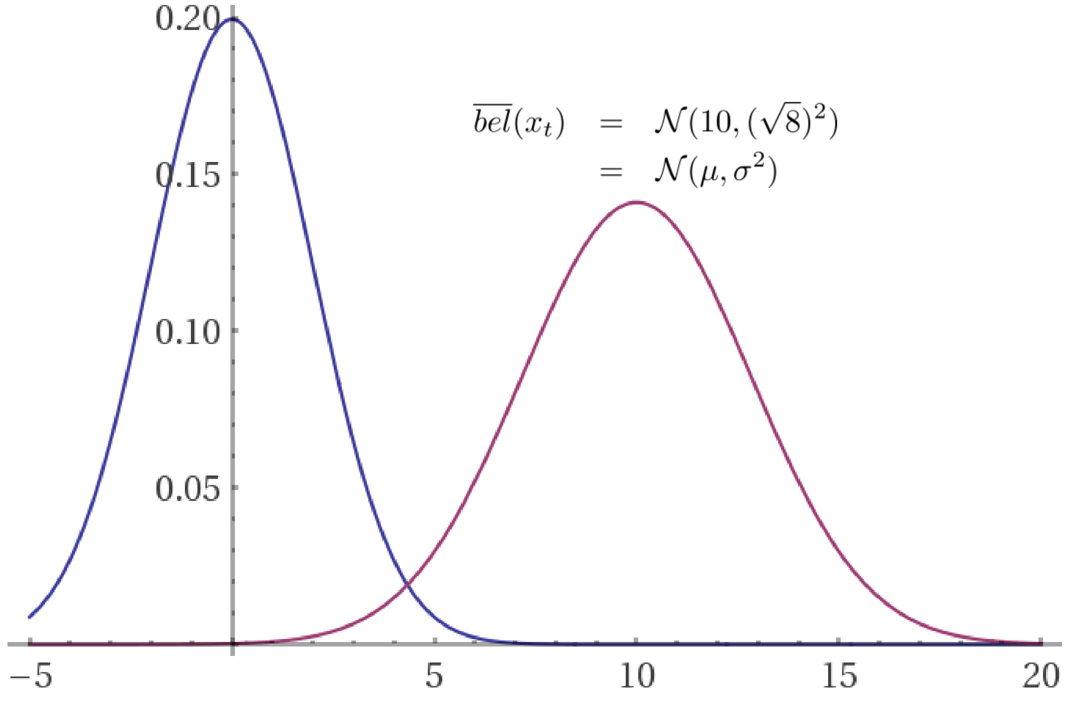
$$\begin{aligned}
\mu &= \mathbb{E}[x_t | z_{0:t-1}, u_{0:t-1}] \\
&= \mathbb{E}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathbb{E}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\
&= \mu_C + u_{t-1}
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}]
\end{aligned}$$

Recall: covariance neglects addition of constant terms, i.e. Cov(X+b) = Cov(X)

Kalman Filter with 1D state: the propagation/prediction step

$$\begin{aligned} bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\ &= \mathcal{N}(\mu_C, \sigma_C^2) \end{aligned}$$



$$\begin{aligned} \overline{bel}(x_t) &= \mathcal{N}(10, (\sqrt{8})^2) \\ &= \mathcal{N}(\mu, \sigma^2) \end{aligned}$$

Suppose that the dynamics model is

$$x_t = x_{t-1} + u_{t-1} + w_{t-1} \quad \text{with} \quad w_{t-1} \sim \mathcal{N}(0, q^2)$$

and you applied the command $u_{t-1} = 10$. Then

$$\begin{aligned} \mu &= \mathbb{E}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \mathbb{E}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathbb{E}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\ &= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\ &= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\ &= \mu_C + u_{t-1} \end{aligned}$$

$$\begin{aligned} \sigma^2 &= \text{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \text{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \text{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \text{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \text{Cov}[w_{t-1} | z_{0:t-1}, u_{0:t-1}] - 2\text{Cov}[x_{t-1}, w_{t-1} | z_{0:t-1}, u_{0:t-1}] \end{aligned}$$

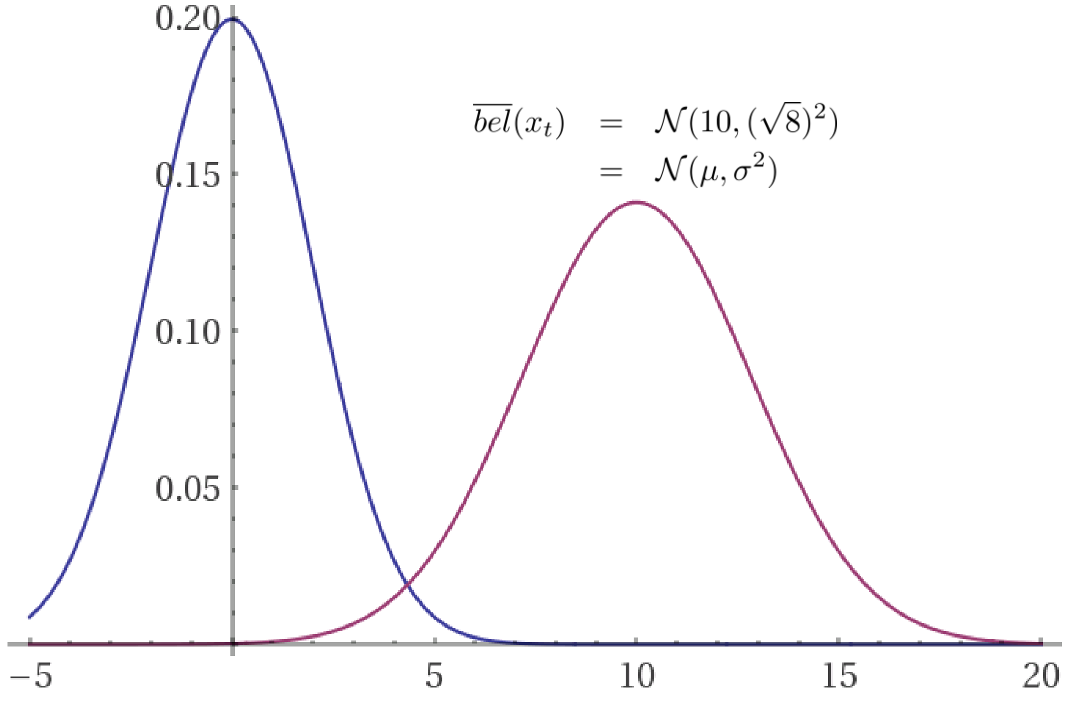
If this algebra doesn’t make sense, see the GraphSLAM lecture.

Recall: Cov(X+Y)=Cov(X)+Cov(Y)-2Cov(X,Y)

Recall: we denote Cov(X,X)=Cov(X) as a shorthand

Kalman Filter with 1D state: the propagation/prediction step

$$\begin{aligned} bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\ &= \mathcal{N}(\mu_C, \sigma_C^2) \end{aligned}$$



$$\begin{aligned} \overline{bel}(x_t) &= \mathcal{N}(10, (\sqrt{8})^2) \\ &= \mathcal{N}(\mu, \sigma^2) \end{aligned}$$

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&= \mathbb{E}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\
&= \mu_C + u_{t-1}
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1} | z_{0:t-1}, u_{0:t-1}] - 2\mathrm{Cov}[x_{t-1}, w_{t-1} | z_{0:t-1}, u_{0:t-1}]
\end{aligned}$$

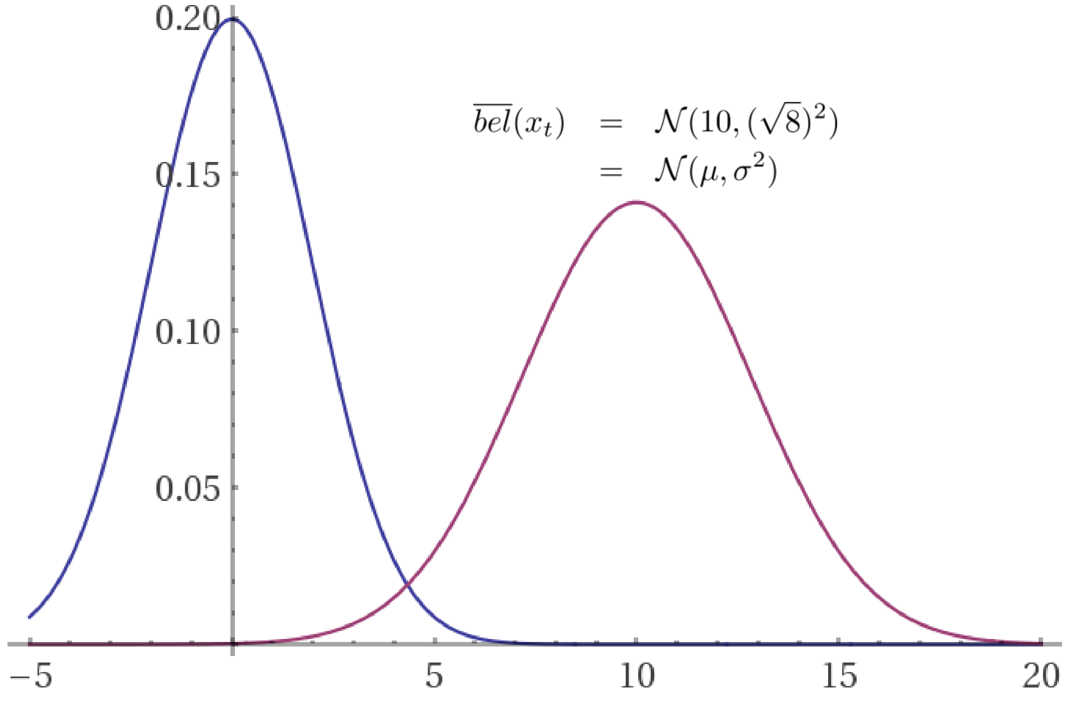
We assumed dynamics noise is independent of past measurement and controls

We assumed noise variables are independent of state. So this covarianc is zero.

Kalman Filter with 1D state: the propagation/prediction step

$$\begin{aligned} bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\ &= \mathcal{N}(\mu_C, \sigma_C^2) \end{aligned}$$

$$\begin{aligned} \overline{bel}(x_t) &= \mathcal{N}(10, (\sqrt{8})^2) \\ &= \mathcal{N}(\mu, \sigma^2) \end{aligned}$$



Suppose that the dynamics model is

$$x_t = x_{t-1} + u_{t-1} + w_{t-1} \quad \text{with} \quad w_{t-1} \sim \mathcal{N}(0, q^2)$$

and you applied the command $u_{t-1} = 10$. Then

$$\begin{aligned} \mu &= \mathbb{E}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \mathbb{E}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathbb{E}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\ &= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\ &= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\ &= \mu_C + u_{t-1} \end{aligned}$$

$$\begin{aligned} \sigma^2 &= \mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1} | z_{0:t-1}, u_{0:t-1}] - 2\mathrm{Cov}[x_{t-1}, w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1}] \end{aligned}$$

Kalman Filter with 1D state: the propagation/prediction step

$$
\begin{aligned}
bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\
&= \mathcal{N}(\mu_C, \sigma_C^2)
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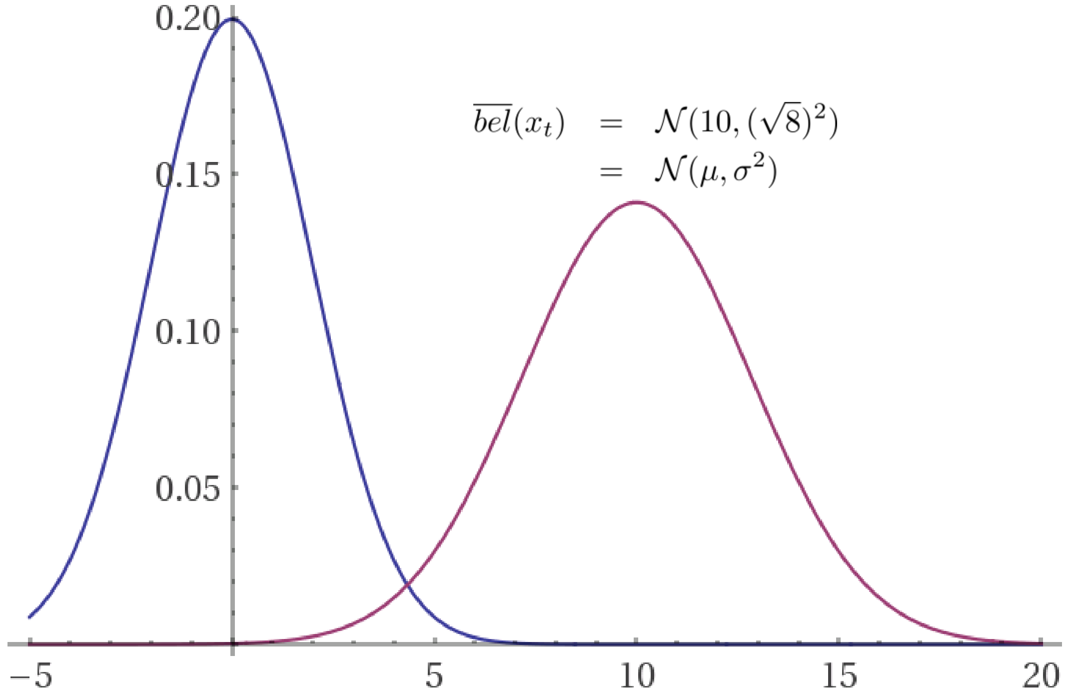
$$



$$
\begin{aligned}
\overline{bel}(x_t) &= \mathcal{N}(10, (\sqrt{8})^2) \\
&= \mathcal{N}(\mu, \sigma^2)
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$$

Suppose that the dynamics model is

$x_t = x_{t-1} + u_{t-1} + w_{t-1}$ with $w_{t-1} \sim \mathcal{N}(0, q^2)$

and you applied the command $u_{t-1} = 10$. Then

$$
\begin{aligned}
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&= \mathbb{E}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
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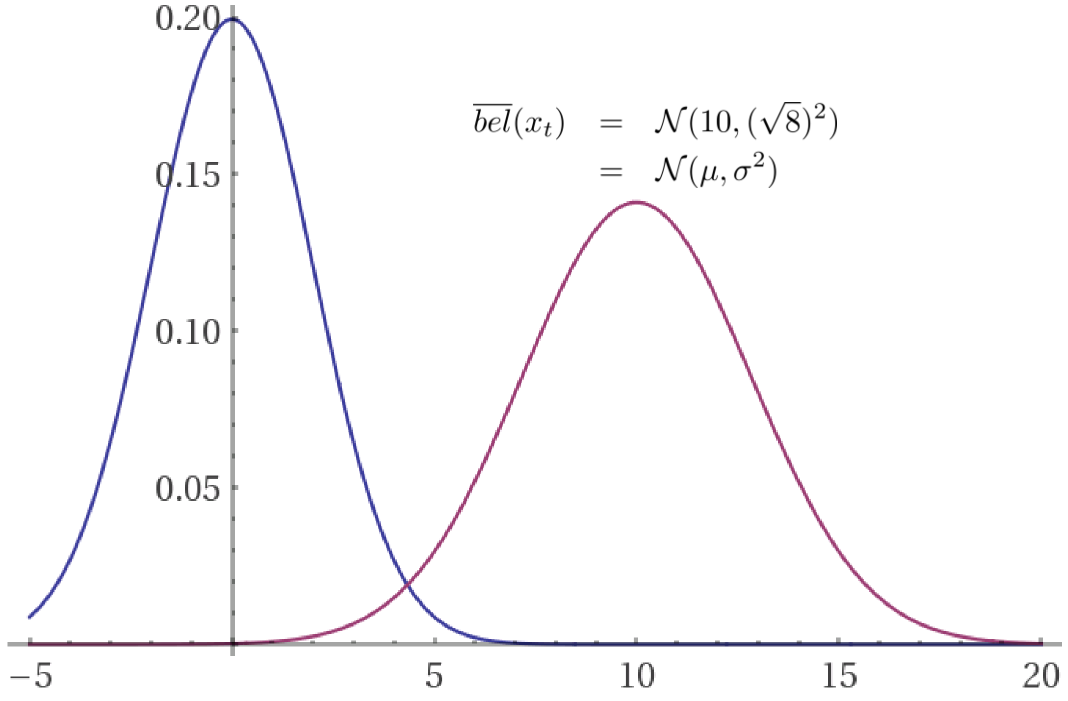
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + u_{t-1} \\
&= \mathbb{E}[x_{t-1} | z_{0:t-1}, u_{0:t-2}] + u_{t-1} \\
&= \mu_C + u_{t-1}
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 &= \mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1} | z_{0:t-1}, u_{0:t-1}] - 2\mathrm{Cov}[x_{t-1}, w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\
&= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1}] \\
&= \mathrm{Cov}[x_{t-1} | z_{0:t-2}, u_{0:t-2}] + \mathrm{Cov}[w_{t-1}]
\end{aligned}
$$

Kalman Filter with 1D state: the propagation/prediction step

$$\begin{aligned} bel(x_{t-1}) &= \mathcal{N}(0, 2^2) \\ &= \mathcal{N}(\mu_C, \sigma_C^2) \end{aligned}$$



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$x_t = x_{t-1} + u_{t-1} + w_{t-1}$ with $w_{t-1} \sim \mathcal{N}(0, q^2)$

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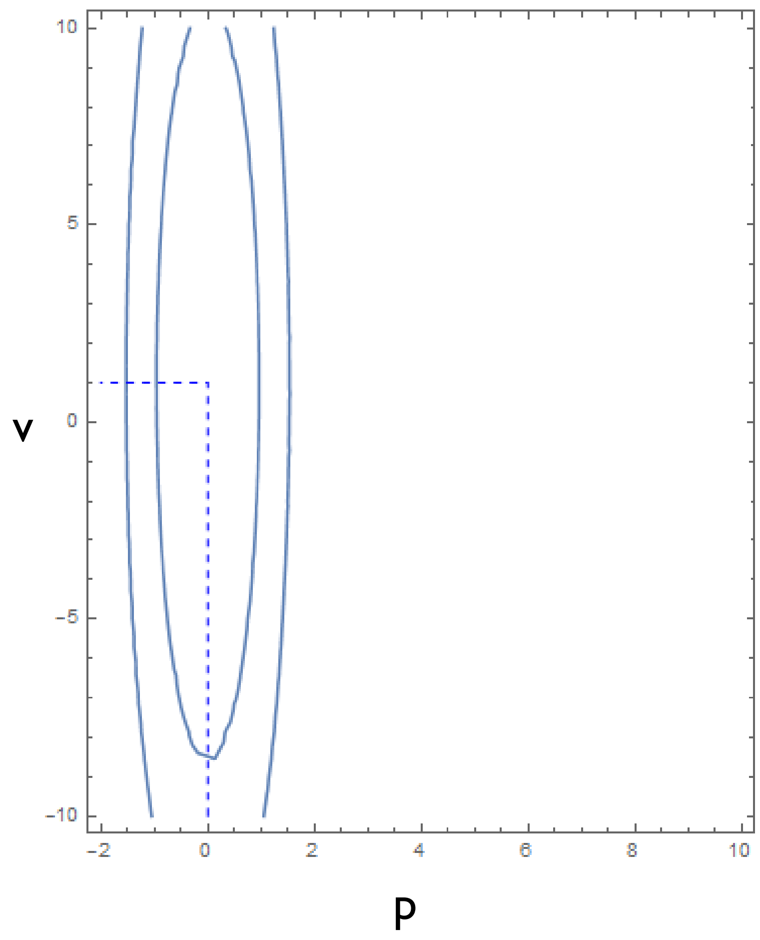
$$\begin{aligned} \sigma^2 &= \mathrm{Cov}[x_t | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} + u_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} + w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1} | z_{0:t-1}, u_{0:t-1}] - 2\mathrm{Cov}[x_{t-1}, w_{t-1} | z_{0:t-1}, u_{0:t-1}] \\ &= \mathrm{Cov}[x_{t-1} | z_{0:t-1}, u_{0:t-1}] + \mathrm{Cov}[w_{t-1}] \\ &= \mathrm{Cov}[x_{t-1} | z_{0:t-2}, u_{0:t-2}] + \mathrm{Cov}[w_{t-1}] \\ &= \sigma_C^2 + q^2 \end{aligned}$$

Kalman Filter with 1D state: the propagation/prediction step

Take home message: uncertainty **increases** after the prediction step, because we are speculating about the future.

Kalman Filter with 2D state



Suppose we have a robot that moves on a 1D line, but we also want to estimate its velocity. Then the 2D state vector is $x_t = [p, v]^T$

Suppose we do not have any control over this robot, i.e. we are just trying to estimate its state through **observations of the position only**. I.e.:

$$z_t = Hx_t + n_t = [1,\ 0]x_t + n_t \quad \text{with} \quad n_t \sim \mathcal{N}(0, r^2)$$

Also suppose that we predict zero acceleration in the near future, so

$$\begin{aligned} p_{t+1} &= p_t + v_t \delta t + w_p(t) \\ v_{t+1} &= v_t + w_v(t) \end{aligned}$$

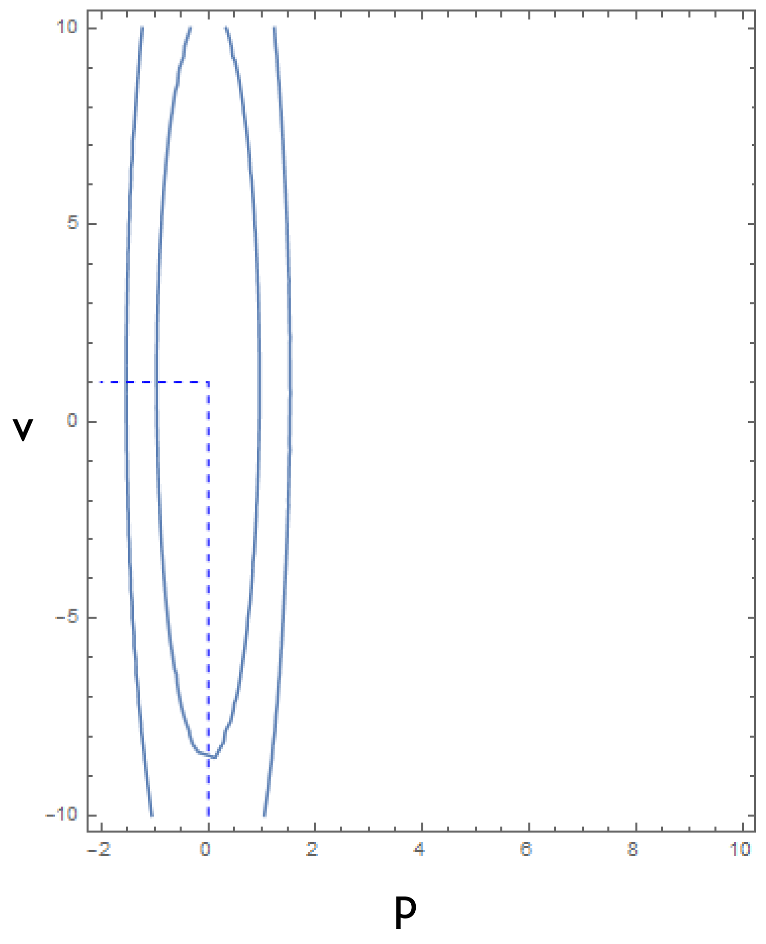
which in vector form is expressed as

$$A = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix}$$

$$x_{t+1} = Ax_t + w_t$$

$$w_t = \begin{bmatrix} w_p(t) \\ w_v(t) \end{bmatrix} \sim \mathcal{N}(0_{2\times 1}, Q)$$

Kalman Filter with 2D state



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Also suppose that we predict zero acceleration in the near future, so

$$\begin{aligned} p_{t+1} &= p_t + v_t \delta t + w_p(t) \\ v_{t+1} &= v_t + w_v(t) \end{aligned}$$

For this example suppose dt=1, r = 1 and $Q = I_{2\times 2}$

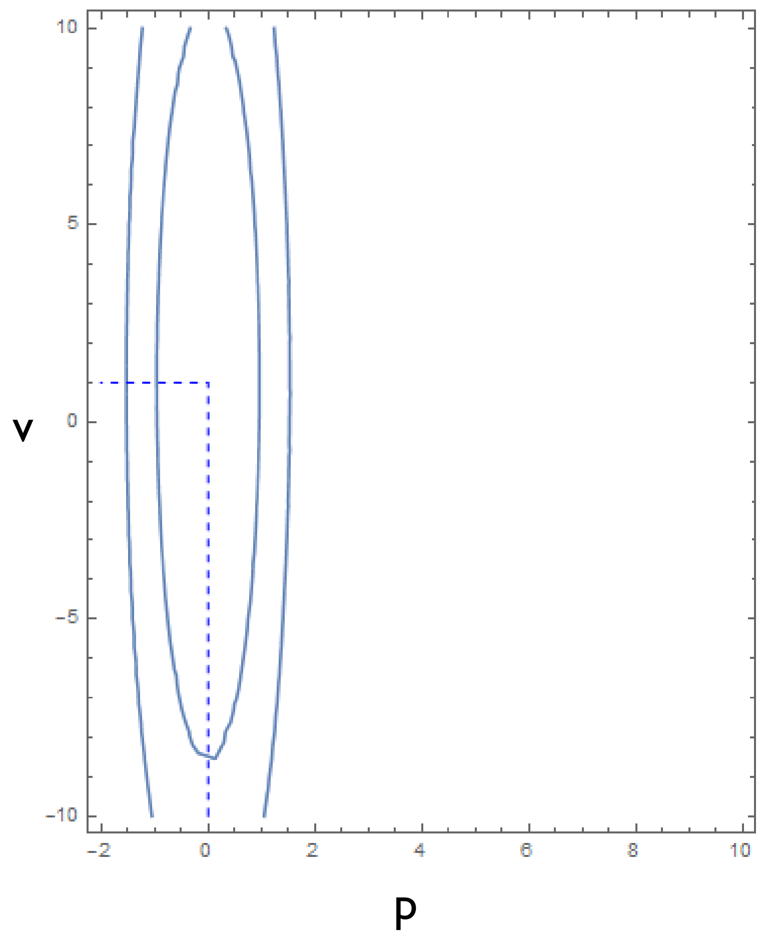
which in vector form is expressed as

$$x_{t+1} = Ax_t + w_t$$

$$A = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix}$$

$$w_t = \begin{bmatrix} w_p(t) \\ w_v(t) \end{bmatrix} \sim \mathcal{N}(0_{2\times 1}, Q)$$

Kalman Filter with 2D state



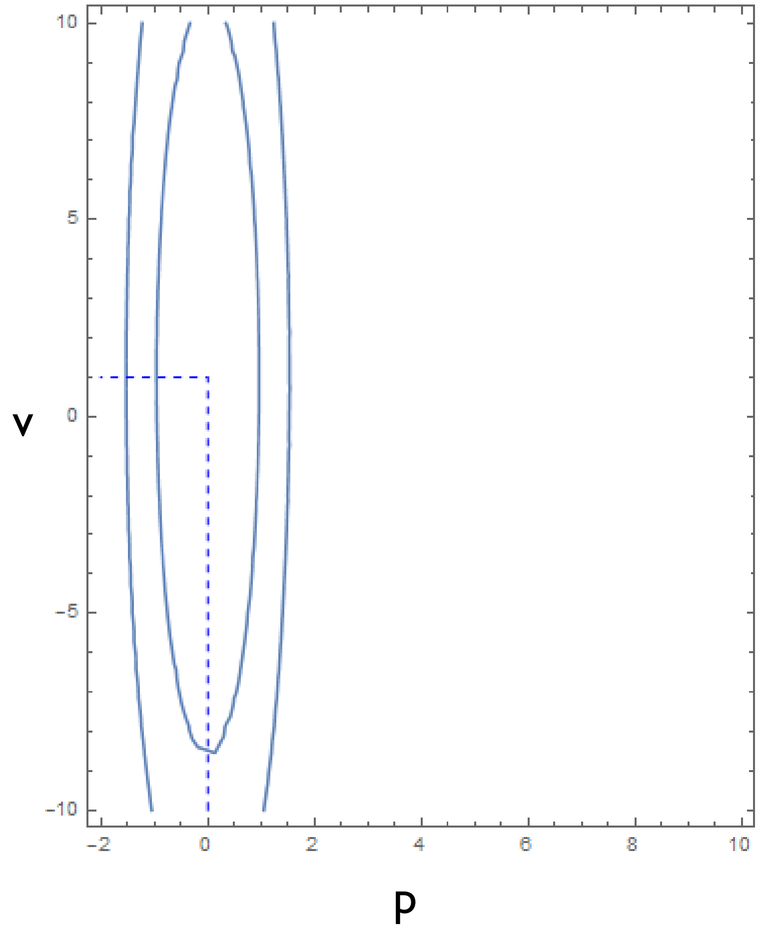
Suppose that at time t the state is distributed as
$p(x_t|z_{0:t}) = \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$ with

$$\mu_{t|t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Sigma_{t|t} = \begin{bmatrix} \sigma_p^2 & \sigma_{pv} \\ \sigma_{pv} & \sigma_v^2 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 \\ 0 & 10^2 \end{bmatrix}$$

In other words, we are confident that in the beginning the position is with high probability (~0.997) within range $3\sigma_p = 3$ of the mean position, 0.

We are not very confident in the velocity, however. We just know a priori that with high probability (~0.997) it is within range $3\sigma_v = 30$ of the mean velocity, 1.

Kalman Filter with 2D state



Suppose that at time t the state is distributed as
$p(x_t|z_{0:t}) = \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$ with

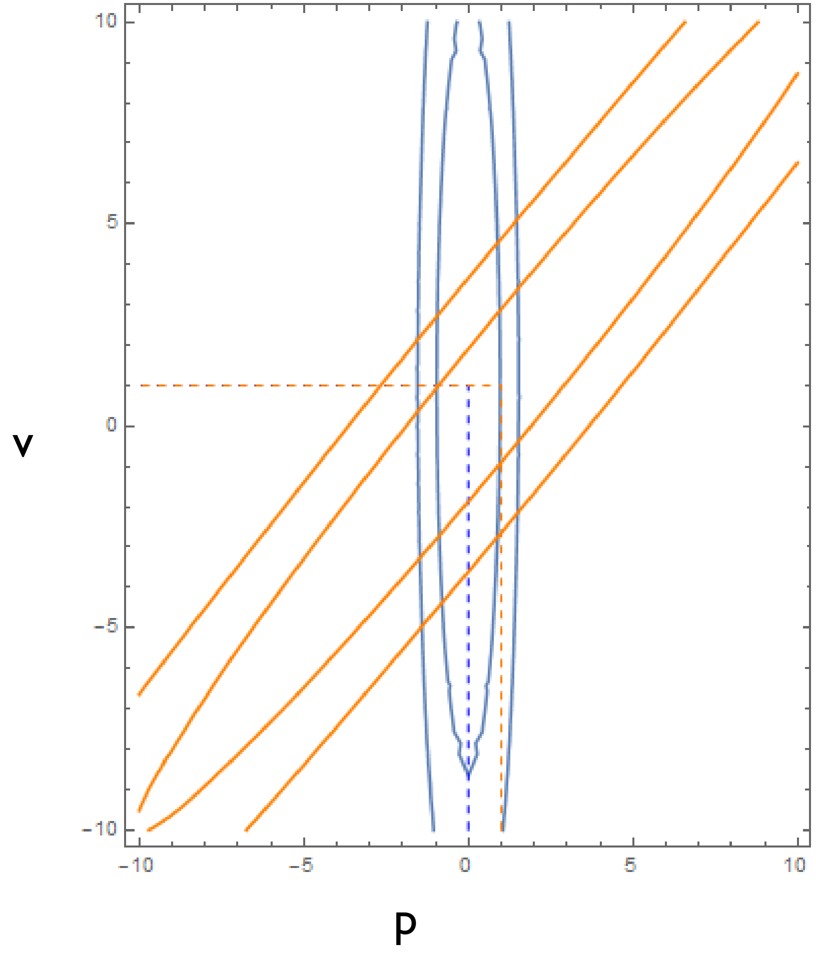
$$\mu_{t|t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Sigma_{t|t} = \begin{bmatrix} \sigma_p^2 & \sigma_{pv} \\ \sigma_{pv} & \sigma_v^2 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 \\ 0 & 10^2 \end{bmatrix}$$

In other words, we are confident that in the beginning the position is with high probability (~0.997) within range $3\sigma_p = 3$ of the mean position, 0.

We are not very confident in the velocity, however. We just know a priori that with high probability (~0.997) it is within range $3\sigma_v = 30$ of the mean velocity, 1.

Notice that when the cross-correlation terms $\sigma_{pv} = 0$ then the ellipse is axis-aligned. This means that the position and velocity are initially uncorrelated.

Kalman Filter with 2D state: the propagation/prediction step

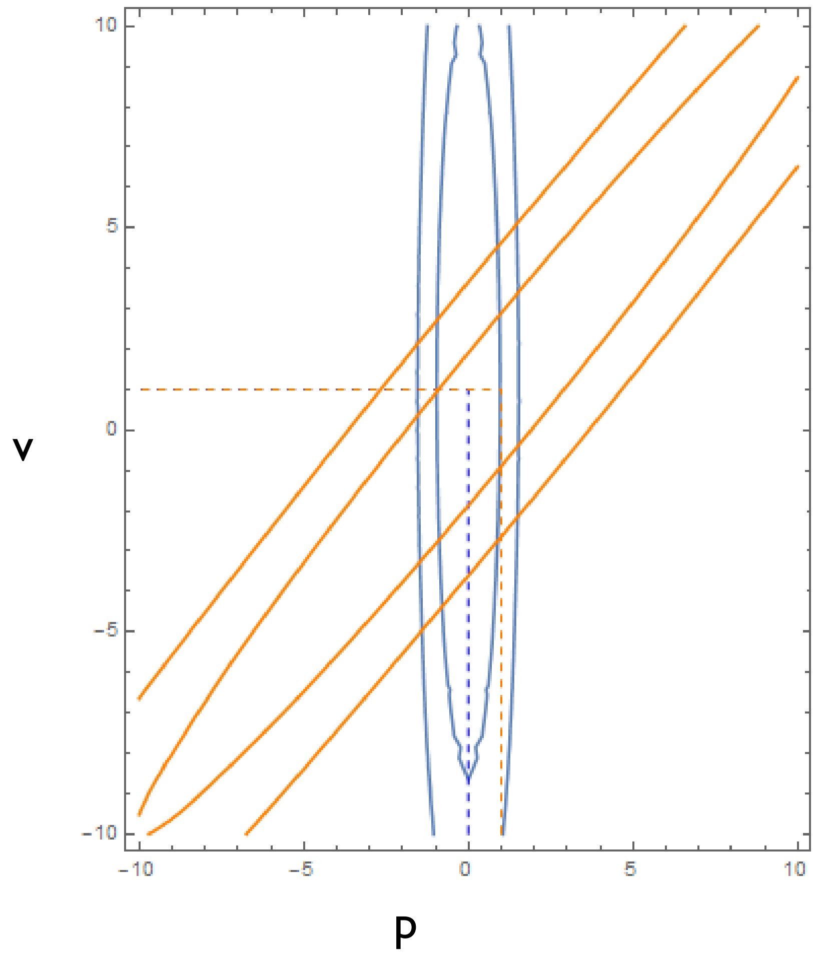


After the prediction step the state is distributed as
$p(x_{t+1}|z_{0:t}) = \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$ with

$$
\begin{aligned}
\mu_{t+1|t} &= \mathbb{E}[x_{t+1}|z_{0:t}] \\
&= \mathbb{E}[Ax_t + w_t|z_{0:t}] \\
&= A\mathbb{E}[x_t + w_t|z_{0:t}] \\
&= A\mathbb{E}[x_t|z_{0:t}] \\
&= A\mu_{t|t} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_{t+1|t} &= \mathrm{Cov}[x_{t+1}|z_{0:t}] \\
&= \mathrm{Cov}[Ax_t + w_t|z_{0:t}] \\
&= \mathrm{Cov}[Ax_t|z_{0:t}] + \mathrm{Cov}[w_t|z_{0:t}] - 2\mathrm{Cov}[Ax_t, w_t|z_{0:t}] \\
&= A\mathrm{Cov}[x_t|z_{0:t}]A^T + \mathrm{Cov}[w_t] \\
&= A\Sigma_{t|t}A^T + Q \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1^2 & 0 \\ 0 & 10^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1^2 & 0 \\ 0 & 1^2 \end{bmatrix} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix}
\end{aligned}
$$

Kalman Filter with 2D state: the propagation/prediction step



After the prediction step the state is distributed as

$p(x_{t+1}|z_{0:t}) = \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$ with

$$
\begin{aligned}
\mu_{t+1|t} &= \mathbb{E}[x_{t+1}|z_{0:t}] \\
&= \mathbb{E}[Ax_t + w_t|z_{0:t}] \\
&= A\mathbb{E}[x_t + w_t|z_{0:t}] \\
&= A\mathbb{E}[x_t|z_{0:t}] \\
&= A\mu_{t|t} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
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$$

$$
\begin{aligned}
\Sigma_{t+1|t} &= \mathrm{Cov}[x_{t+1}|z_{0:t}] \\
&= \mathrm{Cov}[Ax_t + w_t|z_{0:t}] \\
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&= A\mathrm{Cov}[x_t|z_{0:t}]A^T + \mathrm{Cov}[w_t] \\
&= A\Sigma_{t|t}A^T + Q \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1^2 & 0 \\ 0 & 10^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1^2 & 0 \\ 0 & 1^2 \end{bmatrix} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix}
\end{aligned}
$$

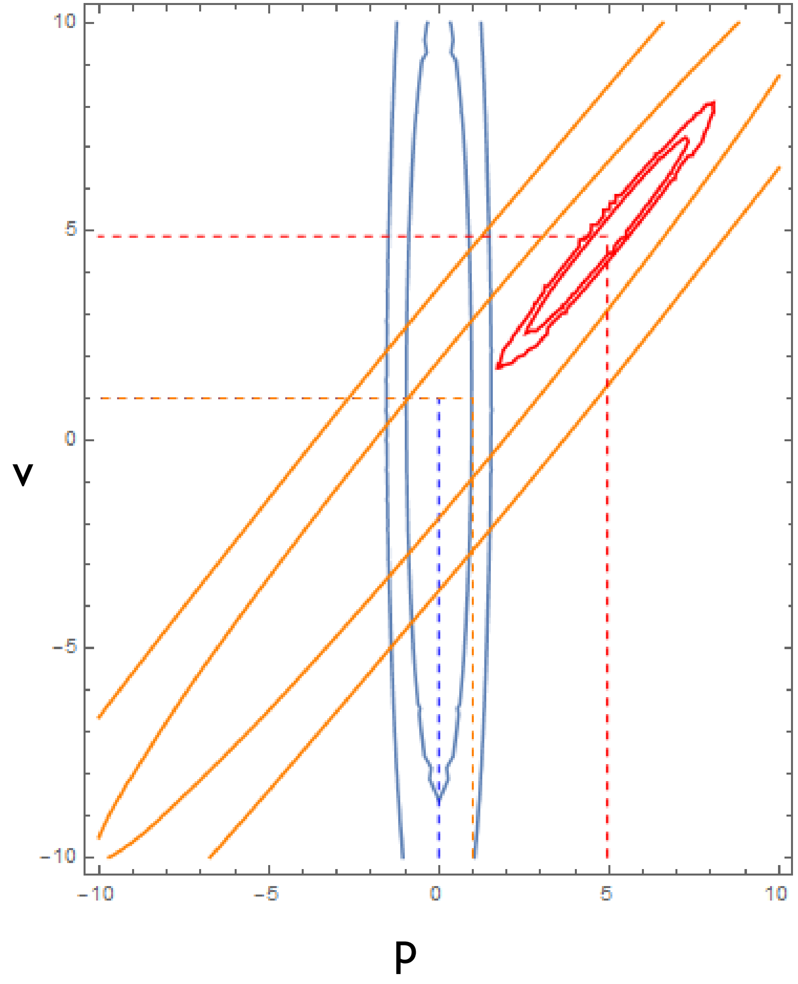
Many things to notice here:

The covariance has nonzero off-diagonal terms, so the position and velocity are now correlated. This is why the orange ellipse is rotated.

Also, the orange ellipse is "larger" than the initial blue ellipse, which means that our uncertainty has increased by speculating for future outcomes.

There is now large uncertainty in the predicted position, since there was large uncertainty in the velocity.

Kalman Filter with 2D state: the update step



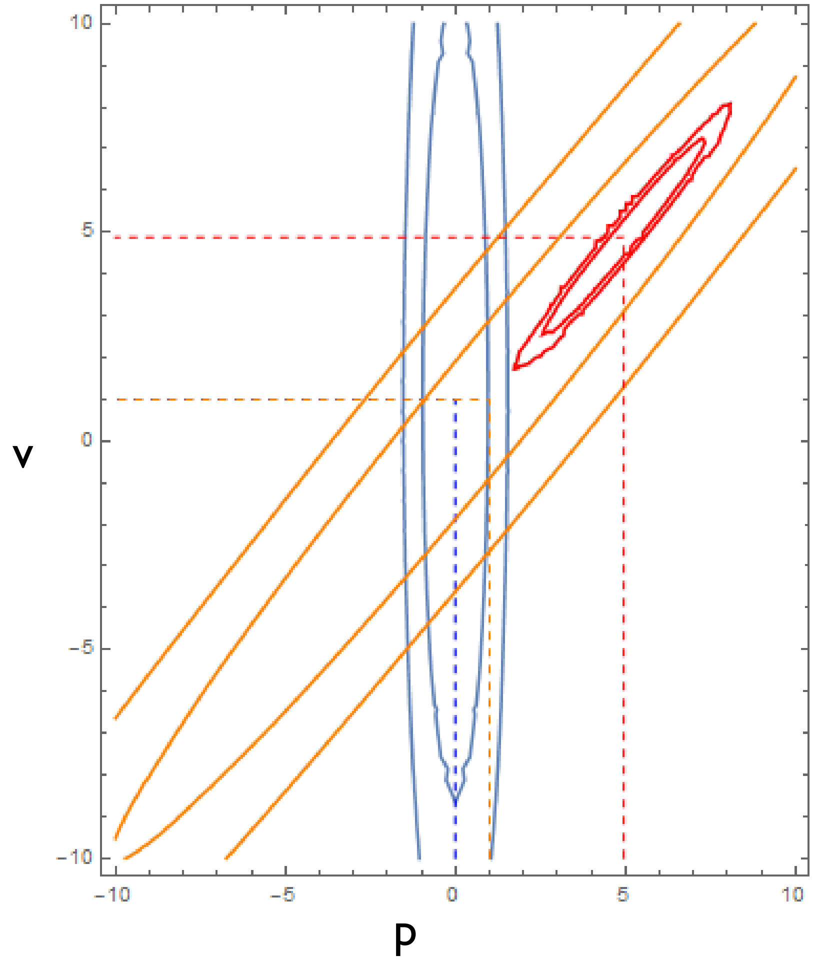
Before the update step the state is distributed as
$p(x_{t+1}|z_{0:t}) = \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$ with $\mu_{t+1|t} = [1,1]^T$ and $\Sigma_{t+1|t} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix}$

At this point we predict that the next measurement of the position is going to be $\mu_{z_{t+1}} = H\mu_{t+1|t} = [1,0]\mu_{t+1|t} = 1$ with uncertainty s_{t+1}^2 which depends on previous uncertainty and measurement uncertainty.

Suppose that we actually measure $\bar{z}_{t+1} = 5$ which means that our mean estimate of the velocity was way off (it was 1).
Therefore, there is a prediction residual/error $\delta z = \bar{z}_{t+1} - z_{t+1} \sim \mathcal{N}(4, s_{t+1}^2)$
How confident are we about this residual?

$$\begin{aligned}
s_{t+1}^2 &= \mathrm{Cov}[\bar{z}_{t+1} - z_{t+1}|z_{0:t}] \\
&= \mathrm{Cov}[z_{t+1}|z_{0:t}] \\
&= \mathrm{Cov}[Hx_{t+1} + n_{t+1}|z_{0:t}] \\
&= H\mathrm{Cov}[x_{t+1}|z_{0:t}]H^T + \mathrm{Cov}[n_{t+1}|z_{0:t}] \\
&= H\mathrm{Cov}[x_{t+1}|z_{0:t}]H^T + \mathrm{Cov}[n_{t+1}] \\
&= H\Sigma_{t+1|t}H^T + r^2 = 102 + 1^2 = 103
\end{aligned}$$

Kalman Filter with 2D state: the update step



Before the update step the state is distributed as $p(x_{t+1}|z_{0:t}) = \mathcal{N}(\mu_{t+1|t}, \Sigma_{t+1|t})$ with $\mu_{t+1|t} = [1,1]^T$ and $\Sigma_{t+1|t} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix}$

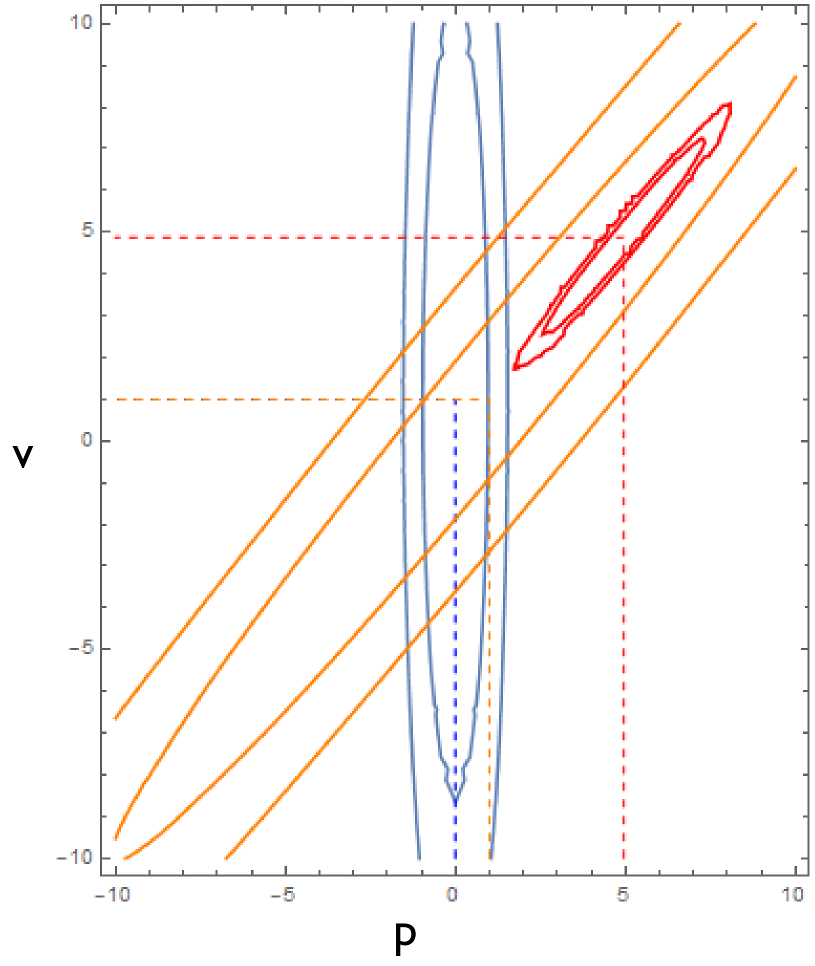
At this point we predict that the next measurement of the position is going to be $\mu_{z_{t+1}} = H\mu_{t+1|t} = [1,0]\mu_{t+1|t} = 1$ with uncertainty s^2_{t+1} which depends on previous state's uncertainty and measurement uncertainty.

Suppose that we actually measure $\bar{z}_{t+1} = 5$ which means that our mean estimate of the velocity was way off (it was 1).
Therefore, there is a prediction residual/error $\delta z = \bar{z}_{t+1} - z_{t+1} \sim \mathcal{N}(4, s^2_{t+1})$
How confident are we about this residual?

$$\begin{aligned} s^2_{t+1} &= \mathrm{Cov}[\bar{z}_{t+1} - z_{t+1}|z_{0:t}] \\ &= \mathrm{Cov}[z_{t+1}|z_{0:t}] \\ &= \mathrm{Cov}[Hx_{t+1} + n_{t+1}|z_{0:t}] \\ &= H\mathrm{Cov}[x_{t+1}|z_{0:t}]H^T + \mathrm{Cov}[n_{t+1}|z_{0:t}] \\ &= H\mathrm{Cov}[x_{t+1}|z_{0:t}]H^T + \mathrm{Cov}[n_{t+1}] \\ &= H\Sigma_{t+1|t}H^T + r^2 = 102 + 1^2 = 103 \end{aligned}$$

This means that our measurement was within a range of $3\sqrt{103}$ from the true position with high probability (~0.997)

Kalman Filter with 2D state: the update step



Kalman Gain:

determines how much the state and the covariance needs to be updated

How do we update our belief based on the noisy measurement? We're not going to provide a proof here (see Probabilistic Robotics, section 3.2), but the updated belief is

$$p(x_{t+1}|z_{0:t+1}) = \mathcal{N}(\mu_{t+1|t+1}, \Sigma_{t+1|t+1})$$

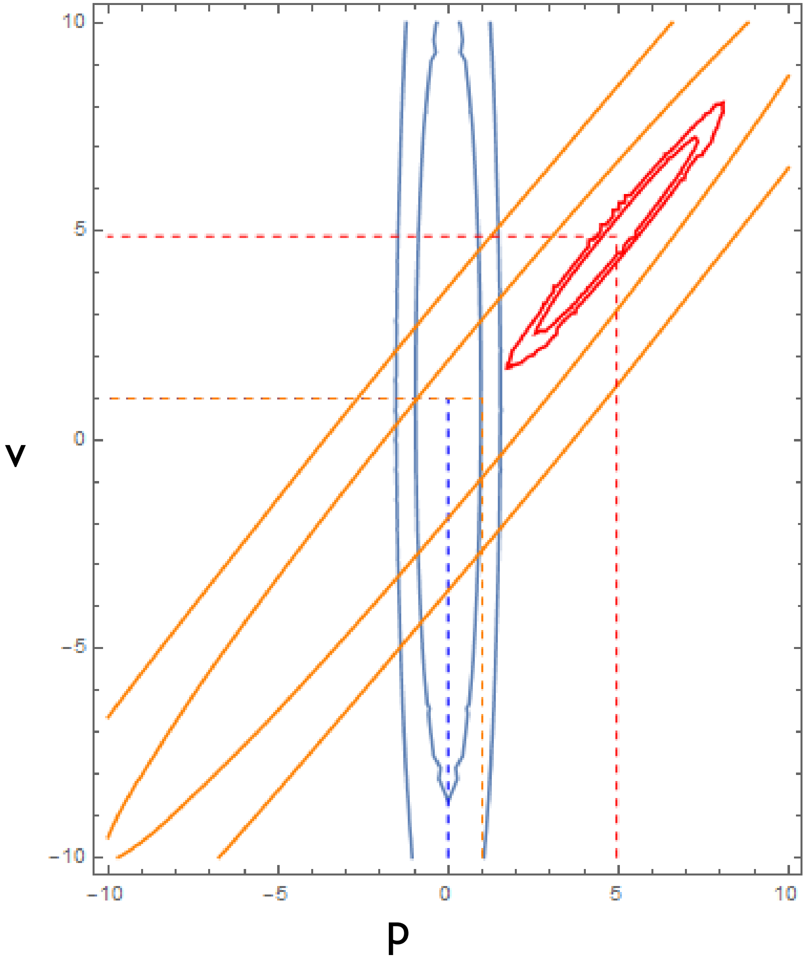
with

$$K_{t+1} = \Sigma_{t+1|t} H^T s_{t+1}^{-2} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 103^{-1} = \begin{bmatrix} 102/103 \\ 100/103 \end{bmatrix}$$

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}\mu_{\delta z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 102/103 \\ 100/103 \end{bmatrix} 4 = \begin{bmatrix} 4.96 \\ 4.88 \end{bmatrix}$$

$$\begin{aligned} \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - KH\Sigma_{t+1|t} \\ &= \Sigma_{t+1|t} - \frac{102}{103}\Sigma_{t+1|t} \\ &= \frac{1}{103}\Sigma_{t+1|t} = \begin{bmatrix} 0.99 & 0.97 \\ 0.97 & 0.98 \end{bmatrix} \end{aligned}$$

Kalman Filter with 2D state: the update step



How do we update our belief based on the noisy measurement? We're not going to provide a proof here (see Probabilistic Robotics, section 3.2), but the updated belief is

$$p(x_{t+1}|z_{0:t+1}) = \mathcal{N}(\mu_{t+1|t+1}, \Sigma_{t+1|t+1})$$

with

Kalman Gain:

$$K_{t+1} = \Sigma_{t+1|t} H^T s_{t+1}^{-2} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 103^{-1} = \begin{bmatrix} 102/103 \\ 100/103 \end{bmatrix}$$

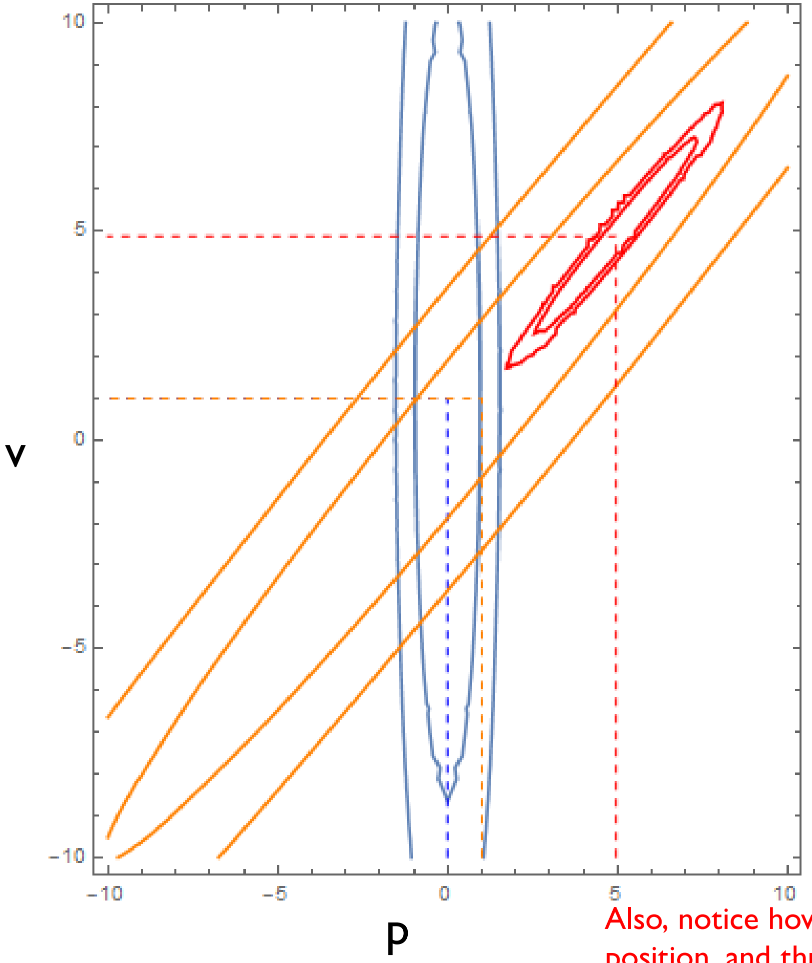
determines how much the state and the covariance needs to be updated

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}\mu_{\delta z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 102/103 \\ 100/103 \end{bmatrix} 4 = \begin{bmatrix} 4.96 \\ 4.88 \end{bmatrix}$$

$$\begin{aligned} \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - KH\Sigma_{t+1|t} \\ &= \Sigma_{t+1|t} - \frac{102}{103}\Sigma_{t+1|t} \\ &= \frac{1}{103}\Sigma_{t+1|t} = \begin{bmatrix} 0.99 & 0.97 \\ 0.97 & 0.98 \end{bmatrix} \end{aligned}$$

After the measurement the covariance was reduced. We are now more confident than both the measurement and the prediction estimate.

Kalman Filter with 2D state: the update step



Also, notice how we MEASURED position, and through correlation, we were able to INFER velocity. This is not always possible.

How do we update our belief based on the noisy measurement? We're not going to provide a proof here (see Probabilistic Robotics, section 3.2), but the updated belief is

$$p(x_{t+1}|z_{0:t+1}) = \mathcal{N}(\mu_{t+1|t+1}, \Sigma_{t+1|t+1})$$

with

$$K_{t+1} = \Sigma_{t+1|t} H^T s_{t+1}^{-2} = \begin{bmatrix} 102 & 100 \\ 100 & 101 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 103^{-1} = \begin{bmatrix} 102/103 \\ 100/103 \end{bmatrix}$$

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$$\begin{aligned} \Sigma_{t+1|t+1} &= \Sigma_{t+1|t} - KH\Sigma_{t+1|t} \\ &= \Sigma_{t+1|t} - \frac{102}{103}\Sigma_{t+1|t} \\ &= \frac{1}{103}\Sigma_{t+1|t} = \begin{bmatrix} 0.99 & 0.97 \\ 0.97 & 0.98 \end{bmatrix} \end{aligned}$$

Dynamics

$$x_{t+1} = Ax_t + Bu_t + Gw_t$$
$$w_t \sim \mathcal{N}(0, Q)$$

Measurements

$$z_t = Hx_t + n_t$$
$$n_t \sim \mathcal{N}(0, R)$$

Kalman Filter in N dimensions

Init

$$bel(x_0) \sim \mathcal{N}(\mu_{0|0}, \Sigma_{0|0})$$

Prediction Step

$$\mu_{t+1|t} = A\mu_{t|t} + Bu_t$$
$$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + GQG^T$$

Update Step

Received measurement $\bar{z}_{t+1}$ but expected to receive $\mu_{z_{t+1}} = H\mu_{t+1|t}$

Prediction residual is a Gaussian random variable $\delta z \sim \mathcal{N}(\bar{z}_{t+1} - \mu_{z_{t+1}}, S_{t+1})$
where the covariance of the residual is $S_{t+1} = H\Sigma_{t+1|t}H^T + R$

Kalman Gain (optimal correction factor): $K_{t+1} = \Sigma_{t+1|t}H^T S_{t+1}^{-1}$

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}(\bar{z}_{t+1} - \mu_{z_{t+1}})$$
$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1}H\Sigma_{t+1|t}$$

Kalman Filter in N dimensions

Dynamics

$$x_{t+1} = Ax_t + Bu_t + Gw_t$$
$$w_t \sim \mathcal{N}(0, Q)$$

Measurements

$$z_t = Hx_t + n_t$$
$$n_t \sim \mathcal{N}(0, R)$$

Init

$$bel(x_0) \sim \mathcal{N}(\mu_{0|0}, \Sigma_{0|0})$$

Prediction Step

$$\mu_{t+1|t} = A\mu_{t|t} + Bu_t$$
$$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + GQG^T$$

Update Step

Received measurement $\bar{z}_{t+1}$ but expected to receive $\mu_{z_{t+1}} = H\mu_{t+1|t}$

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where the covariance of the residual is $S_{t+1} = H\Sigma_{t+1|t}H^T + R$

Kalman Gain (optimal correction factor): $K_{t+1} = \Sigma_{t+1|t}H^T S_{t+1}^{-1}$

Potentially expensive and error-prone operation: matrix inversion O(|z|^2.4)

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}(\bar{z}_{t+1} - \mu_{z_{t+1}})$$
$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1}H\Sigma_{t+1|t}$$

Dynamics

$$x_{t+1} = Ax_t + Bu_t + Gw_t$$
$$w_t \sim \mathcal{N}(0, Q)$$

Kalman Filter in N dimensions

Measurements

$$z_t = Hx_t + n_t$$
$$n_t \sim \mathcal{N}(0, R)$$

Init

$$bel(x_0) \sim \mathcal{N}(\mu_{0|0}, \Sigma_{0|0})$$

Prediction Step

$$\mu_{t+1|t} = A\mu_{t|t} + Bu_t$$
$$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + GQG^T$$

Update Step

Received measurement $\bar{z}_{t+1}$ but expected to receive $\mu_{z_{t+1}} = H\mu_{t+1|t}$

Prediction residual is a Gaussian random variable $\delta z \sim \mathcal{N}(\bar{z}_{t+1} - \mu_{z_{t+1}}, S_{t+1})$
where the covariance of the residual is $S_{t+1} = H\Sigma_{t+1|t}H^T + R$

Kalman Gain (optimal correction factor): $K_{t+1} = \Sigma_{t+1|t}H^T S_{t+1}^{-1}$

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1}(\bar{z}_{t+1} - \mu_{z_{t+1}})$$
$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1}H\Sigma_{t+1|t}$$

Numerical errors may make the covariance non-symmetric at some point. In practice, we either force symmetry, or we decompose the covariance during the update.

See “Factorization methods for discrete sequential estimation” by Gerald Bierman for more info.

Kalman Filter with 4D state

Suppose a cannonball is shot from a cannon, and assume we can somehow measure its position in flight.

Assuming zero drag and resistance from the air, the only force acting on the ball after it is ejected is its weight (suppose mass=1kg).

Then the continuous dynamics of the system are given by $\begin{aligned} \ddot{p_x} &= w_x \\ \ddot{p_y} &= -g + w_y \end{aligned}$ where w is noise in the acceleration.

The discrete-time version of this dynamics model is

$$\begin{aligned}
p_x(t+1) &= p_x(t) + v_x(t)\delta t + w_x(t)\delta t^2/2 \\
p_y(t+1) &= p_y(t) + v_y(t)\delta t + (-g + w_y(t))\delta t^2/2 \\
v_x(t+1) &= v_x(t) + w_x(t)\delta t \\
v_y(t+1) &= v_y(t) + (-g + w_y(t))\delta t
\end{aligned}$$

which can be expressed in matrix form as $x_{t+1} = Ax_t + Bu_t + Gw_t$ where $x_t = [p_x(t),\ p_y(t),\ v_x(t),\ v_y(t)]^T$

$$A = \begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = I_{4\times 4} \qquad u_t = \begin{bmatrix} 0 \\ -g\delta t^2/2 \\ 0 \\ -g\delta t \end{bmatrix} \qquad G = \begin{bmatrix} \delta t^2/2 & 0 \\ 0 & \delta t^2/2 \\ \delta t & 0 \\ 0 & \delta t \end{bmatrix} \qquad w_t \sim \mathcal{N}(0_{2\times 1}, Q) \qquad g = 9.81 m/s^2$$

Since we can measure its position the measurement model is $z_t = Hx_t + n_t$ where $H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $n_t \sim \mathcal{N}(0_{2\times 1}, R)$

Kalman Filter with 4D state

Suppose a cannonball is shot from a cannon, and assume we can somehow measure its position in flight.

Assuming zero drag and resistance from the air, the only force acting on the ball after it is ejected is its weight (suppose mass=1kg).

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The discrete-time version of this dynamics model is

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v_x(t+1) &= v_x(t) + w_x(t)\delta t \\
v_y(t+1) &= v_y(t) + (-g + w_y(t))\delta t
\end{aligned}$$

which can be expressed in matrix form as $x_{t+1} = Ax_t + Bu_t + Gw_t$ where $x_t = [p_x(t),\ p_y(t),\ v_x(t),\ v_y(t)]^T$

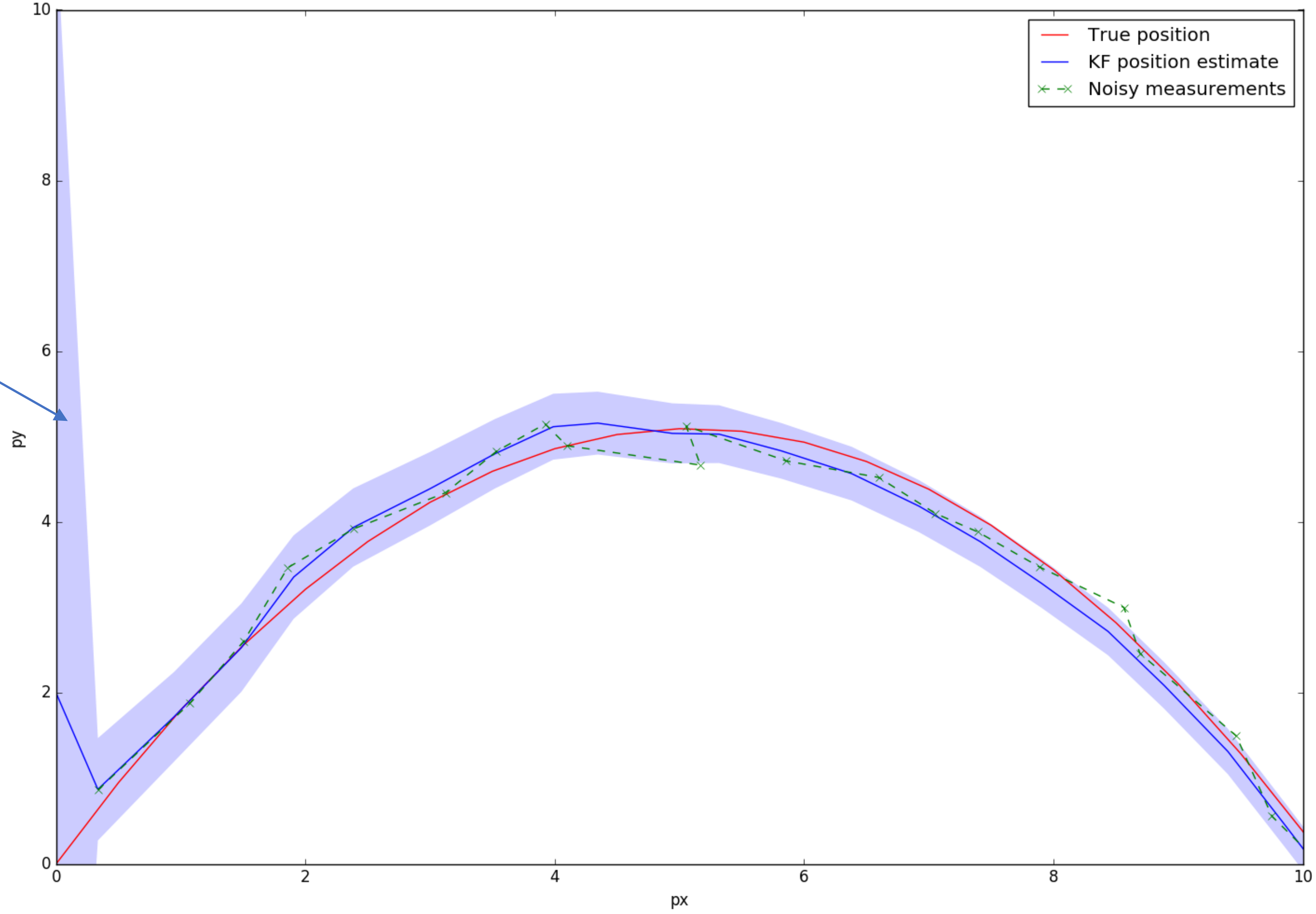
$$A = \begin{bmatrix}1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \qquad B = I_{4\times 4} \qquad u_t = \begin{bmatrix}0 \\ -g\delta t^2/2 \\ 0 \\ -g\delta t\end{bmatrix} \qquad G = \begin{bmatrix}\delta t^2/2 & 0 \\ 0 & \delta t^2/2 \\ \delta t & 0 \\ 0 & \delta t\end{bmatrix} \qquad w_t \sim \mathcal{N}(0_{2\times 1}, Q) \qquad g = 9.81 m/s^2$$

Since we can measure its position the measurement model is $z_t = Hx_t + n_t$ where $H = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{bmatrix}$ and $n_t \sim \mathcal{N}(0_{2\times 1}, R)$

Notice here that we don't actually control the system,
but we include u_t to account for additive constants in the dynamics.

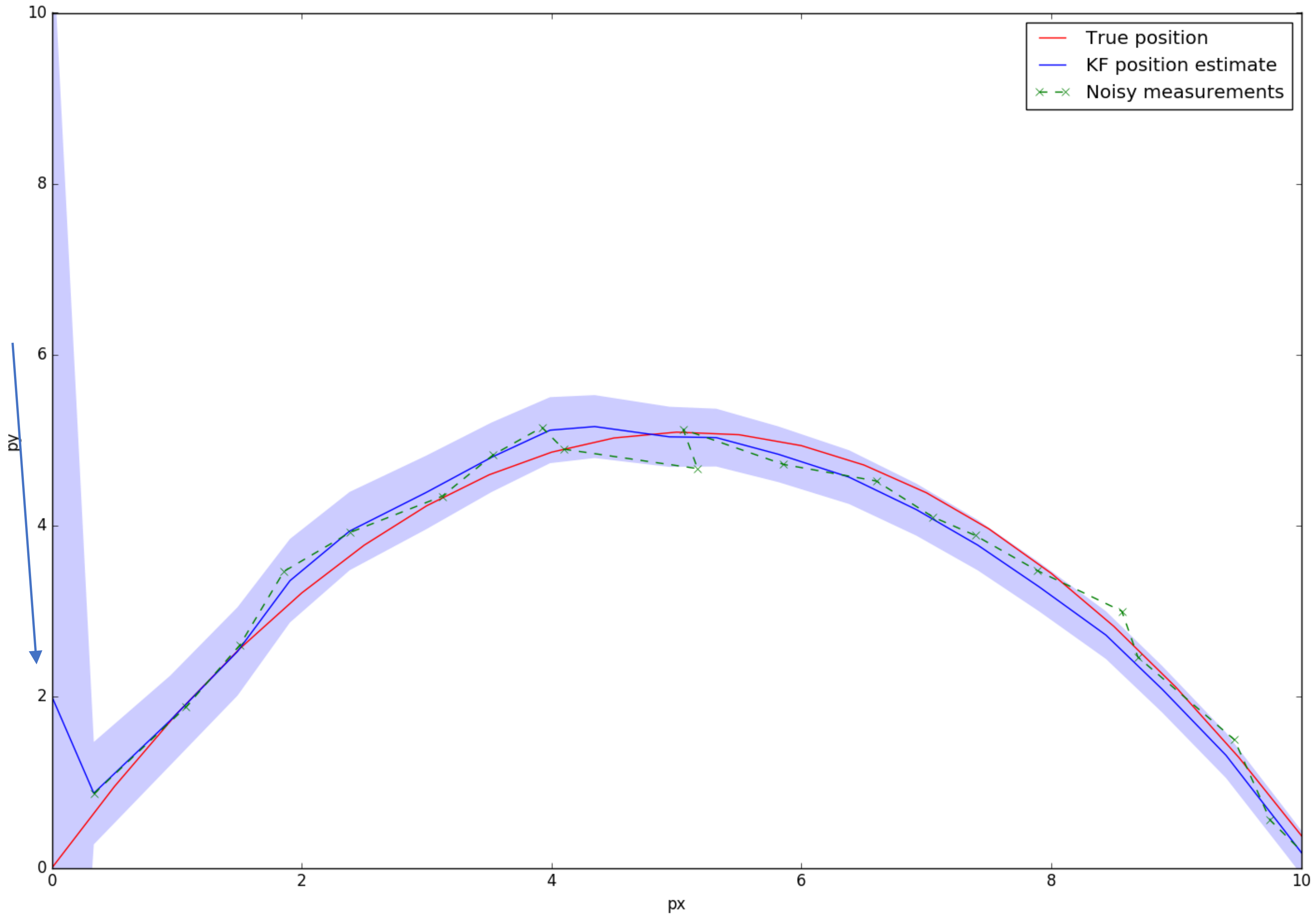
Kalman Filter with 4D state

This is the $3\sigma_{p_y}$ uncertainty around the mean estimate. With probability ~0.997 the true p_y should be within these bounds, as long as the system has been initialized close enough to the true initial state, and as long as the KF assumptions hold.



Kalman Filter with 4D state

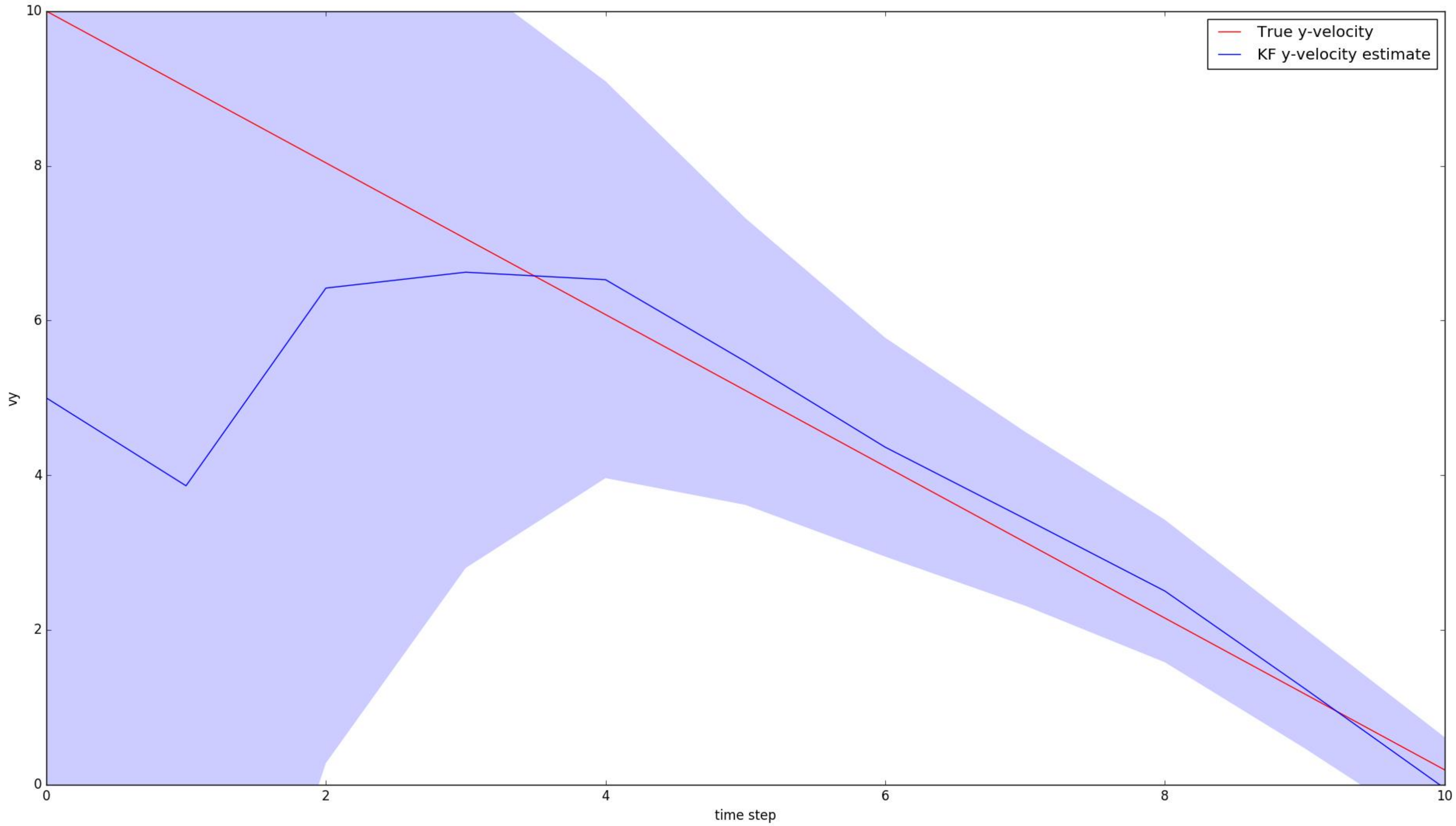
We initialized the KF estimated mean position to be [0, 2] when the true value was [0, 0] and we assigned high uncertainty in the initial position estimate, which was reduced after the first few measurements.



Kalman Filter with 4D state

We initialized the KF estimated mean y-velocity to be 5 when the true value was 10 and we assigned high uncertainty in the initial y-velocity estimate.

Even though we do not measure the velocity directly, through correlation with position, the KF is able to INFER it and the initially large uncertainty shrinks as more measurements are received.



Kalman Filter with 4D state

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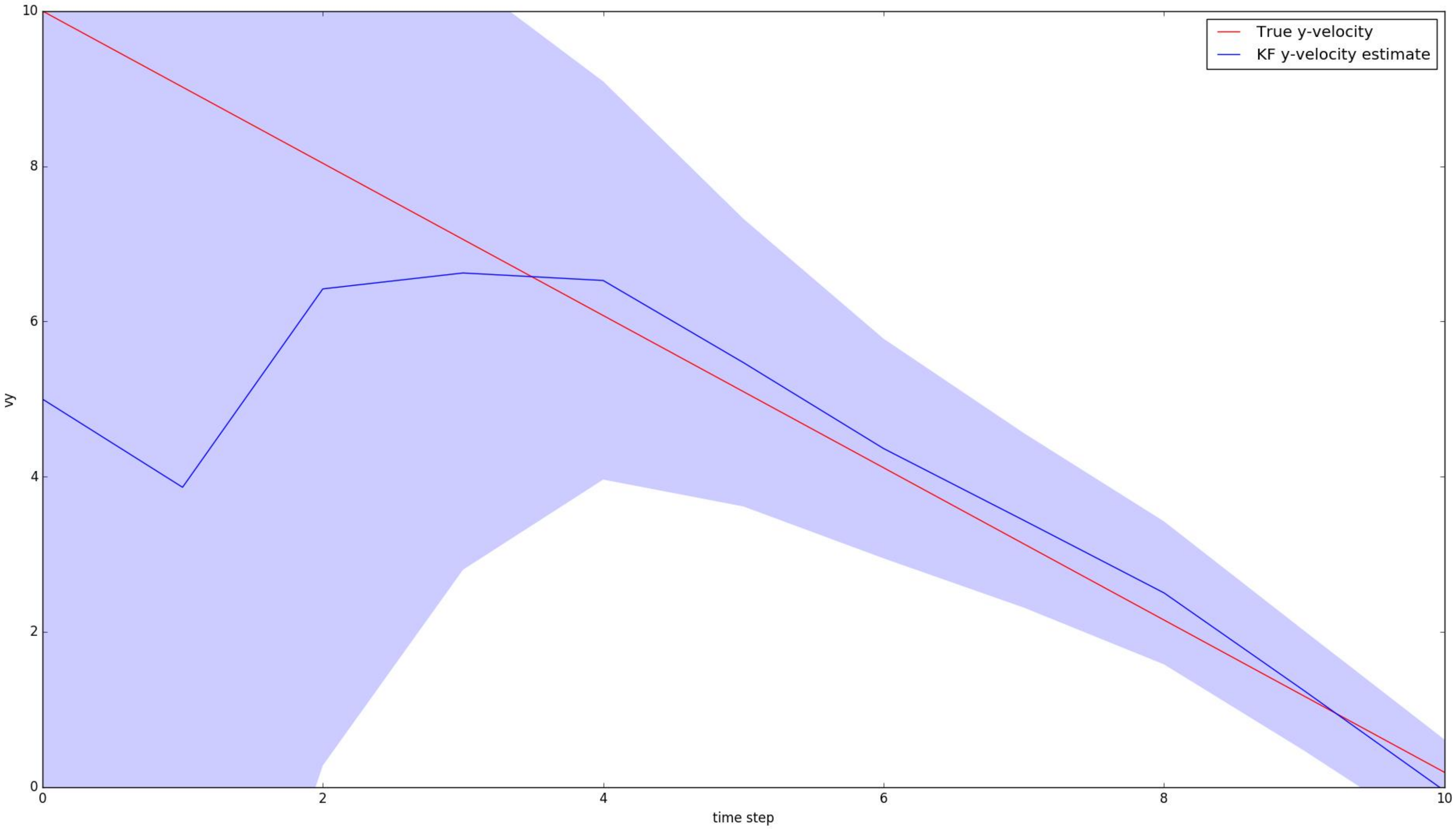

Parameters and code to reproduce this can be found at https://github.com/florianshkurti/comp417/tree/master/filtering_examples

Appendix 1

Claim: $\mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) \propto \mathcal{N}(\mu, \sigma^2)$ where $\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B)$ $\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$

Proof:

$$\mathcal{N}(\mu_A, \sigma_A^2)\mathcal{N}(\mu_B, \sigma_B^2) = \frac{1}{\sqrt{2\pi}\sigma_A\sigma_B}\exp(-0.5(x-\mu_A)^2/\sigma_A^2 - 0.5(x-\mu_B)^2/\sigma_B^2)$$

Define $\beta = \frac{(x-\mu_A)^2}{2\sigma_A^2} + \frac{(x-\mu_B)^2}{2\sigma_B^2}$

$$\beta = \frac{(\sigma_A^2 + \sigma_B^2)x^2 - 2(\mu_A\sigma_B^2 + \mu_B\sigma_A^2)x + \mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2}{2\sigma_A^2\sigma_B^2}$$

$$\beta = \frac{x^2 - 2\frac{\mu_A\sigma_B^2 + \mu_B\sigma_A^2}{\sigma_A^2 + \sigma_B^2}x + \frac{\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2}{\sigma_A^2 + \sigma_B^2}}{2\frac{\sigma_A^2\sigma_B^2}{\sigma_A^2 + \sigma_B^2}}$$

$$\beta = \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2} = \frac{(x-\mu)^2}{2\sigma^2}$$

where

$$\mu = \mu_A\frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2} + \mu_B\frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2} = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B)$$

$$\sigma^2 = \frac{\sigma_A^2\sigma_B^2}{\sigma_A^2 + \sigma_B^2} = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$$