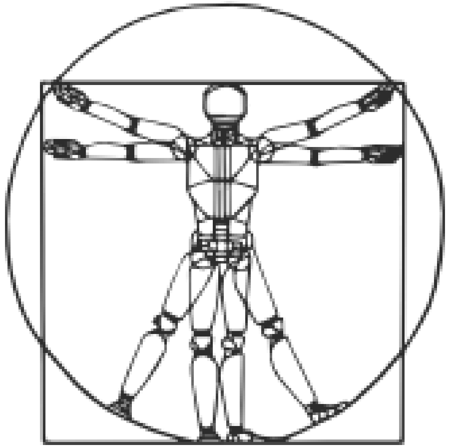

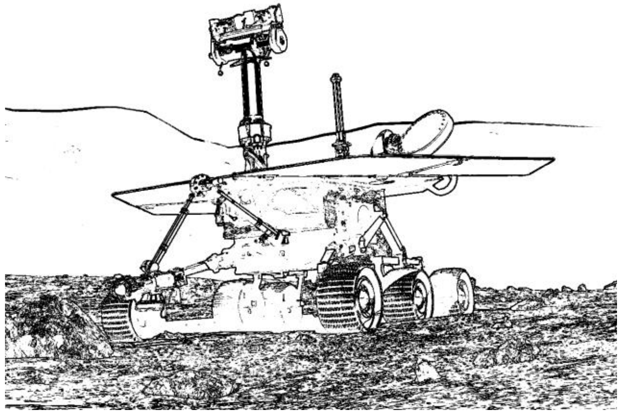

COMP417
Introduction to Robotics and Intelligent Systems

Lecture 7: Linear Quadratic Regulator

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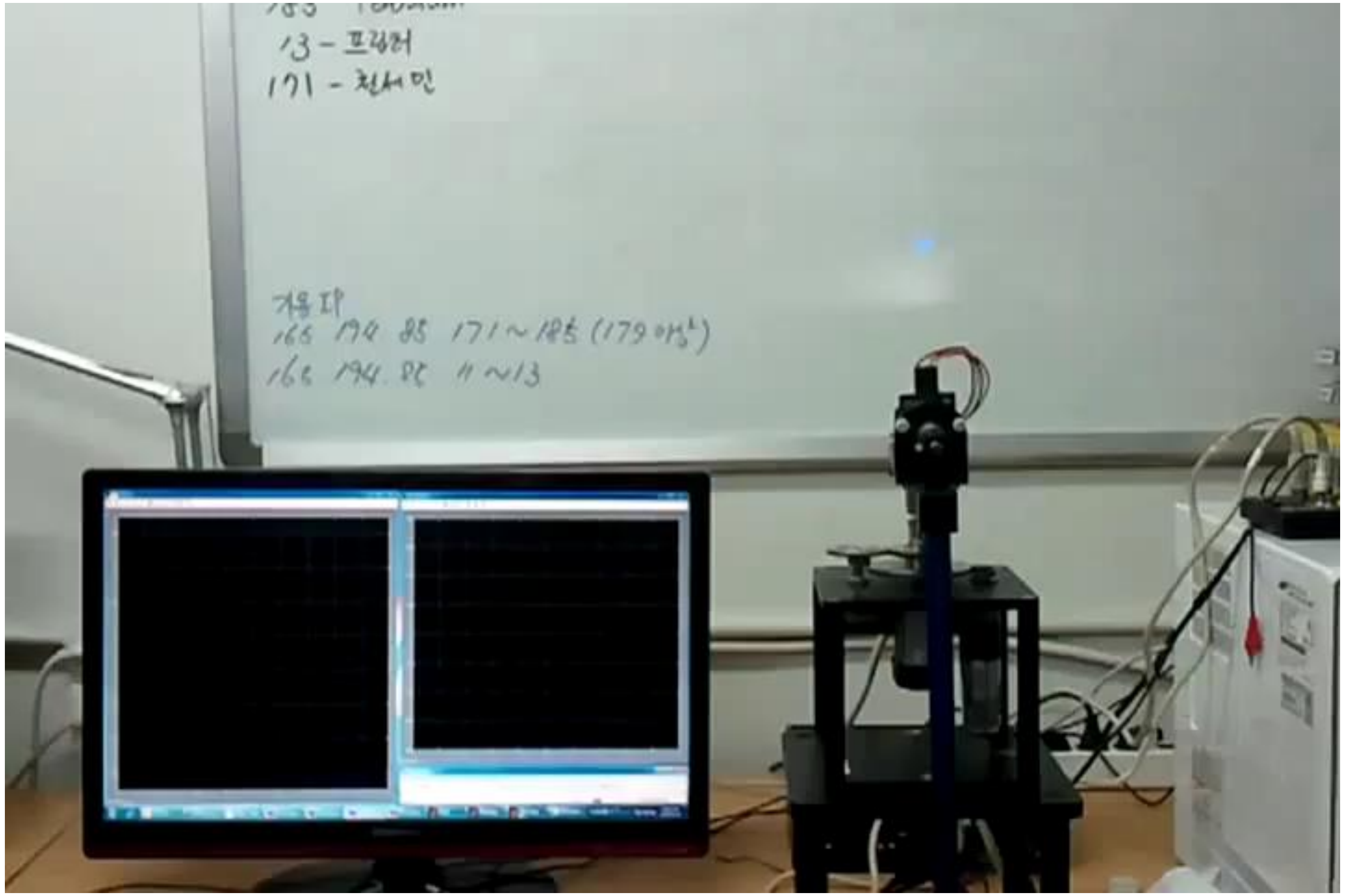
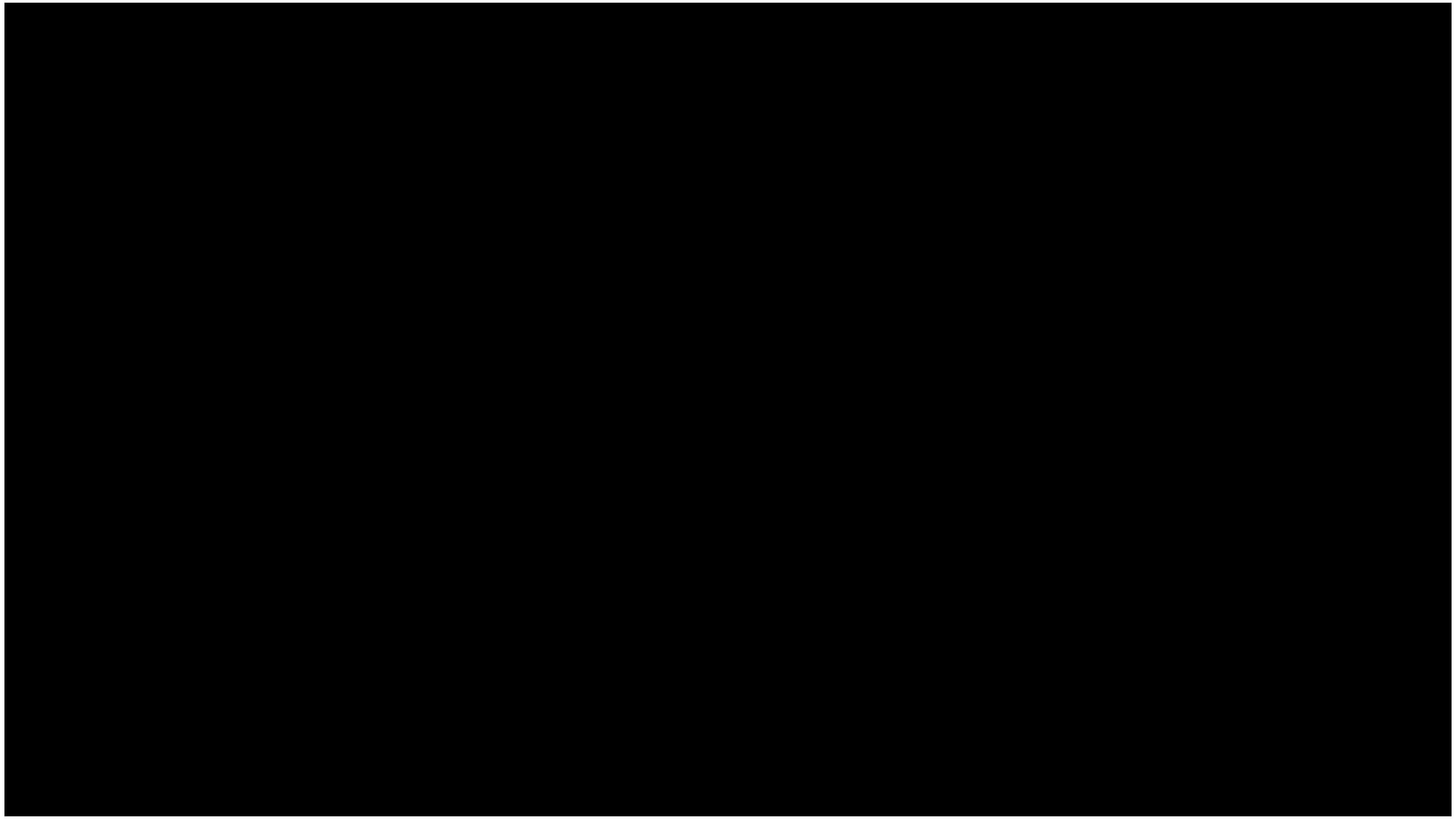
What you can do with LQR controllers



What you can do with LQR controllers



Pieter Abbeel, Helicopter Aerobatics

Why not use PID?

Why not use PID?

- We could, but:

- The gains for PID are good for a small region of state-space.
  - System reaches a state outside this set → becomes unstable
  - PID has no formal guarantees on the size of the set

- We would need to tune PID gains for every control variable.
  - If the state vector has multiple dimensions it becomes harder to tune every control variable in isolation. Need to consider interactions and correlations.

- We would need to tune PID gains for different regions of the state-space and guarantee smooth gain transitions
  - This is called gain scheduling, and it takes a lot of effort and time

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LQR addresses these problems

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LQR: assumptions

- You know the dynamics model of the system
- It is linear: $$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t$$

State at the next time step

Control at the next time step

Q: Why is it useful to have a dynamics model of the system?

LQR: assumptions

- You know the dynamics model of the system
- It is linear: $$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t$$

State at the next time step

Control at the next time step

Q: Why is it useful to have a dynamics model of the system?
A: Because you can predict the state at a future time step,
given a sequence of control inputs

LQR: assumptions

- You know the dynamics model of the system
- It is linear: $\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t$

- There is an instantaneous cost associated with being at state $\mathbf{x}_t$ and taking the action $\mathbf{u}_t$: $g(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{x}_t^T Q \mathbf{x}_t + \mathbf{u}_t^T R \mathbf{u}_t$

Quadratic state cost: Penalizes deviation from the zero vector

Quadratic control cost: Penalizes high control signals

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Square matrices Q and R must be positive definite:

$$Q = Q^T \quad \text{and} \quad \forall x, x^T Q x > 0$$
$$R = R^T \quad \text{and} \quad \forall u, u^T R u > 0$$

i.e. positive cost for ANY nonzero state or control vector

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Note: Recall that a positive definite matrix has positive eigenvalues

LQR: assumptions

- You know the dynamics model of the system
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- You either know the state $\mathbf{x}_t$ directly or you can estimate it

LQR: goal

- Stabilize the system around state $\mathbf{x}_t = \mathbf{0}$ with control $\mathbf{u}_t = \mathbf{0}$
- Then $\mathbf{x}_{t+1} = \mathbf{0}$ and the system will remain at zero.

Which systems are linear?

- Omnidirectional robot

$$
\begin{aligned}
x_{t+1} &= x_t + v_x(t)\delta t \\
y_{t+1} &= y_t + v_y(t)\delta t \\
\theta_{t+1} &= \theta_t + \omega_z(t)\delta t
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\mathbf{x}_{t+1} &= I\mathbf{x}_t + \delta t I \mathbf{u}_t \\
\mathbf{x}_{t+1} &= A\mathbf{x}_t + B\mathbf{u}_t
\end{aligned}
$$

- Simple car?

$$
\begin{aligned}
x_{t+1} &= x_t + v_x(t)\cos(\theta_t)\delta t \\
y_{t+1} &= y_t + v_x(t)\sin(\theta_t)\delta t \\
\theta_{t+1} &= \theta_t + \omega_z\delta t
\end{aligned}
\quad \Rightarrow \quad
\mathbf{x}_{t+1} = I\mathbf{x}_t + \begin{bmatrix} \delta t\cos(\theta_t) & 0 & 0 \\ 0 & \delta t\sin(\theta_t) & 0 \\ 0 & 0 & \delta t \end{bmatrix} \mathbf{u}_t
$$

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B(\mathbf{x}_t)\mathbf{u}_t$$

Which systems are linear?

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$$\begin{aligned} x_{t+1} &= x_t + v_x(t)\delta t \\ y_{t+1} &= y_t + v_y(t)\delta t \\ \theta_{t+1} &= \theta_t + \omega_z(t)\delta t \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{x}_{t+1} &= I\mathbf{x}_t + \delta t I\mathbf{u}_t \\ \mathbf{x}_{t+1} &= A\mathbf{x}_t + B\mathbf{u}_t \end{aligned}$$

- Simple car? NO

$$\begin{aligned} x_{t+1} &= x_t + v_x(t)\text{cos}(\theta_t)\delta t \\ y_{t+1} &= y_t + v_x(t)\text{sin}(\theta_t)\delta t \\ \theta_{t+1} &= \theta_t + \omega_z\delta t \end{aligned} \quad \Rightarrow \quad \mathbf{x}_{t+1} = I\mathbf{x}_t + \begin{bmatrix} \delta t\text{cos}(\theta_t) & 0 & 0 \\ 0 & \delta t\text{sin}(\theta_t) & 0 \\ 0 & 0 & \delta t \end{bmatrix} \mathbf{u}_t$$

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B(\mathbf{x}_t)\mathbf{u}_t$$

Which systems are linear?

- Linearity means if you scale or add to the input the output will also reflect the scaling and addition

- Formally, the dynamics $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$ are linear if

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t$$

Finite-Horizon LQR

- Idea: finding controls is an optimization problem
- Compute the control variables that minimize the cumulative cost

$$
\begin{aligned}
\underset{u_0,\ldots,u_N}{\operatorname{argmin}} & \quad \sum_{t=0}^{t=N} g(\mathbf{x}_t, \mathbf{u}_t) \\
\text{s.t.} & \\
\mathbf{x}_1 & = A\mathbf{x}_0 + B\mathbf{u}_0 \\
\mathbf{x}_2 & = A\mathbf{x}_1 + B\mathbf{u}_1 \\
& \ldots \\
\mathbf{x}_N & = A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1}
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$$

We could solve this as a constrained nonlinear optimization problem. But, there is a better way: we can find a closed-form solution.

Finding the LQR controller in closed-form by recursion

- Let $J_n(\mathbf{x})$ denote the cumulative cost-to-go starting from state x and moving for n time steps.

- I.e. cumulative future cost from now till n more steps

- $J_0(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x}$ is the terminal cost of ending up at state x, with no actions left to do. Let's denote it $J_0(\mathbf{x}) = \mathbf{x}^T P_0 \mathbf{x}$

Q: What is the optimal cumulative cost-to-go function with 1 time step left?

Finding the LQR controller in closed-form by recursion

$$J_0(\mathbf{x}) = \mathbf{x}^T P_0 \mathbf{x}$$

$$J_1(\mathbf{x}) = \min_{\mathbf{u}} [\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + J_0(A\mathbf{x} + B\mathbf{u})]$$

Bellman update (a.k.a. Dynamic Programming)

Finding the LQR controller in closed-form by recursion

$$J_0(\mathbf{x}) = \mathbf{x}^T P_0 \mathbf{x}$$

$$\begin{aligned} J_1(\mathbf{x}) &= \min_{\mathbf{u}} [\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + J_0(A\mathbf{x} + B\mathbf{u})] \\ &= \min_{\mathbf{u}} [\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + (A\mathbf{x} + B\mathbf{u})^T P_0 (A\mathbf{x} + B\mathbf{u})] \end{aligned}$$

Q: How do we optimize a multivariable function with respect to some variables (in our case, the controls)?

Finding the LQR controller in closed-form by recursion

$$J_0(\mathbf{x}) = \mathbf{x}^T P_0 \mathbf{x}$$

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J_1(\mathbf{x}) &= \min_{\mathbf{u}} [\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + J_0(A\mathbf{x} + B\mathbf{u})] \\
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&= \mathbf{x}^T Q \mathbf{x} + \mathbf{x}^T A^T P_0 A \mathbf{x} + \min_{\mathbf{u}} [\mathbf{u}^T R \mathbf{u} + 2\mathbf{u}^T B^T P_0 A \mathbf{x} + \mathbf{u}^T B^T P_0 B \mathbf{u}]
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&= \mathbf{x}^T Q \mathbf{x} + \mathbf{x}^T A^T P_0 A \mathbf{x} + \min_{\mathbf{u}} [\underbrace{\mathbf{u}^T R \mathbf{u}}_{\text{Quadratic term in u}} + \underbrace{2\mathbf{u}^T B^T P_0 A \mathbf{x}}_{\text{Linear term in u}} + \underbrace{\mathbf{u}^T B^T P_0 B \mathbf{u}}_{\text{Quadratic term in u}}]
\end{aligned}$$

A: Take the partial derivative w.r.t. controls and set it to zero. That will give you a critical point.

Finding the LQR controller in closed-form by recursion

$$J_1(\mathbf{x}) \quad = \quad \mathbf{x}^T Q\mathbf{x} + \mathbf{x}^T A^T P_0 A\mathbf{x} + \min_{\mathbf{u}}[\mathbf{u}^T R\mathbf{u} + 2\mathbf{u}^T B^T P_0 A\mathbf{x} + \mathbf{u}^T B^T P_0 B\mathbf{u}]$$

From calculus/algebra:

$$\frac{\partial}{\partial \mathbf{u}}(\mathbf{u}^T M\mathbf{u}) = (M + M^T)\mathbf{u}$$

$$\frac{\partial}{\partial \mathbf{u}}(\mathbf{u}^T M\mathbf{b}) = M\mathbf{b}$$

If M is symmetric:

$$\frac{\partial}{\partial \mathbf{u}}(\mathbf{u}^T M\mathbf{u}) = 2M\mathbf{u}$$

Finding the LQR controller in closed-form by recursion

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The minimum is attained at:

$$2R\mathbf{u} + 2B^T P_0 A\mathbf{x} + 2B^T P_0 B\mathbf{u} = \mathbf{0}$$

$$(R + B^T P_0 B)\mathbf{u} = -B^T P_0 A\mathbf{x}$$

Q: Is this matrix invertible? Recall R, Po are positive definite matrices.

From calculus/algebra:

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Q: Is this matrix invertible? Recall R, Po are positive definite matrices.

$R + B^T P_0 B$ is positive definite, so it is invertible

Finding the LQR controller in closed-form by recursion

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So, the optimal control for the last time step is:

$$\mathbf{u} = -(R + B^T P_0 B)^{-1} B^T P_0 A\mathbf{x}$$

$$\mathbf{u} = K_1 \mathbf{x}$$

Linear controller in terms of the state

Finding the LQR controller in closed-form by recursion

$$J_1(\mathbf{x}) \quad = \quad \mathbf{x}^T Q\mathbf{x} + \mathbf{x}^T A^T P_0 A\mathbf{x} + \min_{\mathbf{u}}[\mathbf{u}^T R\mathbf{u} + 2\mathbf{u}^T B^T P_0 A\mathbf{x} + \mathbf{u}^T B^T P_0 B\mathbf{u}]$$

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Linear controller in terms of the state

We computed the location of the minimum.
Now, plug it back in and compute the
minimum value

Finding the LQR controller in closed-form by recursion

$$J_0(\mathbf{x}) = \mathbf{x}^T P_0 \mathbf{x}$$

$$\begin{aligned}
J_1(\mathbf{x}) &= \mathbf{x}^T Q \mathbf{x} + \mathbf{x}^T A^T P_0 A \mathbf{x} + \min_{\mathbf{u}} [\mathbf{u}^T R \mathbf{u} + 2\mathbf{u}^T B^T P_0 A \mathbf{x} + \mathbf{u}^T B^T P_0 B \mathbf{u}] \\
&= \mathbf{x}^T Q \mathbf{x} + \mathbf{x}^T A^T P_0 A \mathbf{x} + [\mathbf{x}^T K_1^T R K_1 \mathbf{x} + 2\mathbf{x}^T K_1^T B^T P_0 A \mathbf{x} + \mathbf{x}^T K_1^T B^T P_0 B K_1 \mathbf{x}] \\
&= \mathbf{x}^T (Q + K_1^T R K_1 + (A + BK_1)^T P_0 (A + BK_1)) \mathbf{x}
\end{aligned}$$

Q: Why is this a big deal?
A: The cost-to-go function remains quadratic after the first recursive step.

Finding the LQR controller in closed-form by recursion

$$J_0(\mathbf{x}) = \mathbf{x}^T P_0 \mathbf{x}$$

$$\mathbf{u} = -(R + B^T P_0 B)^{-1} B^T P_0 A \mathbf{x}$$
$$\mathbf{u} = K_1 \mathbf{x}$$

$$\begin{aligned} J_1(\mathbf{x}) &= \mathbf{x}^T (Q + K_1^T R K_1 + (A + BK_1)^T P_0 (A + BK_1)) \mathbf{x} \\ &= \mathbf{x}^T P_1 \mathbf{x} \end{aligned}$$

...

In fact the recursive steps generalize

$$\mathbf{u} = -(R + B^T P_{n-1} B)^{-1} B^T P_{n-1} A \mathbf{x}$$
$$\mathbf{u} = K_n \mathbf{x}$$

$$\begin{aligned} J_n(\mathbf{x}) &= \mathbf{x}^T (Q + K_n^T R K_n + (A + BK_n)^T P_{n-1} (A + BK_n)) \mathbf{x} \\ &= \mathbf{x}^T P_n \mathbf{x} \end{aligned}$$

Finite-Horizon LQR: algorithm summary

$$P_0 = Q$$

// n is the # of steps left

for n = 1…N

$$K_n = -(R + B^T P_{n-1} B)^{-1} B^T P_{n-1} A$$

$$P_n = Q + K_n^T R K_n + (A + BK_n)^T P_{n-1} (A + BK_n)$$

Optimal controller for i-step horizon is $\mathbf{u}(\mathbf{x}) = K_i \mathbf{x}$ with cost-to-go $J_i(\mathbf{x}) = \mathbf{x}^T P_i \mathbf{x}$

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Matrix gains are precomputed based on the dynamics model and the instantaneous cost

Then you execute the policy, issuing commands based on state feedback

Optimal controller for i-step horizon is $\mathbf{u}(\mathbf{x}) = K_i \mathbf{x}$ with cost-to-go $J_i(\mathbf{x}) = \mathbf{x}^T P_i \mathbf{x}$

A state-dependent controller is called a policy

Finite-Horizon LQR: algorithm summary

Potential problem for states of dimension >> 100:
Matrix inversion is expensive: O(n^2.3) for the best known algorithm and O(n^3) for Gaussian Elimination.

$$P_0 = Q$$

// n is the # of steps left

for n = 1…N

$$K_n = -(R + B^T P_{n-1} B)^{-1} B^T P_{n-1} A$$

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Optimal controller for i-step horizon is $\mathbf{u}(\mathbf{x}) = K_i \mathbf{x}$ with cost-to-go $J_i(\mathbf{x}) = \mathbf{x}^T P_i \mathbf{x}$

LQR extensions: nonlinear dynamics

What can we do when $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$?

We want to stabilize the system around state $\mathbf{x}_t = \mathbf{0}$
But with nonlinear dynamics we do not know if $\mathbf{u}_t = \mathbf{0}$ will keep the system at the zero state.

LQR extensions: nonlinear dynamics

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→ Need to compute $\mathbf{u}^*$ such that $\mathbf{0}_{t+1} = f(\mathbf{0}_t, \mathbf{u}^*)$

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→ Need to compute $\mathbf{u}^*$ such that $\mathbf{0}_{t+1} = f(\mathbf{0}_t, \mathbf{u}^*)$

Taylor expansion: linearize the nonlinear dynamics around the point $(\mathbf{0}, \mathbf{u}^*)$

$$\mathbf{x}_{t+1} \simeq f(\mathbf{0}, \mathbf{u}^*) + \frac{\partial f}{\partial \mathbf{x}}(\mathbf{0}, \mathbf{u}^*)(\mathbf{x}_t - \mathbf{0}) + \frac{\partial f}{\partial \mathbf{u}}(\mathbf{0}, \mathbf{u}^*)(\mathbf{u}_t - \mathbf{u}^*)$$

$$\mathbf{x}_{t+1} \simeq A\mathbf{x}_t + B(\mathbf{u}_t - \mathbf{u}^*)$$

LQR extensions: nonlinear dynamics

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$$\mathbf{x}_{t+1} \simeq f(\mathbf{0}, \mathbf{u}^*) + \frac{\partial f}{\partial \mathbf{x}}(\mathbf{0}, \mathbf{u}^*)(\mathbf{x}_t - \mathbf{0}) + \frac{\partial f}{\partial \mathbf{u}}(\mathbf{0}, \mathbf{u}^*)(\mathbf{u}_t - \mathbf{u}^*)$$

$$\mathbf{x}_{t+1} \simeq A\mathbf{x}_t + B(\mathbf{u}_t - \mathbf{u}^*)$$

Create new variable $\bar{\mathbf{u}}_t = \mathbf{u}_t - \mathbf{u}^*$ and apply LQR iterations to the linear system $\mathbf{x}_{t+1} = A\mathbf{x}_t + B\bar{\mathbf{u}}_t$

LQR extensions: nonlinear dynamics

What can we do when $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$?

We want to stabilize the system around state $\mathbf{x}^* = \mathbf{0}$
But with nonlinear dynamics we do not know if $\mathbf{u}^* = \mathbf{0}$ will keep the system at the zero state.
→ Need to compute $\mathbf{u}^*$ such that $\mathbf{0}_{t+1} = f(\mathbf{0}_t, \mathbf{u}^*)$

Taylor expansion: linearize the nonlinear dynamics around the point $(\mathbf{0}, \mathbf{u}^*)$

$$\mathbf{x}_{t+1} \simeq f(\mathbf{0}, \mathbf{u}^*) + \frac{\partial f}{\partial \mathbf{x}}(\mathbf{0}, \mathbf{u}^*)(\mathbf{x}_t - \mathbf{0}) + \frac{\partial f}{\partial \mathbf{u}}(\mathbf{0}, \mathbf{u}^*)(\mathbf{u}_t - \mathbf{u}^*)$$

$$\mathbf{x}_{t+1} \simeq A\mathbf{x}_t + B(\mathbf{u}_t - \mathbf{u}^*)$$

Create new variable $\bar{\mathbf{u}}_t = \mathbf{u}_t - \mathbf{u}^*$ and apply LQR iterations to the

Unfortunately with the linearization you pay a price: You have to provide an initial state close to the zero state, otherwise you might not get good results

LQR extensions: non-quadratic cost

What can we do when $g(\mathbf{x}_t, \mathbf{u}_t)$ is not quadratic?

Same trick as before: use Taylor expansion to get the quadratic terms, but matrices involved are not guaranteed to be positive-definite, so this doesn't always work.

LQR extensions: time-varying systems

- What can we do when $\mathbf{x}_{t+1} = A_t \mathbf{x}_t + B_t \mathbf{u}_t$ and $g(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{x}_t^T Q \mathbf{x}_t + \mathbf{u}_t^T R \mathbf{u}_t$?
- Turns out, the proof and the algorithm are almost the same

$$P_0 \quad = \quad Q_N$$

// n is the # of steps left

for n = 1…N

$$K_n = -(R_{N-n} + B_{N-n}^T P_{n-1} B_{N-n})^{-1} B_{N-n}^T P_{n-1} A_{N-n}$$

$$P_n \quad = \quad Q_{N-n} + K_n^T R_{N-n} K_n + (A_{N-n} + B_{N-n} K_n)^T P_{n-1} (A_{N-n} + B_{N-n} K_n)$$

Optimal controller for i-step horizon is $\mathbf{u}(\mathbf{x}) = K_i \mathbf{x}$ with cost-to-go $J_i(\mathbf{x}) = \mathbf{x}^T P_i \mathbf{x}$

LQR extensions: trajectory following

- You are given a reference trajectory (not just path, but states and times, or states and controls) that needs to be approximated

$$\mathbf{x}_0^*, \mathbf{x}_1^*, \ldots, \mathbf{x}_N^* \qquad \mathbf{u}_0^*, \mathbf{u}_1^*, \ldots, \mathbf{u}_N^*$$

Linearize the nonlinear dynamics $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$ around the reference point $(\mathbf{x}_t^*, \mathbf{u}_t^*)$

$$\mathbf{x}_{t+1} \simeq f(\mathbf{x}_t^*, \mathbf{u}_t^*) + \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_t^*, \mathbf{u}_t^*)(\mathbf{x}_t - \mathbf{x}_t^*) + \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}_t^*, \mathbf{u}_t^*)(\mathbf{u}_t - \mathbf{u}_t^*)$$

$$\begin{aligned} \bar{\mathbf{x}}_{t+1} &\simeq A_t \bar{\mathbf{x}}_t + B_t \bar{\mathbf{u}}_t \\ g(\mathbf{x}_t, \mathbf{u}_t) &= \bar{\mathbf{x}}_t^T Q \bar{\mathbf{x}}_t + \bar{\mathbf{u}}_t^T R \bar{\mathbf{u}}_t \end{aligned} \quad \text{where} \quad \begin{aligned} \bar{\mathbf{x}}_t &= \mathbf{x}_t - \mathbf{x}_t^* \\ \bar{\mathbf{u}}_t &= \mathbf{u}_t - \mathbf{u}_t^* \end{aligned}$$

Trajectory following can be implemented as a time-varying LQR approximation. Not always clear if this is the best way though.

LQR example #1: omnidirectional vehicle with friction

- Similar to double integrator dynamical system, but with friction:

$$\underbrace{m\ddot{\mathbf{p}}}_{\substack{\text{Force applied}\\ \text{to the vehicle}}} = \underbrace{\mathbf{u}}_{\substack{\text{Control applied}\\ \text{to the vehicle}}} - \underbrace{\alpha\dot{\mathbf{p}}}_{\substack{\text{Friction opposed to}\\ \text{motion}}}$$

LQR example #1: omnidirectional vehicle with friction

- Similar to double integrator dynamical system, but with friction:

$$m\ddot{\mathbf{p}} = \mathbf{u} - \alpha\dot{\mathbf{p}}$$

- Set $\dot{\mathbf{p}} = \mathbf{v}$ and then you get:

$$m\dot{\mathbf{v}} = \mathbf{u} - \alpha\mathbf{v}$$

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- Set $\dot{\mathbf{p}} = \mathbf{v}$ and then you get:

$$m\dot{\mathbf{v}} = \mathbf{u} - \alpha\mathbf{v}$$

- We discretize by setting

$$\frac{\mathbf{p}_{t+1} - \mathbf{p}_t}{\delta t} \simeq \mathbf{v}_t \qquad\qquad m\frac{\mathbf{v}_{t+1} - \mathbf{v}_t}{\delta t} \simeq \mathbf{u}_t - \alpha\mathbf{v}_t$$

LQR example #1: omnidirectional vehicle with friction

$$\frac{\mathbf{p}_{t+1} - \mathbf{p}_t}{\delta t} \simeq \mathbf{v}_t \qquad\qquad m\frac{\mathbf{v}_{t+1} - \mathbf{v}_t}{\delta t} \simeq \mathbf{u}_t - \alpha \mathbf{v}_t$$

- Define the state vector $\mathbf{x}_t = \begin{bmatrix} \mathbf{p}_t \\ \mathbf{v}_t \end{bmatrix}$

Q: How can we express this as a linear system?

LQR example #1: omnidirectional vehicle with friction

$$\frac{\mathbf{p}_{t+1} - \mathbf{p}_t}{\delta t} \simeq \mathbf{v}_t \qquad\qquad m\frac{\mathbf{v}_{t+1} - \mathbf{v}_t}{\delta t} \simeq \mathbf{u}_t - \alpha \mathbf{v}_t$$

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$$\mathbf{x}_{t+1} = \begin{bmatrix} \mathbf{p}_{t+1} \\ \mathbf{v}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_t + \delta t \mathbf{v}_t \\ \mathbf{v}_t + \frac{\delta t}{m}\mathbf{u}_t - \frac{\alpha \delta t}{m}\mathbf{v}_t \end{bmatrix} = \begin{bmatrix} \mathbf{p}_t + \delta t \mathbf{v}_t \\ \mathbf{v}_t - \frac{\alpha \delta t}{m}\mathbf{v}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\delta t}{m} & 0 \\ 0 & \frac{\delta t}{m} \end{bmatrix} \mathbf{u}_t$$

LQR example #1: omnidirectional vehicle with friction

$$\frac{\mathbf{p}_{t+1} - \mathbf{p}_t}{\delta t} \simeq \mathbf{v}_t \qquad\qquad m\frac{\mathbf{v}_{t+1} - \mathbf{v}_t}{\delta t} \simeq \mathbf{u}_t - \alpha \mathbf{v}_t$$

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LQR example #1: omnidirectional vehicle with friction

$$\frac{\mathbf{p}_{t+1} - \mathbf{p}_t}{\delta t} \simeq \mathbf{v}_t \qquad m\frac{\mathbf{v}_{t+1} - \mathbf{v}_t}{\delta t} \simeq \mathbf{u}_t - \alpha \mathbf{v}_t$$

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LQR example #1: omnidirectional vehicle with friction

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- Define the instantaneous cost function

$$\begin{aligned} g(\mathbf{x}, \mathbf{u}) &= \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \\ &= \mathbf{x}^T \mathbf{x} + \rho \mathbf{u}^T \mathbf{u} \\ &= ||\mathbf{x}||^2 + \rho ||\mathbf{u}||^2 \end{aligned}$$

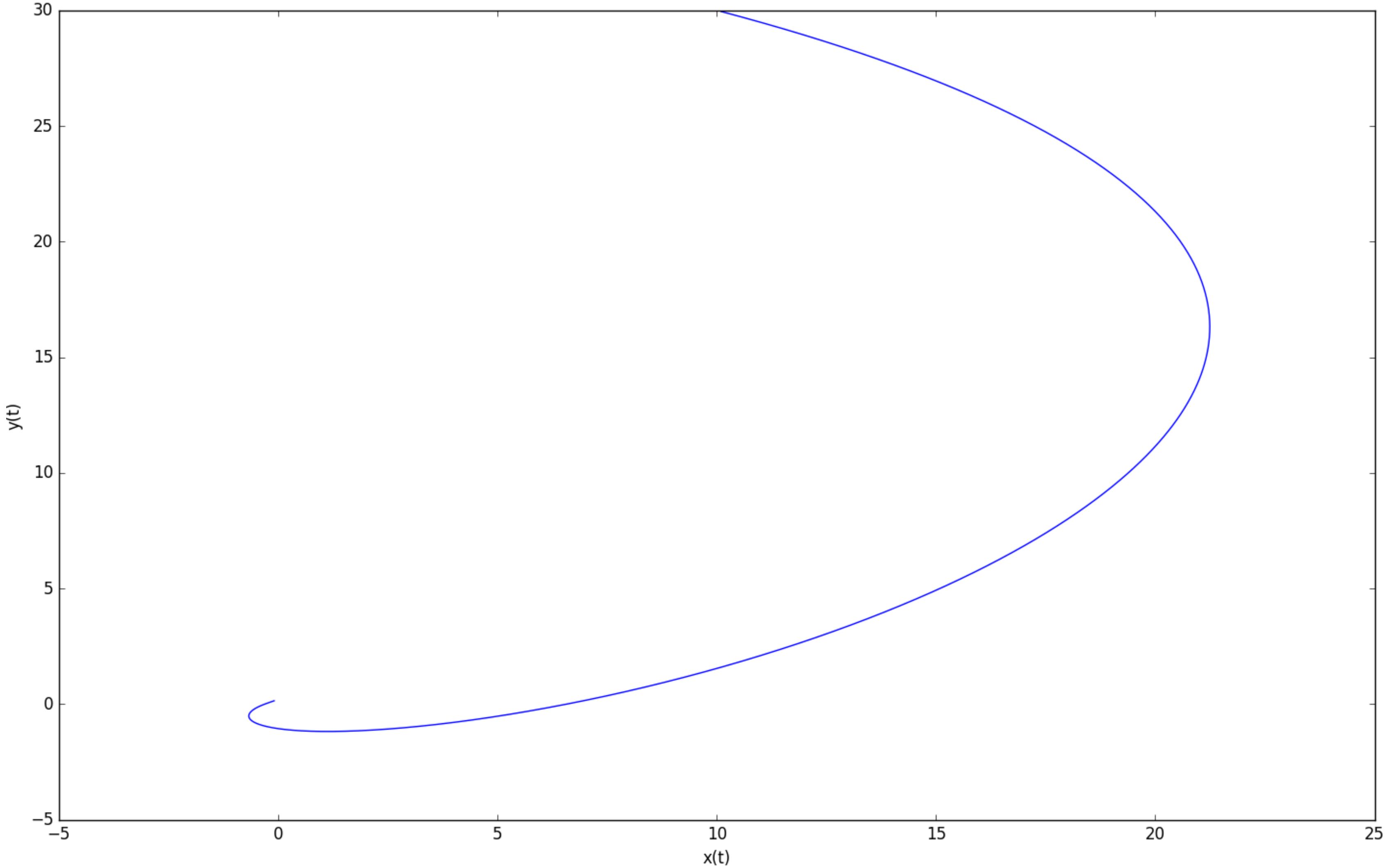
LQR example #1: omnidirectional vehicle with friction

With initial state

$$\mathbf{x}_0 = \begin{bmatrix} 10 \\ 30 \\ 10 \\ -5 \end{bmatrix}$$

Instantaneous cost function

$$g(\mathbf{x}, \mathbf{u}) = ||\mathbf{x}||^2 + 100||\mathbf{u}||^2$$



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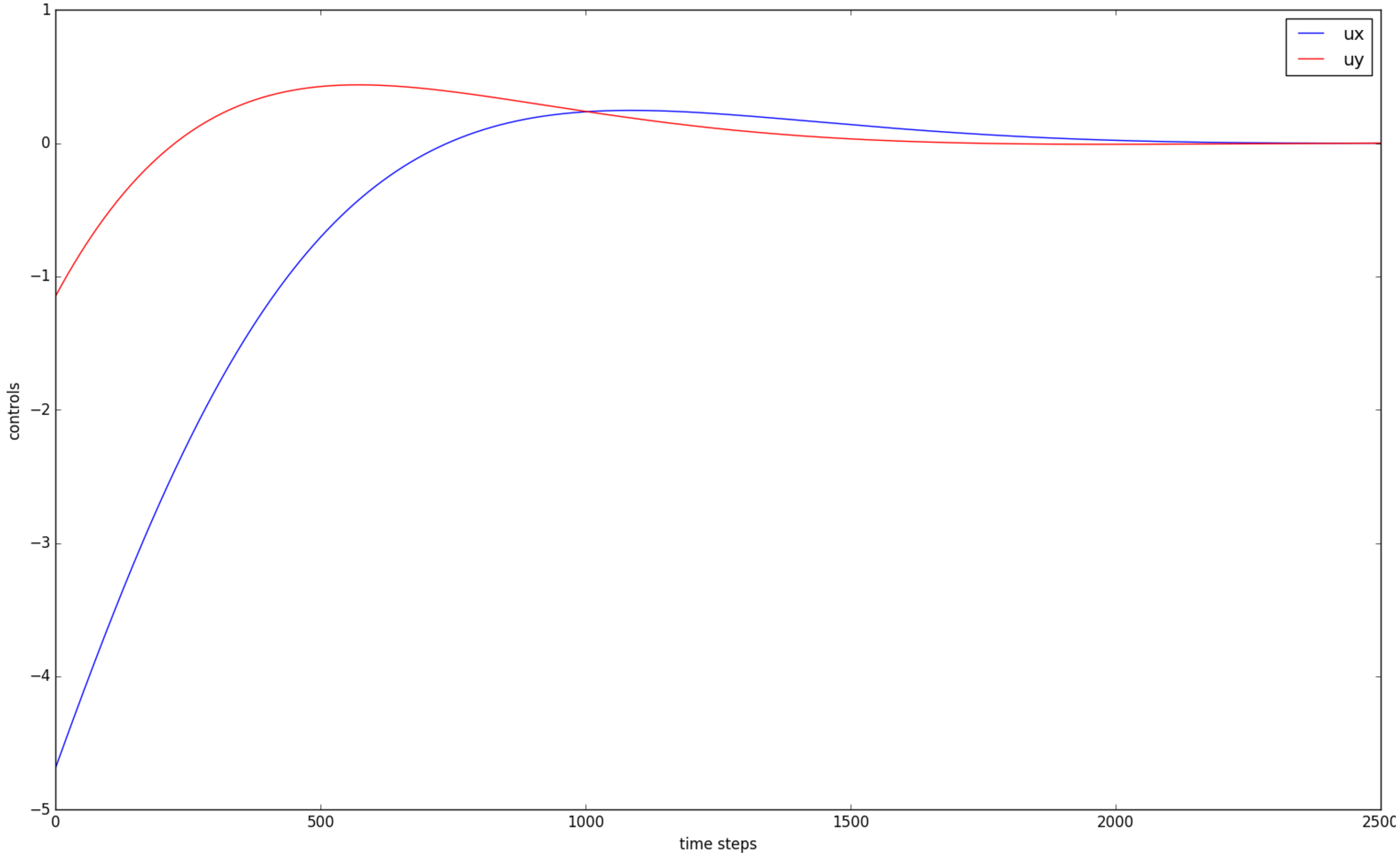

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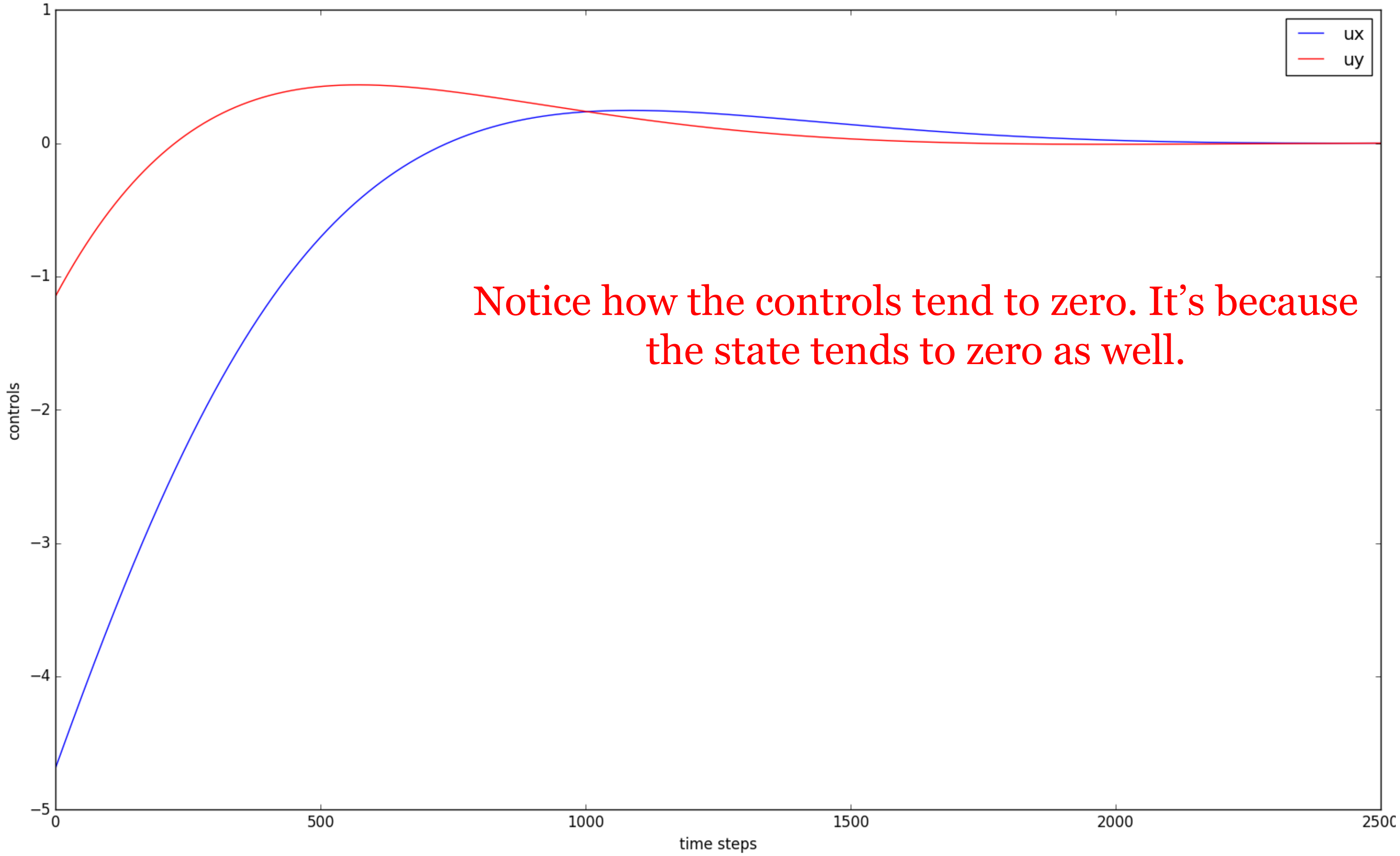

Notice how the controls tend to zero. It's because the state tends to zero as well.

LQR example #1: omnidirectional vehicle with friction

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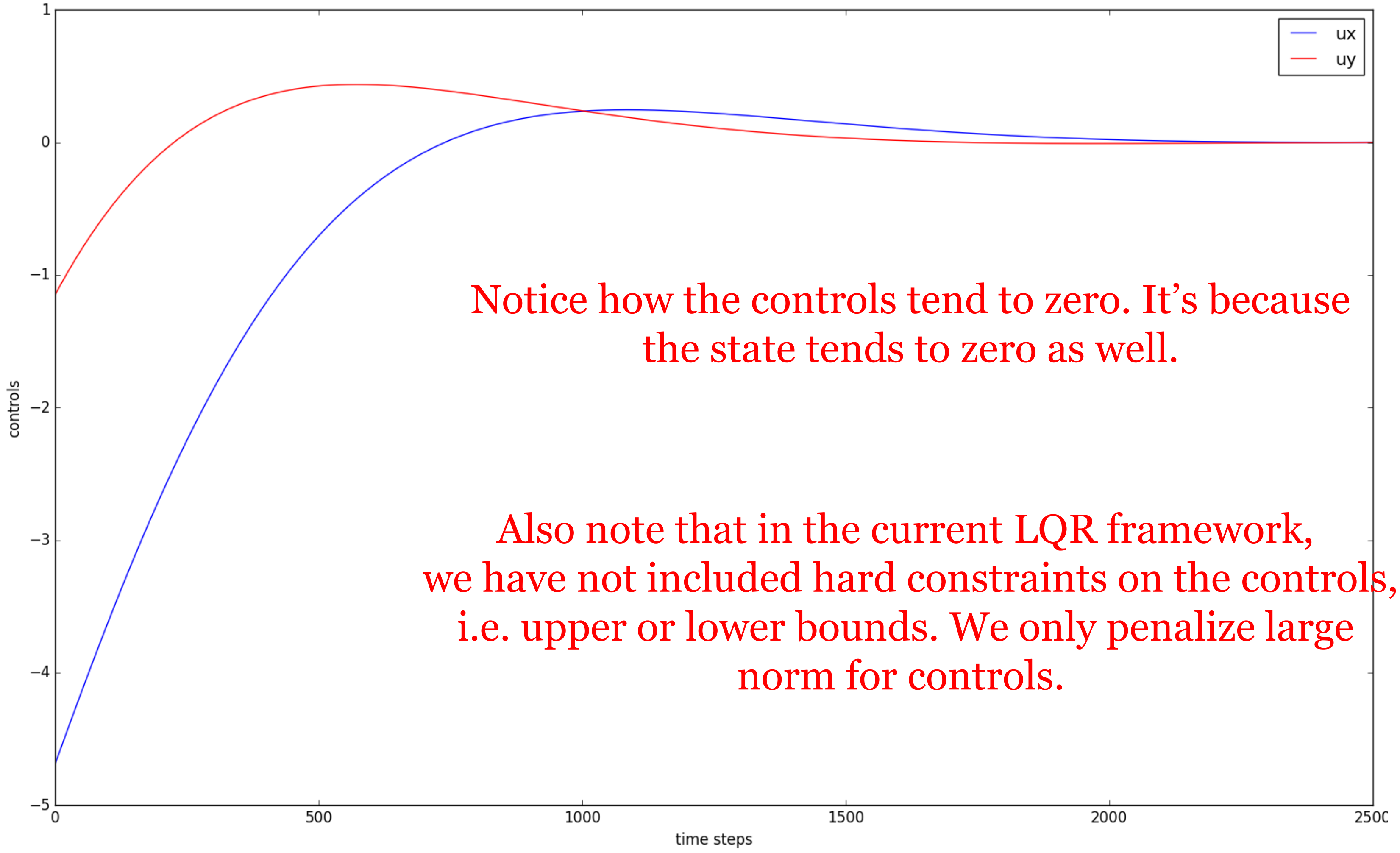
Instantaneous cost function

$$g(\mathbf{x}, \mathbf{u}) = ||\mathbf{x}||^2 + 100||\mathbf{u}||^2$$



LQR example #1: omnidirectional vehicle with friction

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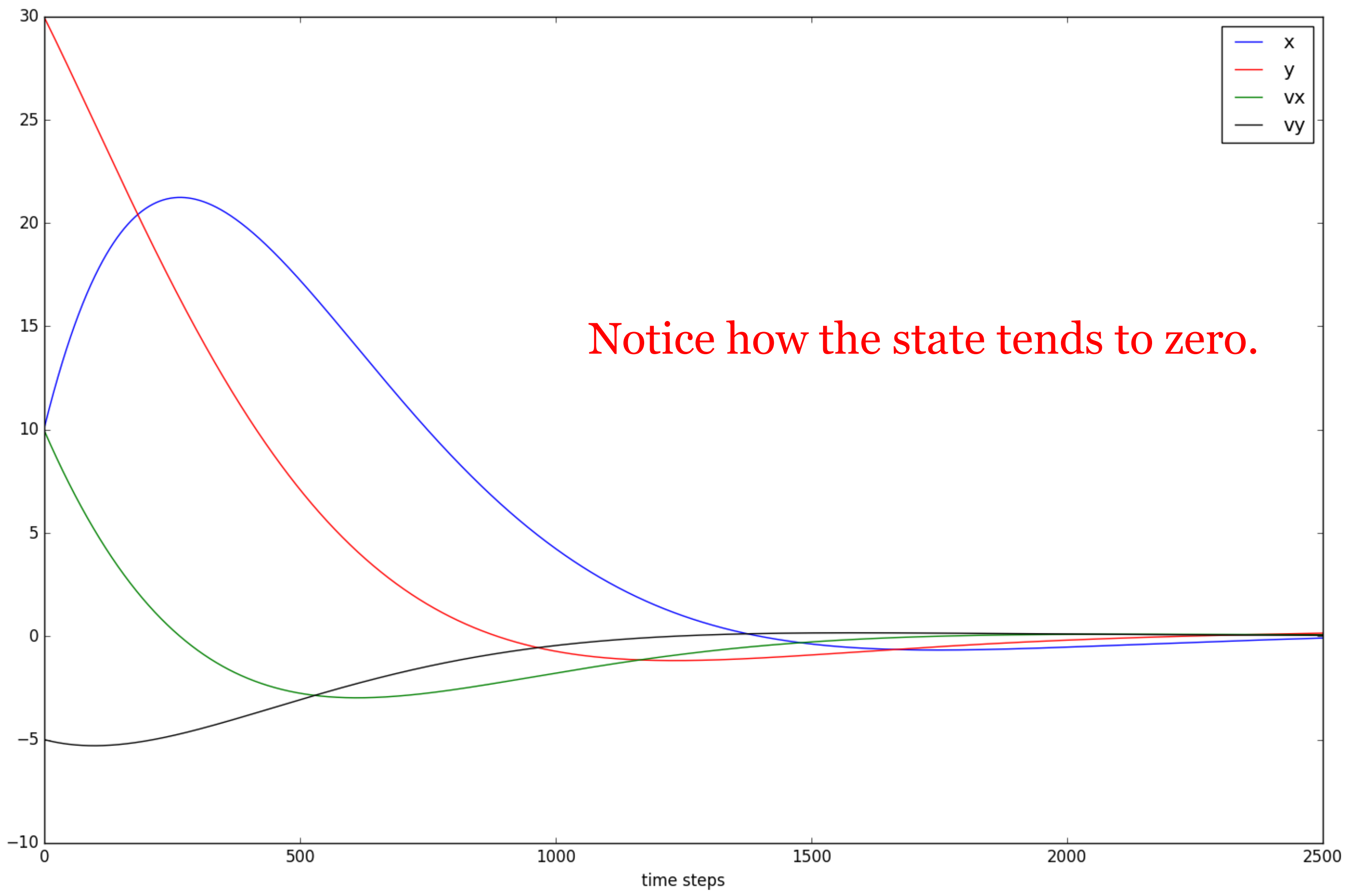
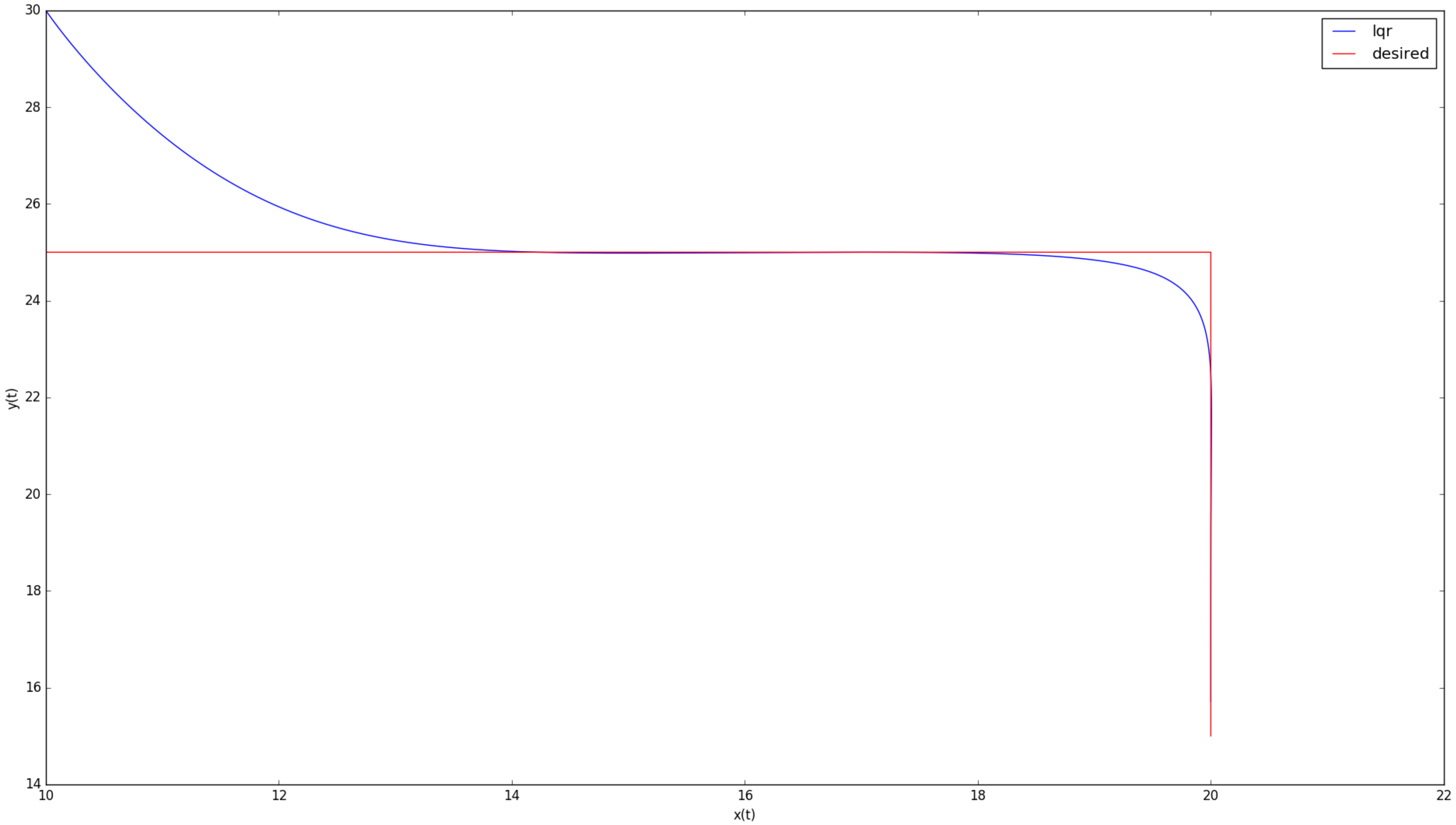


LQR example #2: trajectory following for omnidirectional vehicle



LQR example #2: trajectory following for omnidirectional vehicle

$$\mathbf{x}_{t+1} = \begin{bmatrix} \mathbf{p}_{t+1} \\ \mathbf{v}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1 - \alpha\delta t/m & 0 \\ 0 & 0 & 0 & 1 - \alpha\delta t/m \end{bmatrix}}_{\mathbf{A}} \mathbf{x}_t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\delta t}{m} & 0 \\ 0 & \frac{\delta t}{m} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}_t$$

We are given a desired trajectory $\mathbf{p}_0^*, \mathbf{p}_1^*, \ldots, \mathbf{p}_T^*$

Instantaneous cost $g(\mathbf{x}_t, \mathbf{u}_t) = (\mathbf{p}_t - \mathbf{p}_t^*)^T Q (\mathbf{p}_t - \mathbf{p}_t^*) + \mathbf{u}_t^T R \mathbf{u}_t$

LQR example #2:
trajectory following for omnidirectional vehicle

$$\mathbf{x}_{t+1} = \begin{bmatrix} \mathbf{p}_{t+1} \\ \mathbf{v}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1-\alpha\delta t/m & 0 \\ 0 & 0 & 0 & 1-\alpha\delta t/m \end{bmatrix}}_{\mathbf{A}} \mathbf{x}_t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\delta t}{m} & 0 \\ 0 & \frac{\delta t}{m} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}_t$$

Define

$$\begin{aligned} \bar{\mathbf{x}}_{t+1} &= \mathbf{x}_{t+1} - \mathbf{x}^*_{t+1} \\ &= A\mathbf{x}_t + B\mathbf{u}_t - \mathbf{x}^*_{t+1} \\ &= A\bar{\mathbf{x}}_t + B\mathbf{u}_t - \mathbf{x}^*_{t+1} + A\mathbf{x}^*_t \end{aligned}$$

(We want $\bar{\mathbf{x}}_{t+1} = \bar{A}\bar{\mathbf{x}}_t + \bar{B}\mathbf{u}_t$)

LQR example #2: trajectory following for omnidirectional vehicle

$$\mathbf{x}_{t+1} = \begin{bmatrix} \mathbf{p}_{t+1} \\ \mathbf{v}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1-\alpha\delta t/m & 0 \\ 0 & 0 & 0 & 1-\alpha\delta t/m \end{bmatrix}}_{\mathbf{A}} \mathbf{x}_t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\delta t}{m} & 0 \\ 0 & \frac{\delta t}{m} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}_t$$

$$\begin{aligned} \text{Define} \quad \bar{\mathbf{x}}_{t+1} &= \mathbf{x}_{t+1} - \mathbf{x}^*_{t+1} \\ &= A\mathbf{x}_t + B\mathbf{u}_t - \mathbf{x}^*_{t+1} \\ &= A\bar{\mathbf{x}}_t + B\mathbf{u}_t - \mathbf{x}^*_{t+1} + A\mathbf{x}^*_t \end{aligned}$$

(We want $\bar{\mathbf{x}}_{t+1} = \bar{A}\bar{\mathbf{x}}_t + \bar{B}\mathbf{u}_t$)

Need to get rid of this additive term

LQR example #2: trajectory following for omnidirectional vehicle

$$\mathbf{x}_{t+1} = \begin{bmatrix} \mathbf{p}_{t+1} \\ \mathbf{v}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1-\alpha\delta t/m & 0 \\ 0 & 0 & 0 & 1-\alpha\delta t/m \end{bmatrix}}_{\mathbf{A}} \mathbf{x}_t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\delta t}{m} & 0 \\ 0 & \frac{\delta t}{m} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}_t$$

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(We want $\bar{\mathbf{x}}_{t+1} = \bar{A}\bar{\mathbf{x}}_t + \bar{B}\mathbf{u}_t$)

Need to get rid of this additive term

Redefine state: $\mathbf{z}_{t+1} = \begin{bmatrix} \bar{\mathbf{x}}_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{u}_t = \bar{A}\mathbf{z}_t + \bar{B}\mathbf{u}_t$

LQR example #2: trajectory following for omnidirectional vehicle

$$\mathbf{x}_{t+1} = \begin{bmatrix} \mathbf{p}_{t+1} \\ \mathbf{v}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \delta t & 0 \\ 0 & 1 & 0 & \delta t \\ 0 & 0 & 1-\alpha\delta t/m & 0 \\ 0 & 0 & 0 & 1-\alpha\delta t/m \end{bmatrix}}_{\mathbf{A}} \mathbf{x}_t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\delta t}{m} & 0 \\ 0 & \frac{\delta t}{m} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}_t$$

Define
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(We want $\bar{\mathbf{x}}_{t+1} = \bar{A}\bar{\mathbf{x}}_t + \bar{B}\mathbf{u}_t$)

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Redefine cost function: $g(\mathbf{z}_t, \mathbf{u}_t) = \mathbf{z}_t^T \bar{Q} \mathbf{z}_t + \mathbf{u}_t^T R \mathbf{u}_t$

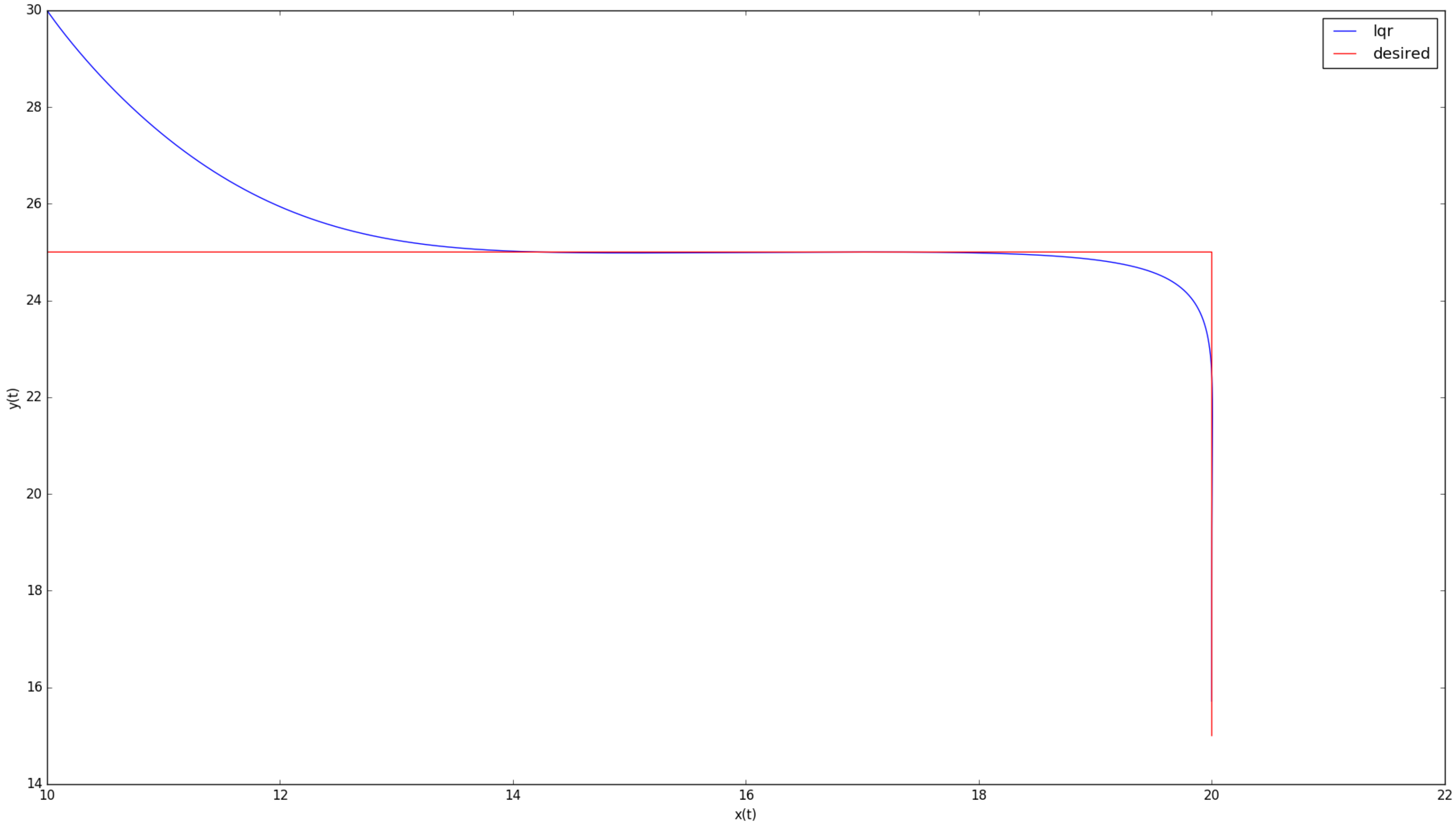
LQR example #2: trajectory following for omnidirectional vehicle

With initial state

$$\mathbf{z}_0 = \begin{bmatrix} 10 \\ 30 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Instantaneous cost function

$$g(\mathbf{z}, \mathbf{u}) = ||\mathbf{z}||^2 + ||\mathbf{u}||^2$$



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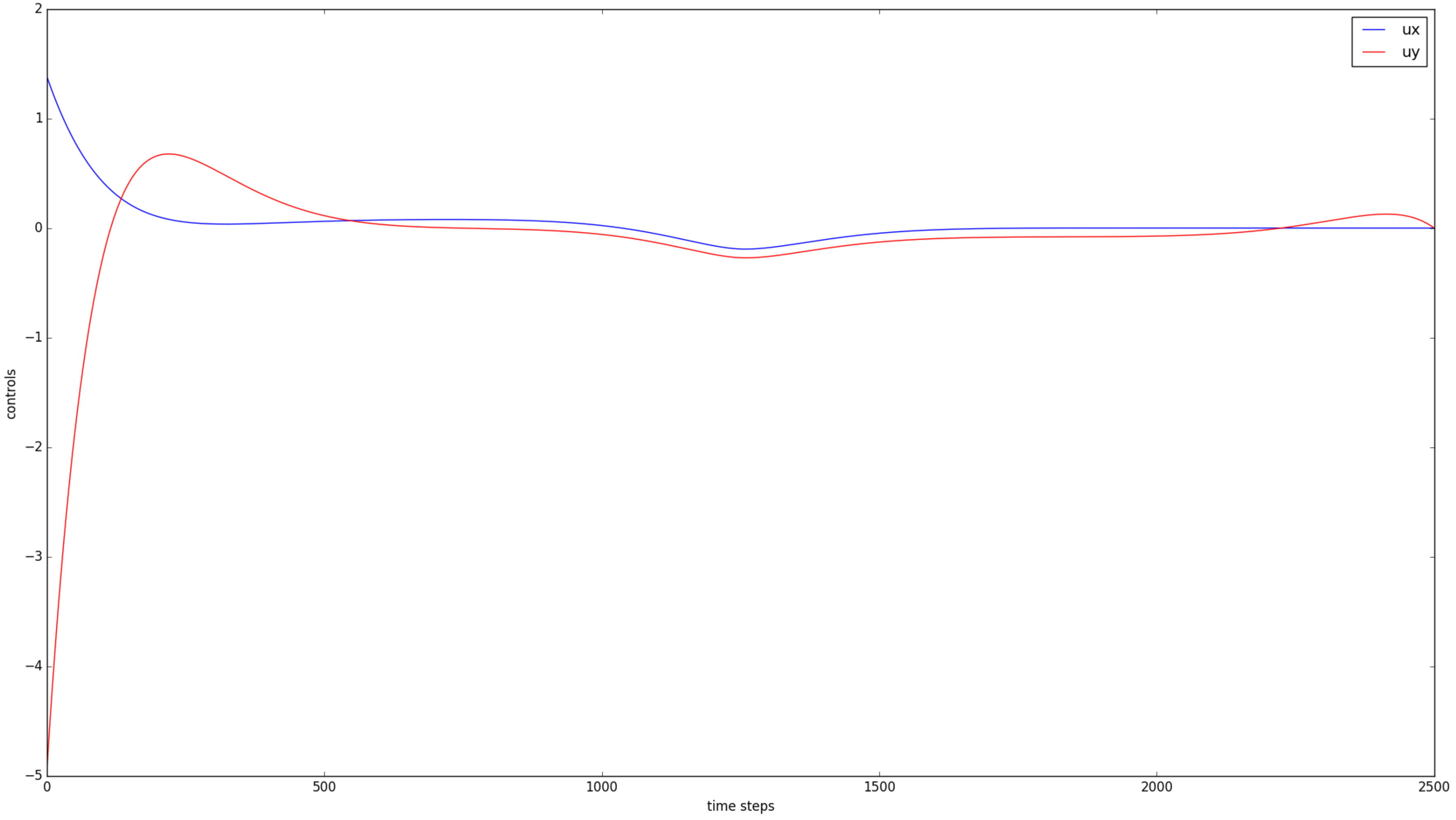

LQR examples: code to replicate these results

- https://github.com/florianshkurti/comp417.git
- Look under comp417/lqr_examples/python

LQR summary

- Advantages:
  - If system is linear LQR gives the optimal controller that takes the system from state A to state B

- Drawbacks:
  - Need to specify final time at which goal should be reached
  - What if you don’t allow enough time to reach the goal?
  - How can you include obstacles or constraints in the specification?