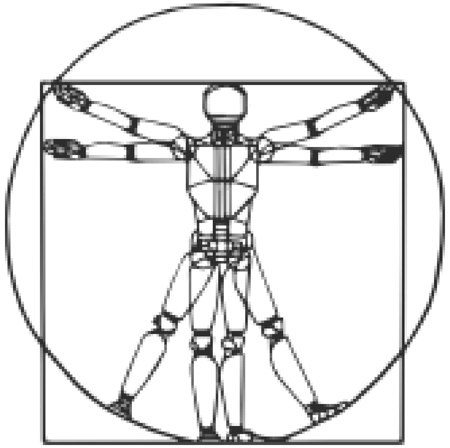

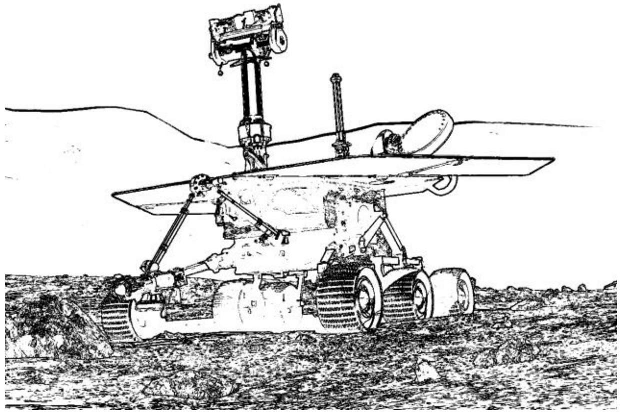

COMP417
Introduction to Robotics and Intelligent Systems

Lecture 3: Kinematics

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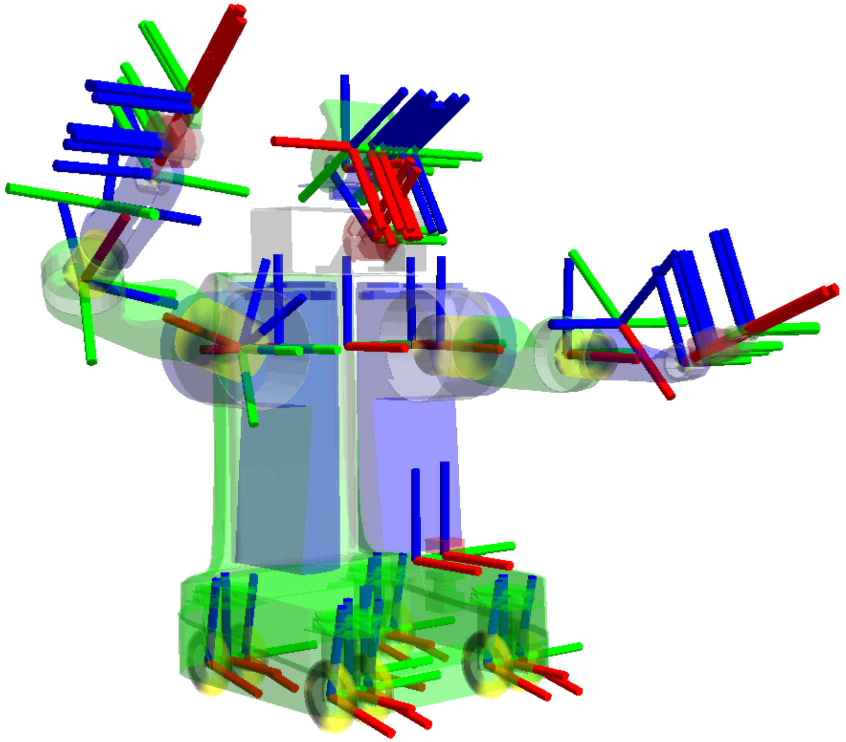
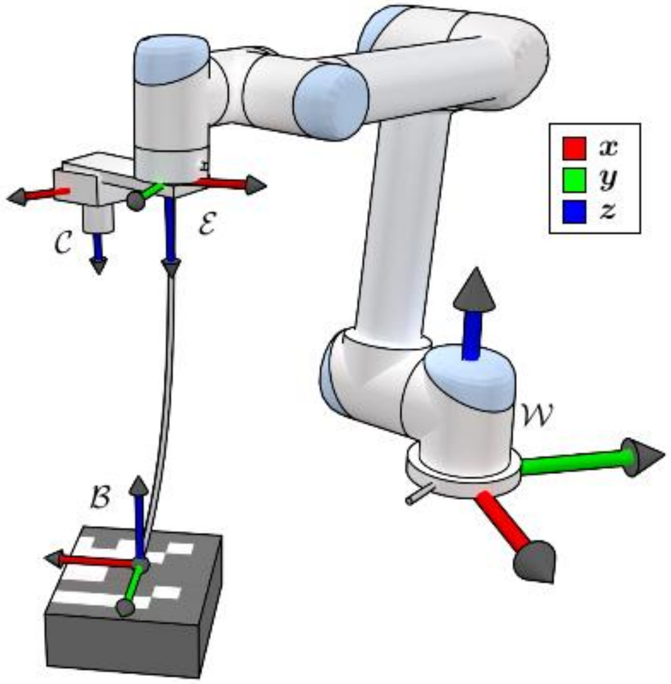
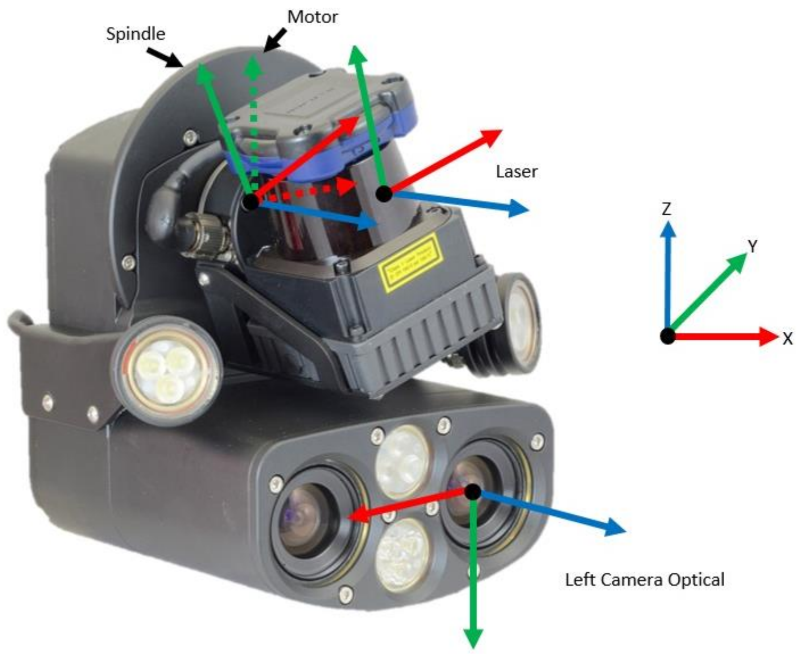
Today’s slides borrow parts of Paul Furgale’s “Representing robot pose” presentation:

http://paulfurgale.info/news/2014/6/9/representing-robot-pose-the-good-the-bad-and-the-ugly

3D frames of reference are everywhere in robotics





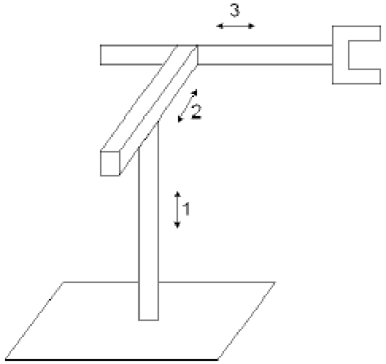


Manipulators

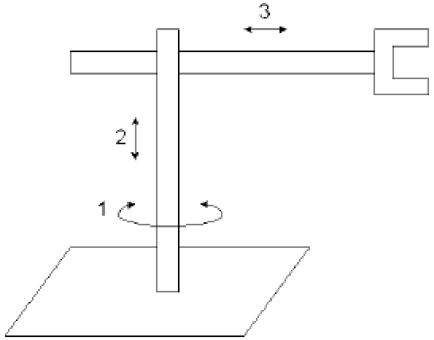
- Robot arms, industrial robot
  - Rigid bodies (links) connected by joints
  - Joints: revolute or prismatic
  - Drive: electric or hydraulic
  - End-effector (tool) mounted on a flange or plate secured to the wrist joint of robot

Manipulators

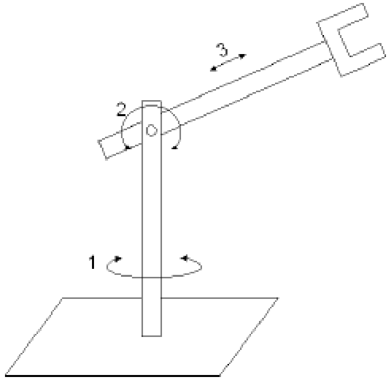
- Robot Configuration:



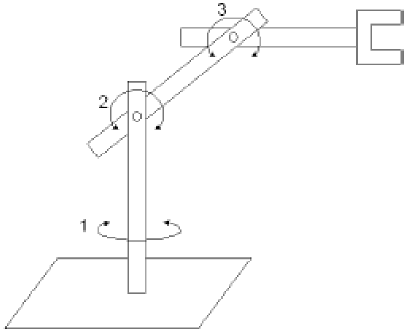
Cartesian: PPP



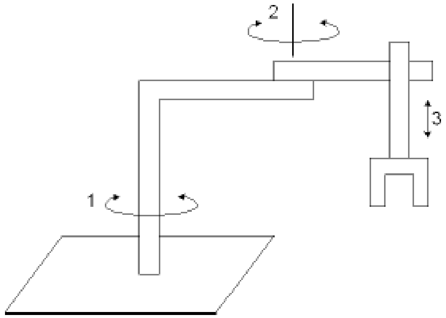
Cylindrical: RPP



Spherical: RRP

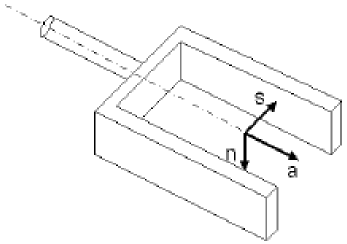


Articulated: RRR



SCARA: RRP

(Selective Compliance Assembly Robot Arm)



Hand coordinate:

n: normal vector; **s**: sliding vector;

a: approach vector, normal to the tool mounting plate

Manipulators

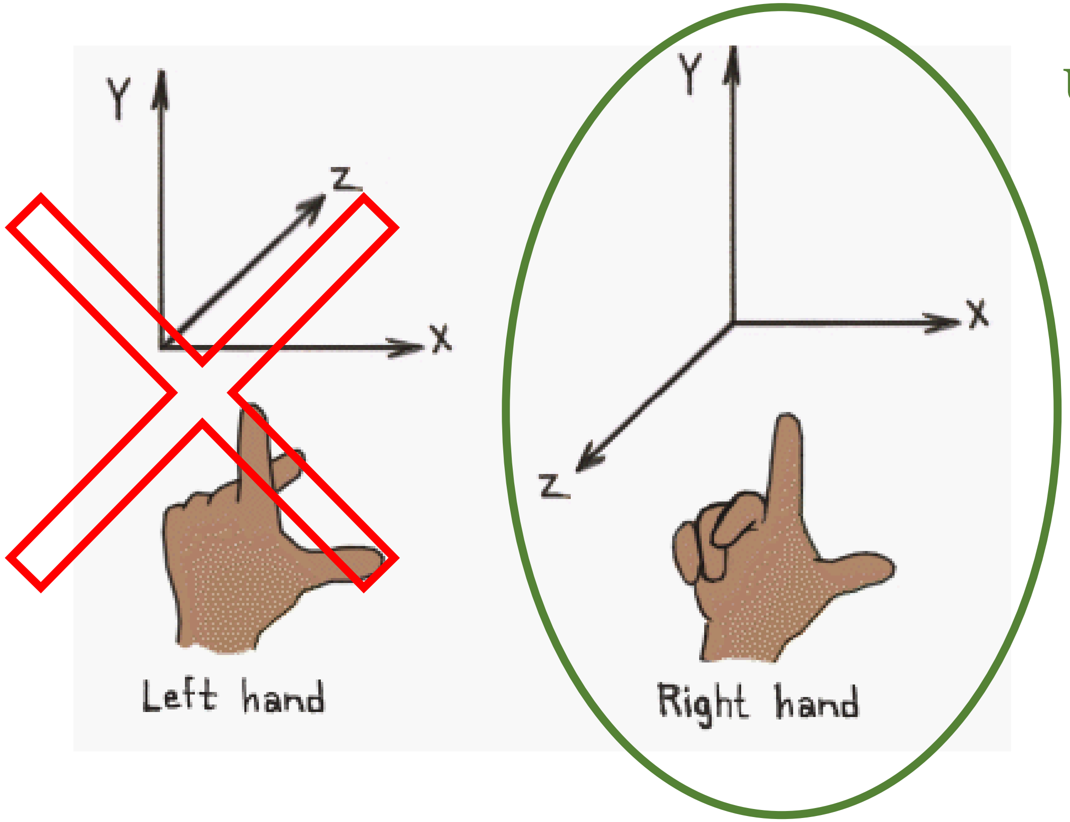
- Motion Control Methods
  - Point to point control
    - a sequence of discrete points
    - spot welding, pick-and-place, loading & unloading
  - Continuous path control
    - follow a prescribed path, controlled-path motion
    - Spray painting, Arc welding, Gluing

Manipulators

- Robot Specifications
  - Number of Axes
    - Major axes, (1-3) => Position the wrist
    - Minor axes, (4-6) => Orient the tool
    - Redundant, (7-n) => reaching around obstacles, avoiding undesirable configuration
  - Degree of Freedom (DOF)
  - Workspace
  - Payload (load capacity)
  - Precision v.s. Repeatability

Which one is more important?

Right-handed vs left-handed frames



Unless otherwise specified, we use right-handed frames in robotics

Why do we need to use so many frames?

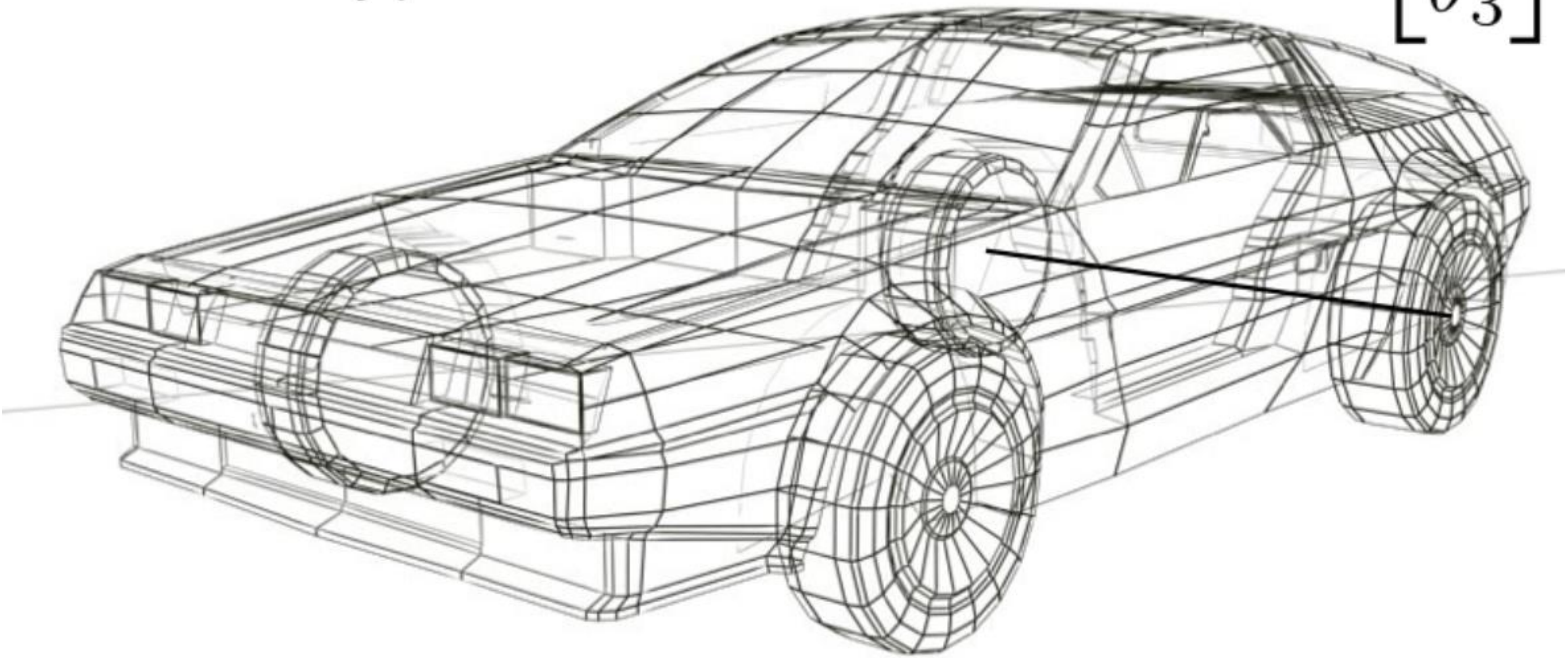
- Because we want to reason and express quantities relative to their local configuration.
- For example: “grab the bottle behind the cereal bowl”
- This class is about defining and representing frames of reference and reasoning about how to express quantities in one frame to quantities in the other.

Rigid-body motion

- Motion that can be described by a rotation and translation.
- All the parts making up the body move in unison, and there are no deformations.
- Representing rotations, translations, and vectors in a given frame of reference is often a source of frustration and bugs in robot software because there are so many options.

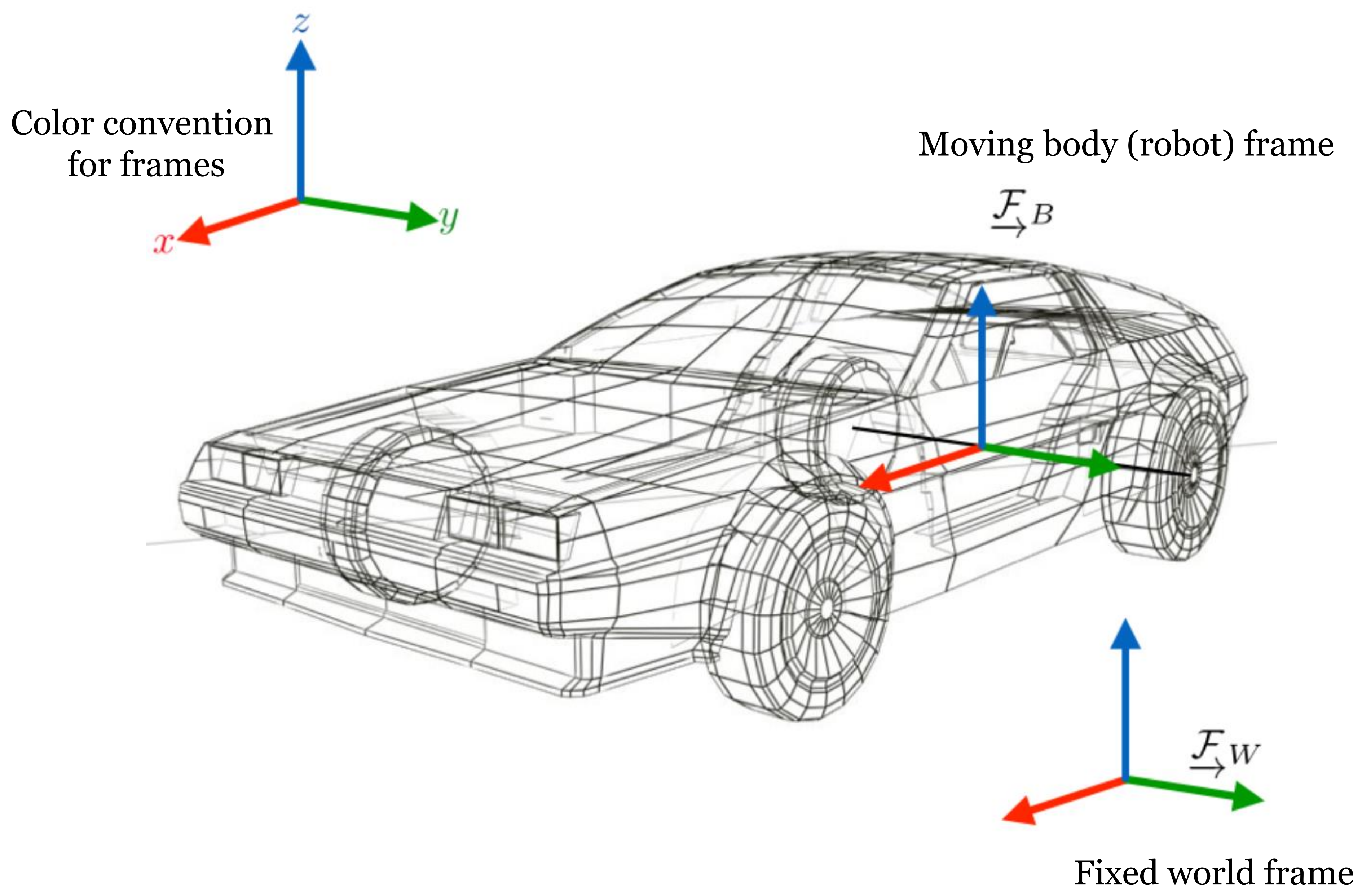
- The “three number” problem
- You are given three numbers representing *the orientation of the robot*
- How many possibilities are there?

$$\mathbf{C} \leftarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$



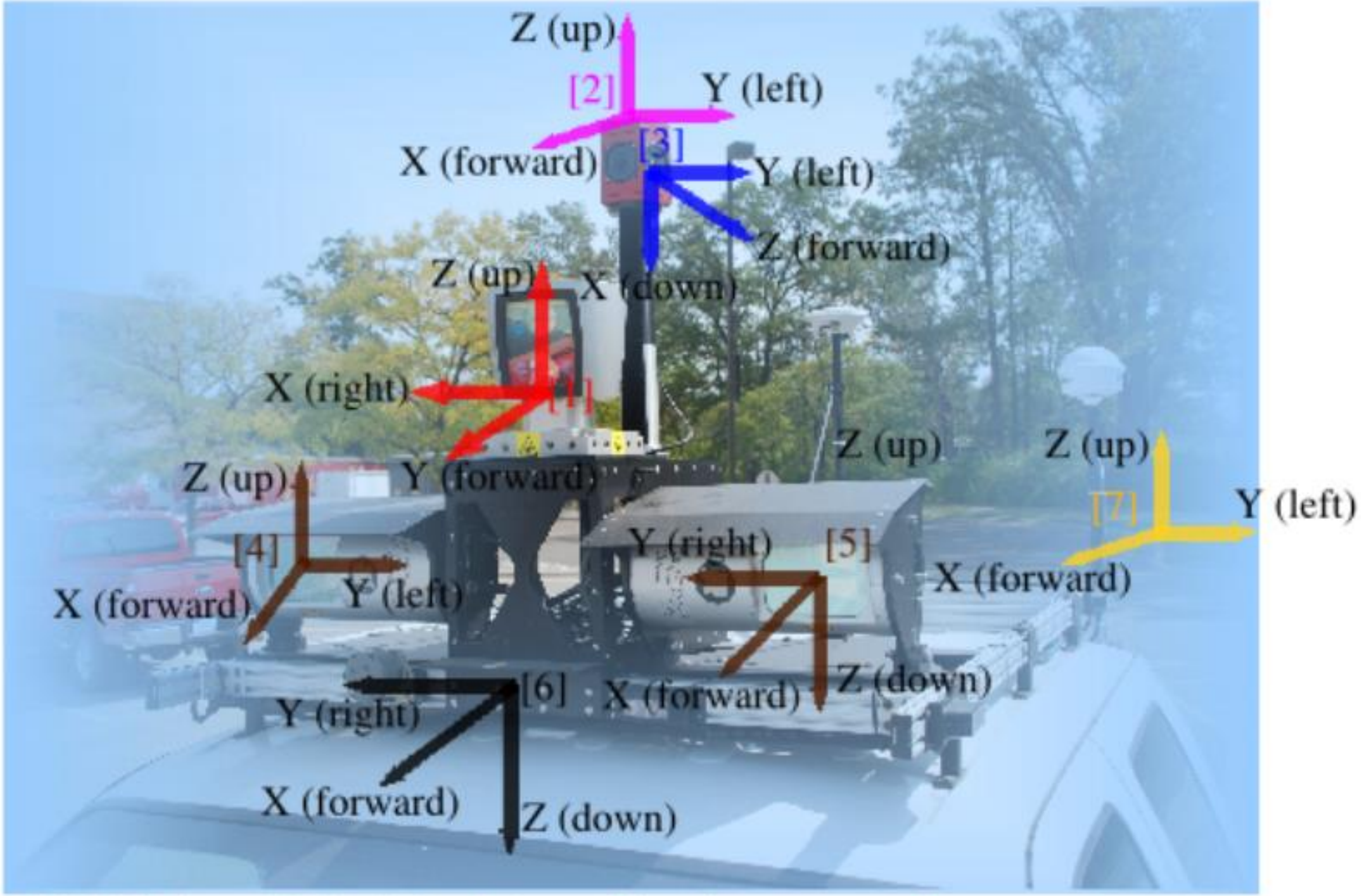
Super sweet wireframe Delorian from http://nanahra.blogspot.com

The answer is meaningless unless I provide a definition of the coordinate frames



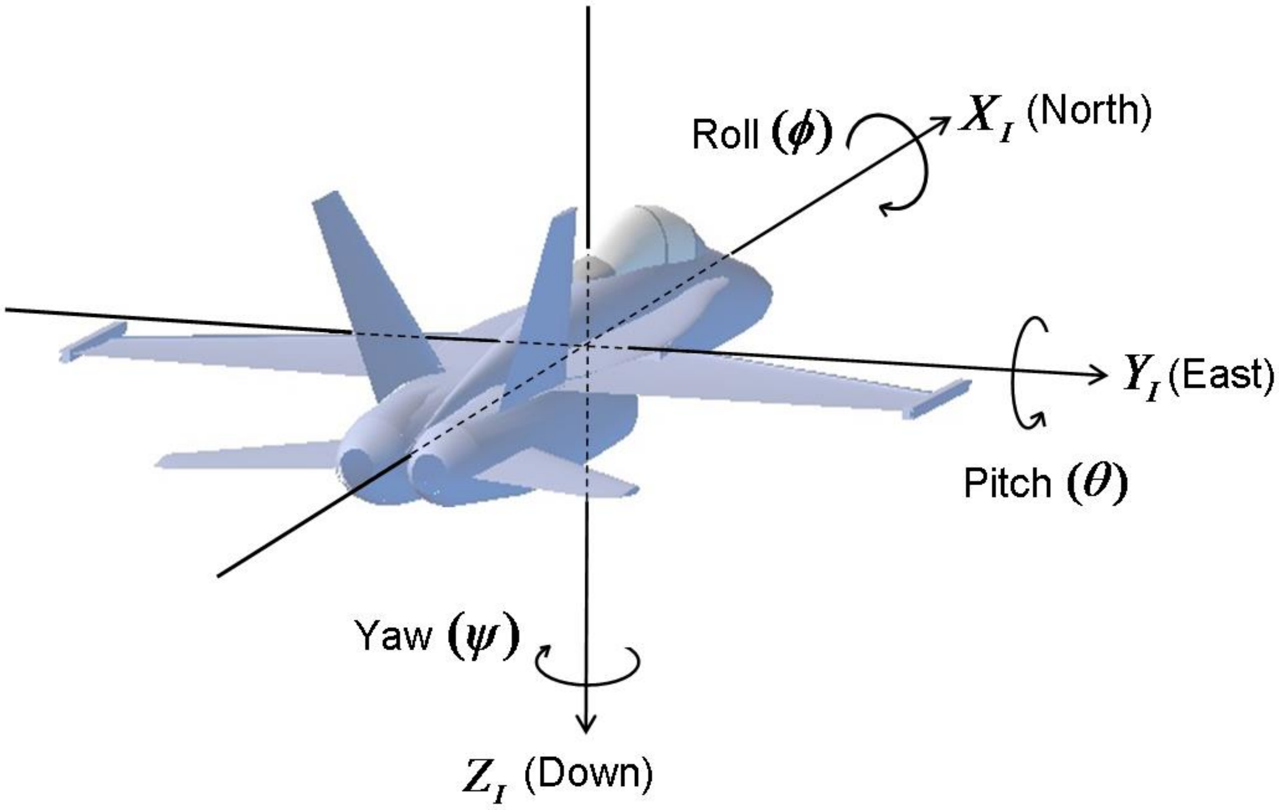
z
Color convention for frames
x
y
Moving body (robot) frame
$\underrightarrow{\mathcal{F}}_B$
$\underrightarrow{\mathcal{F}}_W$
Fixed world frame

Always provide a frame diagram



[1] Velodyne, [2] Ladybug3 (actual location: center of camera system),
[3] Ladybug3 Camera 5, [4] Right Riegl, [5] Left Riegl,
[6] Body Frame (actual location: center of rear axle)
[7] Local Frame (Angle between the X-axis and East is known)

Representing Rotations in 3D: Euler Angles



Specification ambiguities in Euler Angles

- Need to specify the axes which each angle refers to.

- There are **12 different valid combinations** of fundamental rotations. Here are the possible axes:

- z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y
- x-y-z, y-z-x, z-y-x, x-z-y, z-y-x, y-x-z

Specification ambiguities in Euler Angles

- Need to specify the axes which each angle refers to.

- There are **12 different valid combinations** of fundamental rotations. Here are the possible axes:

- z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y
- x-y-z, y-z-x, z-y-x, x-z-y, z-y-x, y-x-z

- E.g.: x-y-z rotation with Euler angles (θ, ϕ, ψ) means the rotation can be expressed as a sequence of simple rotations $R_x(\theta)R_y(\phi)R_z(\psi)$

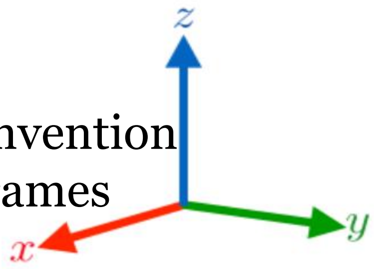
Specification ambiguities in Euler Angles

Simple rotations can be counter-clockwise or clockwise. This gives **another 2 possibilities.**

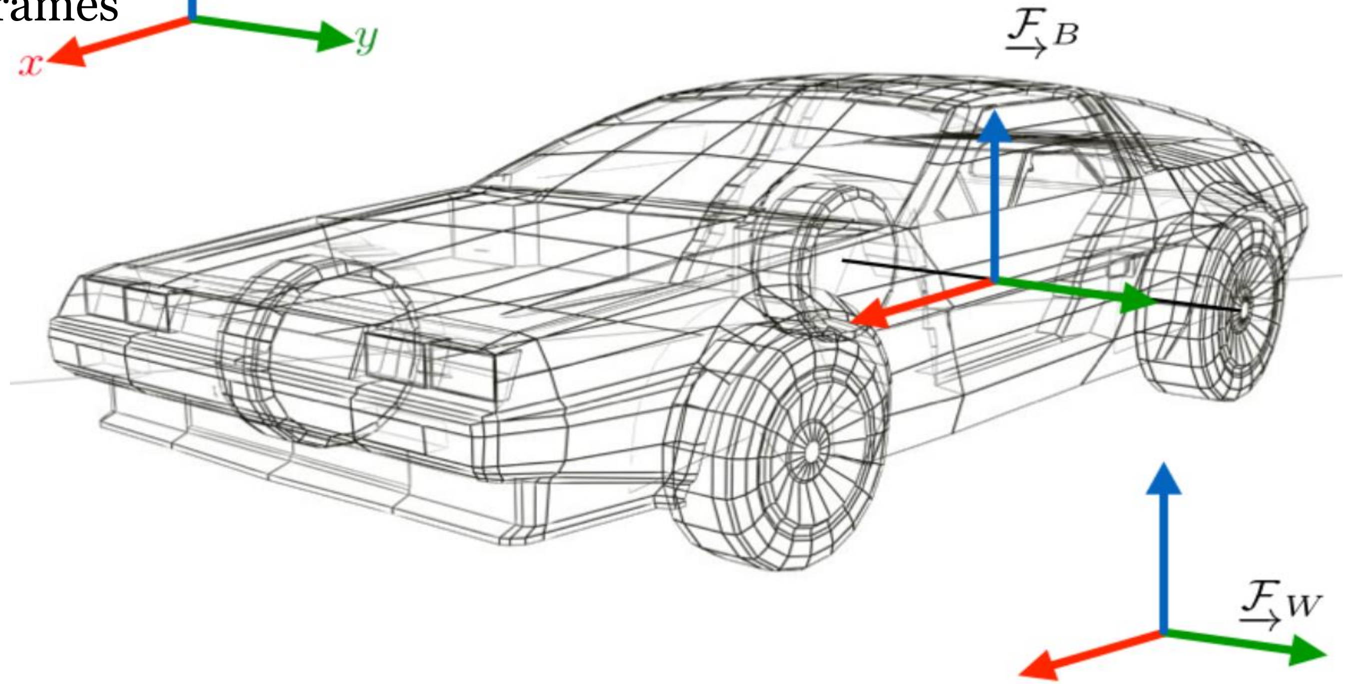
$$\mathbf{R}_z(\alpha) := \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C}_z(\alpha) := \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Specification ambiguities in Euler Angles

Color convention for frames



Moving body (robot) frame



Fixed world frame

You need to specify whether the rotation rotates from the world frame to the body frame, or the other way around.

Another 2 possibilities. More possibilities if you have more frames.

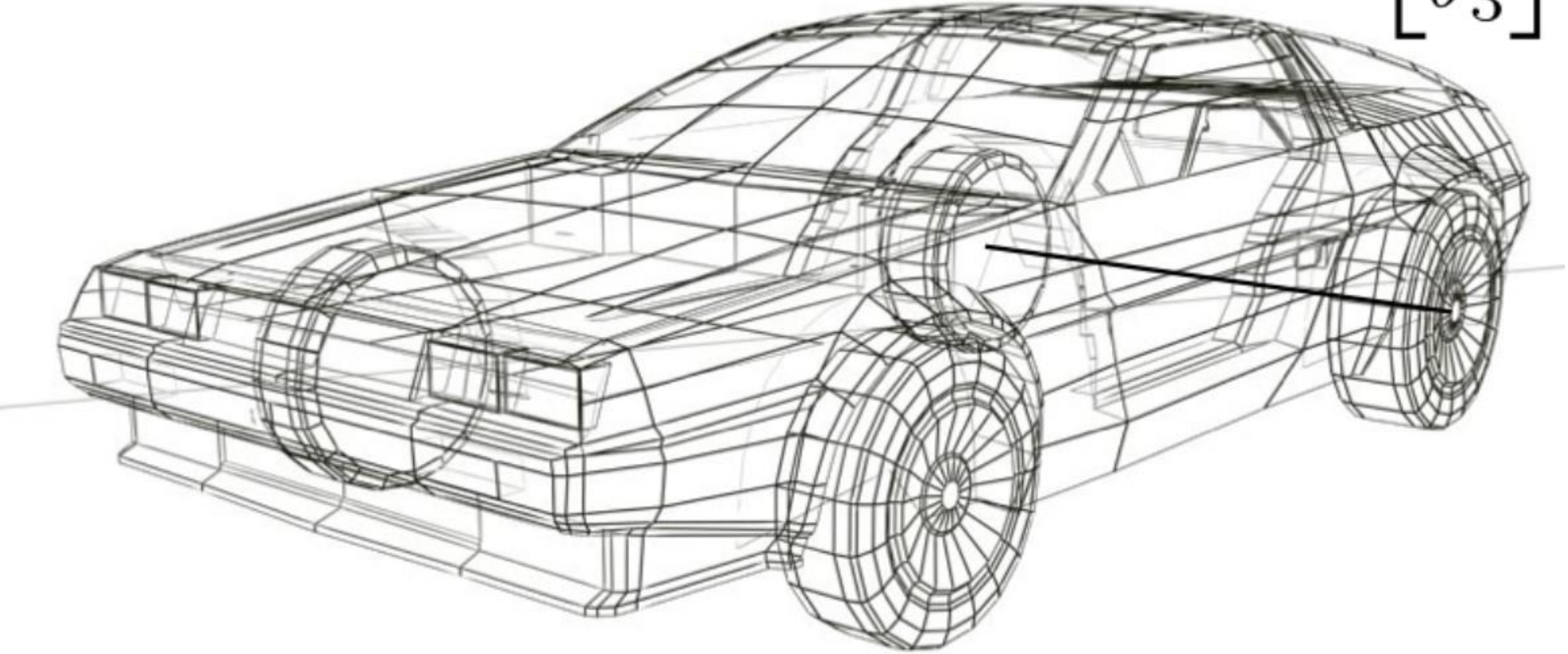
Degrees or radians? Another 2 possibilities

Specification ambiguities in Euler Angles

- Need to specify the ordering of the three parameters.
- 1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2, 3-2-1
- **Another 6 different valid combinations**

- The “three number” problem
- You are given three numbers representing *the orientation of the robot*
- How many possibilities are there?

$$\mathbf{C} \leftarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

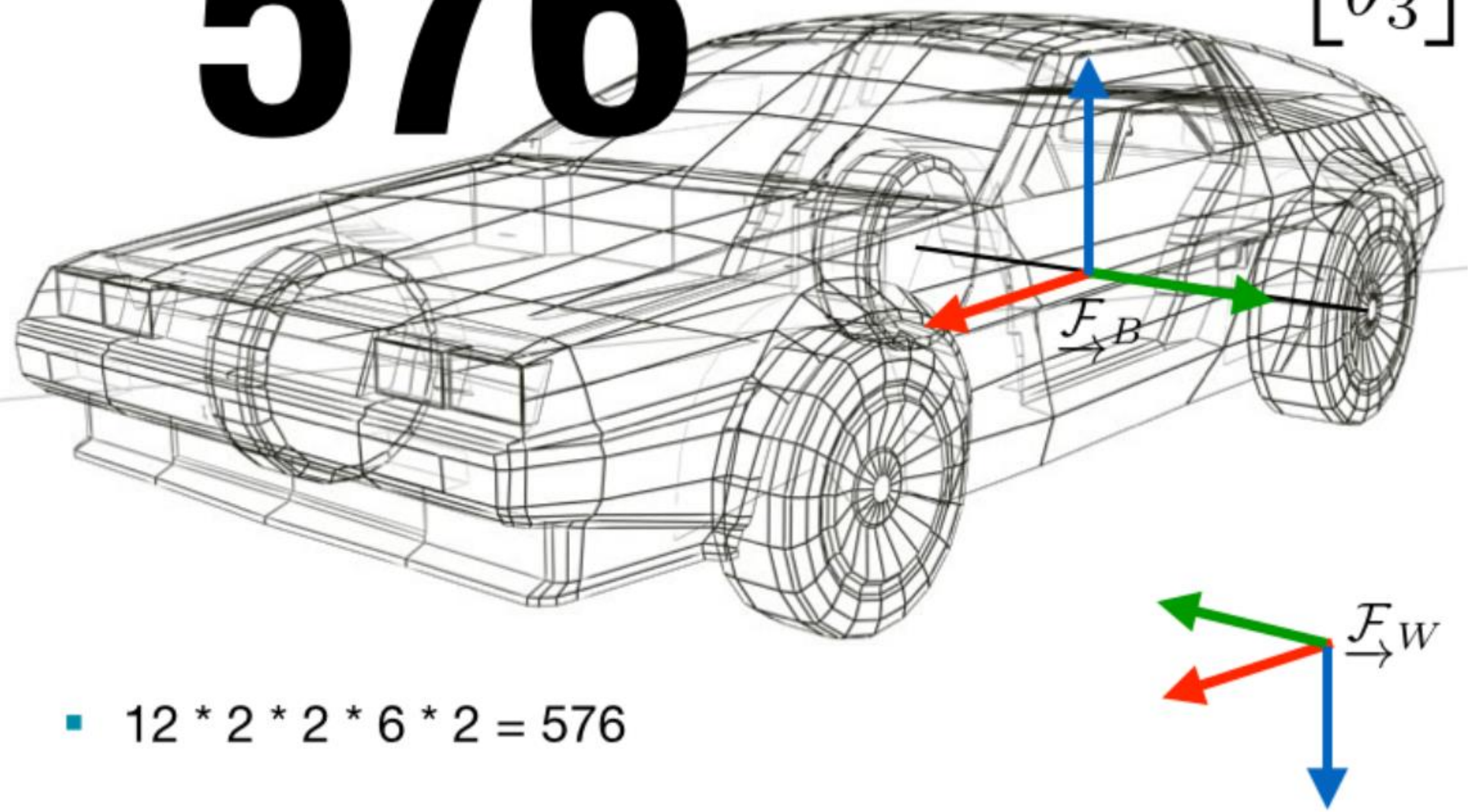


Super sweet wireframe Delorian from http://nanahra.blogspot.com

- The “three number” problem
- How many possibilities are there?

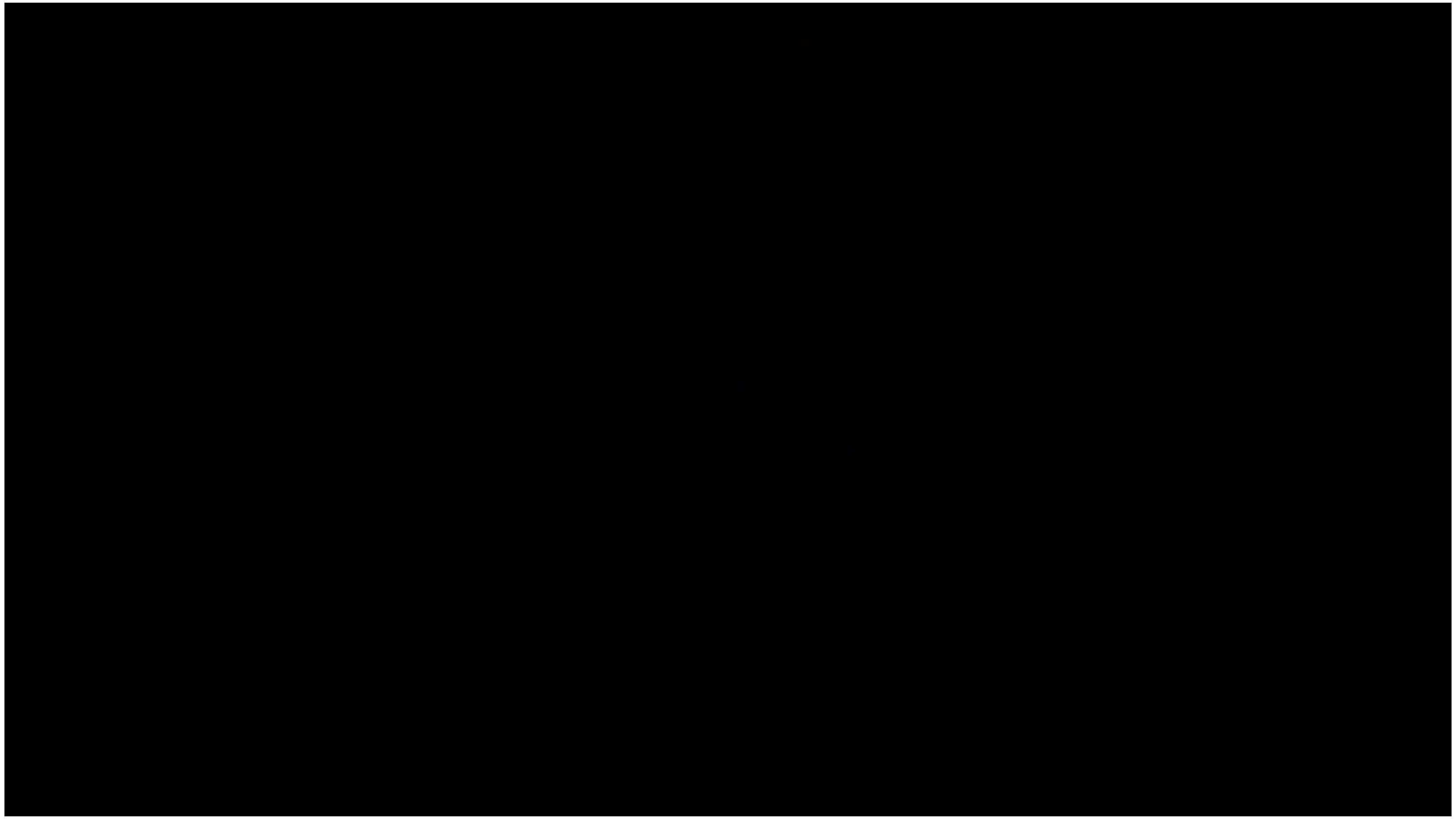
576

$$\mathbf{C} \leftarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$



- 12 * 2 * 2 * 6 * 2 = 576

Another problem with Euler angles: Gimbal Lock



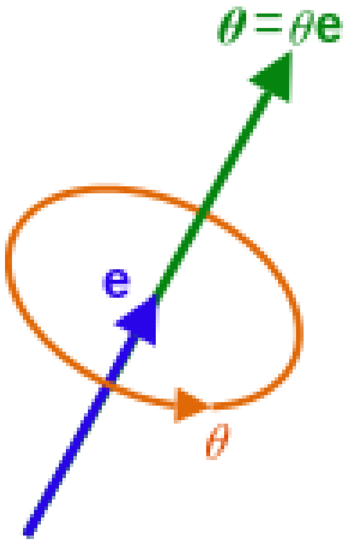
Another problem with Euler angles: Gimbal Lock

- Why should roboticists care about this?
- Because when it happens Euler angle representations lose one degree of freedom.
- They cannot represent the entire range of rotations any more.
- They get “locked” into a subset of the space of possible rotations.

So, we need other representations aside from Euler angles.

Even though they are a minimal representation.

Representing Rotations in 3D: Axis-Angle



- 4-number representation (angle, 3D axis)
- 2 ambiguities: (-angle, -axis) is the same as (angle, axis)

Representing Rotations in 3D: Rotation Matrix

- The royalty of rotation representations
- 3x3-number representation, very redundant
- No ambiguities, as long as source frame and target frame are specified correctly. For example, define your notation this way:
- Rotation from Body frame to World frame: $\mathbf{R}_{BW}$
- Or you can define it this way: ${}^{W}_{B}\mathbf{R}$

Inverse Rotation Matrix

$$^{W}_{B}\mathbf{R}^{-1} = {}^{W}_{B}\mathbf{R}^{t} = {}^{B}_{W}\mathbf{R}$$

Rotation matrices are orthogonal matrices: their transpose is their inverse and they do not change the length of a vector, they just rotate it in space.

$$^{W}_{B}\mathbf{R}^{t}\,{}^{W}_{B}\mathbf{R} = \mathbf{I}$$

Converting axis-angle to rotation matrix

- Given angle theta and axis v the equivalent rotation matrix is

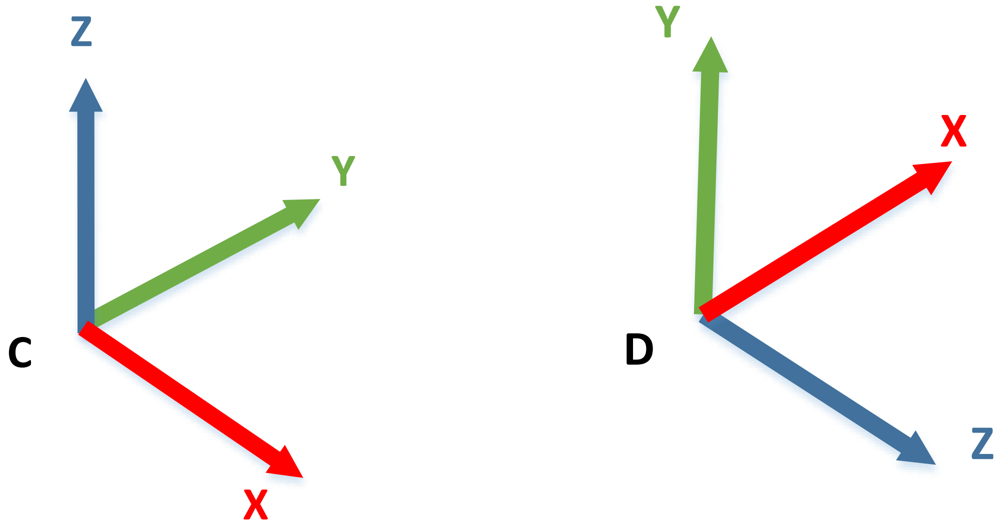
$$\mathbf{R} = \mathbf{I}\cos\theta + (1 - \cos\theta)\mathbf{v}\mathbf{v}^t + [\mathbf{v}]_\times$$

- Where I is the 3x3 identity and

$$[\mathbf{a}]_\times \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

- This is called the “Rodrigues formula”

Example: finding a rotation matrix that rotates one vector to another



$$ {}^{D}_{C}\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} $$

This matrix transforms the x-axis of frame C to the z-axis of frame D. Same for y and z axes.

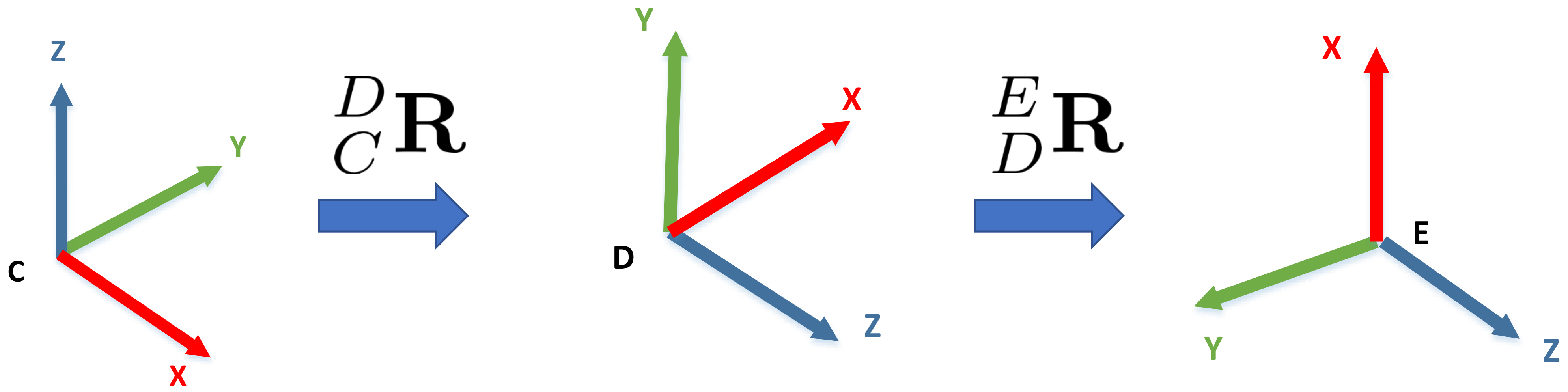
Rotation multiplication vs addition: 3D vs 2D

- In 2D adding angles with wraparound at 360 degrees is a valid operation.

- Rotation matrices can be added, but the result is not necessarily a valid rotation. Rotations are not closed under the operation of addition.

- Rotations are closed under the operation of multiplication. To compose a sequence of simple rotations we need to multiply them.

Compound rotations



$$
{}^E_C\mathbf{R} = {}^E_D\mathbf{R}\,{}^D_C\mathbf{R}
$$

Representing Rotations in 3D: Quaternions

- Based on axis-angle representation, but more computationally efficient.

- The main workhorse of rotation representations.

- Used almost everywhere in robotics, aerospace, aviation.

- Very important to master in this course. You will need it for the first assignment and for working with ROS in general.

Converting axis-angle to quaternion

- Given angle theta and axis v the equivalent quaternion representation is

$$\mathbf{q} = [\sin(\theta/2)v_1,\ \sin(\theta/2)v_2,\ \sin(\theta/2)v_3,\ \cos(\theta/2)]$$
$$\mathbf{q} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + w$$

- Just like in the case of rotation matrices we denote the source and target frames of the rotation quaternion: ${}^{W}_{B}\mathbf{q}$

Converting axis-angle to quaternion

- Given angle theta and a unit axis v, the equivalent quaternion representation is:

$$\mathbf{q} = [\sin(\theta/2)v_1,\ \sin(\theta/2)v_2,\ \sin(\theta/2)v_3,\ \cos(\theta/2)]$$
$$\mathbf{q} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + w$$

- Just like in the case of rotation matrices we denote the source and target frames of the rotation quaternion: ${}^{W}_{B}\mathbf{q}$

- We always work with unit length (normalized) quaternions.

Examples of quaternions

- 90 degree rotation about the z-axis

$$\mathbf{q} = [0,\ 0,\ \sin(\pi/4)v_3,\ \cos(\pi/4)]$$

Quaternion multiplication

- Defined algebraically by

$$Q = q_0 + q_1 i + q_2 j + q_3 k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k,\ jk = i,\ ki = j$$

and usually denoted by the circular cross symbol. For example:

$${}^W_F\mathbf{q} = {}^W_C\mathbf{q} \otimes {}^C_F\mathbf{q}$$

Quaternion multiplication

$$ {}^{W}_{F}\mathbf{q} = {}^{W}_{C}\mathbf{q} \otimes {}^{C}_{F}\mathbf{q} $$

Direct correspondence with matrix multiplication:

$$ {}^{W}_{F}\mathbf{R}(\mathbf{q}) = {}^{W}_{C}\mathbf{R}(\mathbf{q}) \otimes {}^{C}_{F}\mathbf{R}(\mathbf{q}) $$

NOTE: the quaternion to matrix conversion will not be given here.
It is usually present in all numerical algebra libraries. At the moment we'll take it for granted.

Quaternion inversion

$$\mathbf{q}^{-1} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k} + w$$

$$[0, 0, 0, 1] = \mathbf{q}^{-1} \otimes \mathbf{q}$$

Direct correspondence with matrix inversion:

$$\mathbf{I} = \mathbf{R}(\mathbf{q}^{-1})\mathbf{R}(\mathbf{q})$$

$$\mathbf{I} = \mathbf{R}(\mathbf{q})^{-1}\mathbf{R}(\mathbf{q})$$

Example: updating orientation based on angular velocity

- If the angular velocity of the Body frame is ${}^{B}\omega$ and the body-to-world rotation at time t is ${}^{W}_{B}\mathbf{q}(t)$

- Then, at time t+dt the new body-to-world rotation will be

$$ {}^{W}_{B(t+dt)}\mathbf{q} = {}^{W}_{B(t)}\mathbf{q} \otimes {}^{B(t)}_{B(t+dt)}\mathbf{q} $$

where ${}^{B(t)}_{B(t+dt)}\mathbf{q}$ has unit axis $\frac{{}^{B}\omega}{||{}^{B}\omega||}$ and angle $||{}^{B}\omega||dt$

Main ambiguities of quaternion representation

- The ones inherited from the axis-angle representation, but also:

  - Even with unit-length quaternions, there are choices
  - Parameter ordering
    - We won't consider arbitrary ordering
    - We do have to decide on scalar first or scalar last

$$Q = w + xi + yj + zk$$

$$\mathbf{q} := \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \quad \text{Scalar First}$$

Main ambiguities of quaternion representation

- The ones inherited from the axis-angle representation, but also:

  - Even with unit-length quaternions, there are choices
  - Parameter ordering
    - We won't consider arbitrary ordering
    - We do have to decide on scalar first or scalar last

$$Q = w + xi + yj + zk$$

$$\mathbf{q} := \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{Scalar Last}$$

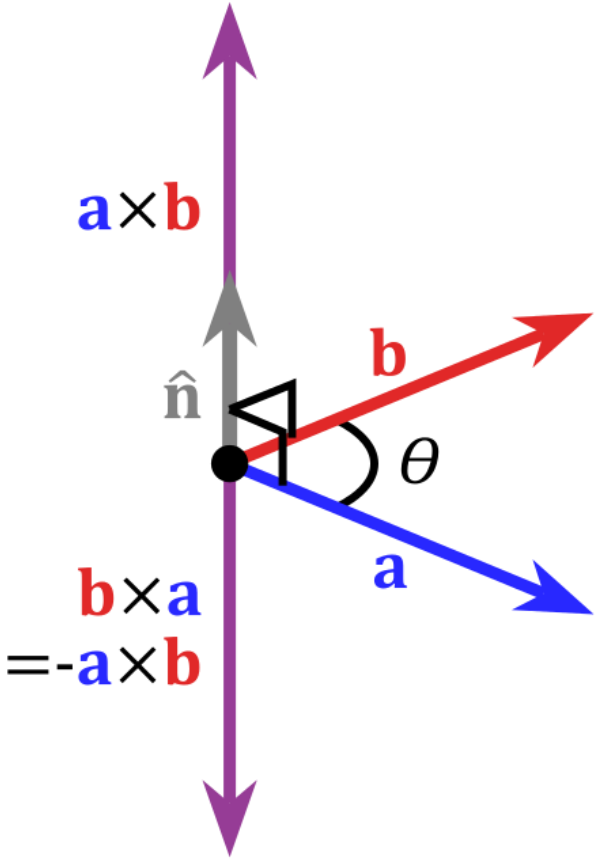
Be clear about your orientation representation.

Example: finding quaternion that rotates one vector into another

- Suppose you have a vector in frame A, and a vector in frame B
- You want to find a quaternion that transforms ${}^{A}\mathbf{v}$ to ${}^{B}\mathbf{v}$
- Idea: use axis-angle and convert it to quaternion
- Can rotate from ${}^{A}\mathbf{v}$ to ${}^{B}\mathbf{v}$ along an axis that is

  perpendicular to both of them. How do we find that?

Cross Product



$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

Example: finding quaternion that rotates one vector into another

$$\mathbf{v}_{\text{rot axis}} = {}^{A}\mathbf{v} \times {}^{B}\mathbf{v}$$ is perpendicular to both of them

$$\theta_{\text{rot angle}} = \text{acos}({}^{A}\mathbf{v} \cdot {}^{B}\mathbf{v})$$

Assuming the two vectors are unit length

Rotating a vector via a quaternion

- Let ${}^{A}\mathbf{v}$ be given and a quaternion ${}^{B}_{A}\mathbf{q}$
- To obtain ${}^{B}\mathbf{v}$ you have two choices:
- Either use the rotation matrix $\quad {}^{B}\mathbf{v} = {}^{B}_{A}\mathbf{R}(\mathbf{q})\,{}^{A}\mathbf{v}$
- Or use quaternion multiplication directly

$$[{}^{B}\mathbf{v}, 0] = {}^{B}_{A}\mathbf{q} \otimes [{}^{A}\mathbf{v}, 0] \otimes {}^{A}_{B}\mathbf{q}$$

Transforming points from one frame to another

- **VERY IMPORTANT AND USEFUL**

- Suppose you have a point in the Body frame, ${}^{B}\mathbf{p}$ which you want to transform/express in the World frame. Then you can do any of the two following options:

$$ {}^{W}\mathbf{p} = {}^{W}_{B}\mathbf{R}\,{}^{B}\mathbf{p} + {}^{W}\mathbf{t}_{WB} \qquad\qquad {}^{W}\mathbf{p} = {}^{W}_{B}\mathbf{R}({}^{B}\mathbf{p} - {}^{B}\mathbf{t}_{BW}) $$

- Think of it as first rotating the point to be in the World frame and then adding to it the translation from Body to World.

Transforming vectors from one frame to another

- **VERY IMPORTANT AND USEFUL**

- Suppose you have a vector in the Body frame, ${}^{B}\mathbf{v}$ which you want to transform/express in the World frame. Then

$$
{}^{W}\mathbf{v} = {}^{W}_{B}\mathbf{R}\,{}^{B}\mathbf{v}
$$

Combining rotations and translation into one transformation

- **VERY IMPORTANT AND USEFUL**

- Many times we combine the rotation and translation of a rigid motion into a 4x4 homogeneous matrix

$$
{}^{W}_{B}\mathbf{T} = \begin{bmatrix} {}^{W}_{B}\mathbf{R} & {}^{W}\mathbf{t}_{WB} \\ \mathbf{0} & 1 \end{bmatrix}
$$

Main advantage of homogeneous transformations: easy composition

$$ {}^{W}_{B}\mathbf{T} = {}^{W}_{A}\mathbf{T}\,{}^{A}_{B}\mathbf{T} $$

Composing rigid motions now becomes a series of matrix multiplications

Inverting a homogeneous transformation

- Be careful:

$$ {}^{W}_{B}\mathbf{T}^{-1} \neq {}^{W}_{A}\mathbf{T}^{t} $$

as was the case with rotation matrices.

Suggested minimum documentation

- Frame diagram.
- Full description of how to build a transformation matrix from the provided scalars and down to the scalar level.
- A clear statement of which transformation matrix it is.

The resulting matrix, $\boldsymbol{T}_{\mathtt{WB}}$, represents the pose of the robot body frame, $\underrightarrow{\mathcal{F}}_{\mathtt{B}}$, with respect to the world frame, $\underrightarrow{\mathcal{F}}_{\mathtt{W}}$, such that a point in the body frame, ${}_{\mathtt{B}}\mathbf{p}$, can be transformed into the world frame by

$$ {}_{\mathtt{W}}\mathbf{p} = \boldsymbol{T}_{\mathtt{WB}}\,{}_{\mathtt{B}}\mathbf{p}. \tag{1} $$

Let's talk about code.

- Code has the same requirements as notation
- Rotation matrices have two frame decorations:
  - to
  - from
- Coordinates of vectors have three decorations:
  - to
  - from
  - expressed in

Let’s talk about code.

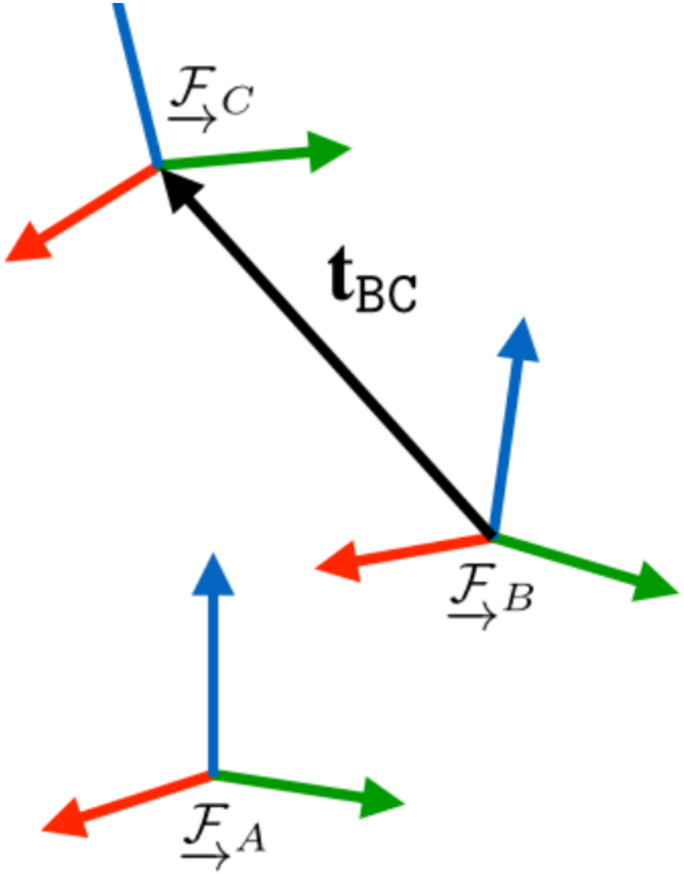
Physical quantity

$${}_{A}\mathbf{t}_{BC}$$

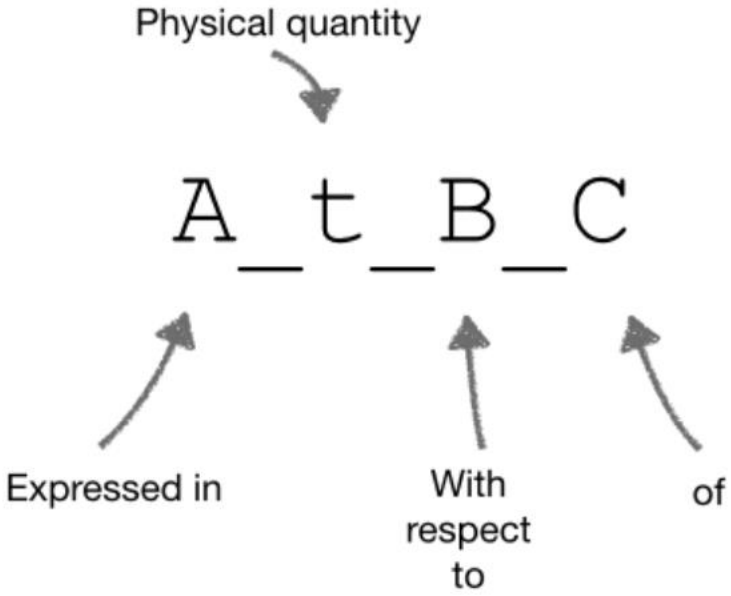
Expressed in

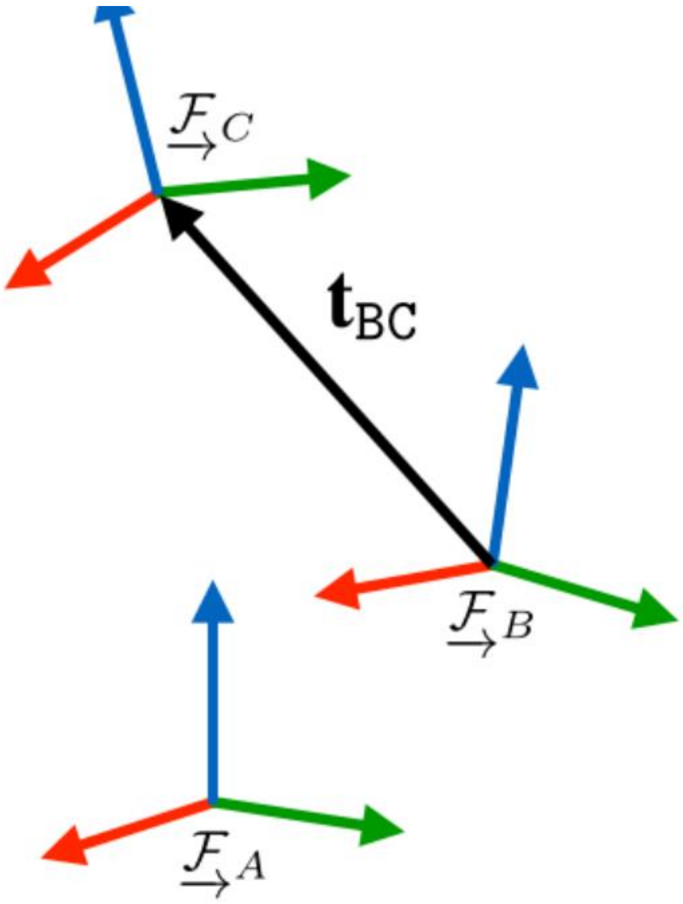
With respect to

of



Let's talk about code.





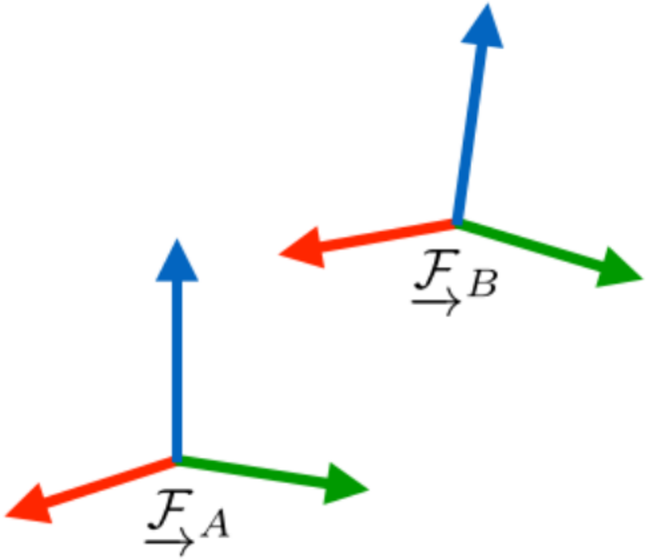
Let’s talk about code.

$$\mathbf{C}_{\mathtt{AB}}$$

into

rotates vectors from

$${}_{\mathtt{A}}\mathbf{v}_{\mathtt{BC}} = \mathbf{C}_{\mathtt{AB}}\,{}_{\mathtt{B}}\mathbf{v}_{\mathtt{BC}}$$



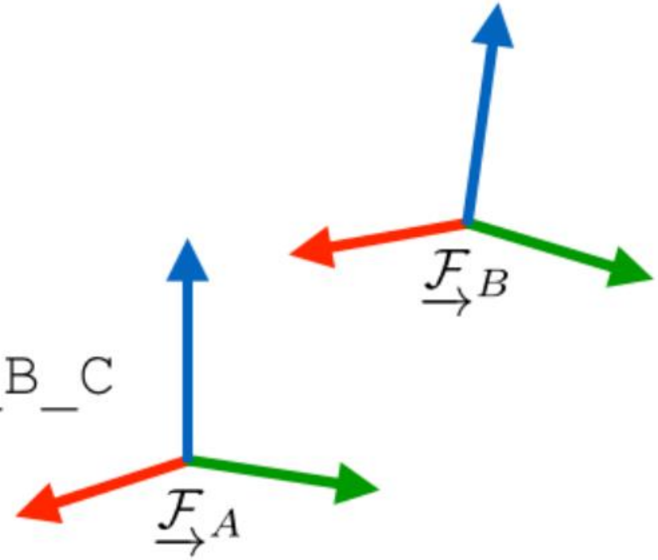
Let's talk about code.

```
C_A_B
```

into

rotates vectors from

```
A_v_B_C = C_A_B * B_v_B_C
```



Let's talk about code.

- Comments

```
/// Coordinate frames in this function:
///    - C : The camera frame, indexed by time, k.
///    - W : The world frame.
Point pointToCamera( const Transformation& T_W_Ckm1,
                     const Transformation& T_Ckm1_Ck,
                     const Transformation& T_Ck_Ckp1,
                     const Point& W_p ) {

  Transformation T_Ckp1_W = (T_W_Ckm1 * T_Ckm1_Ck * T_Ck_Ckp1).inverse();
  return T_Ckp1_W * W_p

}
```

$\boldsymbol{T}_{\mathtt{W}\mathtt{C}_{k-1}}$

$\boldsymbol{T}_{\mathtt{C}_{k-1}\mathtt{C}_{k}}$

$\boldsymbol{T}_{\mathtt{C}_{k}\mathtt{C}_{k+1}}$

${}_{\mathtt{W}}\mathbf{p}$

Let’s talk about code.

Choose an expressive coding style.
Explain it clearly.
Stick with it.