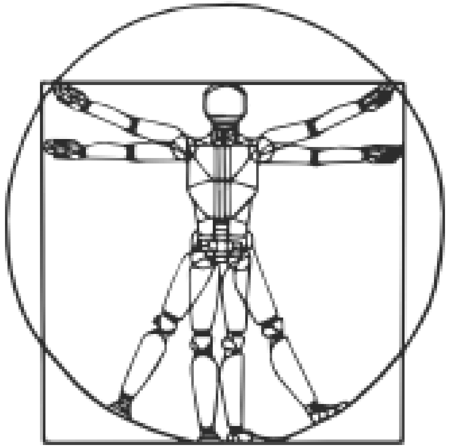

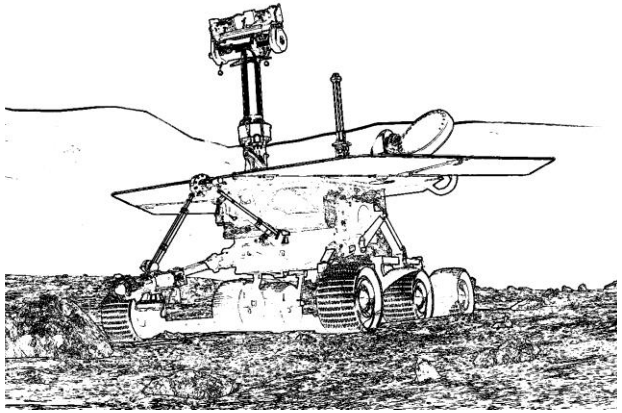

COMP417
Introduction to Robotics and Intelligent Systems

Lecture 12: Least Squares Estimation

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Estimating parameters of probability models

- In the occupancy grid mapping problem we wanted to compute $p(\mathbf{m}|\mathbf{z}_{1:t}, \mathbf{x}_{1:t})$ over all possible maps.

- We can see this problem as a specific instance within a category of problems where we are given data (observations) and we want to "explain" or fit the data using a parametric function.

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- We can see this problem as a specific instance within a category of problems where we are given data (observations) and we want to "explain" or fit the data using a parametric function.

- There are typically three ways to work with this type of problems:
  1. Maximum Likelihood parameter estimation (MLE)
     - Least Squares
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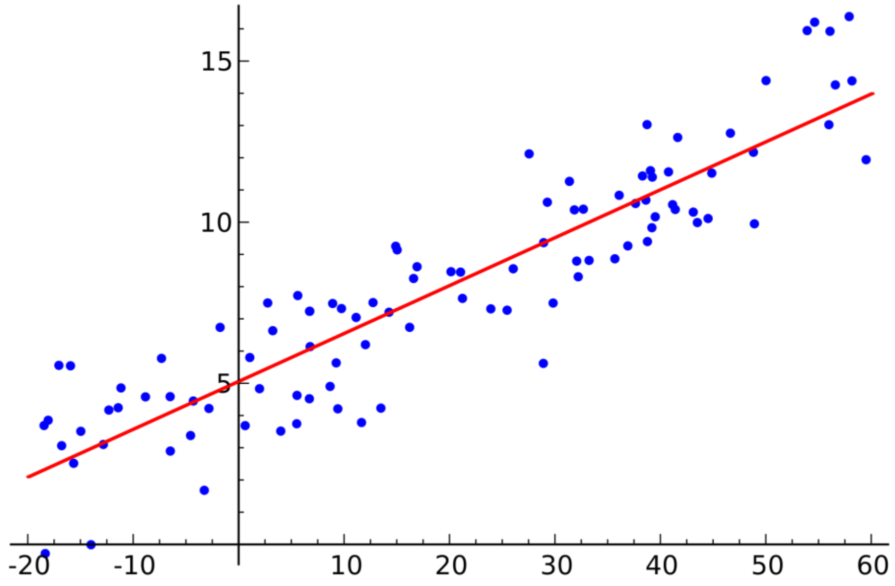
  3. Bayesian parameter distribution estimation

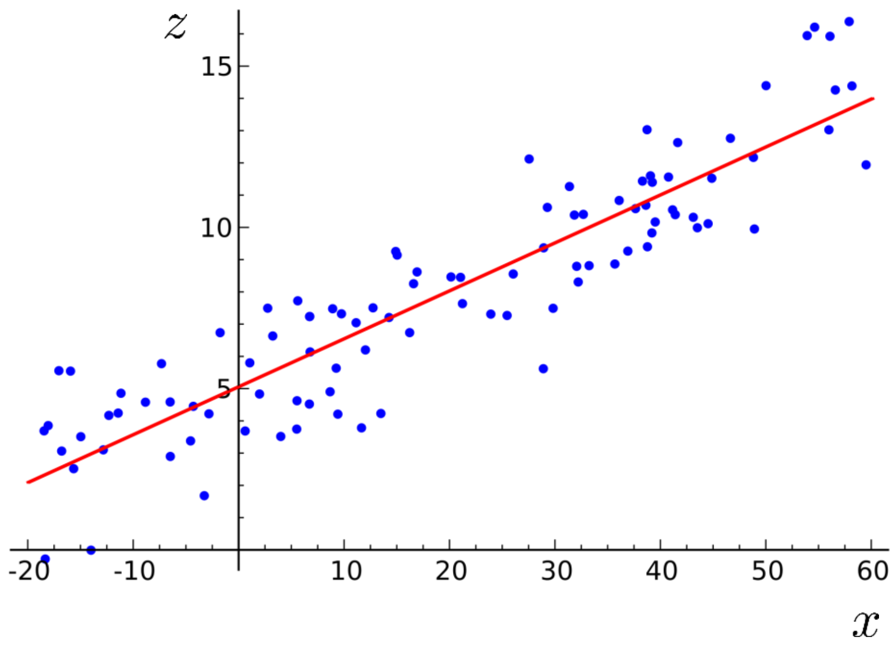
Least Squares Parameter Estimation



We are given data points $(\mathbf{x}_1, \mathbf{z}_1), \ldots, (\mathbf{x}_N, \mathbf{z}_N)$

We **think** that the data was generated by a parametric function $\mathbf{z} = \mathbf{h}(\boldsymbol{\theta}, \mathbf{x})$

Least Squares Parameter Estimation



Example: we think that the 2D data was generated by a line $z = \theta_0 + \theta_1 x$ whose parameters we do not know, and was corrupted by noise.

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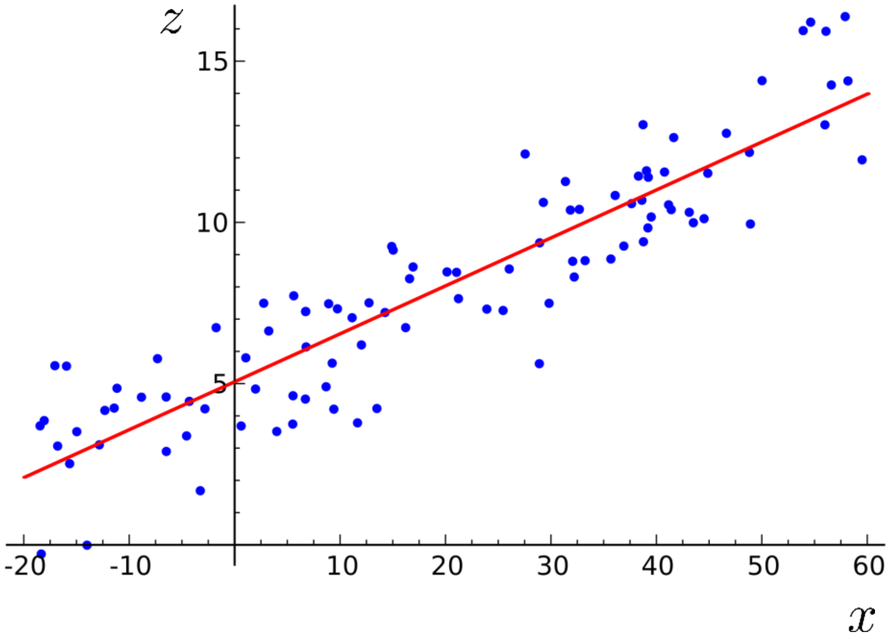

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We are given data points $(\mathbf{x}_1, \mathbf{z}_1), \ldots, (\mathbf{x}_N, \mathbf{z}_N)$

We **think** that the data was generated by a parametric function $\mathbf{z} = \mathbf{h}(\boldsymbol{\theta}, \mathbf{x})$

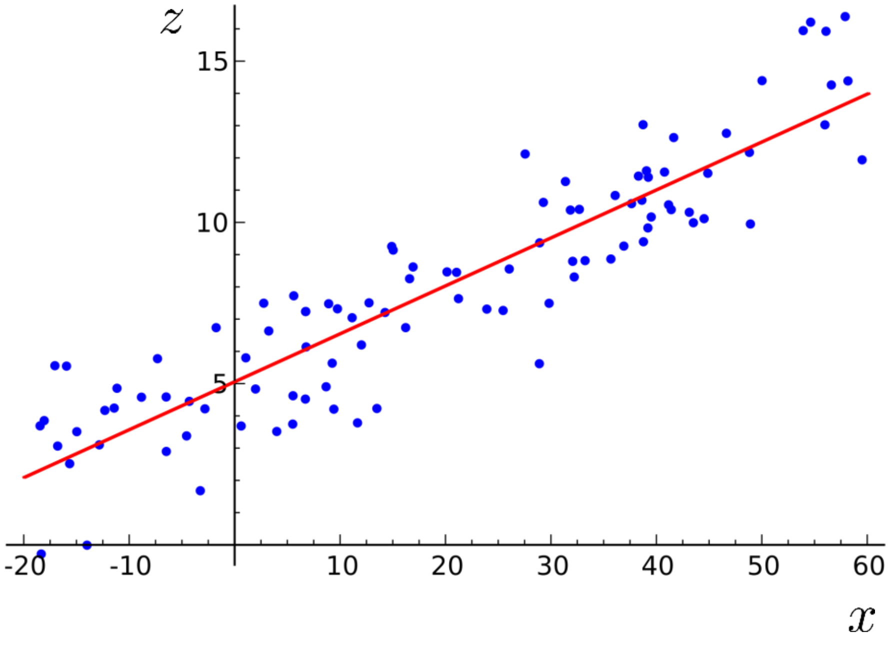
This parametric model will have a fitting error:

$$e(\boldsymbol{\theta}) = \sum_{i=1}^{N} ||\mathbf{z}_i - \mathbf{h}(\boldsymbol{\theta}, \mathbf{x}_i)||^2$$

The least-squares estimator is:

$$\boldsymbol{\theta}_{LS} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\ e(\boldsymbol{\theta})$$

Linear Least Squares Parameter Estimation



Example: we think that the 2D data was generated by a line $z = \theta_0 + \theta_1 x$ whose parameters we do not know.

We are given data points $(\mathbf{x}_1, \mathbf{z}_1), ..., (\mathbf{x}_N, \mathbf{z}_N)$

We **think** that the data was generated by a linear parametric function $\mathbf{z} = \mathbf{h}(\boldsymbol{\theta}, \mathbf{x}) = \mathbf{H}_{\mathbf{x}}\boldsymbol{\theta}$
where $\mathbf{H}_{\mathbf{x}}$ is a matrix whose elements depend on $\mathbf{x}$

This parametric model will have a fitting error:

$$e(\boldsymbol{\theta}) = \sum_{i=1}^{N} ||\mathbf{z}_i - \mathbf{H}_{\mathbf{x}_i}\boldsymbol{\theta}||^2$$

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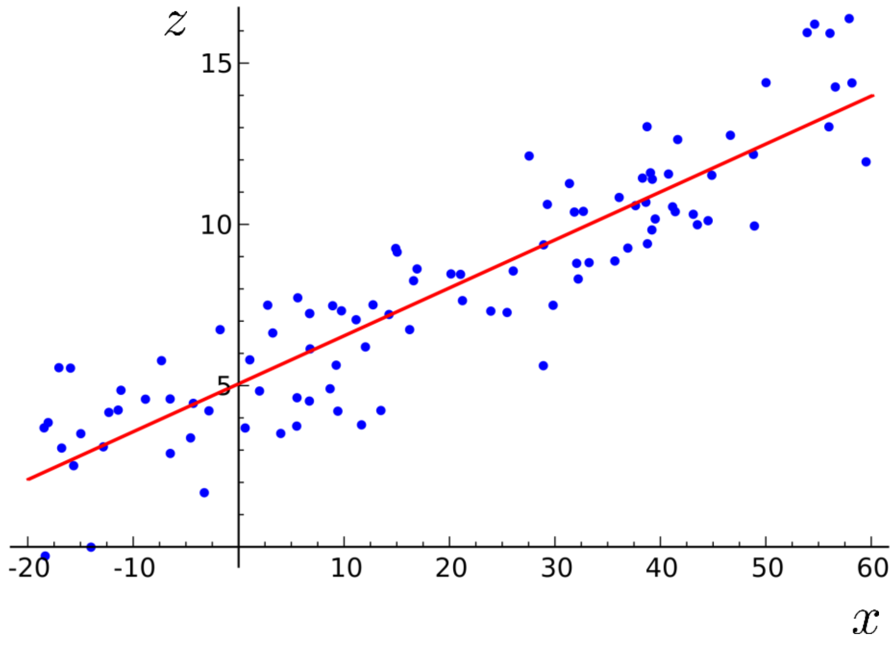
Linear Least Squares Parameter Estimation



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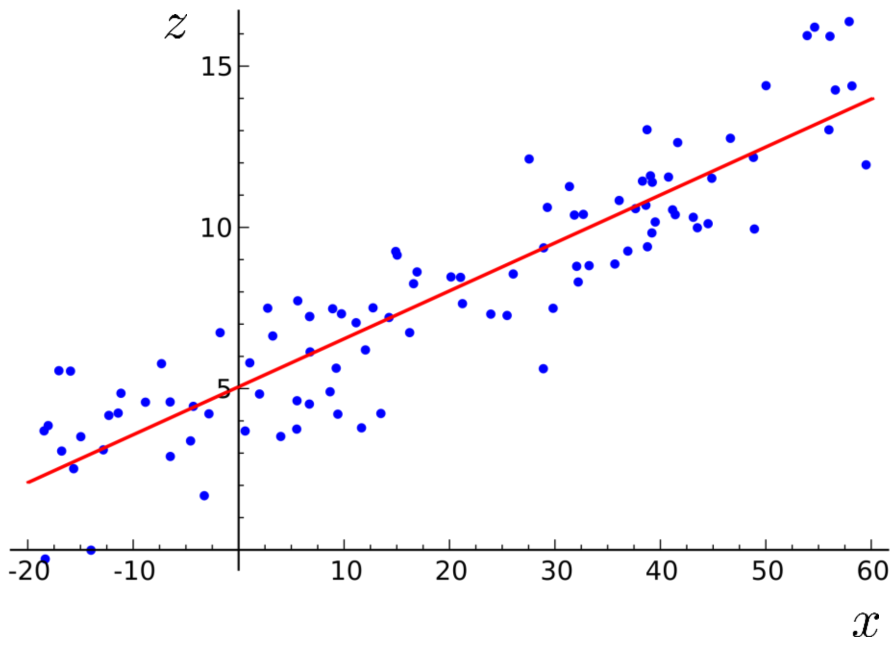
This parametric model will have a fitting error:

$$
\begin{aligned}
e(\boldsymbol{\theta}) &= \sum_{i=1}^{N} ||\mathbf{z}_i - \mathbf{H}_{\mathbf{x}_i}\boldsymbol{\theta}||^2 \\
&= \sum_{i=1}^{N} \mathbf{z}_i^T \mathbf{z}_i - 2\boldsymbol{\theta}^T \mathbf{H}_{\mathbf{x}_i}^T \mathbf{z}_i + \boldsymbol{\theta}^T \mathbf{H}_{\mathbf{x}_i}^T \mathbf{H}_{\mathbf{x}_i} \boldsymbol{\theta}
\end{aligned}
$$

Linear Least Squares Parameter Estimation



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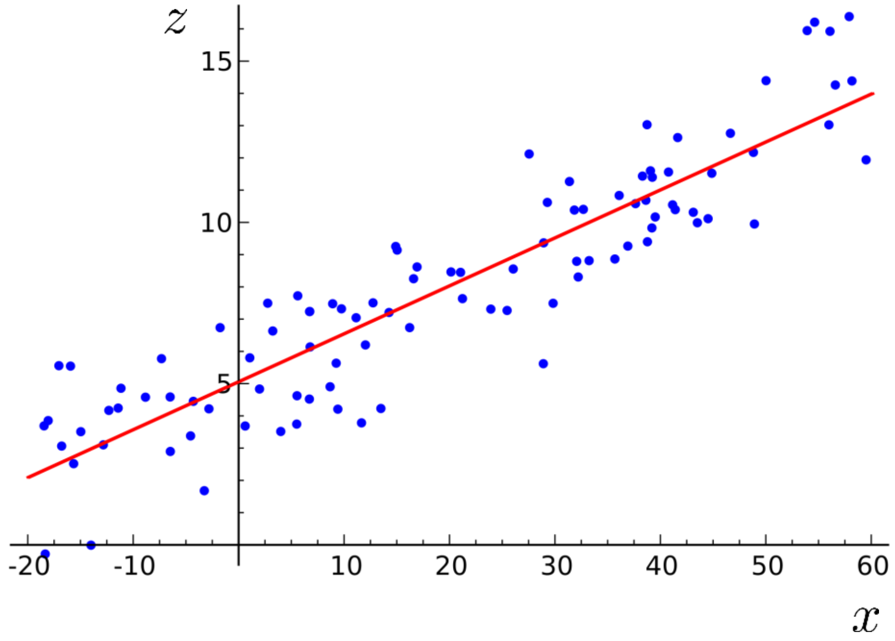
$$\begin{aligned} e(\boldsymbol{\theta}) &= \sum_{i=1}^{N} ||\mathbf{z}_i - \mathbf{H}_{\mathbf{x}_i}\boldsymbol{\theta}||^2 \\ &= \sum_{i=1}^{N} \mathbf{z}_i^T \mathbf{z}_i - 2\boldsymbol{\theta}^T \mathbf{H}_{\mathbf{x}_i}^T \mathbf{z}_i + \boldsymbol{\theta}^T \mathbf{H}_{\mathbf{x}_i}^T \mathbf{H}_{\mathbf{x}_i}\boldsymbol{\theta} \end{aligned}$$

The least-squares estimator minimizes the error:

$$\frac{\partial e(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{0} \Leftrightarrow -2\sum_{i=1}^{N} \mathbf{H}_{\mathbf{x}_i}^T \mathbf{z}_i + 2\mathbf{H}_{\mathbf{x}_i}^T \mathbf{H}_{\mathbf{x}_i}\boldsymbol{\theta} = \mathbf{0} \Leftrightarrow \left[\sum_{i=1}^{N} \mathbf{H}_{\mathbf{x}_i}^T \mathbf{H}_{\mathbf{x}_i}\right]\boldsymbol{\theta} = \sum_{i=1}^{N} \mathbf{H}_{\mathbf{x}_i}^T \mathbf{z}_i$$

Linear Least Squares Parameter Estimation



Example: we think that the 2D data was generated by a line $z = \theta_0 + \theta_1 x$ whose parameters we do not know.

We are given data points $(\mathbf{x}_1, \mathbf{z}_1), ..., (\mathbf{x}_N, \mathbf{z}_N)$

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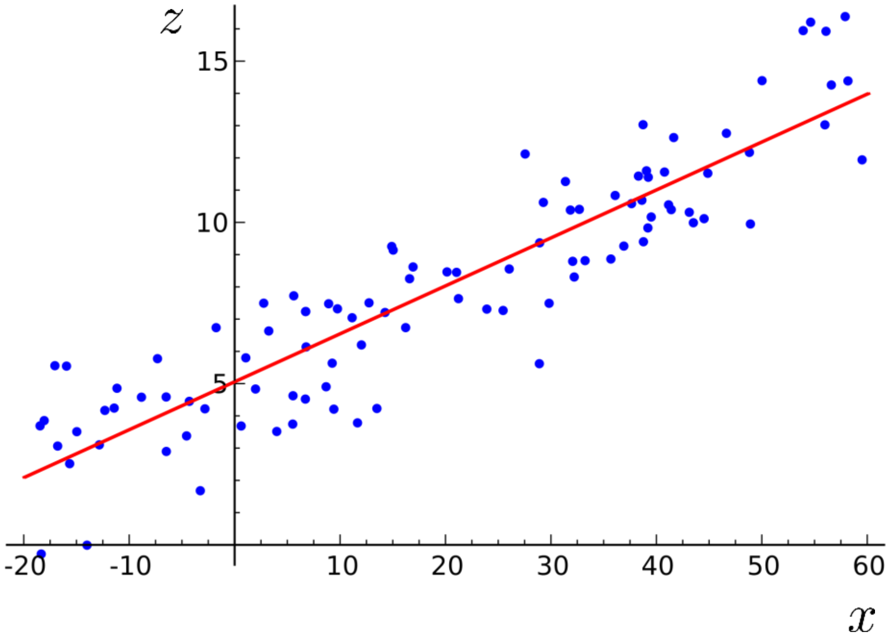
$$\boldsymbol{\theta}_{LS} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\, e(\boldsymbol{\theta}) \Leftrightarrow \left[\sum_{i=1}^{N} \mathbf{H}_{\mathbf{x}_i}^T \mathbf{H}_{\mathbf{x}_i}\right] \boldsymbol{\theta}_{LS} = \sum_{i=1}^{N} \mathbf{H}_{\mathbf{x}_i}^T \mathbf{z}_i$$

Example #1: Linear Least Squares



Example: we think that the 2D data was generated by a line $z = \theta_0 + \theta_1 x$ whose parameters we do not know.

We are given 2D data points $(x_1, z_1), ..., (x_N, z_N)$

We **think** that the data was generated by a linear parametric function $z = h(\boldsymbol{\theta}, x) = [1 \quad x]\boldsymbol{\theta} = \theta_0 + \theta_1 x$

This parametric model will have a fitting error:

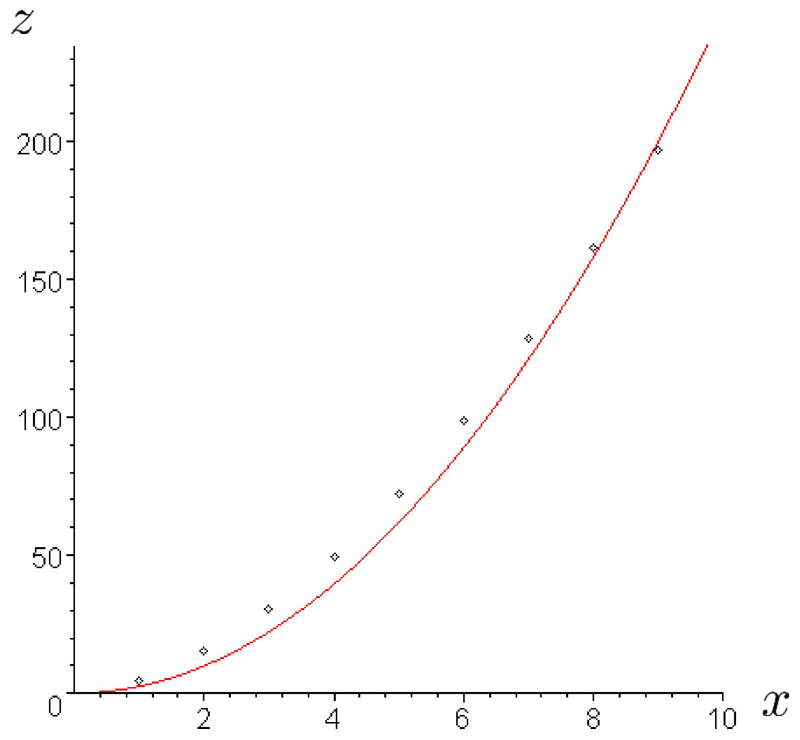
$$e(\theta_0, \theta_1) \quad = \quad \sum_{i=1}^{N} (z_i - \theta_0 - \theta_1 x_i)^2$$

The least-squares estimator minimizes the error:

$$\boldsymbol{\theta}_{LS} = \underset{\theta_0, \theta_1}{\operatorname{argmin}}\ e(\theta_0, \theta_1) \Leftrightarrow \left[\sum_{i=1}^{N} \begin{bmatrix} 1 \\ x_i \end{bmatrix} \begin{bmatrix} 1 & x_i \end{bmatrix} \right] \boldsymbol{\theta}_{LS} = \sum_{i=1}^{N} \begin{bmatrix} 1 \\ x_i \end{bmatrix} z_i$$

Which is a linear system of 2 equations. If we have at least two data points we can solve for $\boldsymbol{\theta}_{LS}$ to define the line.

Example #2: Linear Least Squares



Example: we think that the 2D data was generated by a quadratic $z = \theta_0 + \theta_1 x + \theta_2 x^2$ whose parameters we do not know.

We are given 2D data points $(x_1, z_1), \ldots, (x_N, z_N)$

We **think** that the data was generated by a linear parametric function $z = h(\boldsymbol{\theta}, x) = [1 \quad x \quad x^2]\boldsymbol{\theta} = \theta_0 + \theta_1 x + \theta_2 x^2$

This parametric model will have a fitting error:

$$e(\theta_0, \theta_1, \theta_2) \quad = \quad \sum_{i=1}^{N} (z_i - \theta_0 - \theta_1 x_i - \theta_2 x_i^2)^2$$

The least-squares estimator minimizes the error:

$$\boldsymbol{\theta}_{LS} = \underset{\theta_0, \theta_1, \theta_2}{\operatorname{argmin}}\; e(\theta_0, \theta_1, \theta_2) \Leftrightarrow \left[\sum_{i=1}^{N} \begin{bmatrix} 1 \\ x_i \\ x_i^2 \end{bmatrix} [1 \quad x_i \quad x_i^2]\right] \boldsymbol{\theta}_{LS} = \sum_{i=1}^{N} \begin{bmatrix} 1 \\ x_i \\ x_i^2 \end{bmatrix} z_i$$

Which is a linear system of 3 equations. If we have at least three data points we can solve for $\boldsymbol{\theta}_{LS}$ to define the quadratic.

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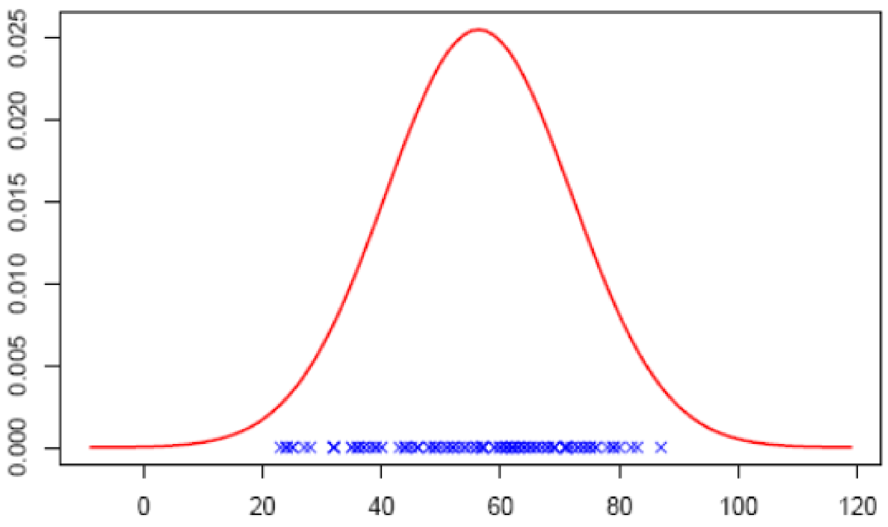
Maximum Likelihood Parameter Estimation



We are given data points $\mathbf{d}_{1:N} = \mathbf{d}_1, ..., \mathbf{d}_N$

We **think** the data has been generated from a probability distribution $p(\mathbf{d}_{1:N}|\boldsymbol{\theta})$

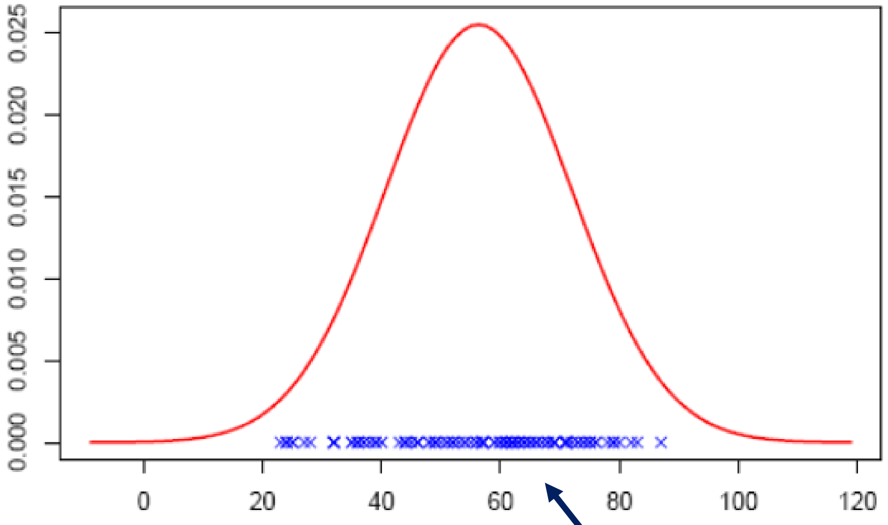
We want to find the parameter of the model that maximizes the likelihood function of the data

$$L(\boldsymbol{\theta}) = p(\mathbf{d}_{1:N}|\boldsymbol{\theta})$$

which is a function of theta, **not** a probability distribution.

$$\boldsymbol{\theta}_{MLE} = \underset{\boldsymbol{\theta}}{\text{argmax}}\, p(\mathbf{d}_{1:N}|\boldsymbol{\theta})$$

Maximum Likelihood Parameter Estimation



Data points

$$\boldsymbol{\theta}_{MLE} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\ p(\mathbf{d}_{1:N}|\boldsymbol{\theta})$$

Find the parameters of the model that maximize the likelihood function of the data

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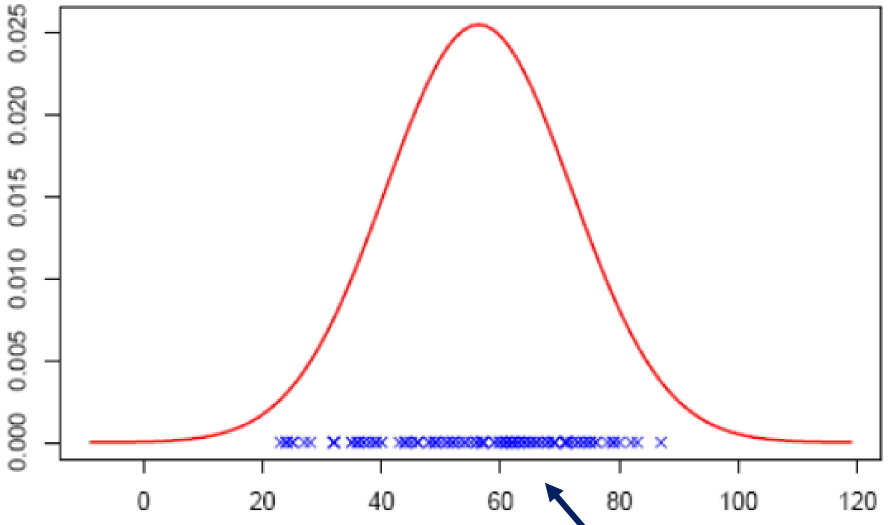
Example: assume we know that 1D data points were generated <u>independently</u> from a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, but we don't know the mean and variance. The likelihood function of the data is

Maximum Likelihood Parameter Estimation



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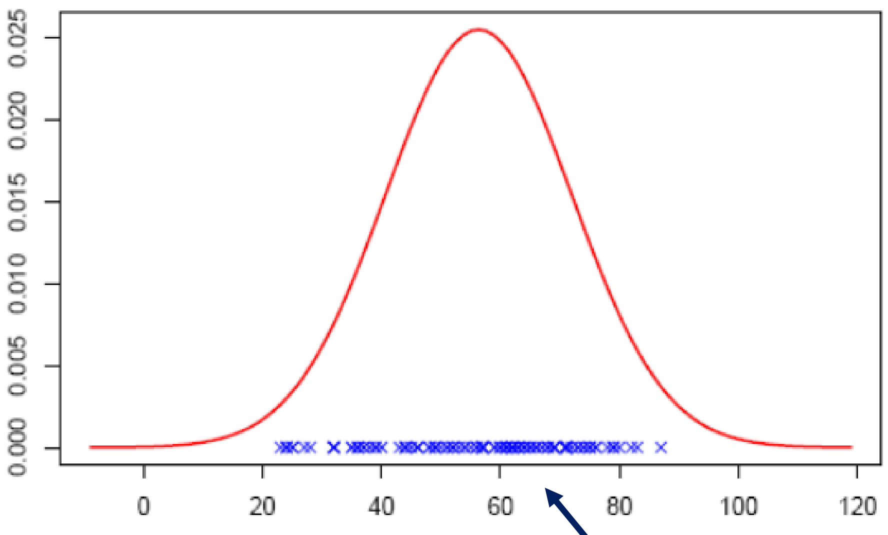
$$L(\mu, \sigma) = p(\mathbf{d}_{1:N}|\mu, \sigma) = \prod_{i=1}^{N} p(d_i|\mu, \sigma) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma}} \exp(-0.5(d_i - \mu)^2/\sigma^2)$$

Maximum Likelihood Parameter Estimation



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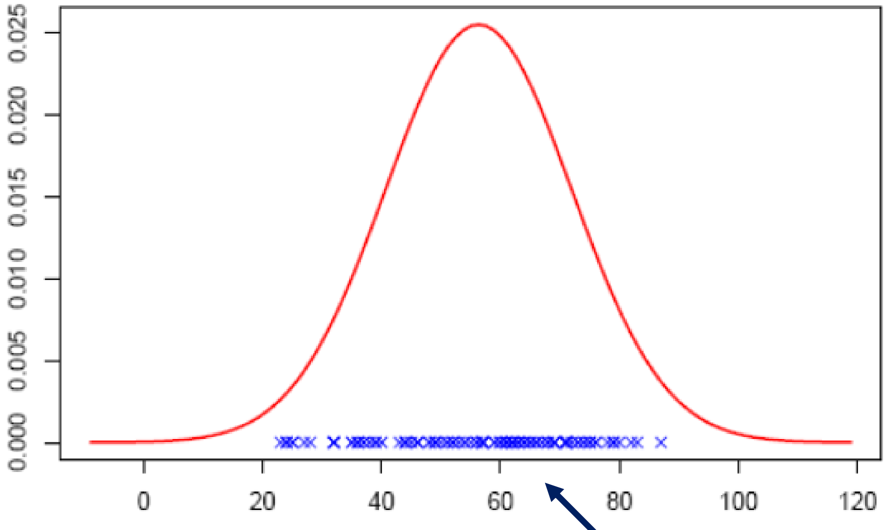
$$L(\mu,\sigma) = p(\mathbf{d}_{1:N}|\mu,\sigma) = \prod_{i=1}^{N} p(d_i|\mu,\sigma) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp(-0.5(d_i-\mu)^2/\sigma^2)$$

And the maximum-likelihood parameter estimates are

$$(\mu,\sigma)_{MLE} = \underset{\mu,\sigma}{\operatorname{argmax}}\; p(\mathbf{d}_{1:N}|\mu,\sigma) = \underset{\mu,\sigma}{\operatorname{argmax}}\; \log p(\mathbf{d}_{1:N}|\mu,\sigma) = \underset{\mu,\sigma}{\operatorname{argmax}} \sum_{i=1}^{N} \log p(d_i|\mu,\sigma)$$

Maximum Likelihood Parameter Estimation



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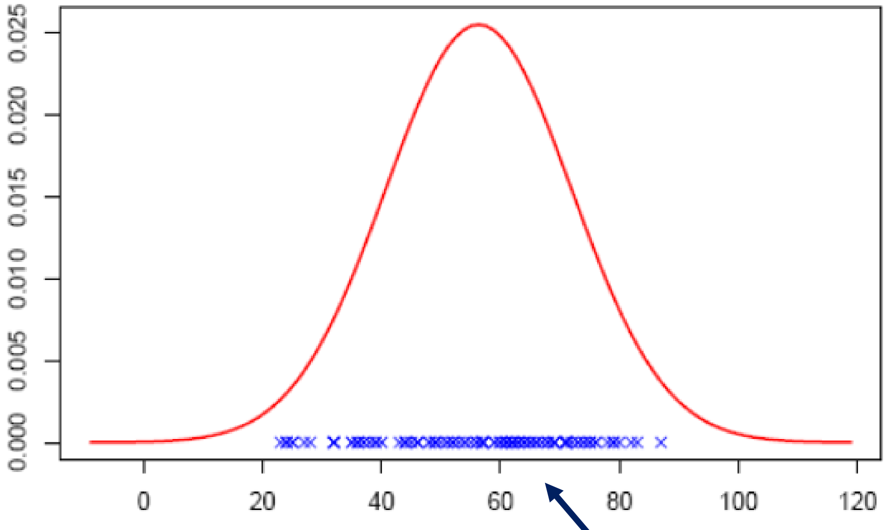
Example: assume we know that 1D data points were generated <u>independently</u> from a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, but we don't know the mean and variance. The likelihood function of the data is

$$L(\mu,\sigma) = p(\mathbf{d}_{1:N}|\mu,\sigma) = \prod_{i=1}^{N} p(d_i|\mu,\sigma) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma}\exp(-0.5(d_i-\mu)^2/\sigma^2)$$

And the maximum-likelihood parameter estimates are

$$(\mu,\sigma)_{MLE} = \underset{\mu,\sigma}{\operatorname{argmax}} \sum_{i=1}^{N} \log p(d_i|\mu,\sigma) = \underset{\mu,\sigma}{\operatorname{argmax}} \left[-N\log(\sqrt{2\pi}\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(d_i-\mu)^2\right]$$

Maximum Likelihood Parameter Estimation



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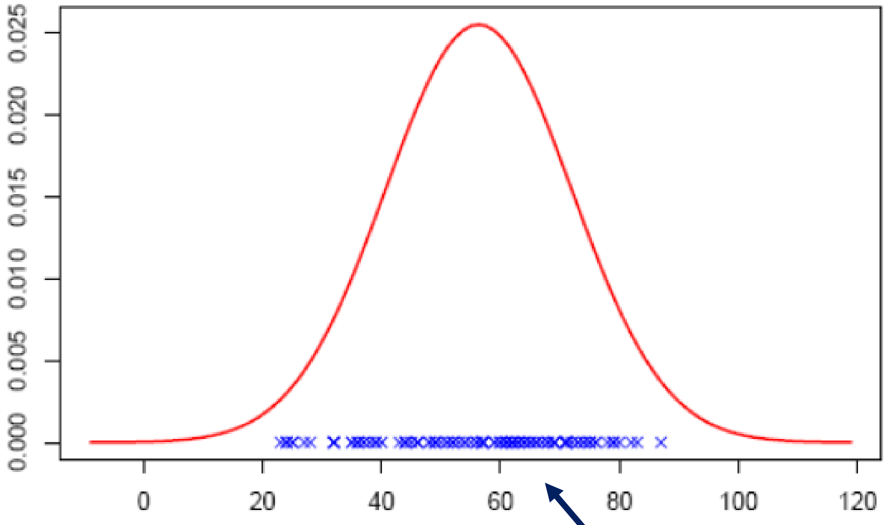
Set partial derivatives w.r.t. μ and σ to zero

$$\mu_{MLE} = \sum_{i=1}^{N} d_i/N$$

$$\sigma^2_{MLE} = \frac{1}{N}\sum_{i=1}^{N}(d_i-\mu_{MLE})^2$$

Least Squares as Maximum Likelihood



$$\boldsymbol{\theta}_{MLE} = \underset{\boldsymbol{\theta}}{\text{argmax}}\ p(\mathbf{d}_{1:N}|\boldsymbol{\theta})$$

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Least squares estimation occurs in maximum likelihood with Gaussian models of data

Estimating parameters of probability models

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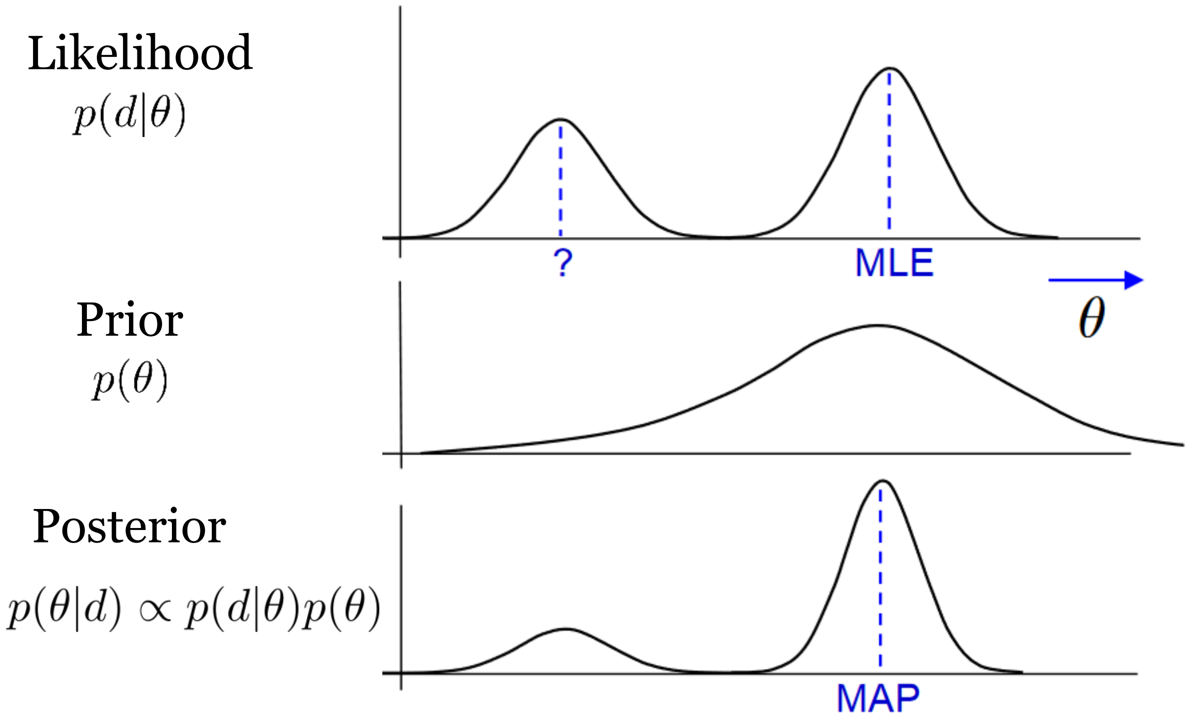
Maximum A Posteriori Parameter Estimation



$$\begin{aligned}
\boldsymbol{\theta}_{MAP} &= \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\; p(\boldsymbol{\theta}|\mathbf{d}_{1:N}) \\
&= \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \left[\frac{p(\mathbf{d}_{1:N}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{d}_{1:N})}\right] \\
&= \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\; [p(\mathbf{d}_{1:N}|\boldsymbol{\theta})p(\boldsymbol{\theta})]
\end{aligned}$$

Almost the same as MLE, but with a prior distribution on the parameters

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Bayesian parameter estimation

- Both MLE and MAP estimators give you a single **point estimate**.
- But there might be many parameters that are compatible with the data.
- Instead of point estimates, compute a **distribution of estimates** that explain the data

- Bayesian parameter estimation:

$$p(\boldsymbol{\theta}|\mathbf{d}_{1:N}) = \frac{p(\mathbf{d}_{1:N}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{d}_{1:N})}$$

The probability of the data is usually hard to compute. But it does not depend on the parameter theta, so it is treated as a normalizing factor, and we can still compute how the posterior varies with theta.

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- Instead of point estimates, compute a **distribution of estimates** that explain the data

- Bayesian parameter estimation:

$$p(\boldsymbol{\theta}|\mathbf{d}_{1:N}) = \frac{p(\mathbf{d}_{1:N}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{d}_{1:N})}$$

- This is what we used in occupancy grid mapping, when we approximated

$$p(\mathbf{m}|\mathbf{z}_{1:t}, \mathbf{x}_{1:t})$$