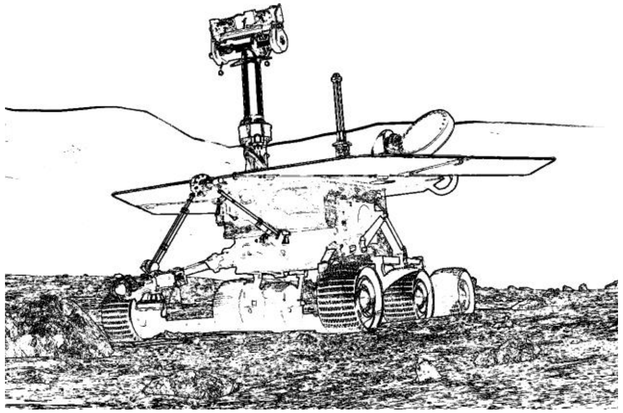


CSC477
Introduction to Mobile Robotics

Florian Shkurti

Week #10: Particle Filters

Recommended reading

- Lesson 3 in https://www.udacity.com/course/artificial-intelligence-for-robotics--cs373

- Chapters 4.3 and 8.3 in the Probabilistic Robotics textbook

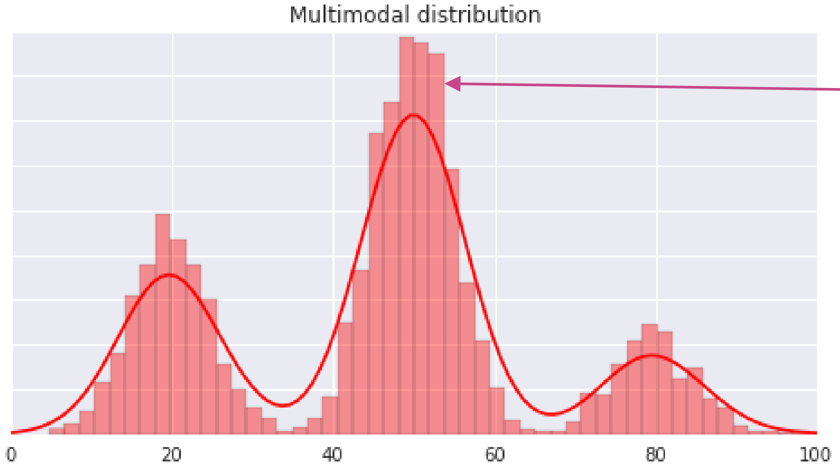
KF vs EKF vs PF

	Kalman Filter	Extended Kalman Filter	Particle Filter
Dynamics model	Linear	Nonlinear	Nonlinear
Sensor model	Linear	Nonlinear	Nonlinear
Noise	Gaussian (Unimodal)	Gaussian (Unimodal)	Multimodal

One peak

Multiple peaks

How can we represent multimodal distributions?



https://www.opsclarity.com

Idea #1: Histograms

Advantages: the higher the number of bars the better the approximation is

Disadvantages: exponential dependence on number of dimensions

Note: this approach is called the Histogram Filter. It is useful for low-dimensional systems but we will not study it in this class.

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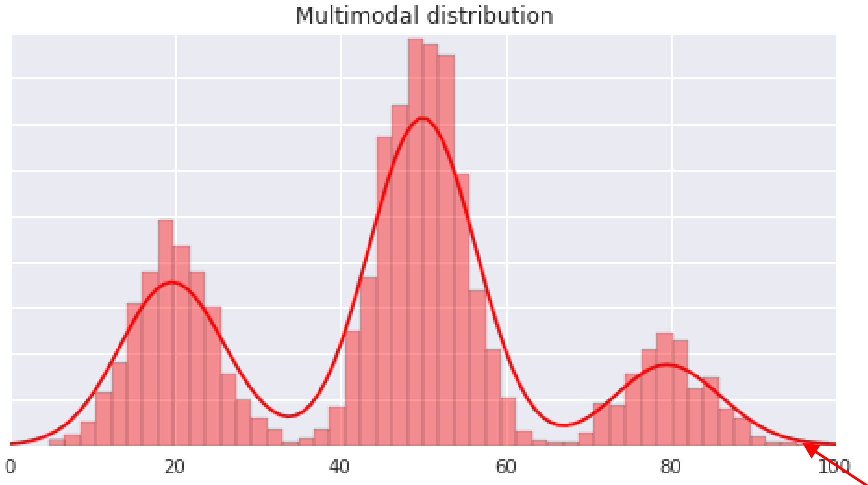

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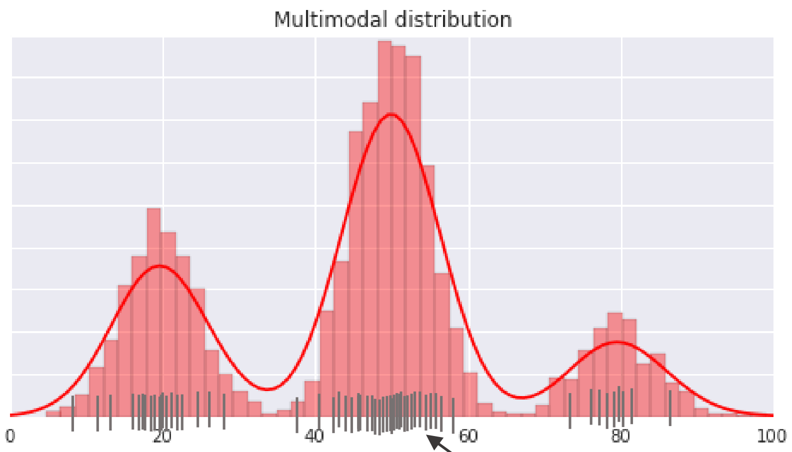

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Idea #2: Function Approximation

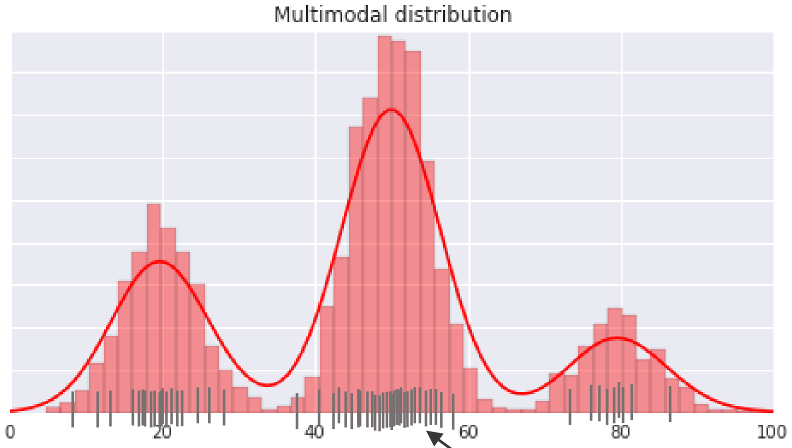
Idea #3: Weighted Particles

$$\{(x^{[1]}, w^{[1]}), \ldots, (x^{[M]}, w^{[M]})\}$$

Advantages: easy to predict/update by treating each particle as a separate hypothesis whose weight is updated.

Disadvantages: need enough particles to "cover" the distribution

How can we represent multimodal distributions?



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Higher density of particles means higher probability mass

$$
\begin{aligned}
bel(x_t) &= p(x_t | z_{0:t}, u_{0:t-1}) \\
&= \sum_{m=1}^{M} \begin{cases} w^{[m]}/W & \text{if } x_t = x_t^{[m]} \\ 0 & \text{o.w.} \end{cases}
\end{aligned}
$$

W = sum of all particles' weights

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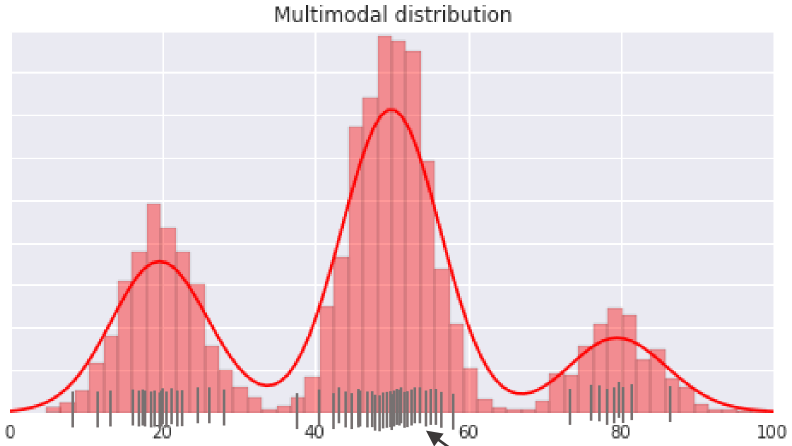

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Higher density of particles means higher probability mass

Want particles to be drawn from the belief at time t:

$$x_t^{[m]} \sim p(x_t | z_{0:t}, u_{0:t-1})$$

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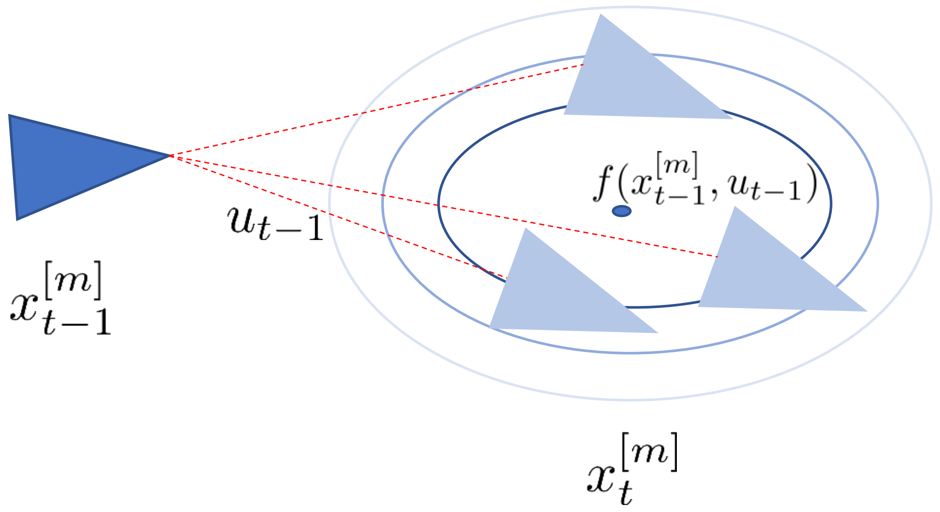
Particle propagation/prediction



Simulate what is going to happen to the particle at the next time step by drawing a sample from the next state specified in the dynamics (a.k.a. one-step simulator)

$$x_t^{[m]} \sim p(x_t \mid x_{t-1}^{[m]}, u_{t-1})$$

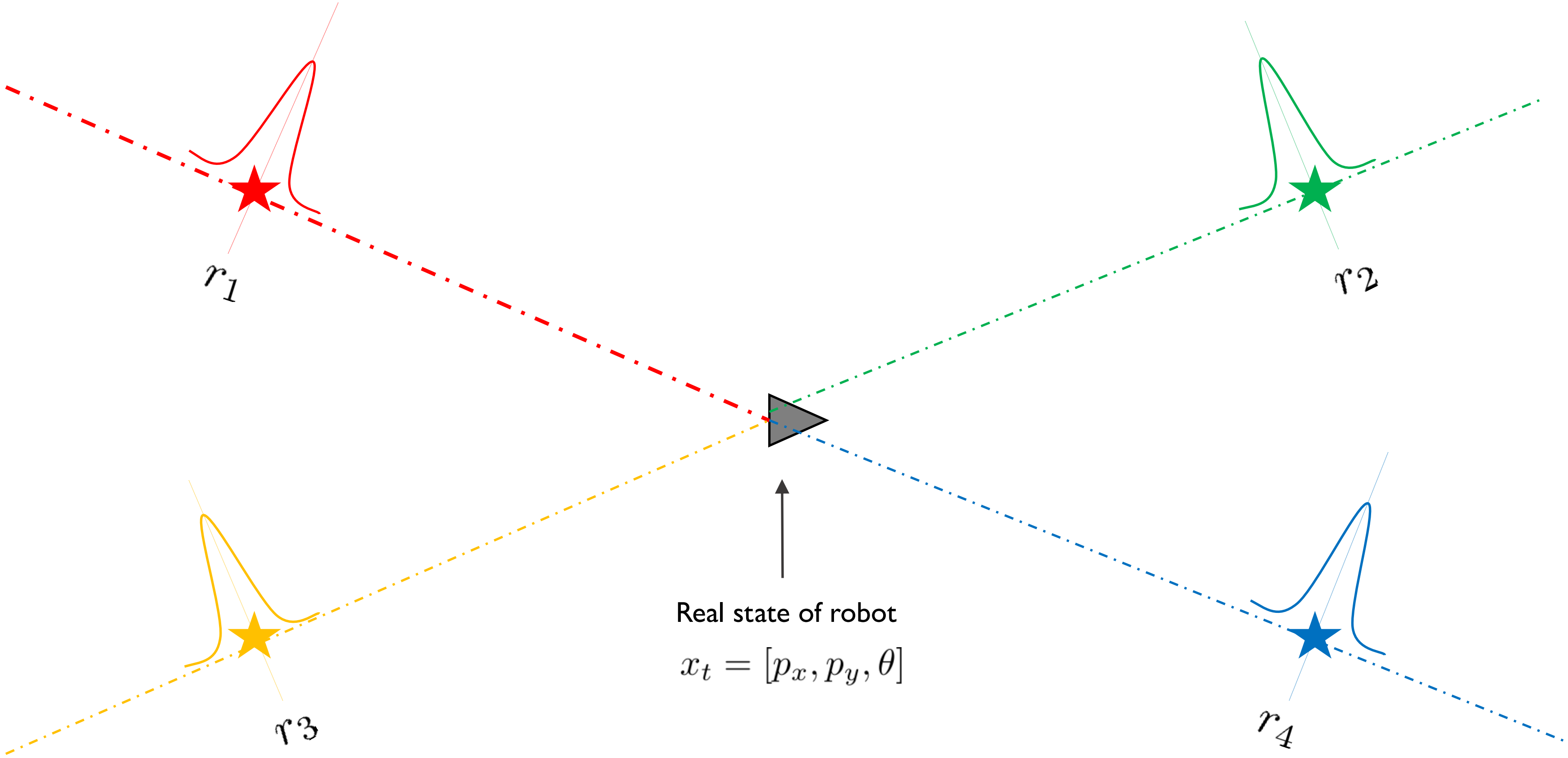
Usually

$$x_t^{[m]} = f(x_{t-1}^{[m]}, u_{t-1}) + w_{t-1}$$

$$w_{t-1} \sim \mathcal{N}(0, Q)$$

Particle Update

How to update particle weights after an observation

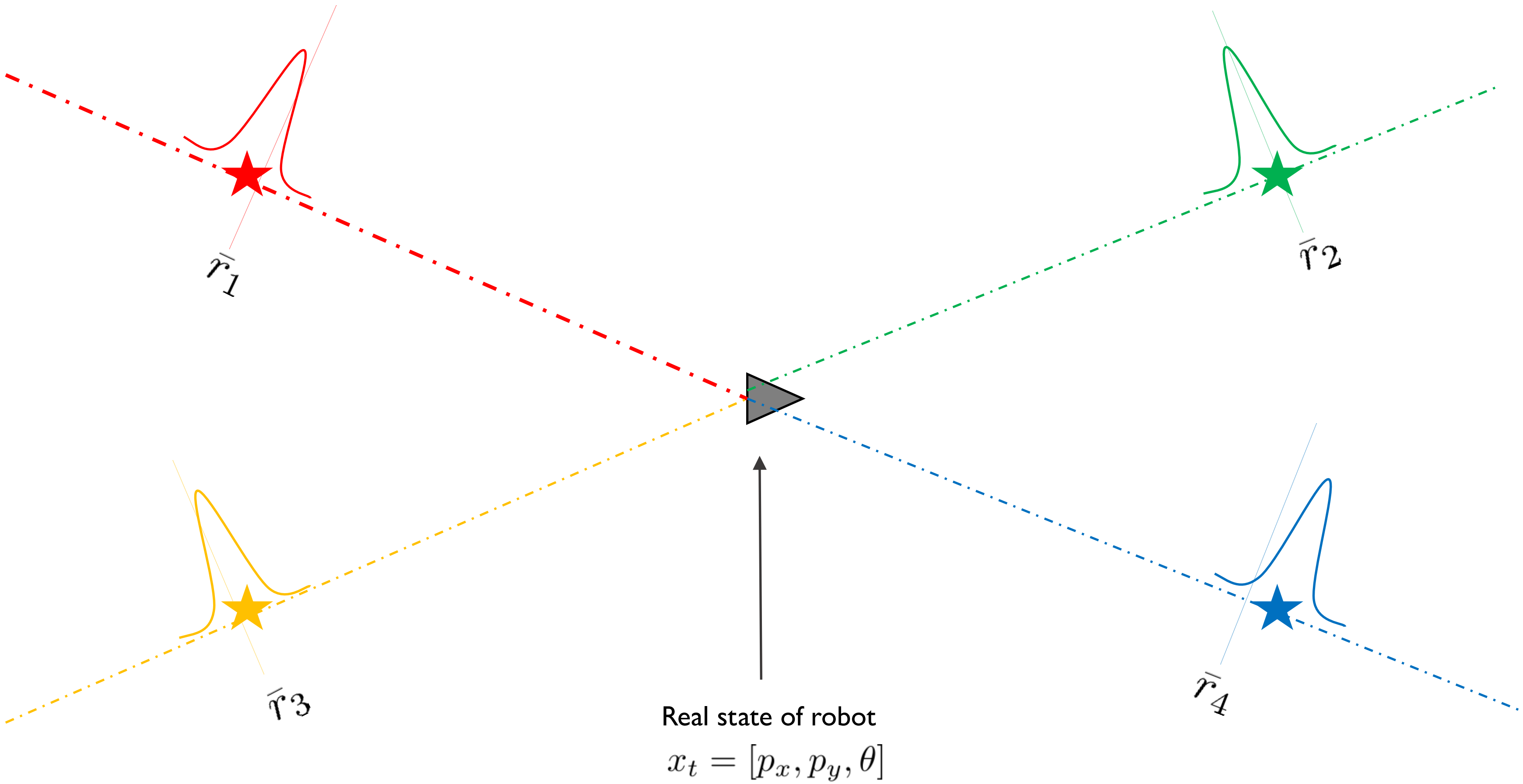


Measurement model $z_t = h(x_t) + n_t = [r_1, r_2, r_3, r_4] + n_t$

with $r_i = \sqrt{(p_x - l_x^{(i)})^2 + (p_y - l_y^{(i)})^2}$ and $n_i \sim \mathcal{N}(0, \sigma^2)$

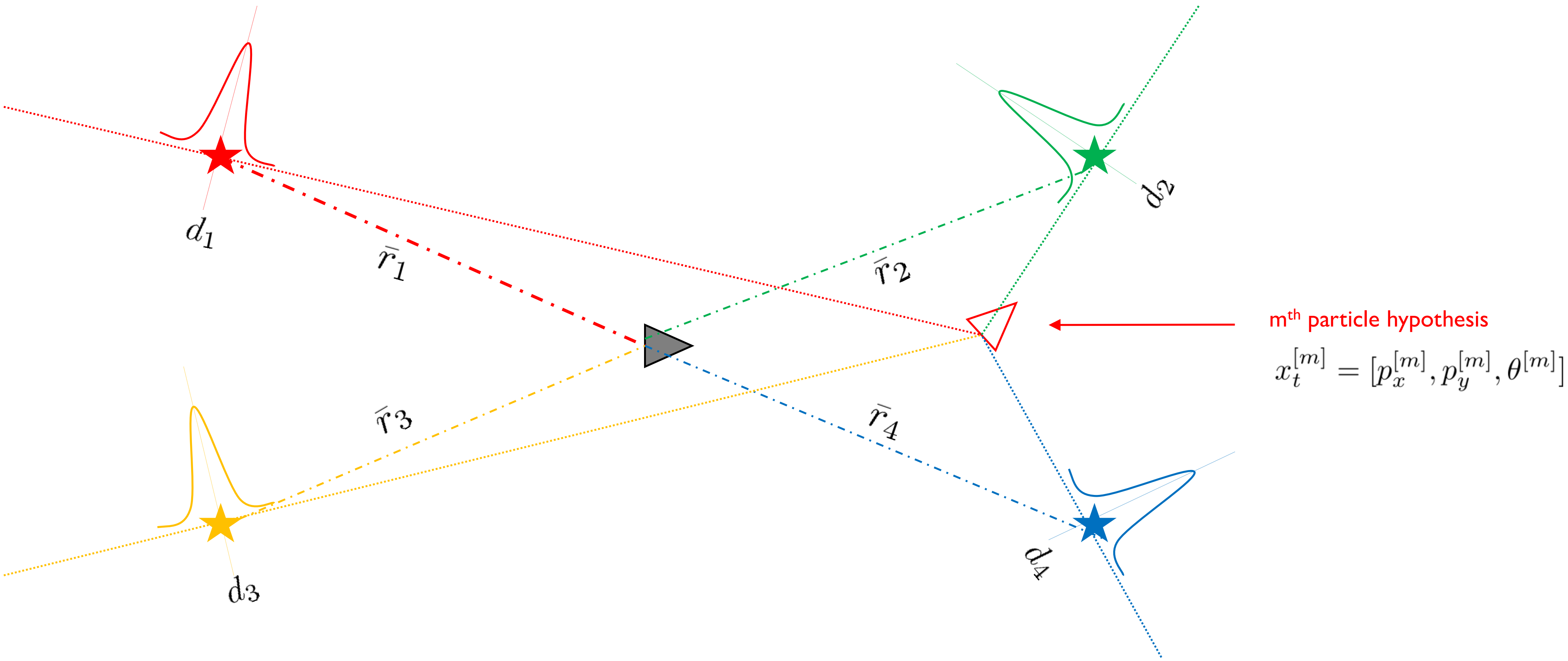
$$\Bigg\} \quad p(z_t | x_t) = \mathcal{N}(z_t;\ r_{1:4}, \sigma^2 \mathbb{I}_4)$$

How to update particle weights after an observation



Actual measurement received: $\bar{z}_t = [\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4]$

How to update particle weights after an observation

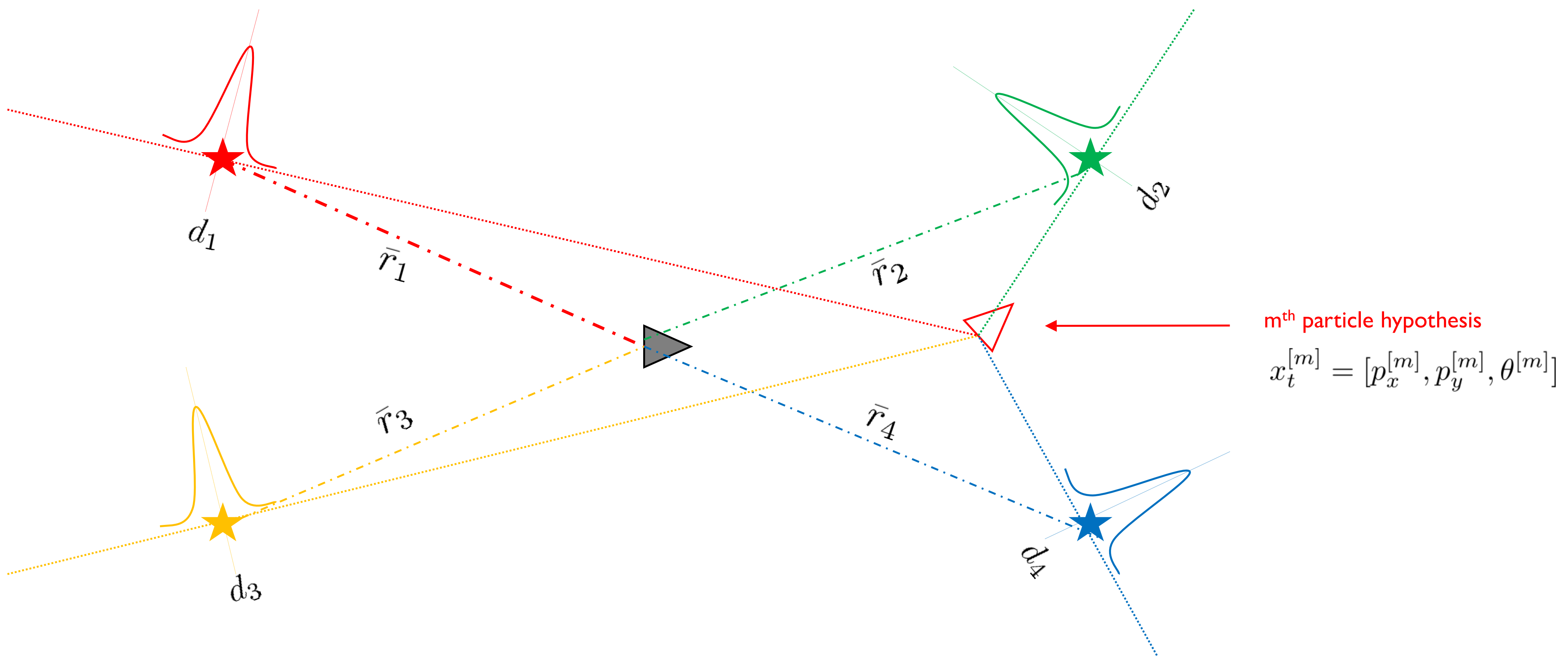


Measurement model $z_t = h(x_t^{[m]}) + n_t = [d_1, d_2, d_3, d_4] + n_t$

with $d_i = \sqrt{(p_x^{[m]} - l_x^{(i)})^2 + (p_y^{[m]} - l_y^{(i)})^2}$ and $n_i \sim \mathcal{N}(0, \sigma^2)$

$$p(z_t | x_t^{[m]}) = \mathcal{N}(z_t;\ d_{1:4}, \sigma^2 \mathbb{I}_4)$$

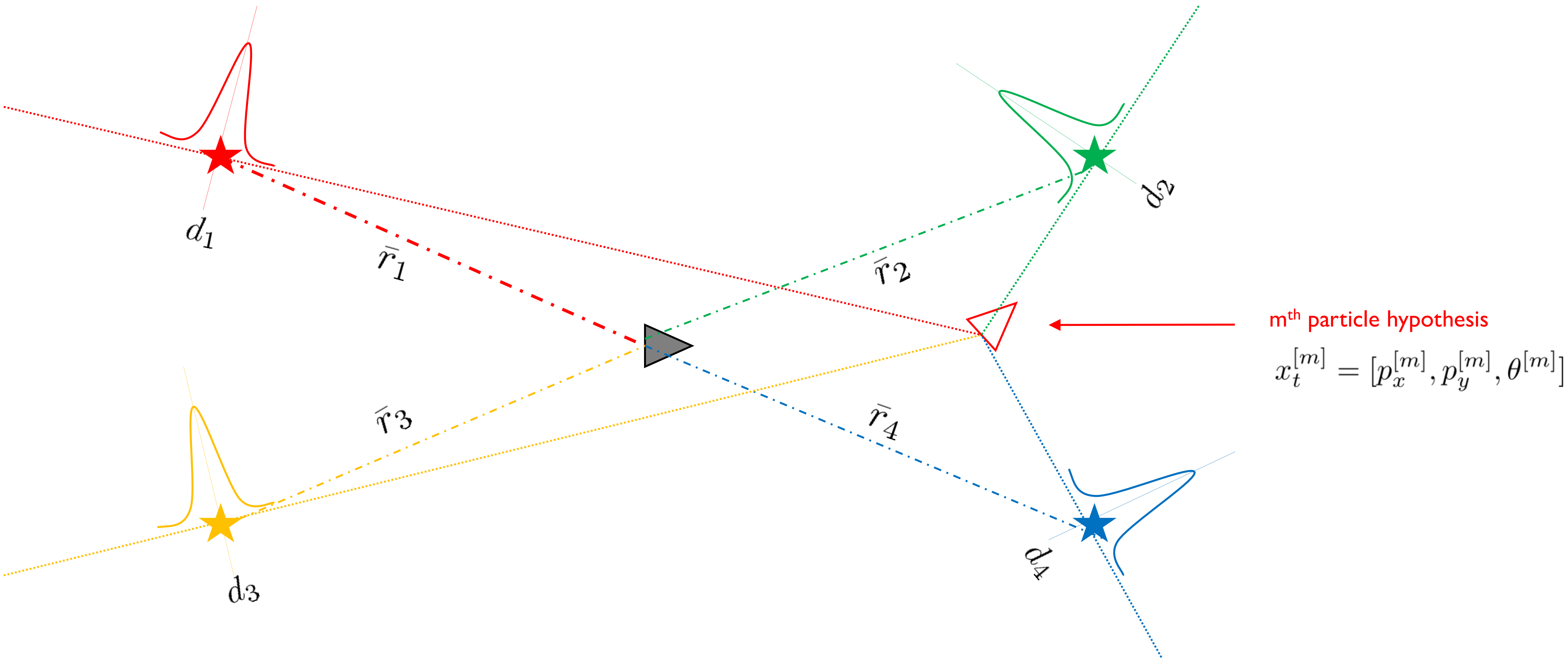
How to update particle weights after an observation



Q: What is the probability of the actual measurement given the state hypothesized by the particle?

A: $p(\bar{z}_t | x_t^{[m]}) = \mathcal{N}(\bar{z}_t;\ d_{1:4}, \sigma^2 \mathbb{I}_4) = \eta \exp(-||\bar{z}_t - d_{1:4}||^2 / \sigma^2)$

How to update particle weights after an observation

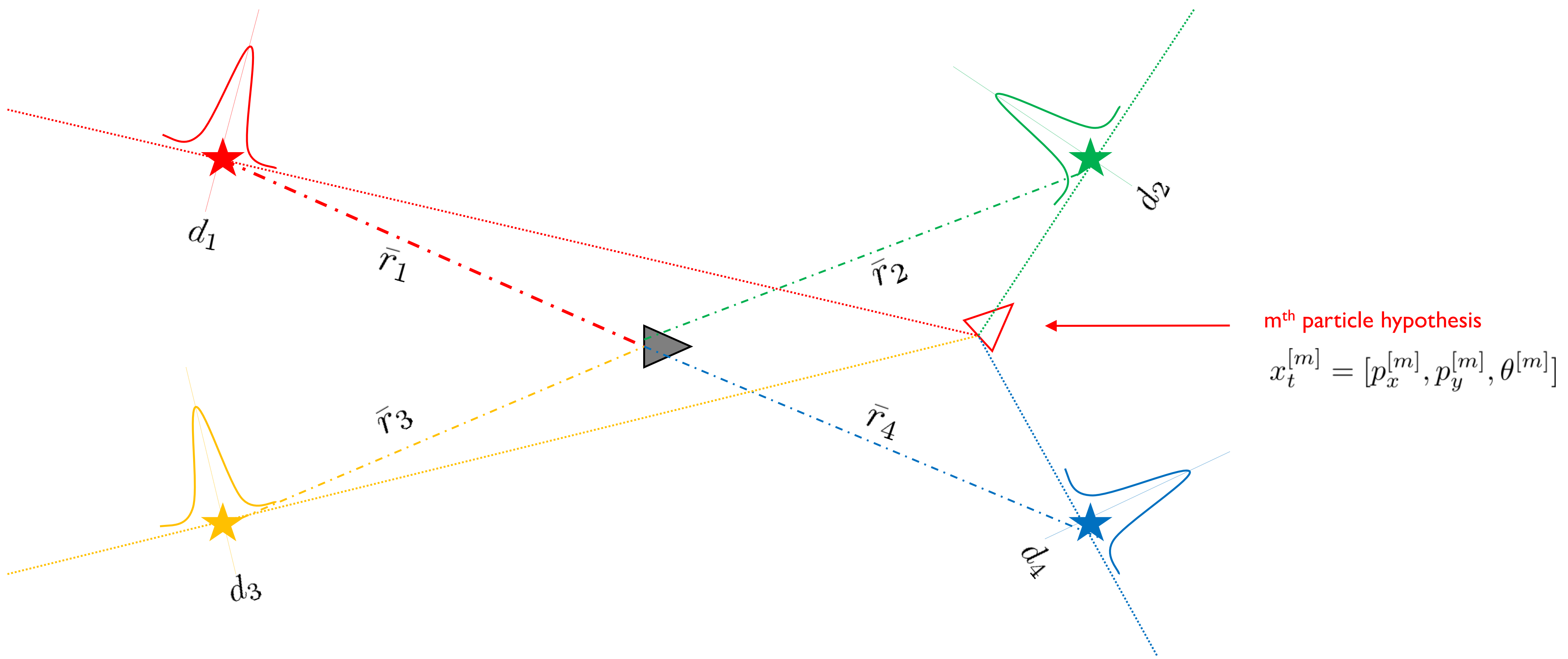


Q: What is the probability of the actual measurement given the state hypothesized by the particle?

Assuming range measurements are conditionally independent given state

A: $p(\bar{z}_t | x_t^{[m]}) = \prod_{i=1}^{4} p(\bar{r}_i | x_t^{[m]}) = \prod_{i=1}^{4} \mathcal{N}(\bar{r}_i;\ d_i, \sigma^2) = \prod_{i=1}^{4} \eta \exp(-(\bar{r}_i - d_i)/\sigma^2)$

How to update particle weights after an observation



Q: What is the probability of the actual measurement given the state hypothesized by the particle?

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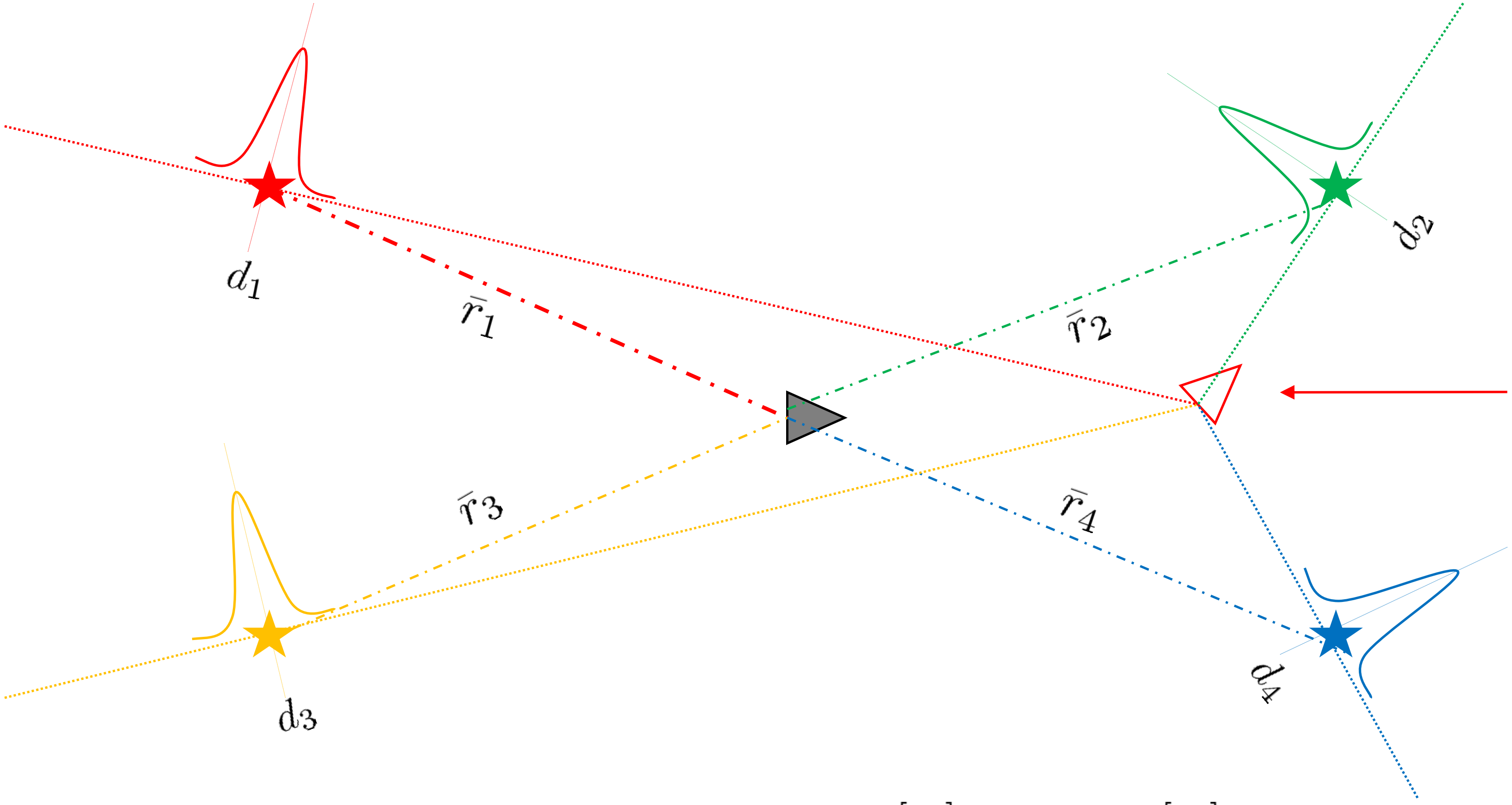
In the figure above this probability would be low and this particle would be unlikely.

How to update particle weights after an observation



m[th] particle hypothesis

$$x_t^{[m]} = [p_x^{[m]}, p_y^{[m]}, \theta^{[m]}]$$

Particle's (unnormalized) weight $\quad w_t^{[m]} \propto p(\bar{z}_t | x_t^{[m]})$

See Appendix 1 for why this choice was made for the weight

The distribution of the particles has not been updated yet. We only updated their weights. To update the distribution of particles we need to do resampling

The distribution of the particles has not been updated yet. We only updated their weights. To update the distribution of particles we need to do <u>resampling</u>

Sample particles with repetition/replacement, according to their updated weights.

Resampling Particles

- Main goal: Get rid of unlikely particles (with too low weights) and focus on most likely particles (a.k.a. survival of the fittest).

- Main mechanism: Sample new set of particles from existing set, with replacement (repetition), so that same particle can be sampled more than once. Sample old particle i with probability $\propto \mathrm{weight}_i$

- Many possible ways to implement it. Here we present two algorithms.

Resampling Particles: Algorithm #1

```
new_particles = []
sample u ~ Uniform[0,1]
idx = int( u * (N-1) )
beta = 0
max_w = max(weights)

for each of the N particles:
    sample v ~ Uniform[0,1]
    beta += v * 2* max_w

    while beta > weights[idx]:
        beta -= weights[idx]
        idx = (idx + 1) % N

    p = particles[idx].copy()
    new_particles.append(p)
```

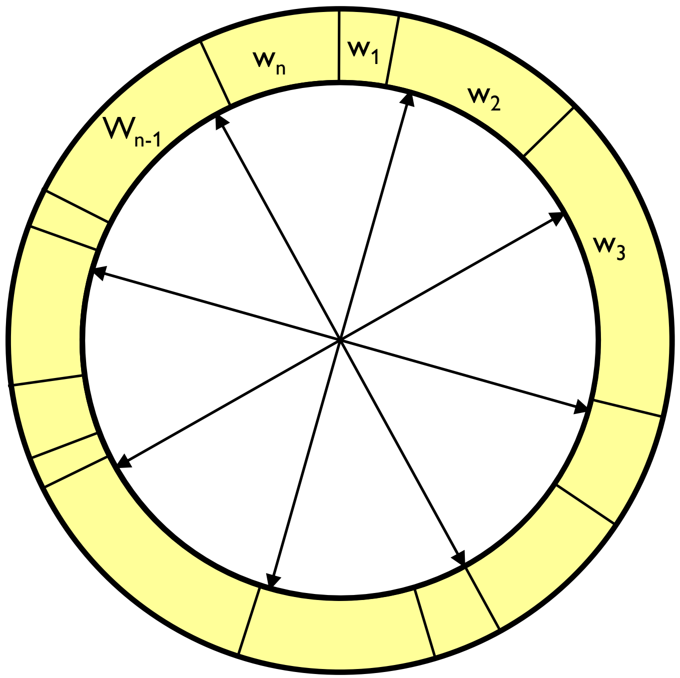
Resampling Particles: Algorithm #2

```
new_particles = []
sample r ~ Uniform[0, 1/N]
c = weights[0]
idx = 0

for n = 1…N:
    u = r + (n-1)/N

    while u > c:
        idx = idx + 1
        c = c + weights[idx]

    p = particles[idx].copy()
    new_particles.append(p)
```



- Stochastic universal sampling
- Systematic resampling
- Linear time complexity

Resampling Particles: Example

- Suppose we only have 5 particles:

Particle index	**Normalized weight**
1	0.1
2	0.2
3	0.4
4	0.1
5	0.2

Q: What is the probability that after a round of resampling the highest probability particle (#3) is not sampled?

A: $0.6^5 \simeq 0.077$

i.e. there is nonzero probability that we will lose the highest-probability particle → it will happen eventually

Resampling Particles: Example

- Suppose we only have 5 particles:

Particle index	Normalized weight
1	0.1
2	0.2
3	0.4
4	0.1
5	0.2

Q: What is the probability that after a round of resampling the highest-probability particle (#3) is not sampled?

A: $0.6^5 \simeq 0.077$

Q: What is the probability that after a round of resampling one of the lowest-probability particles (#1) is not sampled?

A: $0.9^5 \simeq 0.59$

Resampling Particles: Consequences

- Weak particles very likely do not survive.

⇓

- Variance among the set of particles **decreases**, due to mostly sampling strong particles (i.e. loss of particle diversity).

⇓

- Loss of particle diversity implies **increased variance** of the approximation error between the particles and the true distribution**.**

⇓

- Particle deprivation: there are no particles in the vicinity of the correct state

How to address particle deprivation

- Idea #1: don’t resample when only a few particles contribute
- Idea #2: inject random particles during resampling
- Idea #3: increase the number of particles (may be impractical depending on the computational complexity of the system)

How to address particle deprivation

- Idea #1: don't resample when only a few particles contribute
  - Effective sample size: $N_{\text{eff}} = \frac{1}{\sum_{i=1}^{N} w_i^2}$
  - When all particles have equal, normalized weights (1/N) then $N_{\text{eff}} = N$
  - When a single particle carries the entire weight then $N_{\text{eff}} = 1/N$ and we have loss of particle diversity.
  - Resample only when $N_{\text{eff}} > N_{\text{thresh}}$

- Idea #2: inject random particles during resampling
- Idea #3: increase the number of particles (may be impractical depending on the computational complexity of the system)

How to address particle deprivation

- Idea #1: don’t resample when only a few particles contribute

- Idea #2: inject random particles during resampling
  - A small percentage of the particles’ states should be set randomly
    - Pro: simple to code, reduces (but does not fix) particle deprivation
    - Con: incorrect posterior estimation even when there are infinitely many particles

- Idea #3: increase the number of particles (may be impractical depending on the computational complexity of the system)

Particle Filter Algorithm

Actual observation and control received

$\text{ParticleFilter}(\bar{z}_t, u_{t-1})$

$\bar{S}_t = \{\} \quad \bar{W}_t = \{\}$

for particle index $m = 1 ... M$

sample $x_t^{[m]} \sim p(x_t | x_{t-1}^{[m]}, u_{t-1})$

$w_t^{[m]} = p(\bar{z}_t | x_t^{[m]})$

$\bar{S}_t.\text{append}(x_t^{[m]})$

$\bar{W}_t.\text{append}(w_t^{[m]})$

$S_t = \{\}$

for particle index $m = 1 ... M$

sample particle i from $\bar{S}_t$ with probability $\propto w_t^{[i]}$

$S_t.\text{append}(x_t^{[m]})$

return S_t

Particle Filter Algorithm

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$S_t.\text{append}(x_t^{[m]})$

return S_t

Particle propagation/prediction:
noise needs to be added in order to make particles differentiate from each other.

If propagation is deterministic then particles are going to collapse to a single particle after a few resampling steps.

Particle Filter Algorithm

$\text{ParticleFilter}(\bar{z}_t, u_{t-1})$

$\bar{S}_t = \{\} \qquad \bar{W}_t = \{\}$

for particle index $\quad m = 1 \ldots M$

sample $\quad x_t^{[m]} \sim p(x_t | x_{t-1}^{[m]}, u_{t-1})$

$w_t^{[m]} = p(\bar{z}_t | x_t^{[m]})$

Weight computation as measurement likelihood.
For each particle we compute the probability of the actual observation given the state is at that particle.

$\bar{S}_t.\text{append}(x_t^{[m]})$

$\bar{W}_t.\text{append}(w_t^{[m]})$

$S_t = \{\}$

for particle index $\quad m = 1 \ldots M$

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Particle Filter Algorithm

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$S_t = \{\}$

for particle index $m = 1 \ldots M$

sample particle i from $\bar{S}_t$ with probability $\propto w_t^{[i]}$

Resampling step

$S_t.\text{append}(x_t^{[m]})$

Note: particle deprivation heuristics are not shown here

return S_t

Particle Filter Algorithm

$\text{ParticleFilter}(\bar{z}_t, u_{t-1})$

$\bar{S}_t = \{\} \quad \bar{W}_t = \{\}$

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$S_t = \{\}$

for particle index $\quad m = 1...M$

sample particle i from $\bar{S}_t$ with probability $\propto w_t^{[i]}$

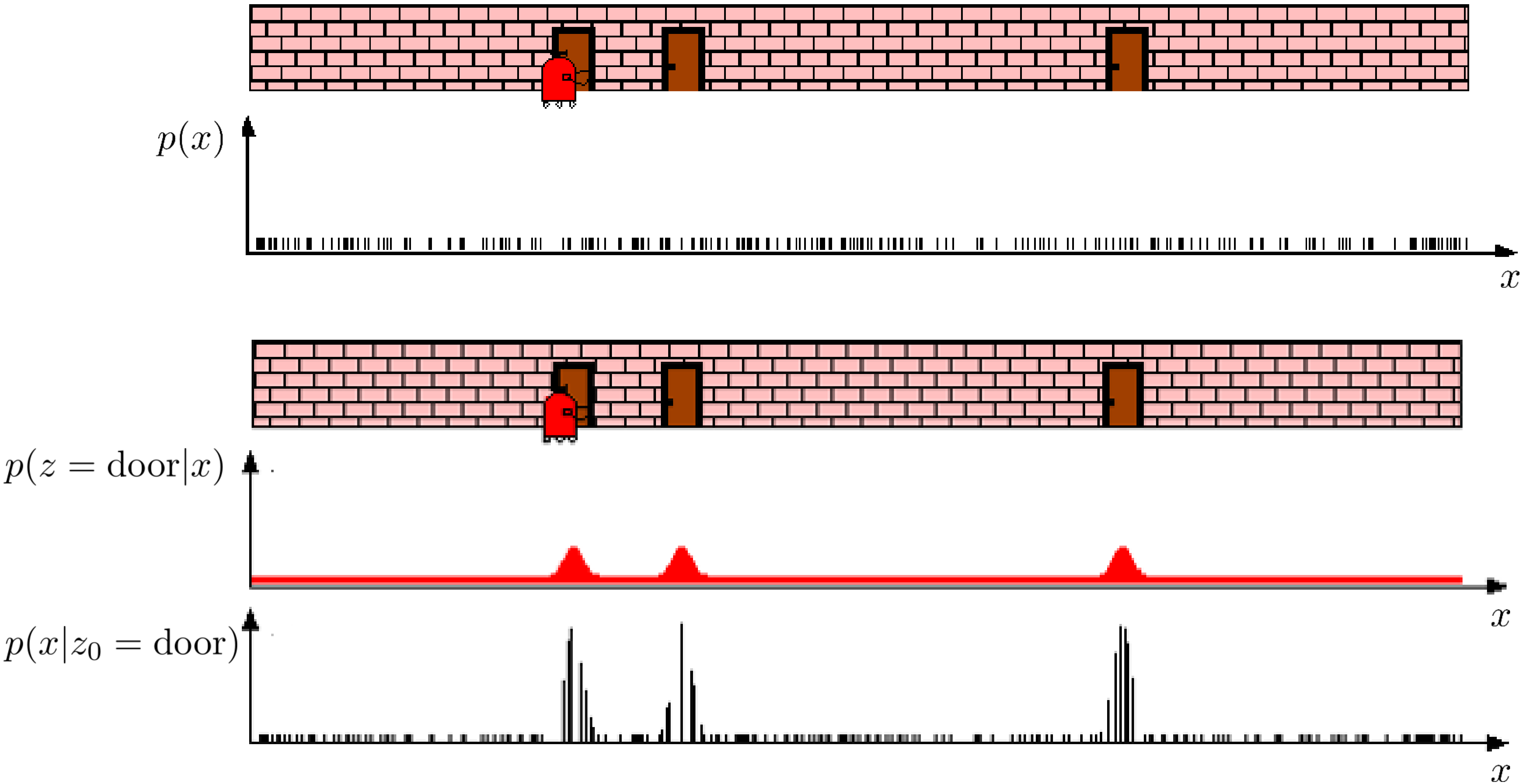
$S_t.\text{append}(x_t^{[m]})$

return S_t

Note: here we work with a fixed number of particles but in many applications, such as localization, you could work with a reduced number of particles after the particles have converged to the true estimate.

Such implementations of particle filters are called adaptive. An example is the KLD-sampling adaptive particle filter, which is not going to be covered here.

Examples: 1D Localization



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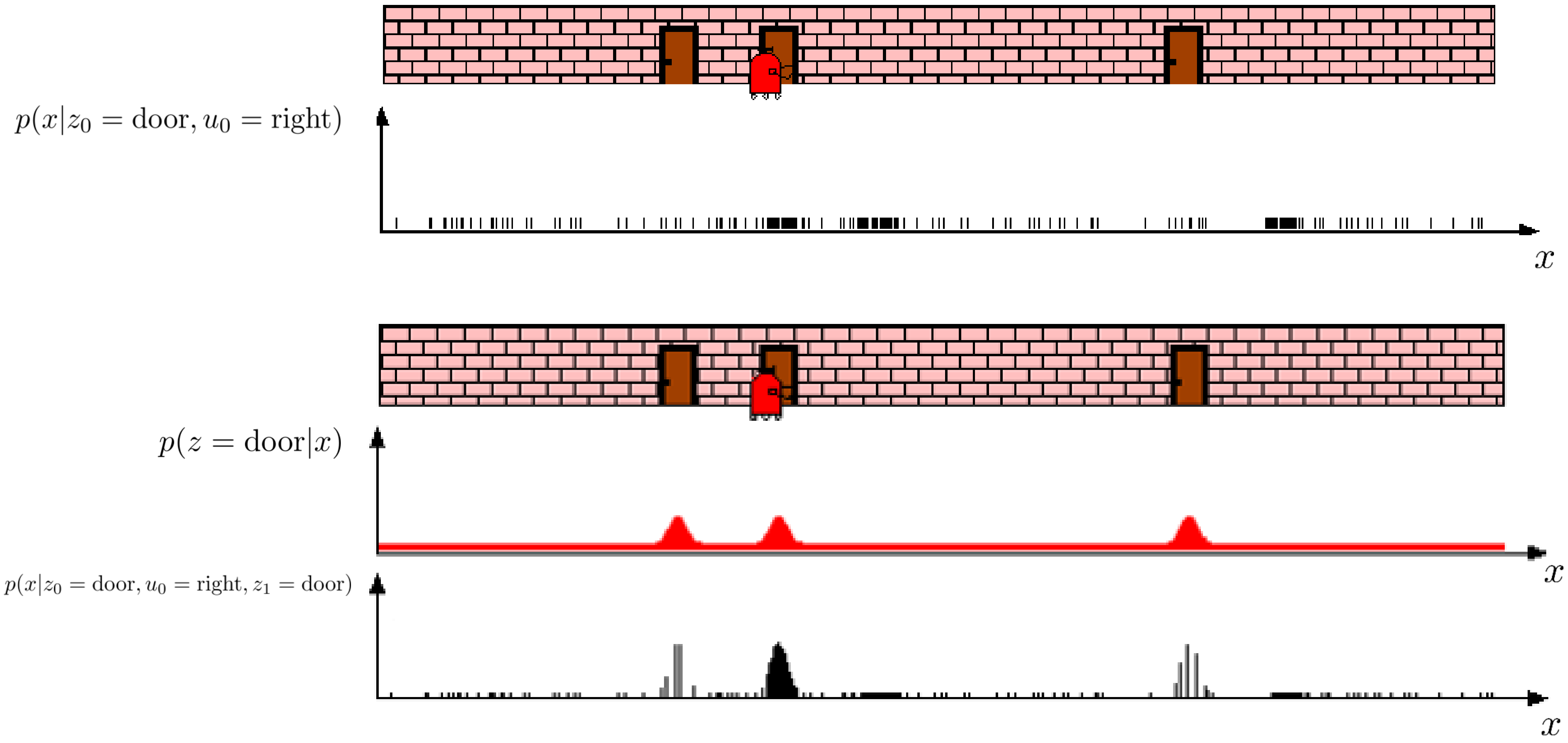
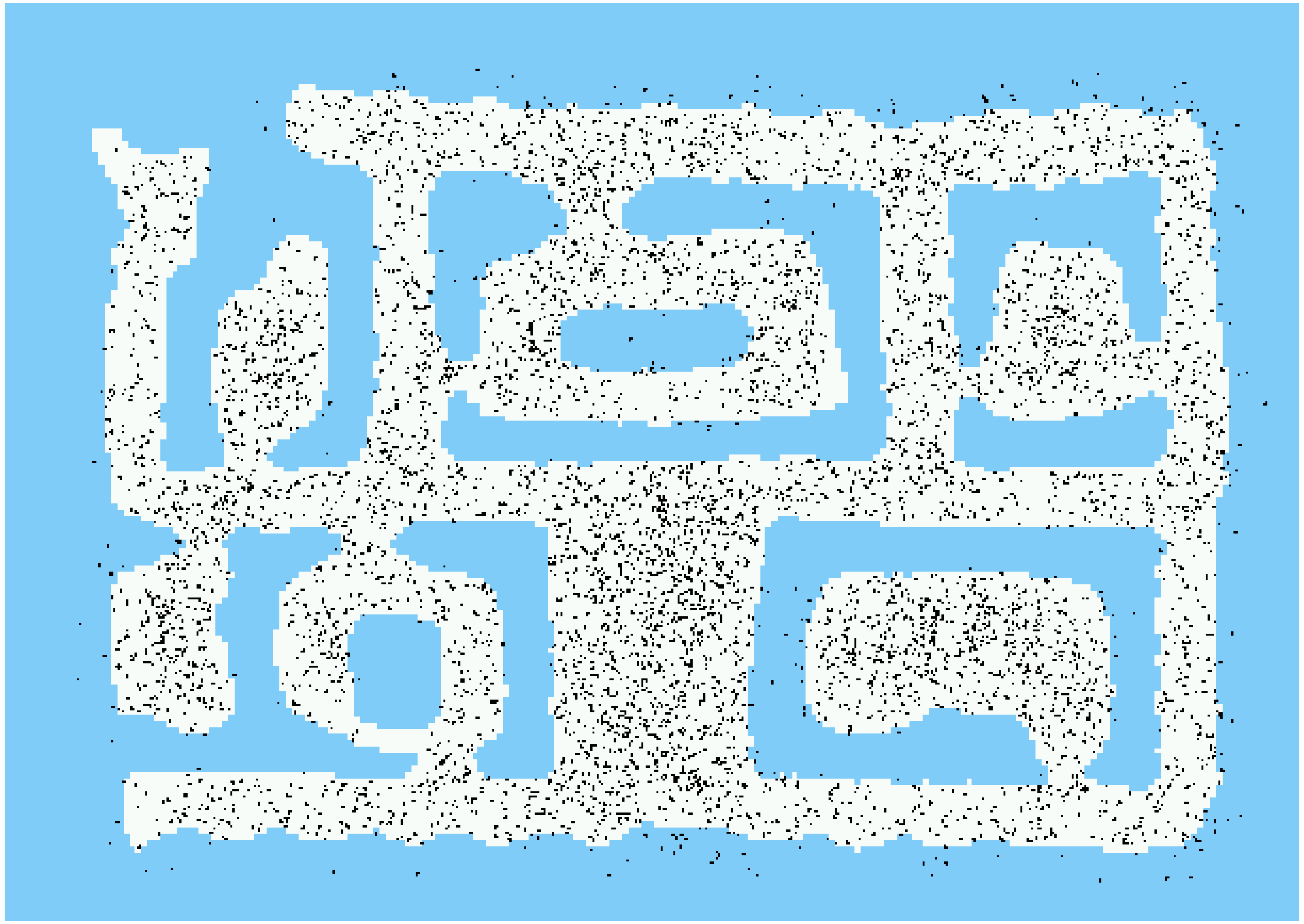


Examples: Monte Carlo Localization



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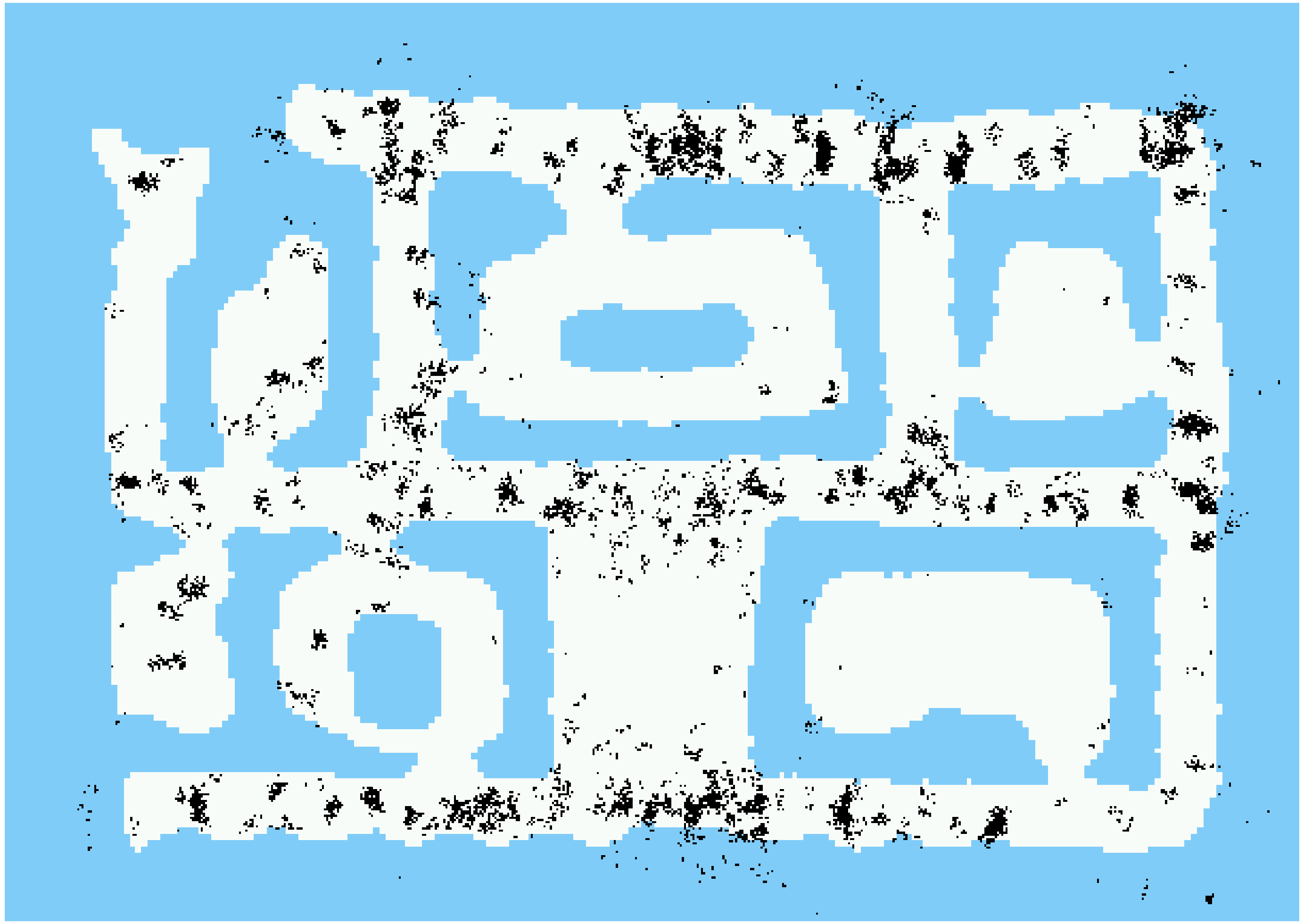
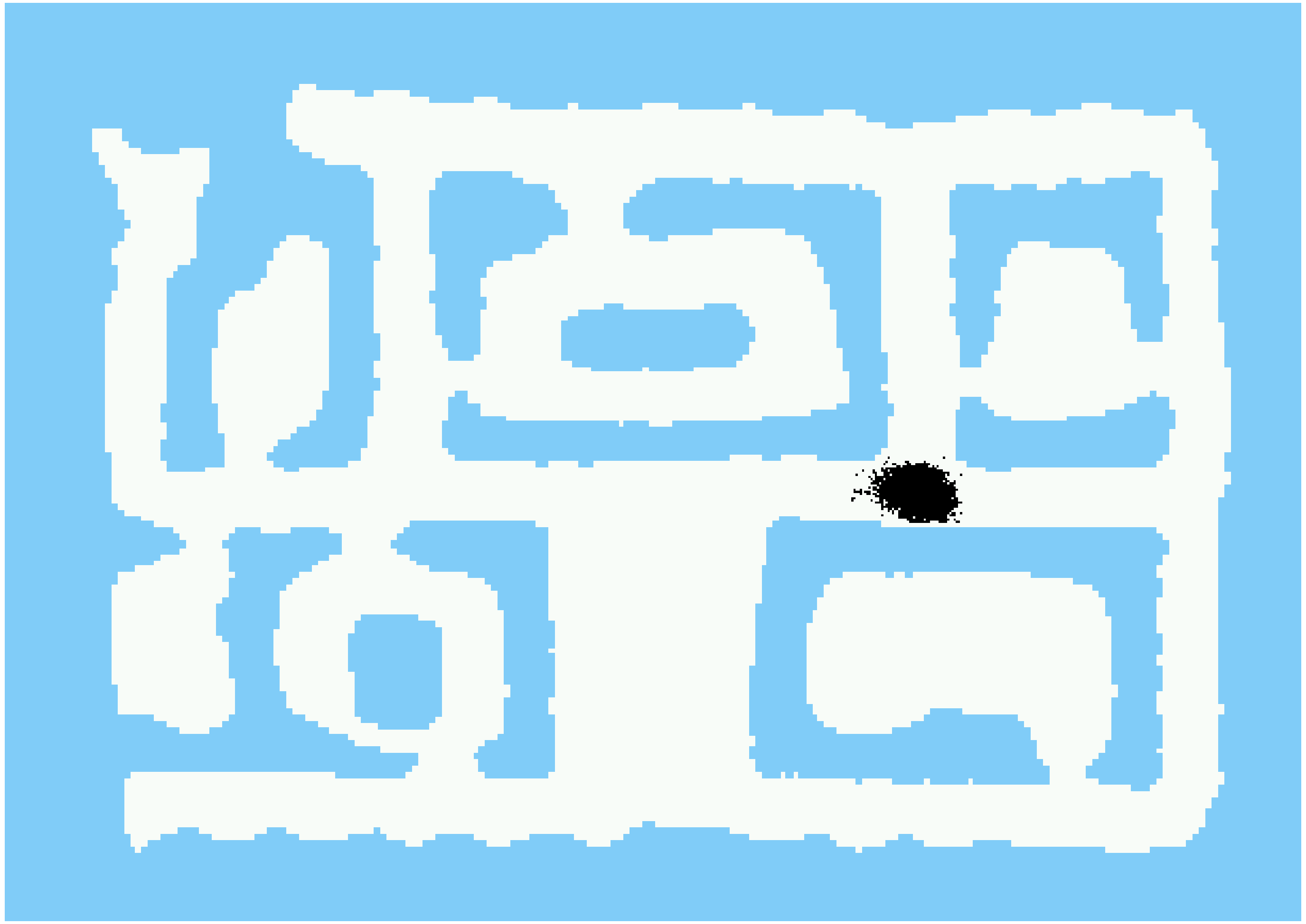


After incorporating 10 ultrasound scans

Examples: Monte Carlo Localization



After incorporating 65 ultrasound scans

Using Ceiling Maps for Localization

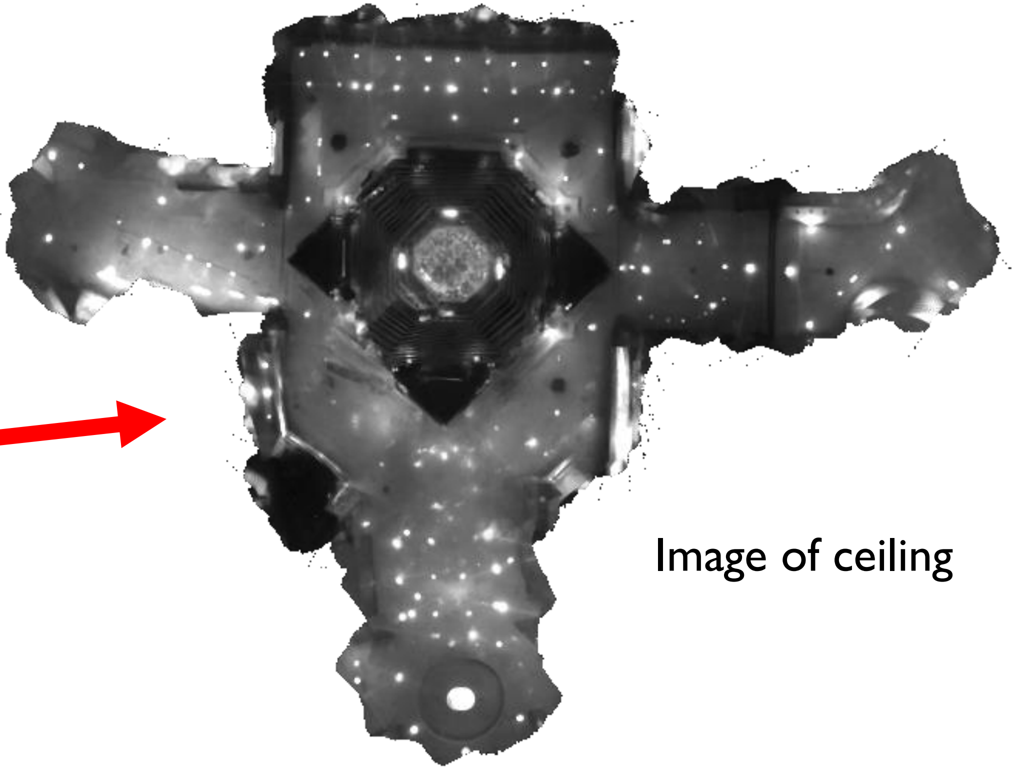
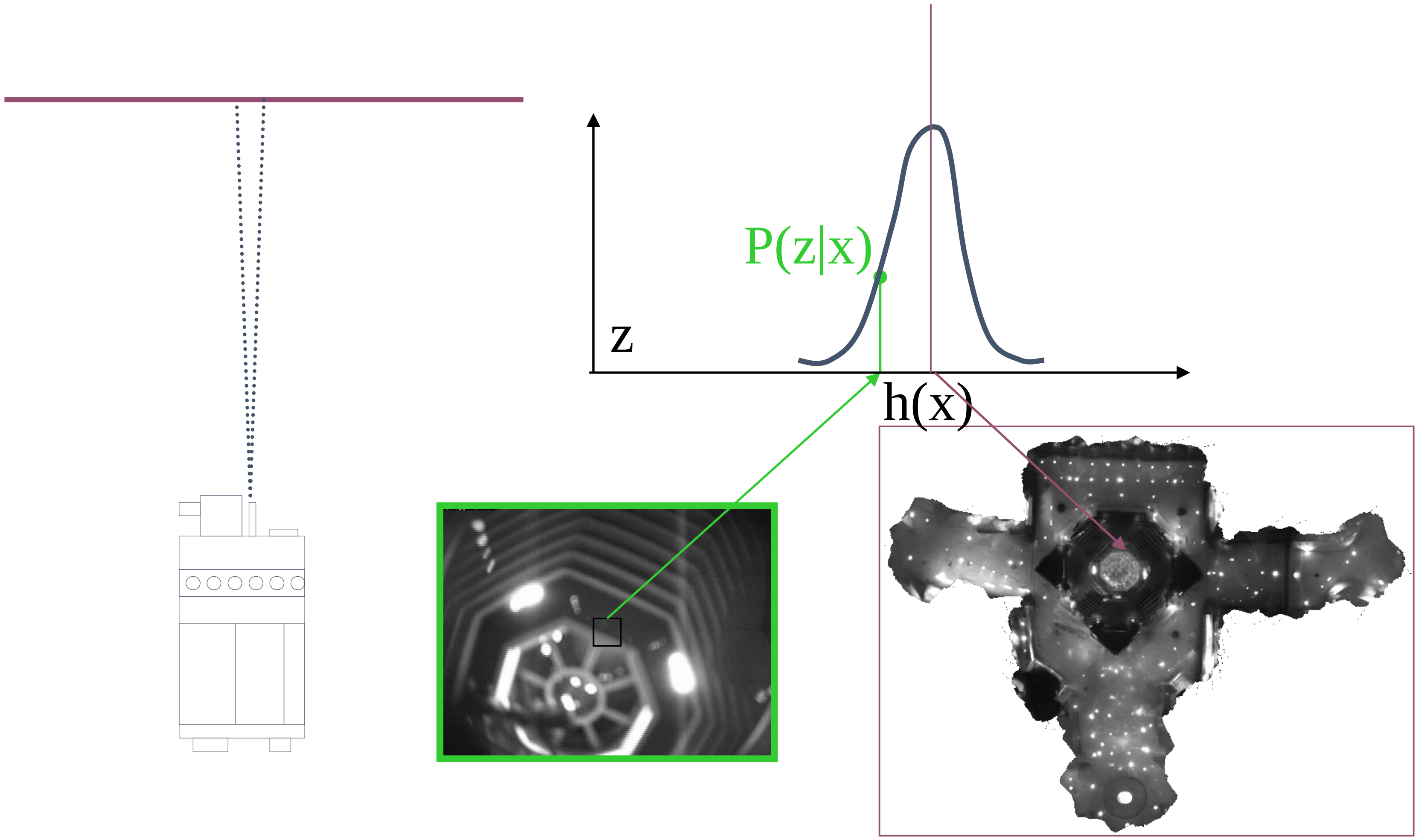
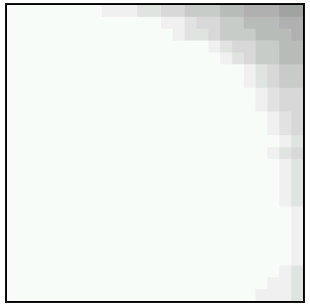


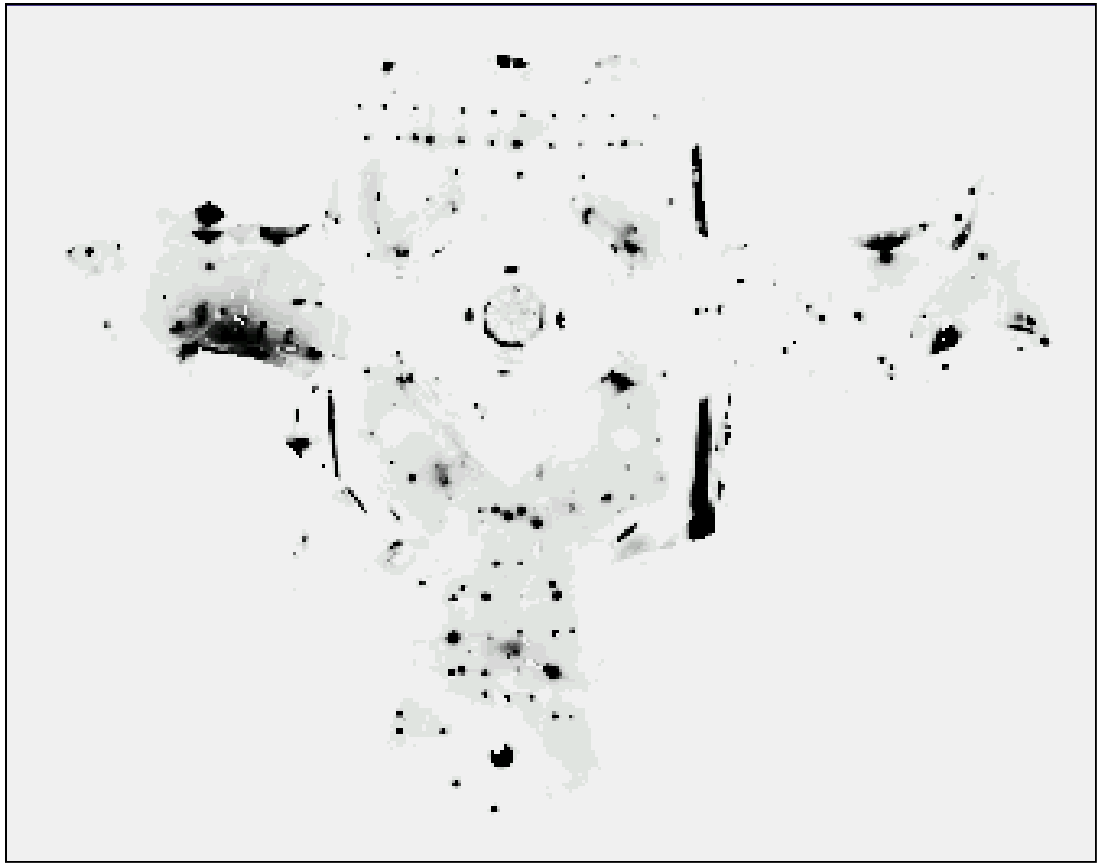
Image of ceiling

Vision-based Localization



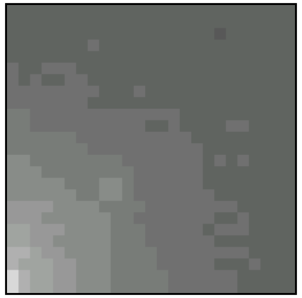
Under a Light

Measurement z: 

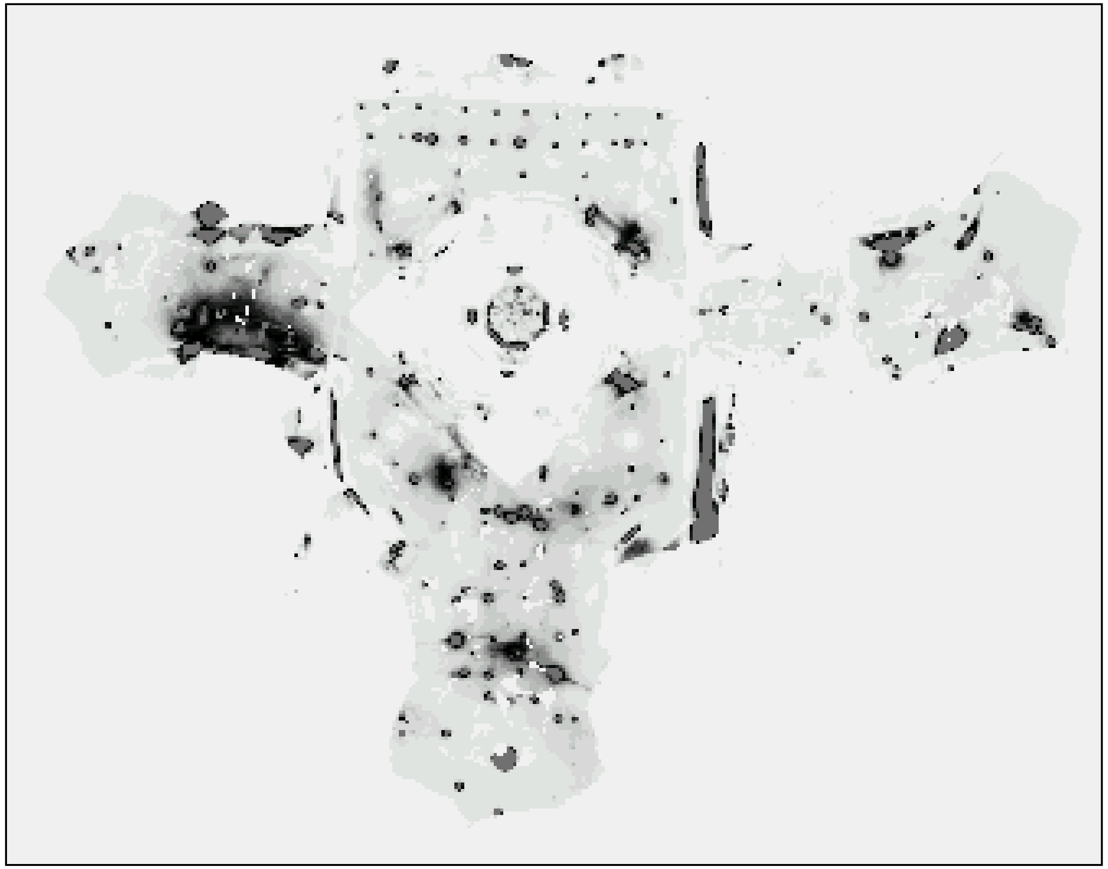
***P(z|x)*:** 

Next to a Light

Measurement z:



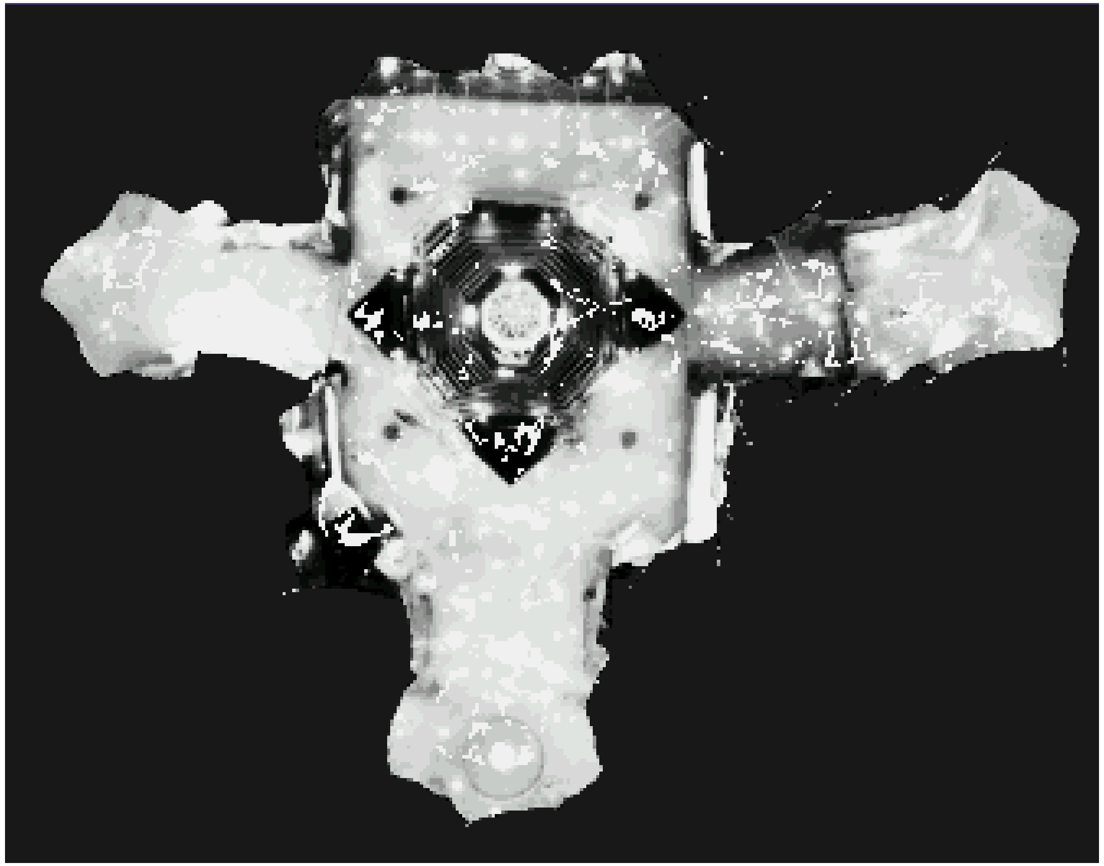
***P(z|x)*:**



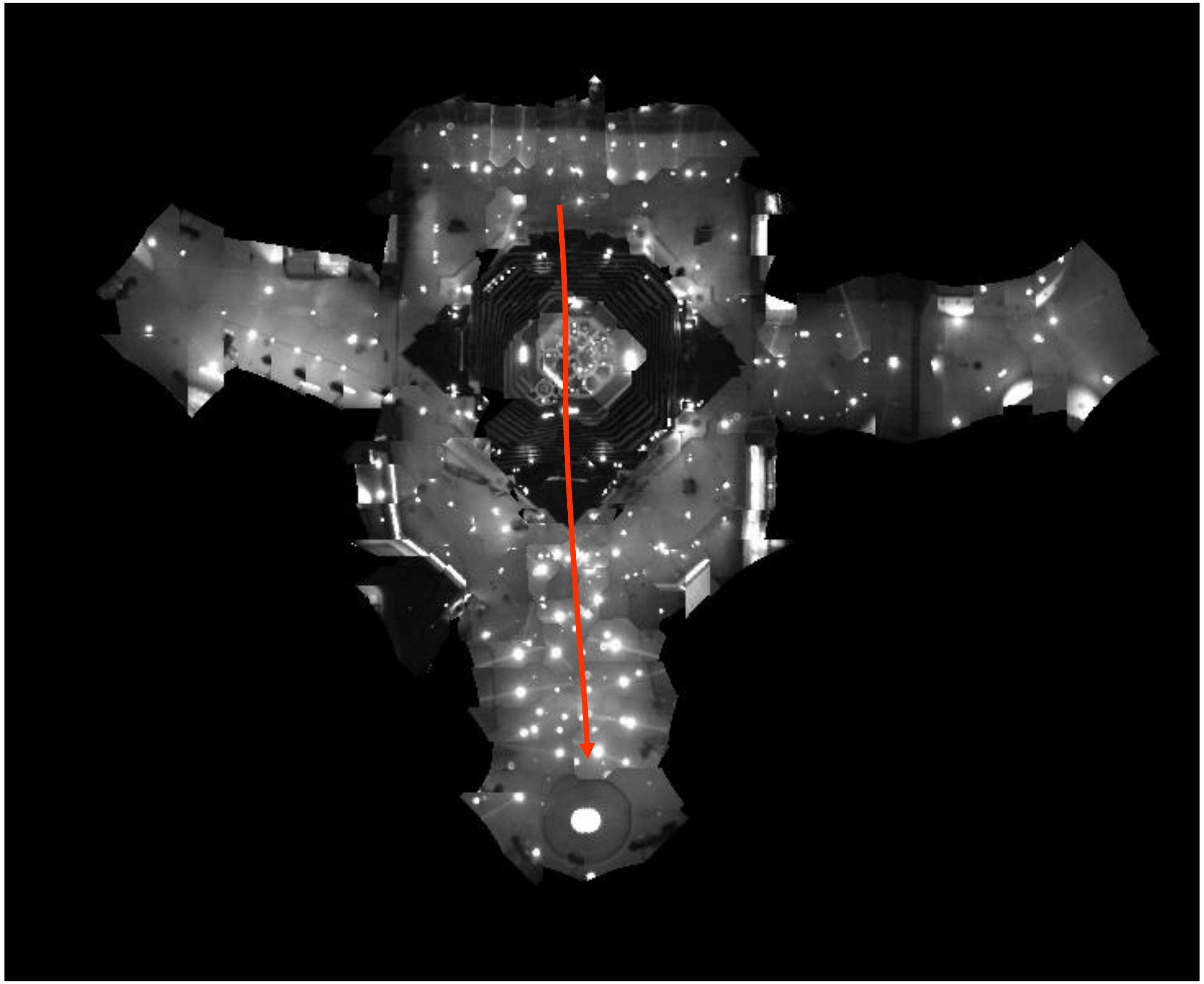
Elsewhere

Measurement z:

***P(z|x)*:**



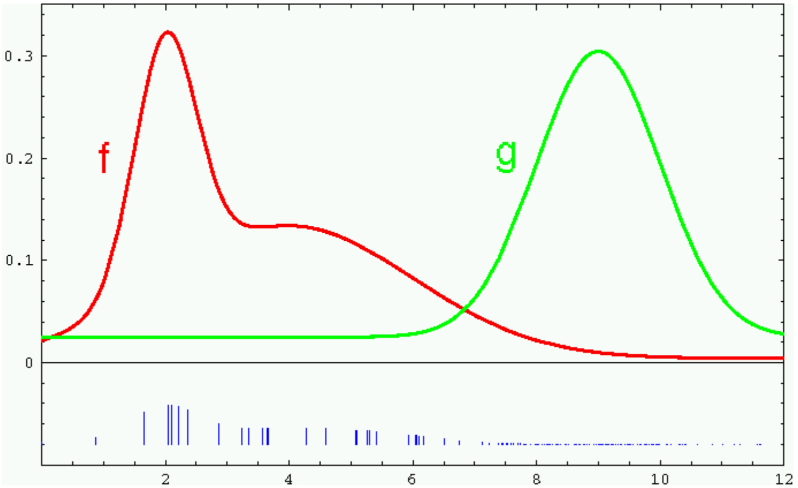
Global Localization Using Vision



Appendix 1

- Why did we choose $w_t^{[m]} \propto p(z_t | x_t^{[m]})$ as the importance weight for particle m?

- Main trick: **importance sampling**, i.e. how to estimate properties/statistics of one distribution (f) given samples from another distribution (g)



For example, suppose we want to estimate the expected value of f given only samples from g.

Appendix 1

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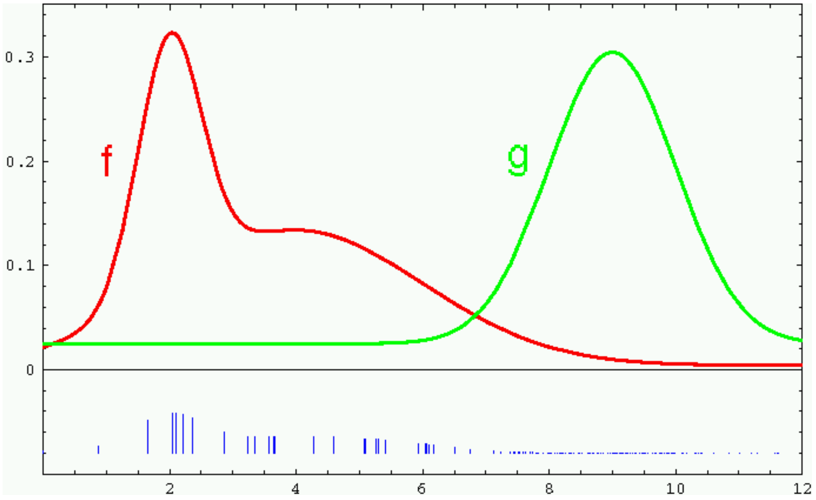

For example, suppose we want to estimate the expected value of f given only samples from g.

$$
\begin{aligned}
\mathbb{E}_{x\sim f(x)}[x] &= \int x f(x) dx \\
&= \int \frac{g(x)}{g(x)} x f(x) dx \\
&= \int \frac{x f(x)}{g(x)} g(x) dx \\
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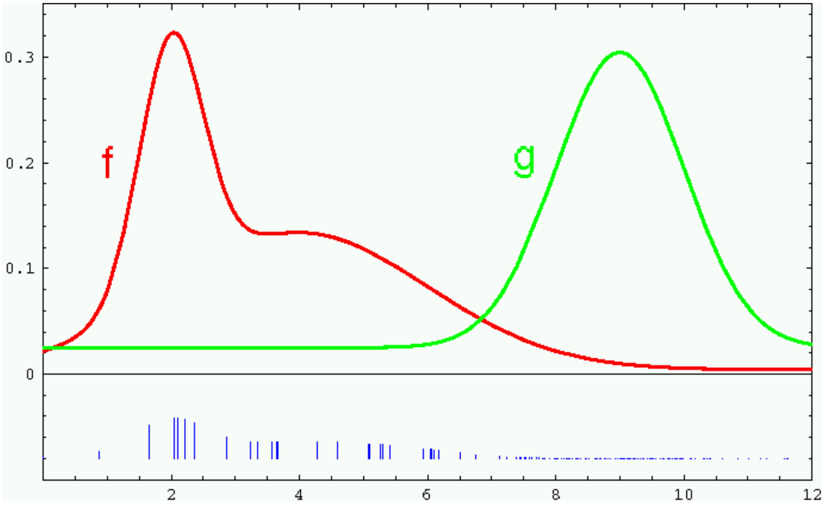
Appendix 1

- Why did we choose $w_t^{[m]} \propto p(z_t | x_t^{[m]})$ as the importance weight for particle m?

- Main trick: **importance sampling**, i.e. how to estimate properties/statistics of one distribution (f) given samples from another distribution (g)



For example, suppose we want to estimate the expected value of f given only samples from g.

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Weights describe the mismatch between the two distributions, i.e. how to reweigh samples to obtain statistics of f from samples of g

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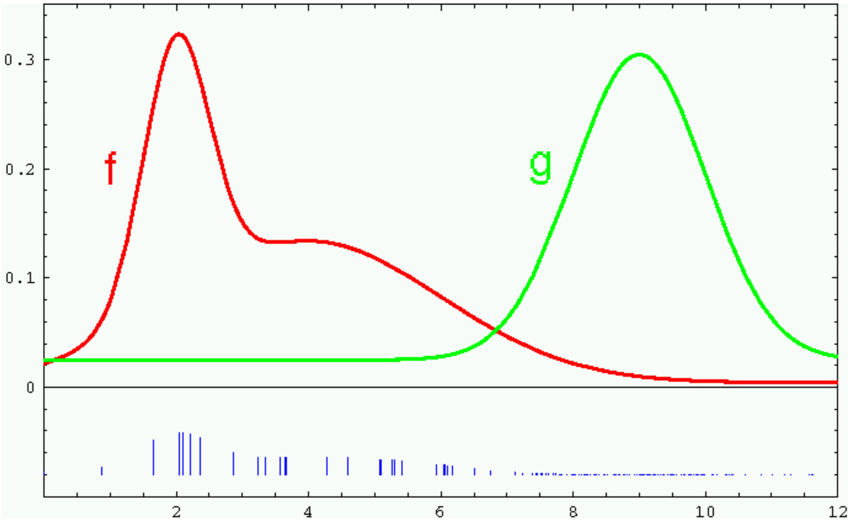

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In the case of particle filters

$f(x_t) = p(x_t|z_{0:t}, u_{0:t-1}) = bel(x_t)$

Posterior belief after update

$g(x_t) = p(x_t|z_{0:t-1}, u_{0:t-1}) = \overline{bel}(x_t)$

Belief after propagation, before update

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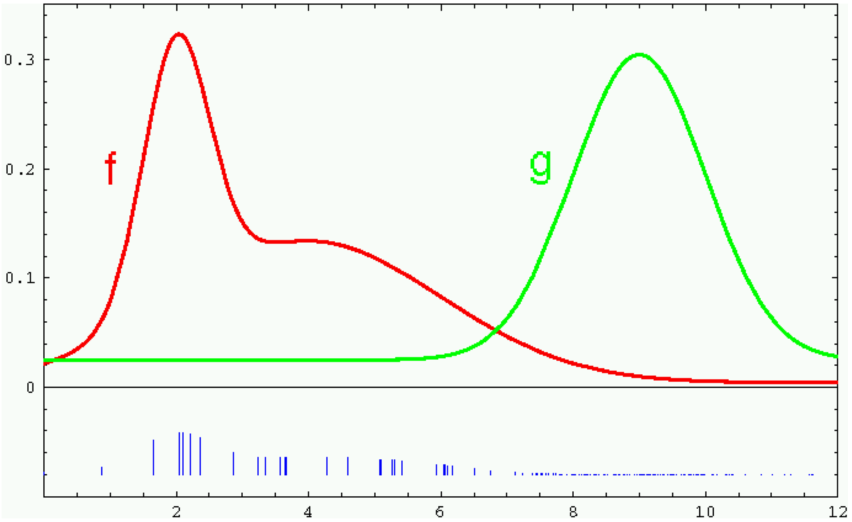

$$
\begin{aligned}
w(x_t^{[m]}) &= \frac{f(x_t^{[m]})}{g(x_t^{[m]})} \\
&\propto \frac{p(z_t | x_t^{[m]})\; p(x_t^{[m]} | x_{t-1}^{[m]}, u_{t-1})\; bel(x_{t-1}^{[m]})}{p(x_t^{[m]} | x_{t-1}^{[m]}, u_{t-1})\; bel(x_{t-1}^{[m]})} \\
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Posterior belief after update $\qquad$ Belief after propagation, before update