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Supplemental Material: Smooth Interpolating Curves with Local Control and Monotone Alternating Curvature

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Abstract

In this document, we present the proof of Proposition 3.1. We also illustrate how the property is used in our work for control points with curvature objectives of opposing signs.

1. Proof of Proposition 3.1

The equation of the normal clothoid shell function is given by

$$s_n : \Phi \in [0, 2\pi] \to 2\Phi \begin{pmatrix} \int_0^1 \cos(\Phi(-u^2+2u))\,du \\ \int_0^1 \sin(\Phi(-u^2+2u))\,du \end{pmatrix} \tag{1}$$

We defined the clothoid shell of the point p_i with tangent angle condition θ_i and curvature κ_i through the normal shell clothoid s_n via the formula

$$s_i(\Phi) = p_i + \frac{1}{\kappa_i} R(\theta_i) s_n(\Phi - \theta_i).$$

This formula is valid for $\Phi \in [\theta_i, 2\pi + \theta_i]$, but the function can be extended to $\Phi \in \mathbb{R}$ by taking the angle corresponding to the angle in $[\theta_i, 2\pi+\theta_i]$. Therefore s_i is discontinuous on $\mathbb{R}$ at $\theta_i + 2k\pi$ with $k \in \mathbb{Z}$.

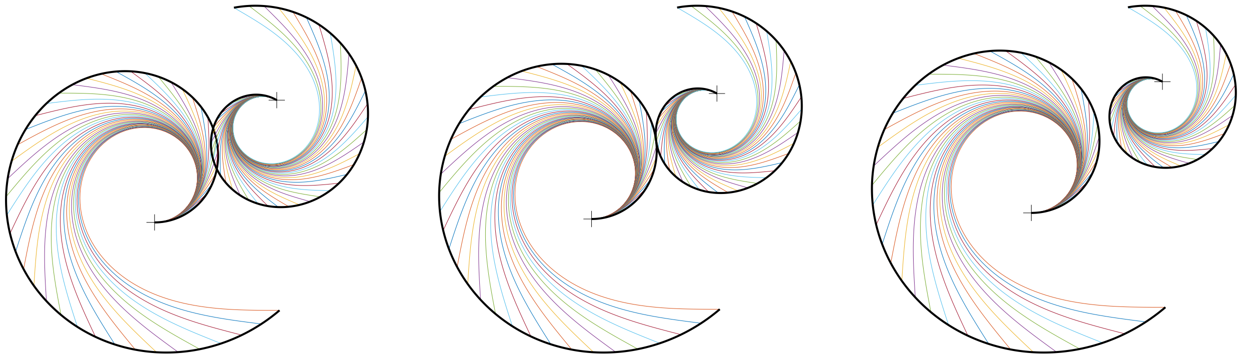
Proposition 3.1. Let p_0, p_1 be two distinct points and θ_0 and θ_1 two angles. There exist curvature bounds K_0, K_1, such that for any $\kappa_0 \geq K_0$, $\kappa_1 \geq K_1$, there exists an angle Φ such that the line segment $[s_0(\Phi),\, s_1(\Phi+\pi)]$ makes an angle Φ with the x-axis, where the starting conditions for the two clothoids are $(p_0, \theta_0, \kappa_0)$ and $(p_1, \theta_1, \kappa_1)$, respectively.

Proof. Let s_0 and s_1 be the clothoid shells related to points $(p_0, \theta_0, \kappa_0)$ and $(p_1, \theta_1, \kappa_1)$, respectively. Let Σ_0 and Σ_1 be the traces of curves s_0 and s_1 and $L_{\Sigma_0,\Sigma_1} = \bigcup_{p\in\Sigma_0, q\in\Sigma_1} [p,q]$ be the set of segments whose two end points are in Σ_0 and Σ_1, respectively. Let $[\Phi_a, \Phi_b]$ be the set of angles between the x-axis and the lines $[p,q] \in L_{\Sigma_0,\Sigma_1}$. This set is simply connected and can be noted as an interval because Σ_0 and Σ_1 are simply connected.

We know that for all Φ in $[0, 2\pi]$, $\|s_i(\Phi) - p_i\| \leq \frac{4\pi}{\kappa_i}$. Therefore, when both curvatures κ_0 and κ_1 converge towards $+\infty$, s_0 and s_1 converge uniformly towards p_0 and p_1 and the set $[\Phi_a, \Phi_b]$ is actually converging to the singleton $\{\Phi_{0,1}\}$, which is the angle formed by the x-axis and the segment $[p_0, p_1]$. We define $v(\Phi) := s_1(\Phi - \pi) - s_0(\Phi)$. Proving Proposition 3.1 is equivalent to proving that there exists Φ such that $v(\Phi)$ makes an angle Φ with the x-axis. We distinguish between several cases:

- If $\Phi_{0,1} = \theta_0 = \theta_1 - \pi$, then $v(\theta_0) = s_1(\theta_1) - s_0(\theta_0) = p_1 - p_0$ makes an angle θ_0 with the x-axis.
- Otherwise, we recall that s_i is continuous except on the set of points $\{\theta_i + 2k\pi \mid k \in \mathbb{Z}\}$. We have again two cases to consider:
  - If $\Phi_{0,1}$ is not equal to $\theta_1 - \pi$ or θ_0, then there exists an $\varepsilon > 0$ such that there are K_0 and K_1, such that for $\kappa_0 > K_0$, $\kappa_1 > K_1$, $[\Phi_a, \Phi_b] \subset [\Phi_{0,1} - \varepsilon, \Phi_{0,1} + \varepsilon]$ and both $\theta_1 - \pi$ and θ_0 are not in $[\Phi_a, \Phi_b]$. Therefore, both s_0 and $s_1 \circ \tau_{-\pi}$ are continuous on such an interval $[\Phi_a, \Phi_b]$ ($\tau_{-\pi}$ is a shift by $-\pi$). By continuity, there is $\Phi \in [\Phi_a, \Phi_b]$ such that $v(\Phi)$ makes an angle Φ with the x-axis.
  - We suppose that $\Phi_{0,1} = \theta_0$ but $\Phi_{0,1} \neq \theta_1 - \pi$ (by symmetry of the problem, the case in which $\Phi_{0,1} = \theta_1 - \pi$ and $\Phi_{0,1} \neq \theta_0$ is similar). Without loss of generality, since the situation is the same up to a rotation and translation, we can consider $\theta_0 = 0 = \Phi_{0,1}$ and $p_0 = (0,0)$. In this setting, p_1 has necessarily a positive x-coordinate and vanishing y-coordinate.
    We consider $\varepsilon > 0$ such that $\varepsilon < \theta_1 \pm \pi$, where $\theta_1 \pm \pi$ is the value of the angle $\theta_1 + \pi$ in $[0, 2\pi]$. Let K_0, K_1 be such that $\forall \kappa_0 > K_0, \kappa_1 > K_1$, $[\Phi_a, \Phi_b] \subset [-\varepsilon, \varepsilon]$. We cannot conclude as before because we do not have continuity of s_0 at 0. We must prove that the angle with the x-axis of $v(\Phi) = s_1(\Phi \pm \pi) - s_0(\Phi)$ is positive for $\Phi \in [0, \varepsilon]$. To do so, we prove that $s_1(\pi)$ has a y-coordinate (noted $s_1(\pi)_y$) that is always positive. Indeed, we know that if $\theta_1 \in [0, \pi)$, then

$$s_1(\pi)_y = \frac{2(\pi - \theta_1)}{\kappa_1} \int_0^1 \sin((\pi - \theta_1)(-u^2 + 2u) + \theta_1)\, du > 0$$



Intersecting clothoid shells
No solution

Tangential clothoid shells
2-arcs clothoid transition

Disjoint clothoid shells
Clothoid-line-clothoid transition

Figure 1: *Clothoid shells (black curves) for clothoid-line-clothoid (CLC) transition between points with given curvatures of opposite signs.* $p_0 = (0,0)$, $\theta_0 = 0$, $\kappa_0 = 1$, $p_1 = (3,3)$, $\theta_1 = -\frac{\pi}{6}$ *and* κ_1 *is equal to* 1.5 *(left),* 1.655 *(middle) and* 2.0 *(right). The function* s_1 *has been produced with positive curvature: we are looking for the point* $p_{e,1} = s_1(\Phi)$ *that makes* $v(\Phi)$ *have an angle* Φ *with the x-axis. But the second part of the CLC transition goes from* $p_{e,1}$ *to* p_1 *and therefore has curvature* $-\kappa_1$ *at* p_1*. Colored curves represent the shortest clothoids joining* (p,θ,K) *and a line of slope angle* Φ*.*

and if $\theta_1 \in (\pi, 2\pi)$,

$$s_1(\pi)_y = \frac{2(3\pi-\theta_1)}{\kappa_1}\int_0^1 \sin((3\pi-\theta_1)(-u^2+2u)+\theta_1)du > 0$$

- If $\theta_1 \in [0,\pi)$: since $u \in [0,1] \to -u^2+2u$ is an increasing function with values in $[0,1]$, we know that $\forall\theta_1 \in [0,\pi)$, $\forall u \in [0,1]$, $0 \leq \theta_1 \leq (\pi-\theta_1)(-u^2+2u)+\theta_1 \leq \pi$. This means that, $\forall\theta_1 \in [0,\pi)$, $\forall u \in [0,1]$, $\sin((\pi-\theta_1)(-u^2+2u)+\theta_1) \geq 0$. Since the sine function is not constantly equal to 0 on $[\theta_1,\pi]$, this shows that $\int_0^1 \sin((\pi-\theta_1)(-u^2+2u)+\theta_1)du > 0$.
- If $\theta_1 \in (\pi,2\pi)$, let us take $\Theta_1 = \theta_1 - \pi \in (0,\pi)$. We want to prove that $\int_0^1 \sin((3\pi-\theta_1)(-u^2+2u)+\theta_1)\,du > 0$, i.e., $\int_0^1 \sin((2\pi-\Theta_1)(-u^2+2u)+\Theta_1+\pi)\,du > 0$, i.e., that

$$\int_0^1 \sin((2\pi-\Theta_1)(-u^2+2u)+\Theta_1)\,du < 0 \qquad (2)$$

We therefore show that:

$$\begin{aligned}
&\int_0^1 \sin((2\pi-\Theta_1)(-u^2+2u)+\Theta_1)\,du\\
&= \int_0^1 \sin(-(2\pi-\Theta_1)(u^2-2u+1)+2\pi-\Theta_1+\Theta_1)\,du\\
&= -\int_0^1 \sin((2\pi-\Theta_1)(1-u)^2)\,du\\
&= -\int_0^1 \sin((2\pi-\Theta_1)u^2)\,du\\
&= -\frac{\int_0^{\sqrt{2\pi-\Theta_1}} \sin(x^2)\,dx}{\sqrt{2\pi-\Theta_1}}\\
&= -\frac{S\left(\sqrt{2\pi-\Theta_1}\right)}{\sqrt{2\pi-\Theta_1}},
\end{aligned}$$

where S is the Fresnel integral $S : t \to \int_0^t \sin(x^2)\,dx$. It is

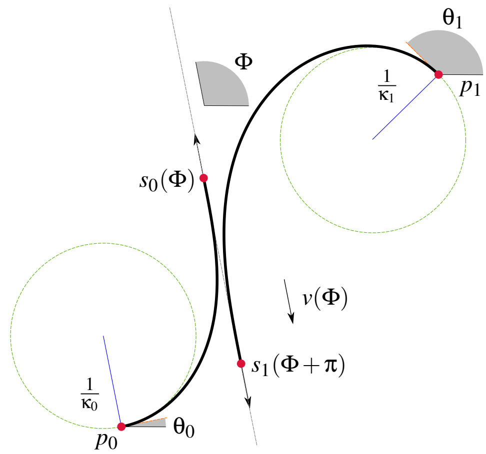


Figure 2: *In this example, the order of the point is reversed on the transition line. This case is not considered as valid for Proposition 3.1 because* $v(\Phi)$ *makes an angle* $\Phi+\pi$ *with the x-axis.*

known that S is positive on $\mathbb{R}^+_{>0}$, therefore we have proved the statement in (2).

Hence the angle ψ between the x-axis and $v(0) = s_1(\pi) - s_0(0)$ is positive. This angle ψ is smaller than ε since $[\Phi_a,\Phi_b] \subset [-\varepsilon,\varepsilon]$ and $s_1 \circ \tau_{-\pi}$ is continuous on $[0,\varepsilon]$. By continuity, there exists $\Phi \in [0,\psi] \subset]0,\varepsilon]$ such that $v(\Phi) = s_1(\Phi-\pi) - s_0(\Phi)$ makes an angle Φ with the x-axis.

□

Interpretations of Proposition 3.1. In our method, the tangent is computed based on a hybrid circular-elliptical interpolation of three successive points p_{i-1}, p_i and p_{i+1} [Yuk20]. Therefore, the angles θ_0 and θ_1 are always such that the vector $p_1 - p_0$ is making a positive angle smaller than $\pi/2$ with the line (p_0, θ_0) (respectively, the vector $p_0 - p_1$ and the line $(p_1, \pi + \theta_1)$). This situation is illustrated in Figure 1. We can see in this figure that three cases can happen:

1. The clothoid shells intersect twice. The angle with the x-axis of $v(\Phi)$ therefore makes a whole round and is never equal to Φ.
2. The clothoid shells are tangent. The unique intersection point p is such that there exists Φ such that $s_0(\Phi) = s_1(\Phi + \pi) = p$. Therefore, the line part of the CLC transition is vanishes, and we have a 2-arcs clothoid transition.
3. The clothoid shells are disjoint. Let ψ be the angle that $v(\Phi)$ makes with the x-axis. In this situation, the difference $\psi - \Phi$ is first positive around $\Phi = 0$ and then negative around $\Phi = \pi$. Therefore there is a Φ such that $\psi - \Phi = 0$.

Note that in the case of same-sign curvature, the situation can be different. It is possible to find K_0, K_1 such that there exists a CLC transition from (p_0, θ_0, K_0) to (p_1, θ_1, K_1), but not every $\kappa_0 > K_0$ and $\kappa_1 > K_1$ makes the existence of a CLC transition persist (Section 5.2 in the paper).

It is possible that the only Φ such that $s_0(\Phi)$ and $s_1(\Phi + \pi)$ end on the same tangent line is not suitable for our CLC transition. In Fig. 2, the order of the points is reversed. However, $v(\Phi)$ makes an angle $\Phi + \pi$ with the x-axis. This case is not considered as valid in our proof, and hence further increase of the curvature is required. Therefore, Proposition 3.1 states that there exists a *valid* CLC transition, without eluding the case of reverse order of the points on the transition line.

References

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