Chebyshev method for the implied volatility in options pricing

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This dissertation is submitted for the degree of
Bachelor

EPFL January 2022

Acknowledgements

I would like to thank my supervisors PhD. Hysan Lam and Professor Daniel Kressner for their consistent support and guidance during the running of this project.

Abstract

The implied volatility is a crucial element of financial toolbox, since it is used for quoting and the hedging of options as well as for model calibration. In contrast to the Black-Scholes formula its inverse, the implied volatility, is not explicitly given and numerical approximation is required. In this paper,we expose a bivariate interpolation method, using Chebyshev polynomials, for arbitrary functions on a rectangular domain and we use it to approximate the implied volatility. This is supposed to yield a closed-form approximation of the implied volatility, which is easy to implement. We also prove a subexponential error decay of the interpolation method. This allows us to obtain an accuracy close to machine precision with polynomials of a low degree.

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Chapter 1

Bivariate Chebyshev interpolation

1.1 Derivation of two-dimensional Chebyshev interpolation from one-dimensional

In this first part our aim is to derive the bivariate Chebyshev interpolation from the one-dimensional one.
Lets recall the one-dimensional Chebyshev interpolation for a function f on $[-1,1]$. We have that :

$$f(x) \approx I_N(x) := \sum_{j=0}^{N} a_j T_j(x) \text{ with } a_j = \frac{2^{\mathbb{1}_{0<j<N}}}{N} \sum_{k=0}^{N}{}'' f(x_k) T_j(x_k)$$

for $N+1$ the number of interpolated points, $x_k,\ k = 0,...,N$ the Chebyshev nodes given by $x_k = cos(\frac{k\pi}{N})$ and $T_j(x)$ the nth Chebyshev polynomial given by $T_j(x) = cos(j.cos^{-1}(x))$. Moreover, $\sum''$ indicates that the first and the last summand are halved.

Lemma 1.1.1. *The univariate Chebyshev interpolation admits a bivariate extension. A function $f : [-1,1]^2 \to \mathbb{R}$ can be approximated by the interpolation*

$$f(x,y) \approx I^{N_1,N_2}(x,y) := \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{ij} T_i(x) T_j(y)$$

with

$$a_{ij} = \frac{2^{\mathbb{1}_{0<i<N_1}}}{N_1} \frac{2^{\mathbb{1}_{0<j<N_2}}}{N_2} \sum_{k=0}^{N_1}{}'' \sum_{l=0}^{N_2}{}'' f(x_k, y_l) T_i(x_k) T_j(y_l)$$

where N_1, N_2 and $x_k, y_l,\ k = 0,...,N_1.\ l = 0,...,N_2$ are the number of interpolated points and the Chebyshev nodes along the x-axis, respectively the y-axis.

Proof. The main idea in this demonstration is to first fix one variable (say x here), interpolate in y using the 1D Chebyshev interpolation. Then interpolate in x.
For $y \in [-1,1]$ and x fixed we can write :

$$f(x,y) \approx \sum_{i=0}^{N_2} c_i(x)T_i(y) \text{ with } c_i : [-1,1] \to \mathbb{R} \text{ defined by } c_i(x) = \sum_{i=0}^{N_1} c_{ij}T_i(y_l).$$

From the 1D Chebyshev interpolation, we must have :

$$c_i(x) = \frac{2^{\mathbb{1}_{0<j<N_2}}}{N_2} \sum_{l=0}^{N_2}{}'' f(x,y_l)T_i(y_l)$$

Now, we can approximate $f(x,y_l)$ by interpolating in x. By fixing $l \in \{0,...,N_2\}$ we find

$$f(x,y_l) \approx \sum_{j=0}^{N_1} b_jT_j(x) \text{ with } b_j = \frac{2^{\mathbb{1}_{0<j<N_1}}}{N_1} \sum_{k=0}^{N_1}{}'' f(x_k,y_l)T_j(x_k).$$

Then by nesting every previous part we can derive the main formula :

$$\begin{aligned}
f(x,y) &\approx \sum_{i=0}^{N_2} \underbrace{\left[\frac{2^{\mathbb{1}_{0<i<N_2}}}{N_2} \sum_{k=0}^{N_2}{}'' \underbrace{\left[\sum_{j=0}^{N_1} \underbrace{\left[\frac{2^{\mathbb{1}_{0<j<N_1}}}{N_1} \sum_{l=0}^{N_1}{}'' f(x_k,y_l)T_j(x_k) \right]}_{b_j} T_j(x) \right]}_{\approx f(x,y_l)} T_i(y_l) \right]}_{c_i(x)} T_i(y) \\
&\approx \sum_{i=0}^{N_2} \sum_{j=0}^{N_1} \underbrace{\left[\frac{2^{\mathbb{1}_{0<i<N_2}}}{N_2} \frac{2^{\mathbb{1}_{0<j<N_1}}}{N_1} \sum_{k=0}^{N_2}{}'' \sum_{l=0}^{N_1}{}'' f(x_k,y_l)T_j(x_k)T_i(y_l) \right]}_{a_{ij}} T_j(x)T_i(y) \\
&\approx \sum_{i=0}^{N_2} \sum_{j=0}^{N_1} a_{ij}T_j(x)T_i(y)
\end{aligned}$$

$\square$

1.2 Implementation of two-dimensional Chebyshev interpolation on an arbitrary rectangular domain

Our main objective is to analyse the approximation of implied volatility by the Chebyshev method. To this end, we have implemented a matlab function `f_interp.m` that calculates the Chebyshev interpolated function from a 2D function.
`f_interp.m`:

- **Input:**
  - N_1 : number of interpolation nodes on the x-axis.
  - N_2 : number of interpolation nodes on the y-axis.
  - $Vimp$: reference points for the function to interpolate.
  - xev : vector of Chebyshev nodes on the x-axis.
  - yev : vector of Chebyshev nodes on the y-axis.
  - x, y : variables.
  - a, b, c, d : bounds of intervals.

- **Output:**

$$-I^{N_1,N_2}(x,y) := \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{ij}T_i(x)T_j(y) : \text{f interpolated.}$$

In order to compute the interpolated function of a function f, we need to compute its coefficient (i.e. the coefficients a_{ij} from Lemma 1.1.1). This is why we have also implemented, separately, a function `a_ij.m`. In fact, this function receives the inputs $Vimp, xev, yev, a, b, c, d$ from the function `f_interp.m`. The additional inputs i and j correspond to the fact that we calculate the coefficient a_{ij}. We also use the function `T_n.m` which computes the n-th Chebyshev polynomial.

1.3 Numerical test of the method

To examine the accuracy of our method, we approximated different functions and calculated the error for the interpolation that is given by the absolute value between the reference functions and the interpolated one. To be sure of the validity of our results, we also interpolated each function using the chebfun2 algorithm [7], which we assume to be accurate. For the sake of clarity and simplicity we defined for the numerical examples:

$$N_1 = N_2 = 40; \;\; a = c = -1; \; b = d = 1;$$

$$xev_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{i \cdot \pi}{N_1}\right); \quad yev_i = \frac{c+d}{2} + \frac{d-c}{2}\cos\left(\frac{i \cdot \pi}{N_2}\right) \quad \forall i = 1, \ldots, 40$$

We also use a meshgrid of 100 equidistant points between [a,b] and [c,d] to compute the interpolated function. Let's start by looking at the results obtained with the following function:

$$f : [-1,1]^2 \longrightarrow \mathbb{R}$$

$$f(x,y) = xe^{-x^2-y^2}$$

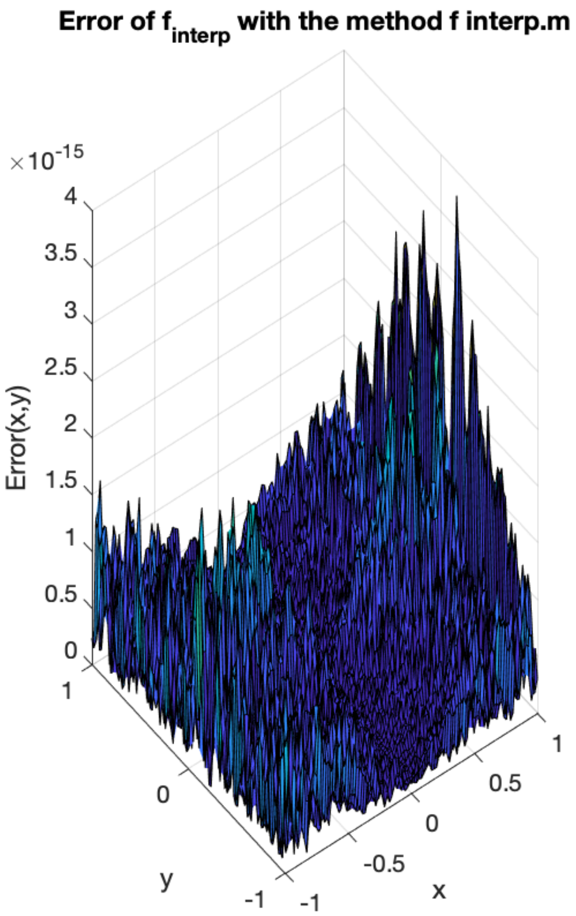

Fig. 1.1 Interpolation error for our method and f.

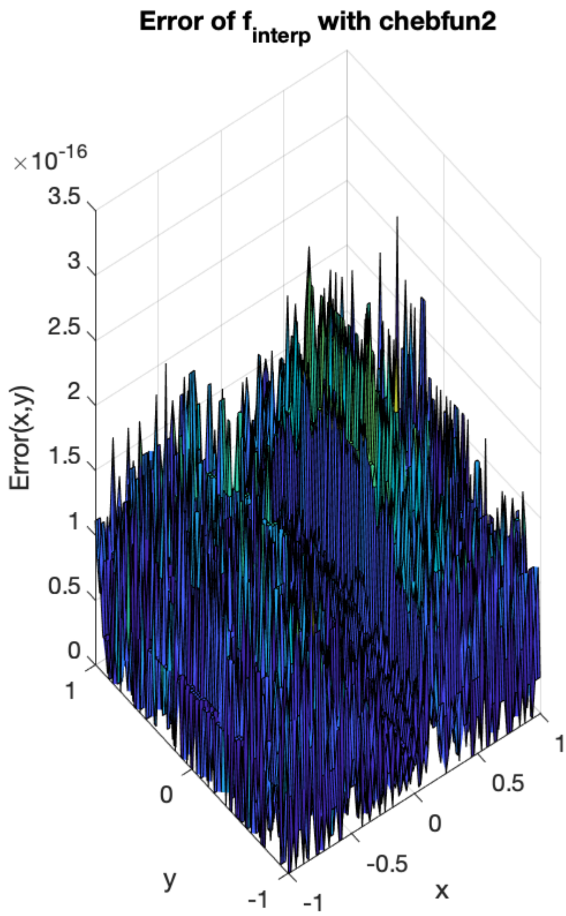

Fig. 1.2 Interpolation error for chebfun2 [7] method and f.

In addition, we analysed the accuracy for different number of chebyshev nodes. We have calculated for $N_1 = N_2 \in \{15, 20, 25, 30, 35, 40, 45, 50\}$ and compared it with chebfun2 algorithm.

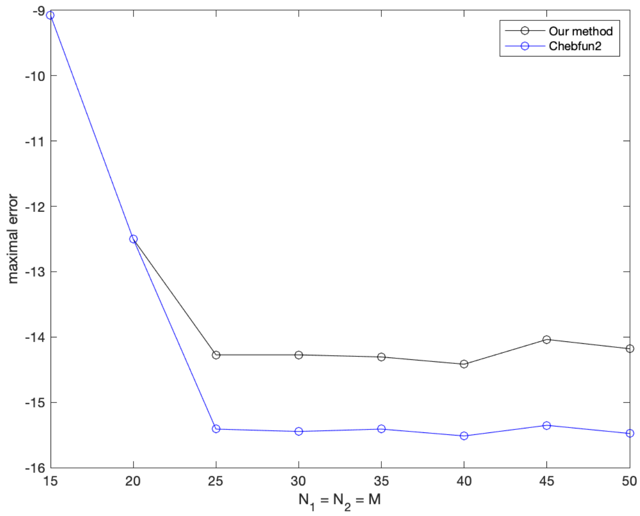

Fig. 1.3 Interpolation error decay for f ($\log_{10}$ basis).

What we can conclude from these results is that the method we have implemented is about 10 times less accurate than the chebfun2 algorithm, but we still have very good accuracy. The greater the number of Chebyshev nodes, the greater the accuracy of our method. This satisfies the error decay property of the Chebyshev interpolation.

Now we look an oscillating function to support our previous argument on the error decay and the accuracy of our method:

$$g : [-1,1]^2 \longrightarrow \mathbb{R}$$

$$g(x,y) = 20 + x^2 + y^2 - 10(\cos(2\pi x) + \cos(2\pi y))$$

The following results show that, we still are very accurate, even if the function is less simple.

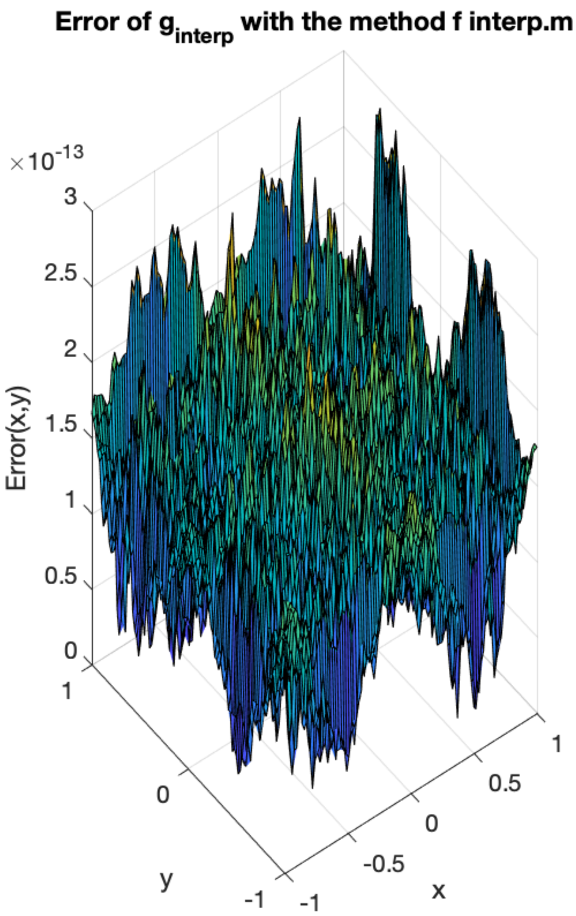

Fig. 1.4 Interpolation error for our method and g.

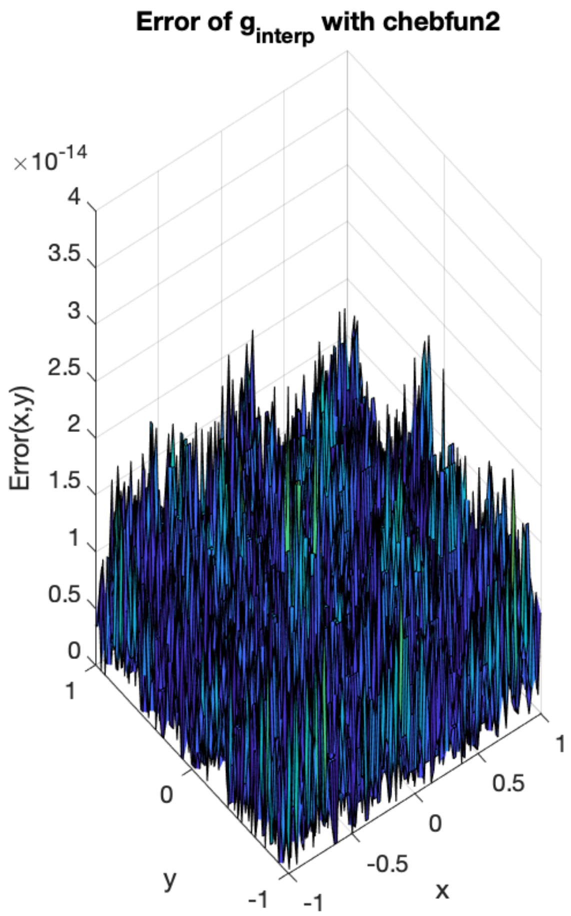

Fig. 1.5 Interpolation error for chebfun2 [7] method and g.

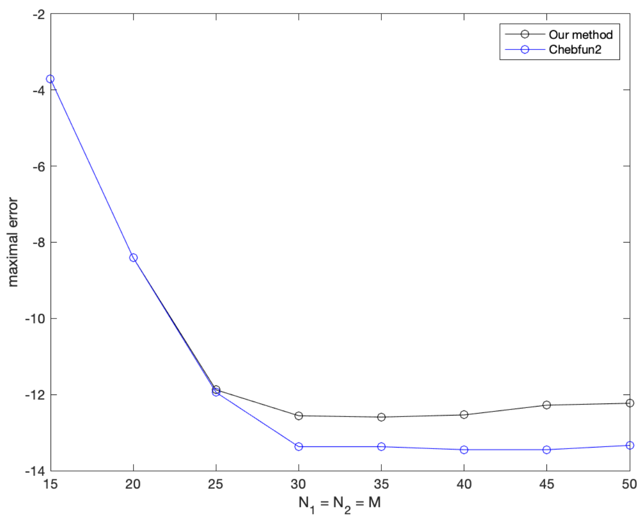

Fig. 1.6 Interpolation error decay for g ($\log_{10}$ basis).

Chapter 2

Interpolation experiment with the Chebyshev method for implied volatility

2.1 Introduction to financial materials

2.1.1 The Black-Scholes Model

The Black-Scholes model is an important concept in modern financial theory. This mathematical equation estimates the theoretical price of European options.

The formula requires five parameters which are :

- $S_0 =$ current stock price
- $K =$ strike price of the option
- $T =$ time to maturity (expressed in year)
- $r =$ risk-free interest rate
- $\sigma =$ volatility of the option price

And we have the Black-scholes equation :

$$C(S_0, K, T, r, \sigma) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where Φ denotes the Normal cumulative distribution function and d_1, d_2 are defined as :

$$d_1 = \frac{\log\left(\frac{S_0}{Ke^{-rT}}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\log\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

Notice that our goal is to interpolate the implied volatility thanks to this kind of function and the computational effort to interpolate a function depending on five variables is quite challenging. It is why we will use the Jäckel's (2015) [3] formula of the normalized call (Black-Scholes) price defined by :

$$c(x,v) = e^{\frac{x}{2}}\Phi\left(\frac{x}{v}+\frac{v}{2}\right) - e^{-\frac{x}{2}}\Phi\left(\frac{x}{v}-\frac{v}{2}\right) = \frac{C(S_0,K,T,r,\sigma)}{\sqrt{s_0 e^{-rT}K}}$$

with

$$x = \log\left(\frac{S_0 e^{rT}}{K}\right) = rT + \log\left(\frac{S_0}{K}\right)$$
$$v = \sigma\sqrt{T}$$

In this context x measures the *moneyness* and v corresponds to the *time-scaled volatility*.

2.2 Chebyshev method applied to the implied volatility

We now want to use our method implemented in Section 1 for implied volatility. However, the implied volatility function is not explicitly given. In fact, as we saw earlier, the implied volatility can be found using the normalized Black-Scholes equation.

$$c(x,v) = e^{\frac{x}{2}}\Phi\left(\frac{x}{v}+\frac{v}{2}\right) - e^{-\frac{x}{2}}\Phi\left(\frac{x}{v}-\frac{v}{2}\right)$$

From this equation, we will calculate reference points for the implied volatility $v(x,c)$ using Jackel's algorithm on Python since it is actually one of the most accurate method (see [5]). First, we take an interval for the value of moneyness $x \in [x_{min}, x_{max}]$. For the call price c, the maximum range depends on x because for $x < 0$ the upper bound is $e^{\frac{x}{2}}$. The intuitive approach is to choose $\xi \in [\xi_{min}, \xi_{max}]$ with $c = \xi e^{\frac{x}{2}}$ for a given moneyness $x \in [x_{min}, x_{max}]$. Thus, we have for each x located in the interval $[x_{min}, x_{max}]$ that $c \in [\xi_{min} e^{\frac{x}{2}}, \xi_{max} e^{\frac{x}{2}}]$. ξ_{min} has to be sufficiently close to 0 in order to avoid promising results when calculating the 2D-Chebyshev interpolation. Since we have determined our domains for the moneyness x and the call price c, we can now apply our method. We do this in two steps.

1. The interpolation stage:
   We take N_1 and N_2 chebyshev nodes in $[-1,1]$ and get

$$\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{N_1})^T \in [-1,1]^{N_1}$$

and

$$\tilde{c}_i = (\tilde{c}_{i,1}, \ldots, \tilde{c}_{i,N_2})^T \in [-1,1]^{N_2} \quad \text{for each} \quad i \in 1, \ldots, N_1$$

Then, we use the following transformations in order to scale the chebyshev nodes from $[-1,1]$ to the real domain of x and c we found above. The first transformation is

$$T_1 : [-1,1] \to [x_{min}, x_{max}]$$

$$T_1(x) := x_{min} + \frac{1}{2}(x+1)(x_{max} - x_{min}).$$

From this linear transformation, we transform our N_1, $\tilde{x}_i$ nodes and obtain $x_i = T_1(\tilde{x}_i)$ for $i = 1, \ldots, N_1$. The second transformation, for a fixed $x_i \in [x_{min}, x_{max}]$ given by the previous transformation, is

$$T_2 : [-1,1] \to [\xi_{min} e^{\frac{x}{2}}, \xi_{max} e^{\frac{x}{2}}]$$

$$T_2(c) := \xi_{min} e^{\frac{x_i}{2}} + \frac{1}{2}(c+1)(\xi_{max} - \xi_{min}) e^{\frac{x_i}{2}}.$$

Notice that for a given $x_i \in [x_{min}, x_{max}]$, the transformation is linear. Now, we transform each $\tilde{c}_{i,j} \in [-1,1]$ to get $c_{i,j} = T_2(\tilde{c}_{i,j}) \in [\xi_{min} e^{\frac{x}{2}}, \xi_{max} e^{\frac{x}{2}}]$ and that $\forall i = 1, \ldots, N_1; j = 1, \ldots, N_2$. At this time we have a $N_1 \times N_2$ area which is not rectangular but is a parallelogram. Now, we compute reference implied volatility points for this area. To this end, we apply the Jackël's function
`normalised_implied_volatility_from_a_transformed_rational_guess` from the library `py_lets_be_rational.py` [4] for each pair of $(x_i, c_{i,j})$. We then have $N_1 \times N_2$ implied volatility points $v_{imp_{i,j}}$. Let call V_{imp} the matrix such that $(V_{imp})_{i,j} = v_{imp_{i,j}}$. We compute the approximated implied volatility thanks to our Chebyshev interpolation implemented method. Thus we get the interpolated function [4]

$$V_{imp,Chebyshev}(x,y) := \texttt{f\_interp}(N_1,\ N_2,\ V_{imp},\ \tilde{x}_{cheb},\ \tilde{c}_{cheb},\ x,\ y,\ a,\ b,\ c,\ d)$$

where $\begin{cases} \tilde{x}_{cheb} = & \text{the vector } (\tilde{x}_1, \ldots, \tilde{x}_{N_1})^T \\ \tilde{c}_{cheb} = & \text{the matrix where } (\tilde{c}_{cheb})_{i,j} = \tilde{c}_{i,j} \ \ \forall i = 1, \ldots, N_1 \text{ and } j = 1, \ldots, N_2 \\ a = c = \text{-1} \\ b = d = 1 \end{cases}$

2. The test stage:
Now our goal is to check if our method is enough accurate to approximate the implied volatility. Then, we determine the interpolation error defining N equidistant points in $[x_{min}, x_{max}]$. Call $x_{equi} = (x_{eq_1}, \ldots, x_{eq_N})^T \in [x_{min}, x_{max}]^N$. We also define, for each x_{eq_i}, N

equidistant points that forms the vector $v_{eq_i} = (v_{eq_{i,1}}, ..., v_{eq_{i,N}})^T \in [v_{min}(x_{eq_i}), v_{max}(x_{eq_i})]^N$ where $v_{min}(x_{eq_i}), v_{max}(x_{eq_i})$ are computed thanks to the Jackël's function `normalised_implied_volatility_from_a_transformed_rational_guess` [4]. We have $v_{min}(x_{eq_i}) = v(\xi_{min} e^{\frac{x_{eq_i}}{2}}, x_{eq_i})$ and $v_{max}(x_{eq_i}) = v(\xi_{max} e^{\frac{x_{eq_i}}{2}}, x_{eq_i})$. Let V_{equi} be the matrix given by $(V_{equi})_{i,j} = v_{eq_{i,j}}$.
At this time we compute reference normalized call price equidistant points using the Jackël's function `normalised_black` [4]. This leads to a grid of $N \times N$ of normalized call price points that we put into the matrix C_{equi}. In order to call our build interpolated function, we need to transform these points to $[-1,1]$. So we use T_1^{-1} and T_2^{-1} on x_{equi} and C_{equi} respectively and get $\tilde{x}_{equi}$ and $\tilde{C}_{equi}$. Note that T_1^{-1} and T_2^{-1} are linear since T_1 and T_2 are. Now we just have to call the build interpolated function

$$V_{imp,Chebyshev}(x,y)$$

for

$$x = \tilde{x}_{equi} \qquad \text{and} \qquad y = \tilde{C}_{equi}.$$

which gives us

$$V_{imp,Chebyshev}(\tilde{x}_{equi}, \tilde{C}_{equi}) = V_{equi,Chebyshev}.$$

And thus define the interpolated error as

$$error := |V_{equi} - V_{equi,Chebyshev}|$$

which is the absolute value of the reference implied volatility computed from Jackël algorithm and the interpolated implied volatility.

2.3 Numerical example

For a numerical example, we fix $x_{min} = -5,\ \ x_{max} = 0,\ \ \xi_{min} = 0.05$ and $\xi_{max} = 0.8$. Also, we fix $N = 100$ and $N_1 = N_2 = 51$.

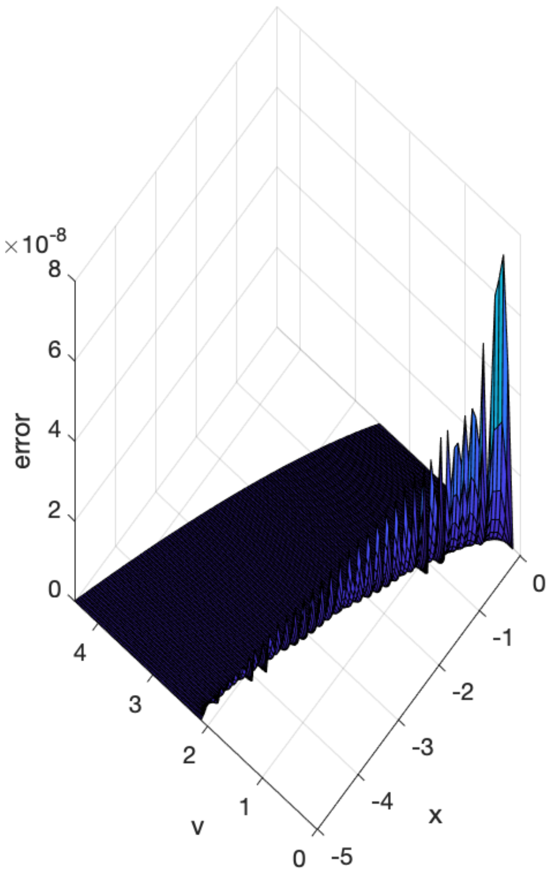

Fig. 2.1 Error of the approximation of the implied volatility using our method.

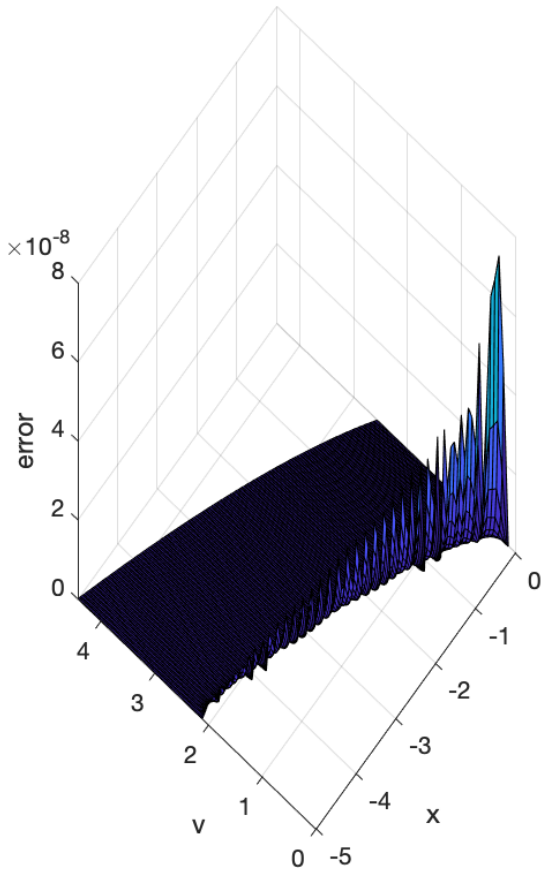

Fig. 2.2 Error of the approximation of the implied volatility using chebfun2 [7].

We also compute the maximal error of our method and the chebfun2 algorithm [7] for $N_1 = N_2 \in \{10, 40, 70, 100, 130\}$ in order to verify the decay of the error.

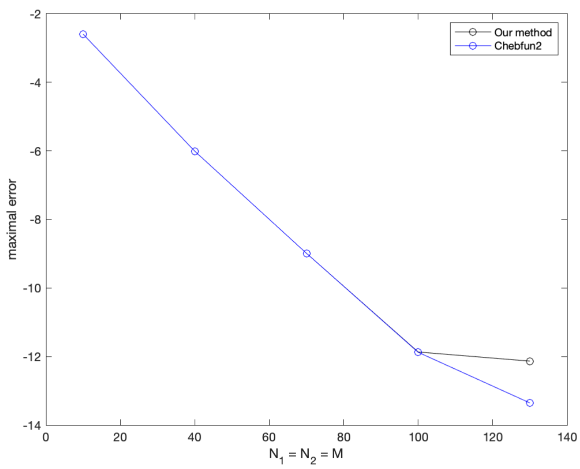

Fig. 2.3 Interpolation error decay for the implied volatility ($\log_{10}$ basis).

Notice that the Chebyshev method performs very well. The maximal error lies below a level of 10^{-8} for $N_1 = N_2 = 51$ and decreases exponentially fast in M.

Chapter 3

Optimisation of the approximation and analysis of the interpolation error

3.1 Scaling the function and splitting the domain for a better approximation of the implied volatility

In the first numerical example, we chose an interpolation domain for the implied volatility that was restricted but not optimal. Now we look for the optimal domain. The implied volatility is not analytic at $c(x) = 0$ and $c(x) = e^{\frac{x}{2}}$. Extending the domain towards the maximal domain decreases the rate of convergence. To reduce this impact, we exploit the limit behaviour of the call price. Therefore we need to restricted the interval to $0 < v_{min}(x) < v_{max}(x) < \infty$ with call prices $0 < c_{min}(x) < c_{max}(x) < e^{\frac{x}{2}}$ for each moneyness $x \in [x_{min}, x_{max}]$. By exploiting the limit behavior of the normalized call price function, we can find the best domain to approximate the implied volatility. The idea is to split the current domain into three new domains.

$$D_1 := [c_{min}(x), c_1(x)], \quad D_2 := [c_1(x), c_2(x)], \quad D_3 := [c_2(x), c_{max}(x)]$$

with corresponding implied volatility

$$0 < v_{min}(x) < v_1(x) < v_2(x) < v_{max}(x) < \infty.$$

After that we will approximate the implied volatility on each domain. The derivation of the domains is based on Jackël's method [3]. For each domain, we compute the 2D-Chebyshev interpolation. Moreover, where call prices are flat, the implied volatility becomes very steep. Hence, a direct polynomial interpolation is not well-suited. Fortunately, by exploiting the asymptotic behaviour of the call price function explained in the paper [5], we dodge this issue.

On each domain, we use the scaling functions $\phi_{i,x} : D_i \longrightarrow [-1,1], \quad \forall i = 1,2,3$ defined in the paper [5] which, for each $x \in [x_{min}, x_{max}]$, transforms the call price to $[-1,1]$. Finally, we get the interpolations

$$I_i^{N_1,N_2} : [-1,1]^2 \longrightarrow \mathbb{R}$$

$$I_i^{N_1,N_2}(\phi_{i,x}(c), \varphi(x)) \approx v(x,c), \qquad \text{where } i \text{ satisfies } c \in D_i$$

For the sake of clarity, we clarify the functions used for the splitting and the scaling.

3.1.1 Splitting functions for the domains

$$D_1 := [c_{min}(x), c_1(x)], \quad D_2 := [c_1(x), c_2(x)], \quad D_3 := [c_2(x), c_{max}(x)]$$

are defined assuming that we have the corresponding implied volatility functions (for x in $[x_{min}, x_{max}]$).

- $v_{min}(x) = 0.001 - 0.03x$.
- $v_1(x) = 0.25 - 0.4x$.
- $v_2(x) = 2 - 0.4x$.
- $v_{max}(x) = 6$.

These functions are found by exploiting the limit behavior of the normalized call price function as described in [5].

3.1.2 Scaling functions for the interpolation

First we have

$$\varphi : [x_{min}, x_{max}] \longrightarrow [-1,1]$$

$$\varphi(x) := 1 - 2 \cdot \frac{x_{max} - x}{x_{max} - x_{min}}.$$

Then we have for each domain and for the data we used before (i.e. $x \in [-5, 0]$:

- $\mathbf{D_1}$:
  We use two different interpolations for the following areas:
  Area I: For $x \in [-5, -0.0348]$ and $c \in [c_{min}(x), c_1(x)]$ we have

$$\phi_{1,x}(c) := 2 \cdot \frac{\tilde{\phi}_1(c) - \tilde{\phi}_1(c_{min}(x))}{1 - \tilde{\phi}_1(c_{min}(x))} - 1.$$

where

$$\tilde{\phi_1} : [0, c_1(x)] \longrightarrow [-1, 1]$$

$$\tilde{\phi_1}(c) = \begin{cases} 2 \cdot \left(-\frac{2}{(x-\delta)^2} \cdot \ln(c) + \frac{2}{(x-\delta)^2} \cdot \ln(c_1(x)) + 1 \right)^{-\frac{1}{2}} - 1 & \text{if } c > 0 \\ -1 & \text{else} \end{cases}$$

Area I': For $x \in [-0.0348, 0]$ and $c \in [c_{min}(x), c_1(x)]$ we also use the transformation $\phi_{1,x}(c)$.

- **D$_2$** : For $x \in [-5, 0]$ and $c \in [c_1(x), c_2(x)]$ we have

$$\phi_{2,x}(c) := 2 \cdot \frac{c - c_1(x)}{c_2(x) - c_1(x)} - 1.$$

- **D$_3$** : For $x \in [-5, 0]$ and $c \in [c_2(x), c_{max}(x)]$ we have

$$\phi_{3,x}(c) := \frac{2\tilde{\phi_3}(c)}{\tilde{\phi_3}(c_{max}(x))} - 1.$$

where

$$\tilde{\phi_3} : [c_2(x), e^{\frac{x}{2}}] \longrightarrow [0, \infty]$$

$$\tilde{\phi_3}(c) = \begin{cases} \left(-8 \cdot \ln \left(\frac{e^{\frac{x}{2}} - c}{e^{\frac{x}{2}} - c_2(x)} \right) \right)^{\frac{1}{2}} & \text{if } c < e^{\frac{x}{2}} \\ \infty & \text{else} \end{cases}$$

3.1.3 Numerical experiment

After finding the scaling and splitting functions for each domain, we can calculate the implied volatility using the Chebyshev method that we implemented in the first section. We can achieve different levels of accuracy depending on the number of Chebyshev nodes on each domain. In this paper, we calculate the implied volatility interpolation function for a medium level of accuracy. As in the paper [5] we have:

- **D$_1$ (Area I)** : $N_1 = 46, \ \ N_2 = 79.$
- **D$_1$ (Area I')** : $N_1 = 51, \ \ N_2 = 39.$
- **D$_2$** : $N_1 = 36, \ \ N_2 = 33.$
- **D$_3$** : $N_1 = 17, \ \ N_2 = 14.$

We obtain the following results for our method against the chebfun2 algorithm [7]

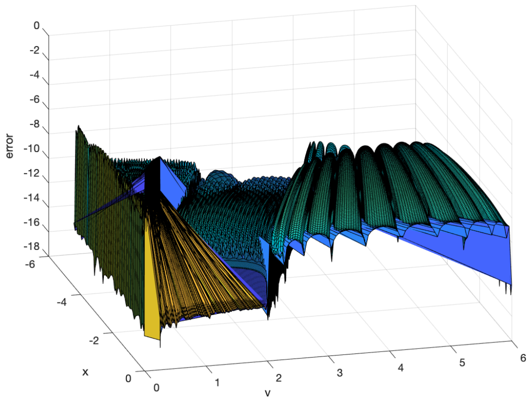

Fig. 3.1 Error of the approximation of the implied volatility using our method ($\log_{10}$ basis).

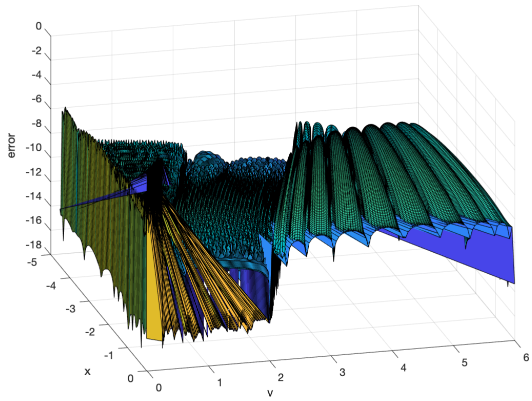

Fig. 3.2 Error of the approximation of the implied volatility using chebfun2 [7] ($\log_{10}$ basis).

3.2 Error estimate for Chebyshev interpolation of analytic functions

3.2.1 Prerequisites for the following results

In this section we will prove one important theorem which allow to give an upper bound for the error of Chebyshev interpolation of analytic functions. In order to fully understand the notions mentioned in the following. Here are some recall.

Definition 1 (Bernstein Ellipse ε_ρ)**.**

$$\varepsilon_\rho := \{z \in \mathbb{C} \mid z = \frac{1}{2}(u+u^{-1}),\ u = \rho e^{i\theta},\ \rho > 1,\ 0 \leq \theta < 2\pi\}$$

For $d \in \mathbb{N}$, $\overrightarrow{\varepsilon}_\rho := \varepsilon_{\rho_1} \times \ldots \times \varepsilon_{\rho_d}$ *where* $\rho := (\rho_1, \ldots, \rho_d)^T \in \mathbb{R}^d$.

Definition 2 (Tensorized Chebyshev polynomials)**.** *For* $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}_0^3$ *and* $\boldsymbol{x} = (x_1, x_2, x_3) \in [-1,1]^3$. *The tensorized Chebyshev polynomials is given by*

$$T_\mu(\boldsymbol{x}) := T_{\mu_1}(x_1) T_{\mu_2}(x_2) T_{\mu_3}(x_3)$$

.

Lemma 3.2.1 (Bound for Tensorized Chebyshev polynomials)**.** *1. For all $x \in [-1,1]$ and $n,m \in \mathbb{N}_0$*

$$|T_m^{(n)}(x)| \leq \tau_m^{(n)} \text{ where } \tau_m^{(n)} := \prod_{i=0}^{n-1} \frac{m^2 - i^2}{2i+1} \tag{3.1}$$

2. For all $\boldsymbol{x} \in [-1,1]^3$

$$|T_\mu(\boldsymbol{x}) - \overrightarrow{\prod}^{(m)}[T_\mu](\boldsymbol{x})| = 0, \ \forall \mu \in \mathbb{N}_0^3: \ max_{1\leq i\leq 3}\, \mu_i \leq m-1. \tag{3.2}$$

$$|T_\mu(\boldsymbol{x}) - \overrightarrow{\prod}^{(m)}[T_\mu](\boldsymbol{x})| \leq 2, \ \forall \mu \in \mathbb{N}_0^3: \ max_{1\leq i\leq 3}\, \mu_i > m-1. \tag{3.3}$$

Proof. The proof of this lemma is given in the Sauter-Schwab book [Corollary 7.3.1]. [6] □

3.2.2 Lemma for the main theorem on the bound of the error between analytic function and its Chebyshev interpolation

We introduce an important Lemma of Sauter, S. and C. Schwab. [6] for the interpolation error.

Lemma 3.2.2. *Let $d = 3$, $Q = [-1,1]^d$ and let a function $f \in C^0(Q)$ be given that can be extended to an analytic function $f^\star$ on $\overrightarrow{\varepsilon}_\rho$ with $\rho_i > 1$, $1 \leq i \leq 3$. Then for the Chebyshev interpolation $p_m = \overrightarrow{\prod}^{(m)}[f]$ the error estimate :*

$$\|f - p_m\|_{C^0(Q)} \leq \sqrt{d}2^{\frac{d}{2}+1}\rho_{min}^{-m}(1-\rho_{min}^{-2})^{-\frac{d}{2}}M_\rho(f)$$

Holds with

$$M_\rho(f) := max_{z\in\overrightarrow{\varepsilon}_\rho}|f^\star(z)|$$

Proof. The main idea of the proof is to use an interpolation error estimates for Chebyshev polynomials and a theorem of approximation of analytic functions. In particular we will use the fact that Chebyshev polynomials are dense with respect to a L^2 metric so that will allow us to show the result for this subspace of Chebyshev polynomials.

To this aim we introduce the inner product, $\forall f,g \in C^0(Q)$

$$\langle f,g\rangle_\rho := \int_{\overrightarrow{\varepsilon}_\rho} \frac{f(\mathbf{z})\overline{g(\mathbf{z})}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} d\mathbf{z}$$

and the Hilbert space

$$L^2(\overrightarrow{\varepsilon}_\rho) := \{f \ : \ f \text{ is analytic in } \overrightarrow{\varepsilon}_\rho \text{ and } \|f\|_\rho := \langle f,f\rangle_\rho^{\frac{1}{2}} < \infty\}$$

In this proof we will assume that this space is a Hilbert space. It is, in fact, a separable, closed subspace of the Lebesgue space $L^2(Q)$. Notice that there are two important properties on this space.

- First, the evaluation at points on $L^2(\overrightarrow{\varepsilon}_\rho)$ is well defined and the associated operator is continuous in a way that $\exists C$ a constant such that

$$sup_{\mathbf{z}\in\overrightarrow{\varepsilon}_\rho}|f(\mathbf{z})| \leq C\|f\|_\rho \qquad \forall f \in L^2(\overrightarrow{\varepsilon}_\rho).$$

- Second, $L^2(\overrightarrow{\varepsilon}_\rho)$ is a Hilbert space which bring powerfull properties of functional analysis.

We can now begin the real part of the proof. Define the scaled Chebyshev polynomials as follow

$$\tilde{T}_\mu(\mathbf{z}) := c_\mu T_\mu(\mathbf{z}) \qquad \text{where} \qquad c_\mu := \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \prod_{i=1}^{d} \left(\rho_i^{2\mu_i} + \rho_i^{-2\mu_i}\right)^{-\frac{1}{2}} \qquad \forall \mu \in \mathbb{N}_0^d.$$

With this scaling we obtain that the system $(\tilde{T}_\mu)_{\mu\in\mathbb{N}_0^d}$ is a complete orthonormal system for $L^2(\overrightarrow{\varepsilon}_\rho)$ with respect to the inner product $\langle .,.\rangle_\rho$. For $\mu,\delta \in \mathbb{N}_0^d$ we have

$$\begin{aligned}
\langle \tilde{T}_\mu, \tilde{T}_\delta\rangle_\rho &= \int_{\overrightarrow{\varepsilon}_\rho} \frac{\tilde{T}_\mu(\mathbf{z})\overline{\tilde{T}_\delta(\mathbf{z})}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} d\mathbf{z} \\
&= \int_{\overrightarrow{\varepsilon}_\rho} \frac{c_\mu T_\mu(\mathbf{z})\overline{c_\delta T_\delta(\mathbf{z})}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} d\mathbf{z} \\
&= \int_{\overrightarrow{\varepsilon}_\rho} \frac{c_\mu T_\mu(\mathbf{z}) c_\delta \overline{T_\delta(\mathbf{z})}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} d\mathbf{z} \\
&= \int_{\overrightarrow{\varepsilon}_\rho} c_\mu c_\delta \frac{T_{\mu_1}(z_1)T_{\mu_2}(z_2)T_{\mu_3}(z_3)\overline{T_{\delta_1}(z_1)T_{\delta_2}(z_2)T_{\delta_3}(z_3)}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} dz_1dz_2dz_3 \\
&= c_\mu c_\delta \underbrace{\prod_{i=1}^{d} \int_{\rho_i} \frac{T_{\mu_i}(z_i)\overline{T_{\delta_i}(z_i)}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} dz_i}_{(3.4)}
\end{aligned}$$

We know from the base case that

$$\begin{cases} c_\mu c_\delta & \text{if } \mu_i = \delta_i \quad \forall i = 1,\dots,d \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

3.4 follow from the fact that if $n \neq m$ then $\langle T_n(z), T_m(z)\rangle_w = 0$ for $\langle .,.\rangle_w$ the inner product

$$\langle f(z), g(z)\rangle_w = \int_{-1}^{1} \frac{f(z)g(z)}{w(z)} dz \qquad \text{where} \qquad w(z) = \sqrt{1-z^2}.$$

We have shown that the system is orthonormal.
Moreover, $(\tilde{T}_\mu)_{\mu\in\mathbb{N}_0^d}$ is complete because Chebyshev polynomials are dense (with respect to the L^2 metric) in the space of analytic continuous function. The subspace they span is the one of all polynomials and we know thanks to Stone-Weierstrass's theorem that the subspace of polynomials is dense in the space of continuous function.
Now, using that $L^2(\overrightarrow{\varepsilon}_\rho)$ is a Hilbert space, $\forall E$ a bounded functional on $L^2(\overrightarrow{\varepsilon}_\rho)$

$$|E(f)| \leq \underbrace{\|E\|_\rho}_{\text{operator norm}} \|f\|_\rho \tag{3.5}$$

Notice that $\|E\|_\rho$ satisfies

$$\|E\|_\rho = \sup_{f\in L^2(\overrightarrow{\varepsilon}_\rho)\setminus\{0\}} = \frac{|E(f)|}{\|f\|_\rho} = \sqrt{\sum_{\mu\in\mathbb{N}_0^d} |E(\tilde{T}_\mu)|^2}$$

Let E be the error of the Chebyshev interpolation at a point $\mathbf{x} \in Q$ i.e.,

$$E(f) = f(\mathbf{x}) - p_m(\mathbf{x})$$

Using that

1. $E(p) = 0 \; \forall p \in \mathbb{P}_{m-1} := \text{span}\{x^i \;:\; i = 0,\dots,m-1\}$ and
2. Lemma 3.2.2, we have

$$
\begin{aligned}
\|E\|_\rho^2 &= \sum_{\mu\in\mathbb{N}_0^d} |E(\tilde{T}_\mu)|^2 = \sum_{\mu\in\mathbb{N}_0^d} c_\mu^2 |E(T_\mu)|^2 \overset{(1)}{=} \sum_{\mu\in\mathbb{N}_0^d,\ |\mu|_\infty\geq m} c_\mu^2 |E(T_\mu)|^2 \\
&\overset{(2)}{\leq} 4 \sum_{\mu\in\mathbb{N}_0^d,\ |\mu|_\infty\geq m} c_\mu^2 = 4 \sum_{\mu\in\mathbb{N}_0^d,\ \mu_i\geq m} \left(\left(\frac{2}{\pi}\right)^d \prod_{j=1}^d \left(\rho_j^{2\mu_j} + \rho_j^{-2\mu_j}\right)^{-1} \right) \\
&\leq 4 \left(\frac{2}{\pi}\right)^d \sum_{i=1}^d \left(\sum_{\mu\in\mathbb{N}_0^d,\ \mu_i\geq m} \prod_{j=1}^d \left(\rho_j^{2\mu_j} + \rho_j^{-2\mu_j}\right)^{-1} \right) \\
&\leq 4 \left(\frac{2}{\pi}\right)^d \sum_{i=1}^d \left(\sum_{\mu\in\mathbb{N}_0^d,\ \mu_i\geq m} \prod_{j=1}^d \rho_j^{-2\mu_j} \right) \\
&\leq 4 \left(\frac{2}{\pi}\right)^d \sum_{i=1}^d \rho_i^{-2m} \left(\sum_{\mu\in\mathbb{N}_0^d,\ \mu_i\geq m} \rho_i^{-2(\mu_i-m)} \prod_{j=1,\ j\neq i}^d \rho_j^{-2\mu_j} \right) \\
&\leq 4 \left(\frac{2}{\pi}\right)^d \sum_{i=1}^d \rho_i^{-2m} \left(\sum_{\mu\in\mathbb{N}_0^d} \prod_{j=1}^d \rho_j^{-2\mu_j} \right) \\
&\leq 4 \left(\frac{2}{\pi}\right)^d \rho_{min}^{-2m} d \sum_{\mu\in\mathbb{N}_0^d} \rho_{min}^{-2|\mu|} = 4 \left(\frac{2}{\pi}\right)^d \rho_{min}^{-2m} d (1-\rho_{min}^{-2})^{-d}
\end{aligned}
$$

In order to use (3.5), we still need to find a bound for $\|f\|_\rho$. We have

$$
\|f\|_\rho^2 = \int_{\overrightarrow{\varepsilon}_\rho} \frac{f(\mathbf{z})\overline{f(\mathbf{z})}}{\prod_{i=1}^d \sqrt{|1-z_i^2|}} d\mathbf{z} \leq \left(sup_{\mathbf{z}\in\overrightarrow{\varepsilon}_\rho} |f(\mathbf{z})| \right)^2 \|1\|_\rho^2
$$

It follows from $\pi^{\frac{d}{2}}\tilde{T}_0 = 1$ and the orthonormality of the system $(\tilde{T}_\mu)_{\mu\in\mathbb{N}_0^d}$ that

$$
\|f\|_\rho^2 \leq \pi^d M_\rho^d(f)
$$

Thus we obtain

$$
\|E(f)\| = \|f-p_m\|_{C^0(Q)} \leq \sqrt{d} 2^{\frac{d}{2}+1} \rho_{min}^{-m} (1-\rho_{min}^{-2})^{-\frac{d}{2}} M_\rho(f)
$$

□

3.2.3 Theorem of the error between analytic function and its Chebyshev interpolation

The following theorem is the theoretical foundation of the high efficiency of the approximation method. Thanks to the analyticity of the Black-Scholes call price and the scaling functions, we gather that the convergence is sub-exponential in the number of nodal points.

Theorem 3.2.3. *Let $\phi_i^{-1}(\tilde{c},\tilde{x})$ be analytically continuable to some open region around $[-1,1]^2$ and let $0 < \phi_i^{-1}([-1,1],\tilde{x}) < e^{\frac{x}{2}}$ for each $x \in [-1,1]$. Then there exist constants $\rho_1, \rho_2 > 1$, $V > 0$ such that for $\tilde{v}(\tilde{c},\tilde{x}) := v(\phi_i^{-1}(\tilde{c},\tilde{x}), \varphi^{-1}(\tilde{x}))$ and its bivariate Chebyshev interpolation $I_i^{N_1^i,N_2^i}(\tilde{c},\tilde{x}) := \sum_{j=0}^{N_1^i-1}\sum_{k=0}^{N_2^i-1} a_{jk}T_j(\tilde{c})T_k(\tilde{x})$ we have :*

$$\max_{(\tilde{c},\tilde{x})\in[-1,1]^2} |\tilde{v}(\tilde{c},\tilde{x}) - I_i^{N_1^i,N_2^i}(\tilde{c},\tilde{x})| \leq 4V \left(\frac{\rho_1^{-2(N_1-1)} + \rho_2^{-2(N_2-1)}}{(1-\rho_1^{-2})(1-\rho_2^{-2})} \right)^{\frac{1}{2}}$$

Proof. First, notice that we know from Complex Analysis that analytic $\Leftrightarrow$ holomorphic. We will use this equivalence throughout the proof and use mainly the properties of holomorphic functions. In order to prove this theorem, we apply the previous lemma to the function $\tilde{v}(\tilde{c},\tilde{x}) := v(\phi_i^{-1}(\tilde{c},\tilde{x}), \varphi^{-1}(\tilde{x}))$. Thus, we need to show that this function is analytically continuable and bounded on $\varepsilon_{\rho_1} \times \varepsilon_{\rho_2}$ for some ρ_1, ρ_2 satisfying the lemma.
First we show the analyticity of $\tilde{v}$ with respect to the variable $\tilde{c}$ for $\tilde{x} \in [-1,1]$ fixed. [Gaß et al. (2015)] [2] already shown that the call price $\tilde{c}$ is analytic. Now, for $\tilde{x} \in [-1,1]$ fixed, we can deduce that $\tilde{v}$ is holomorphic with respect to $\tilde{c} \in [-1,1]$ since the last is holomorphic and we have a theorem that says that the inverse of a bijective holomoprhic function is also holomorphic. In fact, previously, we saw that $\tilde{c}$ and $\tilde{v}$ are related by the fact that they are inverse one from an other.
To complete the analyticity of $\tilde{v}$, we show that it is analytic with respect to $\tilde{x}$ for $\tilde{c} \in [-1,1]$ fixed. So let $\tilde{c} \in [-1,1]$ and define

$$F(x,v) := c(x,v) - \phi_i(\tilde{c},x)$$

that is the normalized call price function minus the scaled call price c on the domain D_i. The function $v(\phi_i(\tilde{c},x),x)$ is implicitly given by the solution of $F(x,v) = 0$. By construction, $\phi_i(\tilde{c},x)$ is holomorphic and by the discussion above, the call price $c(x,v)$ is also holomorphic $\Longrightarrow$

$F(x,v)$ is holomorphic in some open region arround x. Moreover, we have

$$\begin{aligned}
\left|\frac{\partial}{\partial v}F(x,v)\right| &= \left|\frac{\partial}{\partial v}\left(c(x,v)-\phi_i(\tilde{c},x)\right)\right| \\
&= \left|\frac{\partial}{\partial v}c(x,v)\right| \\
&= \left|\frac{\partial}{\partial v}\left(e^{\frac{x}{2}}\Phi\left(\frac{x}{v}+\frac{v}{2}\right)-e^{-\frac{x}{2}}\Phi\left(\frac{x}{v}-\frac{v}{2}\right)\right)\right| \\
&\overset{(*)}{=} \left|e^{\frac{x}{2}}\frac{\partial}{\partial v}\left(\frac{x}{v}+\frac{v}{2}\right)f\left(\frac{x}{v}+\frac{v}{2}\right)-e^{-\frac{x}{2}}\frac{\partial}{\partial v}\left(\frac{x}{v}-\frac{v}{2}\right)f\left(\frac{x}{v}-\frac{v}{2}\right)\right| \\
&= \left|e^{\frac{x}{2}}\left(-\frac{x}{v^2}+\frac{1}{2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{v}+\frac{v}{2}\right)^2}-e^{-\frac{x}{2}}\left(-\frac{x}{v^2}-\frac{1}{2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{v}-\frac{v}{2}\right)^2}\right| \\
&= \left|\left(-\frac{x}{v^2}+\frac{1}{2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2v^2}-\frac{v^2}{8}}-\left(-\frac{x}{v^2}-\frac{1}{2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2v^2}-\frac{v^2}{8}}\right| \\
&= \left|\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2v^2}-\frac{v^2}{8}}\right| > 0 \qquad \text{as } v>0.
\end{aligned}$$

In $(*)$ we defined f to be the PDF of the normal distribution. i.e $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Now, since $\left|\frac{\partial}{\partial v}F(x,v)\right|>0$ in some open region we can use the Complex implicit function theorem (see Theorem 7.6 of Fritzsche and Grauert (2012)) [1] $\implies$ there exits a unique function $v(\phi_i(\tilde{c},x),x)$ that is holomorphic arround x.
Thus $\tilde{v}$ is holomorphic in $G_1\times G_2$ where G_1,G_2 are open regions of $[-1,1] \implies \exists\rho_1,\rho_2$, given by the lemma, such that $G_1\subset\varepsilon_{\rho_1}$ and $G_2\subset\varepsilon_{\rho_2}$. As $\tilde{v}$ is continuous in $[-1,1]^2$, we just have to take small ρ_1,ρ_2 to get the boundedness.
Thus from the lemma we have

$$\begin{aligned}
\text{error} &= \|\tilde{v}(\tilde{c},\tilde{x})-I_i^{N_1^i,N_2^i}(\tilde{c},\tilde{x})\|_{C^0([-1,1])} \\
&= \max_{(\tilde{c},\tilde{x})\in[-1,1]^2}|\tilde{v}(\tilde{c},\tilde{x})-I_i^{N_1^i,N_2^i}(\tilde{c},\tilde{x})| \\
&\leq 4V\left(\frac{\rho_1^{-2(N_1-1)}+\rho_2^{-2(N_2-1)}}{(1-\rho_1^{-2})(1-\rho_2^{-2})}\right)^{\frac{1}{2}}
\end{aligned}$$

□

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