2naire

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1 Explanation

Let $n \in \mathbb{N}$ and $t \in [0,1]$. Let i $\in \{1, \ldots, n\}$ and $U_i \sim$ Uniforme$(t, 1)$.
Y_i is the expected winning with i questions left.

$$Y_i = \mathbb{P}(U_i < x) + \mathbb{P}(U_i \geq x) \cdot \mathbb{E}[U_i | U_i \geq x] \cdot 2 \cdot Y_{i-1}$$

Here is the idea. x is a threshold. If U_i is less then x, we take whatever we currently have. In the opposite case, we calculate the expected value that we win considering that the probability is above x. If we get it, the expected winning will be 2 time the expected winning with i-1 questions left. The only thing left is to find the best x to maximize the expected winning.
If $x <= t$ then $\mathbb{P}(U_i < x <= t) = 0$ and $\mathbb{E}[U_i | U_i \geq x] = \mathbb{E}[U_i] = \frac{1+t}{2}$

$$Y_i = 0 + 1 \cdot \frac{1+t}{2} \cdot 2 \cdot Y_{i-1} = (1+t) \cdot Y_{i-1}$$

If $x > t$ then

$$Y_i = \frac{x-t}{1-t} + (1 - \frac{x-t}{1-t}) \cdot \int_x^1 \frac{s}{1-x} \text{ ds } \cdot 2 \cdot Y_{i-1}$$
$$\frac{x-t}{1-t} + (\frac{1-t}{1-t} - \frac{x-t}{1-t}) \cdot \frac{1-x^2}{2 \cdot (1-x)} \cdot 2 \cdot Y_{i-1}$$
$$\frac{x-t}{1-t} + \frac{1-x^2}{1-t} \cdot Y_{i-1}$$
$$\frac{-x^2 \cdot Y_{i-1} + x + Y_{i-1} - t}{1-t}$$

Deriving the polynomial and finding the zero of the fonction gives us the optimal point.

$$\frac{-2x \cdot Y_{i-1} + 1}{1-t} \Rightarrow x = \frac{1}{2 \cdot Y_{i-1}}$$

We therefore get

$$Y_i = \begin{cases} \frac{\frac{1}{4Y_{i-1}} + Y_{i-1} - t}{1-t} : \text{if } \frac{1}{2Y_{i-1}} > t \\ (1+t) \cdot Y_{i-1} : \text{otherwise} \end{cases}$$

And of course $Y_0 = 1$