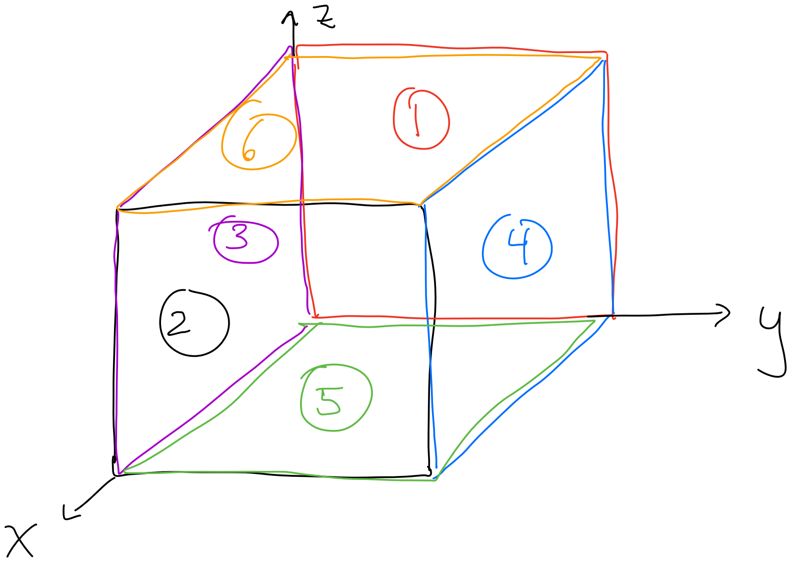


$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \sigma$$

$$\begin{cases} \rho \dfrac{\partial^2 u_1}{\partial t^2} = \dfrac{\partial}{\partial x}\sigma_{xx} + \dfrac{\partial}{\partial y}\sigma_{xy} + \dfrac{\partial}{\partial z}\sigma_{xz} \\[2ex] \rho \dfrac{\partial^2 u_2}{\partial t^2} = \dfrac{\partial}{\partial x}\sigma_{xy} + \dfrac{\partial}{\partial y}\sigma_{yy} + \dfrac{\partial}{\partial z}\sigma_{yz} \\[2ex] \rho \dfrac{\partial^2 u_3}{\partial t^2} = \dfrac{\partial}{\partial x}\sigma_{xz} + \dfrac{\partial}{\partial y}\sigma_{yz} + \dfrac{\partial}{\partial z}\sigma_{zz} \end{cases}$$

$$C_{ijkl} = K\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)$$

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\begin{cases} \sigma_{xx} = \lambda\left[\dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} + \dfrac{\partial u_3}{\partial z}\right] + 2\mu \dfrac{\partial u_1}{\partial x} \\[2ex] \sigma_{xy} = \mu\left[\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x}\right] \end{cases}$$

$$\begin{cases} \sigma_{xz} = \mu\left[\dfrac{\partial u_1}{\partial z} + \dfrac{\partial u_3}{\partial x}\right] \\ \sigma_{yy} = \lambda\left[\dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} + \dfrac{\partial u_3}{\partial z}\right] + 2\mu\dfrac{\partial u_2}{\partial y} \\ \sigma_{yz} = \mu\left[\dfrac{\partial u_2}{\partial z} + \dfrac{\partial u_3}{\partial y}\right] \\ \sigma_{zz} = \lambda\left[\dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} + \dfrac{\partial u_3}{\partial z}\right] + 2\mu\dfrac{\partial u_3}{\partial z} \end{cases}$$

$$\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right] + 2\mu\frac{\partial u_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right]\right) + \frac{\partial}{\partial z}\left(\mu\left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right]\right) + f_1$$

$$\rho\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu\left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right]\right) + \frac{\partial}{\partial y}\left(\lambda\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right] + 2\mu\frac{\partial u_2}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right]\right) + f_2$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right] \right) + \frac{\partial}{\partial y}\left(\mu \left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \right] \right) + \frac{\partial}{\partial z}\left(\lambda \left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right] + 2\mu \frac{\partial u_3}{\partial z} \right) + f_3$$

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \lambda \left[D_{2x} u_1 + D_x D_y u_2 + D_x D_z u_3 \right] + 2\mu D_{2x} u_1 + \mu \left[D_{2y} u_1 + D_y D_x u_2 \right] + \mu \left[D_{2z} u_1 + D_z D_x u_3 \right] + f_1$$

$$\left(K = \lambda + \frac{2}{3}\mu \right)$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = \mu \left[D_x D_y u_1 + D_{2x} u_2 \right] + \lambda \left[D_y D_x u_1 + D_{2y} u_2 + D_y D_z u_3 \right]$$

$$+ 2\mu \left[D_{zy} u_2 \right]$$

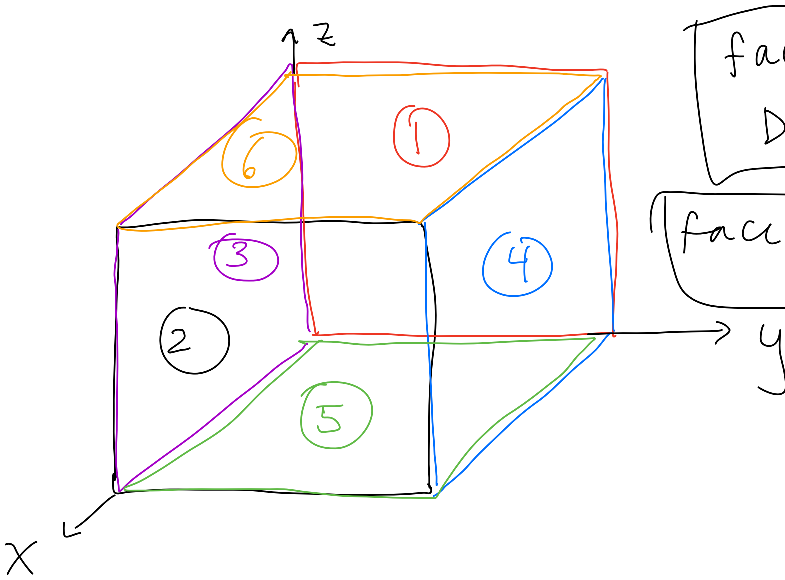
$$+ \mu \left[D_{zz} u_2 + D_z D_y u_3 \right] + f_2$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \mu \left[D_x D_z u_1 + D_{zx} u_3 \right]$$

$$+ \mu \left[D_y D_z u_2 + D_{zy} u_3 \right]$$

$$+ \lambda \left[D_z D_x u_1 + D_z D_y u_2 + D_{zz} u_3 \right]$$

$$+ 2\mu D_{zz} u_3 + f_3$$



faces 1 & 2 : Dirichlet

faces 3 – 6 : Traction

$$\vec{T}_1 = \sigma \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_{xx} \\ -\sigma_{xy} \\ -\sigma_{xz} \end{bmatrix}$$

$$\vec{T}_2 = \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{bmatrix}$$

$$T_3 = -T_4 = \sigma \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_{xy} \\ -\sigma_{yy} \\ -\sigma_{yz} \end{bmatrix}$$

$$T_5 = -T_6 = \sigma \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\sigma_{xz} \\ -\sigma_{yz} \\ -\sigma_{zz} \end{bmatrix}$$

Traction operators:

$$T_{JL} = n_I C_{IJKL} \frac{\partial}{\partial K}$$

on faces 3/4:

$T_{11} = -/+ \mu \frac{\partial}{\partial y}$ $\quad T_{13} = 0$

$T_{12} = -/+ \mu \frac{\partial}{\partial x}$

$T_{21} = -/+ \lambda \frac{\partial}{\partial x}$

$T_{22} = -/+ (\lambda + 2\mu) \frac{\partial}{\partial y}$

$T_{23} = -/+ \lambda \frac{\partial}{\partial z}$

$T_{31} = 0$

$T_{32} = -/+ \mu \frac{\partial}{\partial z}$

$T_{33} = -/+ \mu \frac{\partial}{\partial y}$

on faces 5/6:

$T_{11} = -/+ \mu \frac{\partial}{\partial z}$ $\qquad T_{12} = 0$

$T_{13} = -/+ \mu \frac{\partial}{\partial x}$

$T_{21} = 0$

$T_{22} = -/+ \mu \frac{\partial}{\partial z}$

$T_{23} = -/+ \mu \frac{\partial}{\partial y}$

$T_{31} = -/+ \lambda \frac{\partial}{\partial x}$

$T_{32} = -/+ \lambda \frac{\partial}{\partial y}$

$T_{33} = -/+ (\lambda + 2\mu) \frac{\partial}{\partial z}$

Boundary Conditions

face 1 : $u_1 = g_1^1, \quad u_2 = g_2^1, \quad u_3 = g_3^1$

face 2 : $u_1 = g_1^2, \quad u_2 = g_2^2, \quad u_3 = g_3^2$

face 3 :

$$T_{11}^3 u_1 + T_{12}^3 u_2 = g_1^3$$

$$-\sigma_{xy} = g_1^3 \Rightarrow -\mu\left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right] = g_1^3$$

$$-\sigma_{yy} = g_2^3 \Rightarrow -\lambda\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right] - 2\mu\frac{\partial u_2}{\partial y} = g_2^3$$

$$= T_{21}^3 u_1 + T_{22}^3 u_2 + T_{23}^3 u_3 = g_2^3$$

$$-\sigma_{yz} = g_3^3 \Rightarrow -\mu\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right] = g_3^3$$

$$T_{32}^3 u_2 + T_{33}^3 u_3$$

face 4 :

$$\sigma_{xy} = g_1^4 \Rightarrow \mu\left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right] = g_1^4$$

$$T_{11}^4 u_1 + T_{12}^4 u_2 = g_1^4$$

$$\sigma_{yy} = g_2^4 \Rightarrow \lambda\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right] + 2\mu\frac{\partial u_2}{\partial y} = g_2^4$$

$$T_{21}^4 u_1 + T_{22}^4 u_2 + T_{23}^4 u_3 = g_2^4$$

$$\sigma_{yz} = g_3^4 \Rightarrow \mu\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right] = g_3^4$$

$$T_{32}^4 u_2 + T_{33}^4 u_3 = g_3^4$$

face 5:

$$-\sigma_{xz} = g_1^5 \Rightarrow \overset{T_{11}^5 u_1 + T_{12}^5 u_2 + T_{13}^5 u_3}{-\mu\left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right]} = g_1^5$$

$$-\sigma_{yz} = g_2^5 \Rightarrow \overbrace{-\mu\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right]}^{T_{21}^5 u_1 + T_{22}^5 u_2 + T_{23}^5 u_3} = g_2^5$$

$$-\sigma_{zz} = g_3^5 \Rightarrow \underbrace{-\lambda\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right] - 2\mu\frac{\partial u_3}{\partial z} = g_3^5}_{T_{31}^5 u_1 + T_{32}^5 u_2 + T_{33}^5 u_3 = g_3^5}$$

face 6:

$$\sigma_{xz} = g_1^6 \Rightarrow \underbrace{\mu\left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right]}_{T_{11}^6 u_1 + T_{12}^6 u_2 + T_{13}^6 u_3 = g_1^6} = g_1^6$$

$$\sigma_{yz} = g_2^6 \Rightarrow \mu\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right] = g_2^6$$

$$\sigma_{zz} = g_3^6 \Rightarrow \lambda\left[\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right] + 2\mu\frac{\partial u_3}{\partial z} = g_3^6$$

etc.

SAT terms for traction B.C.

$$SAT_1 = -H^{-1}\left[e_3 H_3 \left(e_3^T\left[T_{11}^3 u_1 + T_{12}^3 u_2 + \cancelto{0}{T_{13}^3 u_3}\right] - g_1^3\right)\right]$$

$$- H^{-1}\left[e_4 H_4 \left(e_4^T\left[T_{11}^4 u_1 + T_{12}^4 u_2 + \cancelto{0}{T_{13}^4 u_3}\right] - g_1^4\right)\right]$$

$$- H^{-1}\left[e_5 H_5 \left(e_5^T\left[T_{11}^5 u_1 + T_{12}^5 u_2 + T_{13}^5 u_3\right] - g_1^5\right)\right]$$

$$- H^{-1}\left[e_6 H_6 \left(e_6^T\left[T_{11}^6 u_1 + T_{12}^6 u_2 + T_{13}^6 u_3\right] - g_1^6\right)\right]$$

$$SAT_2 = -H^{-1}\left[e_3 H_3 \left(e_3^T\left[T_{21}^3 u_1 + T_{22}^3 u_2 + T_{23}^3 u_3\right] - g_2^3\right)\right]$$

$$- H^{-1}\left[e_4 H_4 \left(e_4^T\left[T_{21}^4 u_1 + T_{22}^4 u_2 + T_{23}^4 u_3\right] - g_2^4\right)\right]$$

$$- H^{-1}\left[e_5 H_5 \left(e_5^T\left[T_{21}^5 u_1 + T_{22}^5 u_2 + T_{23}^5 u_3\right] - g_2^5\right)\right]$$

$$- H^{-1}\left[e_6 H_6 \left(e_6^T\left[T_{21}^6 u_1 + T_{22}^6 u_2 + T_{23}^6 u_3\right] - g_2^6\right)\right]$$

$$SAT_3 = -H^{-1}\left[e_3 H_3 \left(e_3^T\left[T_{31}^3 u_1 + T_{32}^3 u_2 + T_{33}^3 u_3\right] - g_3^3\right)\right]$$

$$- H^{-1}\left[e_4 H_4 \left(e_4^T\left[T_{31}^4 u_1 + T_{32}^4 u_2 + T_{33}^4 u_3\right] - g_3^4\right)\right]$$

$$- H^{-1}\left[e_5 H_5\left(e_5^T\left[T^5_{31} u_1 + T^5_{32} u_2 + T^5_{33} u_3\right] - g_3^5\right]\right.$$

$$- H^{-1}\left[e_6 H_6\left(e_6^T\left[T^6_{31} u_1 + T^6_{32} u_2 + T^6_{33} u_3\right] - g_3^6\right]\right.$$

SAT terms for Dirichlet BC.

$$\widetilde{SAT}_1 = H^{-1}(T_{L1} - Z_{L1})^T e_1 H_1 (e_1^T u_L - g_L) + H^{-1}(T_{L1} - Z_{L1})^T e_2 H_2 (e_2^T u_L - g_L)$$

$$\begin{aligned}
= \; & H^{-1}(T_{11} - Z_{11})^T e_1 H_1 (e_1^T u_1 - g_1^1) \\
& + H^{-1}(T_{21} - Z_{21})^T e_1 H_1 (e_1^T u_2 - g_2^1) \\
& + H^{-1}(T_{31} - Z_{31})^T e_1 H_1 (e_1^T u_3 - g_3^1) \\
& + H^{-1}(T_{11} - Z_{11})^T e_2 H_2 (e_2^T u_1 - g_1^2) \\
& + H^{-1}(T_{21} - Z_{21})^T e_2 H_2 (e_2^T u_2 - g_2^2) \\
& + H^{-1}(T_{31} - Z_{31})^T e_2 H_2 (e_2^T u_3 - g_3^2)
\end{aligned}$$

$$\begin{aligned}
\widetilde{SAT}_2 = \; & H^{-1}(T_{12} - Z_{12})^T e_1 H_1 (e_1^T u_1 - g_1^1) \\
& + H^{-1}(T_{22} - Z_{22})^T e_1 H_1 (e_1^T u_2 - g_2^1) \\
& + H^{-1}(T_{32} - Z_{32})^T e_1 H_1 (e_1^T u_3 - g_3^1)
\end{aligned}$$

$$+ H^{-1}(T_{12} - Z_{12})^T e_2 H_2 (e_2^T u_1 - g_1^2)$$
$$+ H^{-1}(T_{22} - Z_{22})^T e_2 H_2 (e_2^T u_2 - g_2^2)$$
$$+ H^{-1}(T_{32} - Z_{32})^T e_2 H_2 (e_2^T u_3 - g_3^2)$$

$$\widetilde{SAT}_3 = H^{-1}(T_{13} - Z_{13})^T e_1 H_1 (e_1^T u_1 - g_1^1)$$
$$+ H^{-1}(T_{23} - Z_{23})^T e_1 H_1 (e_1^T u_2 - g_2^1)$$
$$+ H^{-1}(T_{33} - Z_{33})^T e_1 H_1 (e_1^T u_3 - g_3^1)$$
$$+ H^{-1}(T_{13} - Z_{13})^T e_2 H_2 (e_2^T u_1 - g_1^2)$$
$$+ H^{-1}(T_{23} - Z_{23})^T e_2 H_2 (e_2^T u_2 - g_2^2)$$
$$+ H^{-1}(T_{33} - Z_{33})^T e_2 H_2 (e_2^T u_3 - g_3^2)$$

Check discrete PDE:

Almquist:

$$\rho \ddot{u}_J = D_{IK}(C_{IJKL}) u_L + f_J + SAT_J$$

$\Rightarrow$

$$\rho \ddot{u}_1 = \underbrace{D_{iK} C_{i1KL} u_L} + f_J + SAT_j$$

$$= D_{1K} C_{11KL} u_L + D_{2K} C_{21KL} u_L + D_{3K} C_{31KL} u_L$$

$$\rho \ddot{u}_1 = D_{11} C_{1111} u_1 + D_{12} C_{1122} u_2 + D_{13} C_{1133} u_3$$
$$+ D_{21} C_{2112} u_2 + D_{22} C_{2121} u_1$$
$$+ D_{31} C_{3113} u_3 + D_{33} C_{3131} u_1$$

$$\rho \ddot{u}_2 = D_{iK}(C_{i2Kl} u_l)$$

$$= D_{1K}(C_{12Kl} u_l) + D_{2K}(C_{22Kl} u_l) + D_{3K}(C_{32Kl} u_l)$$

$$= D_{11}(C_{121l} u_l) + D_{12}(C_{122l} u_l) + D_{13}(C_{123l} u_l)$$
$$+ D_{21}(C_{221l} u_l) + D_{22}(C_{222l} u_l) + C_{23}(C_{223l} u_l)$$
$$+ D_{31}(C_{321l} u_l) + D_{32}(C_{322l} u_l) + D_{33}(C_{323l} u_l)$$

$$\rho \ddot{u}_2 = D_{11} C_{1212} u_2 + D_{12} C_{1221} u_1$$
$$+ D_{21} C_{2211} u_1 + D_{22} C_{2222} u_2 + C_{23} C_{2233} u_3$$
$$+ D_{32} C_{3223} u_3 + D_{33} C_{3232} u_2$$

$$\rho \ddot{u}_3 =$$