IEOR 4106, HMWK 3, Professor Sigman

1. Consider the *Rat in the Open Maze*; 4 rooms, and the outside (state 0) (state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$), but now the transition probabilities have been changed: The new 1-step transition matrix is given by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5/12 & 7/12 & 0 \\ 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 & 0 \end{pmatrix}.$$

   (a) Solve for the four values for $E(T_{i,0})$, the expected number of moves until the rat escapes given it starts in Room i, $1 \le i \le 4$. (Hint: Condition on the first move.)

   (b) Compute the probability that the rat, if initially in Room 2, has escaped by $(\le)$ time/move $n = 10$.

2. Consider from class lecture, the weather Markov chain $X_n = (W_{n-1}, W_n)$ where $W_n \in \{0, 1\}$ (1 = rain, 0 = no rain), and the labeling is given by

$$\begin{aligned} 0 &= (0,0) \\ 1 &= (0,1) \\ 2 &= (1,0) \\ 3 &= (1,1); \end{aligned}$$

   the state space is thus $\mathcal{S} = \{0, 1, 2, 3\}$. Suppose that the transition probabilities are given by the 1-step transition matrix

$$P = \begin{pmatrix} 0.1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}.$$

   Today is Monday: Compute the probability that it does not rain 2 days from now (Wed), given that it rained today (Monday) but did not rain yesterday (Sunday).

3. Consider modeling the weather where we now we include (in the state space) the previous two days weather instead of only the day before as we just did in Problem 2 above. Letting W_n denote weather on the n^{th} day (0 = no rain, 1 = rain), let $X_n = (W_{n-2}, W_{n-1}, W_n)$. There are 8 states, and we will relabel them 0—7 as: $(0,0,0) = 0$, $(1,0,0) = 1$, $(0,1,0) = 2$, $(0,0,1) = 3$, $(1,1,0) = 4$, $(1,0,1) = 5$, $(0,1,1) = 6$, $(1,1,1) = 7$. We will *assume* it forms a Markov chain, and that the transition probabilities are given by: If it has rained for the past 3 days, then it will rain today with probability 0.6; if it did not rain on any of the past 3 days, then it will rain today with probability 0.20. In any other case assume that the weather today will with probability 0.8 be the same as the weather yesterday. **Derive the transition matrix**.

4. *Continuation:*

   Given that it did not rain today, did not rain yesterday and did not rain the day before yesterday, compute the probability that it does rain 2 days from now.

5. For the Gambler's Ruin Problem: Suppose $N = 8$.

   (a) With $p = 0.6$ and initially $i = 4$ dollars, compute the probability that the Gambler stops gambling after $n = 2, 5, 10$ gambles.

   (b) *Continuation:* Find the minimum number of gambles n required so that after n gambles the probability of stopping is > 0.5.

   (c) Compute (a) and (b) again if $p = 0.4$. Explain why the answers are the same as before.

6. Consider a Markov chain $\{X_n : n \geq 0\}$ with $\mathcal{S} = \{0, 1, 2\}$, and transition matrix

$$P = \begin{pmatrix} 1/3 & 1/6 & 1/2 \\ 0 & 1/7 & 6/7 \\ 1/4 & 0 & 3/4 \end{pmatrix}.$$

   (a) Compute $E(X_3 \,|X_0 = i)$, for each of $i = 0, 1, 2$.

   (b) Suppose that (independently) X_0 is chosen randomly with $P(X_0 = 0) = 1/4,\ P(X_0 = 1) = 1/8,\ P(X_0 = 2) = 5/8$. Compute $E(X_3)$.

7. Each of the following transition matrices is for a Markov chain. For each, find the communication classes for breaking down the state space, $\mathcal{S} = C_1 \cup C_2 \cup \cdots$ and for each class C_k tell if it is recurrent or transient.

   (a)

$$P = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/2 \\ 7/8 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/3 & 0 & 2/3 \end{pmatrix}.$$

   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (b)

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3/10 & 1/10 & 4/10 & 2/10 \\ 0 & 6/13 & 0 & 7/13 \end{pmatrix}.$$

   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (c)

$$P = \begin{pmatrix} 1/7 & 0 & 2/7 & 0 & 4/7 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 2/9 & 0 & 4/9 & 0 & 3/9 \\ 0 & 1/6 & 0 & 5/6 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/2 \end{pmatrix}.$$

   $\mathcal{S} = \{0, 1, 2, 3, 4\}$.

(d)

$$P = \begin{pmatrix} 1/9 & 2/9 & 1/9 & 1/9 & 4/9 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 6/7 & 0 & 0 & 1/7 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 \\ 2/11 & 0 & 5/11 & 0 & 4/11 \end{pmatrix}.$$

$\mathcal{S} = \{0, 1, 2, 3, 4\}$.

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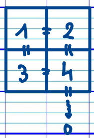
HW 3: Stochastic

Exercice 1:

Rat in an open maze : $\mathcal{E} = \{0,1,2,3,4\}$



$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5/12 & 7/12 & 0 \\ 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ \boxed{1/2 & 0 & 1/4 & 1/4 & 0} \end{pmatrix} \end{array}$$

future j

present i

rows sum to 1

a. We have $Z_i := \min\{n \geq 1,\ X_n = 0 \mid X_0 = i\}$ minimum number of moves needed to escape

We are searching for $E[Z_i]$.

$$\begin{aligned} E[Z_1] &= E[\min\{n \geq 1,\ X_n = 0 \mid X_0 = 1\}] \\ &= E[Z_1 \mid X_1 = 2]\,\mathbb{P}(X_1 = 2) + E[Z_1 \mid X_1 = 3]\,\mathbb{P}(X_1 = 3) \\ &= (1 + E[Z_2])\frac{5}{12} + (1 + E[Z_3])\frac{7}{12} \\ &= 1 + \frac{5}{12}E[Z_2] + \frac{7}{12}E[Z_3] \end{aligned}$$

We proceed similarly for the others Z_i. We finally get the following system:

$$\begin{cases} E[Z_1] = 1 + \frac{5}{12}E[Z_2] + \frac{7}{12}E[Z_3] \\ E[Z_2] = 1 + \frac{3}{4}E[Z_1] + \frac{1}{4}E[Z_4] \\ E[Z_3] = 1 + \frac{1}{2}E[Z_1] + \frac{1}{4}E[Z_4] \\ E[Z_4] = 1 + \frac{1}{4}E[Z_2] + \frac{1}{4}E[Z_3] \end{cases} \rightarrow \begin{cases} E[Z_1] = \frac{213}{19} \\ E[Z_2] = \frac{208}{19} \\ E[Z_3] = \frac{184}{19} \\ E[Z_4] = \frac{117}{19} \end{cases}$$

b) $\mathbb{P}(\text{Rat escapes by time } 10 \mid X_0 = 2) = \mathbb{P}(X_{10} = 0 \mid X_0 = 2) = P^{10}_{2,0}$ ≤

So we need to compute P^{10} to get $P^{10}_{2,0}$.

10 set transition matrix

$$P^{(10)} = P^{10} = \frac{1}{15925248}\begin{pmatrix} 15\,925\,248 & 0 & 0 & 0 & 0 \\ 9271164 & 4\,053\,839 & 0 & 0 & 2600245 \\ 10\,092\,294 & 0 & 2\,548\,434 & 3284520 & 0 \\ 10\,913\,424 & 0 & 2\,189\,680 & 2\,822\,144 & 0 \\ 12\,555\,684 & 2\,052\,825 & 0 & 0 & 1316\,739 \end{pmatrix}$$

thanks to Chapman Kolmogorov

exact value with WolframAlpha

So : $\mathbb{P}(\text{Rat escapes by time } 10 \mid X_0 = 2) = \underbrace{\pi_0^2}_{(0,0,1,0,0)} \cdot P^{10} = \dfrac{10\,092\,294}{15925248} = 63.32\,\%$

Exercise 2:

- Let's consider the following Markov Chain: $X_n = (W_{n-1}, W_n),\ n \geq 1$

With support: $\mathcal{S} = \{\underset{0}{(0,0)}, \underset{1}{(0,1)}, \underset{2}{(1,0)}, \underset{3}{(1,1)}\}$

$0 : (0,0) \xrightarrow{0.1} (0,0) = 0$, $\quad 0 : (0,0) \xrightarrow{0.9} (0,1) = 1$

$2 : (1,0) \xrightarrow{0.9} (0,0) = 0$, $\quad 2 : (1,0) \xrightarrow{0.1} (0,1) = 1$

$1 : (0,1) \xrightarrow{0.5} (1,0) = 2$, $\quad 1 : (0,1) \xrightarrow{0.5} (1,1) = 3$

$3 : (1,1) \xrightarrow{0.2} (1,0) = 2$, $\quad 3 : (1,1) \xrightarrow{0.8} (1,1) = 3$

$$\text{So } P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

- $\mathbb{P}(\text{No rain on wednesday} \mid \text{rain on monday and no rain saturday}) = \mathbb{P}(X_{n+2} = \underset{(0,0)}{0} \text{ or } \underset{(1,0)}{2} \mid X_n = \underset{(1,0)}{1})$

$= P^{(2)}_{1,0} + P^{(2)}_{1,2}$

$= 0.45 + 0.1$

$= 0.55$

with : $P^{(2)} = P^2 = \begin{pmatrix} 0.01 & 0.09 & 0.45 & 0.45 \\ \boxed{0.45} & 0.05 & \boxed{0.1} & 0.4 \\ 0.09 & 0.81 & 0.05 & 0.05 \\ 0.18 & 0.02 & 0.16 & 0.64 \end{pmatrix}$

Exercise 3:

- Let's consider the following Markov Chain: $X_n = (W_{n-2}, W_{n-1}, W_n),\ n \geq 2$

With support: $\mathcal{S} = \{\underset{0}{(0,0,0)}, \underset{1}{(1,0,0)}, \underset{2}{(0,1,0)}, \underset{3}{(0,0,1)}, \underset{4}{(1,1,0)}, \underset{5}{(1,0,1)}, \underset{6}{(0,1,1)}, \underset{7}{(1,1,1)}\}$

$\mathbb{P}(\text{Rain today} \mid \text{rained for 3 days}) = 0.6$

$7 = (1,1,1) \xrightarrow{0.6} (1,1,1) = 7$, $\quad 7 = (1,1,1) \xrightarrow{0.4} (1,1,0) = 4$ $\Big\}$ $P_{7,7} = 0.6$ and $P_{7,4} = 0.4$

state 7

$\mathbb{P}(\text{Rain today} \mid \text{no rain for 3 days}) = 0.2$

$0 = (0,0,0) \xrightarrow{0.2} (0,0,1) = 3$

$0 = (0,0,0) \xrightarrow{0.8} (0,0,0) = 0$

$\Big\} \; P_{0,0} = 0.8$ and $P_{0,3} = 0.2$ | state 0

$\mathbb{P}(\text{Rain today} \mid \text{Rain yesterday}) = 0.8$

$\mathbb{P}(\text{No rain today} \mid \text{No rain yesterday}) = 0.8$

$\Big\} \; \mathbb{P}(W_n = i \mid W_{n-1} = i) = 0.8$, where $i = 0$ or 1

So, state $1 : (1,0,0) \xrightarrow{0.8} (0,0,0) = 0$ ie $P_{1,0} = 0.8$

$\xrightarrow{0.2} (0,0,1) = 3$ ie $P_{1,3} = 0.2$

state $2 : (0,1,0) \xrightarrow{0.8} (1,0,0) = 1$ ie $P_{2,1} = 0.8$

$\xrightarrow{0.2} (1,0,1) = 5$ ie $P_{2,5} = 0.2$

state $3 : (0,0,1) \xrightarrow{0.8} (0,1,1) = 6$ ie $P_{3,6} = 0.8$

$\xrightarrow{0.2} (0,1,0) = 2$ ie $P_{3,2} = 0.2$

state $4 : (1,1,0) \xrightarrow{0.8} (1,0,0) = 1$ ie $P_{4,1} = 0.8$

$\xrightarrow{0.2} (1,0,1) = 5$ ie $P_{4,5} = 0.2$

state $5 : (1,0,1) \xrightarrow{0.8} (0,1,1) = 6$ ie $P_{5,6} = 0.8$

$\xrightarrow{0.2} (0,1,0) = 2$ ie $P_{5,2} = 0.2$

state $6 : (0,1,1) \xrightarrow{0.8} (1,1,1) = 7$ ie $P_{6,7} = 0.8$

$\xrightarrow{0.2} (1,1,0) = 4$ ie $P_{6,4} = 0.2$

So $P = $
$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0.8 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\
1 & 0.8 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\
2 & 0 & 0.8 & 0 & 0 & 0 & 0.2 & 0 & 0 \\
3 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 \\
4 & 0 & 0.8 & 0 & 0 & 0 & 0.2 & 0 & 0 \\
5 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 \\
6 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.8 \\
7 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.6
\end{array}$$

Exercice 4:

- The problem can be modeled as : $\begin{cases} W_n = 0 & \text{today} \\ W_{n-1} = 0 & \text{yesterday} \\ W_{n-2} = 0 & \text{the day before yesterday} \end{cases}$

We want:

$$\mathbb{P}(W_{n+2} = 1 \mid W_n = 0, W_{n-1} = 0, W_{n-2} = 0)$$
$$= \mathbb{P}(W_{n+2} = 1 \mid X_n = 0)$$
$$= \mathbb{P}(X_{n+2} = (0,0,1) \text{ or } (0,1,1) \text{ or } (1,1,1) \text{ or } (1,0,1) \mid X_n = 0)$$
$$= \mathbb{P}(X_{n+2} = 3 \text{ or } 6 \text{ or } 7 \text{ or } 5 \mid X_n = 0)$$
$$= P^{(2)}_{0,3} + P^{(2)}_{0,6} + P^{(2)}_{0,7} + P^{(2)}_{0,5}$$
$$= 0.16 + 0.16 + 0 + 0$$
$$= 32\%$$

P^2

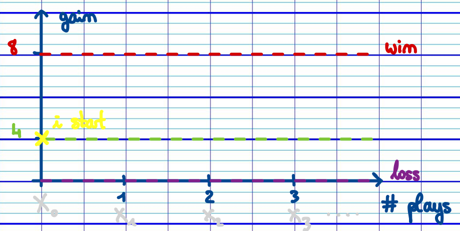
```python
P = np.array([
    [0.8, 0,   0,   0.2, 0,   0,   0,   0],
    [0.8, 0,   0,   0.2, 0,   0,   0,   0],
    [0,   0.8, 0,   0,   0,   0.2, 0,   0],
    [0,   0,   0.2, 0,   0,   0,   0.8, 0],
    [0,   0.8, 0,   0,   0,   0.2, 0,   0],
    [0,   0,   0.2, 0,   0,   0,   0.8, 0],
    [0,   0,   0,   0,   0.2, 0,   0,   0.8],
    [0,   0,   0,   0,   0,   0.4, 0,   0.6]
])

# Compute P²
P_squared = np.dot(P, P)

# Print P²
print(P_squared)
```

```
[[0.64 0.   0.04 0.16 0.   0.   0.16 0.  ]
 [0.64 0.   0.04 0.16 0.   0.   0.16 0.  ]
 [0.64 0.   0.04 0.16 0.   0.   0.16 0.  ]
 [0.   0.16 0.   0.   0.16 0.04 0.   0.64]
 [0.64 0.   0.04 0.16 0.   0.   0.16 0.  ]
 [0.   0.16 0.   0.   0.16 0.04 0.   0.64]
 [0.   0.16 0.   0.   0.   0.36 0.   0.48]
 [0.   0.   0.08 0.   0.   0.24 0.32 0.36]]
```

Exercice 5:



- We need to compute $E[X_3 \mid X_0 = i]$

We know that $\mathbb{P}(X_n \text{ hits } N \text{ before } 0 \mid X_0 = i) = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq q \\ \dfrac{i}{N} & \text{if } p = q \end{cases}$

a) Let $\begin{cases} p = 0.6 \\ i = 4 \end{cases}$

We have $P = \underbrace{\begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}}_{N+1} \Big\updownarrow N+1$: $\begin{cases} P_{i,i+1} = p \\ P_{i,i-1} = 1-p \end{cases}$ and $\begin{cases} P_{0,0} = 1 \\ P_{N,N} = 1 \end{cases}$

We start at stake: $\Pi_0 = (0,0,0,0,1,0,0,0,0)$

$\underline{n=2}$: So $\Pi_2 = \Pi_0 P^2$

$= (0, 0, 0.16, 0, 0.48, 0, 0.36, 0, 0)$

And $P(\text{stop after } n=2) = \Pi_2(0) + \Pi_2(N) = 0$

```
# Compute pi_2 (state distribution after 2 steps)
pi_2 = np.dot(initial_distribution, np.linalg.matrix_power(P, 2))

# Print the result
print(f"State distribution after 2 steps: {pi_2}")
```

```
State distribution after 2 steps: [0.   0.   0.16 0.   0.48 0.   0.36 0.   0.  ]
```

$\underline{n=5}$: So $\Pi_5 = \Pi_0 P^5$

$= (0.0256, 0.06144, 0, 0.2304, 0, 0.3456, 0, 0.20736, 0.1296)$

And $P(\text{stop after } n=5) = \Pi_5(0) + \Pi_5(N) = 0.1552$

```
# Compute pi_5 (state distribution after 5 steps)
pi_5 = np.dot(initial_distribution, np.linalg.matrix_power(P, 5))

# Print the result
print(f"State distribution after 5 steps: {pi_5}")
```

```
State distribution after 2 steps: [0.0256  0.06144 0.      0.2304  0.      0.3456  0.      0.20736 0.1296 ]
```

$\underline{n=10}$: So $\Pi_{10} = \Pi_0 P^{10}$

$= (0.1847, \ldots, 0.444)$

And $P(\text{stop after } n=10) = \Pi_{10}(0) + \Pi_{10}(N) = 0.5323$

```
# Compute pi_10 (state distribution after 10 steps)
pi_10 = np.dot(initial_distribution, np.linalg.matrix_power(P, 10))

# Print the result
print(f"State distribution after 10 steps: {pi_10}")
```

```
State distribution after 10 steps: [0.08780677 0.         0.08705802 0.         0.18473288 0.
 0.19588055 0.         0.44452178]
```

b) We want to find n so that $P(\text{gambler stops after } n \text{ games}) > 0.5$

$\rightarrow \Pi_n(0) + \Pi_n(N) > 0.5$

$\rightarrow \Pi_0 P^{(n)}(0) + \Pi_0 P^{(n)}(N) > 0.5$

$\rightarrow n > 9 \text{ or } n \geq 10$

For $n = 10$, $P(\text{gambler stops after } 10 \text{ games}) = 0.5323$

```
def stopping_probability(n):
    current_distribution = np.dot(initial_distribution, np.linalg.matrix_power(P, n))
    return current_distribution[0] + current_distribution[N]  # Probability of reachi

n = 1
while True:
    prob = stopping_probability(n)
    if prob > 0.5:
        break
    n += 1

print(f"Minimum number of steps n: {n}, Stopping probability: {prob}")
```

```
Minimum number of steps n: 10, Stopping probability: 0.5323285504
```

It's basically the same thing but flipping the problem around

c) a. We start at state : $\pi_0 = (0,0,0,0,1,0,0,0,0)$ and $p = 0.4$

$\underline{n=2}$: So $\pi_2 = \pi_0 P^2$

$= (0, \dots, 0)$

And $P(\text{stop after } n=2) = \pi_2(0) + \pi_2(N) = 0$

```python
# Compute pi_2 (state distribution after 2 steps)
pi_2 = np.dot(initial_distribution, np.linalg.matrix_power(P, 2))

# Print the result
print(f"State distribution after 2 steps: {pi_2}")
```

```
State distribution after 2 steps: [0.   0.   0.36 0.   0.48 0.   0.16 0.   0.  ]
```

$\underline{n=5}$: So $\pi_5 = \pi_0 P^5$

$= (0.1296, \dots, 0.0256)$

And $P(\text{stop after } n=5) = \pi_5(0) + \pi_5(N) = 0.1552$

```python
# Compute pi_5 (state distribution after 5 steps)
pi_5 = np.dot(initial_distribution, np.linalg.matrix_power(P, 5))

# Print the result
print(f"State distribution after 5 steps: {pi_5}")
```

```
State distribution after 5 steps: [0.1296  0.20736 0.      0.3456  0.      0.2304  0.      0.06144 0.0256 ]
```

$\underline{n=10}$: So $\pi_{10} = \pi_0 P^{10}$

$= (0.4452, \dots, 0.0878)$

And $P(\text{stop after } n=10) = \pi_{10}(0) + \pi_{10}(N) = 0.533$

```python
# Compute pi_10 (state distribution after 10 steps)
pi_10 = np.dot(initial_distribution, np.linalg.matrix_power(P, 10))

# Print the result
print(f"State distribution after 10 steps: {pi_10}")
```

```
State distribution after 10 steps: [0.44452178 0.         0.19588055 0.         0.18473288 0.
 0.08705802 0.         0.08780677]
```

b. We want to find n so that $P(\text{gambler stops after } n \text{ games}) > 0.5$

$\rightarrow \pi_n(0) + \pi_n(N) > 0.5$

$\rightarrow \pi_0 P^{(n)}(0) + \pi_0 P^{(n)}(N) > 0.5$

$\rightarrow n > 9$ or $n \geq 10$

For $n = 10$, $P(\text{gambler stops after 10 games}) = 0.5323$

```python
def stopping_probability(n):
    current_distribution = np.dot(initial_distribution, np.linalg.matrix_power(P, n))
    return current_distribution[0] + current_distribution[N]  # Probability of reachi

n = 1
while True:
    prob = stopping_probability(n)
    if prob > 0.5:
        break
    n += 1

print(f"Minimum number of steps n: {n}, Stopping probability: {prob}")
```

```
Minimum number of steps n: 10, Stopping probability: 0.5323285503999999
```

```python
P = np.array([
    [1/3, 1/6, 1/2],
    [0, 1/7, 6/7],
    [1/4, 0, 3/4]
])

P_3 = np.linalg.matrix_power(P, 3)

states = np.array([0, 1, 2])

E_X3_given_0 = np.dot(P_3[0], states)
E_X3_given_1 = np.dot(P_3[1], states)
E_X3_given_2 = np.dot(P_3[2], states)

print(f"E(X3 | X0 = 0) = {E_X3_given_0}")
print(f"E(X3 | X0 = 1) = {E_X3_given_1}")
print(f"E(X3 | X0 = 2) = {E_X3_given_2}")
```

```
E(X3 | X0 = 0) = 1.4496409674981103
E(X3 | X0 = 1) = 1.4358600583090377
E(X3 | X0 = 2) = 1.4246031746031744
```

Exercice 6 :

a) $E[X_3 | X_0 = i] = \sum_{x \in S} x \, \mathbb{P}(X_3 = x | X_0 = i)$

$= \sum_{x \in S} x \, P^{(3)}_{ix}$

$= 0 . P^3_{i0} + 1 . P^3_{i1} + 2 . P^3_{i2}$

where $P^3 = \frac{1}{592\,704} \begin{pmatrix} 148\,078 & 30\,044 & 414\,582 \\ 155\,736 & 22\,896 & 414\,072 \\ 155\,379 & 30\,282 & 407\,043 \end{pmatrix}$

So : $E[X_3 | X_0 = 0] = \frac{1}{592\,704} (30\,044 + 2 \times 414\,582) = 1.4496$

$E[X_3 | X_0 = 1] = \frac{1}{592\,704} (22\,896 + 2 \times 414\,072) = 1.4359$

$E[X_3 | X_0 = 2] = \frac{1}{592\,704} (30\,282 + 2 \times 407\,043) = 1.4246$

to make sure

b) We have $\begin{cases} \mathbb{P}(X_0 = 0) = 1/4 \\ \mathbb{P}(X_0 = 1) = 1/8 \\ \mathbb{P}(X_0 = 2) = 5/8 \end{cases}$

Where :

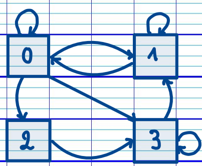
$E[X_3] = E[X_3 | X_0 = 0] \mathbb{P}(X_0 = 0) + E[X_3 | X_0 = 1] \mathbb{P}(X_0 = 1) + E[X_3 | X_0 = 2] \mathbb{P}(X_0 = 2)$

$= 1.4496 \times 1/4 + 1.4359 \times 1/8 + 1.4246 \times 5/8$

$= 1.4322$

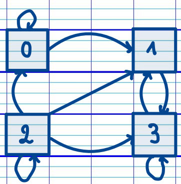
Exercice 7 :

a) $P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/4 & 1/8 & 1/8 & 1/2 \\ 1 & 7/8 & 1/8 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1/3 & 0 & 2/3 \end{array}$.



$C_1 = \{0,1,2,3\}$, recurrent

b) $P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/4 & 3/4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 3/10 & 1/10 & 4/10 & 2/10 \\ 3 & 0 & 6/13 & 0 & 7/13 \end{array}$



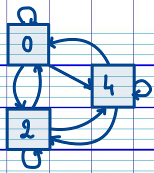
$C_1 = \{1,3\}$, recurrent

$C_2 = \{0\}$, transient

$C_3 = \{2\}$, transient

c)

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
1/7 & 0 & 2/7 & 0 & 4/7 \\
0 & 3/4 & 0 & 1/4 & 0 \\
2/9 & 0 & 4/9 & 0 & 3/9 \\
0 & 1/6 & 0 & 5/6 & 0 \\
1/4 & 0 & 1/4 & 0 & 1/2
\end{pmatrix} \end{array}
$$

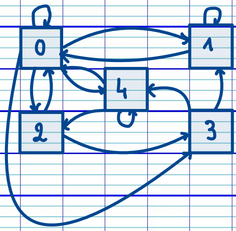


$C_1 = \{0, 2, 4\}$, recurrent

$C_2 = \{1, 3\}$, recurrent

d)

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
1/9 & 2/9 & 1/9 & 1/9 & 4/9 \\
1/2 & 1/2 & 0 & 0 & 0 \\
6/7 & 0 & 0 & 1/7 & 0 \\
0 & 1/3 & 0 & 0 & 2/3 \\
2/11 & 0 & 5/11 & 0 & 4/11
\end{pmatrix} \end{array}
$$



$C_1 = \{0, 1, 2, 3, 4\}$, recurrent