IEOR 4106, HMWK 2, Professor Sigman

1. *State-dependent Gambler's ruin problem:* Consider the Gambler's ruin problem with $N = 3$; $\mathcal{S} = \{0, 1, 2, 3\}$, but for which the value of p depends upon i, $1 \leq i \leq 2$: That is, if $X_n = 1$, then (independent of the past) the probability that the Gambler wins \$1 is 1/4, the probability the Gambler loses \$1 is 3/4.
   If $X_n = 2$, then (independent of the past) the probability that the Gambler wins \$1 is 2/7, the probability the Gambler loses \$1 is 5/7. Let $P_i(3)$ denote the probability that the Gambler, starting with $X_0 = i \in \{1, 2\}$, will reach 3 before 0.
   Explicitly solve for the $P_i(3), 1 \leq i \leq 2$.

2. *Continuation: the general case.* Solve now the general $N = 3$ case; when 1/4 and 2/7 are replaced by $0 < p_1 < 1$ and $0 < p_2 < 1$.

3. For the simple random walk ($0 < p < 1$) with $R_0 = 0$, derive the following probabilities:

   (a) $P(R_2 = 0)$, $P(R_3 = 0)$, $P(R_4 = 0)$

   (b) $P(R_3 = -1)$

   (c) $P(R_{2m} = 0)$ for $m \geq 1$.

4. Beth wishes to move to a new location to live, each location a distinct neighborhood located at points along $\mathbb{Z}$. She lives now at location 0. She is most interested in the locations on the negative axis, $i \leq -1$, but wishes to at least look at some on the positive axis as well before choosing. She decides to look at locations according to a simple random walk with $p = 0.3$ and $q = 1 - p = 0.7$. (This way, she also might be able to visit some places more than once.)

   (a) What is the probability that she visits location 6 before location -6?

   (b) What is the probability that she visits location -6 before location 6?

   (c) What is the probability that she would ever visit location -6?

   (d) What is the probability that she would ever visit location 6?

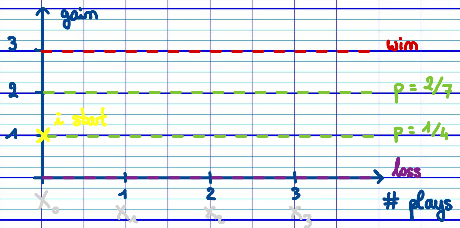
   (e) What is Beth's expected location after the random walk has taken her 10 steps?

   (f) What is the probability that Beth never visits any location $i \geq 1$?

   (g) Beth is now at location 5. She realizes that she forgot something at her original location 0, hence hopes to return there if possible. What is the probability that she will return? What is the probability that she visits location 10 before returning?

HW2: Stochastic

Exercise 1:

1) 

We want to calculate: $P_i(3) = \mathbb{P}(\text{gambler hits } 3 \text{ before } 0 \mid X_0 = i)$

Using " $P_i(3) = pP_{i+1}(3) + qP_{i-1}(3)$ " : $\begin{cases} \text{State } 1: & P_1(3) = \frac{1}{4}P_2(3) + \frac{3}{4} \times 0 = \frac{1}{4}P_2(3) \\ \text{State } 2: & P_2(3) = \frac{2}{7} \times 1 + \frac{5}{7}P_1(3) = \frac{2}{7} + \frac{5}{7}P_1(3) \end{cases}$

So we have : $P_1(3) = 2/23$

$P_2(3) = 8/23$

Exercise 2:

Continuing... : $\begin{cases} \text{State } 1: & P_1(3) = p_1 P_2(3) \\ \text{State } 2: & P_2(3) = p_2 + (1-p_2)P_1(3) \end{cases}$

$\begin{cases} P_1 - p_1 P_2 = 0 \\ (1-p_2)P_1 - P_2 = p_2 \end{cases}$

$\Leftrightarrow \begin{cases} P_1 - p_1 P_2 = 0 \\ 0 - (-1 + p_1(1-p_2))P_2 = p_2 \end{cases}$

$\Leftrightarrow \begin{cases} P_1 = -p_1 P_2 \\ P_2 = \dfrac{p_2}{1 - p_1(1-p_2)} \end{cases}$

So we have : $\begin{cases} P_1(3) = \dfrac{p_1 p_2}{1 - p_1(1-p_2)} \\ P_2(3) = \dfrac{p_2}{1 - p_1(1-p_2)} \end{cases}$

Exercise 3:

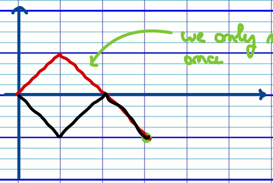
a) $P(R_2 = 0)$: "probability of going up then down or down then up"

$= 2p(1-p)$

$P(R_3 = 0) = 0$ because it is impossible

$$P(R_4=0) = \overbrace{\binom{4}{2}}^{\text{\# of possible paths (go up twice out of the 4 tries)}} \overbrace{p^2}^{\text{go up twice in total}} \overbrace{(1-p)^2}^{\text{go down twice in total}} = 6p^2(1-p)^2$$

b) $P(R_3=-1) = \binom{3}{1} p(1-p)^2 = 3p(1-p)^2$

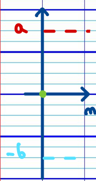


we only need to win once

c) $P(R_{2m}=0)$ = "probability of winning m times and loosing m times in total")

$$= \binom{2m}{m} p^m (1-p)^m$$

Exercice 4:



$\begin{cases} R_0 = 0 \\ R_n = \sum_{k=1}^{n} \Delta_k, \ n \geq 1 \end{cases}$, $p(a) = P(R_n \text{ hits } a \text{ before } -b \mid R_0=0) = \begin{cases} \dfrac{1-(q/p)^b}{1-(q/p)^{a+b}} & \text{if } p \neq \frac{1}{2} \\ \dfrac{b}{a+b} & \text{if } p = \frac{1}{2} \end{cases}$

a) $P(\text{"Beth visit 6 before -6"}) = p(6) = \dfrac{1-(q/p)^6}{1-(q/p)^{12}} = 0.0062$

b) $P(\text{"Beth visit -6 before 6"}) = 1 - P(\text{"Beth visit 6 before -6"}) = 0.994$

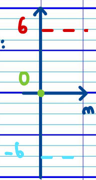
c) $P(\text{ever visit } -6) = 1 - P_0(\infty) = 1$, negative drift

d) $P(\text{ever visit } 6) = P(Max \geq 6) = \left(\dfrac{p}{q}\right)^6 = 0.0062$

e) $E[R_{10}] = 10 \times \overbrace{(2p-1)}^{E[\Delta]} = -4$

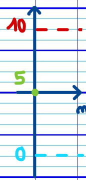
f) $P(\text{"Beth never goes} \geq 1\text{"}) = P(\text{"Beth's max is 0"}) = P(M=0) = \left(\dfrac{p}{q}\right)^0 \left(1-\dfrac{p}{q}\right) = 0.5714$

g) Initially:

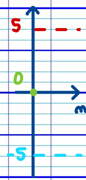


$\begin{cases} a = 6 \\ b = 6 \end{cases}$

$\longrightarrow$ Now:



translating to:



$\begin{cases} a = 5 \\ b = 5 \end{cases}$

So $P(\text{"Beth going to 10 before 0 from 5"}) = p(5) = \dfrac{1-(q/p)^b}{1-(q/p)^{a+b}} = 0.014$

And $\mathbb{P}$("Beth going to 0 from 5") = 1, negative drift