IEOR 4404 Simulation	**Fall 2024**	

Homework 1

Instructor: Henry Lam *Due: September 13*

Problem 1. A pair of fair dice is rolled. Let E denote the event that the sum of the dice is equal to 7
(a) Show that E is independent of the event that the first die lands on 4
(b) Show that E is independent of the event that the second die lands on 3 .

Problem 2. Let A, B, C be events such that $P(A) = 0.2, P(B) = 0.3, P(C) = 0.4$ Find the probability that at least one of the events A and B occurs if
(a) A and B are mutually exclusive;
(b) A and B are independent.
Find the probability that all of the events A, B, C occurs if
(c) A, B, C are independent;
(d) A, B, C are mutually exclusive.

Problem 3. A total number of n children of different heights are placed in a line at random. You select the first child from the line, and walk with her/him along the line, until you encounter a child who is taller, or until you reach the end of the line. If you do encounter a taller child, you also have him/her to accompany you further along the line, until you encounter yet again a taller child or reach the end of the line, etc. Let the random variable X denote the number of children to be selected from the line. What is the expected value of X?

Problem 4. If the density function of X equals

$$f(x) = \begin{cases} ce^{-2x}, & \text{for } 0 \leq x < \infty. \\ 0, & \text{for } x < 0. \end{cases} \tag{1}$$

a) Find the value c.
b) What is $P\{X > 2\}$?

Problem 5. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} xe^{-(x+y)}, & \text{for } x > 0, y > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

a) Compute the density of X.
b) Compute the density of Y.
c) Are X and Y independent?

Problem 6. Let $X \sim \text{Geo}(p)$ for some p in (0,1); that is, X is a geometric random variable with the parameter denoting probability of success $= p$. Is X memoryless? i.e., for all positive integers m and n, is

$$P(X > m + n \mid X > m) = P(X > n)?$$

Problem 7. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30. What is the probability that you will have to wait longer than 10 minutes? If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

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Homework 1:

Problem 1:

a) E : "sum of dices is equal to 7"

$\Omega_E = \{(1,6), (2,5), \dots, (6,1)\}$

We want to show that E is $\perp$ (independant) of A : "1st die lands on 4"

We want to prove that $P(E \cap A) = P(E).P(A)$

For $P(E)$: We have $card(\Omega_E) = 6$

So $\boxed{P(E)} = \frac{6}{6^2} = \boxed{\frac{1}{6}}$

both dice have 6 possibilities

For $P(A)$: $\Omega_A = \{(4,1), (4,2), \dots, (4,6)\}$

Where $card(\Omega_A) = 6$

So $\boxed{P(A)} = \frac{6}{6^2} = \boxed{\frac{1}{6}}$

$P(E \cap A)$: $E \cap A$: "sum of dices is equal to 7" and "1st die lands on 4"

: "sum of dices is equal to 7 and 1st die lands on 4"

$\Omega_{E \cap A} = \{(4,3)\}$ ie $card(\Omega_{E \cap A}) = 1$

And $\boxed{P(E \cap A)} = \frac{1}{6^2} = \boxed{\frac{1}{36}}$

Let's check the independence :

$$\left.\begin{array}{l} P(E).P(A) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \\ P(E \cap A) = \frac{1}{36} \end{array}\right\}$$ ie $P(A).P(E) = P(A \cap E)$
so A and E are independent

b) E is the same.

Let's define B: "2nd die lands on 3"

$\Omega_B = \{(1,3), (2,3), \dots, (6,3)\}$

Where $card(\Omega_B) = 6$

$\boxed{P(B)} = \frac{6}{6^2} = \boxed{\frac{1}{6}}$

Let's check the independence :

$$\left.\begin{array}{l} P(E).P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \\ P(E \cap B) = \frac{1}{36} \end{array}\right\} \text{ie } P(B).P(E) = P(B \cap E)$$

so B and E are independent

Problem 2:

a) A and B mutually exclusive $\longrightarrow$ $P(A \cap B) = 0$

Probability that at least A/B occurs $\longrightarrow$ $P(A \cup B)$

$$\boxed{P(A \cup B)} = P(A) + P(B) - \underbrace{\cancel{P(A \cap B)}}_{=0}$$

$$= \boxed{0.5}$$

b) $A \perp B \longrightarrow P(A \cap B) = P(A).P(B)$

$$\boxed{P(A \cup B)} = P(A) + P(B) - P(A).P(B)$$

$$= \boxed{0.44}$$

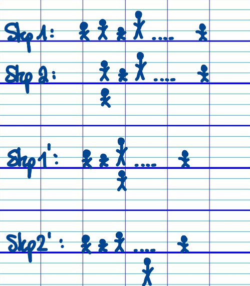
c) $A \perp B \perp C \longrightarrow P(A \cap B \cap C) = P(A).P(B).P(C)$

Probability that all A, B & C occurs $\longrightarrow$ $P(A \cap B \cap C)$

$$\boxed{P(A \cap B \cap C)} = P(A).P(B).P(C) = \boxed{0{,}024}$$

d) A, B and C mutually exclusive $\longrightarrow$ $\boxed{P(A \cap B \cap C) = 0}$

Problem 3:



First child is always wanted

2^{nd} child selected if taller than 1^{st}

3^{rd} child selected if taller than 1^{st} and 2^{nd}

⋮

etc

Let's note H_i the height of child i so that $H \sim \mathcal{N}(m, \sigma^2)$

The probability of child i being taller than all the previous $i-1$ child is equal to their height being the maximum of all $i-1$ child.

$P(\text{select child } i) = P(H_i \geq H_{i-1}, H_i \geq H_{i-2}, \dots, H_i \geq H_1) = P(H_i \geq \max_{i-1 \geq k \geq 1}(H_k))$

Moreover we have : $P(H_i = \max_{i \geq k \geq 1}(H_k)) = \frac{1}{i}$ (any child can be the tallest and iid)

But $H_i \geq \max_{i-1 \geq k \geq 1}(H_k) \rightarrow H_i = \max_{i \geq k \geq 1}(H_k)$

since H_i is the tallest

So $P(\text{select child } i) = \frac{1}{i}$

We have :

$\boxed{E[X]} = \sum_{i=1}^{m} P(\text{select child } i) = \boxed{\sum_{i=1}^{m} \frac{1}{i}}$ which is the harmonic series.

Problem 4:

a) By definition: $\int_{\mathbb{R}} f(x)\,dx = 1 \leftrightarrow \int_0^{+\infty} ce^{-2x}\,dx = 1$

$\leftrightarrow c\left[\frac{e^{-2x}}{-2}\right]_0^{+\infty} = 1$

$\leftrightarrow c\left(0 - \left(-\frac{1}{2}\right)\right) = 1$

$\leftrightarrow \boxed{c = 2}$

b) $\boxed{P(X > 2)} = \int_2^{+\infty} f(x)\,dx = 2\int_2^{+\infty} e^{-2x}\,dx = 2\left[\frac{e^{-2x}}{-2}\right]_2^{+\infty} = \boxed{e^{-4}}$

Problem 5:

a) $\boxed{f_X(x)} = \int_{\mathbb{R}} f(x,y)\,dy = x\int_0^{+\infty} e^{-(x+y)}\,dy = xe^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{+\infty} = \boxed{\begin{cases} xe^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}}$

b) $\boxed{f_Y(y)} = \int_{\mathbb{R}} f(x,y)\,dx = e^{-y}\int_0^{+\infty} xe^{-x}\,dx = e^{-y}\left(\left[-e^{-x}x\right]_0^{+\infty} + \int_0^{+\infty} e^{-x}\,dx\right)$

$= e^{-y}\left(0 + \left[e^{-x}\right]_0^{+\infty}\right) = \boxed{\begin{cases} e^{-y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}}$

c) $f(x,y) = f_X(x) \cdot f_Y(y) \rightarrow X \perp Y$

$\boxed{f_X(x) \cdot f_Y(y)} = \begin{cases} xe^{-x} \times e^{-y} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} xe^{-(x+y)} \\ 0 \end{cases} = \boxed{f(x,y)}$, so $X \perp Y$

$x, y > 0$

Problem 6 :

- Memoryless $\Leftarrow$ $m,n \in \mathbb{N}$, $\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > n)$

$X \sim Geo(p)$ so $\mathbb{P}(X = i) = (1-p)^{i-1} p$

$$\boxed{\mathbb{P}(X > i)} = \mathbb{P}(X = i+1 \cup X = i+2 \cup \cdots)$$
$$= \mathbb{P}(X = i+1) + \mathbb{P}(X = i+2) + \cdots$$
$$= \sum_{k=i+1}^{\infty} \mathbb{P}(X = k)$$
$$= p \sum (1-p)^{k-1}$$
$$= p \sum_{k=1}^{+\infty} (1-p)^{k+i-1}$$
$$= p(1-p)^i \sum_{k=0}^{+\infty} (1-p)^k$$
$$= \cancel{p}(1-p)^i \frac{1}{\cancel{1-(1-p)}}$$
$$= \boxed{(1-p)^i}$$

$$\boxed{\mathbb{P}(X > m+n \mid X > m)} = \frac{\mathbb{P}(X > m+n \cap X > m)}{\mathbb{P}(X > m)}$$
$$= \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)}$$
$$= \frac{(1-p)^{m+n}}{(1-p)^m}$$
$$= (1-p)^n$$
$$= \boxed{\mathbb{P}(X > n)}$$

so $\boxed{X \text{ is memoryless}}$

Problem 7 :

1) Let's note T as the bus arrival time so that $f_T(t) = \frac{1}{30}$ for $0 \leq t \leq 30$

So the probability of waiting more than 10 min is:

$$\boxed{P(T > 10)} = \int_{10}^{30} f_T(t)\, dt$$

bus comes at 10:30 worst case

$$= \frac{1}{30} \int_{10}^{30} 1\, dt$$
$$= \boxed{\frac{2}{3}}$$

2) The second problem can be formalized as:

$$\boxed{\mathbb{P}(T > 25 \mid T > 15)} = \frac{\mathbb{P}(T > 25 \cap T > 15)}{\mathbb{P}(T > 15)} = \frac{\mathbb{P}(T > 25)}{\mathbb{P}(T > 15)} = \frac{5/\cancel{30}}{15/\cancel{30}} = \boxed{\frac{1}{3}}$$