Optimization Project Group 34

Alexandre Duhamel
Donghan Li
Shujun Liu
Shuo Kong

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1 Definitions

Before formulating the problem, we need to define the sets, parameters, and constants:

- Z: the set of regions (zip codes).
- F_i: the set of existing facilities in region i.
- L_i: the set of potential locations for new facilities in region i.
- c_{ij}: current capacity of existing facility j in region i.
- c^*_{ij}: current capacity for children aged 0-5 of facility j in region i
- $c^{\max}_{ij}$: maximum capacity allowed for facility j after expansion (500 slots).
- Z^H: the set of high-demand regions.
- Z^N: the set of normal-demand regions.
- p_i: population of children aged 0-12 in region i.
- p^*_i: population of children aged 0-5 in region i.
- D: minimum allowed distance between facilities (0.06 miles).
- d_{ijk}: distance between existing facility j and potential location k in region i.
- $d_{ikk'}$: distance between potential locations k and k' in region i.
- M: a sufficiently large constant used in the Big M method.

2 Problem 1: Problem of Budget

New York State (NYS) faces a critical shortage of licensed child care facilities, with many regions designated as "child care deserts." This term refers to areas where the number of available child care slots is significantly lower than the demand, particularly in regions where more than 60% of parents are employed or the average income is below $60,000. To address this issue, the NYS government aims to increase child care capacity statewide, ensuring that all regions have sufficient access to these services.

In this initial problem, we are tasked with formulating an ideal solution to eliminate child care deserts by determining the minimum funding required to either expand existing facilities or build new ones. We focus on two main approaches:

- **Expansion of Existing Facilities**: By expanding the capacities of current facilities, we can increase child care slots within existing infrastructures, up to a maximum of 500 slots or 1.2 times the current capacity. This expansion is cost-effective, as it leverages current resources.
- **Construction of New Facilities**: In areas where expansion alone is insufficient, new child care facilities can be built. We consider three facility sizes—small, medium, and large—each with different associated capacities and costs.

Objective:

The primary objective is to minimize the total funding required to ensure that each area has sufficient capacity to avoid classification as a child care desert. Our model will calculate the necessary expansion or new construction efforts to meet this goal, providing a comprehensive budget plan for NYS.

Policy Constraints and Considerations:

Several policy considerations guide our approach to this problem:

- **Child Care Desert Classification**: For high-demand areas, child care slots must cover at least 50% of the child population. In normal-demand areas, the requirement is at least 33%.
- **Increased Capacity for Ages 0-5**: Acknowledging the heightened demand for child care for children under 5, NYS policy mandates that slots for this age group cover at least two-thirds of the population aged 0-5.
- **Expansion Cost Dependencies**: The expansion cost for each facility is proportional to the scale of the increase, and additional costs are applied for slots designated for children under 5.

This problem represents the first step in a broader analysis to eliminate child care deserts in NYS. By considering these policy constraints and budgeting effectively, our model aims to provide a feasible and financially sustainable approach for the state to meet its child care goals. In the sections that follow, we define our decision variables, objective function, and constraints in detail, ultimately presenting a solution that optimizes the allocation of funds to maximize child care access across New York State.

2.1 Decision Variables

There are two parts for the decision variables:

First, we need to decide for each existing facility, how many new slots are needed for children aged 0-5 and children aged 5-12. For facility j in region i, we need to decide x_{ij} and y_{ij}, where x_{ij} is for new slots of ages 0-5 and y_{ij} is for new slots of ages 5-12. Mathematically,

$$x_{ij} \geq 0, x_{ij} \in \mathbb{N}, \forall i \in Z, \forall j \in F_i$$
$$y_{ij} \geq 0, y_{ij} \in \mathbb{N}, \forall i \in Z, \forall j \in F_i$$

Second, we need to decide for each new location, whether to build a new facility. Here we need to introduce three binary variables s_{ik}, m_{ik} and l_{ik} to decide whether we build a new small facility, a new medium facility, or a new large facility, respectively. Mathematically,

$$\begin{aligned} s_{ij} &\in \{0,1\}, \forall i \in Z, \forall k \in L_i \\ m_{ij} &\in \{0,1\}, \forall i \in Z, \forall k \in L_i \\ l_{ij} &\in \{0,1\}, \forall i \in Z, \forall k \in L_i \end{aligned}$$

2.2 Objective

Our objective is to minimize the total amount of funding of the program.

For expanding existing facilities, the cost is proportional to the proportion of the increase in capacity, plus the additional cost for newly created slots for 0-5 ages. The capacity after expansion is $c_{ij} + x_{ij} + y_{ij}$, and the proportion of the increase is

$$w_{ij} = \frac{x_{ij} + y_{ij}}{c_{ij}}$$

The cost for expanding this facility will be:

$$\text{Cost of Expansion} = (20000 + 200c_{ij}) \cdot w_{ij} + 100x_{ij}, \forall i \in Z, \forall j \in F_i$$

For building new facilities, the cost will be:

$$\text{Cost of New Building} = 65000s_{ik} + 95000m_{ik} + 115000l_{ik}, \forall i \in Z, \forall k \in L_i$$

Therefore, the total cost can be formulated as follows:

$$\text{Cost} = \sum_{i \in Z} \sum_{j \in F_i} ((20000 + 200c_{ij}) \cdot w_{ij} + 100x_{ij}) + \sum_{i \in Z} \sum_{k \in L_i} (65000s_{ik} + 95000m_{ik} + 115000l_{ik})$$

2.3 Constraints

There are four types of constraints:

Capacity Requirements
The first constraint is to make sure that no region is classified as child care desert:

$$\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} 100s_{ik} + 200m_{ik} + 400l_{ik} \geq \frac{1}{2} p_i, \forall i \in Z^H$$

$$\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} 100s_{ik} + 200m_{ik} + 400l_{ik} \geq \frac{1}{3} p_i, \forall i \in Z^N$$

Enough Slots for Children Aged 0-5
The second constraint is to make sure the number of slots for children aged 0-5 is at least **two-thirds** of the population of children aged 0-5:

$$\sum_{j \in F_i} (c^*_{ij} + x_{ij}) + \sum_{k \in L_i} 50s_{ik} + 100m_{ik} + 200l_{ik} \geq \frac{2}{3} p^*_i, \forall i \in Z$$

Maximum Expansion Requirement
The third constraint is specified for the expansions of existing facilities. Firstly, for each facility, if its current

capacity is no more than 500 [1], we need to make sure that its capacity after expansion is still no more than 500:

$$c_{ij} + x_{ij} + y_{ij} \leq 500, \forall i \in Z, \forall j \in F_i$$

Besides, for each facility, the proportion of expansion is no more than 20% of current capacity:

$$w_{ij} = \frac{x_{ij} + y_{ij}}{c_{ij}} \leq \frac{1}{5} \implies 5(x_{ij} + y_{ij}) \leq c_{ij}, \forall i \in Z, \forall j \in F_i$$

Range of Decision Variables
The fourth constraint is about the range of decision variables: x_{ij} and y_{ij} must be integers, and s_{ij}, m_{ij}, l_{ij} must be binary. Moreover, for each location, we can at most build one new facility, which requires $s_{ij} + m_{ij} + l_{ij} \leq 1$:

$$\begin{aligned}
x_{ij} \geq 0, x_{ij} \in \mathbb{N} \quad & \forall i \in Z, \forall j \in F_i \\
y_{ij} \geq 0, y_{ij} \in \mathbb{N} \quad & \forall i \in Z, \forall j \in F_i \\
s_{ij} \in \{0,1\} \quad & \forall i \in Z, \forall k \in L_i \\
m_{ij} \in \{0,1\} \quad & \forall i \in Z, \forall k \in L_i \\
l_{ij} \in \{0,1\} \quad & \forall i \in Z, \forall k \in L_i \\
s_{ij} + m_{ij} + l_{ij} \leq 1 \quad & \forall i \in Z, \forall k \in L_i
\end{aligned}$$

2.4 Official Formulation

Given the definitions in Part 2.1, we can give the mathematical formulation of this problem as:

$$\begin{aligned}
\min \quad & \sum_{i \in Z} \sum_{j \in F_i} ((20000 + 200c_{ij}) \cdot w_{ij} + 100x_{ij}) + \sum_{i \in Z} \sum_{k \in L_i} (65000s_{ik} + 95000m_{ik} + 115000l_{ik}) \\
\text{s.t.} \quad & \sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} 100s_{ik} + 200m_{ik} + 400l_{ik} \geq \frac{1}{2}p_i, \forall i \in Z^H \\
& \sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} 100s_{ik} + 200m_{ik} + 400l_{ik} \geq \frac{1}{3}p_i, \forall i \in Z^N \\
& \sum_{j \in F_i} (c^*_{ij} + x_{ij}) + \sum_{k \in L_i} 50s_{ik} + 100m_{ik} + 200l_{ik} \geq \frac{2}{3}p^*_i, \forall i \in Z \\
& c_{ij} + x_{ij} + y_{ij} \leq 500, \forall i \in Z, \forall j \in F_i \\
& 5(x_{ij} + y_{ij}) \leq c_{ij}, \forall i \in Z, \forall j \in F_i \\
& s_{ij} + m_{ij} + l_{ij} \leq 1 \quad \forall i \in Z, \forall k \in L_i \\
& x_{ij}, y_{ij} \in \mathbb{N} \quad \forall i \in Z, \forall j \in F_i \\
& s_{ij}, m_{ij}, l_{ij} \in \{0,1\} \quad \forall i \in Z, \forall k \in L_i
\end{aligned}$$

2.5 Optimal Solutions

Based on the previous analysis, we found the optimal solutions. In order to eliminate child care desert regions and meet the special policy requirements for children under 5, while restricting the expanding rate for every existing facilities, our minimum funding will be $312,411,445.21.

[1] For facilities with current capacity greater than 500, they can no longer be expanded. Therefore, we should exclude these facilities from $F_i, \forall i \in Z$, which means not defining x and y for these facilities since we cannot make "decisions" on these facilities.

Given the constraints, we created 49,459 new slots for children under 5 in 1,990 existing facilities across New York State, and 1,315 new slots for children aged 5-12 in 78 existing facilities. In addition, we built 231 new small child care facilities, 157 new medium facilities and 2,263 new large facilities across New York State, creating 479,850 new slots for children under 5 and 959,700 new slots in total for all children.

The following graphs demonstrate the change the program brings. In order to make the graphs more intuitive, we only showed the changes in New York City. In each graph, the size of the bubble indicates the number of slots[2], and the color indicates the ratio.

[2]The sizes of bubbles are scaled in each graph, and represent the relative relationships

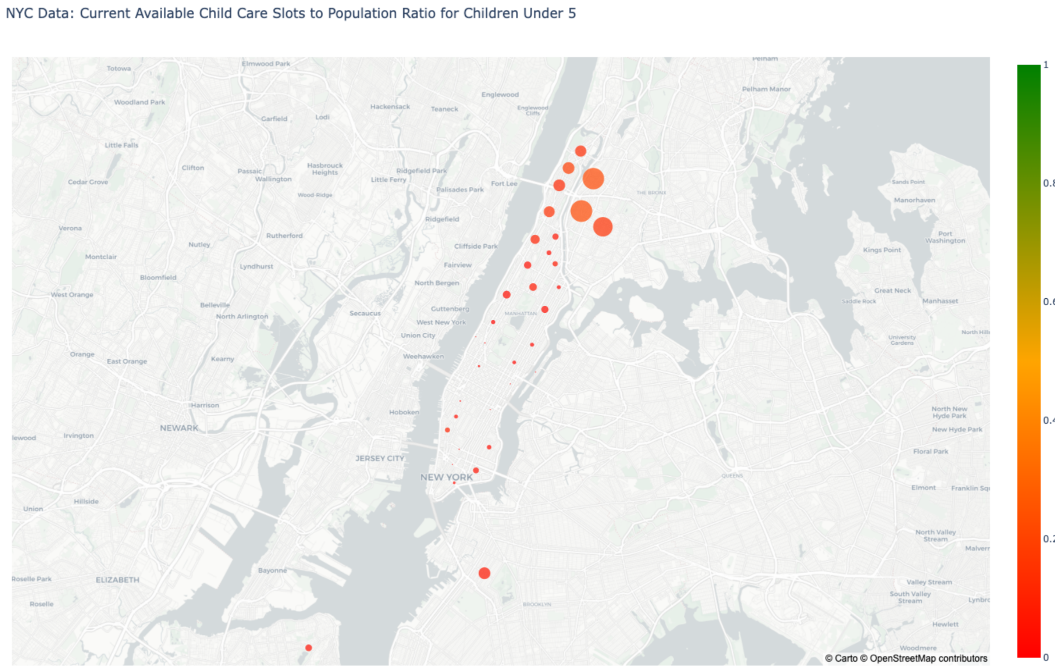

Figure 1: Available Child Care Slots to Population Ratio: Children Under 5 (Before)

Figure 2: Available Child Care Slots to Population Ratio: Children Under 5 (After)

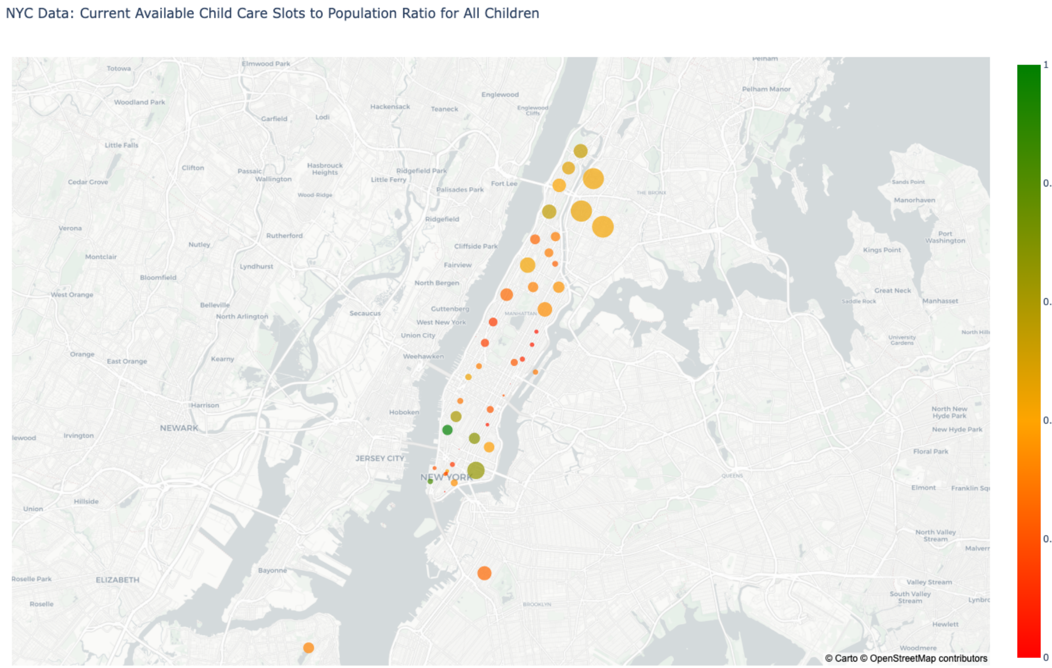

Figure 3: Available Child Care Slots to Population Ratio: All Children (Before)

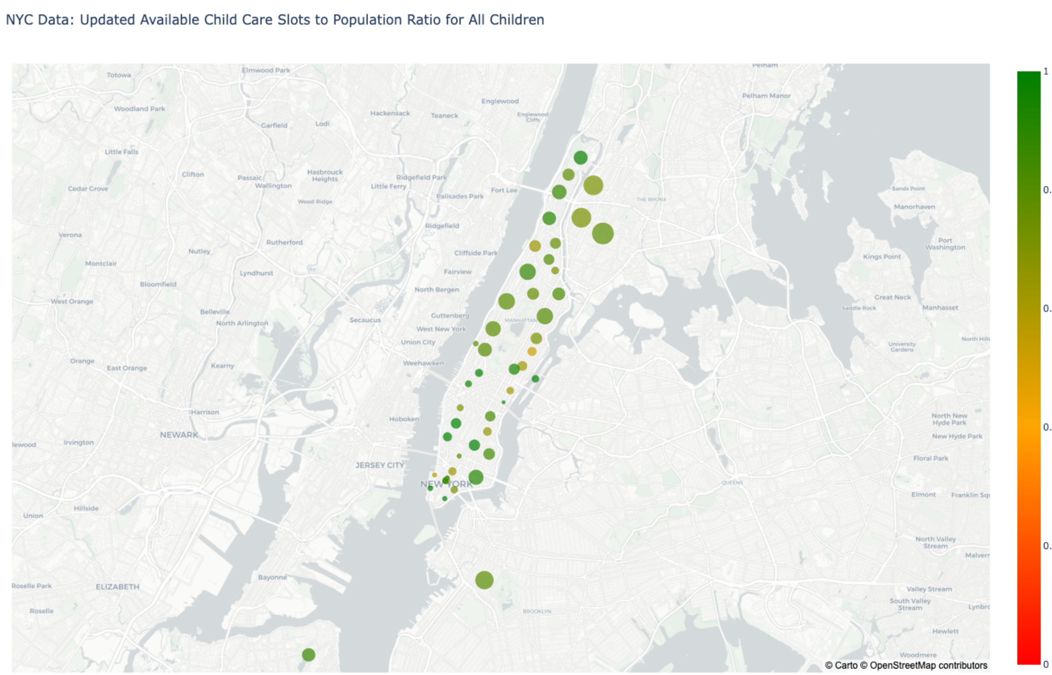

Figure 4: Available Child Care Slots to Population Ratio: All Children (After)

3 Problem 2: Incorporating Realistic Expansion and Distance Constraints

In the first part of our analysis, we developed a model to address child care deserts by expanding existing facilities and constructing new ones to meet policy requirements across New York State. However, this initial model did not fully account for the complexities involved in facility expansion and site selection. In this part, we enhance our model by incorporating more realistic considerations provided by NYS officials, namely:

- **Variable Expansion Costs:** Recognizing that larger expansions are more costly due to space limitations and logistical challenges, we introduce a tiered cost structure where the marginal cost increases with the scale of expansion.
- **Distance Limitations:** To prevent over-concentration of child care facilities in specific regions and ensure equitable access, we impose a minimum distance constraint between facilities within each area.

Objective:

Our objective in this enhanced model is to determine the minimum amount of funding required to eliminate child care deserts and meet policy requirements, while accounting for realistic expansion costs and distance constraints.

Justification for Decisions:

- **Tiered Expansion Costs:** We adopted a piecewise linear cost function for expanding existing facilities to reflect the increasing difficulty and expense associated with larger expansions. This approach aligns with the officials' recommendation and provides a more accurate estimation of expansion costs.
- **Expansion Limits:** We set a maximum expansion limit of 20% of current capacity for existing facilities, as costs become prohibitively high beyond this point, discouraging excessive expansions.
- **Distance Constraints:** By enforcing a minimum distance of 0.06 miles between any two facilities within the same area, we ensure a more even distribution of child care services, avoiding clusters and better serving the needs of all communities.

In the following sections, we define the decision variables, formulate the objective function, outline the constraints, and present the optimal solutions obtained from the enhanced model.

3.1 Decision Variables

There are two parts for the decision variables:

Expansion Variables for Existing Facilities

For each existing facility j in region i, we need to decide:

$$\begin{aligned}
&x_{ij} \geq 0, \quad x_{ij} \in \mathbb{N}, \quad \forall i \in Z, \forall j \in F_i \quad \text{(additional slots for ages 0-5)} \\
&y_{ij} \geq 0, \quad y_{ij} \in \mathbb{N}, \quad \forall i \in Z, \forall j \in F_i \quad \text{(additional slots for ages 5-12)} \\
&e_{ij} \in \{0,1\}, \quad \forall i \in Z, \forall j \in F_i \quad \text{(1 if facility } j \text{ is expanded)}
\end{aligned}$$

The proportion of expansion for facility j in region i is:

$$w_{ij} = \frac{x_{ij} + y_{ij}}{c_{ij}}$$

To build the Piecewise Costs Model, we "split" w_{ij} into three expansion tiers, where:

$$w_{ij} = w_{ij}^{(1)} + w_{ij}^{(2)} + w_{ij}^{(3)}$$

$$0 \le w_{ij}^{(1)} \le 10\%,\ 0 \le w_{ij}^{(2)} \le 5\%,\ 0 \le w_{ij}^{(3)} \le 5\%$$

We also need to introduce binary variables $\delta_{1,ij}$, $\delta_{1,ij}$ to indicate the expansion tier of w_{ij}:

$$\delta_{1,ij},\ \delta_{2,ij} \in \{0,1\}, \quad \forall i \in Z, \forall j \in F_i$$

$$\begin{cases} \delta_{1,ij} = 0,\ \delta_{2,ij} = 0, & \text{corresponding to} \quad 0 \le w_{ij} \le 10\% \\ \delta_{1,ij} = 1,\ \delta_{2,ij} = 0, & \text{corresponding to} \quad 10\% \le w_{ij} \le 15\% \\ \delta_{1,ij} = 1,\ \delta_{2,ij} = 1, & \text{corresponding to} \quad 15\% \le w_{ij} \le 20\% \end{cases}$$

Variables for New Facilities

For each potential location k in region i, we need to decide:

$$s_{ik} \in \{0,1\}, \quad \forall i \in Z, \forall k \in L_i \quad \text{(1 if a small facility is built at location } k\text{)}$$
$$m_{ik} \in \{0,1\}, \quad \forall i \in Z, \forall k \in L_i \quad \text{(1 if a medium facility is built at location } k\text{)}$$
$$l_{ik} \in \{0,1\}, \quad \forall i \in Z, \forall k \in L_i \quad \text{(1 if a large facility is built at location } k\text{)}$$

3.2 Objective

Our objective is to minimize the total amount of funding of the program.

Cost of Expanding Existing Facilities

The cost for expanding a facility equals to the sum of costs of three expansion tiers.

$$\text{Cost}_{ij}^{\text{Expansion}} = \text{Cost}_{ij}^{\text{Tier 1}} + \text{Cost}_{ij}^{\text{Tier 2}} + \text{Cost}_{ij}^{\text{Tier 3}}$$

- **Tier 1 (up to 10% expansion):**

$$\text{Cost}_{ij}^{\text{Tier 1}} = e_{ij} \cdot (20000 + 200c_{ij}) \cdot w_{ij}^{(1)}$$

- **Tier 2 (10% to 15% expansion):**

$$\text{Cost}_{ij}^{\text{Tier 2}} = e_{ij} \cdot (20000 + 400c_{ij}) \cdot w_{ij}^{(2)}$$

- **Tier 3 (15% to 20% expansion):**

$$\text{Cost}_{ij}^{\text{Tier 3}} = e_{ij} \cdot (20000 + 1000c_{ij}) \cdot w_{ij}^{(3)}$$

Additionally, there is an extra cost for new slots for ages 0-5:

$$\text{Additional Cost}_{ij} = 100x_{ij}$$

Cost of Building New Facilities

The cost for building a new facility at location k in region i is:

$$\text{Cost}_{ik}^{\text{New Facility}} = 65000s_{ik} + 95000m_{ik} + 115000l_{ik}$$

Total Cost

Therefore, the total cost can be formulated as:

$$\text{Minimize} \quad \sum_{i \in Z} \left[\sum_{j \in F_i} \left(\text{Cost}_{ij}^{\text{Expansion}} + \text{Additional Cost}_{ij} \right) + \sum_{k \in L_i} \text{Cost}_{ik}^{\text{New Facility}} \right]$$

3.3 Constraints

There are several types of constraints:

Expansion Tier Constraints

For each existing facility j in region i:

$$
\begin{aligned}
& w_{ij} \leq M e_{ij} \\
& 10\% \cdot \delta_{1,ij} \ \leq w_{ij}^{(1)} \leq 10\% \\
& 5\% \cdot \delta_{2,ij} \ \leq w_{ij}^{(2)} \leq 5\% \cdot \delta_{1,ij} \\
& 0 \leq w_{ij}^{(3)} \leq 5\% \cdot \delta_{2,ij} \\
& \delta_{1,ij}, \delta_{2,ij} \in \{0,1\}, \forall i \in Z, \forall j \in F_i \\
& \delta_{1,ij} \geq \delta_{2,ij}
\end{aligned}
$$

Linking Expansion Variables

Ensure that if a facility is expanded, $e_{ij} = 1$:

$$
\begin{aligned}
x_{ij} + y_{ij} &\geq \varepsilon e_{ij} \\
x_{ij} + y_{ij} &\leq M e_{ij}
\end{aligned}
$$

Capacity Requirements

Eliminating Child Care Deserts

For high demand regions $i \in Z^H$:

$$\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} 100 s_{ik} + 200 m_{ik} + 400 l_{ik} \geq \frac{1}{2} p_i, \forall i \in Z^H$$

For normal demand regions $i \in Z^N$:

$$\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} 100 s_{ik} + 200 m_{ik} + 400 l_{ik} \geq \frac{1}{3} p_i, \forall i \in Z^N$$

Enough Slots for Children Aged 0-5

For each region i:

$$\sum_{j \in F_i} (c_{ij}^* + x_{ij}) + \sum_{k \in L_i} 50 s_{ik} + 100 m_{ik} + 200 l_{ik} \geq \frac{2}{3} p_i^*, \forall i \in Z$$

Expansion Limits

For each existing facility j in region i:

$$
\begin{aligned}
& c_{ij} + x_{ij} + y_{ij} \leq c_{ij}^{\max}, \quad \forall i \in Z, \forall j \in F_i \\
& w_{ij} \leq 0.20, \quad \forall i \in Z, \forall j \in F_i
\end{aligned}
$$

where $c_{ij}^{\max} = 500$ if $c_{ij} \leq 500$. Otherwise $c_{ij}^{\max} = c_{ij}$

Facility Size Constraints

At each potential new facility location k in region i:

$$s_{ik} + m_{ik} + l_{ik} \leq 1, \quad \forall i \in Z, \forall k \in L_i$$

Distance Constraints

Between Existing and New Facilities

For existing facility j and potential location k in region i:

$$\text{If } d_{ijk} \leq D \implies s_{ik} + m_{ik} + l_{ik} \leq 0$$

Between New Facilities

For potential locations k and k' in region i:

$$\text{If } d_{ikk'} \leq D \implies s_{ik} + m_{ik} + l_{ik} + s_{ik'} + m_{ik'} + l_{ik'} \leq 1$$

Range of Decision Variables

All decision variables must satisfy:

$$\begin{aligned}
&x_{ij}, y_{ij} \geq 0, \quad x_{ij}, y_{ij} \in \mathbb{N}, \quad \forall i \in Z, \forall j \in F_i \\
&e_{ij}, t_{1,ij}, t_{2,ij}, t_{3,ij} \in \{0, 1\}, \quad \forall i \in Z, \forall j \in F_i \\
&s_{ik}, m_{ik}, l_{ik} \in \{0, 1\}, \quad \forall i \in Z, \forall k \in L_i
\end{aligned}$$

3.4 Official Formulation

Given the definitions in Part 3.1, we can give the mathematical formulation of this problem as:

$$\text{Minimize} \quad \sum_{i\in Z}\left[\sum_{j\in F_i}\left(\text{Cost}_{ij}^{\text{Expansion}} + \text{Additional Cost}_{ij}\right) + \sum_{k\in L_i}\text{Cost}_{ik}^{\text{New Facility}}\right]$$

s.t.

$$w_{ij} \le Me_{ij}, \forall i \in Z, \forall j \in F_i$$

$$10\% \cdot \delta_{1,ij} \le w_{ij}^{(1)} \le 10\%, \forall i \in Z, \forall j \in F_i$$

$$5\% \cdot \delta_{2,ij} \le w_{ij}^{(2)} \le 5\% \cdot \delta_{1,ij}, \forall i \in Z, \forall j \in F_i$$

$$0 \le w_{ij}^{(3)} \le 5\% \cdot \delta_{2,ij}, \forall i \in Z, \forall j \in F_i$$

$$\delta_{1,ij} \ge \delta_{2,ij}, \forall i \in Z, \forall j \in F_i$$

$$x_{ij} + y_{ij} \ge \varepsilon e_{ij}, \forall i \in Z, \forall j \in F_i$$

$$x_{ij} + y_{ij} \le Me_{ij}, \forall i \in Z, \forall j \in F_i$$

$$\sum_{j\in F_i}(c_{ij} + x_{ij} + y_{ij}) + \sum_{k\in L_i} 100s_{ik} + 200m_{ik} + 400l_{ik} \ge \frac{1}{2}p_i, \forall i \in Z^H$$

$$\sum_{j\in F_i}(c_{ij} + x_{ij} + y_{ij}) + \sum_{k\in L_i} 100s_{ik} + 200m_{ik} + 400l_{ik} \ge \frac{1}{3}p_i, \forall i \in Z^N$$

$$\sum_{j\in F_i}(c_{ij}^* + x_{ij}) + \sum_{k\in L_i} 50s_{ik} + 100m_{ik} + 200l_{ik} \ge \frac{2}{3}p_i^*, \forall i \in Z$$

$$c_{ij} + x_{ij} + y_{ij} \le c_{ij}^{\max}, \forall i \in Z, \forall j \in F_i$$

$$w_{ij} \le 0.20, \forall i \in Z, \forall j \in F_i$$

$$s_{ik} + m_{ik} + l_{ik} \le 1, \forall i \in Z, \forall k \in L_i$$

$$s_{ik} + m_{ik} + l_{ik} \le 0, \forall i \in z, \forall j \in F_i, \forall k \in L_i, \text{if } d_{ijk} \le D,$$

$$s_{ik} + m_{ik} + l_{ik} + s_{ik'} + m_{ik'} + l_{ik'} \le 1, \forall i \in z, \forall k \in L_i, k' \in L_i, \text{if } d_{ikk'} \le D,$$

$$w_{ij} = \frac{x_{ij} + y_{ij}}{c_{ij}}, \forall i \in Z, \forall j \in F_i$$

$$w_{ij} = w_{ij}^{(1)} + w_{ij}^{(2)} + w_{ij}^{(3)}, \forall i \in Z, \forall j \in F_i$$

$$x_{ij}, y_{ij} \ge 0, \quad x_{ij}, y_{ij} \in \mathbb{N}, \forall i \in Z, \forall j \in F_i$$

$$e_{ij}, \delta_{1,ij}, \delta_{2,ij} \in \{0,1\}, \forall i \in Z, \forall j \in F_i$$

$$s_{ik}, m_{ik}, l_{ik} \in \{0,1\}, \quad \forall i \in Z, \forall k \in L_i$$

3.5 Optimal Solutions

In Part 2, we aimed to determine the minimal funding required to eliminate child care deserts and meet the policy requirements across New York State, while adhering to the specified constraints.

Minimal Funding Required:

The optimization model indicates that the minimal funding required is approximately $337,288,899.90.

Expansion of Existing Facilities:

- **Number of Existing Facilities Expanded for Ages 0-5:** 24 facilities
- **Number of Existing Facilities Expanded for Ages 5-12:** 0 facilities

- **Total New Slots Added via Expansion:**
  - **For Ages 0-5:** 23 slots
  - **For Ages 5-12:** 0 slots

Construction of New Facilities:

- **Number of New Facilities Built:**
  - **Small Facilities:** 299
  - **Medium Facilities:** 260
  - **Large Facilities:** 2,540
- **Total New Slots Created in New Facilities:**
  - **For Ages 0-5:** 548,950 slots
  - **For All Ages:** 1,097,900 slots

Total New Slots Created:

- **Total New Slots for Ages 0-5 (Expansion + New Facilities):** 23 + 548,950 = **548,973 slots**
- **Total New Slots for All Ages (Expansion + New Facilities):** 23 + 1,097,900 = **1,097,923 slots**

Key Findings:

- The vast majority of new capacity is achieved through the construction of new facilities, particularly large ones.
- Expansion of existing facilities contributes minimally to the overall increase in capacity.
- The optimization model prioritizes building new facilities as a cost-effective strategy to meet child care needs.

Conclusion:

With the proposed funding of approximately $337 million, New York State can significantly increase child care capacity, effectively eliminate child care deserts, and meet policy requirements. The construction of new facilities, especially large ones, is the primary driver of this capacity increase.

4 Problem 3: The Problem of Fairness

As New York State (NYS) continues to address the shortage of licensed child care through targeted funding and expansion, equitable distribution of resources has emerged as a key concern. Beyond merely increasing the number of child care slots, the government seeks to ensure that all areas within NYS have equitable access to child care, preventing significant disparities in child care availability between regions. This emphasis on fairness is driven by the need for a balanced approach, where no community is left significantly underserved compared to others.

In this problem, we are tasked with optimizing child care access across NYS under a fairness constraint, ensuring that the child care slot-to-population ratio between any two areas differs by no more than 0.1. To achieve this, the government has allocated a budget of \$1 billion, which will be used to expand existing facilities and build new ones. This budget not only addresses the elimination of child care deserts but also enforces an even distribution of resources.

Objective and Fairness Measures

The primary objective in this part is to maximize the state's social coverage index, a weighted measure of child care coverage that prioritizes children under 5. This index uses a 2:1 weighting in favor of younger children, reflecting the higher demand for child care among this age group and the importance of early childhood care in promoting social well-being. The social coverage index, therefore, is calculated as:

$$\text{Maximize} \quad \left(\frac{2}{3} \times \text{Coverage}_{0-5} + \frac{1}{3} \times \text{Coverage}_{0-12}\right)$$

where Coverage_{0-5} and Coverage_{0-12} represent the coverage ratios for children aged 0-5 and all children aged 0-12, respectively.

Constraints and Budget Limitations

To ensure the fairness of child care distribution, we impose the fairness constraint, which mandates that the coverage ratio between any two areas does not exceed a 0.1 difference. This constraint ensures that child care accessibility is distributed relatively evenly across the state, preventing regions from being disproportionately advantaged or disadvantaged.

The \$1 billion budget also constrains our decisions, necessitating a careful allocation of funds to maximize coverage while respecting the fairness constraint. Expansion costs and the structure of new facilities are therefore modeled to achieve the optimal distribution of child care slots without exceeding the budget limit.

Challenges and Feasibility

Our analysis shows that enforcing the strict fairness constraint across NYS while staying within the budget may result in infeasibility due to the substantial costs required to equalize access in all regions. Therefore, we explore the maximum baseline coverage ratio that can be achieved within the budget and expansion limitation is approximately 0.7. Through this approach, although sacrificing the fairness constraint, NYS can optimize the social coverage index within the \$1 billion budeget.

4.1 Objective

We aim to maximize the weighted sum of child care coverage for two groups in NYS, which is:

$$\text{Maximize} \quad \left(\frac{2}{3} \times \text{Coverage}_{0-5} + \frac{1}{3} \times \text{Coverage}_{0-12}\right)$$

Where Coverage_{0-5} represents the child care coverage for children aged 0 - 5, and Coverage_{0-12} represents the child care coverage for all children, i.e., aged 0 - 12. The child care coverage for both age groups can be calculated as:

$$\text{Coverage}_{0-5} = \frac{1}{\sum_{i \in Z} p_i^*} \times \sum_{i \in Z} \left[\sum_{j \in F_i} (c_{ij}^* + x_{ij}) + \sum_{k \in L_i} (50 s_{ik} + 100 m_{ik} + 200 l_{ik}) \right]$$

And

$$\text{Coverage}_{0-12} = \frac{1}{\sum_{i \in Z} p_i} \times \sum_{i \in Z} \left[\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} (100 s_{ik} + 200 m_{ik} + 400 l_{ik}) \right]$$

Here we are using the state-level child care coverage in our objective.

4.2 Constraints

Budget Constraint
Total budget for the problem is 1 billion, which is to say:

$$\sum_{i \in Z} \left[\sum_{j \in F_i} \left(\text{Cost}_{ij}^{\text{Expansion}} + \text{Additional Cost}_{ij} \right) + \sum_{k \in L_i} \text{Cost}_{ik}^{\text{New Facility}} \right] \leq 1 \text{ Billion}$$

where $\text{Cost}_{ij}^{\text{Expansion}}$, $\text{Additional Cost}_{ij}$ and $\text{Cost}_{ik}^{\text{New Facility}}$ follows the formulation of Part 3.2. To make the budget formulation meaningful, we also keep all the constraints in Part 3.3, which prevent creating a child care desert as well as maintain the distance limitation between facilities.

Fairness Constraint
To ensure that the difference in the child care coverage between any two areas does not exceed 0.1, we have:

$$\begin{aligned} &\frac{1}{p_i} \cdot \left[\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} (100 s_{ik} + 200 m_{ik} + 400 l_{ik}) \right] \\ -&\frac{1}{p_n} \cdot \left[\sum_{j \in F_n} (c_{nj} + x_{nj} + y_{nj}) + \sum_{k \in L_n} (100 s_{nk} + 200 m_{nk} + 400 l_{nk}) \right] \leq 0.1, \forall i, n \in Z, i \neq n \end{aligned}$$

4.3 Official Formulation

Based on the definitions in both Part 3 and Part 4, we summarize he mathematical formulation of Problem 3 as:

$$\begin{aligned} \text{Maximize} \quad & \left(\frac{2}{3} \times \text{Coverage}_{0-5} + \frac{1}{3} \times \text{Coverage}_{0-12} \right) \\ \text{s.t.} \quad & \\ & \text{All constraints in Problem 2} \\ & \sum_{i \in Z} \left[\sum_{j \in F_i} \left(\text{Cost}_{ij}^{\text{Expansion}} + \text{Additional Cost}_{ij} \right) + \sum_{k \in L_i} \text{Cost}_{ik}^{\text{New Facility}} \right] \leq 10^9 \\ & \frac{1}{p_i} \cdot \left[\sum_{j \in F_i} (c_{ij} + x_{ij} + y_{ij}) + \sum_{k \in L_i} (100 s_{ik} + 200 m_{ik} + 400 l_{ik}) \right] \\ & -\frac{1}{p_n} \cdot \left[\sum_{j \in F_n} (c_{nj} + x_{nj} + y_{nj}) + \sum_{k \in L_n} (100 s_{nk} + 200 m_{nk} + 400 l_{nk}) \right] \leq 0.1, \forall i, n \in Z, i \neq n \end{aligned}$$

4.4 Optimal Solution

We can take an alternative way to solve this problem. According to the optimal solution of Problem 2, we calculate the child care coverage ratio of each region and look at its statistics, as shown in Table 1:

	Coverage Index
count	1005
mean	0.7123
std	0.6925
min	0.2924
25%	0.4736
50%	0.6050
75%	0.7844
90%	1.0117
99%	2.2492
max	18.1818

Table 1: Statistics of Child Care Coverage Index of Each Region Before Part 3

The Fairness Constraints requires the differences of coverage index between any two of regions do not exceed 0.1. This means we have to expand existing facilities or building new ones to ensure the coverage index of each region is at least close to the region with currently the highest coverage index after we optimize problem 2. After investigating the distribution of current coverage index across regions, we found that if we strictly follow the Fairness Constraint, obviously we need to make every coverage ratio greater than or equal to 18.08. However, it exceeds the maximum expansion potentiality. **Therefore, Problem 3 is infeasible**.

If we can compromise the Fairness Constraint while improving the minimum coverage ratio as much as possible, we calculate that the maximum baseline of coverage ratio which can make this problem feasible is approximately 0.7. That is to say, we can make every coverage ratio greater than or equal to at most 0.7 in order to maintain the total cost within $1 billion and the expansion within its maximum potentiality. Here the optimal social coverage index is 1.74.