IEOR 4106, HMWK 6, Professor Sigman

1. Vehicles arrive to a bridge according to a Poisson process at rate $\lambda = 10$ per hour. With probability $p = 0.70$ any such vehicle is, independently, a car; otherwise (probability $q = 1 - p = 0.3$) it is a truck.

   (a) What is the expected value and variance of the number of vehicles that arrive to the bridge by time $t = 2$ hours? What is the expected value and variance of the number of cars that arrive by time $t = 2$ hours?

   (b) What is the probability that 5 trucks arrive before the first car?

   (c) What is the probability that 4 trucks arrive between 2 and 4 PM, and 2 trucks arrive between 5 and 7 PM?

   (d) What is the probability that 4 trucks arrive between 2 and 4 PM, and 2 cars arrive between 5 and 7 PM?

   (e) Given that 6 cars arrived between 2 and 4 PM, what is the probability that 6 trucks arrived in that same time interval?

   (f) Given that 1 car arrived between 2 and 4 PM, what is the probability that it arrived between 2 and $2:30$PM?

   (g) Given that 2 cars arrived between 2 and 4 PM, what is the probability that they both arrived between 2 and $2:30$PM?

   (h) Given that $n \geq 1$ cars arrived between 2 and 4 PM, what is the probability that they all arrived between 2 and $2:30$PM?

2. M/G/∞ Queue.

   According to a Poisson process at rate $\lambda = 20$ per day, a company rents housing units and rents each one, independently of other units, for H days, where H has an exponential distribution with $E(H) = 30$ (days).

   Assume that initially (time $t = 0$) no units are rented yet. (And also assume that the company has an unlimited number of houses to rent.)

   (a) Compute the expected number of units rented at times $t = 10$, $t = 20$ and $t = 100$ days.

   (b) What is the long-run time-average number of units rented out by the company? Letting $X(t) =$ the number of units rented at time t, the limit is defined by

   $$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds.$$

   (c) Repeat (a), (b) when H has a uniform distribution on $(20, 40)$.

   (d) For the uniform distribution in (c) for H, compute the probability that at time $t = 1$ there are 0 units rented. Repeat for the case when H has an exponential distribution with $E(H) = 30$ (days).

   (e) Repeat (a) and (b) when H is constant at size 30 days: $P(H = 30) = 1$.

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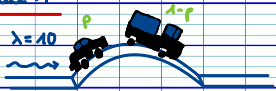
HW6: Stochastic

Exercice 1:

a) 

car $\sim PP(\lambda p)$, $\lambda_c = 7$

truck $\sim PP(\lambda q)$, $\lambda_t = 3$

Total vehicules: $\begin{cases} \underbrace{E[N] = \lambda t}_{\sim PP(\lambda)} = 20 \text{ vehicules} \\ Var(N) = \lambda t = 20 \end{cases}$

Total cars: $\begin{cases} E[C] = pE[N] = 14 \text{ cars} \\ Var(C) = 14 \end{cases}$, $C \sim PP(\lambda p)$ by superposition principle

b) Time until 1st car $\sim exp(\lambda_c)$

Time until 5th truck $\sim gamma(\lambda_t, 5)$

So $P(\text{5 trucks arrive before 1st car}) = \left(\frac{\lambda_t}{\lambda_t + \lambda_c}\right)^5 = 0.243\%$

probability that truck arrives 1st, 5 times + memoryless property

c) $P(\text{4 trucks } [2PM, 4PM] \text{ and 2 trucks } [5PM, 7PM]) = P(N_1 = 4) \times P(N_2 = 2) = \left(\frac{e^{-6} \times 6^4}{4!}\right)\left(\frac{e^{-6} \times 6^2}{2!}\right)$ ($\lambda_t t$)

$= 0.597\%$

d) $P(\text{2 trucks } [2PM, 4PM] \text{ and 2 cars } [5PM, 7PM]) = P(\text{4 truck}) \times P(\text{2 cars}) = \left(\frac{e^{-6} \times 6^4}{4!}\right)\left(\frac{e^{-14} \times 14^2}{2!}\right)$

$= 0.0011\%$

e) Trucks and cars arrive independently

$P(\text{6 cars } [2PM, 4PM] \mid \text{6 trucks } [2PM, 4PM]) = \frac{e^{-6} \times 6^6}{6!} = 16.06\%$

f) $P(\text{1 car } [2PM, 2{:}30PM] \mid \text{1 car } [2PM, 4PM]) = \frac{0.5}{2} = 0.25$, uniform distribution of arrival times

g) $P(\text{2 car } [2PM, 2{:}30PM] \mid \text{2 car } [2PM, 4PM]) = \left(\frac{0.5}{2}\right)^2 = 0.0625$

h) $P(\text{m car } [2PM, 2{:}30PM] \mid \text{m car } [2PM, 4PM]) = 0.25^m$

Exercise 2:

a) $E[H] = 30$ so $\mu = \frac{1}{30}$ per day

We know that

$$E[X(t)] = \alpha(t) = \lambda \int_0^t \overbrace{\mathbb{P}(H > u)}^{\mathbb{P}(H > t-u)} du = \lambda \int_0^t e^{-\mu(t-u)} du = \lambda \frac{1}{\mu}(1 - e^{-\mu t})$$

So $E[X(t)] = \frac{\lambda}{\mu}(1 - e^{-\mu t}) = \begin{cases} 170.08 \text{ for } t = 10 \\ 291.95 \text{ for } t = 20 \\ 578.60 \text{ for } t = 100 \end{cases}$

b) $\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)\,ds \longrightarrow \rho = \frac{\lambda}{\mu} = 600$

c) For $H \sim \text{Uniform}(20, 40)$ $\begin{cases} f_H(h) = \frac{1}{40-20} = \frac{1}{20} \text{ for } h \in (20, 40) \\ F_H(h) = \frac{h-20}{20} \end{cases}$

For $t \leq 20$: $E[X(t)] = \lambda \int_0^t \mathbb{P}(H > t-s)\,ds = \lambda \int_0^t 1\,ds = 20t$

At $t = 20$: $E[X(t)] = 200$

For $20 < t \leq 40$: $E[X(t)] = \lambda \int_0^t \mathbb{P}(H > t-s)\,ds = \lambda \left(\int_0^{t-20} 1\,ds + \int_{t-20}^t 1 - \frac{t-s-20}{20}\,ds \right) = \lambda \left(20 + \frac{(t-20)(40-t)}{20} \right)$

At $t = 40$: $E[X(t)] = 400$

For $40 < t$: $E[X(t)] = \lambda \int_0^t \overbrace{\mathbb{P}(H > t-s)}^{=0 \text{ for } t-s > 40}\,ds = \lambda \times 30 = 600$

$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)\,ds \longrightarrow \rho = \frac{\lambda}{\mu} = 600$

d) 1. $H \sim \text{Uniform}(20, 40)$

The expected number of units rented at $t = 1$: $\alpha(1) = \lambda \times 1 = 20$

$\mathbb{P}(X(1) = 0) = e^{-\alpha(1)} \frac{\alpha(1)^0}{0!} = e^{-20}$

2. $H \sim \text{Exp}(1/30)$

$\alpha(1) = \frac{\lambda}{\mu}(1 - e^{-\mu t}) = 19.68$

$\mathbb{P}(X(1) = 0) = e^{-\alpha(1)} \frac{\alpha(1)^0}{0!} = 2.85 \times 10^{-9}$

e) $H \sim$ constant (30)

For $t \leq 30$: $E[X(t)] = \lambda \int^{t} 1\, ds = \lambda t$

For $t > 30$: $E[X(t)] = \lambda \int_{t-30}^{t} 1\, ds = \lambda(t-(t-30)) = 30\lambda$

So: $\begin{cases} \text{For } t = 10: E[X(t)] = 200 \\ \text{For } t = 20: E[X(t)] = 400 \\ \text{For } t = 100: E[X(t)] = 600 \end{cases}$

Still: $\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)\, ds \longrightarrow \rho = \frac{\lambda}{\mu} = 600$