IEOR E4004: Optimization Models and Methods
Instructor: Dr. Yaren Bilge Kaya
Due: 12/1/2024, 11:59 pm

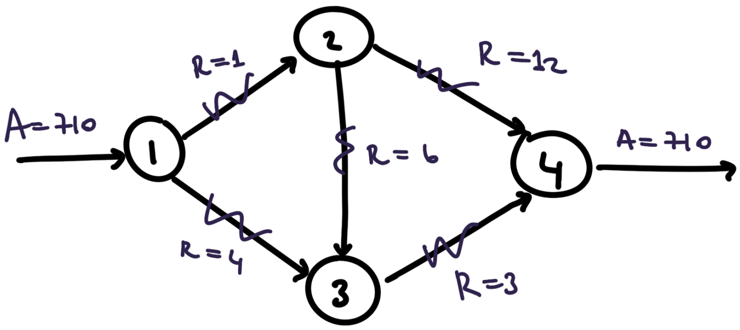
Assignment 5

November 25, 2024

Question 1. *(50 points) In an electrical network, the power loss that happens when an electrical current of I amperes flows through a resistance of R ohms is I^2R watts. In the figure below, 710 amperes of current must be sent from node 1 to node 4. The current flowing through each node must satisfy conservation of flow. For example, the flow entering the network (node 1), which is 710 amperes, should equal the flow exiting node 1, that is, the flow through the 1-ohm resistor + the flow through the 4-ohm resistor (note that there are five total resistors in the network). Amazingly, the physical laws of conservation determine the flow through the network that minimizes the power loss.*

1. *(30 points) Formulate a quadratic program (QP) that, when solved, will yield the current flowing through each resistor.*

2. *(20 points) Write out a .lp file, and use Gurobi to solve the QP you formulated. What is the optimal flow across the network? What is the optimized power loss?*



1) Set $N = \{1,2,3,4\}$ and $E = \{(1,2),(1,3),(2,3),\ldots\}$

Decision variables: I_{ij} := current flowing from point i to point j

Objective function: $\min \sum_{(i,j)\in E} I_{ij}^2 R_{ij}$ for R_{ij} resistance from $i \to j$

Constraints: 1. Flow conservation: $\sum_{\text{inflow to node } i} I_{in} = \sum_{\text{outflow to node } i} I_{out}$

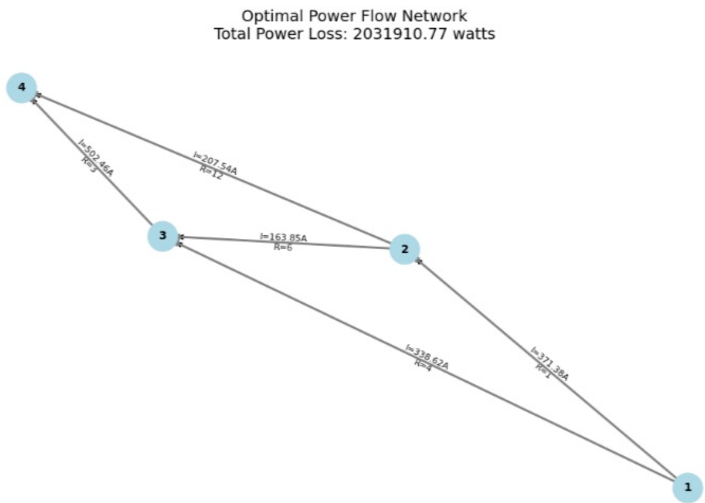
$\Leftrightarrow \sum_j I_{ji} - \sum_j I_{ij} = A_i, \ \forall i \in N$

with $A_1 = 710$, $A_2 = A_3 = 0$, $A_4 = -710$

2. Boundary conditions: $\begin{cases} \sum_j I_{1j} = 710 \\ \sum_j I_{j4} = 710 \end{cases}$

3. Non-negativity: $I_{ij} \geq 0, \ \forall (i,j) \in E$

2)
```
Optimal solution found:
Optimized Power Loss: 2031910.77 watts
Current on edge (1, 2): 371.38 amps
Current on edge (1, 3): 338.62 amps
Current on edge (2, 3): 163.85 amps
Current on edge (2, 4): 207.54 amps
Current on edge (3, 4): 502.46 amps
```



```python
# Network edges and resistances
edges = [(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)]
resistance = {(1, 2): 1, (1, 3): 4, (2, 3): 6, (2, 4): 12, (3, 4): 3}

# Nodes
nodes = [1, 2, 3, 4]
source = 1
sink = 4
flow_demand = 710

# Create a Gurobi model
model = Model("Power_Loss_Minimization")

# Variables: Current on each edge
I = model.addVars(edges, lb=-GRB.INFINITY, vtype=GRB.CONTINUOUS, name="I")

# Objective: Minimize total power loss
objective = QuadExpr()
for (i, j) in edges:
    objective += resistance[(i, j)] * I[i, j] * I[i, j]  # R * I^2
model.setObjective(objective, GRB.MINIMIZE)

# Constraints: Flow conservation at each node
for node in nodes:
    inflow = sum(I[i, j] for i, j in edges if j == node)  # Incoming current
    outflow = sum(I[i, j] for i, j in edges if i == node)  # Outgoing current

    # Apply flow conservation
    if node == source:
        model.addConstr(outflow - inflow == flow_demand, name=f"flow_conservation_source_{node}")
    elif node == sink:
        model.addConstr(outflow - inflow == -flow_demand, name=f"flow_conservation_sink_{node}")
    else:
        model.addConstr(outflow - inflow == 0, name=f"flow_conservation_{node}")

# Solve the model
model.optimize()
```

```
Gurobi Optimizer version 11.0.3 build v11.0.3rc0 (mac64[arm] - Darwin 23.4.0 23E224)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 4 rows, 5 columns and 10 nonzeros
Model fingerprint: 0xdfa0db5b
Model has 5 quadratic objective terms
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [0e+00, 0e+00]
  QObjective range [2e+00, 2e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [7e+02, 7e+02]
Presolve removed 1 rows and 0 columns
Presolve time: 0.01s
Presolved: 3 rows, 5 columns, 7 nonzeros
Presolved model has 5 quadratic objective terms
Ordering time: 0.00s

Barrier statistics:
 Free vars  : 5
 AA' NZ     : 2.000e+00
 Factor NZ  : 6.000e+00
 Factor Ops : 1.400e+01 (less than 1 second per iteration)
 Threads    : 1

                  Objective                Residual
Iter       Primal          Dual         Primal    Dual     Compl     Time
   0   2.09802387e+06 -2.09802387e+06  1.71e-13 4.15e+02  0.00e+00     0s
   1   2.03191077e+06  2.03190671e+06  1.14e-13 4.15e-04  0.00e+00     0s
   2   2.03191077e+06  2.03191077e+06  0.00e+00 4.15e-10  0.00e+00     0s

Barrier solved model in 2 iterations and 0.01 seconds (0.00 work units)
Optimal objective 2.03191077e+06
```

IEOR E4004: Optimization Models and Methods
Instructor: Dr. Yaren Bilge Kaya
Due: 12/1/2024, 11:59 pm

IEOR Industrial Engineering & Operations Research COLUMBIA | ENGINEERING

Assignment 5

November 25, 2024

Question 2. *(30 points) Without using any software to solve, prove that the optimal solution of the following LP is* $x^* = [1, 1, 1, 1]$.

$$\begin{array}{ll} \textit{minimize} & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \textit{subject to} & -x_1 - 6x_2 + x_3 + 3x_4 \le -3 \\ & -x_1 - 2x_2 + 7x_3 + x_4 \le 5 \\ & 0x_1 + 3x_2 - 10x_3 - x_4 \le -8 \\ & -6x_1 - 11x_2 - 2x_3 + 12x_4 \le -7 \\ & x_1 + 6x_2 - x_3 - 3x_4 \le 4 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{array}$$

- We must verify feasability and optimality.

Feasability:

$$\begin{array}{rl} -x_1 - 6x_2 + x_3 + 3x_4 = -3 & \le -3 \\ -x_1 - 2x_2 + 7x_3 + x_4 = 5 & \le 5 \\ 0x_1 + 3x_2 - 10x_3 - x_4 = -8 & \le -8 \\ -6x_1 - 11x_2 - 2x_3 + 12x_4 = -7 & \le -7 \\ x_1 + 6x_2 - x_3 - 3x_4 = 3 & \le 4 \\ x_1, x_2, x_3, x_4 = 1 & \ge 0 \end{array}$$

everything is verified ✓

and optimal value is 64

Optimality: Using the complementary slackness theorem

Let's write the dual of the LP: $A = \begin{bmatrix} -1 & -6 & 3 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix}$, $b = \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix}$ and $c = \begin{bmatrix} 47 \\ 93 \\ 17 \\ -93 \end{bmatrix}$

Primal: $\begin{cases} \min c^T x & \text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ Ax \le b \\ x \ge 0 \end{cases}$, so dual: $\begin{cases} \max b^T y & \text{where } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \\ A^T y \ge c \\ y \ge 0 \end{cases}$

$$\Leftrightarrow \max -3y_1 + 5y_2 - 8y_3 - 7y_4 + 4y_5$$

$$\text{st} \begin{cases} -y_1 - y_2 + 0y_3 - 6y_4 + y_5 \ge 47 \\ -6y_1 - 2y_2 + 3y_3 - 11y_4 + 6y_5 \ge 93 \\ y_1 + 7y_2 - 10y_3 - 2y_4 - y_5 \ge 17 \\ 3y_1 + y_2 - y_3 + 12y_4 - 3y_5 \ge -93 \end{cases}, \text{ and } y_i \le 0 \ \forall i$$

Primal (Max)	Dual (Min)
i-th constraint $\le$	variable $y_i \ge 0$
i-th constraint $\ge$	variable $y_i \le 0$
i-th constraint $=$	variable y_i unrestricted
$x_i \ge 0$	i-th constraint $\ge$
$x_i \le 0$	i-th constraint $\le$
x_i unrestricted	i-th constraint $=$

Looking at the primal, only the last equation is not binding so: $\begin{cases} y_1, y_2, y_3, y_4 \le 0 \\ y_5 = 0 \end{cases}$

So the dual becomes: $\max -3y_1 + 5y_2 - 8y_3 - 7y_4$

$$\text{st}\begin{cases} -y_1 - y_2 + 0y_3 - 6y_4 = 47 \\ -6y_1 - 2y_2 + 3y_3 - 11y_4 = 93 \\ y_1 + 7y_2 - 10y_3 - 2y_4 = 17 \\ 3y_1 + y_2 - y_3 + 12y_4 = -93 \\ y_1, y_2, y_3, y_4 \leq 0 \end{cases} \Leftrightarrow \begin{cases} y_1 + y_2 - 0y_3 + 6y_4 = -47 \\ 4y_2 + 3y_3 + 25y_4 = -189 \\ 6y_2 - 10y_3 - 8y_4 = -30 \\ -2y_2 - y_3 - 6y_4 = 48 \\ y_1, y_2, y_3, y_4 \leq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1 + y_2 - 0y_3 + 6y_4 = -47 \\ y_2 + \frac{3}{4}y_3 - \frac{25}{4}y_4 = -189/4 \\ -29y_3 - 91y_4 = -695 \\ 1y_3 - 13y_4 = 93 \\ y_1, y_2, y_3, y_4 \leq 0 \end{cases} \Leftrightarrow \begin{cases} y_1 + y_2 - 0y_3 + 6y_4 = -47 \\ y_2 + \frac{3}{4}y_3 - \frac{25}{4}y_4 = -189/4 \\ y_3 + \frac{91}{29}y_4 = \frac{695}{29} \\ -\frac{143}{29}y_4 = \frac{1001}{29} \\ y_1, y_2, y_3, y_4 \leq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1 = -3 \\ y_2 = -2 \\ y_3 = -2 \\ y_4 = -7 \\ y_1, y_2, y_3, y_4 \leq 0 \end{cases}$$

These values of y satisfy the dual and satisfy the complementary slackness conditions:

$x_1 = 1 > 0$, the corresponding dual constraint $-y_1 - y_2 + 0y_3 - 6y_4 = 47$ is binding.
$x_2 = 1 > 0$, the corresponding dual constraint $-6y_1 - 2y_2 + 3y_3 - 11y_4 = 93$ is binding.
$x_3 = 1 > 0$, the corresponding dual constraint $y_1 + 7y_2 - 10y_3 - 2y_4 = 17$ is binding.
$x_4 = 1 > 0$, the corresponding dual constraint $3y_1 + y_2 - y_3 + 12y_4 = -93$ is binding.

The non-binding primal constraint corresponds to $y_5 = 0$

Therefore, the solution is optimal as it satisfies the complementary slackness condition (strong duality).

- In Fine, the proposed solution is both feasible and optimal.

IEOR E4004: Optimization Models and Methods
Instructor: Dr. Yaren Bilge Kaya
Due: 12/1/2024, 11:59 pm

Assignment 5

November 25, 2024

Question 3. *(20 points) Find the Hessian matrix for the following functions:*

1) $$f(x) = x_1^2 + 4x_2^2 + x_1 x_2$$

2) $$f(x, y, z) = x^2 + 3y^2 + 5z^2 - 8xy - 5yz$$

Are the above functions convex?

1) $f(x) = x_1^2 + 4x_2^2 + x_1x_2$

Let's compute: $\begin{cases} \dfrac{\partial f}{\partial x_1}(x) = 2x_1 + x_2 \\ \dfrac{\partial f}{\partial x_2}(x) = 8x_2 + x_1 \end{cases}$ and $\begin{cases} \dfrac{\partial^2 f}{\partial x_1^2}(x) = 2 \\ \dfrac{\partial^2 f}{\partial x_2^2}(x) = 8 \\ \dfrac{\partial f}{\partial x_1 \partial x_2}(x) = 1 \\ \dfrac{\partial f}{\partial x_2 \partial x_1}(x) = 1 \end{cases}$ (black theorem verified for a C^2 function), so the Hessian matrix is: $H_f = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$

If H_f is positive definite, then f is convex.

Characteristic equation: $\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 8-\lambda \end{pmatrix} = 0 \Leftrightarrow (2-\lambda)(8-\lambda) - 1 = 0 \Leftrightarrow \lambda = 5 \pm \sqrt{10} > 0$

Since both eigenvalues are positive, f is convex.

2) $f(x,y,z) = x^2 + 3y^2 + 5z^2 - 8xy - 5yz$

Let's compute: $\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = 2x - 8y \\ \dfrac{\partial f}{\partial y}(x,y,z) = 6y - 8x - 5z \\ \dfrac{\partial f}{\partial z}(x,y,z) = 10z - 5y \end{cases}$ and $\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(x,y,z) = 2 \\ \dfrac{\partial^2 f}{\partial y^2}(x,y,z) = 6 \\ \dfrac{\partial f}{\partial z^2}(x,y,z) = 10 \\ \dfrac{\partial f}{\partial x \partial y}(x,y,z) = -8 \\ \dfrac{\partial f}{\partial y \partial z}(x,y,z) = -5 \\ \dfrac{\partial f}{\partial x \partial z}(x,y,z) = 0 \end{cases}$, so the Hessian matrix is: $H_f = \begin{bmatrix} 2 & -8 & 0 \\ -8 & 6 & -5 \\ 0 & -5 & 10 \end{bmatrix}$

Let's use successive principal minors theorem.

We want : $2 = \det(2) > 0$

$\cancel{12 - 64 = \det\begin{pmatrix} 2 & -8 \\ -8 & 6 \end{pmatrix}} > 0$

doesn't matter $= \det\begin{pmatrix} 2 & -8 & 0 \\ -8 & 6 & -5 \\ 0 & -5 & 10 \end{pmatrix} > 0$

H_f is not positive definite, we cannot conclude on the convexity of f.

We can still try to verify using the definition of convexity: $\forall v_1, v_2$ and $\lambda \in [0,1]$, f : $f(\lambda v_1 + (1-\lambda)v_2)$
$\leq \lambda f(v_1) + (1-\lambda) f(v_2)$

Consider 2 points $\begin{cases} A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; f(A) = 1+3-8 = -4 \\ B = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \; f(B) = 1+3-8 = -4 \end{cases}$

Choosing $\lambda = \frac{1}{2}$: $\begin{cases} x_C = \lambda x_A + (1-\lambda) x_B = 0 \\ y_C = \lambda y_A + (1-\lambda) y_B = 0 \\ z_C = \lambda z_A + (1-\lambda) z_B = 0 \end{cases}$, so $f(C) = 0$

We have $\lambda f(A) + (1-\lambda) f(B) = -4$

So, $f(C) > \lambda f(A) + (1-\lambda) f(B)$

ie, we found a counter-example.

f is not convex.