## **IEOR 4404 Simulation**

Fall 2024

Homework 5

Instructor: Henry Lam

**Problem 1** For Unif(0,1) random variables  $U_1, U_2, \ldots$  define

$$N_1 = \text{Minimum} \left\{ n : \sum_{i=1}^n U_i > 1 \right\}.$$

That is,  $N_1$  is equal to the number of random numbers that must be summed to exceed 1. Write a pseudo-code for generating  $N_1$ . Give a point estimate and a 95% confidence interval for  $\mathbb{E}[N_1]$ , by generating 100, 1000 and 10,000 copies of  $N_1$  respectively on a computer.

Do the same for  $N_2$ , the number of random numbers that must be summed to exceed 2. That is,

$$N_2 = \text{Minimum} \left\{ n : \sum_{i=1}^n U_i > 2 \right\},$$

From your results, do you think  $E[N_2]$  increases from  $E[N_1]$ ? Explain briefly.

**Problem 2** Write a pseudo-code to generate a simulation run that is appropriate for numerically approximating the integral

$$\int_0^\infty \int_0^{x^2} e^{-(x+y)} \sin(xy) dy dx.$$

Implement 1000 simulation runs on a computer. Give a point estimate and a 95% confidence interval for the value of this integral.

**Problem 3** Suppose we estimate  $\alpha = \mathbb{E}[X]$  via

$$\hat{\alpha}_n = \frac{1}{n} \sum_{j=1}^n X_j,$$

where the  $X_j$ 's are i.i.d. simulation replications of X. Assume that Var(X) = 2. Find the smallest value n such that  $\hat{\alpha}_n$  is within 0.05 units of  $\alpha$  with 95% confidence. Briefly justify your answer.

**Problem 4** To estimate  $\theta$ , we generate 20 i.i.d. replications each having mean  $\theta$ . Suppose the obtained values are 102, 112, 131, 107, 114, 95, 133, 145, 139, 117, 93, 111, 124, 122, 136, 141, 119, 122, 151, 143.

- a) Find a 95% confidence interval for  $\theta$ .
- b) How many additional replications do you think we will have to generate if we want to be 99% certain that our final estimate of  $\theta$  is correct to within  $\pm 0.5$ ?

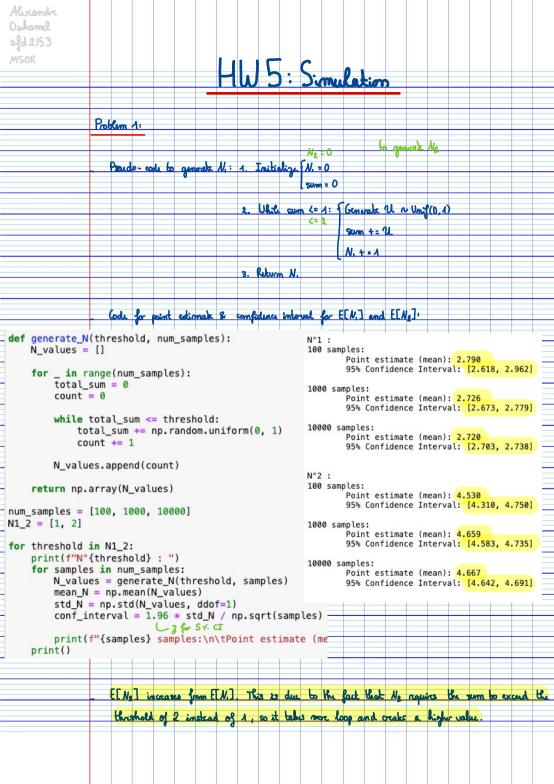
**Problem 5** Let  $X_1, ..., X_{40}$  be i.i.d. copies of random variable X generated from a computer routine. Let

$$p = \mathbb{P}((X-1)^2 < 30)$$

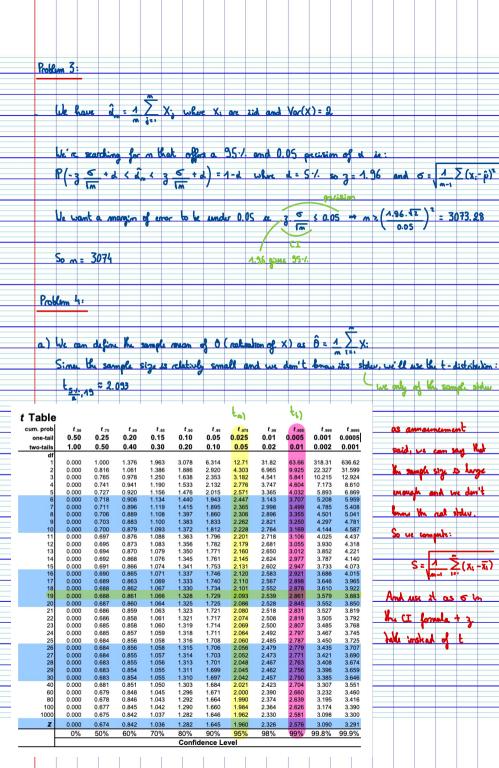
Give a point estimate and a 95% confidence interval of p if, among all the values of the  $X_i$ , 2 appears 11 times, 4 appears 5 times, 5 appears 10 times, 6 appears 2 times, 8 appears 7 times, 9 appears 4 times and 24 appears 1 time.

**Problem 6** Suppose that the following are i.i.d. copies of random variable X generated from a computer routine: 5, 4, 9, 6, 21, 12, 7, 14, 17, 11, 20, 7, 10, 21, 15, 26, 9, 13, 8, 6. We want to estimate  $\mathbb{E}[X^2]$ .

- (a) How many additional replications will we need to generate if we want to be 99% certain that our estimate has a margin of error 0.1?
- (b) What assumptions have you made in part (a)?



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Problem 2:
                  Pseudo-code using Mank - Carlo: 1. Initialize (N = soundes wanted
                                                         Zenax = Apper value of 2 for 100
                                              2. For i = 1 to N: General XN Vniform (0, 2
                                                               Generale Y N Uniform (0, X2)
                                                                Compute: {(2,y) = e = 2-9 sin(24)
                                                                Stom += 1(x,y)
                                             3. Return
                  The idea or that e-(200) sin (200) digla & (200) sin (200) digla & Zanov
                 Here is the code:
 def monte_carlo_integral(num_samples, x_max):
     results = []
     for in range(num samples):
         x = np.random.uniform(0, x_max)
         y = np.random.uniform(0, x**2)
         f_xy = np.exp(-(x + y)) * np.sin(x * y)
          results.append(f xy)
     estimate = (x_max / num_samples) * np.sum(results)
     return estimate, np.array(results)
                                 I tried multiple values until I fund one when he output stabilizes (ho less Kam
 num samples = 1000
 x max = 80
 point_estimate, results = monte_carlo_integral(num_samples, x_max)
 std_dev = np.std(results, ddof=1)
z score = 1.96
 margin_of_error = z_score * std_dev / np.sqrt(num_samples)
conf_interval = (point_estimate - margin_of_error, point_estimate + margin_of_error) =
 print(f"Point Estimate: {point_estimate:.3f}")
print(f"95% Confidence Interval: {conf_interval}")
Point Estimate: 0.096
 95% Confidence Interval: (0.09537085834560255, 0.09666106448183479)
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Here is the code to compace the value of the 35% CT:
 data = [102, 112, 131, 107, 114, 95, 133, 145, 139, 117, 93, 111, 124, 122, 136, 141, 119, 122, 151, 143]
 n = len(data)
sample mean = np.mean(data)
 sample std = np.std(data, ddof=1)
 confidence level 95 = 0.95
 alpha 95 = 1 - confidence level 95
t_{95} = 2.093
 margin_of_error_95 = t_95 * sample_std / np.sqrt(n)
ci_95 = (sample_mean - margin_of_error_95, sample_mean + margin_of_error_95)
print(f"95% Confidence Interval: {ci_95}")
95% Confidence Interval: (114.98745566576802, 130.71254433423195)
                      margin_of_error_99 = 0.5
                 ()
                      confidence_level_99 = 0.99
                      alpha_99 = 1 - confidence_level_99
                      t 99 = 2.861
                      - تستمو لمانا ( required_n = (t_99 * sample_std / margin_of_error_99) ** 2
                      required_n = np.ceil(required_n)
                                                                                                 CKUCISC
                      additional replications = required n - n
                      print(f"Additional replications needed: {additional_replications}");
                      Additional replications needed: 9221.0
                 Protein 5:
                       (X-1)^2 < 30 \leftrightarrow \sqrt{(X-1)^2} < \sqrt{30}
                                    ↔ | X-1| < 30
                                    ← 1-30 < X < 1+30
                                    4 -4.477 < X < 6.477
                     So X must be in the interval: ]-4.477, 6.477[
                     The given values that lit in this interval on: 12 4.5,64 with respective occurrences: 11,5,10,24
                     For a lotal values that satisfy the interval : 11+ S+10+2=28
                                                # walus that salogy
                     Je can estimate p as
                                                                             = 0.7
                                                 Total # of observations
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variance of Bermoulli trial
                                                       either et s in or a
                                             ρ(1-p)
                The 95.1. CI is: CI = 6 + 3
                                                     = 0.7 ± 0.1420 = 0.55798. 0.84201
                                                0.1420
             Poblem 6.
             a) X2 = 1 X2 for X; in X7 = 125, 16, 81, ... }
                                                                  I used the to bable for the same
                 So point of estimate of E[x2]: p. = 1 \ X;2
                 ie the margin of error is t. s
                                                                  But we can compute S and use M
                                               = MOE = 0.4
                 ite the provious problems:
                                                                   3 hable of we assume there are en
                 Code to compute it:
data = np.array([5, 4, 9, 6, 21, 12, 7, 14, 17, 11, 20, 7, 10, 21, 15, 26, 9, 13, 8, 6])
x squared = data**2
n = len(data)
mean x squared = np.mean(x squared)
std_x_squared = np.std(x_squared, ddof=1)
margin_of_error = 0.1
confidence_level = 0.99
alpha = 1 - confidence_level
t_critical = 2.8609
required_n = (t_critical * std_x_squared / margin_of_error) ** 2
required_n = np.ceil(required_n)
additional_replications = required_n - n
print(f"Point estimate of E[X^2]: {mean x squared:.3f}")
print(f"Sample standard deviation of X^2: {std_x_squared:.3f}")
print(f"Number of additional replications needed: {additional_replications}")
Point estimate of E[X^2]: 181.950
Sample standard deviation of X^2: 180.031
Number of additional replications needed: 26527826.0
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					V	or (X2	) ( +	00													
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