IEOR 4106, HMWK 7, Professor Sigman

Things to recall when doing this homework: Poisson Arrivals See Time Averages (PASTA) (Page 27–28, in Lecture Notes 13, "Introduction to Continuous-Time Markov Chains"). Little's Law ($l = \lambda w$).

1. *Printer with jams*: Jobs arrive to a computer printer according to a Poisson process at rate λ. Jobs are printed one at a time requiring iid printing times that are exponentially distributed with rate μ. Jobs that arrive when the server is busy printing wait in a FIFO queue before entering service.

   Additionally, independently, the printer *jams* at times $\{s_n : n \geq 1\}$ that form a Poisson process at rate γ: Whenever a jam occurs the job being printed (if any) is removed (and lost), and the printer continues printing the remaining jobs. If the printer is empty of jobs when a jam occurs, then the jam has no effect (i.e., the printer instantly resets). Let $X(t)$ denote the number of jobs at the printer at time t.

   (a) For a given time t, suppose that $X(t) \geq 1$, and let R denote how long it will take until either the job in service is completed or lost (due to a jam). Argue that $R \sim exp(\mu + \gamma)$.

   (b) Argue that $\{X(t)\}$ is a Birth and Death process; give the birth rates $\{\lambda_i\}$ and the death rates $\{\mu_i\}$. Give a rate diagram.

   (c) Set up and solve the Birth and Death Balance Equations for the limiting distribution ($\lambda_j P_j = \mu_{j+1} P_{j+1},\ j \geq 0$); what condition on the parameters $\lambda,\ \mu,\ \gamma$ is necessary and sufficient for positive recurrence?

   (d) Explain why $\{X(t)\}$ **is the same Birth and Death process as for a regular FIFO M/M/1 queue** with the same arrival rate λ, but a different service-time rate (denoted by $\overline{\mu}$); give its value.

   (e) What is the average *sojourn time* (in the printing facility) of a job ?

   (f) What proportion of *time* is the printer printing?

   (g) What proportion of *arriving customers* find the printer printing?

   (h) What proportion of *jam times* $\{s_n : n \geq 1\}$ cause a removal of a job?

2. *Inventory model*: A retailer starting with $B \geq 2$ headphones, as its inventory level, sells them one at a time according to demand which forms a Poisson process at rate λ: At Poisson arrival time t_n (n^{th} demand request), the inventory level drops down by 1 if the inventory is non-empty. If the inventory level is empty at a request time, then nothing happens; the demand request is "lost". As soon as the inventory level drops down to 0 (initiated by an arrival finding the inventory level equal to 1), it will be re-stocked back up to B after an exponential amount of time L (called the *lead time*) at rate γ, independent of the past. During those L time units, all demand is lost. Let $X(t)$ denote the inventory level at time t. The state space is $\mathcal{S} = \{0, 1, \ldots, B\}$.

   (a) Argue that $\{X(t)\}$ forms a CTMC (but not a Birth and Death process, why?); find both the holding time rates a_j and the embedded MC transition matrix $P = (P_{i,j})$.

   (b) Give a rate diagram for this CTMC.

   (c) Set up and solve the Global Balance Equations for the limiting distribution. "rate out of state j = rate into state j, for each $0 \leq j \leq B$". Here are some of the equations to help you:

$$\begin{aligned}\gamma P_0 &= \lambda P_1\\ \lambda P_1 &= \lambda P_2\\ &\vdots\\ \lambda P_B &= ?\end{aligned}$$

(d) Find the long-run average inventory level;

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)ds.$$

(e) What proportion of *time* is the inventory level empty?

(f) What proportion of *demand requests* are lost?

(g) What proportion of *demand requests* initiate a re-stock?

(h) Try using $l = \lambda w$ to re-derive (d). HINT: Customers are the headphones.

3. Consider a company that is open for an amount of time U_1, then closed for an amount of time C_1. Then independently this starts all over: Open for U_2, closed for C_2 and so on forever with $U_n, C_n,\ n \geq 1$. Assume that $\{(U_n, C_n) : n \geq 1\}$ are iid, where the U_n are continuous Uniform rvs over $(0, 5)$ and the C_n are exponential rvs at rate $1/4$ (time is in days).

   (a) Compute the rate (long run number of times per unit time) that the company opens up.

   (b) Compute the rate (long run number of times per unit time) that the company closes down.

   (c) Compute the the long-run proportion of time that the company is open/closed.

4. *Train dispatching problem; different model*: Passengers arrive to a train platform according to a *Poisson process* at rate μ. A train departs every T time units ($T > 0$ is a constant), taking all passengers who arrived during the T time units. Suppose further that the train company incurs a cost at the constant rate of \$$c$ per unit time that each passenger waits, and also incurs a fixed cost of \$$K$ each time a train departs. This process continues over and over. Our objective in what follows is to compute (using Renewal Reward) the long-run cost rate for the train company. Observe that the cycle lengths are deterministic of length T; $X_n = T,\ n \geq 0$.

   (a) On average, how many passengers get on a train?

   (b) On average, what is the waiting time of a passenger? (Hint: Condition on how many Poisson arrivals occur by time T; recall the use of order statistics, etc.)

   (c) What is the expected waiting cost per cycle *per passenger*? What is thus $E(R)$?

   (d) Now compute $E(R)/E(X) =$ the long-run cost rate for the train company.

   (e) Find the optimal value of T; the one that minimizes cost.

5. *Machine Repair Model:* Independently, four (4) machines have iid lifetimes exponential at rate μ. When any one of them breaks down, it attends a FIFO **2**-server repair facility that has iid service times exponential at rate λ, and one queue (all together) to wait in.

   (a) Draw the rate diagram for the $B\&D$ process $\{X(t) : t \geq 0\}$ defined by $X(t) =$ the number of working machines at time t.

   (b) Give the $B\&D$ balance equations and solve.

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HW 7: Stochastic

Exercice 1:

a) Consider a time t such that $X(t) \geq 1$, meaning there is a job currently in service. The remaining time until the job in service is either completed or lost is governed by two independent exp "clocks": - completion clock with rate μ
- jam clock with rate γ

The event that ends the current job is the minimum of these two exponential times.
The minimum of two independent exponentials with rates μ and γ is an exponential of rate $\mu+\gamma$.
So, $R \sim exp(\mu+\lambda)$

b) We have a Birth & Death process such that the transition rates in between states:
- $i \to i+1: \lambda$, $\forall i \geq 0$
- $i \to i-1: \mu+\gamma$, $\forall i \geq 1$ (no departures from state 0)

c) The stationary distribution $\{\Pi_j\}$ is $\forall j \geq 0$: $\lambda \Pi_j = (\mu+\gamma)\Pi_{j+1} \to \Pi_{j+1} = \frac{\lambda}{(\mu+\gamma)}\Pi_j \Rightarrow \Pi_j = \left(\frac{\lambda}{(\mu+\gamma)}\right)^j \Pi_0$

Also:

$$\sum_{j=0}^{+\infty} \Pi_j = 1 \to \Pi_0 \sum_{j=0}^{+\infty}\left(\frac{\lambda}{(\mu+\gamma)}\right)^j = 1 \to \Pi_0 \frac{1}{1-\left(\frac{\lambda}{(\mu+\gamma)}\right)} = 1 \quad \text{we need } \frac{\lambda}{\mu+\gamma} < 1$$

Thus:

$$\Pi_j = \left(\frac{\lambda}{(\mu+\gamma)}\right)^j\left(1-\left(\frac{\lambda}{(\mu+\gamma)}\right)\right), \forall j \geq 1 \quad \text{and} \quad \Pi_0 = 1-\left(\frac{\lambda}{(\mu+\gamma)}\right)$$

d) It's the same as an M/M/1 queue with a service rate $\bar{\mu} = \mu+\gamma$

e) The average sojourn time for such an M/M/1 queue is: $E[H_i] = \frac{1}{a_i} = \frac{1}{(\mu+\gamma)-\lambda}$

f) The proportion of the time the printer is printing is analogue to the server utilization rate for an M/M/1 queue: $\rho = \frac{\lambda}{\mu+\gamma}$

h) The proportion of arriving customers that find the printer printing is the same as the proportion of the time the printer is printing. We confirm this using PASTA: $\rho = \frac{\lambda}{\mu + \gamma}$

d) Since jams form a Poisson process independent of the queue, PASTA applies yet again and $\rho = \frac{\lambda}{\mu + \gamma}$

Exercise 2:

a) The process $\{X(t)\}$ is a CTMC because transitions occur due to Poisson events (demands) and exp lead times (restock).

The states are: $\mathcal{E} = \{0, \dots, B\}$

The transition rates are:
- $i \to i-1 : \lambda$
- $0 \to B : \gamma$

This is not a B&D process since you can jump from state 0 to B directly, without having to go through 1.

The embedded chain is: $\begin{cases} \mathbb{P}(0 \to B) = \mathbb{P}(\text{restock}) = 1 \\ \mathbb{P}(i \to i-1) = \mathbb{P}(\text{demand}) = 1 \end{cases}$

b)

$$\begin{tikzcd} B \arrow[r, "\lambda"] & B-1 \arrow[r, "\lambda"] & B-2 \arrow[r, dotted] & 1 \arrow[r, "\lambda"] & 0 \arrow[llll, bend left, "\gamma"] \end{tikzcd}$$

c) From 0: $\left.\begin{array}{l} \text{Outflow} = \gamma \pi_0 \\ \text{Inflow} = \lambda \pi_1 \end{array}\right\}$ at balance: $\gamma \pi_0 = \lambda \pi_1 \Rightarrow \pi_1 = \frac{\gamma}{\lambda} \pi_0$

From 1: $\left.\begin{array}{l} \text{Outflow} = \lambda \pi_1 \\ \text{Inflow} = \lambda \pi_2 \end{array}\right\}$ at balance: $\lambda \pi_1 = \lambda \pi_2 \Rightarrow \pi_2 = \pi_1$

Iterating the process: $\pi_1 = \pi_2 = \dots = \pi_B$

So we have $\sum_{i=1}^{B} \pi_i = 1 \Rightarrow \pi_0 + B.\pi_1 = 1$

so $\pi_0 + B.\frac{\gamma}{\lambda}\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{1 + B.\frac{\gamma}{\lambda}} = \frac{\lambda}{\lambda + B.\gamma}$

So: $\begin{cases} \pi_0 = \frac{\lambda}{\lambda + B.\gamma} \\ \pi_1 = \pi_2 = \dots = \pi_B = \frac{\gamma}{\lambda}\pi_0 = \frac{\gamma}{\lambda} \frac{\lambda}{\lambda + B.\gamma} = \frac{\gamma}{\lambda + B.\gamma} \end{cases}$

d) $\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)\,ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(\sum_{j=0}^{B} j\, \mathbb{1}\{X(s)=j\}\right) ds$

$$= \lim_{t\to\infty} \frac{1}{t} \sum_{j=0}^{B} \left(\int_0^t j\, \mathbb{1}\{X(s)=j\}\, ds\right)$$

$$= \sum_{j=0}^{B} \lim_{t\to\infty} \left(\frac{1}{t}\int_0^t j\, \mathbb{1}\{X(s)=j\}\, ds\right)$$

$$= \sum_{j=0}^{B} j\pi_j \qquad \text{PASTA}$$

$$= 0.\pi_0 + \sum_{j=1}^{B} j\pi_j$$

$$= \sum_{j=1}^{B} j \frac{\gamma}{\lambda + B.\gamma}$$

$$= \frac{\gamma}{\lambda + B.\gamma} \frac{B(B+1)}{2}$$

e) Proportion of the time the inventory is empty $= \pi_0 = \frac{\lambda}{\lambda + B.\gamma}$

f) Proportion of the demand request that is lost = Proportion of the time the inventory is empty

$$= \pi_0 = \frac{\lambda}{\lambda + B.\gamma}$$

g) Proportion of demand requests to initiate a restock = Probability inventory is 1 at an arrival

$$= \pi_1 = \frac{\gamma}{\lambda + B.\gamma}$$

h) Using Little's Law: $E[X]$ = (arrival rate of headphones) $\times\, \omega$

$$= B\pi_0\gamma \times \omega$$

$\omega = \frac{B+1}{2\lambda}$ as the expected time the headphone spends in the system

(stock goes from B to 0 in $\frac{B}{\lambda}$ time

$$= B\pi_0\gamma \times \frac{B+1}{2\lambda}$$

$$= \frac{\gamma}{\lambda + B.\gamma} \frac{B(B+1)}{2}$$

Exercise 3:

a) We have: $\begin{cases} U_n \sim Unif(0,5) \\ C_n \sim Exp(1/4) \end{cases}$ so $\begin{cases} E[U] = \frac{0+5}{2} = 2.5 \text{ days} \\ E[C] = \frac{1}{1/4} = 4 \text{ days} \end{cases}$ and each cycle length $= E[U+C] = 6.5$

Long-run rate of opening = # of cycles per unit of time (each cycle has one opening)

$= \frac{1}{E[U+C]}$

$= \frac{1}{6.5}$

b) Long-run rate of closing = # of cycles per unit of time (each cycle has one closing)

$= \frac{1}{E[U+C]}$

$= \frac{1}{6.5}$

c) Long-run proportion of the time opened $= \frac{E[U]}{E[U]+E[C]} = \frac{2.5}{6.5} = \frac{5}{13}$

Long-run proportion of the time closed $= \frac{E[C]}{E[U]+E[C]} = 1 -$ long-run proportion of the time opened

$= \frac{8}{13}$

Exercise 4:

a) passengers arrive to a train platform according to a renewal process at rate μ

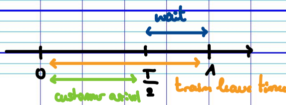
Costs: 1. \$nc per unit of time that n passengers are waiting

2. fixed cost \$K each time the train departs

On average, the number of passengers per train is μT (homogeneous PP).

b) Arrivals are uniformly distributed over $[0,T]$ in expectation (order statistic of a PP), so the average arrival time is $\frac{T}{2}$.

Since the train leaves at T: $E[\text{waiting time}] = A - \frac{T}{2} = \frac{T}{2}$ for A = train leave time $= T$



c) The expected waiting cost per cycle per passenger $= c \times E[\text{waiting}] = c\frac{T}{2}$

excluding fixed cost

d) So total cost per cycle on average $= c\frac{T}{2}\mu T$, where μT is the number of customers on the interval

Including the fixed cost $= c\frac{T^2}{2}\mu + K = E[R]$

So:

Long-run cost rate $= \dfrac{E[R]}{E[X]} = \dfrac{c\frac{T^2}{2}\mu + K}{T} = c\frac{T}{2}\mu + \frac{K}{T}$

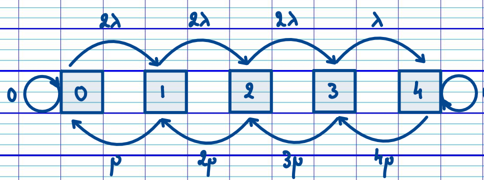
X is the length of a cycle hence $X_n = T$ is deterministic

e) $\dfrac{d}{dT}\left(c\frac{T}{2}\mu + \frac{K}{T}\right) = 0 \rightarrow \dfrac{c\mu}{2} - \dfrac{K}{T^2} = 0 \rightarrow T = \sqrt{\dfrac{2K}{c\mu}}$ we exclude $T<0$

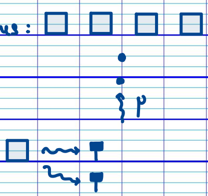
also: $\dfrac{d^2}{dT^2}\left(c\frac{T}{2}\mu + \frac{K}{T}\right) = \dfrac{K}{T^3} > 0$ it's a minimum

Exercise 5:

a) 4 machines:



Repair:



Let $\begin{cases} X(t) = \#\text{ of working machines at time } t \\ \mathcal{S} = \{0,1,2,3,4\} \end{cases}$

Transitions: $\begin{cases} \text{machine fails, } i \rightarrow i-1 \text{ with rate } i\mu \\ \text{machine repaired, } i \rightarrow i+1 \text{ with rate } \min(\underbrace{4-i}_{\#\text{ of broken machines}}, \underbrace{2}_{\text{2 server repair facility maxed out}})\lambda \end{cases}$

# working machines	repair rate
0	2λ
1	2λ
2	2λ
3	λ
4	0

b) $0 \rightarrow 1$: $\pi_0 \times 2\lambda = \pi_1 \times \mu \rightarrow \pi_1 = \frac{2\lambda}{\mu}\pi_0$

$1 \rightarrow 2$: $\pi_1 \times 2\lambda = \pi_2 \times 2\mu \rightarrow \pi_2 = \frac{\lambda}{\mu}\pi_1 = \frac{2\lambda^2}{\mu^2}\pi_0$

$2 \rightarrow 3$: $\pi_2 \times 2\lambda = \pi_3 \times 3\mu \rightarrow \pi_3 = \frac{2\lambda}{3\mu}\pi_2 = \frac{4\lambda^3}{3\mu^3}\pi_0$

$3 \rightarrow 4$: $\pi_3 \times \lambda = \pi_4 \times 4\mu \rightarrow \pi_4 = \frac{\lambda}{4\mu}\pi_3 = \frac{\lambda^4}{3\mu^4}\pi_0$

and $\sum_{i=0}^{4} \pi_i = 1$

$\Leftrightarrow \pi_0\left(1 + \frac{2\lambda}{\mu} + \frac{2\lambda^2}{\mu^2} + \frac{4\lambda^3}{3\mu^3} + \frac{\lambda^4}{3\mu^4}\right) = 1$

$\Leftrightarrow \pi_0 = \dfrac{1}{1 + \frac{2\lambda}{\mu} + \frac{2\lambda^2}{\mu^2} + \frac{4\lambda^3}{3\mu^3} + \frac{\lambda^4}{3\mu^4}}$

$$
S_0 : \begin{cases}
\pi_0 = \dfrac{1}{1 + \dfrac{2\lambda}{\mu} + \dfrac{2\lambda^2}{\mu^2} + \dfrac{4\lambda^3}{3\mu^3} + \dfrac{\lambda^4}{3\mu^4}} \\[2ex]
\pi_1 = \dfrac{2\lambda}{\mu}\pi_0 \\[1ex]
\pi_2 = \dfrac{\lambda}{\mu}\pi_1 = \dfrac{2\lambda^2}{\mu^2}\pi_0 \\[1ex]
\pi_3 = \dfrac{2\lambda}{3\mu}\pi_2 = \dfrac{4\lambda^3}{3\mu^3}\pi_0 \\[1ex]
\pi_4 = \dfrac{\lambda}{4\mu}\pi_3 = \dfrac{\lambda^4}{3\mu^4}\pi_0
\end{cases}
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