An Event-B Specification of

SquareRoot

This Event-B system is based on a model that appeared in the book: System Modelling & Design Using Event-B by Ken Robinson.

This project implements an integer square root algorithm. The algorithm performs a binary search of a value x such that x*x = input, ie x will become the square root.

CONTEXT SquareRootDefinition 4 a 1 i 1

This is the mathematical definition of the function SQRT.

EXTENDS Theories

CONSTANTS 1.1

SQRT

AXIOMS

axm1: $\mathrm{SQRT} \in \mathbb{N} \rightarrow \mathbb{N}$

axm2: $\forall m, n \cdot m \in \mathbb{N} \land n \in \mathbb{N} \Rightarrow (m = \mathrm{SQRT}(n) \Leftrightarrow m * m \leq n \land (m + 1) * (m + 1) > n)$

THEOREM

thm1: $\forall n \cdot n \in \mathbb{N} \Rightarrow \mathrm{SQRT}(n) * \mathrm{SQRT}(n) \leq n \land (\mathrm{SQRT}(n) + 1) * (\mathrm{SQRT}(n) + 1) > n$

thm2: $\forall n \cdot n \in \mathbb{N} \Rightarrow n = \mathrm{SQRT}(n * n)$

END

Helpful theorems when proving the square root algorithm.

AXIOMS

axm1: $\forall n \cdot n \in \mathbb{N} \Rightarrow (\exists m \cdot m \in \mathbb{N} \land (n = 2 * m \lor n = 2 * m + 1))$ Every natural number is either even or odd.

THEOREM

thm2: $\forall n \cdot n \in \mathbb{N} \Rightarrow n < (n + 1) * (n + 1)$ Every natural number is less than the square of its successor.

thm3: $\forall m, n \cdot m \in \mathbb{N} \land n \in \mathbb{N} \land n > m \Rightarrow (m + n) \div 2 < n$ The mean of any pair of unequal natural numbers is less than the larger of the pair.

thm4: $\forall m, n \cdot m \in \mathbb{N} \land n \in \mathbb{N} \land n > m \Rightarrow (m + n) \div 2 \geq m$
The mean of any pair of natural numbers is greater than or equal to the smaller of the pair.

END

We first define the sequence: setInput,SquareRoot,getResult. The value calculated is entirely desribed using the mathematical description.

SEES SquareRootDefinition

VARIABLES 3.1

input	The value to calculate the square root for.
input_valid	True when a number has been supplied
result	This is the calculated result
result_valid	True when result = sqrt(num)

INVARIANTS

inv_1: $input \in \mathbb{N}$
inv_2: $input\_valid \in \mathrm{BOOL}$
inv_3: $result \in \mathbb{N}$
inv_4: $result\_valid \in \mathrm{BOOL}$
inv_5: $input\_valid = \mathrm{TRUE} \land result\_valid = \mathrm{TRUE} \Rightarrow result = \mathrm{SQRT}(input)$

EVENT INITIALISATION
THEN

act_1: $input := 0$
act_2: $input\_valid := \mathrm{FALSE}$
act_3: $result := 0$
act_4: $result\_valid := \mathrm{FALSE}$

END

EVENT setInput 3.2
ANY

v

WHERE

grd_1: $v \in \mathbb{N}$
grd_2: $input\_valid = \mathrm{FALSE}$
grd_3: $result\_valid = \mathrm{FALSE}$

THEN

act_1: $input := v$
act_2: $input\_valid := \mathrm{TRUE}$

END

EVENT SquareRoot 3.3
WHERE

grd_1: $input\_valid = \mathrm{TRUE}$
grd_2: $result\_valid = \mathrm{FALSE}$

THEN

act_1: $result := \mathrm{SQRT}(input)$
An alternative is to specify a non-deterministic assignment:
sqrt :| (sqrt' ∈ ℕ ∧ sqrt' ∗ sqrt' ≤ num ∧ num < (sqrt' + 1) ∗ (sqrt' + 1))
act_2: $result\_valid := \mathrm{TRUE}$

END

EVENT getResult $\boxed{3.4}$

ANY

out_result

WHERE

grd_1: $result\_valid = \mathrm{TRUE}$

grd_2: $out\_result = result$

END

REFINEMENT SquareRoot_R1_AddIncrementalImprovements 28 a 2 i 1 u [4]

Add a lower and upper bound of the correct answer. Each improvement will narrow the bounds until, eventually, we have a single number as the answer.

u `Improve/vrt/VAR`

REFINES SquareRoot
SEES SquareRootDefinition

VARIABLES [4.1]

low	When improving we have a lower bound of the correct answer.
high	And an upper bound of the correct answer.

INVARIANTS

`inv1_1:`	$low \in \mathbb{N}$
`inv1_2:`	$high \in \mathbb{N}$
`inv1_3:`	$low + 1 \leq high$
`inv1_4:`	$low * low \leq input$
`inv1_5:`	$input < high * high$
`inv1_6:`	$low < high$
theorem	`thm1_1:`
	$low + 1 \neq high \Rightarrow low < (low + high) \div 2$
theorem	`thm1_2:`
	$(low + high) \div 2 < high$
theorem	`thm1_3:`
	$high - low > 0$
theorem	`thm1_4:`
	$high - low \in \mathbb{N}$

VARIANTS

$high - low$

The variant guarantees that the span must decrease in each step. Eventually the span will be zero and a number has been found.

EVENT INITIALISATION
EXTENDS INITIALISATION
THEN

`init1_1:`	$low := 0$
`init1_2:`	$high := 1$

END

EVENT setInput [4.2]
When num is set, we specify that a lower and upper bound is magically selected in some way, that enables the improvement step to work. We do not yet know, how this performed.

EXTENDS setInput
THEN

`act1_1:`	$low :\mid low' \in \mathbb{N} \wedge low' * low' \leq v$
`act1_2:`	$high :\mid high' \in \mathbb{N} \wedge v < high' * high'$

END

EVENT SquareRoot 4.3

We detect the terminating case, when low+1=high, then $low = \sqrt{num}$

REFINES SquareRoot

WHERE

grd1_1: $input\_valid = \mathrm{TRUE}$
grd1_2: $result\_valid = \mathrm{FALSE}$
grd1_3: $low + 1 = high$
thm1_1: $low * low \leq input$
thm1_2: $input < high * high$

THEN

act1_1: $result := low$
act1_2: $result\_valid := \mathrm{TRUE}$

END

CONVERGENT EVENT Improve 4.4

The improve event magically selects an l and h, that are an improvement on the existing bounds. We do not know how this is done, but we specify the result here.

ANY

l
h

WHERE

grd1_1: $low + 1 \neq high$
grd1_2: $l \in \mathbb{N} \wedge low \leq l \wedge l * l \leq input$
grd1_3: $h \in \mathbb{N} \wedge h \leq high \wedge input < h * h$
grd1_4: $l + 1 \leq h$
grd1_5: $h - 1 < high - low$

THEN

act1_1: $low := l$
act1_2: $high := h$

END

REFINEMENT SquareRoot_R2_WithImproveLowerOrUpper 10 a 2 u **5**

We now split the improve event into improveLowerBound and improveUpperBound.

u `ImproveLowerBound/grd1_5/GRD`
u `ImproveUpperBound/grd1_5/GRD`

REFINES SquareRoot_R1_AddIncrementalImprovements
SEES SquareRootDefinition

VARIABLES **5.1**

EVENT ImproveLowerBound **5.2**
REFINES Improve
ANY
 m
WHERE
 grd2_1: $low + 1 \neq high$
 grd2_2: $m \in \mathbb{N}$
 grd2_3: $low < m \land m < high$
 grd2_4: $m * m \leq input$ m is a better lower bound!
WITH
 l: $l = m$
 h: $h = high$
THEN
 act2_1: $low := m$
END

EVENT ImproveUpperBound **5.3**
REFINES Improve
ANY
 m
WHERE
 grd2_1: $low + 1 \neq high$
 grd2_2: $m \in \mathbb{N}$
 grd2_3: $low < m \land m < high$
 grd2_4: $m * m > input$ m is a better upper bound!
WITH
 l: $l = low$
 h: $h = m$
THEN
 act2_1: $high := m$
END

REFINEMENT SquareRoot_R3_AddDivisionToFindM 6 a 6

We now pick a suitable middle value by dividing by 2.

REFINES SquareRoot_R2_WithImproveLowerOrUpper
SEES SquareRootDefinition

VARIABLES 6.1

EVENT ImproveLowerBound 6.2
REFINES ImproveLowerBound
ANY

m

WHERE

grd3_1: $low + 1 \neq high$
grd3_2: $m = (low + high) \div 2$
grd3_3: $m * m \leq input$ m is a better lower bound!

THEN

act3_1: $low := m$

END

EVENT ImproveUpperBound 6.3
REFINES ImproveUpperBound
ANY

m

WHERE

grd3_1: $low + 1 \neq high$
grd3_2: $m = (low + high) \div 2$
grd3_3: $m * m > input$

THEN

act3_1: $high := m$ m is a better upper bound!

END

REFINEMENT SquareRoot_R4_WithMiddleInVariable 20 a 7

We now store the middle value in a variable.

REFINES SquareRoot_R3_AddDivisionToFindM
SEES SquareRootDefinition

VARIABLES 7.1

mid Track each middle value to find next bound.,

INVARIANTS

inv1: $mid = (low + high) \div 2$
inv2: $mid \in \mathbb{N}$

EVENT INITIALISATION
THEN

init4_1: $input := 0$
init4_2: $input\_valid := \mathrm{FALSE}$
init4_3: $low := 0$
init4_4: $mid := 0$
init4_5: $high := 1$
init4_6: $result := 0$
init4_7: $result\_valid := \mathrm{FALSE}$

END

EVENT setInput 7.2
REFINES setInput
ANY

v

WHERE

grd4_1: $v \in \mathbb{N}$
grd4_2: $input\_valid = \mathrm{FALSE}$
grd4_3: $result\_valid = \mathrm{FALSE}$

THEN

init4_0: $input := v$
init4_1: $low := 0$
init4_2: $high := v + 1$
init4_3: $mid := (v + 1) \div 2$
init4_4: $input\_valid := \mathrm{TRUE}$

END

EVENT ImproveLowerBound 7.3
REFINES ImproveLowerBound
WHERE

grd4_1: $low + 1 \neq high$
grd4_2: $mid * mid \leq input$

WITH

m: $m = mid$ mid is a better value for low

THEN

act4_1: $low := mid$
act4_2: $mid := (mid + high) \div 2$

END

EVENT ImproveUpperBound $\boxed{7.4}$
REFINES ImproveUpperBound
WHERE

grd4_1: $low + 1 \neq high$
grd4_2: $mid * mid > input$

WITH

m: $m = mid$ mid is a better value for high

THEN

act4_1: $high := mid$
act4_2: $mid := (low + mid) \div 2$

END