Proving Beta-Binomial Conjugation

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1 Introduction

Beta-Binomial conjugation is demonstrated by using the Beta distribution as the prior distribution for the success probability p in a Binomial distribution.

2 Proof

Assume $X \sim \text{Binomial}(n, p)$, with the prior distribution of p as $\text{Beta}(\alpha, \beta)$. The joint probability mass function is given by:

$$f(x, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \tag{1}$$

where $B(\alpha, \beta)$ is the Beta function.

The posterior distribution of p is obtained by applying Bayes' theorem as:

$$\begin{aligned}
Posterior(\theta|x) &= \frac{\mathbb{P}(x|\theta) \times \text{Prior}(\theta)}{Prior(x)} \\
&\propto \mathbb{P}(x|\theta) \times \text{Prior}(\theta) \\
&= Binomial(n, \theta) \times Beta(\alpha, \beta) \\
&= \binom{n}{x} \theta^x (1-\theta)^{n-x} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \qquad (since\ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{1}{B(\alpha, \beta)}) \\
&= k \times \theta^x (1-\theta)^{n-x} \times \theta^{\alpha-1} (1-\theta)^{\beta-1} \qquad (since\ \binom{n}{x} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = k) \\
&\propto \theta^x (1-\theta)^{n-x} \times \theta^{\alpha-1} (1-\theta)^{\beta-1} \\
&\propto \theta^{x+\alpha-1} (1-\theta)^{n-x+\beta-1}
\end{aligned}$$

This distribution is proportional to the probability density function of a Beta distribution with parameters $\alpha + x$ and $\beta + n - x$. Therefore, the posterior distribution is:

$$\theta|x \sim \text{Beta}(\alpha + x, \beta + n - x)$$

3 More details

$$\begin{aligned}
Posterior(\theta|x) &= \frac{\mathbb{P}(x|\theta) \times \text{Prior}(\theta)}{Prior(x)} \\
&= \frac{\mathbb{P}(x, \theta)}{\int_0^1 \mathbb{P}(x, \theta)\, d\theta}
\end{aligned}$$

with:

- $\mathbb{P}(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, where $\binom{n}{x} = \frac{n!}{(n-x)!x!} = \frac{\Gamma(n+1)}{\Gamma(n-x+1)\Gamma(x+1)}$
- $\mathbb{P}(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \theta^{\alpha-1}(1-\theta)^{\beta-1}$

thus,

$$\begin{aligned}\mathbb{P}(x,\theta) &= \frac{\Gamma(a+b)\Gamma(n+1)}{\Gamma(a)\Gamma(b)\Gamma(n-x+1)\Gamma(x+1)} \times \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1} \\ &= \gamma \times \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1}\end{aligned} \tag{2}$$

using (2), we have:

$$\begin{aligned}\mathbb{P}(x) &= \int_0^1 \mathbb{P}(x,\theta)\, d\theta \\ &= \int_0^1 \frac{\Gamma(a+b)\Gamma(n+1)}{\Gamma(a)\Gamma(b)\Gamma(n-x+1)\Gamma(x+1)} \times \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1}\, d\theta \\ &= \int_0^1 \gamma \times \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1}\, d\theta \\ &= \gamma \times \int_0^1 \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1}\, d\theta \\ &= \gamma \times \frac{\Gamma(\alpha+x)\Gamma(n+\beta-x)}{\Gamma(\alpha+\beta+n)} \times \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(n+\beta-x)}\theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1}\, d\theta \\ &= \gamma \times \frac{\Gamma(\alpha+x)\Gamma(n+\beta-x)}{\Gamma(\alpha+\beta+n)} \qquad \text{since we have the probability density function of Beta}(\alpha+x, \beta+n-x)\end{aligned}$$

so,

$$\begin{aligned}\mathbb{P}(\theta|x) &= \frac{\gamma \times \Gamma(\alpha+\beta+n)}{\gamma \times \Gamma(\alpha+x)\Gamma(n+\beta-x)} \times \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1} \\ &= Beta(x+\alpha, \beta+n-x)\end{aligned}$$

4 Conclusion

The posterior distribution is of the same form as the prior distribution, thus proving Beta-Binomial conjugation.