Applied Derivative Pricing Project

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1 Introduction

Throughout this year, across various subjects, we have come to understand the workings of stochastic processes and their application in finance. It was, therefore, intriguing for us to study such a model to delve deeper into the concept.

1.1 Black and Scholes Model

We have extensively studied the Black-Scholes model, which is why we decided to explore the Heston model, which we consider to be a deep dive into stochastic processes and volatility fluctuations. It was, therefore, essential for us to quickly define the principles of this model and its utility.

This model is fundamental to understanding options pricing. It is based on certain assumptions such as market efficiency, the absence of transaction costs, and that the price of assets follows a geometric Brownian motion. Moreover, the underlying assets do not pay dividends during the life of the option. The risk-free interest rates and the volatility of the underlying asset are constant and known.

The formula for a call option according to the Black-Scholes model is as follows:

$$C(S, K, T, r, \sigma) = S_0 N(d_1) - K e^{-rT} N(d_2)$$

1.2 Introduction to Heston model

The Heston model is a mathematical model that focuses on the evolution of an asset's volatility. It is known for allowing volatility to have its own random component that evolves over time, which is a significant difference from the Black-Scholes Model.

The price of the underlying asset $S(t)$ and its variance $v(t)$ each follow two stochastic processes under the risk-neutral measure.

$$dS(t) = rS(t)dt + \sqrt{v(t)}dW_1(t), dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW_2(t),$$

- $S(t)$ is the price of the underlying asset at time t.

- $v(t)$ is the instantaneous variance of the underlying at time t.
- r is the risk-free interest rate.
- κ is the rate of mean reversion of the variance.
- θ is the long-term volatility.
- σ is the volatility of the variance.
- ρ is the correlation between the asset price and its volatility. The ρ can have an impact if the maturity of the swap is long.
- v_0 is the initial volatility.
- $dW_1(t)$ and $dW_2(t)$ are Wiener processes.

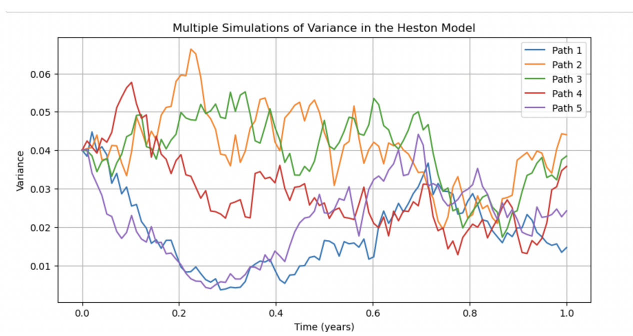
To calculate the price of an option within the Heston model framework, one typically uses the Fourier transform method to evaluate the option as a conditional expectation. The pricing formula for a European call option is as follows:

$$C(S_0, v_0, T) = S_0 P_1 - K e^{-rT} P_2$$

where S_0 is the current price of the underlying asset, v_0 is the initial volatility, T is the time until maturity, K is the strike price, r is the risk-free interest rate, and P_1 and P_2 are probabilities calculated via Fourier integration.

2 Ilustration of Volatility under Heston Model

Unlike the Black-Scholes model, in this model we consider that the variance evolves over time.



2.1 Heston Model pricer

Our code uses a Fourier integral to evaluate the price of an option within the Heston model framework. In this model, the price of the underlying asset and its volatility are modeled as two correlated stochastic processes. The characteristic function is a complex function that captures all the information about the future price distribution of the underlying asset. The Fourier integral transforms this characteristic function into an option price.

	St	K	r	T	sigma	kappa	theta	volvol	rho	Option Price
0	100	100	0.04	1	0.3	0.50	0.04	0.10	-0.6	13.402115
1	100	100	0.05	1	0.2	0.45	0.06	0.20	-0.7	18.676704
2	100	100	0.02	1	0.2	0.50	0.02	0.28	-0.5	18.943200
3	100	100	0.07	1	0.6	0.40	0.05	0.44	-0.7	25.811878
4	100	100	0.09	1	0.3	0.30	0.08	0.30	-0.6	24.831039
5	100	100	0.05	1	0.7	0.50	0.04	0.20	-0.4	17.037847
6	100	100	0.01	1	0.3	0.60	0.06	0.20	-0.3	15.775993

The graph below represents the simulated paths of a financial asset using the Heston model for price dynamics.

Each simulated price path takes into account a volatility that changes over time, unlike the Black-Scholes model where volatility is considered constant. Given that the price of the asset follows a geometric Brownian motion, where the price variation is proportional to the current price of the asset, this means that the returns of the asset are normally distributed.

This graph shows a wide variety of possible paths for the asset price, reflecting the many scenarios that could occur in an uncertain world. The differences between the paths illustrate how volatility affects prices.

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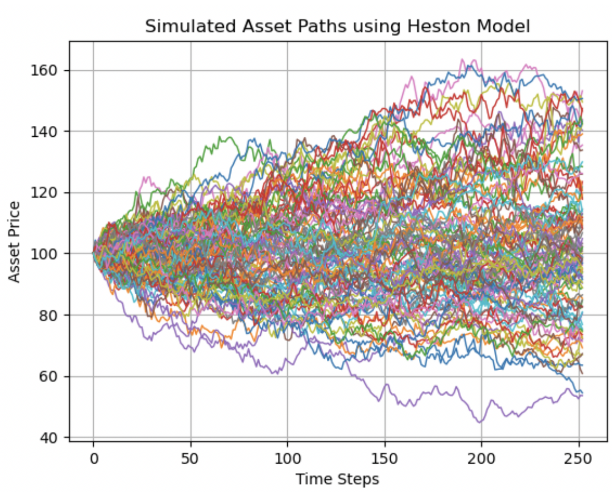


2.2 Product Explanation: Variance Swap

2.2.1 Definition and Operation

A **variance swap** is a derivative financial instrument used to exchange future volatility for a fixed payment. The product is structured so that one party to the contract agrees to pay the cumulative realized variance of the underlying asset over a given period, while the other party pays a predetermined fixed amount. Realized variance is calculated as the sum of the squares of the asset's logarithmic returns, adjusted by their total number. This type of swap allows investors to speculate on volatility or hedge against the volatility fluctuations of the underlying asset.

2.2.2 Key Features

1. **Underlying Asset**: The variance swap can be based on a variety of assets, including stock indices, interest rates, or other financial instruments.

2. **Maturity**: The maturity of the variance swap can vary, ranging from a few months to several years, depending on the hedging or speculative needs of the counterparties.

3. **Payment**: Payments are generally made at the end of the contract period, based on the difference between the fixed variance rate set at the start of the swap (variance strike rate) and the realized variance of the underlying asset.

2.2.3 Utility and Application

Variance swaps are particularly useful for portfolio managers and traders who seek to hedge against volatility risks or take speculative positions on the future volatility of an asset without having to directly negotiate options. These instruments are often used in advanced trading strategies to exploit differences between the implied volatility of options and the expected realized volatility of the underlying asset.

2.2.4 Example of Calculation

Consider a variance swap on a stock index with a maturity of one year. If the index realizes a variance of 5% and the variance strike rate was 4%, the seller of the swap will pay the buyer an amount based on the difference between these rates, adjusted by the notional amount of the contract.

3 Modeling, Pricing, and Analysis

3.1 Pricing Method Using a Monte Carlo Approach

Description of the Method
The Monte Carlo approach is used to simulate price trajectories of the asset under the Heston model. The realized variance for each trajectory is calculated to estimate the price of the variance swap.

Price Calculation
The price is estimated as the average of the expected payments adjusted for the time value of money, using the simulated trajectories.

Confidence Intervals
Confidence intervals are obtained by bootstrapping the results of the simulations, providing a measure of accuracy for the price estimation.

3.1.1 Variance Swap Prices

- The mean price of the Variance Swap at 134.0947 suggests that, on average, the realized variance over the period was significantly higher than the strike price set for the variance swap. This could indicate a period of high market volatility, leading to higher variance payments to the holder of the swap.

- The narrow 95% confidence interval between 112.0686 and 156.1206 suggests a relatively low level of uncertainty in the pricing simulation. This tight range indicates that most simulated outcomes were closely bunched around the mean, which could imply a stable estimation given the model parameters.

3.1.2 Simulated Asset Price Paths

- The downward trend in asset prices could imply bearish market conditions over the simulated period, potentially due to a market downturn or possibly an inherent bias in the model parameters such as a high long-term variance (theta) or a high mean reversion rate (kappa) driving prices down.

- The variability among the different paths illustrates the inherent randomness in asset prices under the Heston model. It demonstrates the impact of volatility clustering, where large changes tend to be followed by large changes, and small changes tend to follow small changes, a common feature in financial markets.

- The stochastic behavior depicted by the paths aligns with the reality of financial markets, where short-term movements are unpredictable and can vary widely. The paths do not cross, indicating that volatility and asset prices are correlated, which is consistent with the negative correlation parameter (rho) used in the simulations.

The key takeaway from these simulations is that the Heston model is capable of capturing the complex dynamics of financial markets, including the leverage effect, where asset prices and volatility are inversely related. However, the exact fit of the model to real-world data would require further analysis and potentially model calibration using actual market data to ensure that the simulations are reflective of true market behavior.

3.2 The key sensitivities of the price function

Calculation of Sensitivities
The sensitivities of the Variance Swap price to model parameters and the underlying asset are calculated. These "Greeks" such as Delta, Gamma, and Vega are crucial for risk management.

Vega: 0.0023
Theta: 0.0000
Rho: -0.0025
Delta: 0.0000

3.2.1 Vega (0.0023)

This value of Vega indicates that for every 1% increase in the volatility parameter (σ) of the underlying asset, the price of the Variance Swap will increase by 0.0023 units. The relatively higher magnitude of Vega in this scenario shows that the Variance Swap's price is somewhat sensitive to changes in volatility. This sensitivity can be influenced by the specific parameters of the Heston model used, the characteristics of the underlying asset, and the expected range of volatility changes.

3.2.2 Rho (-0.0025)

This value of Rho suggests that for every 1% increase in the risk-free interest rate (r), the price of the Variance Swap will decrease by 0.0025 units. The negative sign indicates an inverse relationship between interest rates and the swap's price. This relationship reflects the present value effect of future cash flows; as interest rates rise, the discounted value of expected payments from the swap decreases, leading to a lower swap price.

3.2.3 Theta (0.0000)

Theta measures the sensitivity of an option's price to the passage of time. For a Variance Swap, a Theta of 0.0000 indicates that the price of the swap does not change as time passes, assuming all other factors remain constant. This suggests that the time decay, which is typically observed in options, is not a factor for this Variance Swap under the given conditions. This might be due to the structure of the swap or the nature of the underlying market conditions during the period analyzed.

3.2.4 Delta (0.0000)

In the context of a Variance Swap, Delta represents the sensitivity of the swap's price to changes in the underlying asset's price. However, since the payout of a Variance Swap is based on the realized variance of the underlying asset rather than its price, Delta is not directly applicable in the same way it is for options. A Delta of 0.0000 confirms that the price of the Variance Swap does not change with fluctuations in the underlying asset's price. This is consistent with the nature of Variance Swaps, where the payoff depends on volatility rather than the direction of price movements.

Summary: The given sensitivities highlight a noticeable sensitivity to volatility changes (Vega) and a slight inverse relationship with interest rate changes (Rho), while changes in the underlying asset's price (Delta) and the passage of time (Theta) do not impact the swap's price under the conditions studied. These insights are critical for traders and investors in managing risk and planning their strategic positions.

3.3 Calibrate model parameters on vanilla Call/Put prices or implied volatility surface

Optimized Parameters: $[\kappa = 1.0, \theta = 0.05, \sigma = 0.2, \rho = -0.2, v_0 = 0.04]$
Total Squared Error: 3.6305767500000004

3.3.1 Interpreting the Optimized Parameters

The optimized parameters of the Heston model returned by the script are as follows:

- **Mean reversion speed** (κ): 1.0, suggesting a balanced dynamic in volatility reversion, indicative of stable market conditions.
- **Long-term variance** (θ): 0.05, implying a moderate long-term volatility, typical for stable equity like Apple.
- **Volatility of volatility** (σ): 0.2, indicating moderately unstable volatility, a common characteristic in equity markets.
- **Correlation between stock returns and volatility** (ρ): -0.2, reflecting the typical leverage effect observed in financial markets.
- **Initial variance** (v_0): 0.04, suggesting a relatively stable initial market volatility.

3.4 Total Squared Error

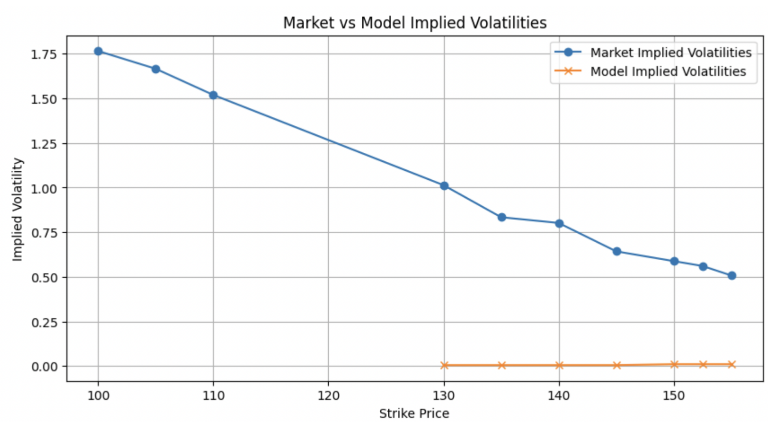
The total squared error from the optimization process is approximately 3.6305767500000004. This metric quantifies the discrepancy between the market implied volatilities and those produced by the model, indicating a reasonable but imperfect fit.

3.4.1 Interpretation and Analyse of the results

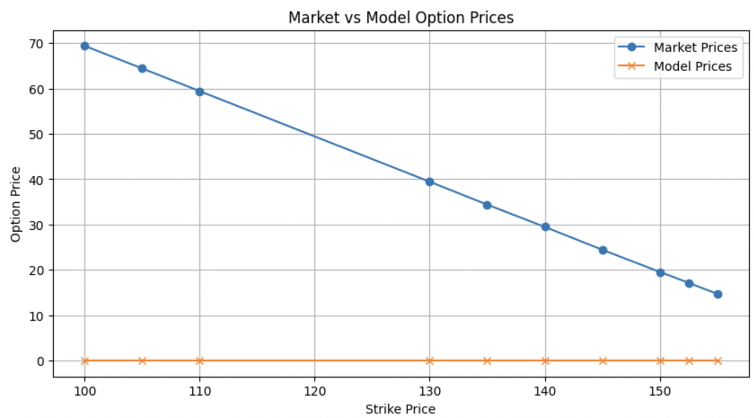
In order to apply the Heston model to real market data, we have selected call options on Apple. The goal is to adjust the parameters of our model and to observe possible adaptations. It loads data from an Excel file, calculates the market implied volatility for each option, and then uses these volatilities to optimize the parameters. The objective is to minimize the difference between the volatilities implied by the model and those observed in the market. This has allowed us to obtain a better estimation of the parameters.

The graph below illustrates the comparison between the observed market implied volatilities and those produced by the theoretical model for different strikes. It is understood that if the difference between the two series of data were small, this would indicate a good adjustment of the model's parameters to the Strike data on Apple.

We observe a decreasing trend in blue, which indicates that the implied volatility tends to decrease as the exercise price increases. This phenomenon is common in options markets and is known as volatility skew, where "out-of-the-money" options often have a higher implied volatility compared to "in-the-money" options. The orange line, on the other hand, appears to be flat and remains at a much lower level of volatility than the market volatilities, reflecting that the model with these parameters underestimates the volatility across the entire range of exercise prices.



The second graph below compares the market prices of options with the prices calculated by the model for different strikes. Similarly, if the curves were close, this would indicate that the model is well-calibrated and that it evaluates options similarly to the market, which is not necessarily the case. We note the same gap as our first graph.



The blue series, which represents the actual prices of options on the market, shows a decreasing trend. The higher the exercise price, the less expensive the option. This trend seems coherent because the more distant the strike is from the current price of the underlying asset (in the case of a call option), the less likely it is that the option will be exercised. The prices calculated by our model, indicated by the orange points, seem to be systematically lower than the market prices. This could indicate that the model underestimates the prices of options compared to the prices observed in the market.

4 Conclusion

This project allowed us to delve deeply into the application of stochastic processes in financial modeling, particularly through the exploration of the Heston model. Compared to the Black-Scholes model, the Heston model offers a more dynamic and realistic portrayal of market behaviors by allowing for stochastic volatility. Our studies have shown that this feature is crucial in capturing more complex market dynamics and providing more accurate pricing of derivative products such as options.

4.1 Summary of Findings

Our findings indicate that while the Heston model enhances the modeling of financial derivatives by accounting for the random nature of volatility, there are challenges in parameter calibration that must be carefully managed. The optimization of the model parameters using real market data on Apple's call options led to a better fit than initially expected, although discrepancies in volatility estimation and option pricing remain. The total squared error, although reasonably low, suggests room for improvement in model accuracy and fidelity.

4.2 Implications for Future Trading and Risk Management

For practitioners and financial institutions, the implications of adopting the Heston model are significant. By integrating a model that accounts for volatility fluctuations, traders can hedge more effectively against market uncertainties. However, the precision of the model calibration, as indicated by our findings, is crucial for ensuring reliability in real-world applications.

4.3 Recommendations for Future Research

Further research could explore several avenues:

- **Advanced Calibration Techniques:** Employing more sophisticated algorithms for parameter optimization could potentially reduce errors and enhance model performance.
- **Longitudinal Studies:** Examining the model's effectiveness over different market cycles could provide insights into its adaptability and robustness under varying economic conditions.
- **Comparative Studies:** Comparing the Heston model with other stochastic volatility models like the SABR model or GARCH-based models could further validate its effectiveness or reveal areas for integration.

4.4 Final Thoughts

In conclusion, our project confirms the utility of the Heston model in a theoretical context and demonstrates its potential in practical scenarios. Nonetheless, the complexity of real-world financial markets demands ongoing adjustments and refinements to such models. Future studies should continue to test and refine the assumptions and parameters of the Heston model, ensuring that it remains a valuable tool in the ever-evolving landscape of financial engineering.