Bayesian hypothesis testing as a mixture estimation model*

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Abstract. We consider a novel paradigm for Bayesian testing of hypotheses and Bayesian model comparison. Instead of the traditional comparison of posterior probabilities of the competing hypotheses, given the data, we consider the hypotheses as components of a mixture model. We therefore replace the original testing problem with an estimation one that focus on the probability or weight of a given hypothesis within a mixture model as the parameter of interest and the posterior distribution of this weight as the outcome of the test. A major differentiating feature of this approach is that that generic improper priors are acceptable. For example, a reference Beta $\mathcal{B}(a_0, a_0)$ prior on the mixture weight parameter can be used for the common problem of testing two contrasting hypotheses. In this case, the sensitivity of the posterior estimates of the weights to the choice of a_0 vanishes as the sample size increases, leading to a consistent procedure and a suggested default choice of $a_0 = 0.5$. Another feature of this easily implemented alternative to the classical Bayesian solution is that the speeds of convergence of the posterior mean of the weight and of the corresponding posterior probability are quite similar.

Key words and phrases: Noninformative prior, Mixture of distributions, Bayesian analysis, testing statistical hypotheses, Dirichlet prior, Posterior probability.

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1. INTRODUCTION

1.1 A open problem

Statistical testing of hypotheses and the related model choice problem are central issues for statistical inference. Perspectives and methods relating to these issues have been developed over the past two centuries, but it remains an object of study and debate, in particular because the most standard approach based on p-values is open to misue and abuse, as highlighted by a recent ASA warning (Wasserstein and Lazar, 2016), and also because the classical (frequentist) and Bayesian paradigms (Neyman and Pearson, 1933; Jeffreys, 1939; Berger and Sellke, 1987; Casella and Berger, 1987; Gigerenzer, 1991; Berger, 2003; Mayo and Cox, 2006; Gelman, 2008) are at odds, both conceptually and practically.

From the perspectives of Neyman-Pearson and Fisher, tests are constructed as a competition beween so-called null and alternative hypotheses, and typically evaluated with respect to their ability to control the type I error, i.e., the probability of falsely rejecting the null hypothesis in favour of the alternative. These procedures therefore handle the two hypotheses differently, with an imbalance that makes the subsequent action of accepting the null hypothesis problematic. The ASA statement (Wasserstein and Lazar, 2016) recommends against basing decisions solely on the p-value and advocates the use of supplementary indicators such as those obtained by Bayesian methods.

However, from a Bayesian perspective, the handling of hypothesis testing is also problematic, for different reasons (Jeffreys, 1939; Bernardo, 1980; Berger, 1985; Aitkin, 1991; Berger and Jefferys, 1992; De Santis and Spezzaferri, 1997; Bayarri and Garcia-Donato, 2007; Christensen et al., 2011; Johnson and Rossell, 2010; Gelman et al., 2013a; Robert, 2014). In particular, we consider that the issue of non-informative Bayesian hypothesis testing is still mostly unresolved both theoretically and in practice, despite having produced much debate and proposals; witness the specific case of the Lindley or Jeffreys-Lindley paradox (Lindley, 1957; Shafer, 1982; DeGroot, 1982; Robert, 1993; Lad, 2003; Spanos, 2013; Sprenger, 2013; Robert, 2014) and discussions about pseudo-Bayes factors (Aitkin, 1991; Berger and Pericchi, 1996; O'Hagan, 1995, 1997; Aitkin, 2010; Gelman et al., 2013b).

There are similar difficulties with Bayesian model selection. Several perspectives can be defended even for the canonical problem of comparing models with the aim of choosing one of them. For instance, Hoeting et al. (1999) propose a model averaging approach; Christensen et al. (2011) argue that this is a decision issue that pertains to testing; Robert (2001) express this as a model index estimation setting; Gelman et al. (2013a) prefer to rely on more exploratory predictive tools, and (Park and Casella, 2008; Rockova and George, 2016) restrict the inference to a maximisation problem in a sparse model or variable selection context.

1.2 The traditional Bayesian framework

Under all the frameworks described above, a generally accepted perspective is that hypothesis testing and model selection do not primarily seek to identify which alternative or model is "true" (if any). From a Bayesian perspective, hypotheses can be formulated as models and hypothesis testing can therefore be viewed as a form of model selection, in which the aim to compare several potential statistical models in order to identify the model that is most strongly supported by the data (see, e.g. Berger and Jefferys, 1992; Madigan and Raftery, 1994; Balasubramanian, 1997; MacKay, 2002; Consonni et al., 2013). For this reason, we will use the terms hypotheses and models interchangeably in the context of hypothesis testing and model choice.

The most common approaches to Bayesian hypothesis testing in practice are posterior probabilities of the model given the data (see, e.g., Robert, 2001), the Bayes factor (Jeffreys, 1939) and its approximations such as the Bayesian information criterion (BIC) and the Deviance in-

formation criterion (DIC) (Schwarz, 1978; Csiszár and Shields, 2000; Spiegelhalter et al., 2002; Forbes et al., 2006; Plummer, 2008) and posterior predictive tools and their variants (Gelman et al., 2013a; Vehtari and Ojanen, 2012). For example, consider two families of models, one for each of the hypotheses under comparison,

$$\mathfrak{M}_1: x \sim f_1(x|\theta_1), \ \theta_1 \in \Theta_1 \quad \text{and} \quad \mathfrak{M}_2: x \sim f_2(x|\theta_2), \ \theta_2 \in \Theta_2.$$

Following Berger (1985) and Robert (2001), a standard Bayesian approach is to associate with each of those models a prior distribution,

$$\theta_1 \sim \pi_1(\theta_1)$$
 and $\theta_2 \sim \pi_2(\theta_2)$,

and to compute the marginal or integrated likelihoods

$$m_1(x) = \int_{\Theta_1} f_1(x|\theta_1) \, \pi_1(\theta_1) \, d\theta_1$$
 and $m_2(x) = \int_{\Theta_2} f_2(x|\theta_2) \, \pi_1(\theta_2) \, d\theta_2$

either through the Bayes factor or through the posterior probability, respectively:

$$\mathfrak{B}_{12} = \frac{m_1(x)}{m_2(x)}, \quad \mathbb{P}(\mathfrak{M}_1|x) = \frac{\omega_1 m_1(x)}{\omega_1 m_1(x) + \omega_2 m_2(x)}, \quad \omega_1 + \omega_2 = 1, \quad \omega_i \ge 0$$

Note that the latter quantity depends on the prior weights ω_i of both models. The Bayesian decision step proceeds by comparing the Bayes factor \mathfrak{B}_{12} to the threshold value of one or comparing the posterior probability $\mathbb{P}(\mathfrak{M}_1|x)$ to a bound derived from a 0–1 loss function or a "golden" bound like $\alpha=0.05$ inspired from frequentist practice (Berger and Sellke, 1987; Berger et al., 1997, 1999; Berger, 2003; Ziliak and McCloskey, 2008). As a general rule, when comparing more than two models, the model with the largest posterior probability is selected, but this rule is highly dependent on the prior modelling, even with large datasets, which makes it hard to promote as the default solution in practical studies.

1.3 Our alternative proposal

Given that the difficulties associated with the traditional handling of posterior probabilities for Bayesian testing and model selection are well documented and comprehensively reviewed (Vehtari and Lampinen, 2002) and Vehtari and Ojanen (2012), we do not replicate or dwell further on this discussion, nor do we attempt to resolve the attendant problems. Instead, we propose a different approach, which we argue provides a convergent and naturally interpretable solution, a measure of uncertainty on the outcome, a wider range of prior modelling, and straightforward calibration tools.

The proposed approach is described here in the context of a hypothesis test or model selection problem with two alternatives. We represent the distribution of each individual observation as a two-component mixture between both models \mathfrak{M}_1 and \mathfrak{M}_2 . The resulting mixture model is thus an encompassing model, as it contains both models under comparison as special cases. While those special cases are extreme cases which weights are located at the boundaries of the interval (0,1), the posterior distribution of this weight parameter does concentrate on one of those boundaries for enough data from the corresponding model. This concentration property follows from the results of Rousseau and Mengersen (2011), in which they established that overfitted mixtures (i.e., mixtures with more components than are supported by the data) can be consistently estimated, despite the parameter sitting on one (or several) boundary(ies) of the parameter space. The outcome of our analysis is a posterior distribution over the parameters of the mixture, including the component weights.

While mixtures are natural tools for classification and clustering, which can separate a sample into observations associated with each model, we argue that the posterior distribution on the weights provides a relevant indicator of the strength of support for each model given the data. Given a sufficiently large sample from model \mathfrak{M}_1 , say, this distribution will almost surely concentrate near 0. Starting from this theoretical garantee, we can calibrate the degree of concentration near 0 versus 1 for the current sample size by comparing the posterior distribution to posteriors associated with simulated samples from models \mathfrak{M}_1 and \mathfrak{M}_2 , respectively. Even though we fundamentally object to returning a decision about "the" model corresponding to the data (McShane et al., 2018), and thus would like to halt the statistician's input at returning the above posterior, it is furthermore straightforward to produce a posterior median estimate of the component weight that can be compared with realisations from each model. Quite obviously, the mixture posterior produced by a standard Bayesian analysis (Frühwirth-Schnatter, 2006) also provides information about the model parameters and the presence of potential outliers in the data.

With regard to the classical approach to Bayesian hypothesis testing, this mixture representation is not equivalent to the use of a posterior probability. In fact, a posterior estimate of the mixture weight cannot be viewed as a proxy to the numerical value of this posterior probability, which we do not see as a worthwhile tool for testing for reasons given below. As mentioned in the previous paragraph, this new tool can be calibrated in its own right, while allowing for a degree of uncertainty in the hypothesis evaluation, which is not the case for the Bayes factor. In particular, while posterior probabilities are scaled against the (0,1) interval, it can be argued (Fraser, 2011; Fraser et al., 2009, 2016) that they cannot be taken at face value because of their lack of frequentist coverage and hence need to be calibrated as well. Furthermore, the mixture approach offers the valuable feature of limiting the number of parameters in the model and hence is in keeping with Occam's razor, see, e.g., Jefferys and Berger (1992); Rasmussen and Ghahramani (2001).

The plan of the paper is as follows. Section 2 provides a description of the mixture model specifically created for this setting and presents a simple example of its implementation. Section 3 expands Rousseau and Mengersen (2011) to provide conditions on the hyperameters of the mixture model that are sufficient to achieve convergence. The performance of the mixture approach is then illustrated through three further examples in Section 4 and concluding remarks about the general applicability of the method are made in Section 5.

2. TESTING PROBLEMS AS ESTIMATING MIXTURE MODELS

2.1 A new paradigm for testing

Following from the above, given two classes of statistical models, \mathfrak{M}_1 and \mathfrak{M}_2 , which may correspond to a hypothesis to be tested and its alternative, respectively, it is always possible to embed both models within an encompassing mixture model

(1)
$$\mathfrak{M}_{\alpha}: x \sim \alpha f_1(x|\theta_1) + (1-\alpha)f_2(x|\theta_2), \ 0 \le \alpha \le 1.$$

Indeed, both models correspond to very special cases of the mixture model, one for $\alpha = 1$ and the other for $\alpha = 0$ (with a slight notational inconsistency in the indices).¹

$$x \sim f_{\alpha}(x) \propto f_1(x|\theta_1)^{\alpha} f_2(x|\theta_2)^{1-\alpha}$$

is a conceivable alternative. However, such alternatives are less practical to manage, starting with the issue of the intractable normalizing constant. Note also that when f_1 and f_2 are Gaussian densities, the Geometric mixture remains Gaussian for all values of α . Similar drawbacks can be found with harmonic mixtures.

¹The choice of possible encompassing models is obviously unlimited: for instance, a Geometric mixture (Meng, 2016, personnal communication)

When considering a sample (x_1, \ldots, x_n) from one of the two models, the mixture representation still holds at the likelihood level, namely the likelihood for each model is a special case of the weighted sum of both likelihoods. However, this is not directly appealing for estimation purposes since it corresponds to a mixture with a single observation. See however O'Neill and Kypraios (2014) for a computational solution based upon this representation.

What we propose in this paper is to draw inference on the individual mixture representation (1), acting as if each observation was individually and independently² produced by the mixture model. Hence α represents the probability that a new observations is sampled from f_1 belonging to model \mathfrak{M}_1 . The approach proposed here therefore aims at answering the question: what is the proportion of the data which support one model, which has a definite predictive flavour to the testing problem.

Here are five advantages we see about this approach.

First, if the data were indeed generated from model \mathfrak{M}_1 , then the Bayesian estimate of the weight α and the posterior probability of model \mathfrak{M}_1 produce equally convergent indicators of preference for this model (see Section 3). Moreover, the posterior distribution of α evaluates more thoroughly the strength of the support for a given model than the single figure outcome of a Bayes factor or of a posterior probability, while the variability of the posterior distribution on α allows for a more thorough assessment of the strength of the support of one model against the other. Indeed, the approach allows for the possibility that, for a finite dataset, one model, both models or neither model could be acceptable, as illustrated in Section 4.

Second, the mixture approach also removes the need for artificial prior probabilities on model indices, ω_1 and ω_2 . These priors are rarely discussed in a classical Bayesian approach, even though they linearly impact on the posterior probabilities. Under the new approach, prior modelling only involves selecting an operational prior on α , for intance a Beta $\mathcal{B}(a_0, a_0)$ distribution, with a wide range of acceptable values for the hyperparameter a_0 , as demonstrated in Section 3. While the value of a_0 impacts the posterior distribution of α , it can be argued that (a) it nonetheless leads to an accumulation of the mass near 1 or 0; (b) a sensitivity analysis on the impact of a_0 is straightforward to carry out; and (c) in most settings the approach can be easily calibrated by a parametric bootstrap experiment, so the prior predictive error can be directly estimated and can drive the choice of a_0 if need be.

Third, the problematic computation (Chen et al., 2000; Marin and Robert, 2011) of the marginal likelihoods is bypassed, since standard algorithms are available for Bayesian mixture estimation (Richardson and Green, 1997; Berkhof et al., 2003; Frühwirth-Schnatter, 2006; Lee et al., 2009). Moroever, the (simultaneously conceptual and computational) difficulty of "label switching" (Celeux et al., 2000; Stephens, 2000; Jasra et al., 2005) that plagues both Bayesian estimation and Bayesian computation for most mixture models completely vanishes in this particular context, since components are no longer exchangeable in the current framework. In particular, we compute neither a Bayes factor³ nor a posterior probability related with the substitute mixture model and we hence avoid the difficulty of recovering the modes of the posterior distribution (Berkhof et al., 2003; Lee et al., 2009; Rodriguez and Walker, 2014). Our perspective is completely centred on estimating the parameters of a mixture model where both components are always identifiable.

Fourth, the extension to a finite collection of models to be compared is straightforward, as this simply involves a larger number of components. The mixture approach allows consideration of

²Dependent observations like Markov chains can be modeled by a straightforward extension of (1) where both terms in the mixture are conditional on the relevant past observations.

³Using a Bayes factor to test for the number of components in the mixture (1) as in Richardson and Green (1997) would be possible. However, the outcome would fail to answer the original question of selecting between both (or more) models.

all these models at once rather than engaging in costly pairwise comparisons. It is thus possible to eliminate the least likely models from simulations, since they will not be explored by the corresponding computational algorithm (Carlin and Chib, 1995; Richardson and Green, 1997).

Finally, while standard (proper and informative) prior modeling can be painlessly reproduced in this novel setting, non-informative (improper) priors are also permitted, provided both models under comparison are first reparameterised so that they share parameters with common meaning. For instance, in the special case when all parameters make sense in both models,⁴ the mixture model (1) can read as

$$\mathfrak{M}_{\alpha}: \ x \sim \alpha f_1(x|\theta) + (1-\alpha)f_2(x|\theta), 0 \le \alpha \le 1.$$

For instance, if θ is a location parameter, a flat prior $\pi(\theta) \propto 1$ can be used with no foundational difficulty, in contrast to the traditional testing case (DeGroot, 1973; Berger et al., 1998).

2.2 A Normal Example

In order to illustrate the proposed approach, consider a hypothesis test between a Normal $\mathcal{N}(\theta_1, 1)$ and a Normal $\mathcal{N}(\theta_2, 2)$ distribution. We construct the mixture so that the same location parameter θ is used in both the Normal $\mathcal{N}(\theta, 1)$ and the Normal $\mathcal{N}(\theta, 2)$ distribution. This allows the use of Jeffreys' (1939) noninformative prior $\pi(\theta) = 1$, in contrast with the corresponding Bayes factor. We thus embed the test in the mixture of Normal models, $\alpha \mathcal{N}(\theta, 1) + (1-\alpha)\mathcal{N}(\theta, 2)$, and adopt a Beta $\mathcal{B}(a_0, a_0)$ prior on α . In this case, considering the posterior distribution on (α, θ) , conditional on the allocation vector ζ , leads to conditional independence between θ and α :

 $\theta | \mathbf{x}, \zeta \sim \mathcal{N}\left(\frac{n_1\bar{x}_1 + .5n_2\bar{x}_2}{n_1 + .5n_2}, \frac{1}{n_1 + .5n_2}\right), \quad \alpha | \zeta \sim \mathcal{B}e(a_0 + n_1, a_0 + n_2),$

where n_i and \bar{x}_i denote the number of observations and the empirical mean of the observations allocated to component i, respectively (with the convention that $n_i\bar{x}_i=0$ when $n_i=0$. Since this conditional posterior distribution is well-defined for every possible value of ζ and since the distribution ζ has a finite support, $\pi(\theta|x)$ is proper.

Note that for this example, the conditional evidence $\pi(x|\zeta)$ can easily be derived in closed form, which means that a random walk on the allocation space $\{1,2\}^n$ could be implemented. We did not follow that direction, as it seemed unlikely such a random walk would have been more efficient than a Metropolis–Hastings algorithm on the parameter space only.

In order to evaluate the convergence of the estimates of the mixture weights, we simulated $100 \mathcal{N}(0,1)$ datasets. Figure 1 displays the range of the posterior means and medians of α when either a_0 or n varies, showing the concentration effect (with a lingering impact of a_o) when n increases. We also included the posterior probability of \mathfrak{M}_1 in the comparison, derived from the Bayes factor

$$\mathfrak{B}_{12} = 2^{n-1/2} / \exp 1/4 \sum_{i=1}^{n} (x_i - \bar{x})^2,$$

with equal prior weights, even though it is not formally well defined since it is based on an improper prior. The shrinkage of the posterior expectations towards 0.5 motivate the use the posterior median instead of the posterior mean as the relevant estimator of α . The same concentration phenomenon occurs for the $\mathcal{N}(0,2)$ case, as illustrated in Figure 2 for a single $\mathcal{N}(0,2)$ dataset.

⁴While this may sound like an extremely restrictive requirement in a traditional mixture model, we stress here that the presence of common meaning parameters becomes quite natural within a testing setting. To wit, when comparing two different models for the *same* data, moments like $\mathbb{E}[X^{\gamma}]$ are defined in terms of the observed data and hence *should* be the *same* for both models. Reparametrising the models in terms of those common meaning moments does lead to a mixture model with some and maybe *all* common parameters.

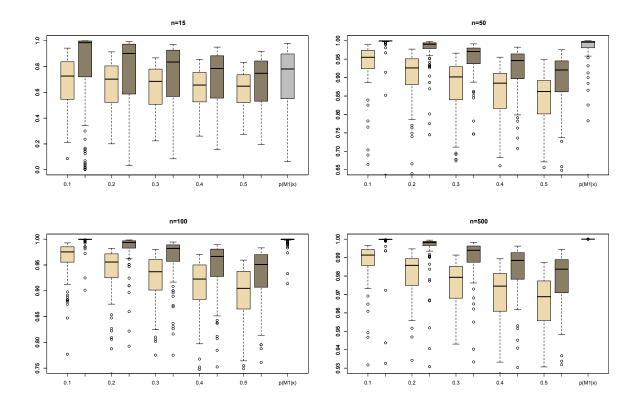


Fig 1: Normal Example: Boxplots of the posterior means (wheat) and medians of α (dark wheat), compared with a boxplot of the exact posterior probabilities of \mathfrak{M}_0 (gray) for a $\mathcal{N}(0,1)$ sample, derived from 100 datasets for sample sizes equal to 15, 50, 100, 500. Each posterior approximation is based on 10^4 MCMC iterations.

3. ASYMPTOTIC CONSISTENCY

In this Section we study the asymptotic properties of our mixture testing procedure. More precisely we study the asymptotic behaviour of the posterior distribution of α and we prove that the posterior on α concentrates on the *true* value of α in the sense that if model \mathfrak{M}_1 is correct the posterior distribution concentrates on $\alpha=1$, if model \mathfrak{M}_2 is correct then the posterior distribution concentrates on $\alpha=0$ and if neither are correct then the posterior concentrate on the value of α which minimizes the Kullback-Leibler divergence. This shows that the posterior on α leads to a consistent testing procedure. Moreover we also study the separation rate associated to such procedure when the models are embedded and we show that the approach leads to optimal separation rate, contrarywise to the Bayes factor which has an extra $\sqrt{\log n}$ factor.

To do so we consider two different cases. In the first case, the two models, \mathfrak{M}_1 and \mathfrak{M}_2 , are well separated while, in the second case, model \mathfrak{M}_1 is a submodel of \mathfrak{M}_2 . We denote by π the prior distribution on (α, θ) with $\theta = (\theta_1, \theta_2)$ and assume that $\theta_j \in \Theta_j \subset \mathbb{R}^{d_j}$. We first prove that, under weak regularity conditions on each model, we can obtain posterior concentration rates for the marginal density $f_{\theta,\alpha}(\cdot) = \alpha f_{1,\theta_1}(\cdot) + (1-\alpha) f_{2,\theta_2}(\cdot)$. Let $\mathbf{x}^n = (x_1, \dots, x_n)$ be a n sample with true density f^* .

Proposition 1 Assume that, for all $C_1 > 0$, there exist Θ_n a subset of $\Theta_1 \times \Theta_2$ and $B_0, B_1 \ge 0$ such that

(2)
$$\pi \left[\Theta_n^c \right] \le n^{-C_1}, \quad \Theta_n \subset \{ \|\theta_1\| + \|\theta_2\| \le B_0 n^{B_1} \}$$

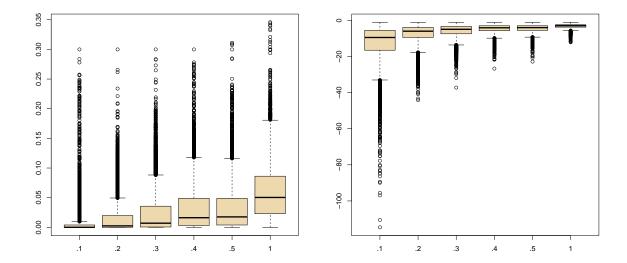


Fig 2: Normal Example: (left) Posterior distributions of the mixture weight α and (right) of their logarithmic transform $\log\{\alpha\}$ under a Beta $\mathcal{B}(a_0, a_0)$ prior when $a_0 = .1, .2, .3, .4, .5, 1$ and for a single Normal $\mathcal{N}(0, 2)$ sample of 10^3 observations. The MCMC outcome is based on 10^4 iterations.

and that there exist $H \ge 0$ and $L, \delta > 0$ such that, for j = 1, 2,

(3)
$$\sup_{\theta, \theta' \in \Theta_n} \|f_{j,\theta_j} - f_{j,\theta'_j}\|_1 \le Ln^H \|\theta_j - \theta'_j\|, \quad \theta = (\theta_1, \theta_2), \ \theta' = (\theta'_1, \theta'_2),$$
$$\forall \|\theta_j - \theta_j^*\| \le \delta; \quad KL(f_{j,\theta_j}, f_{j,\theta_j^*}) \lesssim \|\theta_j - \theta_j^*\|.$$

We then have that, when $f^* = f_{\theta^*,\alpha^*}$, with $\alpha^* \in [0,1]$, there exists M > 0 such that

$$\pi\left[(\alpha,\theta); \|f_{\theta,\alpha} - f^*\|_1 > M\sqrt{\log n/n} |\mathbf{x}^n\right] = o_p(1).$$

The proof of Proposition 1 is a direct consequence of Theorem 2.1 of Ghosal et al. (2000) and is thus omitted here. Condition (3) is a weak regularity condition on each of the candidate models. Combined with condition (2) it allows consideration of noncompact parameter sets in the usual way; see, for instance, Ghosal et al. (2000). It is satisfied in all examples considered in this paper. We build on Proposition 1 to describe the asymptotic behaviour of the posterior distribution on the parameters. It is possible to sharpen the above posterior concentration rate into M_n/\sqrt{n} for any sequence M_n going to infinity by controlling the local entropy and obtaining precise upper bounds on neighbourhoods of f^* . This is not useful in the case of separated models but becomes more important in the context of embedded models. Although we do not treat this here, following Kleijn and van der Vaart (2006), if the true distribution f_0 does not belong to the embedding model $f_{\theta,\alpha}$, then the posterior will concentrate on f^* which minimizes the Kullback-Leibler divergence between f_0 and $f_{\theta,\alpha}$, at a similar rate.

3.1 The case of separated models

Assume that both models are separated in the sense that there is identifiability:

(4)
$$\forall \alpha, \alpha' \in [0, 1], \quad \forall \theta_j, \theta'_j, \ j = 1, 2 \quad P_{\theta, \alpha} = P_{\theta', \alpha'} \quad \Rightarrow \alpha = \alpha', \quad \theta = \theta',$$

where $P_{\theta,\alpha}$ denotes the distribution associated with $f_{\theta,\alpha}$. We assume that (4) also holds on the boundary of $\Theta_1 \times \Theta_2$. In other words, the following

$$\inf_{\theta_1 \in \Theta_1} \inf_{\theta_2 \in \Theta_2} \|f_{1,\theta_1} - f_{2,\theta_2}\|_1 > 0$$

holds. We also assume that, for all $\theta_j^* \in \Theta_j$, j = 1, 2, if P_{θ_j} converges in the weak topology to $P_{\theta_j^*}$, then θ_j converges in the Euclidean topology to θ_j^* . The following result then holds:

Theorem 1 Assume that (4) is satisfied, together with (2) and (3), then for all $\epsilon > 0$

$$\pi[|\alpha - \alpha^*| > \epsilon | \mathbf{x}^n] = o_p(1).$$

In addition, assume that the mapping $\theta_j \to f_{j,\theta_j}$ is twice continuously differentiable in a neighbourhood of θ_i^* , j = 1, 2, and that

$$f_{1,\theta_1^*} - f_{2,\theta_2^*}, \nabla f_{1,\theta_1^*}, \nabla f_{2,\theta_2^*}$$

are linearly independent as functions of y and that there exists $\delta > 0$ such that

$$\nabla f_{1,\theta_1^*}, \nabla f_{2,\theta_2^*}, \sup_{|\theta_1-\theta_1^*|<\delta} |D^2 f_{1,\theta_1}|, \sup_{|\theta_2-\theta_2^*|<\delta} |D^2 f_{2,\theta_2}| \in L_1.$$

Then

(5)
$$\pi \left[|\alpha - \alpha^*| > M \sqrt{\log n/n} |\mathbf{x}^n| \right] = o_p(1).$$

Theorem 1 allows for the interpretation of the quantity α under the posterior distribution. In particular, if the data \mathbf{x}^n are generated from model \mathfrak{M}_1 (resp. \mathfrak{M}_2), then the posterior distribution on α concentrates around $\alpha = 1$ (resp. around $\alpha = 0$), which establishes the consistency of our mixture approach.

We now consider the embedded case.

3.2 Embedded case

In this Section we assume that \mathfrak{M}_1 is a submodel of \mathfrak{M}_2 , in the sense that $\theta_2 = (\theta_1, \psi)$ with $\psi \in \mathcal{S} \subset \mathbb{R}^{d_{\psi}}$ and that $f_{2,\theta_2} \in \mathfrak{M}_1$ when $\theta_2 = (\theta_1, \psi_0)$ for some given value ψ_0 , say $\psi_0 = 0$. Condition (4) is no longer verified for all α 's: we assume however that it is verified for all $\alpha, \alpha^* \in [0, 1)$ and that $\theta_2^* = (\theta_1^*, \psi^*)$ satisfies $\psi^* \neq 0$. In this case, under the same conditions as in Theorem 1, we immediately obtain the posterior concentration rate $\sqrt{\log n/n}$ for estimating α when $\alpha^* \in [0, 1)$ and $\psi^* \neq 0$ and Theorem 1 implies that (5) holds, which in turns implies that if $\alpha^* = 1$, i.e. if the distribution comes from model \mathfrak{M}_2 ,

$$\pi \left[\alpha > M \sqrt{\log n/n} | \mathbf{x}^n \right] = o_p(1).$$

We now treat the case where $\psi^* = 0$; in other words, f^* is in model \mathfrak{M}_1 .

As in Rousseau and Mengersen (2011), we consider both possible paths to approximate f^* : either α goes to 1 or ψ goes to $\psi_0 = 0$. In the first case, called path 1, $(\alpha^*, \theta^*) = (1, \theta_1^*, \theta_1^*, \psi)$ with $\psi \in \mathcal{S}$; in the second, called path 2, $(\alpha^*, \theta^*) = (\alpha, \theta_1^*, \theta_1^*, 0)$ with $\alpha \in [0, 1]$. In either case, we write P^* as the distribution and denote $F^*g = \int f^*(x)g(x)d\mu(x)$ for any integrable function g. For sparsity reasons, we consider the following structure for the prior on (α, θ) :

$$\pi(\alpha, \theta) = \pi_{\alpha}(\alpha)\pi_1(\theta_1)\pi_{\psi}(\psi), \quad \theta_2 = (\theta_1, \psi).$$

This means that the parameter θ_1 is common to both models, i.e., that θ_2 shares the parameter θ_1 with f_{1,θ_1} .

Condition (4) is replaced by

(6)
$$P_{\theta,\alpha} = P^* \quad \Rightarrow \alpha = 1, \quad \theta_1 = \theta_1^*, \quad \theta_2 = (\theta_1^*, \psi) \quad \text{or} \quad \alpha \le 1, \quad \theta_1 = \theta_1^*, \quad \theta_2 = (\theta_1^*, 0)$$

Let Θ^* be the above parameter set.

As in the case of separated models, the posterior distribution concentrates on Θ^* . We now describe more precisely the asymptotic behaviour of the posterior distribution, using Rousseau and Mengersen (2011). We cannot apply directly Theorem 1 of Rousseau and Mengersen (2011), hence the following result is an adaptation of it. We require the following assumptions with $f^* = f_{1,\theta_1^*}$. For the sake of simplicity, we assume that Θ_1 and \mathcal{S} are compact. Extension to non compact sets can be handled similarly to Rousseau and Mengersen (2011).

B1 Regularity: Assume that $\theta_1 \to f_{1,\theta_1}$ and $\theta_2 \to f_{2,\theta_2}$ are 3 times continuously differentiable and that

$$F^* \left(\frac{\bar{f}_{1,\theta_1^*}^3}{\underline{f}_{1,\theta_1^*}^3} \right) < +\infty, \quad \bar{f}_{1,\theta_1^*} = \sup_{|\theta_1 - \theta_1^*| < \delta} f_{1,\theta_1}, \quad \underline{f}_{1,\theta_1^*} = \inf_{|\theta_1 - \theta_1^*| < \delta} f_{1,\theta_1}$$

$$F^* \left(\frac{\sup_{|\theta_1 - \theta_1^*| < \delta} |\nabla f_{1,\theta_1^*}|^3}{\underline{f}_{1,\theta_1^*}^3} \right) < +\infty, \quad F^* \left(\frac{|\nabla f_{1,\theta_1^*}|^4}{f_{1,\theta_1^*}^4} \right) < +\infty,$$

$$F^* \left(\frac{\sup_{|\theta_1 - \theta_1^*| < \delta} |D^2 f_{1,\theta_1^*}|^2}{\underline{f}_{1,\theta_1^*}^2} \right) < +\infty, \quad F^* \left(\frac{\sup_{|\theta_1 - \theta_1^*| < \delta} |D^3 f_{1,\theta_1^*}|}{\underline{f}_{1,\theta_1^*}} \right) < +\infty$$

B2 Integrability: There exists $S_0 \subset S \cap \{|\psi| > \delta_0\}$, for some positive δ_0 and satisfying $Leb(S_0) > 0$, and such that for all $\psi \in S_0$,

$$F^*\left(\frac{\sup_{|\theta_1-\theta_1^*|<\delta} f_{2,\theta_1,\psi}}{f_{1,\theta_1^*}^4}\right) < +\infty, \quad F^*\left(\frac{\sup_{|\theta_1-\theta_1^*|<\delta} f_{2,\theta_1,\psi}^3}{f_{1,\theta_1^*}^3}\right) < +\infty,$$

B3 Stronger identifiability : Set

$$\nabla f_{2,\theta_{1}^{*},\psi^{*}}(x) = \left(\nabla_{\theta_{1}} f_{2,\theta_{1}^{*},\psi^{*}}(x)^{\mathrm{T}}, \nabla_{\psi} f_{2,\theta_{1}^{*},\psi^{*}}(x)^{\mathrm{T}}\right)^{\mathrm{T}}.$$

Then for all $\psi \in \mathcal{S}$ with $\psi \neq 0$, if $\eta_0 \in \mathbb{R}$, $\eta_1 \in \mathbb{R}^{d_1}$

(7)
$$\eta_0(f_{1,\theta_1^*} - f_{2,\theta_1^*,\psi}) + \eta_1^{\mathrm{T}} [\nabla_{\theta_1} f_{1,\theta_1^*} - \nabla_{\theta_1} f_{2,\theta_1^*,\psi}(x)] = 0 \quad \Leftrightarrow \eta_1 = 0, \ \eta_2 = 0$$

Assumptions B1-B3 are similar, but weaker, to Rousseau and Mengersen (2011)'s set of conditions and in fact B3 is milder than the strong identifiability condition imposed in that paper. Hence these conditions are satisfied for a wide range of regular models.

We can now state the main theorem:

Theorem 2 Given the model

$$f_{\theta_1,\psi,\alpha} = \alpha f_{1,\theta_1} + (1-\alpha) f_{2,\theta_1,\psi},$$

assume that the data comprise the n sample $\mathbf{x}^n = (x_1, \dots, x_n)$ issued from f_{1,θ_1^*} for some $\theta_1^* \in \Theta_1$, and that assumptions B1 - B3 are satisfied. Then for all sequence M_n going to infinity,

(8)
$$\pi \left[(\alpha, \theta); \|f_{\theta,\alpha} - f^*\|_1 > M_n / \sqrt{n} |\mathbf{x}^n| \right] = o_p(1).$$

If the prior π_{α} on α is a Beta $\mathcal{B}(a_1, a_2)$ distribution, with $a_2 < d_{\psi}$, and if the prior $\pi_1 \pi_{\psi}$ is absolutely continuous with positive and continuous density at $(\theta_1^*, 0)$, then for all M_n going to infinity,

$$\pi \left[|\alpha - 1| > M_n / \sqrt{n} |\mathbf{x}^n| \right] = o_p(1).$$

If $a_2 > d_{\psi}$, then for any $e_n = o(1)$,

$$\pi\left[|\alpha - 1| < e_n|\mathbf{x}^n\right] = o_p(1).$$

Note that the phase transition on the behaviour of the posterior distribution is $a_2 < d_{\psi}$ versus $a_2 > d_{\psi}$, which is not quite the same as in Rousseau and Mengersen (2011).

Theorems 2 and 1 imply that testing decisions can be taken based on the posterior distribution of $1 - \alpha$ when $a_2 < d_{\psi}$. Indeed, in this case if one considers a testing approach of the form: H_0 is rejected if $\pi(1 - \alpha > M_n/\sqrt{n}|\mathbf{x}^n) \ge 1/2$ for some sequence M_n large or increasing to infinity, then this testing procedure is consistent under both the null and the alternative.

In contrast to the Bayes factor which converges to 0 under the alternative model \mathfrak{M}_2 exponentially quickly, the convergence rate of α to $\alpha^* \neq 1$ is of order $1/\sqrt{n}$. However this does not mean that the separation rate of the procedure based on the mixture model is worse than that of the Bayes factor. On the contrary, while it is well known that the Bayes factor leads to a separation rate of order $\sqrt{\log n}/\sqrt{n}$ in parametric models, we show in the following theorem that our approach can lead to a testing procedure with a better separation rate of order $1/\sqrt{n}$.

To prove the following result we need to strengthen slightly assumption B3:

B4 second order identifiability condition:

Set $D_{\psi}^2 f_{2,\theta_1,0}$ as the second derivative of $f_{2,\theta_1,\psi}$ with respect to ψ calculated at $\theta = (\theta_1,0)$ Then for all $\theta_1 \in \Theta_1$ if $\eta_1 \in \mathbb{R}^{d_1}, \eta_2, \eta_3 \in \mathbb{R}^{d_{\psi}}$

(9)
$$\eta_1^{\mathrm{T}} \nabla_{\theta_1} f_{1,\theta_1} + \eta_2^{\mathrm{T}} \nabla_{\psi} f_{2,\theta_1,0}(x) + \eta_3^{\mathrm{T}} D_{\psi}^2 f_{2,\theta_1,0} \eta_3 = 0 \quad \Leftrightarrow \eta_1 = 0, \, \eta_2 = \eta_3 = 0$$

Note that condition B4 is very similar to the strong identifiability condition of Rousseau and Mengersen (2011).

Theorem 3 Given the model

$$f_{\theta,\alpha} = f_{\theta_1,\psi,\alpha} = \alpha f_{1,\theta_1} + (1-\alpha) f_{2,\theta_1,\psi}, \quad \theta = (\theta_1,\psi)$$

assume that the data comprise the n sample $\mathbf{x}^n = (x_1, \dots, x_n)$ issued from $f_n^* = f_{2,\theta_{1,n},\psi_n}$ for some some sequence $\theta_{1,n} \in \Theta_1$ and $\psi_n \in \mathcal{S}$ Let assumptions B1 - B4 be satisfied. Moreover if the prior $\pi_{\theta_1,\psi}$ is absolutely continuous with positive and continuous density on Θ and if the prior π_{α} on α is a Beta $\mathcal{B}(a_1,a_2)$ distribution then there exists M' > 0 such that

$$\sup_{\theta_{1,n}\in\Theta_1, \|\psi_n\|\geq M_n/\sqrt{n}} E_{\theta_{1,n},\psi_n} \pi \left[|\alpha-1| \leq M' M_n^2 / \sqrt{n} |\mathbf{x}^n| \right] = o(1)$$

for any sequence M_n going to infinity such that $M_n^2 = o(\sqrt{n})$.

Theorem 3 implies in particular that if the testing procedure is: H_0 is rejected as soon as $\pi(1-\alpha > M_0/\sqrt{n}|\mathbf{x}^n) \leq 1/2$ with M_0 an arbitrarily large constant then the separation rate is of order $\sqrt{M_0}/\sqrt{n}$. Although Theorem 3 holds for any value of a_2 and d_{ψ} , for the testing procedure to make sense one needs to choose $a_2 < d_{\psi}$, since, otherwise, for any $e_n = o(1)$, the posterior distribution $\pi(1-\alpha > e_n|\mathbf{x}^n) = o_p(1)$ under H_0 . Calibrating the procedure by a prior predictive approach under both H_0 and H_1 will lead to a consistent testing procedure.

4. ILLUSTRATIONS

In this Section, we present three further examples that demonstrate the performance of the mixture estimation approach and provide confirmation of the consistency results obtained in Section 3. The first follows from the example given in Section 2.2 and is a direct application of Theorem 1. The second is cast in a nonparametric setting and is an application of Theorem 2. The third example is a case study that illustrates the hypothesis testing approach in a regression setup.

Example 4.1 Inspired by Marin et al. (2014), we oppose the Normal $\mathcal{N}(\mu, 1)$ model to the double-exponential $\mathcal{L}(\mu, \sqrt{2})$ model. The scale $\sqrt{2}$ is intentionally chosen to make both distributions share the same variance. As in the Normal case of Section 2.2, the location parameter μ can be shared by both models and allows for the use of the flat Jeffreys' prior. As in the example in Section 2.2, Beta distributions $\mathcal{B}(a_0, a_0)$ are compared with respect to their hyperparameter a_0 . However, whereas in the previous example we illustrated that the posterior distribution of the weight of the true model converged to 1, we now consider a setting in which neither model is correct. We achieve this feature by using a $\mathcal{N}(0, .7^2)$ distribution to simulate the data as it corresponds to neither model \mathfrak{M}_1 nor to model \mathfrak{M}_2 . In this specific case, both posterior means and medians of α fail to concentrate near 0 and 1 as the sample size increases, as shown in Figure 3. Thus in the majority of cases in this experiment, the outcome indicates that neither of both models is favored by the data. This example does not exactly follow the assumptions of Theorem 1 since the Laplace distribution is not differentiable everywhere. However, it is both almost surely differentiable and differentiable in quadratic mean, so we expect to see the same types of behaviour as predicted by Theorem 1.

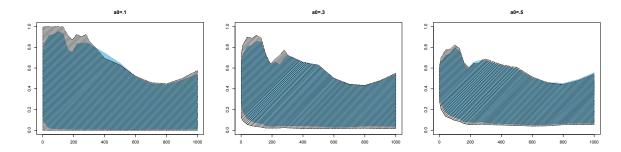


Fig 3: **Example 4.1:** Ranges of posterior means (skyblue) and medians (dotted) of the weight α of model $\mathcal{N}(\theta, 1)$ over 100 $\mathcal{N}(0, .7^2)$ datasets for sample sizes from 1 to 1000. Each estimate is based on a Beta prior with $a_0 = .1, .3, .5$, respectively, and 10^4 MCMC iterations.

In this example, the Bayes factor associated with Jeffreys' prior is defined as

$$\mathfrak{B}_{12} = \frac{\exp\left\{-\sum_{i=1}^{n} (x_i - \bar{x})^2 / 2\right\}}{(\sqrt{2\pi})^{n-1} \sqrt{n}} / \int_{-\infty}^{\infty} \frac{\exp\left\{-\sum_{i=1}^{n} |x_i - \mu| / \sqrt{2}\right\}}{(2\sqrt{2})^n} d\mu$$

where the denominator is available in closed form. As above, since the prior is improper, it is formally undefined, even though the classical Bayesian approach argues in favour of using the same prior on both μ 's. Nonetheless, we employ it in order to compare Bayes estimators of α with the posterior probability of the model being a $\mathcal{N}(\mu, 1)$ distribution. Based on a Monte Carlo experiment involving 100 replicas of a $\mathcal{N}(0, .7^2)$ dataset, Figure 4 demonstrates the reluctance of the estimates of α to approach 0 or 1, while $\mathbb{P}(\mathfrak{M}_1|\mathbf{x})$ varies over the whole range between 0 and 1 for all sample sizes considered here. While this is a weakly informative indication,

the right hand side of Figure 4 shows that, on average, the posterior estimates of α converge toward a value between .1 and .4 for all a_0 while the posterior probabilities converge to .6. In this respect, both criteria offer a similar interpretation about the data because neither α nor $P(\mathfrak{M}_1|x)$ provide definitive support for either model.

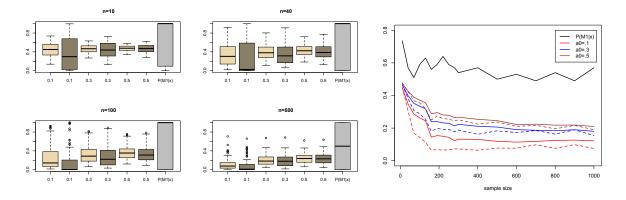


Fig 4: Example 4.1: (left) Boxplot of the posterior means (wheat) and medians (dark wheat) of α , and of the posterior probabilities of model $\mathcal{N}(\mu, 1)$ over 100 $\mathcal{N}(0, .7^2)$ datasets for sample sizes n = 10, 40, 100, 500; (right) averages of the posterior means and posterior medians of α against the posterior probabilities $\mathbb{P}(\mathfrak{M}_1|\mathbf{x})$ for sample sizes going from 1 to 1000. Each posterior approximation is based on 10^4 Metropolis-Hastings iterations.

Example 4.2 In this example we investigate a nonparametric goodness-of-fit problem of testing if the data come from a Gaussian distribution or not. We represent non Gaussian distributions as nonparametric mixtures of Gaussian distributions so that our encompassing model becomes (with an abuse of notations)

$$\mathfrak{M}_{\alpha}: \ \alpha \mathcal{N}(\mu_1, \sigma_1^2) + (1 - \alpha) \int_{\mathbb{R}} \mathcal{N}(\mu, \sigma_1^2) dP(\mu)$$

where we consider a prior distribution on (μ_1, σ_1^2, P) defined by

$$\mu_1 | \sigma_0^2 \sim \mathcal{N}(0, \tau^2 \sigma_1^2), \quad \sigma_1^2 \sim IG(b_1, b_2), \quad P \sim DP(M, \mathcal{N}(0, \sigma_1^2 \tau^2))$$

where DP(M,G) denotes the Dirichlet process with base measure MG and $IG(b_1,b_2)$ the inverse Gamma distribution with parameter (b_1,b_2) . This model defines a standard nonparametric prior distribution on the density f of the observations.

Although the model does not follow the theory developed in Section 3 since it is restricted to the parametric case, the general theory on nonparametric mixture models implies that the posterior distribution on f concentrates under \mathfrak{M}_{α} around the true density in Hellinger or L_1 ; see, for instance, Kruijer et al. (2010) or Ghosal and van der Vaart (2007). This implies that if the true distribution with density f_0 is not Gaussian, i.e.

$$\inf_{\mu,\sigma} \|f_0 - \varphi_{\mu,\sigma}\|_1 = \delta > 0,$$

where $\varphi_{\mu,\sigma}$ is the density of a $\mathcal{N}(\mu,\sigma^2)$ random variable, then the posterior probability $\Pi(\alpha > 1 - \delta/2 - \epsilon | \mathbf{x}^n)$ for all $\epsilon > 0$, goes to 0 almost surely under f_0 . This is a consequence of

$$||f_0 - \alpha \varphi_{\mu_1, \sigma_1} + (1 - \alpha) \int_{\mathbb{R}} \varphi_{\mu, \sigma_1} dP(\mu)||_1 \ge ||f_0 - \alpha \varphi_{\mu_1, \sigma_1}||_1 - (1 - \alpha) \ge ||f_0 - \varphi_{\mu_1, \sigma_1}||_1 - 2(1 - \alpha).$$

The convergence under $f_0 = \varphi_{\mu_0,\sigma_0}$ is more intricate but the following heuristic argument gives us some hints on how to choose the hyperparameters: using Scricciolo (2011) we find that the posterior distribution concentrates around f_0 at the rate $\sqrt{\log n}/\sqrt{n}$. In Nguyen (2013), it is proved that for nonparametric location mixtures the posterior distribution on the mixing density is Wasserstein consistent. Here the model is a location mixture of Gaussians, but the common scale is also unknown, and we conjecture the result of Nguyen (2013) still holds in our case. Hence assuming that the posterior distribution of $Q_{\alpha} = (\alpha \delta_{(\mu_1)} + (1 - \alpha)P) \times \delta_{(\sigma_1)}$ converges in L_2 -Wasserstein distance to $\delta_{(\mu_0)} \times \delta_{(\sigma_0)}$, we consider a Taylor expansion and we obtain (see Nguyen (2013) and Rousseau and Mengersen (2011))

$$(\log n/n)^{1/2} \gtrsim \|\varphi_{\mu_0,\sigma_0} - \alpha \varphi_{\mu_1,\sigma_1} + (1-\alpha) \int_{\mathbb{R}} \varphi_{\mu,\sigma_1} dP(\mu) \|_1$$

$$= \left\| \frac{1}{2} \left(L_{\mu}^{"} [E_{Q_{\alpha}}(\mu - \mu_0)^2 + 2\sigma_0(\sigma_1 - \sigma_0)] + (\sigma_1 - \sigma_0)^2 L_{\sigma,\sigma}^{"} + 2(\bar{\mu} - \mu_0)(\sigma_1 - \sigma_0) L_{\sigma,\mu}^{"} \right) + (\bar{\mu} - \mu_0) \nabla_{\mu} \varphi + o(u_n) \|_1$$

where $\bar{\mu}=E_{Q_{\alpha}}(\mu)$, $u_n=|\bar{\mu}-\mu_1|+|E_{Q_{\alpha}}(\mu-\mu_0)^2+2\sigma_0(\sigma_1-\sigma_0)|+(\sigma_1-\sigma_0)^2$ and L_{μ}^n , $L_{\sigma,\mu}^n$ and $L_{\sigma,\sigma}^n$) are the second derivative of $\varphi_{\mu,\sigma}$ with respect to μ , (μ,σ) and σ respectively. By linear independence, this leads to $|u_n|\lesssim \sqrt{\log n/n}$. In particular, the prior mass of this event if $1-\alpha<\epsilon$ is bounded by a term of order $(\log n/n)^{1+M_0(1-e)/4+(a_0\wedge 1/2)/4}$ for any 1>e>0, which is $o(n^{-1-a_0/2})$ as soon as $M_0+a_0\wedge 1/2>2a_0$. Hence, using the same argument as in Rousseau and Mengersen (2011) under the Gaussian model the posterior distribution on α will concentrate around 1.

This reasoning leads us to consider hyperparameters satisfying $M_0 + a_0 \wedge 1/2 > 2a_0$, for instance: $a_0 = 1, M_0 > 3/2$ or $a_0 = 1/2$ and $M_0 = 1$. We implemented an MCMC algorithm using a marginal representation for the mixture, that is, integrating out the parameters μ_1, σ_1^2 and P and sampling purely α and the allocation random variables in the data augmentation scheme. The output of this implementation is illustrated in Figure 5 for both Normal and non-Normal (t distributed) samples, showing a departure away from $\alpha = 1$ for the later, the slower the decrease the larger the degree of freedom.

Example 4.3 In this last example we demonstrate that the theory and methodology corresponding to Theorem 1 can be extended to the regression case under the assumption that the design is random. We consider a binary response setup, using the R dataset about diabetes in Pima Indian women (R Development Core Team, 2006) as a benchmark (as in Marin and Robert, 2007). The dataset contains a random sample of 200 women tested for diabetes according to WHO criteria. The response variable y is "Yes" or "No", for presence or absence of diabetes and the explanatory variable x is restricted here to the body mass index (bmi) weight in kg/(height in m)². For this problem, either logistic or probit regression models could be suitable, so we compare these fits via our method. If $y = (y_1 \ y_2 \dots y_n)$ is the vector of binary responses and $X = [I_n \ x_1]$ is the $n \times 2$ matrix of corresponding explanatory variables, the models in competition can be defined as $(i = 1, \dots, n)$

(10)
$$\mathfrak{M}_{1}: y_{i} \mid \mathbf{x}^{i}, \theta_{1} \sim \mathcal{B}(1, p_{i}) \quad \text{where} \quad p_{i} = \frac{\exp(\mathbf{x}^{i}\theta_{1})}{1 + \exp(\mathbf{x}^{i}\theta_{1})}$$
$$\mathfrak{M}_{2}: y_{i} \mid \mathbf{x}^{i}, \theta_{2} \sim \mathcal{B}(1, q_{i}) \quad \text{where} \quad q_{i} = \Phi(\mathbf{x}^{i}\theta_{2})$$

where $\mathbf{x}^i = (1 \ x_{i1})$ is the vector of explanatory variables and where θ_j , j = 1, 2, is a 2×1 vector made of the intercept and of the regression coefficient under either \mathfrak{M}_1 or \mathfrak{M}_2 . We

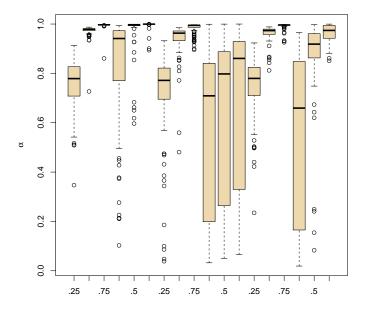


Fig 5: Example 4.2: Boxplot of the posterior 25%, 50%, and 75% quantiles of the mixture weight α of the Normal component for 100 replications of N simulations from (a) a standard normal distribution (N=100); (b) a standard normal distribution (N=1000); (c) a t_6 distribution (N=100); (d) a t_6 distribution (N=100); (e) a t_9 distribution (N=100); (g) a t_9 distribution (N=100). All values are based on $2\,10^4$ Metropolis-Hastings iterations and 100 replications of the MCMC runs.

once again consider the case where both models share the same parameter. However, for this generalised linear model there is no moment equation that relates θ_1 and θ_2 , so we adopt a local reparameterisation strategy by rescaling the parameters of the probit model \mathfrak{M}_2 so that the MLE's of both models coincide. This strategy follows from the remark by Choudhury et al. (2007) regarding the connection between the Normal cdf and a logistic function

$$\Phi(\mathbf{x}^i \theta_2) \approx \frac{\exp(k\mathbf{x}^i \theta_2)}{1 + \exp(k\mathbf{x}^i \theta_2)}$$

and we attempt to find the best estimate of k to make both parameters coherent. Given

$$(k_0, k_1) = (\widehat{\theta_{01}}/\widehat{\theta_{02}}, \widehat{\theta_{11}}/\widehat{\theta_{12}}),$$

which denote ratios of the maximum likelihood estimates of the logistic model parameters to those for the probit model, we redefine q_i in (10) as

(11)
$$q_i = \Phi(\mathbf{x}^i(\kappa^{-1}\theta)),$$

$$\kappa^{-1}\theta = (\theta_0/k_0, \theta_1/k_1).$$

Once the mixture model is thus parameterised, we set our now standard Beta $\mathcal{B}(a_0, a_0)$ on the weight of \mathfrak{M}_1 , α , and choose the default g-prior on the regression parameter (see, e.g., Marin and Robert, 2007, Chapter 4), so that

$$\theta \sim \mathcal{N}_2(0, n(X^{\mathrm{T}}X)^{-1}).$$

Table 1
Dataset Pima.tr: Posterior medians of the mixture model parameters.

	Logis	tic model parameters	Probi	t model parameters
α	θ_0	$ heta_1$	$\frac{\theta_0}{k_0}$	$\frac{\theta_1}{k_1}$
.352	-4.06	.103	-2.51	.064
.427	-4.03	.103	-2.49	.064
.440	-4.02	.102	-2.49	.063
.456	-4.01	.102	-2.48	.063
.449	-4.05	.103	-2.51	.064
	.352 .427 .440 .456	$\begin{array}{ccc} \alpha & \theta_0 \\ .352 & -4.06 \\ .427 & -4.03 \\ .440 & -4.02 \\ .456 & -4.01 \end{array}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$.352 -4.06 .103 -2.51 .427 -4.03 .103 -2.49 .440 -4.02 .102 -2.49 .456 -4.01 .102 -2.48

Table 2

Simulated dataset: Posterior medians of the mixture model parameters.

	\mathfrak{M}^1_lpha					\mathfrak{M}^2_lpha				
True model:	logistic with $\theta_1 = (5, 1.5)$					probit with $\theta_2 = (3.5, .8)$				
a_0	α	θ_0	$ heta_1$	$\frac{\theta_0}{k_0}$	$\frac{\theta_1}{k_1}$	α	θ_0	$ heta_1$	$\frac{\theta_0}{k_0}$	$\frac{\theta_1}{k_1}$
.1	.998	4.940	1.480	2.460	.640	.003	7.617	1.777	3.547	.786
.2	.972	4.935	1.490	2.459	.650	.039	7.606	1.778	3.542	.787
.3	.918	4.942	1.484	2.463	.646	.088	7.624	1.781	3.550	.788
.4	.872	4.945	1.485	2.464	.646	.141	7.616	1.791	3.547	.792
.5	.836	4.947	1.489	2.465	.648	.186	7.596	1.782	3.537	.788

In a Gibbs representation (not implemented here), the full conditional posterior distributions given the allocation vector ζ are $\alpha \sim \mathcal{B}(a_0 + n_1, a_0 + n_2)$ and

(12)
$$\pi(\theta \mid \mathbf{y}, X, \zeta) \propto \frac{\exp\left\{\sum_{i} \mathbb{I}_{\zeta_{i}=1} y_{i} \mathbf{x}^{i} \theta\right\}}{\prod_{i;\zeta_{i}=1} [1 + \exp(\mathbf{x}^{i} \theta)]} \exp\left\{-\theta^{T}(X^{T} X) \theta / 2n\right\} \times \prod_{i;\zeta_{i}=2} \Phi(\mathbf{x}^{i} (\kappa^{-1} \theta))^{y_{i}} (1 - \Phi(\mathbf{x}^{i} (\kappa^{-1} \theta)))^{(1-y_{i})}$$

where n_1 and n_2 are the number of observations allocated to the logistic and probit models, respectively. This conditional representation shows that the posterior distribution is then clearly defined, which is obvious when considering that the chosen prior is proper.

For the Pima dataset, the maximum likelihood estimates of the GLMs are $\hat{\theta}_1 = (-4.11, 0.10)$ and $\hat{\theta}_2 = (-2.54, 0.065)$, respectively, and so k = (1.616, 1.617). We compare the outcomes of this Bayesian analysis when $a_0 = .1, .2, .3, .4, .5$ in Table 1. As clearly shown in the Table, the estimates of α are close to 0.5 for all values of a_0 and the estimates of θ_0 and θ_1 are very stable (and quite similar to the MLEs). We note a slight increase of α towards 0.5 as a_0 increases, but do not want to over-interpret the phenomenon. This behaviour leads us to conclude that (a) none or both of the models are appropriate for the Pima Indian data, and (b) the sample size may be insufficiently large to allow discrimination between the logit and the probit models.

To follow up on this last remark, we ran a second experiment with simulated logit and probit datasets and a larger sample size n = 10,000. We used the regression coefficients (5,1.5) for the logit model and (3.5,.8) for the probit model. The estimates of the parameters of both \mathfrak{M}_{α_1} and \mathfrak{M}_{α_2} and for both datasets are presented in Table 2. For every a_0 , the estimates in the true model are quite close to the true values and the posterior estimates of α are close to 1 in the logit case and to 0 in the probit case. For this large setting, there is thus consistency in the selection of the proper model. In addition, Figure 6 shows that when the sample size is large enough, the posterior distribution of α concentrates its mass near 1 and 0 when the data are simulated from a logit and a probit model, respectively.

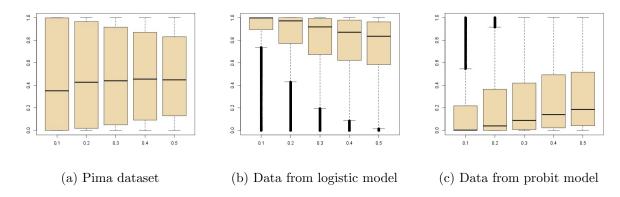


Fig 6: **Example 4.3:** Histograms of the posterior distributions of α in favor of the logistic model based on 10^4 Metropolis-Hastings iterations where $a_0 = .1, .2, .3, .4, .5$.

5. CONCLUSION

Bayesian inference has been used in a very wide and increasing range of contexts over the past thirty years, and many of the applications of the Bayesian paradigm have concentrated on comparing scientific theories and testing hypotheses. Due to the ever increasing complexity of the statistical models handled in such applications, the natural and understandable tendency of practitioners has been to rely on the default solution of the posterior probability (or equivalently of the Bayes factor) without fully understanding the sensitivity of these methods to both prior modeling and posterior calibration (Robert et al., 2011). In this area, objective Bayes solutions remain tentative and have not reached consensus.

The novel approach we have proposed here for Bayesian testing of hypotheses and Bayesian model comparison offers in our opinion many incentives over these established methods. By casting the problem as an encompassing mixture model, not only do we replace the original testing problem with a better controlled estimation target that focuses on the frequency of a given model within the mixture model, but we also allow for posterior variability of this frequency. The posterior distribution of the weights of both components in the mixture offers a setting for deciding about which model is most favored by the data that is at least as intuitive as the sole number corresponding to either the posterior probability or the Bayes factor. The range of acceptance, rejection and indecision conclusions can easily be calibrated by simulation under both models, as well as by deciding on the values of the weights that are extreme enough in favor of one model. The examples provided in this paper have shown that the posterior medians of such weights settle very quickly near the boundary values 1. Although we do not advocate such practice, it is even possible to derive a Bayesian p-value by considering the posterior area under the tail of the distribution of the weight. Moreover, the approach does not induce additional computational strain on the analysis.

Besides decision making, another issue of potential concern about this new approach is the impact of the prior modelling. As demonstrated in our examples, a partly common parameterisation is often feasible and hence allows for reference priors, at least on the common parameters. This proposal thus allows for a partial removal of the prohibition on using improper priors in hypothesis testing (DeGroot, 1973), a problem which has plagued the objective Bayes literature for decades. Concerning the prior on the weight parameter, we analyzed the sensitivity on the resulting posterior distribution of various prior Beta modelings on those weights. While the sensitivity is clearly present, it naturally vanishes as the sample size increases, in agreement with our consistency results, and remains of a moderate magnitude. This leads us to suggest

the default value of $a_0 = 0.5$ in the Beta prior, in connection with both the earlier result of Rousseau and Mengersen (2011) and Jeffreys' prior in the simplest mixture setting.

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APPENDIX 1: PROOFS OF SECTION 3

In this Section we give the proofs of Theorems 1, 2 and 3.

Proof of Theorem 1

Using Proposition 1, we have that

$$\pi\left(A_n|\mathbf{x}^n\right) = 1 + o_p(1)$$

with $A_n = \{(\alpha, \theta); \|f_{\theta,\alpha} - f_{\theta^*,\alpha^*}\|_1 \leq \delta_n\}$ and $\delta_n = M\sqrt{\log n/n}$. Consider a subsequence $\alpha_n, P_{1,\theta_{1n}}, P_{2,\theta_{2n}}$ which converges to α, μ_1, μ_2 where convergence holds in the sense that $\alpha_n \to \alpha$ and $P_{j,\theta_{jn}}$ converges weakly to μ_j . Note that $\mu_j(\mathcal{X}) \leq 1$ by precompacity of the unit ball under the weak topology. At the limit

$$\alpha \mu_1 + (1 - \alpha)\mu_2 = \alpha^* P_{1,\theta_1^*} + (1 - \alpha^*) P_{2,\theta_2^*}$$

The above equality implies that μ_1 and μ_2 are probabilities. Using (4), we obtain that

$$\alpha = \alpha^*, \quad \mu_j = P_{j,\theta_i^*},$$

which implies posterior consistency for α . The proof of (5) follows the same line as in Rousseau and Mengersen (2011). Consider first the case where $\alpha^* \in (0,1)$. Then the posterior distribution on θ concentrates around θ^* .

Writing

$$L' = (f_{1,\theta_1^*} - f_{2,\theta_2^*}, \alpha^* \nabla f_{1,\theta_1^*}, (1 - \alpha^*) \nabla f_{2,\theta_2^*}) := (L_{\alpha}, L_1, L_2)$$

$$L'' = \operatorname{diag}(0, \alpha^* D^2 f_{1,\theta^*}, (1 - \alpha^*) D^2 f_{2,\theta^*})$$
 and $\eta = (\alpha - \alpha^*, \theta_1 - \theta_1^*, \theta_2 - \theta_2^*), \quad \omega = \eta/|\eta|,$

we then have

(13)
$$||f_{\theta,\alpha} - f_{\theta^*,\alpha^*}||_1 = |\eta| \left| \omega^{\mathrm{T}} L' + |\eta| / 2\omega^{\mathrm{T}} L'' \omega + |\eta| \omega_1 \left[\omega_2^T L_1 - \omega_3^T L_2 \right] + o(|\eta|) \right|$$

For all $(\alpha, \theta) \in A_n$, $\eta = (\alpha - \alpha^*, \theta_1 - \theta_1^*, \theta_2 - \theta_2^*)$ goes to 0 and for n large enough there exists $\epsilon > 0$ such that $|\alpha - \alpha^*| + |\theta - \theta^*| \le \epsilon$. We now prove that there exists c > 0 such that for all $(\alpha, \theta) \in A_n$

$$v(\omega) = \left| \omega^{\mathrm{T}} L' + \frac{|\eta|}{2} \omega^{\mathrm{T}} L^{"} \omega + |\eta| \omega_1 \left[\omega_2^{\mathrm{T}} L_2' + \omega_3^{\mathrm{T}} L_3' \right] + o(|\eta|) \right| > c,$$

where ω is defined with respect to α, θ . Were it not the case, there would exist a sequence $(\alpha_n, \theta_n) \in A_n$ such that the associated $v(\omega_n) \leq c_n$ with $c_n = o(1)$. As ω_n belongs to a compact set we could find a subsequence converging to a point $\bar{\omega}$. At the limit we would obtain

$$\bar{\omega}^{\mathrm{T}}L^{'}=0$$

and by linear independence $\bar{\omega} = 0$ which is not possible. Thus for all $(\alpha, \theta) \in A_n$

$$|\alpha - \alpha^*| + |\theta - \theta^*| \leq \delta_n$$
.

Assume now instead that $\alpha^* = 0$. Then define $L' = (L_{\alpha}, L_2)$ and

$$L'' = \operatorname{diag}(0, D^2 f_{2,\theta_2^*})$$
 and $\eta = (\alpha - \alpha^*, \theta_2 - \theta_2^*), \quad \omega = \eta/|\eta|$

and consider a Taylor expansion with θ_1 fixed, $\theta_1^* = \theta_1$ and $|\eta|$ going to 0. This leads to

(14)
$$||f_{\theta,\alpha} - f_{\alpha^*,\theta^*}||_1 = |\eta| \left| \omega^{\mathrm{T}} L' + \frac{|\eta|}{2} \omega^{\mathrm{T}} L'' \omega - |\eta| \omega_1 \omega_3 L_2 \right| + o(|\eta|)$$

in place of (13) and using the same argument as in the case $\alpha^* \in (0,1)$,

$$|\alpha - \alpha^*| + |\theta - \theta^*| \lesssim \delta_n$$
.

Proof of Theorem 2

Recall that $f^* = f_{1,\theta_1^*}$. To prove (8) we must first find a precise lower bound on

$$D_n := \int_{\Omega} \int_{\Theta} e^{l_n(f_{\theta,\alpha}) - l_n(f^*)} d\pi_{\theta}(\theta) d\pi_{\alpha}(\alpha)$$

Consider the approximating set

$$S_n(\epsilon) = \{(\theta, \alpha), \alpha > 1 - 1/\sqrt{n}, |\theta_1 - \theta_1^*| \le 1/\sqrt{n}, |\psi - \bar{\psi}| \le \epsilon\}, \quad \theta = (\theta_1, \theta_2)$$

with $|\bar{\psi}| > 2\epsilon$ some fixed parameter in \mathcal{S} . Using the same computations as in Rousseau and Mengersen (2011), it holds that for all $\delta > 0$ there exists $C_{\delta} > 0$ such that

(15)
$$P^* \left(D_n < e^{-C_\delta} \pi(S_n(\epsilon))/2 \right) < \delta.$$

So that with probability greater than $1 - \delta$, $D_n \gtrsim n^{-(a_2+d_1)/2}$. Denote $B_n = \{(\theta, \alpha); \|f_{\theta,\alpha} - f^*\|_1 \le M_n/\sqrt{n}\}$, we know that

$$\pi(\|f_{\theta,\alpha} - f^*\|_1 \le M_n \sqrt{\log n} / \sqrt{n} |\mathbf{x}^n|) = o_p(1).$$

Let $M_n \leq j \leq M_n \sqrt{\log n}$ and consider the slice $S_n(j) = \{j/\sqrt{n} \leq \|f_{\theta,\alpha} - f^*\|_1 \leq (j+1)/\sqrt{n}\}$. We now upper bound $\pi(S_n(j))$. To do so we split the parameter space into $\alpha < 1 - \delta$ for a fixed arbitrarily small δ , and $\alpha > 1 - \delta$. In the first case, we have that ψ converges to 0 and θ_1 to θ_1^* . A Taylor expansion leads to

(16)
$$\begin{aligned} \|\alpha f_{1,\theta_1} + (1-\alpha) f_{2,\theta_1,\psi} - f_{1,\theta_1^*}\|_1 \\ &= \|(\theta_1 - \theta_1^*)^T \nabla_{\theta} f_{1,\theta_1^*} + (1-\alpha) \psi^T \nabla_{\psi} f_{2,\theta_1^*,0}\|_1 + O(\|\theta_1 - \theta_1^*\|^2 + (1-\alpha)\|\psi\|^2) \end{aligned}$$

Setting $v_n = \|\theta_1 - \theta_1^*\| + \|(1 - \alpha)\psi\|$ and $\eta = (\theta_1 - \theta_1^*, (1 - \alpha)\psi)/v_n$, (16) implies that

$$\|\alpha f_{1,\theta_1} + (1-\alpha)f_{2,\theta_1,\psi} - f_{1,\theta_1^*}\|_1 \ge v_n \|\eta^T \nabla f_{2,\theta_1^*,0}\|_1 + O(v_n^2)$$

and by linear independence of $\nabla f_{2,\theta_1,\psi}$ we obtain that $v_n \leq Cj/\sqrt{n}$ on $S_n(j) \cap \{\alpha \in (0,1-\delta)\}$ and

$$\pi_{n,1} := \pi(S_n(j) \cap \{\alpha \in (0, 1 - \delta)\}) \lesssim j^{d_2} n^{-d_2/2}.$$

Now consider $1 - \alpha \le \delta$. If $1 - \alpha \le Mj/\sqrt{n}$ then $\|\theta_1 - \theta_1^*\| \lesssim j/\sqrt{n}$ which has prior probability bounded by $O((j/\sqrt{n})^{d_1+a_2})$. If $1 - \alpha > Mj/\sqrt{n}$, then ψ goes to 0 and (16) implies that

$$\|\theta_1 - \theta_1^*\| + \|(1 - \alpha)\psi\| \lesssim j/\sqrt{n}$$

which in turns implies that

$$\pi(S_n(j)) \lesssim j^{d_2} n^{-d_2/2} + (j/\sqrt{n})^{d_1 + a_2} + j^{d_2} n^{-d_2/2} \int_{Mj/\sqrt{n}}^{\delta} u^{a_2 - d_{\psi} - 1} du \lesssim j^{d_2} n^{-d_2/2} + (j/\sqrt{n})^{d_1 + a_2} + j^{d_2} n^{-d_2/2} \int_{Mj/\sqrt{n}}^{\delta} u^{a_2 - d_{\psi} - 1} du \lesssim j^{d_2} n^{-d_2/2} + (j/\sqrt{n})^{d_1 + a_2} + j^{d_2} n^{-d_2/2} \int_{Mj/\sqrt{n}}^{\delta} u^{a_2 - d_{\psi} - 1} du \lesssim j^{d_2} n^{-d_2/2} + (j/\sqrt{n})^{d_1 + a_2} + j^{d_2} n^{-d_2/2} \int_{Mj/\sqrt{n}}^{\delta} u^{a_2 - d_{\psi} - 1} du \lesssim j^{d_2} n^{-d_2/2} + (j/\sqrt{n})^{d_1 + a_2} + j^{d_2} n^{-d_2/2} \int_{Mj/\sqrt{n}}^{\delta} u^{a_2 - d_{\psi} - 1} du \lesssim j^{d_2} n^{-d_2/2} + (j/\sqrt{n})^{d_1 + a_2} + + (j/\sqrt{n})^{d_1 + a_$$

Since $a_2 \leq d_{\psi}$,

(17)
$$\frac{\pi(S_n(j))}{\pi(S_n(\epsilon))} \lesssim j^{d_{\psi} + a_2}.$$

Write $S_{\epsilon} = \{\epsilon \leq \|f_{\theta,\alpha} - f^*\|_1 \leq 2\epsilon\}$ Equation (16) implies that if $M_n \sqrt{\log n} / \sqrt{n} \geq \epsilon \geq M_n / \sqrt{n}$, $S_{\epsilon} \subset \{\|\theta_1 - \theta_1^*\| \leq \tau_1 \epsilon\} \cap \{\|(1 - \alpha)\psi\| \leq \tau_1 \epsilon\}$ for some $\tau_1 > 0$. To cover S_{ϵ} by L_1 balls of radius $\epsilon/2$ we consider $\|\theta_1 - \theta_1'\| \leq \zeta \epsilon$ and we split the set into $1 - \alpha \leq \epsilon$ and $1 - \alpha > \epsilon$. Note that by choosing ζ small enough,

$$||f_{\theta,\alpha} - f_{\theta',\alpha'}||_1 \le ||f_{\theta_1,\psi,\alpha} - f_{\theta_1,\psi',\alpha'}||_1 + \epsilon/16$$

= $||(1-\alpha)(f_{2,\theta_1,\psi} - f_{2,\theta_1,0}) - (1-\alpha')(f_{2,\theta_1,\psi'} - f_{2,\theta_1,0})||_1 + \epsilon/16$

If $1 - \alpha > \kappa \epsilon$ for some $\kappa > 0$. Then on S_{ϵ} , $\|\psi\| \leq \tau_1/\kappa$ and by choosing κ large enough,

$$||f_{\theta,\alpha} - f_{\theta',\alpha'}||_1 \le M_1 ||(1-\alpha)\psi - (1-\alpha')\psi'|| + \epsilon/8 \le \epsilon/4$$

by choosing $\|(1-\alpha)\psi - (1-\alpha')\psi'\| \le \zeta\epsilon$ with ζ small enough. If $1-\alpha \le \kappa\epsilon$, choose $|\alpha' - \alpha| \le \epsilon/16$ so that

$$||f_{\theta,\alpha} - f_{\theta',\alpha'}||_1 \le (1-\alpha)||f_{2,\theta_1,\psi} - f_{2,\theta_1,\psi'}||_1 + \epsilon/4 \le (1-\alpha)||\psi - \psi'||M_1 + \epsilon/8 \le \epsilon/2$$

by choosing $\|\psi - \psi'\| \le 1/(4M_1\kappa)$. Hence the local L_1 entropy is bounded by a constant for all $M_n\sqrt{\log n}/\sqrt{n} > \epsilon > M_n/\sqrt{n}$ and using Theorem 2.4 of (Ghosal et al., 2000), we obtain (8).

Now consider $d_{\psi} > a_2$ and let $A_n = \{(\theta, \alpha) \in B_n; 1 - \alpha > z_n/\sqrt{n}\}$ with z_n a sequence increasing to infinity faster than M_n and $B_n = \{\|f_{\theta,\alpha} - f^*\|_1 \le M_n/\sqrt{n}\}$. We prove that $\pi(A_n|\mathbf{x}^n) = o_p(1)$ by proving that $\pi(A_n) = o(n^{-(a_2+d_1)/2})$ and using the lower bound on D_n of order $n^{-(a_2+d_1)/2}$. We split B_n into

$$B_{n,1}(\delta) = B_n \cap \{(\theta, \alpha), \theta = (\theta_1, \psi); \|\psi\| < \delta\}, \quad B_{n,2}(\delta) = B_n \cap B_{n,1}(\delta)^c, \quad \delta > 0$$

To simplify notation we also write $\delta_n = M_n/\sqrt{n}$. First we prove that for all $\delta > 0$, $A_n \cap B_{n,2}(\delta) = \emptyset$, when n is large enough. Let $\delta > 0$, then for any $(\theta, \alpha) \in A_n \cap B_{n,2}(\delta)$, We thus have $\|\psi\| \neq o(1)$, $\alpha = 1 + o(1)$ and $|\theta_1 - \theta_1^*| = o(1)$. Consider a Taylor expansion of $f_{\theta,\alpha}$ around $\alpha = 1$ and $\theta_1 = \theta_1^*$, with ψ fixed. This leads to

$$\begin{split} f_{\theta,\alpha} - f^* &= (\alpha - 1)[f_{1,\theta_1^*} - f_{2,\theta_1^*,\psi}] + (\theta_1 - \theta_1^*)[\nabla_{\theta_1} f_{1,\theta_1^*} - \nabla_{\theta_1} f_{2,\theta_1^*,\psi}(x)] \\ &\quad + \frac{1}{2}(\theta_1 - \theta_1^*)^{\mathrm{T}} \left(\bar{\alpha} D_{\theta_1}^2 f_{1,\bar{\theta}_1} + (1 - \bar{\alpha}) D_{\theta_1}^2 f_{2,\bar{\theta}_1,\psi}\right) (\theta_1 - \theta_1^*) \\ &\quad + (\alpha - 1)(\theta_1 - \theta_1^*)^{\mathrm{T}} [\nabla_{\theta_1} f_{1,\bar{\theta}_1} - \nabla_{\theta_1} f_{2,\bar{\theta}_1,\psi}] \\ &= (\alpha - 1)[f_{1,\theta_1^*} - f_{2,\theta_1^*,\psi}] + (\theta_1 - \theta_1^*)[\nabla_{\theta_1} f_{1,\theta_1^*} - \nabla_{\theta_1} f_{2,\theta_1^*,\psi}(x)] + o(|\alpha - 1| + ||\theta_1 - \theta_1^*||) \end{split}$$

with $\bar{\alpha} \in (0,1)$ and $\bar{\theta}_1 \in (\theta_1, \theta_1^*)$ and the o(1) is uniform over $A_n \cap B_{n,2}(\delta)$. Set $\eta = (\alpha - 1, \theta_1 - \theta_1^*)$ and $x = \eta/|\eta|$ if $|\eta| > 0$. Then

$$||f_{\theta,\alpha} - f^*||_1 = |\eta| \left(x^{\mathrm{T}} L_1(\psi) + o(1) \right), \quad L_1 = \left(f_{1,\theta_1^*} - f_{2,\theta_1^*,\psi}, \nabla_{\theta_1} f_{1,\theta_1^*} - \nabla_{\theta_1} f_{2,\theta_1^*,\psi}(x) \right)$$

We now prove that on $A_n \cap B_{n,2}(\delta)$, $||f_{\theta,\alpha} - f^*||_1 \gtrsim |\eta|$. Assume that it is not the case; then there exist $c_n > 0$ going to 0 and a sequence $(\theta_{1,n}, \alpha_n)$ such that along that subsequence $|x_n^T L_1(\psi_n) + o(1)| \leq c_n$ with $x_n = \eta_n/|\eta_n|$. Since it belongs to a compact set, together with ψ_n , any converging subsequence satisfies at the limit $(\bar{x}, \bar{\psi})$,

$$\bar{x}^{\mathrm{T}}L_1(\bar{\psi}) = 0,$$

which is not possible. Hence $|\alpha-1| \lesssim M_n/\sqrt{n} = o(w_n/\sqrt{n})$, which is not possible so that $A_n \cap B_{n,2}(\delta) = \emptyset$ when n is large enough. We now bound $\pi(A_n \cap B_{n,1}(\delta))$ for $\delta > 0$ small enough but fixed. We consider a Taylor expansion around $\theta^* = (\theta_1^*, 0)$, leaving α fixed. Note that $\nabla_{\theta_1} f_{2,\theta^*} = \nabla_{\theta_1} f_{1,\theta_1^*}$. We have

$$f_{\theta,\alpha} - f^* = (\theta_1 - \theta_1^*)^{\mathrm{T}} \nabla_{\theta_1} f_{1,\theta_1^*} + (1 - \alpha) \psi^{\mathrm{T}} \nabla_{\psi} f_{2,\theta^*} + \frac{1}{2} (\theta - \theta^*)^{\mathrm{T}} H_{\alpha,\bar{\theta}} (\theta - \theta^*)$$

where $H_{\alpha,\bar{\theta}}$ is the block matrix

$$H_{\alpha,\bar{\theta}} = \begin{pmatrix} \alpha D_{\theta_1}^2 f_{1,\bar{\theta}_1} + (1-\alpha) D_{\theta_1,\theta_1}^2 f_{2,\bar{\theta}} & (1-\alpha) D_{\theta_1,\psi}^2 f_{2,\bar{\theta}} \\ (1-\alpha) D_{\psi,\theta_1}^2 f_{2,\bar{\theta}} & (1-\alpha) D_{\psi,\psi}^2 f_{2,\bar{\theta}} \end{pmatrix}$$

Since $H_{\alpha,\bar{\theta}}$ is bounded in L_1 (in the sense that each of its components is bounded as functions in L_1), uniformly in neighbourhoods of θ^* , we have writing $\eta = (\theta_1 - \theta_1^*, (1 - \alpha)\psi)$ and $x = \eta/|\eta|$, that $|\eta| = o(1)$ on $A_n \cap B_{n,1}(\delta)$ and

$$||f_{\theta,\alpha} - f^*||_1 \gtrsim |\eta| \left(x^{\mathrm{T}} \nabla f_{2,\theta^*} + o(1) \right),$$

if ϵ is small enough. Using a similar argument to before, this leads to $|\eta| \lesssim \delta_n$ on $A_n \cap B_{n,1}(\delta)$, so that

$$\pi \left(A_n \cap B_{n,1}(\delta) \right) \lesssim \delta_n^{d_1} \int_{z_n/\sqrt{n}}^1 (\delta_n/u)^{d_\psi} u^{a_2-1} du \lesssim \delta_n^{d_1+d_\psi} z_n^{a_2-d_\psi} n^{(d_\psi-a_2)/2} \lesssim n^{-(d_1+a_2)/2} M_n^{d_2} z_n^{a_2-d_\psi}$$
$$= o(n^{-(d_1+a_2)/2})$$

choosing $M_n = o(z_n^{(d_{\psi} - a_2)/d_2})$, going to infinity (recall that $d_{\psi} > a_2$).

Finally assume that $d_{\psi} < a_2$ and denote $C_n = \{(\theta, \alpha) \in B_n; 1 - \alpha < e_n\}$, then the sam arguments imply that if $\delta > 0$ is small enough, $C_n = C_n \cap B_{n,1}(\delta)$ and

$$\pi\left(C_n \cap B_{n,1}(\delta)\right) \lesssim \delta_n^{d_1} \int_0^{e_n} (\delta_n/u)^{d_{\psi}} u^{a_2-1} du \lesssim \delta_n^{d_1+d_{\psi}} e_n^{a_2-d_{\psi}} \lesssim n^{-d_2/2} M_n^{d_{\psi}} e_n^{a_2-d_{\psi}} = o(n^{-d_2/2})$$

if $M_n^{d_\psi} = o(e_n^{-(a_2-d_\psi)})$. Now, working with $S_n' = \{\|\theta_1 - \theta_1^*\| \le 1/\sqrt{n}, \|\psi\| \le 1/\sqrt{n}, \alpha \in (\bar{\alpha} - e_n', \bar{\alpha} + e_n') \}$ with e_n' going to 0 arbitrarily slowly, we have that with probability going to 1, $D_n \gtrsim \pi(S_n') \gtrsim n^{-d_2/2} e_n'$ so that by choosing e_n' accordingly, $\pi(C_n) = o(n^{-d_2/2}e_n')$ and Theorem 2 is proved .

Proof of Theorem 3

The proof of Theorem 3 proceeds along the same line as the previous proof. Let $f_n^* = f_{2,\theta_{1,n},\psi_n}$ with $\|\psi_n\| = o(1)$; the other case has already been proved in Theorem 1. Recall that

$$\pi \left(\|f_{\theta,\alpha} - f_n^*\|_1 \ge M_0 \sqrt{\log n} / \sqrt{n} |\mathbf{x}^n \right) = o_p(1)$$

if $M_0 > 0$ is large enough. We restrict ourselves to the case where $M_n/\sqrt{n} \le \|\psi_n\| = o(n^{-1/4})$ with since the other case can be treated more easily. We prove first that the posterior concentration rate can be sharpened into

(18)
$$\pi \left(\|f_{\theta,\alpha} - f_n^*\|_1 \ge z_n / \sqrt{n} |\mathbf{x}^n| \right) = o_p(1), \quad \text{for any} \quad z_n \to +\infty.$$

To do so we first obtain a sharp lower bound on D_n . From the regularity assumptions [B1] and [B2] for all $(\alpha, \theta_1, \psi) \in \tilde{S}_n = \{\|\theta_1 - \theta_{1,n}\| \le 1/\sqrt{n}, \|(1-\alpha)\psi - \psi_n\| + \|\psi_n\| \|\psi\| \le 1/\sqrt{n}\}$ and a Taylor expansion with $\theta = (\theta_1, \psi)$ around $\theta_{1,n}$ and 0 (both for ψ and ψ_n) we bound,

$$KL(f_n^*, f_{\theta,\alpha}) \le \int \frac{(f_{\theta_{1,n},\psi_n} - f_{\theta,\alpha})^2}{f_{\theta_{1,n},\psi_n}}(x)dx$$

$$\lesssim \|\theta - \theta_{1,n}\|_2^2 + \|(1 - \alpha)\psi - \psi_n\|^2 + (1 - \alpha)^2 \|\psi\|^2 \|\psi - \psi_n\|^2$$

$$\lesssim \|\theta - \theta_{1,n}\|_2^2 + \|(1 - \alpha)\psi - \psi_n\|^2 + \|\psi_n\| \|\psi\|$$

with a similar inequality for $\int f_n^*(\log f_n^* - \log f_{\theta,\alpha})^2(x)dx$. Hence, with probability bounded by ϵ ,

$$D_n \ge e^{-C_\epsilon} \pi(\tilde{S}_n),$$

for some large positive constant C_{ϵ} . We have the following lower bound on $\pi(\tilde{S}_n)$. Note that $\|(1-\alpha)\psi - \psi_n\| + \|\psi_n\| \|\psi\| \le 1/\sqrt{n}$ implies that $1-\alpha \ge 2\sqrt{n} \|\psi_n\|^2$.

(19)
$$\pi\left(\tilde{S}_n\right) \gtrsim n^{-d_1/2} n^{-d_{\psi}/2} \int_{2\sqrt{n}\|\psi_n\|^2}^{\delta} u^{a_2 - d_{\psi} - 1} du \gtrsim n^{-d_2/2} (\sqrt{n}\|\psi_n\|^2)^{-(d_{\psi} - a_2)_+}.$$

We now bound from above $\pi(S_n(j))$ and control the entropy of $S_n(j)$ for neighbourhoods $S_n(j) = \{j/\sqrt{n} \le ||f_{\theta_1,\psi,\alpha} - f_n^*||_1 \le (j+1)/\sqrt{n}\}.$

We have on $S_n(j)$:

$$\|\alpha f_{1,\theta_1} + (1-\alpha)f_{2,\theta_1,\psi} - f_{2,\theta_1,n},\psi_n\|_1 \le (j+1)/\sqrt{n}$$

We split this set into two subsets $\alpha < 1 - \delta$ for a fixed arbitrarily small δ and $\alpha \ge 1 - \delta$. In the first case, we have as a first approximation

$$\|\alpha f_{1,\theta_1} + (1-\alpha)f_{2,\theta_1,\psi} - f_{1,\theta_{1,n}}\|_1 \lesssim \|\psi_n\| + j/\sqrt{n}$$

which implies in turn that $\|\theta_1 - \theta_{1,n}\| \lesssim \|\psi_n\| + j/\sqrt{n}$ and $\|\psi\| \lesssim \|\psi_n\| + j/\sqrt{n}$ where the constants depend on δ . A more refined Taylor expansion then leads to

$$\|\alpha f_{1,\theta_1} + (1-\alpha) f_{2,\theta_1,\psi} - f_{2,\theta_1,n,\psi_n}\|_1$$

$$= \|(\theta_1 - \theta_n)^T (\nabla_{\theta} f_{1,\theta_{1,n}} + o(1)) + ((1-\alpha)\psi - \psi_n)^T \nabla_{\psi} f_{2,\theta_{1,n},0} + (1-\alpha)(\psi - \psi_n)^T (D_{\psi}^2 f_{2,\theta_{1,n},0} + o(1))(\psi - \psi_n)\|_1$$

Setting $v_n = \|\theta_1 - \theta_n\| + \|(1 - \alpha)\psi - \psi_n\|$ and $\eta = (\theta_1 - \theta_{1,n}, (1 - \alpha)\psi - \psi_n)/v_n$, (20) implies that

$$\|\alpha f_{1,\theta_1} + (1-\alpha)f_{2,\theta_1,\psi} - f_{2,\theta_1,\eta,\psi_n}\|_1 \ge v_n \|\eta^T (\nabla f_{2,\theta_1,\eta,0} + o(1))\|_1 + O(\|\psi_n\|^2 + v_n^2)$$

and by linear independence of $\nabla f_{2,\theta_1,0}$ (assumption B4), $v_n \leq Cj/\sqrt{n}$ on $S_n(j)$ and

$$\pi(S_n(j) \cap \{\alpha \le 1 - \delta\}) \lesssim j^{d_2} n^{-d_2/2}.$$

We now consider the last case: $\alpha > 1 - \delta$. As before, a first crude approximation leads to $||f_{1,\theta_1} - f_{2,\theta_{1,n},\psi_n}||_1 \lesssim 1 - \alpha + j/\sqrt{n}$, which in turns implies that $||\theta_1 - \theta_{1,n}|| + ||\psi_n|| \lesssim 1 - \alpha + j/\sqrt{n}$. In particular if $j/\sqrt{n} = o(||\psi_n||)$, then $1 - \alpha \gtrsim ||\psi_n||$.

Consider first the case where $j/\sqrt{n} \gtrsim ||\psi_n||$. Then

(21)
$$\pi(S_n(j) \cap \{1 - \alpha \le \kappa j / \sqrt{n}\}) \lesssim (j / \sqrt{n})^{d_1 + a_2}.$$

Moreover

$$\|\alpha f_{1,\theta_1} + (1-\alpha)f_{2,\theta_1,\psi} - f_{2,\theta_1,\eta,\psi_n}\|_1 \le \|\alpha f_{1,\theta} + (1-\alpha)f_{2,\theta_1,\psi} - f_{1,\theta_1,\eta}\|_1 + O(\|\psi_n\|)$$

so that when $a_2 \leq d_{\psi}$, (17) holds. When $a_2 > d_{\psi}$, the proof of Theorem 2 implies that there exists C > 0 such that

$$S_n(j) \cap \{1 - \alpha > j/\sqrt{n}\} \subset \{\|\theta_1 - \theta_{1,n}\| + \|(1 - \alpha)\psi - \psi_n\| \le Cj/\sqrt{n}\}$$

and

(22)
$$\pi(S_n(j)) \lesssim (j/\sqrt{n})^{d_2}.$$

Now consider the case where $j/\sqrt{n} = o(\|\psi_n\|)$ and recall that $1 - \alpha \gtrsim \|\psi_n\|$ and $\|\theta_1 - \theta_{1,n}\| = o(1)$. A Taylor expansion with θ_1 around $\theta_{1,n}$ and $1 - \alpha$ around 0 holding ψ fixed and non null leads to

$$\|\alpha f_{1,\theta} + (1-\alpha)f_{2,\theta_1,\psi} - f_{2,\theta_1,n}\psi_n\|_1 = \|-\psi_n^T \nabla_{\psi} f_{2,\theta_1,n}\psi_n + (1-\alpha)(f_{2,\theta_1,n}\psi - f_{2,\theta_1,n}\psi_n) + (\theta_1 - \theta_{1,n})^T \nabla_{\theta} f_{1,\theta_1,n}\|_1 + o(\|\theta_1 - \theta_{1,n}\| + 1/n) + O((1-\alpha)\|\psi_n\|)$$

Set $\eta = (\theta_1 - \theta_{1,n}, 1 - \alpha, \psi_n)$ and $\omega = \eta/||\eta||$, then

$$\|\eta\| ((1-\alpha)(f_{2,\theta_{1,n},\psi}-f_{2,\theta_{1,n},0})+(\theta_1-\theta_{1,n})^T \nabla_{\theta} f_{1,\theta_{1,n}}) \|_1 \lesssim j/\sqrt{n}$$

so that by linear independence of $(f_{2,\theta_{1,n},\psi} - f_{2,\theta_{1,n},0})$, $\nabla_{\theta} f_{1,\theta_{1,n}}$ (assumption B3), $\|\psi_n\| \lesssim j/\sqrt{n}$ which is impossible. Therefore $\|\psi\| = o(1)$. A Taylor expansion then implies that

(23)
$$\|(\theta_{1} - \theta_{1,n})^{T} \nabla_{\theta} f_{1,\theta_{1,n}} + ((1 - \alpha)\psi - \psi_{n})^{T} \nabla_{\psi} f_{2,\theta_{1,n},0} + (1 - \alpha)(\psi - \psi_{n})^{T} D_{\psi}^{2} f_{2,\theta_{1,n},0}(\psi - \psi_{n})/2\|_{1} + o(\|\theta_{1} - \theta_{1,n}\| + (1 - \alpha)\|\psi\|^{2}) \lesssim j/\sqrt{n}$$

When $\|(1-\alpha)\psi - \psi_n\| \le \delta(1-\alpha)\|\psi - \psi_n\|^2$, with δ small, then

$$\|(\theta_1 - \theta_{1,n})^T \nabla_{\theta} f_{1,\theta_{1,n}} + (1 - \alpha)(\psi - \psi_n)^T D_{\psi}^2 f_{2,\theta_{1,n},0}(\psi - \psi_n)/2\|_1 \lesssim j/\sqrt{n},$$

set $\eta = (\theta_1 - \theta_{1.n}, \sqrt{1 - \alpha}(\psi - \psi_n))$ and $\omega = \eta/||\eta||$, assumption B4 implies that

$$\|\theta_1 - \theta_{1,n}\| + (1-\alpha)\|\psi - \psi_n\|^2 \lesssim j/\sqrt{n}, \quad \|(1-\alpha)\psi - \psi_n\| = o(j/\sqrt{n})$$

and since $1 - \alpha \le \delta$ small, $\|\psi - \psi_n\| = \|\psi\|(1 + o(1))$ so that

$$\|\psi\|\|\psi_n\| \lesssim j/\sqrt{n}$$
, and $\frac{(1-\alpha)j}{\sqrt{n}\|\psi_n\|} \gtrsim \|(1-\alpha)\psi\| = \|\psi_n\| + o(j/\sqrt{n})$, $1-\alpha \gtrsim \frac{\sqrt{n}\|\psi_n\|^2}{j}$

The prior mass of this set is bounded above by

$$\pi_{n,2} \lesssim (j/\sqrt{n})^{d_2} \left(\frac{\sqrt{n}\|\psi_n\|^2}{j}\right)^{-(d_{\psi}-a_2)_+}.$$

Similarly when $(1-\alpha)\|\psi-\psi_n\|^2 \leq \delta\|(1-\alpha)\psi-\psi_n\|$, (23) becomes

$$\|(\theta_1 - \theta_{1,n})^T \nabla_{\theta} f_{1,\theta_{1,n}} + ((1 - \alpha)\psi - \psi_n)^T \nabla_{\psi} f_{2,\theta_{1,n},0}\|_1 \lesssim j/\sqrt{n},$$

which in turns implies that $\|\theta_1 - \theta_{1,n}\| + \|(1-\alpha)\psi - \psi_n\| \lesssim j/\sqrt{n}$ and $(1-\alpha)\|\psi - \psi_n\|^2 = o(j/\sqrt{n})$ so that

$$\frac{\alpha}{1-\alpha} \|\psi_n\| \lesssim \sqrt{\frac{j}{\sqrt{n}(1-\alpha)}}$$

and $1-\alpha \gtrsim (\sqrt{n}\|\psi_n\|^2)/j$ and the prior mass of this set is bounded from above by

$$\pi_{n,3} \lesssim (j/\sqrt{n})^{d_2} (\sqrt{n} \|\psi_n\|^2)^{-(d_{\psi}-a_2)_+} j^{(d_{\psi}-a_2)_+}.$$

Finally let $\|(1-\alpha)\psi - \psi_n\| \ge \delta(1-\alpha)\|\psi - \psi_n\|^2 \ge \delta^2 \|(1-\alpha)\psi - \psi_n\|$. Set $\eta_1 = (\theta_1 - \theta_{1,n})/\|\theta_1 - \theta_{1,n}\|$, $\eta_2 = ((1-\alpha)\psi - \psi_n)/\|(1-\alpha)\psi - \psi_n\|$, $\eta_3 = (\psi - \psi_n)/\|\psi - \psi_n\|$ and $u_n = \|\theta_1 - \theta_{1,n}\| + \|(1-\alpha)\psi - \psi_n\| + (1-\alpha)\|\psi - \psi_n\|^2$ and

$$w_1 = \frac{\|\theta_1 - \theta_{1,n}\|}{u_n}, \quad w_2 = \frac{\|(1 - \alpha)\psi - \psi_n\|}{u_n}, \quad w_3 = \frac{(1 - \alpha)\|\psi - \psi_n\|^2}{u_n},$$

Then (w_1, w_2, w_3) belongs to the sphere in \mathbf{R}^3 with radius 1 and for each $j = 1, 2, 3, \eta_j$ belongs to the sphere (with radius 1) in \mathbf{R}^d with $d = d_{\psi}$ or d_1 so that (23) becomes

$$u_n \| w_1 \eta_1^T \nabla_{\theta} f_{1,\theta_{1,n}} + w_2 \eta_2^T \nabla_{\psi} f_{2,\theta_{1,n},0} + w_3 \eta_3^T D_{\psi}^2 f_{2,\theta_{1,n},0} \eta_3 \|_1 \lesssim j / \sqrt{n},$$

Assumption B4 implies that $u_n \lesssim j/\sqrt{n}$. This leads to the same constraints as in the case of $\pi_{n,3}$ so that finally

$$\pi(S_n(j)) \lesssim (j/\sqrt{n})^{d_2} + (j/\sqrt{n})^{d_2} (\sqrt{n} \|\psi_n\|^2)^{-(d_\psi - a_2)} + j^{(d_\psi - a_2)} + \text{ and } \frac{\pi(S_n(j))}{\pi(\tilde{S}_n)} \lesssim j^{d_2 + 2(d_\psi - a_2)} + .$$

We now control the entropy of $S_n(j)$ for $j \leq M_0 \sqrt{\log n}$, i.e. the logarithm of the covering number of $S_n(j)$ by L_1 balls with radius $\zeta j/\sqrt{n}$, $\zeta > 0$ arbitrarily small. Recall that from the above conditions $S_n(j)$ is included in

$$\{\|\theta - \theta_{1,n}\| \le C_1 j/\sqrt{n}\} \cap \{\|(1-\alpha)\psi - \psi_n\| \le C_1 j/\sqrt{n}\} \cap \{(1-\alpha)\|\psi - \psi_n\|^2 \le C_1 j/\sqrt{n}\}$$

for some $C_1 > 0$ large enough, so that if $\|\psi_n\| \lesssim j/\sqrt{n}$, we are back to the proof of Theorem 2, and the local entropy is bounded by a constant. If $j/\sqrt{n} \le \delta \|\psi_n\|$ with δ small, recall from the above computations that $1 - \alpha \ge \tau \sqrt{n} \|\psi_n\|^2/j$ for some $\tau > 0$ and $\|\psi - \psi_n\| \le C_1 j (1 - \alpha)^{-1}/\sqrt{n}$. A Taylor expansion in $\psi, \psi', \theta_1, \theta'_1$ leads to

$$\begin{split} &\|\alpha f_{1,\theta} + (1-\alpha)f_{2,\theta,\psi} - (\alpha' f_{1,\theta'} + (1-\alpha')f_{2,\theta',\psi'})\|_{1} \\ &= \|((1-\alpha)\psi - (1-\alpha')\psi')^{T}\nabla_{\psi}f_{2,\theta_{1},0} + \frac{(1-\alpha)}{2}\psi^{T}D_{\psi}^{2}f_{2,\theta_{1},\bar{\psi}}\psi - \frac{(1-\alpha')}{2}(\psi')^{T}D_{\psi}^{2}f_{2,\theta_{1},\bar{\psi}'}\psi'\| + O(\|\theta_{1}-\theta'_{1}\|) + o(j/\sqrt{n}) \\ &= \|((1-\alpha)\psi - (1-\alpha')\psi')^{T}\nabla_{\psi}f_{2,\theta_{1},0} + \frac{1}{2}[(1-\alpha)\psi - (1-\alpha')\psi']^{T}D_{\psi}^{2}f_{2,\theta_{1},\bar{\psi}}\psi + \frac{(1-\alpha')}{2}(\psi')^{T}D_{\psi}^{2}f_{2,\theta_{1},\bar{\psi}'}(\psi - \psi')\|_{1} \\ &+ O(\|\theta_{1}-\theta'_{1}\|) + o(j/\sqrt{n}) \leq \frac{\zeta j}{\sqrt{n}} \end{split}$$

as soon as $\|\theta - \theta'\| \le \delta \zeta j/(4\sqrt{n})$, $\|(1 - \alpha)\psi - (1 - \alpha')\psi'\| \le \delta \zeta j/(4\sqrt{n})$ and $\|\psi' - \psi\| \le \delta \zeta j/(4\sqrt{n}\|\psi_n\|)$. The number of sets to cover

$$\{\|(1-\alpha)\psi - \psi_n\| \le C_1 j/\sqrt{n}\} \cap \{(1-\alpha)\|\psi - \psi_n\|^2 \le C_1 j/\sqrt{n}\} \cap \{\|\theta_1 - \theta_{1,n}\| \le C_1 j/\sqrt{n}\}$$

is bounded by a constant independent of j and n. Finally combining the lower bound on D_n , the upper bound on $\pi(S_n(j))$, the entropy bounds above and Theorem 2.4 of (Ghosal et al., 2000), we obtain that for all increasing sequence to infinity z_n , uniformly over $\|\psi_n\| \ge M_n/\sqrt{n}$, $\theta_{1,n} \in \Theta_1$

$$E_{f_n^*}\left[\pi\left(\|f_{\alpha,\theta,\psi}-f_n^*\|_1 \ge z_n/\sqrt{n}|\mathbf{x}^n\right)\right] = o(1), \quad f_n^* = f_{2,\theta_{1,n},\psi_n}.$$

From the above computations with $j = z_n$ going to infinity with $z_n = o(\sqrt{n} \|\psi_n\|) = o(n^{1/4})$ we have that $1 - \alpha \ge C_2 \sqrt{n} \|\psi_n\|^2$, so that there exists $M_0 > 0$ for which

$$\sup_{\|\psi\| \ge M_n/\sqrt{n}} \sup_{\theta_1 \in \Theta_1} E_{\theta_1,\psi} \left[\pi (1 - \alpha < M_0 M_n^2 / \sqrt{n} | \mathbf{x}^n) \right] = o(1)$$

and Theorem 3 is proved.

Note that when $a_2 > d_{\psi}$, for any $e_n > 0$ going to 0, the posterior probability of the set $A_n = \{1 - \alpha \le e_n\}$ has posterior probability going to 0 under f_n^* since the prior mass of the event

$$\{\|\theta_1 - \theta_{1,n}\| + \|(1 - \alpha)\psi - \psi_n\| + (1 - \alpha)\|\psi - \psi_n\|^2 \le z_n/\sqrt{n}\} \cap \{(1 - \alpha) \le e_n\}$$

is of order $O(e_n^{a_2-d_\psi}z_n^{d_2}n^{-d_2/2})=o(n^{-d_2/2})$ as soon as $e_n^{a_2-d_\psi}z_n^{d_2}=o(1)$. Since we can choose z_n going to infinity arbitrarily slowly, the result holds.