Midterm Solutions

MACS 40000, Dr. Evans
Thursday, Oct. 27, 10:30am

1. **2-period-lived agent model (15 points).** In this problem, I want you to solve the characterizing equations of the 2-period-lived agent model with endogenous labor, which is just a small version of the model in Chapter 7. Let the lifetime optimization problem of each individual in the economy be to choose labor supply $\{n_{1,t}, n_{2,t+1}\}$ and savings $b_{2,t+1}$ to maximize lifetime utility subject to a budget constraint in every period.

$$\max_{n_1,n_2,b_2} U\big(c_{1,t}, n_{1,t}\big) + \beta U\big(c_{2,t+1}, n_{2,t+1}\big)$$
$$\text{s.t.} \quad c_{1,t} + b_{2,t+1} = w_t n_{1,t}$$
$$\text{and} \quad c_{2,t+1} = (1 + r_{t+1})b_{2,t+1} + w_t n_{2,t+1}$$

And let the utility function in every period be the following,

$$U(c_{s,t}, n_{s,t}) = \frac{\big(c_{s,t}\big)^{1-\sigma} - 1}{1-\sigma} - \chi^n_s \left[\frac{\big(n_{s,t}\big)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right]$$

where $\sigma > 1$ is the coefficient of relative risk aversion, $\chi^n_s > 0$ is an age-specific level parameter influencing the relative disutility of labor, and $\theta > 0$ is the Frisch elasticity of labor supply.
Assume that a representative firm in this economy produced output Y_t every period using aggregate capital K_t and aggregate labor L_t according to the following Cobb-Douglas production function.

$$Y_t = A\big(K_t\big)^\alpha \big(L_t\big)^{1-\alpha} \quad \text{for} \quad A > 0 \quad \text{and} \quad \alpha \in (0,1)$$

Let the profits of a representative firm be given by revenues minus costs each period,

$$PR_t = Y_t - (r_t + \delta)K_t - w_t L_t$$

where $\delta \in [0,1]$ is the rate of depreciation, r_t is the interest rate, and w_t is the wage.

(a) Solve for, show your work, and write down the three Euler equations that characterize the three lifetime decisions of an agent in this problem. Write these functions in terms of labor supply, savings, interest rates, wages, and parameters.

$$w_t\big(w_t n_{1,t} - b_{2,t+1}\big)^{-\sigma} = \chi_1\big(n_{1,t}\big)^{\frac{1}{\theta}}$$
$$w_t\big([1 + r_{t+1}]b_{2,t+1} + w_{t+1}n_{2,t+1}\big)^{-\sigma} = \chi_2\big(n_{2,t+1}\big)^{\frac{1}{\theta}}$$
$$\big(w_t n_{1,t} - b_{2,t+1}\big)^{-\sigma} = \beta(1 + r_{t+1})\big([1 + r_{t+1}]b_{2,t+1} + w_{t+1}n_{2,t+1}\big)^{-\sigma}$$

(b) If the representative firm's production and profit functions are given above, solve for, show your work, and write down the two first order conditions that characterize the optimal capital stock that a firm wants to rent and the optimal labor that the firm wants to hire.

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha}$$
$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha}$$

(c) Write down the market clearing condition for capital and the market clearing condition for labor in this market. (Ignore goods market clearing.)

$$K_t = b_{2,t+1}$$
$$L_t = n_{1,t} + n_{2,t}$$

2. **Feasibility (5 points).** In the two-period-lived agent problem above, suppose that the wage in every period is $w_t = 1$ and the interest rate in every period is $r_t = 0.1$ for all t and the time endowment to each agent each period is $\tilde{l} = 1$. Is the following household decision—$n_{1,t} = 0.2$, $n_{2,t+1} = 0.15$, and $b_{2,t+1} = -0.1$—for a household born in period t feasible? If not, what constraints does it break? (NOTE: This is not a steady-state problem, so you can ignore any effect of saving on the aggregate capital stock.)
Allocation is feasible, all constraints are satisfied, because $n_{1,t}, n_{2,t+1} \in (0,1)$ and $c_{1,t}, c2, t+1 > 0$.

$$c_{1,t} = w_t n_{1,t} - b_{2,t+1} = 0.1$$
$$c_{2,t+1} = (1+r_t)b_{2,t+1} + w_{t+1}n_{2,t+1} = 0.04$$

Because this is not the steady-state problem, it is not a violation that $b_{2,t+1} = K_{t+1} < 0$ through the market clearing condition. We are assuming that the interest rate and the wage are both positive $w_t, r_t > 0$.

3. **Root finder in Python (10 points).** Assume that an individual in the last period of their life $s = S$ in period t has CFE utility of leisure and must decide how much to work n_S. The Euler equation governing this decision is the following.

$$w_t\Big([1+r_t]b_{S,t} + w_t n_{S,t}\Big)^{-\sigma} = \chi^n_S\big(n_{S,t}\big)^{\frac{1}{\theta}}$$

Solve for the optimal labor supply $n_{S,t}$ when $w_t = 1.0$, $r_t = 0.1$, $b_{S,t} = 1.0$, $\sigma = 2.2$, $\chi^n_s = 1.0$, and $\theta = 2.0$. Report both your solution for n and the value of the Euler error (in the difference form, not ratio form).

$n_S = 0.25917345$, and the Euler error $= 0$. (see output from `Midterm.py`).

4. **Minimizer in Python (10 points).** Assume that an individual's time endowment each period is one ($n + l = 1$). Let the period utility of a household be an additively separable function of consumption and leisure,

$$U(c,l) = u(c) + v(l)$$

where $v(l)$ is the utility of leisure l. The constant Frisch elasticity (CFE) disutility of labor functional form is the following.

$$v_{cfe}(l) = -\,\chi_{cfe}\left[\frac{(1-l)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right] = -\chi_{cfe}\left[\frac{(n)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right]$$
$$\text{for} \quad \chi_{cfe} > 0 \quad \text{and} \quad \theta > 0$$

And the constant relative risk aversion functional form is the following.

$$v_{crra}(l) = \chi_{crra}\left[\frac{(l)^{1-\sigma} - 1}{1-\sigma}\right] = \chi_{crra}\left[\frac{(1-n)^{1-\sigma} - 1}{1-\sigma}\right]$$
$$\text{for} \quad \chi_{crra} > 0 \quad \text{and} \quad \sigma > 0$$

Note that you can represent the utility of leisure $v(l)$ as the disutility of labor $v(1-n)$ by simply replacing l with $1-n$.

(a) The marginal disutility of labor $\frac{\partial v(l)}{\partial n}$ governs the household decision of how much to work. Give the expression for the marginal disutility of labor for the CFE specification $v_{cfe}(l)$ and for the CRRA specification $v_{crra}(l)$. [NOTE: Make sure to give me the derivative of the function with respect to n (not l).]

$$\frac{\partial v_{cfe}(l)}{\partial n} = -\chi_{cfe}(n)^{\frac{1}{\theta}}$$
$$\frac{\partial v_{crra}(l)}{\partial n} = -\chi_{crra}(1-n)^{-\sigma}$$

(b) Assume that the Frisch elasticity of labor supply in v_{cfe} is $\theta = 0.5$ and that the level parameter in the CFE case is $\chi_{cfe} = 1.0$. Using 1,000 evenly spaced points from the support of leisure between 0.1 and 0.9, estimate the two CRRA utility of leisure parameters $\chi_{crra} > 0$ and $\sigma > 0$ that minimize the sum of squared deviations between the two marginal disutility of labor functions $\frac{\partial v_{cfe}(l)}{\partial n}$ and $\frac{\partial v'_{crra}(l)}{\partial n}$ from part (a). Report your estimates.

$\chi_{crra} = 0.500629455845$ and $\sigma = 0.354653003991$ (see output from `Midterm.py`).

5. **Plotting in Python (10 points).** Plot the two marginal disutility of labor functions $\frac{\partial v_{cfe}(l)}{\partial n}$ and $\frac{\partial v_{crra}(l)}{\partial n}$ from 4(b) with labor supply n on the x-axis. Make sure your axes are labeled correctly, your plot includes a legend, and your plot has a title.
   See output from `Midterm.py`.

Figure 1: Marginal utilitities