

## Problem Set #3

MACS 40000, Dr. Evans

Due Tuesday, Oct. 18 at 10:30am

1. **Checking feasibility in the steady-state (2 points).** Using the calibration of the  $S$ -period-lived agent model described in Section 6.6 of Chapter 6, write a Python function named `feasible()` that has the following form,

```
b_cnstr, c_cnstr, K_cnstr = feasible(f_params, bvec_guess)
```

where the inputs are a tuple `f_params = (nvec, A, alpha, delta)`, and a guess for the steady-state savings vector `bvec_guess = np.array([scalar2, scalar3, ..., scalarS])`. The outputs should be Boolean (`True` or `False`, 1 or 0) vectors of lengths  $S-1$ ,  $S$ , and 1, respectively. `K_cnstr` should be a singleton Boolean that equals `True` if  $K \leq 0$  for the given `f_params` and `bvec_guess`. The object `c_cnstr` should be a length- $S$  Boolean vector in which the  $s$ th element equals `True` if  $c_s \leq 0$  given `f_params` and `bvec_guess`. And `b_cnstr` is a length- $(S-1)$  Boolean vector that denotes which element of `bvec_guess` is likely responsible for any of the consumption nonnegativity constraint violations identified in `c_cnstr`. If the first element of `c_cnstr` is `True`, then the first element of `b_cnstr` is `True`. If the second element of `c_cnstr` is `True`, then both elements of `b_cnstr` are `True`. And if the last element of `c_cnstr` is `True`, then the last element of `b_cnstr` is `True`.

- (a) Which, if any, of the constraints is violated if you choose an initial guess for steady-state savings of `bvec_guess = np.ones(S-1)`?
- (b) Which, if any, of the constraints is violated if you choose the following initial guess for steady-state savings?

```
bvec_guess = \
    np.array([-0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2])
```

- (c) Which, if any, of the constraints is violated if you choose the following initial guess for steady-state savings?

```
bvec_guess = \
    np.array([-0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
              0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1,
              0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1,
              0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1])
```

- (d) What is a principle or a rule that might help you in this problem to choose a good initial guess? That is, what properties should a feasible initial guess have? [Hint: There are upper bounds and lower bounds on all the savings levels  $\bar{b}_{s+1}$  that you cannot calculate *ex ante*.]
2. **Solve for the steady-state equilibrium (4 points).** Use the calibration from Section 6.6 and the steady-state equilibrium Definition 6.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters for the model `params = (beta, sigma, nvec, L, A, alpha, delta, SS_tol)`, an initial guess of the steady-state savings `bvec_guess`, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`.

The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
    'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
    'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss,
    'EulErr_ss': EulErr_ss, 'RCerr_ss': RCerr_ss,
    'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library. And let the object `EulErr_ss` be a length- $(S - 1)$  vector of the Euler errors from the resulting steady-state solution given in ratio form  $\frac{\beta(1+\bar{r})u'(\bar{c}_{s+1})}{u'(\bar{c}_s)} - 1$  or difference form  $\beta(1 + \bar{r})u'(\bar{c}_{s+1}) - u'(\bar{c}_s)$ . The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by  $\bar{Y} - \bar{C} - \delta\bar{K}$ .

- (a) Solve numerically for the steady-state equilibrium values of  $\{\bar{c}_s\}_{s=1}^S$ ,  $\{\bar{b}_s\}_{s=2}^S$ ,  $\bar{w}$ ,  $\bar{r}$ ,  $\bar{K}$ ,  $\bar{Y}$ ,  $\bar{C}$ , the  $S - 1$  Euler errors and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

- (b) Generate a figure that shows the steady-state distribution of consumption and savings by age  $\{\bar{c}_s\}_{s=1}^S$  and  $\{\bar{b}_s\}_{s=2}^S$ .
- (c) What happens to each of these steady-state values if all households retire sooner? That is, what happens if exogenous labor supply becomes the following?

$$n_s = \begin{cases} 1.0 & \text{if } s \leq \text{round}\left(\frac{S}{2}\right) \\ 0.2 & \text{if } s > \text{round}\left(\frac{S}{2}\right) \end{cases}$$

Specifically, how does this change affect each steady-state value  $\{\bar{c}_s\}_{s=1}^S$ ,  $\{\bar{b}_s\}_{s=2}^S$ ,  $\bar{w}$ , and  $\bar{r}$ ? What is the intuition?

3. **Solve for the non-steady-state equilibrium time path (4 points).** Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy. Let the initial state of the economy be given by the following distribution of savings,

$$\{b_{s,1}\}_{s=2}^S = \{x(s)\bar{b}_s\}_{s=2}^S \quad \text{where} \quad x(s) = \frac{(1.5 - 0.87)}{78} (s - 2) + 0.87$$

where the function of age  $x(s)$  is simply a linear function of age  $s$  that equals 0.87 for  $s = 2$  and equals 1.5 for  $s = S = 80$ . This gives an initial distribution where there is more inequality than in the steady state. The young have less than their steady-state values and the old have more than their steady-state values. You'll have to choose a guess for  $T$  and a time path updating parameter  $\xi \in (0, 1)$ , but I can assure you that  $T < 320$ . Use an  $L^2$  norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of  $\varepsilon = 10^{-9}$ . Use a linear initial guess for the time path of the aggregate capital stock from the initial state  $K_1^1$  to the steady state  $K_T^1$  at time  $T$ .

- (a) Plot the equilibrium time paths of the aggregate capital stock  $\{K_t\}_{t=1}^{T+5}$ , wage  $\{w_t\}_{t=1}^{T+5}$ , and interest rate  $\{r_t\}_{t=1}^{T+5}$ .
- (b) Also plot the equilibrium time path for savings of every person age  $s = 15$  in every period  $\{b_{15,t}\}$ . Are there any periods  $t$  in which  $b_{15,t}$  rises above its steady-state value  $\bar{b}_{15}$ ?
- (c) How many periods did it take for the economy to get within 0.0001 of the steady-state aggregate capital stock  $\bar{K}$ ? What is the period after which the aggregate capital stock never is again farther than 0.0001 away from the steady-state?