Part 1

Completed steps a-d.

Part 2

In this part, we consider a standard 2-period endowment OLG model with notation as designated in the assignment (I use the same notation and do not reproduce the problem setup here for brevity).

Part 2a

Households born in period $t \geq 1$ solve the following optimization problem:

$$\begin{aligned} & \max_{c_{1,t}\ c_{2,t+1}} (1-\beta)\ln(c_{1,t}) + \beta \ln(c_{2,t+1}) \\ s.t. \quad & p_t c_{1,t} + p_{t+1} c_{2,t+1} = p_t e_1 + p_{t+1} e_2 \end{aligned}$$

The first-order condition with respect to $c_{1,t}$ is:

$$[c_{1,t}]: \qquad \frac{1-\beta}{c_{1,t}} = \lambda p_t$$

And that w.r.t. $c_{2,t+1}$ is:

$$[c_{2,t+1}]: \qquad \frac{\beta}{c_{2,t+1}} = \lambda p_{t+1}$$

Where p_t and p_{t+1} are known prices and λ is the Lagrange multiplier on the constraint. The optimal consumption ratio is therefore:

$$\frac{c_{2,t+1}}{c_{1,t}} = \frac{\beta}{1-\beta}\frac{p_t}{p_{t+1}}$$

We can use the budget constraint to solve for each consumption explicitly:

$$\begin{aligned} p_t c_{1,t} + p_{t+1} c_{2,t+1} &= p_t e_1 + p_{t+1} e_2 \\ p_t c_{1,t} + \frac{\beta}{1-\beta} p_t c_{1,t} &= p_t e_1 + p_{t+1} e_2 \\ c_{1,t} &= \frac{p_t e_1 + p_{t+1} e_2}{p_t + \frac{\beta}{1-\beta} p_t} \\ c_{1,t} &= (1-\beta)\left[e_1 + \frac{p_{t+1}}{p_t} e_2\right] \end{aligned}$$

And then optimal consumption in the second period is given by:

$$\begin{aligned} c_{2,t+1} &= \frac{\beta}{1-\beta}\frac{p_t}{p_{t+1}} c_{1,t} \\ &= \frac{\beta}{1-\beta}\frac{p_t}{p_{t+1}} (1-\beta)\left[e_1 + \frac{p_{t+1}}{p_t} e_2\right] \\ &= \beta\left[\frac{p_t}{p_{t+1}} e_1 + e_2\right] \end{aligned}$$

Note that these consumption levels satisfy the intertemporal budget constraint, as required.

Part 2b

Now we solve for the initial old generation's optimal consumption in period 1. As we will see, this will become the source of the market incompleteness and no-trade equilibrium that prevents a first-best allocation from being achieved. The initial old's problem is:

$$\begin{aligned} &\max_{c_{2,1}} \beta \ln (c_{2,1}) \\ s.t. \quad & p_1 c_{2,1} = p_1 e_2 \end{aligned}$$

Clearly, the constraint implies $c_{2,1} = e_2$ (i.e. the initial old generation optimally chooses to consume his entire endowment).

Part 2c

Given what we have established in Part 2b and using market clearing ($c_{2,1} + c_{1,1} = e_1 + e_2$), we know that the generation born at time $t = 1$ must consume:

$$c_{1,1} = e_1$$

In their first period of life. They enter their second period only with their second-period endowment, e_2 (just like the initial old generation), and will optimally choose to consume that. The argument carries through over all periods using a proof by induction (the first step is outlined above, and the n^{th} step works exactly the same way), so that consumption in the competitive equilibrium is:

$$c_{1,t} = e_1$$

and

$$c_{2,t+1} = e_2$$

i.e. this is a no-trade equilibrium where each period each generation eats its endowment that period. These hold for all t. The price process is determined by combining the first-order conditions and imposing the no-trade equilibrium, which results in prices that support this equilibrium:

$$\frac{p_{t+1}}{p_t} = \frac{\beta}{1-\beta}\frac{e_1}{e_2}$$

Imposing $p_1 = 1$:

$$p_2 = \frac{\beta}{1-\beta}\frac{e_1}{e_2}$$

$$\begin{aligned} p_3 &= \frac{\beta}{1-\beta}\frac{e_1}{e_2} p_2 \\ &= \left[\frac{\beta}{1-\beta}\frac{e_1}{e_2}\right]^2 \end{aligned}$$

And in general:

$$p_{t+1} = \left[\frac{\beta}{1-\beta}\frac{e_1}{e_2}\right]^t$$