LXXXIX. *Mathematical Theory of Deflection of Beam.*

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1. *Introduction.*

A LIGHT, straight beam is supported on two smooth supports, and a force at right-angles to the beam is applied at a point midway between the supports.

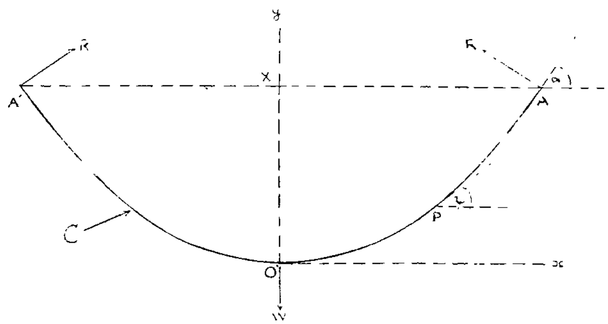
Elementary theory gives the resulting deflection only when this is small compared with the span. Provided, however, that the depth of the beam is small enough, it is possible for the deflection to be comparable with or greater than the span, without the elastic limit being exceeded.

The problem to be treated here is the determination of the relation between the central force, the span and the deflection, when the deflection is not necessarily small compared with the span. Finally, the results obtained are verified experimentally, as described in § 6.

2. *Conditions for Equilibrium.*

The beam will be supposed initially to have rectangular cross-section. The surface formed by points of the beam midway between the top and bottom surfaces will be called " the central surface."

Fig. 1.



Curve C, fig. 1, is the section of the central surface by the plane in which lie the lines of action of the reactions R, R at the supports and the force W applied midway between the supports. Lines of action of R, R, W meet C at A, A′, O ; since the supports are smooth, reactions R, R are normal to C. Ox, Oy are co-ordinate axes respectively parallel and perpendicular to AA′ ; X is mid-point of AA′. Span AA′ and deflection XO will be denoted by 2L and Δ. The tangents to C at P (any point on C between O, A) and A make angles ψ, α with Ox.

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By the elementary theory of Strength of Materials there is at P a bending moment IE/ρ, where I is the second moment of the cross-section through P about the line in which it intersects the central surface, E is the modulus of elasticity of the material considered, and ρ is the radius of curvature of C at P. Hence, x and y being co-ordinates of P,

$$\frac{IE}{\rho}=R\cos\alpha(L-x)+R\sin\alpha(\Delta-y). \quad . \quad . \quad . \quad . \quad (2.1)$$

Resolving forces in direction Oy, for the equilibrium of the beam, we have

$$2R\cos\alpha=W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

3. *Geometrical Equations.*

Differentiation of (2.1) with respect to s, the arc of C measured from O, gives

$$-\frac{IE}{\rho^2}\frac{d\rho}{ds}=-R\cos\alpha\cos\psi-R\sin\alpha\sin\psi=-R\cos(\alpha-\psi).$$

Since

$$\rho=\frac{ds}{d\psi},$$

$$\frac{IE}{\rho^3}\frac{d\rho}{d\psi}=R\cos(\alpha-\psi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

Integration with respect to ψ gives

$$-\frac{IE}{2\rho^2}=-R\sin(\alpha-\psi)+\text{arbitrary constant}.$$

Since, from (2.1),

$$\frac{1}{\rho}=0$$

at A, when $\psi=\alpha$, the arbitrary constant in the above vanishes, and

$$\frac{1}{\rho}=\left\{\frac{2R}{IE}\sin(\alpha-\psi)\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

Now

$$\frac{dx}{d\psi}=\frac{dx}{ds}\rho=\left(\frac{IE}{2R}\right)^{\frac{1}{2}}\cos\psi\,.\,\sin^{-\frac{1}{2}}(\alpha-\psi),$$

$$\frac{dy}{d\psi}=\frac{dy}{ds}\rho=\left(\frac{IE}{2R}\right)^{\frac{1}{2}}\sin\psi\,.\,\sin^{-\frac{1}{2}}(\alpha-\psi).$$

Hence

$$\left.\begin{aligned}\left(\frac{2R}{IE}\right)^{\frac{1}{2}}x&=\int_0^{\psi}\cos\psi\,.\,\sin^{-\frac{1}{2}}(\alpha-\psi)d\psi,\\ \left(\frac{2R}{IE}\right)^{\frac{1}{2}}y&=\int_0^{\psi}\sin\psi\,.\,\sin^{-\frac{1}{2}}(\alpha-\psi)d\psi.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3.3)$$

Writing $\alpha-\psi=\theta$ and observing that

$$\int_0^{\psi}\cos\theta\sin^{-\frac{1}{2}}\theta\,d\theta=2\sin^{\frac{1}{2}}\psi, \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

and putting $$\int_0^{\psi} \sin^{\frac{1}{2}} \theta \, d\theta = \mathrm{P}(\psi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.5)$$

(3.3) may be written

$$\left.\begin{aligned} \left(\frac{2\mathrm{R}}{\mathrm{IE}}\right)^{\frac{1}{2}} x &= 2 \cos \alpha(\sin^{\frac{1}{2}}\alpha - \sin^{\frac{1}{2}}\theta) + \sin \alpha\{\mathrm{P}(\alpha) - \mathrm{P}(\theta)\}, \\ \left(\frac{2\mathrm{R}}{\mathrm{IE}}\right)^{\frac{1}{2}} y &= 2 \sin \alpha(\sin^{\frac{1}{2}}\alpha - \sin^{\frac{1}{2}}\theta) - \cos \alpha\{\mathrm{P}(\alpha) - \mathrm{P}(\theta)\}. \end{aligned}\right\} \quad . \qquad (3.6)$$

In particular, if $\psi = \alpha$,

$$\boxed{\begin{aligned} \left(\frac{2\mathrm{R}}{\mathrm{IE}}\right)^{\frac{1}{2}} \mathrm{L} &= 2 \cos \alpha \sin^{\frac{1}{2}}\alpha + \sin \alpha \, . \, \mathrm{P}(\alpha), \\ \left(\frac{2\mathrm{R}}{\mathrm{IE}}\right)^{\frac{1}{2}} \Delta &= 2 \sin \alpha \sin^{\frac{1}{2}}\alpha - \cos \alpha \, . \, \mathrm{P}(\alpha). \end{aligned}} \left.\vphantom{\begin{aligned} a \\ b \end{aligned}}\right\} \quad . \quad . \quad . \qquad (3.7)$$

The elimination of α between these equations, after the substitution $2\mathrm{R} = \mathrm{W} \sec \alpha$, gives the required relation between W, Δ, L *.

For brevity I shall write

$$\left.\begin{aligned} 2 \cos \alpha \, . \, \sin^{\frac{1}{2}}\alpha + \sin \alpha \, . \, \mathrm{P}(\alpha) &= \mathrm{Q}(\alpha) \\ 2 \sin \alpha \, . \, \sin^{\frac{1}{2}}\alpha - \cos \alpha \, . \, \mathrm{P}(\alpha) &= \mathrm{S}(\alpha), \end{aligned}\right\}, \quad . \quad . \quad . \quad . \qquad (3.8)$$

so that (3.7) may be written

$$\left(\frac{2\mathrm{R}}{\mathrm{IE}}\right)^{\frac{1}{2}} \mathrm{L} = \mathrm{Q}(\alpha), \quad \left(\frac{2\mathrm{R}}{\mathrm{IE}}\right)^{\frac{1}{2}} \Delta = \mathrm{S}(\alpha), \quad . \quad . \quad . \quad . \quad . \qquad (3.9)$$

whence

$$\frac{\Delta}{\mathrm{L}} = \frac{\mathrm{S}(\alpha)}{\mathrm{Q}(\alpha)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.10)$$

and, from (3.9) and (2.2),

$$\left(\frac{\mathrm{W}}{\mathrm{IE} \cos \alpha}\right)^{\frac{1}{2}} \mathrm{L} = \mathrm{Q}(\alpha),$$

whence

$$\frac{\mathrm{WL}^2}{\mathrm{IE}} = \mathrm{Q}(\alpha)^2 \cos \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.11)$$

and, from (3.6), (3.7) and (3.8),

$$\left.\begin{aligned} \frac{\mathrm{Q}(\alpha)}{\mathrm{L}} x &= \mathrm{Q}(\alpha) - 2 \cos \alpha \sin^{\frac{1}{2}}\theta - \sin \alpha \, . \, \mathrm{P}(\theta), \\ \frac{\mathrm{S}(\alpha)}{\Delta} y &= \mathrm{S}(\alpha) - 2 \sin \alpha \sin^{\frac{1}{2}}\theta + \cos \alpha \, . \, \mathrm{P}(\theta). \end{aligned}\right\} \quad . \quad . \quad . \qquad (3.12)$$

These are the equations to the deflection curve in terms of parameter θ.

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* This cannot be done algebraically, but a graphical solution is given in § 4

4. *Numerical Calculations.*

Reading values of $\sin^{\frac{1}{2}}\theta$ at intervals of $2\frac{1}{2}°$, and evaluating P at intervals of 5° by Simpson's Rule, the following results are obtained, correct to two decimal places * :—

a degrees.	P.	Q.	S.	S/Q $=\Delta/L$.	$Q^2 \cos a$ $=WL^2/IE$.
0	0·00	0·00	0·00	0·00	0·00
5	0·02	0·59	0·03	0·06	0·35
10	0·05	0·83	0·10	0·12	0·68
15	0·09	1·01	0·18	0·18	0·99
20	0·14	1·15	0·27	0·24	1·24
25	0·19	1·26	0·38	0·30	1·44
30	0·25	1·35	0·49	0·36	1·58
35	0·31	1·42	0·61	0·43	1·65
40	0·38	1·47	0·74	0·50	1·67
45	0·45	1·51	0·87	0·57	1·61
50	0·53	1·53	1·00	0·65	1·51
55	0·61	1·53	1·14	0·74	1·36
60	0·69	1·53	1·27	0·84	1·17
65	0·77	1·50	1·40	0·94	0·95
70	0·85	1·46	1·54	1·05	0·73
75	0·93	1·41	1·66	1·18	0·52
80	1·02	1·35	1·78	1·32	0·32
85	1·10	1·27	1·88	1·48	0·14
90	1·19	1·19	2·00	1·68	0·00
95	1·28	1·10	2·09	1·90	−0·11
100	1·37	1·00	2·19	2·20	−0·17
105	1·45	0·89	2·28	2·55	−0·21
110	1·54	0·78	2·35	3·00	−0·21
115	1·62	0·66	2·41	3·66	−0·18
120	1·70	0·54	2·46	4·59	−0·14
125	1·78	0·42	2·50	6·03	−0·10
130	1·86	0·30	2·53	8·60	−0·06
135	1·93	0·18	2·55	14·50	−0·02
140	2·00	0·06	2·57	44·20	−0·00
145	2·07	−0·05	2·56	−51·30	−0·00
150	2·13	−0·16	2·55	−16·08	−0·02
155	2·19	−0·25	2·53	−10·02	−0·06
160	2·25	−0·33	2·51	−7·63	−0·10
165	2·29	−0·39	2·48	−6·33	−0·15
170	2·34	−0·41	2·45	−5·91	−0·17
175	2·37	−0·38	2·41	−6·30	−0·15
180	2·38	−0·00	2·38	$-\infty$	−0·00

* Alternatively, P(α) may be expressed as an elliptic integral

$$2\int_{\frac{\pi}{4}-\frac{\alpha}{2}}^{\frac{\pi}{4}} (1-2\sin^2 u)^{\frac{1}{2}}\,du$$

values of which are given in tables.

From this table * are plotted graphs :—

(A) α against Δ/L,

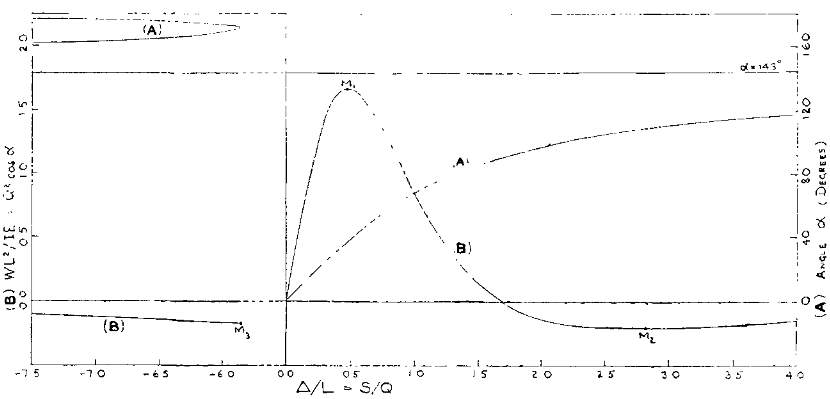
(B) WL^2/IE against Δ/L.

These are shown in fig. 2.

Plotting the graph of Q against α shows that Q is positive for values of α between 0° and 143° (approximately), and negative for values of α between 143° and 180°. In the former case L is positive, in the latter negative, and the beam crosses over itself (fig. 3).

When the beam crosses over itself, the problem is no longer two-dimensional, but, provided the beam is narrow, the theory given here will be approximately true.

Fig. 2.



The following problems may now be solved graphically :—

(1) From measurements of L, Δ, W, I to determine E.

(2) Given L, W, I, E to determine Δ, R, α.

(3) Given Δ, L to determine the deflection curve.

Method :—

(1) Evaluate Δ/L. Read WL^2/IE from graph (B). W, L and I being known, E is then determined.

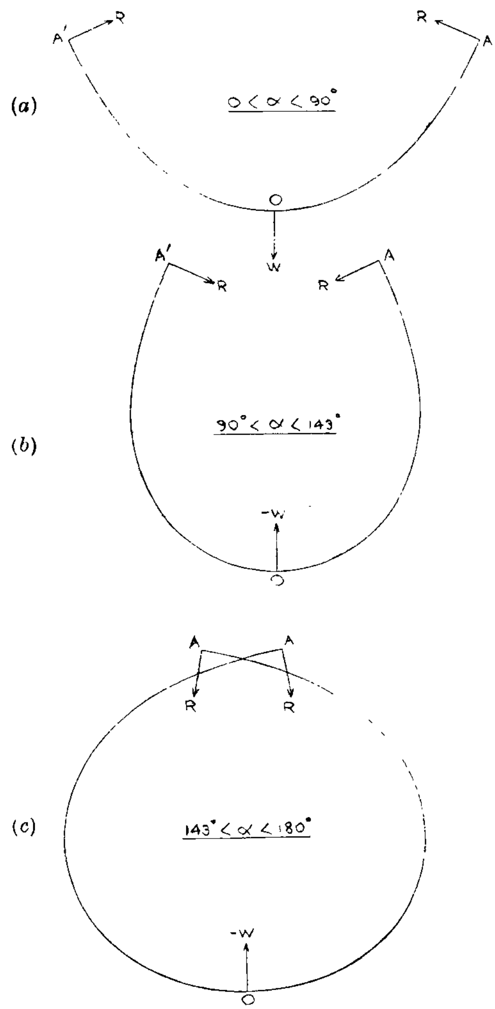
(2) Evaluate WL^2/IE. Read possible values of Δ/L from graph (B) ; L being known, Δ is then determined. Read α from graph (A) ; R is determined by $R=W/2\cos\alpha$.

* Together with additional values :—

α.	S/Q.	$Q^2 \cos \alpha$.
171	−5·85	0·17
172	−5·90	0·17
176	−6·68	0·13
177	−7·21	0·11

(3) Evaluate Δ/L. Read α and $\mathrm{Q}^2 \cos \alpha$ from graphs (A), (B); Q is thus determined; determine S from $\mathrm{S}/\mathrm{Q}=\Delta/\mathrm{L}$. Giving θ values 0, 5°,

Fig. 3.



10°, 15° . . . (not exceeding α), and reading values of $\mathrm{P}(\theta)$ from table of this paragraph, x, y are given by (3.12), enabling curve OA to be plotted; curve OA′ is obtained by reflection of OA in $\mathrm{O}y$.

5. *Properties of Beam with Constant Span.*

(i) $0<\alpha<143°$.—Beam not crossing over itself.

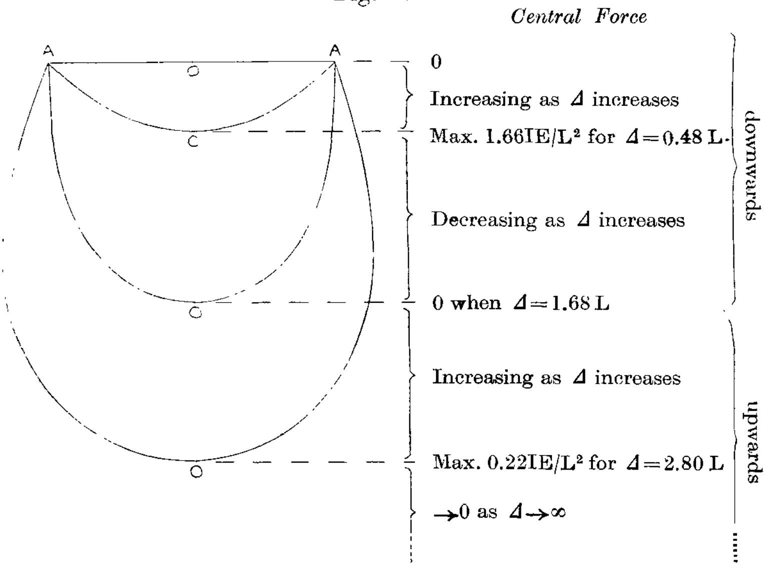
From graph (B), $\mathrm{WL}^2/\mathrm{IE}$ has maximum value 1·66 (at M_1) when $\Delta/\mathrm{L}=0{\cdot}48$, and minimum value $-0{\cdot}22$ (at M_2) when $\Delta/\mathrm{L}=2{\cdot}80$. If L is constant, the beam thus has the properties shown in fig. 4.

(ii) $143^\circ < \alpha < 180^\circ$.—Beam crossing over itself.

From graph (A) it is seen that there are two values of α for a given value of Δ/L; there are thus two possible deflection curves for a given deflection. Also $-\Delta/L$ has a minimum value 5·85; equilibrium is therefore not possible with deflection less than $-5{\cdot}85$ L.

From graph (B), WL^2/IE has minimum value $-0{\cdot}17$ (at M_3), so that maximum value of central force is $0{\cdot}17\ L^2/IE$.

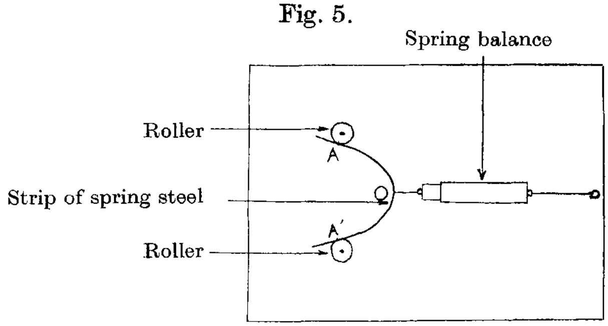
Fig. 4.



6. *Experimental Verification.*

A strip of spring steel, 1 in. $\times$·01 in., was mounted on a horizontal board, pressing against adjustable rollers at A, A′ (ensuring normal reactions at these points), and the force at O was supplied and measured by a spring balance (see fig. 5). The strip was supported by glass rods resting on the

Fig. 5.

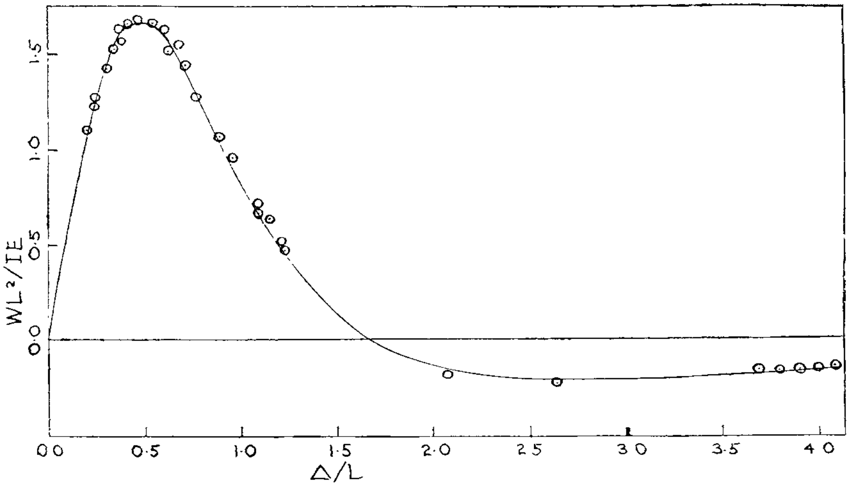


board; these, being free to roll, supported the strip with negligible friction.

By adjusting the positions of the rollers and spring balance, various spans and deflections were obtained, and, taking E for spring steel to be

31×10^6 lb./in.^{2}, corresponding values of WL^2/IE and Δ/L were evaluated and compared with the theoretical graph. Results are shown in fig. 6.

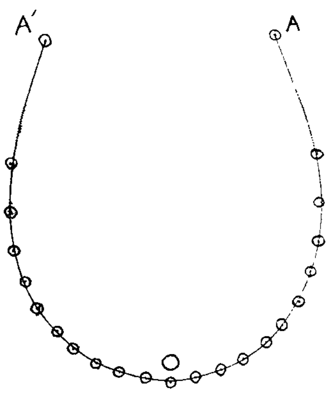
Fig. 6.



Continuous curve obtained from theoretical tables.
Points ⊙ obtained by experiment.

Secondly, the actual curve formed by the strip for given values of Δ, L was compared with the theoretical curve with the same values of Δ, L, as obtained by method given in problem (3) of § 4. The result is shown in fig. 7.

Fig. 7.



Continuous curve is deflection curve obtained by experiment with $\Delta = 3$ in., L=1 in.

Points ⊙ are obtained by theory, for $\theta = 0°$, $10°$, $20°$, $30°$......α when $\Delta = 3$ in., L=1 in.

It will be seen that there is close agreement between experiment and the theory developed here.

In conclusion, I should like to express my thanks to Messrs. Bowling, Barmbv and Densfield, of the Birmingham Central Technical College, for encouragement and assistance given in the preparation of this paper.