LARGE DEFLECTIONS OF BEAMS IN FOUR-POINT BENDING

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Equations are derived for large deflections of slender prismatic beams of linearly elastic materials in four-point bending – and are solved numerically.

1 INTRODUCTION

Classical theory of the bending of beams is based on the assumption that their deflections are small, which makes it possible to derive algebraic solutions for them in terms of applied loads and beam geometry. These solutions are adequate for many engineering purposes. However in a number of practical cases it is desirable (besides being of academic interest) to have theoretical solutions to large deflections of beams of linearly elastic materials.

One solution has already been presented by Freeman (1)*, who derived equations to the large deflection of a slender, simply-supported, centrally loaded beam. As these cannot be solved algebraically, he solved them numerically using graphical methods. More recently, as computers became available, Knowles and Stephenson (2) used Freeman's equations to calculate more accurately the large deflections of beams in three-point bending; and West (3) obtained virtually the same results as Freeman by using a different numerical solution.

Theoretical solutions to large deflections of beams are of especial interest in connection with the use of plastics materials, because of the latter's ability to sustain relatively large strains. Thus Freeman's solution has been invoked in a number of cases involving plastics (2)(4)(5) while West's solution (3) was directly inspired by them. However, solving for deflections in three-point bending has not answered the need for a solution to the large deflection of beams in four-point bending. This is of particular interest in the testing of plastics. We have therefore undertaken the solution for large deflections of slender beams in four-point bending, as presented in this paper.

2 SOLUTION TO THE PROBLEM

2.1 Basic relationships

Consider a long slender prism of rectangular cross-section supported in one of its longitudinal planes of symmetry over a span L at points O and D. Let it be loaded in the same plane by a pair of forces F (symmetrically disposed about its centre) acting at A and B so that the horizontal distances between O and A and between B and D are both equal to a, as shown in Fig. 1. Assume (after Freeman) that the prismatic beam has negligible mass and that there is no friction at the points of load application and at the supports. Thus the forces F and the reactions R at the supports are always normal to the beam surface.

Because of its symmetry, only one half of the beam needs to be considered, but in each half there are two distinct regions:

Region 1

When $0 \leqslant x \leqslant a$, the bending moment M at any point (x, y) is

$$M = R_x y + R_y x = R\,(y \sin\alpha + x\cos\alpha) \quad . \quad . \quad . \quad . \quad (1)$$

where R_x and R_y are the components of R in the x and y directions respectively and α is the angle between the tangent to the deflected centroidal axis of the beam and the horizontal at O (and D).

Equation (1) is the same as that for a point on a beam in three-point bending, so Freeman's equations are directly applicable. In terms of the symbols used in this paper, our origin being at O (instead of his at C), they are

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}} x = 2\cos\alpha \sin^{\frac{1}{2}}(\alpha-\theta) + \sin\alpha \cdot P(\alpha-\theta) \quad . \quad (2)$$

and

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}} y = 2\sin\alpha \sin^{\frac{1}{2}}(\alpha-\theta) - \cos\alpha \cdot P(\alpha-\theta) \quad . \quad (3)$$

where E is modulus of elasticity, assumed the same in tension and compression; I is second moment of area of cross-section of the beam referred to its neutral axis; and θ is angle between tangent to the deflected axis of the beam and the horizontal at (x,y). Lastly, the function P is defined by

$$P(\alpha-\theta) = \int_{\phi=0}^{\phi=\alpha-\theta} \sin^{\frac{1}{2}}\phi \cdot d\phi$$

At A, where $\theta = \beta$, $x = a$ and $y = y_A$, equations (2) and (3) become

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}} a = 2\cos\alpha \sin^{\frac{1}{2}}(\alpha-\beta) + \sin\alpha \cdot P(\alpha-\beta) \qquad (4)$$

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* *References are given in Appendix 2.*

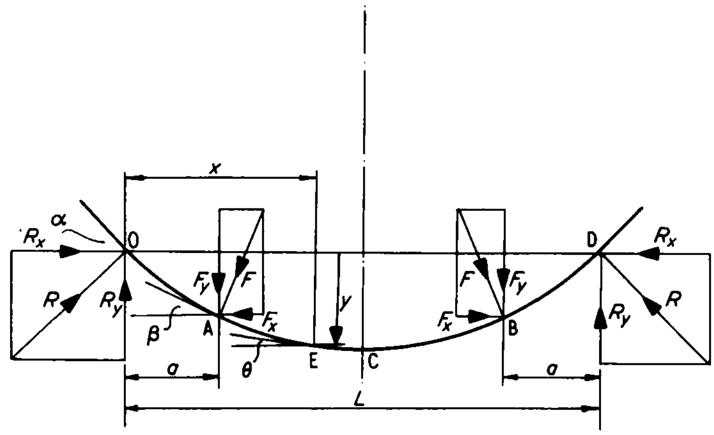

Fig. 1. Diagrammatic representation of a beam in four-point bending

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}} y_A = 2 \sin \alpha \sin^{\frac{1}{2}} (\alpha - \beta) - \cos \alpha \cdot P(\alpha - \beta) \quad . \quad . \quad . \quad (5)$$

Region 2

When $a \leqslant x \leqslant L/2$, the bending moment M at any point (x, y) is

$$M = R_y x + R_x y - F_y(x - a) - F_x(y - y_A) \quad . \quad . \quad . \quad (6)$$

where F_x and F_y are respectively the x and y components of F.

But, for equilibrium

$$F_y = R_y$$

or

$$F \cos \beta = R \cos \alpha \quad . \quad . \quad . \quad (7)$$

Substituting from (7) in (6) and rearranging gives

$$M = R\left\{y \frac{\sin (\alpha - \beta)}{\cos \beta} + a \cos \alpha + y_A \tan \beta \cos \alpha\right\} \quad . \quad (8)$$

Accepting, as usual, the Bernoulli-Euler hypothesis – that plane sections originally normal to the centroidal axis of the beam remain plane and normal to its deformed axis – gives

$$M = \frac{EI}{r} \quad . \quad . \quad . \quad (9)$$

where r is the radius of curvature of the beam at (x, y); $r \gg$ beam thickness.

Combining (8) and (9) and differentiating with respect to s, the distance measured along the beam's deformed axis, leads to

$$-\frac{EI}{r^2} \frac{\mathrm{d}r}{\mathrm{d}s} = R \frac{\mathrm{d}y}{\mathrm{d}s} \cdot \frac{\sin (\alpha - \beta)}{\cos \beta} \quad . \quad . \quad . \quad (10)$$

but

$$\mathrm{d}s = -r \, \mathrm{d}\theta$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \sin \theta$$

which, on substitution in (10), gives

$$\frac{EI}{r^3} \frac{\mathrm{d}r}{\mathrm{d}\theta} = R \sin \theta \cdot \frac{\sin (\alpha - \beta)}{\cos \beta} \quad . \quad . \quad . \quad (11)$$

Integrating (11) with respect to θ gives

$$\frac{EI}{2r^2} = R \cos \theta \cdot \frac{\sin (\alpha - \beta)}{\cos \beta} + C \quad . \quad . \quad . \quad (12)$$

Now, when $\theta = \beta$, compatibility demands that the right-hand sides of equation (11) and of the corresponding equation derived (after Freeman) for the first region, are equal. Thus

$$R \sin (\alpha - \beta) = R \cos \beta \cdot \frac{\sin (\alpha - \beta)}{\cos \beta} + C \quad . \quad . \quad . \quad (13)$$

which means that $C = 0$. Hence, after rearranging, equation (12) becomes

$$\frac{1}{r} = \left[\frac{2R}{EI} \frac{\sin (\alpha - \beta) \cos \theta}{\cos \beta}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad (14)$$

but

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{\mathrm{d}x}{\mathrm{d}s} \cdot r = -\cos \theta \left[\frac{EI}{2R} \frac{\cos \beta}{\sin (\alpha - \beta) \cos \theta}\right]^{\frac{1}{2}} \quad . \quad . \quad (15)$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{\mathrm{d}y}{\mathrm{d}s} \cdot r = -\sin \theta \left[\frac{EI}{2R} \frac{\cos \beta}{\sin (\alpha - \beta) \cos \theta}\right]^{\frac{1}{2}} \quad . \quad . \quad (16)$$

Changing the origin from O to A, so that x and y become $(x - a)$ and $(y - y_A)$, and integrating (15) and (16) with respect to θ leads to

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}}(x-a)=-\int_{\beta}^{\theta}\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\cdot\cos\theta\right\}^{\frac{1}{2}}\mathrm{d}\theta \quad . \quad . \; (17)$$

and

$$\frac{2R}{EI}^{\frac{1}{2}}(y-y_A)=-\int_{\beta}^{\theta}\left\{\frac{\cos\beta}{\sin(\alpha-\beta)\cos\theta}\right\}^{\frac{1}{2}}\sin\theta\,\mathrm{d}\theta \quad . \quad . \quad . \; (18)$$

But

$$\int_{\beta}^{\theta}\frac{\sin\theta}{\cos^{\frac{1}{2}}\theta}\,\mathrm{d}\theta=-\int_{\beta}^{\theta}\frac{\mathrm{d}(\cos\theta)}{\cos^{\frac{1}{2}}\theta}=-2\left(\cos^{\frac{1}{2}}\theta-\cos^{\frac{1}{2}}\beta\right)$$

and letting

$$\int_{\phi=0}^{\phi=\psi}\cos^{\frac{1}{2}}\phi\cdot\mathrm{d}\phi=M(\psi),$$

the values of which are tabulated in Appendix 1, reduces equations (17) and (18) to

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}}(x-a)=\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}[M(\beta)-M(\theta)] \quad . \quad . \; (19)$$

and

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}}(y-y_A)=2\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}\left(\cos^{\frac{1}{2}}\theta-\cos^{\frac{1}{2}}\beta\right) \quad . \quad . \quad . \; (20)$$

When $\theta=0$, $x=L/2$ and $y=y_C$, the deflection at the centre of the beam, equations (19) and (20) become, respectively

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}}(L/2-a)=\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}M(\beta) \quad . \quad . \quad . \quad . \; (21)$$

and

$$\left(\frac{2R}{EI}\right)^{\frac{1}{2}}(y_C-y_A)=2\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}\left(1-\cos^{\frac{1}{2}}\beta\right) \quad . \; (22)$$

Dividing equation (21) by (4), and denoting the ratio a/L by n gives

$$\frac{1}{2n}-1=\frac{\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}M(\beta)}{2\cos\alpha\cdot\sin^{\frac{1}{2}}(\alpha-\beta)+\sin\alpha\cdot P(\alpha-\beta)} \quad . \; (23)$$

2.2 Solution of equations

Equation (23) enables the value of β to be found for any given values of n and α by an iterative process, which amounts to a version of Newton's method of solving non-linear algebraic equations, and was carried out on a CDC 6400 computer. The results, for α ranging from 0 to 75° and for two values of n of particular interest, are given in Table 1. The relationship between α and β is shown graphically for a series of values of n in Fig. 2. It might be noted that $n=0$ corresponds to the case of pure bending – which is not, however, achievable in practice because it implies that $a=0$ and $F=R=\infty$, as can be seen from equations (25) and (26).

Once a relationship has been established between α and β for particular values of n, equations (4) and (5), and (21) and (22) can be solved. Thus from equation (21)

$$\left(\frac{2RL^2}{EI}\right)^{\frac{1}{2}}=\frac{2\,V(\alpha,n)}{(1-2n)} \quad . \; (24)$$

and, substituting from (7), this leads to

$$\frac{FL^2}{EI}=\frac{2\,V^2(\alpha,n)}{(1-2n)^2}\cdot\frac{\cos\alpha}{\cos\beta} \quad . \; (25)$$

where

$$V(\alpha,n)=\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}M(\beta)=\left(\frac{2RL^2}{EI}\right)^{\frac{1}{2}}\frac{1-2n}{2} \quad . \; (26)$$

Dividing equation (5) by (4) gives

$$\frac{y_A}{a}=\frac{U(\alpha,n)}{T(\alpha,n)}$$

or

$$\frac{y_A}{L}=n\,\frac{U(\alpha,n)}{T(\alpha,n)} \quad . \; (27)$$

where

$$T(\alpha,n)=2\cos\alpha\cdot\sin^{\frac{1}{2}}(\alpha-\beta)+\sin\alpha\cdot P(\alpha-\beta)$$
$$=\left(\frac{2R}{EI}\right)^{\frac{1}{2}}a \quad . \; (28)$$

and

$$U(\alpha,n)=2\sin\alpha\cdot\sin^{\frac{1}{2}}(\alpha-\beta)-\cos\alpha\cdot P(\alpha-\beta)$$
$$=\left(\frac{2R}{EI}\right)^{\frac{1}{2}}y_A \quad . \; (29)$$

Dividing equation (22) by (21) gives

$$\frac{2\,(y_C-y_A)}{L\,(1-2n)}=\frac{S\,(\alpha,n)}{V\,(\alpha,n)} \quad . \; (30)$$

where

$$S(\alpha,n)=2\left\{\frac{\cos\beta}{\sin(\alpha-\beta)}\right\}^{\frac{1}{2}}\left(1-\cos^{\frac{1}{2}}\beta\right)$$
$$=\left(\frac{2R}{EI}\right)^{\frac{1}{2}}(y_C-y_A) \quad . \; (31)$$

Finally, substituting from (27) in (30) gives

$$\frac{y_C}{L}=n\,\frac{U(\alpha,n)}{T(\alpha,n)}+\frac{1-2n}{2}\cdot\frac{S(\alpha,n)}{V(\alpha,n)} \quad . \; (32)$$

A range of values of the functions $V(\alpha,n)$, $S(\alpha,n)$,

Table 1 (n = 0.10)

α	β	$V(\alpha, n)$	$S(\alpha, n)$	$T(\alpha, n)$	$U(\alpha, n)$	FL^2/EI	y_C/L
5	4.44	0.78551	0.03048	0.19638	0.01654	1.92667	0.02395
10	8.93	1.12947	0.08826	0.26981	0.04584	3.97411	0.04825
15	13.31	1.32957	0.15552	0.33240	0.08558	5.48348	0.07253
20	17.74	1.50849	0.23635	0.37519	0.13010	7.01570	0.09759
25	22.12	1.63753	0.32220	0.40938	0.18262	8.19807	0.12331
30	26.49	1.73520	0.41229	0.43380	0.23877	9.10417	0.15008
35	30.82	1.80159	0.50320	0.44983	0.29894	9.67488	0.17818
40	35.09	1.83493	0.59056	0.45873	0.36313	9.85103	0.20790
45	39.31	1.84032	0.67252	0.46008	0.43058	9.67221	0.23976
50	43.44	1.81774	0.74536	0.45443	0.50138	9.14030	0.27435
55	47.44	1.76846	0.80565	0.44212	0.57578	8.28873	0.31246
60	51.28	1.69405	0.84976	0.42351	0.65450	7.16917	0.35510
65	54.88	1.59670	0.87394	0.39914	0.73897	5.85248	0.40408
70	58.10	1.47946	0.87415	0.36986	0.83194	4.42679	0.46128
75	60.73	1.34768	0.84705	0.33691	0.93781	3.00424	0.52977

(n = 0.20)

α	β	$V(\alpha, n)$	$S(\alpha, n)$	$T(\alpha, n)$	$U(\alpha, n)$	FL^2/EI	y_C/L
5	3.77	0.44824	0.01475	0.29218	0.02346	1.11435	0.02593
10	7.49	0.62036	0.04061	0.41356	0.06670	2.12348	0.05190
15	11.20	0.75013	0.07369	0.50009	0.12223	3.07820	0.07835
20	14.89	0.85112	0.11154	0.56610	0.18748	3.91315	0.10544
25	18.52	0.92919	0.15221	0.61946	0.26091	4.58476	0.13338
30	22.10	0.98887	0.19436	0.65924	0.34122	5.07773	0.16248
35	25.59	1.03170	0.23641	0.68777	0.42749	5.37093	0.19306
40	28.98	1.05888	0.27684	0.70591	0.51903	5.45477	0.22549
45	32.23	1.07143	0.31415	0.71428	0.61530	5.33125	0.26025
50	35.31	1.07024	0.34680	0.71349	0.71600	5.01248	0.29791
55	38.16	1.05624	0.37325	0.70416	0.82103	4.52135	0.33921
60	40.71	1.03052	0.39199	0.68701	0.93059	3.89185	0.38503
65	42.88	0.99438	0.40159	0.66291	1.04513	3.16795	0.43648
70	44.53	0.94941	0.40088	0.63294	1.16535	2.40254	0.49490
75	45.53	0.89749	0.38911	0.59833	1.29191	1.65339	0.56191

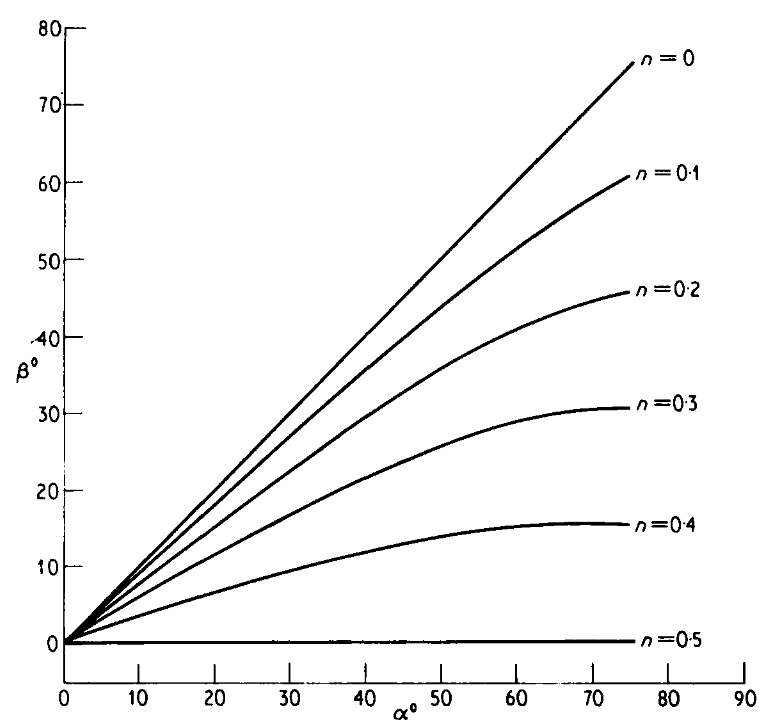

Fig. 2. Curves of α vs β for different values of n derived from the solution of equation (23)

$T(\alpha, n)$ and $U(\alpha, n)$ is given in Table 1 and the results calculated from equations (25) and (32) are plotted in Fig. 3. Similar results from equations (25) and (27) are plotted in Fig. 4.

3 DISCUSSION OF RESULTS

At small values of y_C/L, the curves of FL^2/EI vs y_C/L approximate to a straight line, which corresponds to the standard solution based on the assumption that deflections are small. However, as deflection increases, divergence from the standard small-deflection solution becomes considerable. Thus as deflection increases up to a value of α between 40 and 45°, the corresponding load given by the large-deflection solution becomes progressively smaller than that given by the standard solution. The alternative manifestation of the effect is that, at a given load, the deflection given by the large-deflection solution is larger than that given by the standard solution.

Unfortunately, a simple numerical comparison between the results given by the two solutions is not possible, because the load systems differ in the two cases. In particular, in the small-deflection case the applied loads always act vertically, whereas in the large-deflection solution loads are normal to the beam surface at their point of application. One could compare the (small-deflection) vertical loads with the (large-deflection) vertical components of the loads. However this would ignore the effects of the remaining horizontal components.

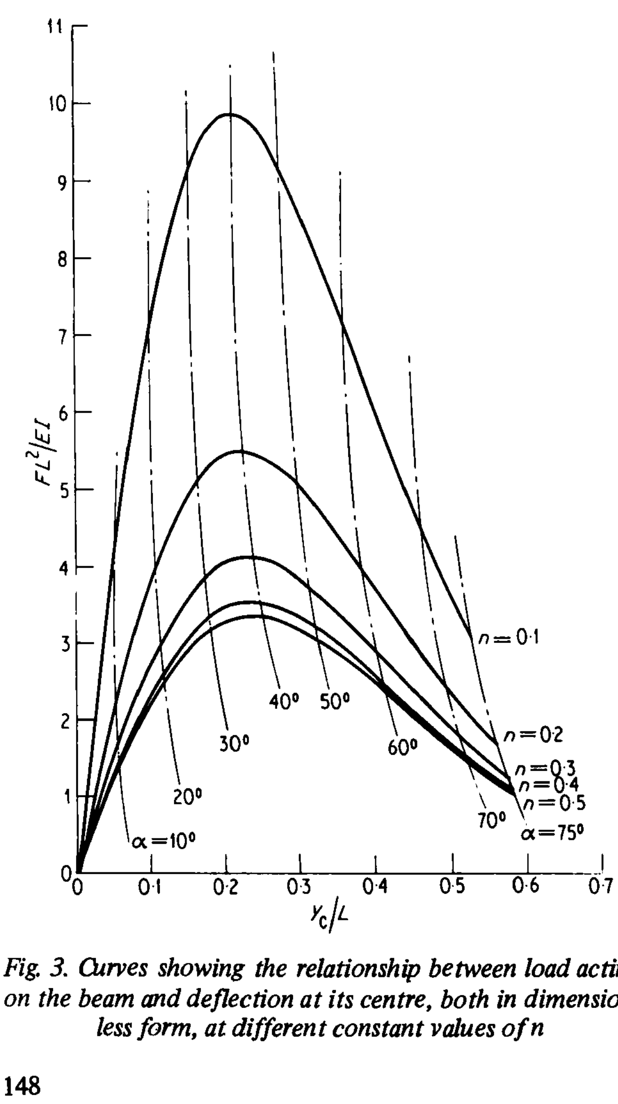

Fig. 3. Curves showing the relationship between load acting on the beam and deflection at its centre, both in dimensionless form, at different constant values of n

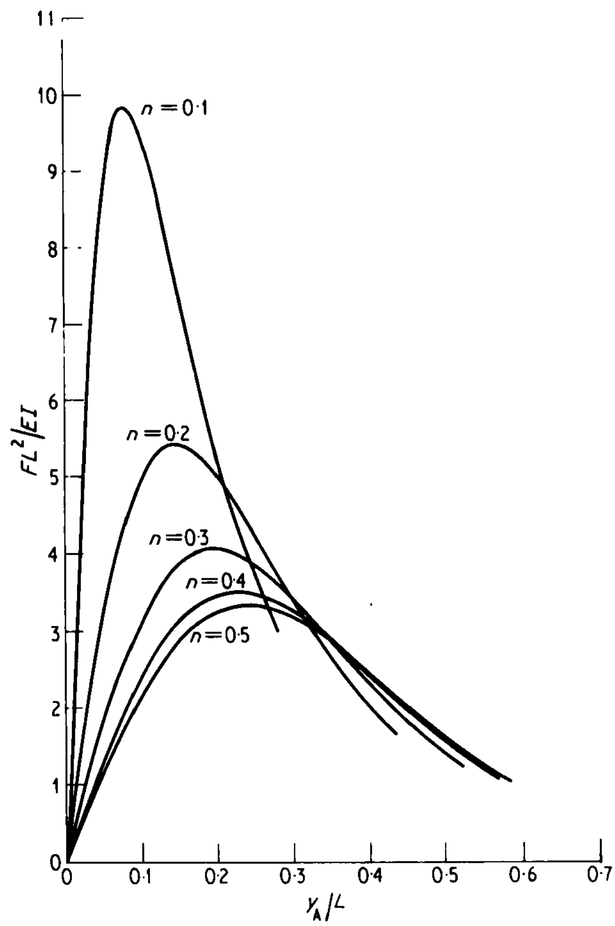

Fig. 4. Curves showing the relationship between load acting on the beam and its deflection at the point of load application, both in dimensionless form, at different constant values of n

Note also that, when deflections are large, the portion of the beam between the two symmetrically applied loads is no longer in pure bending but is under a combination of a bending moment and an axial load, since F_x is not equal to R_x. However, the condition of the central portion being under pure bending is approached closely when a is small, when $\beta \to \alpha$.

Clearly, from Fig. 3, the curves of load vs deflection fall when the deflection ratio y_C/L exceeds about 0.20 – 0.25, which corresponds to $\alpha \approx 40°$. This is due to a rapid increase in the bending moment imposed on the beam by the horizontal components of the applied loads, which is related to changes in beam geometry when α exceeds about 40°. However, deflection ratios above 0.2 are generally of little practical interest.

Note too that, when $n = 0.5$, the results we obtained correspond exactly to Freeman's solution for a beam in three-point bending.

APPENDIX 1

Values of the function

$$M(\psi) = \int_{\phi=0}^{\phi=\psi} \cos^{\frac{1}{2}}\phi \, d\phi$$

obtained using the corrected trapezoidal rule (6).

Angle ψ	$M(\psi)$	Angle ψ	$M(\psi)$	Angle ψ	$M(\psi)$
0	0.000000	30	0.511549	60	0.948025
1	0.017453	31	0.527750	61	0.960273
2	0.034903	32	0.543866	62	0.972329
3	0.052348	33	0.559894	63	0.984188
4	0.069785	34	0.575832	64	0.995846
5	0.087211	35	0.591676	65	1.007298
6	0.104624	36	0.607424	66	1.018537
7	0.122021	37	0.623072	67	1.029558
8	0.139399	38	0.638618	68	1.040354
9	0.156756	39	0.654058	69	1.050920
10	0.174090	40	0.669389	70	1.061249
11	0.191396	41	0.684608	71	1.071332
12	0.208673	42	0.699712	72	1.081163
13	0.225918	43	0.714699	73	1.090734
14	0.243129	44	0.729563	74	1.100035
15	0.260301	45	0.744303	75	1.109057
16	0.277434	46	0.758915	76	1.117789
17	0.294524	47	0.773395	77	1.126222
18	0.311569	48	0.787741	78	1.134341
19	0.328565	49	0.801948	79	1.142133
20	0.345510	50	0.816013	80	1.149583
21	0.362402	51	0.829933	81	1.156673
22	0.379237	52	0.843703	82	1.163382
23	0.396013	53	0.857320	83	1.169686
24	0.412726	54	0.870781	84	1.175557
25	0.429375	55	0.884081	85	1.180959
26	0.445957	56	0.897216	86	1.185845
27	0.462467	57	0.910183	87	1.190154
28	0.478905	58	0.922976	88	1.193793
29	0.495267	59	0.935591	89	1.196603

APPENDIX 2

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