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Initiation of cracks with cohesive force models:
a variational approach

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Abstract

In the spirit of the variational approach of Fracture Mechanics initiated in [Del Piero, G., 1997. One-dimensional ductile-brittle transition, yielding and structured deformations. In: P. Argoul, M. Frémond (Eds.), Proceedings of IUTAM Symposium "Variations de domaines et frontières libres en mécanique", Paris, 1997, Kluwer Academic] and [Francfort, G.A., Marigo, J.-J., 1998. Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids 46 (8), 1319–1342], we define the loss of stability of the elastic response of the body as the criterion of initiation of cracks. The result is very sensitive to the choice of the surface energy density. On one hand, if we adopt the Griffith assumption, then the elastic state is generally always stable. On the other hand, in the case of a surface energy of the Barenblatt type, i.e. a surface energy depending non-trivially on the jump of the displacement and inducing cohesive forces, the elastic response remains stable only if the stress field does not reach a critical value. In the full three-dimensional context of an isotropic material, we prove that this yield stress criterion is equivalent to a maximal traction criterion and a maximal shear criterion if the surface energy density is Fréchet differentiable at the origin. When the surface energy density is only Gâteaux differentiable, we obtain a yield stress criterion based on an intrinsic curve in the Mohr diagram. In any case, the domain of the admissible stress tensors is convex, unbounded in the direction of the hydrostatic pressures and depends only on the extreme eigenvalues of the stress tensor.
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1. Introduction

Griffith's theory of fracture (Griffith, 1920) remains until now the most used although it contains major drawbacks. The most important is probably its incapacity to account for crack initiation in a sound elastic body. Let us recall that this theory is based on two fundamental ingredients: (i) an assumption on the surface energy inducing the absence of any cohesive force between the lips of a crack; (ii) a criterion of propagation formulated in terms of a critical energy

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release rate. The classical methods of the Calculus of Variations reveals that the Griffith criterion of propagation is nothing but a so-called first order optimality condition, see Francfort and Marigo (1998). Specifically, it can be seen as a necessary condition of the state of the body to be a local minimum of the energy. It is obtained by comparing the energy of the body in its initial state with that of the body containing a new small crack surface. If we try to apply it to a sound body, it appears that the release of the potential energy due to a small crack surface tends generally to zero more quickly than the area of the crack. Hence no initiation is possible. That result has been regarded as true since a long time even if a quite general proof is recent, see Chambolle et al. (2005). In order to correct the drawbacks of the Griffith's theory, the first idea was to conserve the Griffith assumption on the surface energy and to simply reinforce his criterion by changing the concept of local minimum by that of global minimum. Unfortunately, even if it becomes thus possible to predict realistic crack initiation in some circumstances, the reinforced criterion appears too strong in general, particularly when surface or body forces are prescribed, see Charlotte et al. (2000). So, if one wants to remain within the variational framework of Fracture Mechanics, the unique solution is to change the form of the surface energy.

By using the concept of local minimum with a surface energy of the Barenblatt type, it is shown in Del Piero (1997) that a bar will break once a critical stress is reached. Of course, this concept of yield stress was introduced by Barenblatt himself Barenblatt (1962) to correct other drawbacks of the Griffith theory, like the inevitable presence of a stress singularity at the crack front. Nevertheless, Barenblatt, like Dugdale (1960) and many other workers (Needleman, 1987, 1992) after, was satisfied to introduce the concept of cohesive forces, but he never made use of it as a criterion of crack initiation. The cause was surely the absence of a genuine variational approach. Several works have been published since, in the same spirit, but all were limited to the one-dimensional context, see Del Piero and Truskinovsky (2001), Charlotte et al. (2000) or Marigo and Truskinovky (2004). In the second part of our paper, we investigate once more the one-dimensional case in order to illustrate all the different concepts of stability introduced in the first part. That also allows to easily compare the stability results obtained in the case of the Griffith surface energy with those related to a Barenblatt-type surface energy. But the principal interest of the paper lies incontestably in the third part where one tackles the questions of initiation of crack in the three-dimensional structures.

Let us mention finally that, since we are only interested in obtaining a criterion of crack initiation, it is not necessary to set the whole evolution problem of the cracking in the body when the loading evolves with the time. In particular, it is unnecessary to account for the reversibility of the cracking, condition which needs additional assumptions and which constitutes a delicate question in the case of Barenblatt surface energy.

The paper is organized as follows. In Section 2, we introduce the ingredients of our variational approach: the different parts of the energy, the definitions of global, local or directional stability, the elastic response of the body. The criterion of initiation of cracks is then defined as the loss of stability of the elastic response. The end of the section is devoted to the derivations of the so-called zero-, first- and second order stability conditions which are used in the sequel. Section 3 is devoted to the treatment of the general one-dimensional case. The main results are Propositions 4 and 6. In Proposition 4, it is proved that the elastic response is locally stable *for any loading* in the case of the Griffith surface energy. On the other hand, in the case of a Barenblatt surface energy, Proposition 6 establishes that it becomes unstable once the stress reaches a critical value σ_c given by the slope at the origin of the surface energy density. Moreover, this stress yield criterion is one of the first order necessary stability conditions for any state of the body, what allows to consider σ_c as the rupture limit stress of the material. In Section 4, the full three-dimensional problem is investigated by assuming that the Barenblatt surface energy density is isotropic and Gâteaux differentiable. We prove in Proposition 9 that the elastic response is locally directionally stable provided that the elastic stress field satisfies a so-called yield stress criterion. The inequality (102) that the stress tensor must satisfy pointwise involves the directional derivatives of the surface energy density at the origin and generalizes the one-dimensional result. Furthermore, an extended analysis shows that this criterion is equivalent to a maximal traction criterion and a maximal shear criterion if the surface energy density is Fréchet differentiable (Proposition 10). When the surface energy density is only Gâteaux differentiable, the criterion is written in terms of an intrinsic curve in the Mohr diagram (Proposition 11), see Salençon (1983). In both cases, the domain of the admissible stress tensors is convex, unbounded in the direction of the hydrostatic pressures and depends only on the extreme eigenvalues of the stress tensor. This part completes the preliminary results presented in Laverne and Marigo (2004).

From the technical standpoint, we essentially use basic tools of the Calculus of Variations, see Ekeland and Temam (1976), Dacorogna (1989) or Dacorogna (1992), and basic concepts of Fracture Mechanics, see Cherepanov (1979), Bui (1978) or Leblond (2000). Throughout, intrinsic vectorial notation are generally used: vectors and tensors are

represented by boldface letters, their components by italic letters. The point $\cdot$ stands for the inner product of vectors or tensors: for instance, $\mathbf{v} \cdot \mathbf{v} = \sum_i v_i v_i$, $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sum_{i,j} \sigma_{ij} \varepsilon_{ij}$. The norm of vectors, tensors or fields is represented by $\| \ \|$.

2. The ingredients of the variational approach

2.1. The energies and the admissible displacement fields

Let Ω be the reference configuration of a body. We assume that Ω is an open, bounded and connected subset of $\mathbb{R}^N$, $1 \leqslant N \leqslant 3$, with a smooth boundary $\partial \Omega$. The open set Ω is the interior of its closure $\overline{\Omega}$, which ensures that the body is free of any preexisting crack.

Let $\mathbf{u}$ be a displacement field of the body, i.e. $\mathbf{u}$ is a vector field defined on $\overline{\Omega}$ with values in $\mathbb{R}^N$. We allow $\mathbf{u}$ to be discontinuous across a $(N-1)$-dimensional oriented set called the *jump set* of $\mathbf{u}$ and denoted by $S_{\mathbf{u}}$. To simplify the presentation, we will assume that $S_{\mathbf{u}}$ is the reunion of a finite number of smooth hypersurfaces. Denoting by $\boldsymbol{\nu}(\mathbf{x})$ the unit normal to $S_{\mathbf{u}}$ at the point $\mathbf{x}$, the jump $[\![\mathbf{u}]\!](\mathbf{x})$ of $\mathbf{u}$ at $\mathbf{x}$ is defined by

$$[\![\mathbf{u}]\!](\mathbf{x}) = \mathbf{u}^+(\mathbf{x}) - \mathbf{u}^-(\mathbf{x}), \quad \mathbf{u}^\pm(\mathbf{x}) = \lim_{\{\mathbf{y} \to \mathbf{0}: \ \mathbf{y} \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\}} \mathbf{u}(\mathbf{x} \pm \mathbf{y}). \tag{1}$$

Assuming that the displacements are small, the (linearized) strain tensor field $\boldsymbol{\varepsilon}(\mathbf{u})$ is defined on the "uncracked" part of the body $\Omega \setminus S_{\mathbf{u}}$ and reads as:

$$2\varepsilon_{ij}(\mathbf{u}) = u_{i,j} + u_{j,i}. \tag{2}$$

The *total mechanical energy* $\mathcal{E}(\mathbf{u})$ associated at the displacement field $\mathbf{u}$ is divided into three parts:

$$\mathcal{E}(\mathbf{u}) = \mathcal{E}_b(\mathbf{u}) + \mathcal{E}_s(\mathbf{u}) + \mathcal{F}(\mathbf{u}). \tag{3}$$

The first part $\mathcal{E}_b(\mathbf{u})$ corresponds to the *strain energy* of the body. We assume that the behavior of the uncracked part of the body is governed by the elastic-type potential Ψ which is a strictly convex function defined on the linear space of $N \times N$ symmetric matrices. Moreover, Ψ is supposed at least twice continuously differentiable and enjoys the growing condition

$$\exists a \in \mathbb{R}, \ \exists \alpha > 0, \ \exists p \in (1, +\infty): \quad \Psi(\varepsilon) \geqslant a + \alpha \|\varepsilon\|^p, \tag{4}$$

where $\|\varepsilon\|$ denotes the euclidean norm of matrices, i.e. $\|\boldsymbol{\varepsilon}\| = \sqrt{\varepsilon_{ij} \varepsilon_{ij}}$. So, the bulk energy reads as

$$\mathcal{E}_b(\mathbf{u}) = \int_{\Omega \setminus S_{\mathbf{u}}} \Psi\big(\boldsymbol{\varepsilon}(\mathbf{u})\big) \, \mathrm{d}x. \tag{5}$$

The second part $\mathcal{E}_s(\mathbf{u})$ of the total mechanical energy represents the *surface energy* due to the discontinuities of the displacement field $\mathbf{u}$. Introducing the surface energy density Φ which may depend on the local orientation $\boldsymbol{\nu}$ of the surface $S_{\mathbf{u}}$ and on the jump $[\![u]\!]$ as well, the total surface energy reads as

$$\mathcal{E}_s(\mathbf{u}) = \int_{S_{\mathbf{u}}} \Phi\big(\boldsymbol{\nu}, [\![\mathbf{u}]\!]\big) \, \mathrm{d}S. \tag{6}$$

In order to prohibit the interpenetration of the lips of the cracks, the normal jump must be non-negative:

$$[\![\mathbf{u}]\!] \cdot \boldsymbol{\nu} \geqslant 0 \quad \text{on } S_{\mathbf{u}}. \tag{7}$$

This condition can also be ensured by imposing that the density surface energy is infinite when the normal jump is negative:

$$\Phi\big(\boldsymbol{\nu}, [\![\mathbf{u}]\!]\big) = +\infty \quad \text{if } [\![\mathbf{u}]\!] \cdot \boldsymbol{\nu} < 0. \tag{8}$$

In the same manner the jump set $S_{\mathbf{u}}$ could be restricted to be some part of the body. For instance, let us assume that the body is made of two materials which divide Ω into two parts Ω_1 and Ω_2 separated by the interface $\mathcal{I} = \partial \Omega_1 \cap \partial \Omega_2$. Then, one can restrict $S_{\mathbf{u}}$ to be only inside the first material domain (or at its boundary) if the second material is

unbreakable, by imposing $\mathcal{E}_s(\mathbf{u}) = +\infty$ if $S_{\mathbf{u}} \not\subset \partial\overline{\Omega}_1$. One can also restrict $S_{\mathbf{u}}$ to be only at the interface as well, by imposing $\mathcal{E}_s(\mathbf{u}) = +\infty$ if $S_{\mathbf{u}} \not\subset \mathcal{I}$.

The third part $\mathcal{F}(\mathbf{u})$ of the total mechanical energy is the *potential of the prescribed bulk or surface forces*. By assuming that the body is submitted to a density $\mathbf{f}$ of bulk forces and to a density $\mathbf{F}$ of surface forces on a part $\partial_F\Omega$ of its boundary, $\mathcal{F}(\mathbf{u})$ reads as:

$$\mathcal{F}(\mathbf{u}) = -\int_{\Omega} \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}x - \int_{\partial_F\Omega} \mathbf{F} \cdot \mathbf{u}\,\mathrm{d}S. \tag{9}$$

Moreover we assume that the densities $\mathbf{f}$ and $\mathbf{F}$ are smooth enough so that $\mathbf{f} \in L^q(\Omega, \mathbb{R}^N)$ and $\mathbf{F} \in L^q(\partial_F\Omega, \mathbb{R}^N)$, with $q = (p-1)/p$. Naturally, it is also possible to insert the kinematic boundary conditions in the definition of $\mathcal{F}(\mathbf{u})$. Indeed, let us assume that the displacements are prescribed on the part $\partial_D\Omega$ of the boundary $\partial\Omega$ et let $\mathbf{U}$ be their value. Since a discontinuity may appear on this part of the boundary, i.e. $S_{\mathbf{u}} \cap \partial_D\Omega \neq \emptyset$, the kinematic condition must be written in terms of the outer trace of $\mathbf{u}$ on $\partial_D\Omega$. If $\mathbf{\nu}$ is chosen as the outer (resp. the inner) normal to $\partial\Omega$, then the kinematic condition reads as

$$\mathbf{u}^+ \text{ (resp. } \mathbf{u}^-) = \mathbf{U} \quad \text{on } \partial_D\Omega. \tag{10}$$

This condition will appear in the potential energy of the loads by imposing

$$\mathcal{F}(\mathbf{u}) = +\infty \quad \text{if } \mathbf{u}^+(\text{resp. } \mathbf{u}^-) \neq \mathbf{U} \text{ on } \partial_D\Omega. \tag{11}$$

Adopting these extended definitions of the energies, we will consider as being admissible any displacement field with finite energy.

Definition 1. For a given loading of the body, a displacement field $\mathbf{u}$ is called an admissible state of the body if $\mathbf{u}$ is piecewise smooth and if its associated total energy $\mathcal{E}(\mathbf{u})$ is finite. The set of all the admissible states is denoted by $\mathcal{D}$. By construction, $\mathcal{D}$ is a convex cone. Moreover, we will distinguish the subset $\mathcal{C}$ of $\mathcal{D}$ containing all continuous admissible states:

$$\mathcal{C} = \{\mathbf{u} \in \mathcal{D}: \ S_{\mathbf{u}} = \emptyset\}. \tag{12}$$

In the definition of local stability below enters a norm for the displacement field which must be chosen in accordance with the properties of the strain energy density. In view of the growing condition (4), the norm of $\mathbf{u}$ is defined as

$$\|\mathbf{u}\| = \left(\frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega \setminus S_{\mathbf{u}}} \|\boldsymbol{\varepsilon}(\mathbf{u})\|^p \,\mathrm{d}x\right)^{1/p} + \frac{1}{\mathrm{vol}(\Omega)} \int_{S_{\mathbf{u}}} \|[\![\mathbf{u}]\!]\|\,\mathrm{d}S, \tag{13}$$

where the volume of Ω is introduced so that the norm is dimensionless. (In (13), three norms appear. The first is the defined norm of the displacement field, the second is the Euclidean norm of matrices and the third is the usual Euclidean norm of vectors, i.e. $\|[\![\mathbf{u}]\!]\| = \sqrt{[\![u_i]\!][\![u_i]\!]}$.)

2.2. The stability definitions

We are now in a position to define the concepts of stability of a state. We will distinguish three types of stability, by starting from the concept of global stability (the strongest requirement) and by finishing by the concept of local directional stability (the weakest one).

Definition 2 *(Global stability).* For a given loading, we say that the displacement field $\mathbf{u}$ is a *globally stable state* of the body if the total energy of the body in this state is less than or equal to the total energy of the body in any state:

$$\forall \mathbf{v} \in \mathcal{D}, \quad \mathcal{E}(\mathbf{u}) \leqslant \mathcal{E}(\mathbf{v}). \tag{14}$$

Definition 3 *(Local stability).* For a given loading, we say that the admissible displacement field **u** is a *locally stable state* if there exists a neighborhood (in the sense of the chosen norm) of **u** such that the total energy of the body in this state is less than or equal to the total energy of the body in any state of this neighborhood:

$$\exists r > 0, \ \forall \mathbf{v} \in \mathcal{D}: \|\mathbf{v} - \mathbf{u}\| \leqslant r, \quad \mathcal{E}(\mathbf{u}) \leqslant \mathcal{E}(\mathbf{v}). \tag{15}$$

Definition 4 *(Local directional stability).* Let **v** be a displacement of norm 1. For a given loading, we say that the admissible displacement field **u** is *locally stable in the direction* **v** if there exists a line segment $[\mathbf{u}, \mathbf{u} + r\mathbf{v}]$ such that the total energy of the body at **u** is less than or equal to the total energy of the body at any of the states of this segment:

$$\exists r > 0, \ \forall h \in [0, r], \quad \mathcal{E}(\mathbf{u}) \leqslant \mathcal{E}(\mathbf{u} + h\mathbf{v}). \tag{16}$$

When **u** is locally stable in any direction, we say that **u** is locally directionally stable.

The reader will note that these notions of stability of a state depend on the type of loading. In other words, the same displacement field may be a stable equilibrium state when the boundary conditions are prescribed by a hard device, but unstable when the loading is prescribed by a soft device.

2.3. The elastic state and the crack initiation criterion

Let us first define the elastic state of the body.

Definition 5 *(Elastic state).* For a given loading, the elastic state of the body is defined as the continuous admissible state $\mathbf{u}^e$ which minimizes the total energy of the body among all continuous admissible states, i.e.

$$\mathbf{u}^e \in \mathcal{C}: \forall \mathbf{v} \in \mathcal{C}, \quad \mathcal{E}(\mathbf{u}^e) \leqslant \mathcal{E}(\mathbf{v}). \tag{17}$$

This definition is of course conform to the usual definition of the equilibrium state of an elastic body according of the theorem of the potential energy minimum: owing to the strict convexity and the growing properties of Ψ, and owing to the smoothness assumptions on the loading, this theorem ensures the existence and the uniqueness of $\mathbf{u}^e$. Moreover, the elastic state satisfies the following variational equation:

$$\int_{\Omega} \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, \mathrm{d}x = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x + \int_{\partial_F \Omega} \mathbf{F} \cdot \mathbf{v} \, \mathrm{d}S, \quad \forall \mathbf{v} \in \mathcal{C}_0, \tag{18}$$

where $\boldsymbol{\sigma}^e = \Psi'(\boldsymbol{\varepsilon}(\mathbf{u}^e))$ denotes the elastic stress field. Of course, the elastic stress field satisfies the equilibrium equations and the natural boundary conditions (in a classical sense or in a weak sense following the smoothness of **f** and **F**):

$$\operatorname{div} \boldsymbol{\sigma}^e + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \qquad \boldsymbol{\sigma}^e \mathbf{n} = \mathbf{F} \quad \text{on } \partial_F \Omega. \tag{19}$$

However, although this theorem says that the elastic response is globally stable under continuous perturbation fields, it does not ensure that the elastic response is a stable state in the sense of any of the three definitions above, because it remains to compare its energy to that of discontinuous states. Therefore, we propose to define the concept of crack initiation as the loss of stability of the elastic response, in the spirit of the stability concepts developed by Nguyen Q.S. Nguyen (2000).

Definition 6 *(Crack initiation).* The loading is said no more compatible with the uncracked state of the body once the elastic response is no more locally stable in at least one direction:

$$\exists \mathbf{v}: \|\mathbf{v}\| = 1, \ \exists \{h_n\}_{n \in \mathbb{N}}: h_n \downarrow 0 \quad \text{such that} \quad \mathcal{E}(\mathbf{u}^e + h_n \mathbf{v}) < \mathcal{E}(\mathbf{u}^e). \tag{20}$$

In practice, we will determine the set of loadings such that the elastic response is locally directionally stable and the boundary of this set will give the critical loadings at which a first discontinuity appears in the body.

2.3.1. The stability conditions

To test the local stability of the elastic state $\mathbf{u}^e$ we have to compare its energy with that of all its neighbors. A neighborhood of $\mathbf{u}^e$ is made of states of the form

$$\mathbf{u}^e + h\mathbf{v} \quad \text{with } 0 < h < r, \ \|\mathbf{v}\| = 1, \tag{21}$$

where r can depend on $\mathbf{v}$ if we test only the directional stability. In order that $\mathbf{u}^e + h\mathbf{v}$ is admissible (i.e. with finite energy), the direction $\mathbf{v}$ must satisfy the following conditions:

$$[\![\mathbf{v}]\!] \cdot \nu \geqslant 0 \quad \text{on } S_{\mathbf{v}}, \qquad \mathbf{v}^+ \ (\text{or } \mathbf{v}^-) = 0 \quad \text{on } \partial_D \Omega. \tag{22}$$

Like $\mathcal{D}$, the set of all the admissible directions, i.e. the set of $\mathbf{v}$ satisfying (22), is a convex cone denoted by $\mathcal{D}_0$. Its subset which is made up with all the continuous functions will be denoted by $\mathcal{C}_0$; $\mathcal{C}_0$ is a linear space.

Let $\mathbf{u}_h = \mathbf{u}^e + h\mathbf{v}$ be a family of admissible states in a neighborhood of $\mathbf{u}^e$ with $h > 0$, $\mathbf{v} \in \mathcal{D}_0$ and $\|\mathbf{v}\| = 1$. When h tends to 0, $\mathbf{u}_h$ converges to $\mathbf{u}^e$. The energy $\mathcal{E}(\mathbf{u}_h)$ can be expanded in powers of h provided that both the elastic potential Ψ and the surface energy density Φ are smooth enough. In such a case, that leads to the following expansion:

$$\mathcal{E}(\mathbf{u}^e + h\mathbf{v}) = \mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) + h\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) + \frac{h^2}{2}\mathcal{E}_2(\mathbf{u}^e; \mathbf{v}) + \mathrm{o}(h^2). \tag{23}$$

In the case of the Griffith assumption on the surface energy, since Φ is not continuous, $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v})$ is not equal to $\mathcal{E}(\mathbf{u}^e)$ but homogeneous of degree 0 in $\mathbf{v}$,

$$\forall \lambda > 0, \quad \mathcal{E}_0(\mathbf{u}^e; \lambda\mathbf{v}) = \mathcal{E}_0(\mathbf{u}^e; \mathbf{v}). \tag{24}$$

In the case of Barenblatt surface energy, we have $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e)$, but the functional $\mathbf{u} \mapsto \mathcal{E}(\mathbf{u})$ is not necessarily differentiable. If we assume that it admits at least directional derivatives, the term $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v})$ is not necessarily linear in $\mathbf{v}$ but at least homogeneous of degree 1; Similarly, $\mathcal{E}_2(\mathbf{u}^e; \mathbf{v})$ is not necessarily quadratic but at least homogeneous of degree 2:

$$\forall \lambda > 0, \quad \mathcal{E}_1(\mathbf{u}^e; \lambda\mathbf{v}) = \lambda\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) \quad \text{and} \quad \mathcal{E}_2(\mathbf{u}^e; \lambda\mathbf{v}) = \lambda^2\mathcal{E}_2(\mathbf{u}^e; \mathbf{v}). \tag{25}$$

In order that $\mathbf{u}^e$ be locally directionally stable, the inequality $\mathcal{E}(\mathbf{u}^e + h\mathbf{v}) \geqslant \mathcal{E}(\mathbf{u}^e)$ must hold for all $h \in (0, r)$ and all $\mathbf{v} \in \mathcal{D}_0$ such that $\|\mathbf{v}\| = 1$. Consequently, passing to the limit in (23) when h tends to 0 leads to $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) \geqslant 0$. This inequality must hold for all the admissible directions of norm 1, but, because of the homogeneity of $\mathcal{E}_0$, it must hold in fact for any admissible direction. Moreover, if $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) > \mathcal{E}(\mathbf{u}^e)$, we can choose h sufficiently small so that $\mathcal{E}(\mathbf{u}_h) \geqslant \mathcal{E}(\mathbf{u}^e)$ and hence $\mathbf{u}^e$ is locally stable in that direction $\mathbf{v}$. We have obtained the so-called zero order stability conditions:

Proposition 1 (SC0). *The elastic state $\mathbf{u}^e$ is locally directionally stable only if it satisfies the zero order stability conditions*:

$$\forall \mathbf{v} \in \mathcal{D}_0, \quad \mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) \geqslant \mathcal{E}(\mathbf{u}^e). \tag{26}$$

The elastic state $\mathbf{u}^e$ is locally stable in the direction $\mathbf{v}$ if $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) > \mathcal{E}(\mathbf{u}^e)$.

Let us now consider a direction $\mathbf{v}$ such that $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e)$. In order that $\mathbf{u}^e$ be locally stable in this direction $\mathbf{v}$, $\mathbf{u}^e$ must satisfy $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) \geqslant 0$. Moreover, if $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) > 0$, we can choose h sufficiently small so that $\mathcal{E}(\mathbf{u}_h) \geqslant \mathcal{E}(\mathbf{u}^e)$ and hence $\mathbf{u}^e$ is locally stable in that direction $\mathbf{v}$. We have obtained the so-called first order stability conditions.

Proposition 2 (SC1). *The elastic state $\mathbf{u}^e$ is locally directionally stable only if $\mathbf{u}^e$ satisfies the first order stability conditions*:

$$\forall \mathbf{v} \in \mathcal{D}_0 : \mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e), \quad \mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) \geqslant 0. \tag{27}$$

The elastic state $\mathbf{u}^e$ is locally stable in the direction $\mathbf{v}$ if $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) > \mathcal{E}(\mathbf{u}^e)$ or if $(\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e)$ and $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) > 0)$.

We can reiterate the process to obtain the so-called second order stability conditions.

Proposition 3 *(SC2). The elastic state $\mathbf{u}^e$ is locally directionally stable only if $\mathbf{u}^e$ satisfies the second order stability conditions*:

$$\forall \mathbf{v} \in \mathcal{D}_0: \mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e) \text{ and } \mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) = 0, \quad \mathcal{E}_2(\mathbf{u}^e; \mathbf{v}) \geqslant 0. \tag{28}$$

The elastic state $\mathbf{u}^e$ is locally stable in the direction $\mathbf{v}$ if $\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) > \mathcal{E}(\mathbf{u}^e)$ or if $(\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e)$ and $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) > 0)$ or if $(\mathcal{E}_0(\mathbf{u}^e; \mathbf{v}) = \mathcal{E}(\mathbf{u}^e), \mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) = 0$ and $\mathcal{E}_2(\mathbf{u}^e; \mathbf{v}) > 0)$.

3. The one-dimensional case

3.1. Setting of the problem

Consider now a homogeneous bar of natural length L and of cross-sectional area S. The bar is made of an elastic breakable material characterized by the elastic potential Ψ and the surface energy density Φ, the properties of which will be given below. Before any loading, the bar is assumed to be sound. Its reference configuration is the interval $\Omega = (0, L)$. When a field u is discontinuous at $x \in [0, L]$, the normal ν is taken equal to $+1$ so that u^+ (resp. u^-) denotes the right (resp. left) limit. The prime will denote the derivative of any function with respect to its scalar variable. We will only consider fields u containing a *finite* number of discontinuity points, the interested reader should refer to Braides et al. (1999) for a treatment of such a problem in the whole space of functions of bounded variations.

The elastic potential Ψ is a twice continuously differentiable and strictly convex function of $\varepsilon = u'$ which grows to infinity like $|\varepsilon|^p$ with $1 < p < +\infty$. Consequently, the norm is chosen as follows:

$$\|u\| = \left(\frac{1}{L} \int_{(0,L) \setminus S_u} |u'|^p \right)^{1/p} + \frac{1}{L} \sum_{S_u} |[\![u]\!]|. \tag{29}$$

The bar is submitted to a linear density of forces $f \in L^q$, $q = p/(p-1)$. The extremity $x = 0$ is fixed whereas the extremity $x = L$ is controlled by a soft device and submitted to a force F. Let u be a displacement field of the bar. Its potential energy $\mathcal{F}(u)$ reads as

$$\mathcal{F}(u) = \begin{cases} -\int_0^L fu \, dx - Fu^+(L) & \text{if } u^-(0) = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{30}$$

For the surface energy density, we will successively consider the models of Griffith and of Barenblatt depicted in Fig. 1.

(1) **Griffith's model:** In the case of the Griffith model Griffith (1920) the surface energy density is given by

$$\Phi\big([\![u]\!]\big) = \begin{cases} 0 & \text{if } [\![u]\!] = 0, \\ G_c & \text{if } [\![u]\!] > 0, \\ +\infty & \text{if } [\![u]\!] < 0. \end{cases} \tag{31}$$

The surface energy of the bar reads then as:

$$\mathcal{E}_s(u) = \sum_{S_u} G_c S = \text{card}(S_u) G_c S, \tag{32}$$

provided that $[\![u]\!] > 0$ on S_u. In (32), $\text{card}(S_u)$ denotes the cardinal of S_u, i.e. the number of points where u is discontinuous.

(2) **Barenblatt's model:** In the case of the Barenblatt model Barenblatt (1962), the surface energy density Φ is a concave function which depends continuously on the positive values of the displacement jump, starts from 0 and progressively grows up to the Griffith's value G_c when the displacement jump grows from 0 to infinity. Specifically, we assume that the surface energy density is twice continuously differentiable and satisfies the following properties:

$$\Phi = +\infty \quad \text{on } (-\infty, 0), \qquad \Phi(0) = 0 \quad \text{and} \quad \Phi'(0+) = \sigma_c > 0, \tag{33}$$

$$\Phi' > 0 \quad \text{and} \quad \Phi'' < 0 \quad \text{on } (0, +\infty), \qquad \Phi(+\infty) = G_c. \tag{34}$$
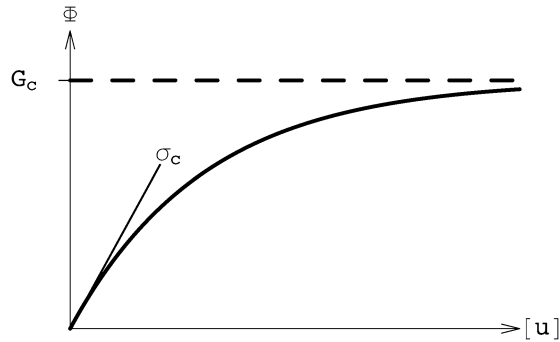
Fig. 1. The Griffith and Barenblatt surface energy densities.

3.2. The elastic state

The elastic response is obtained by minimizing $\mathcal{E}_b(u) + \mathcal{F}(u)$ on $\mathcal{C}$ with

$$\mathcal{E}_b(u) = \int_0^L \Psi\big(u'(x)\big) S \, dx \tag{35}$$

and

$$\mathcal{C} = \big\{ u \in W^{1,p}(0, L) \colon u(0) = 0 \big\}. \tag{36}$$

By virtue of the strict convexity and the coercivity of Ψ the elastic state u^e is unique. Denoting by ε^e and σ^e the elastic strain and stress fields, i.e. $\varepsilon^e = u^{e\prime}$ and $\sigma^e = \Psi'(\varepsilon^e)$, we obtain

$$\sigma^e(x)S = F + \int_x^L f(s) \, ds \tag{37}$$

and

$$u^e(x) = \int_0^x \big(\Psi'\big)^{-1}\big(\sigma^e(s)\big) \, ds. \tag{38}$$

Let us note that the stress field is absolutely continuous on $[0, L]$ and so it reaches its maximum somewhere.

3.3. The case of Griffith's surface energy

Proposition 4 (*Stability of the elastic state*). *In the case of the Griffith surface energy, the elastic state u^e is locally stable for any loading.*

Moreover, the elastic state is globally stable if and only if *the bar is nowhere in traction,* i.e. iff $\sigma^e(x) \leqslant 0$ $\forall x \in [0, L]$.

Proof. According to (17) we may only consider the admissible discontinuous directions v, i.e. v such that $\mathrm{card}(S_v) > 0$ and $[\![v]\!] > 0$ on S_v. The local directional stability is easily checked while remarking that the zero order term $\mathcal{E}_0(u^e; v)$ reads as

$$\mathcal{E}_0\big(u^e; v\big) = \mathcal{E}\big(u^e\big) + G_c S \, \mathrm{card}(S_v) \tag{39}$$

and by using Proposition 1. The local stability proof requires to refine the estimates. The convexity of Ψ gives

$$\sigma^e\big(\varepsilon - \varepsilon^e\big) \leqslant \Psi(\varepsilon) - \Psi\big(\varepsilon^e\big), \quad \forall \varepsilon. \tag{40}$$

By virtue of the equilibrium equations and the boundary conditions satisfied by σ^e, we have

$$\int_{[0,L]\setminus S_v} \sigma^e S v'\,\mathrm{d}x + \mathcal{F}(v) = -\sum_{S_v} \sigma^e S[\![v]\!] \tag{41}$$

for any admissible direction. Furthermore,

$$\mathcal{E}(v) - \mathcal{E}(u^e) \geqslant \int_{[0,L]\setminus S_v} \sigma^e S(v' - u^{e'})\,\mathrm{d}x + \mathcal{F}(v - u^e) + G_c S\,\mathrm{card}(S_v) \tag{42}$$

$$= -\sum_{S_v} \sigma^e S[\![v]\!] + G_c S\,\mathrm{card}(S_v). \tag{43}$$

When the bar is everywhere in compression, since $[\![v]\!] > 0$ on S_v, (43) ensures that $\mathcal{E}(v) \geqslant \mathcal{E}(u^e)$ for all v and hence that u^e is globally stable.

Otherwise, the definition (13) of the norm gives

$$L\|v - u^e\| \geqslant \sum_{S_v} [\![v]\!] \tag{44}$$

and we deduce from (43) that

$$\mathcal{E}(v) - \mathcal{E}(u^e) \geqslant G_c S - \max_{x\in[0,L]} \sigma^e(x) S L \|v - u^e\|. \tag{45}$$

Thus u^e is a minimizer of $\mathcal{E}$ on the ball of center u^e and of radius r with

$$r = \frac{G_c}{\max_{x\in[0,L]} \sigma^e(x) L}, \tag{46}$$

which gives the desired local stability result.

Let us finally prove that the energy is not bounded from below when the loading is such that the bar is somewhere in traction in its elastic state, i.e.

$$\inf_{v\in\mathcal{D}} \mathcal{E}(v) = -\infty \quad \text{if } \max_{[0,L]} \sigma^e > 0. \tag{47}$$

Let $x_1 \in [0, L]$ be such that $\sigma^e(x_1) = \max_{[0,L]} \sigma^e > 0$ and consider then the family of admissible fields $\{v_n\}_{n\in\mathbb{N}}$ defined as follows:

$$v_n' = u^{e'} \quad \text{in } [0, L] \setminus \{x_1\}, \qquad [\![v_n]\!](x_1) = nL. \tag{48}$$

By using (37), we get

$$\mathcal{E}(v_n) = \mathcal{E}(u^e) + G_c S - n\sigma^e(x_1) S L. \tag{49}$$

By passing to the limit when n goes to infinity we obtain (47). $\quad\square$

Proposition 5 *(Other stable states). The other locally directionally stable states are the admissible displacements fields u such that*

(1) $\mathrm{card}(S_u) > 0$;
(2) *the associated stress field $\Psi'(u')$ is equal to the elastic stress field σ^e;*
(3) $S_u \subset \{x \in [0, L]:\ \sigma^e(x) = 0\}$.

Of course, such states exist only for the particular loadings such that the bar is neither in traction everywhere nor in compression everywhere.

Proof. By virtue of the strict convexity of Ψ and of the theorem of the potential energy minimum, the elastic state is the unique locally stable *continuous* state. Let us assume that $u \in \mathcal{D}$ is another locally directionally stable state, then $\mathrm{card}(S_u) > 0$. For any admissible direction, we get:

$$\mathcal{E}_0(u; v) = \mathcal{E}(u) + G_c S \operatorname{card}(S_v \setminus S_u), \tag{50}$$

$$\mathcal{E}_1(u; v) = \int\limits_{[0,L]\setminus(S_u \cup S_v)} \sigma S v' \, dx + \mathcal{F}(v), \tag{51}$$

where $\sigma = \Psi'(u')$ denotes the stress field of the state u. Owing to Proposition 1, u is locally stable in any direction v such that $\operatorname{card}(S_v \setminus S_u) > 0$. So, we have only to consider the directions v such that $S_v \subset S_u$.

Let $v \in C^\infty(0, L)$ be a smooth direction with $v(0) = 0$. Since $S_v = \emptyset$, $\mathcal{E}_0(u; v) = \mathcal{E}(u)$. Then, by virtue of Proposition 2, we must have $\mathcal{E}_1(u; v) \geqslant 0$. By changing v by $-v$, we obtain the converse inequality and so the equality $\mathcal{E}_1(u; v) = 0$. Since this equality must hold for every v in $C^\infty(0, L)$ such that $v(0) = 0$, classical arguments of the Calculus of Variations yield

$$\sigma'S + f = 0 \quad \text{in } (0, L) \setminus S_u, \qquad [\![\sigma]\!] = 0 \quad \text{on } S_u, \qquad \sigma(L) = F \tag{52}$$

and then $\sigma = \sigma^e$. In other words the stress field is necessarily equal to the elastic stress field.

Let v be now an admissible direction such that $S_v = S_u$. Since $\mathcal{E}_0(u; v) = \mathcal{E}(u)$, we must still have $\mathcal{E}_1(u; v) \geqslant 0$. By inserting (52) into (51), the first order stability condition reads as

$$0 \leqslant \int\limits_{[0,L]\setminus S_u} \sigma^e S v' \, dx + \mathcal{F}(v) = -\sum_{S_u} \sigma^e S [\![v]\!]. \tag{53}$$

Since $[\![u]\!] > 0$ on S_u, $[\![u + hv]\!] > 0$ on S_u for h small enough. Consequently, $[\![v]\!]$ is arbitrary on S_u and if v is an admissible direction, so is $-v$. The inequality in (53) must be an equality. Owing to the arbitrariness of $[\![v]\!]$, we necessarily have

$$\sigma^e = 0 \quad \text{on } S_u. \tag{54}$$

We have thus proved that an admissible field u is locally directionally stable only if u satisfies the three conditions stated in the proposition.

Conversely, let us consider u satisfying these conditions. Then $\mathcal{E}_0(u; v) \geqslant \mathcal{E}(u)$ for any admissible direction and the equality holds for any v such that $S_v \subset S_u$. Moreover, for such a direction we have also $\mathcal{E}_1(u; v) = 0$. Let us finally consider the second order stability conditions. Since $\mathcal{E}_2(u; v)$ reads as

$$\mathcal{E}_2(u; v) = \int\limits_{[0,L]\setminus S_u} \Psi''(u') S v'^2 \, dx \tag{55}$$

and since Ψ is strictly convex, we have $\mathcal{E}_2(u; v) > 0$ in any direction of norm 1 such that $S_v \subset S_u$. Thus, u is locally directionally stable. $\quad \square$

Remark 1. The third condition of Proposition 5 ensures that there is no cohesive force at a cut if we adopt the Griffith surface energy.

In conclusion, adopting the Griffith surface energy leads to a paradoxical situation: the elastic response is always locally stable and is never globally stable when the bar is somewhere in traction. Consequently, in order to explain at the same time a certain resistance and a certain threshold of rupture of the bar under a traction loading, there is no other alternative but to consider another type of surface energy.

3.4. The case of Barenblatt's surface energy

Proposition 6 (*Stability of the elastic state and yield stress*). *In the case of a Barenblatt surface energy, the elastic state is locally stable provided that the elastic stress field σ^e is everywhere smaller than the critical stress σ_c given by the slope of the surface energy density at the origin. The elastic state is no more locally directionally stable when the elastic stress field σ^e is equal to or greater than σ_c somewhere. On the other hand, the elastic state is globally stable if and only if the bar is everywhere in compression:*

$$\max_{x \in [0,L]} \sigma^e(x) < \sigma_c \Longleftrightarrow u^e \text{ locally stable}, \tag{56}$$

$$\max_{x \in [0,L]} \sigma^e(x) \leqslant 0 \Longleftrightarrow u^e \text{ globally stable}. \tag{57}$$

Proof. It suffices to consider the admissible discontinuous directions v. Let us first remark that

$$\mathcal{E}_0(u^e; v) = \mathcal{E}(u^e), \tag{58}$$

$$\mathcal{E}_1(u^e; v) = \int_{[0,L] \setminus S_v} \sigma^e S v' \, \mathrm{d}x + \sigma_c S \sum_{S_v} [\![v]\!] + \mathcal{F}(v). \tag{59}$$

After inserting (41) into (59), we obtain

$$\mathcal{E}_1(u^e; v) = \sum_{S_v} (\sigma_c - \sigma^e) S [\![v]\!]. \tag{60}$$

We conclude that u^e is locally directionally stable when $\max_{[0,L]} \sigma^e < \sigma_c$, by virtue of Proposition 2.

Let us consider now the case where the maximal stress is reached at a certain point x_1 and is greater than the critical stress σ_c: $\sigma^e(x_1) = \max_{[0,L]} \sigma^e > \sigma_c$. Let us consider the following admissible direction v of norm 1:

$$v^-(0) = 0, \qquad [\![v]\!] = L \quad \text{on } S_v = \{x_1\}, \qquad v' = 0 \quad \text{in } [0, L] \setminus S_v. \tag{61}$$

We obtain $\mathcal{E}_1(u^e; v) = (\sigma_c - \max_{[0,L]} \sigma^e) S L < 0$ which implies that the elastic state is not locally stable in this direction v.

Let us consider the limit case where $\sigma^e(x_1) = \max_{[0,L]} \sigma^e = \sigma_c$. If we choose the same direction as above, we get $\mathcal{E}_1(u^e; v) = 0$, the second order term reads as

$$\mathcal{E}_2(u^e; v) = \Phi''(0) S L^2 \tag{62}$$

and is negative because of the concavity of Φ. We conclude that u^e is not locally stable in the direction v by virtue of Proposition 3.

To improve the above result of local directional stability in the case where $\max_{[0,L]} \sigma^e < \sigma_c$ we must refine the estimates. Let v be an admissible state. According to (40) and (41), we have

$$\mathcal{E}(v) - \mathcal{E}(u^e) \geqslant \int_{(0,L) \setminus S_v} \sigma^e S(v' - u^{e\prime}) \, \mathrm{d}x + \mathcal{F}(v - u^e) + \sum_{S_v} \Phi([\![v]\!]) S \tag{63}$$

$$= \sum_{S_v} (\Phi([\![v]\!]) - \sigma^e [\![v]\!]) S. \tag{64}$$

If the bar is everywhere in compression, (64) allows to conclude that the elastic state is globally stable. Let us consider the case where the bar is somewhere in traction:

$$0 < \sigma_m^e = \max_{[0,L]} \sigma^e < \sigma_c. \tag{65}$$

Let $r > 0$ and let $\mathbb{B}_r(u^e)$ be the ball of center u^e with radius r:

$$\mathbb{B}_r(u^e) = \{u \in \mathcal{D} : \|u - u^e\| \leqslant r\}. \tag{66}$$

By virtue of the definition of the norm and in view of the jump positivity, for any $v \in \mathbb{B}_r(u^e)$ we have $0 < [\![v]\!] \leqslant r L$ on S_v. Then from the concavity of Φ we get

$$\Phi([\![v]\!]) \geqslant \Phi'([\![v]\!])[\![v]\!] \geqslant \Phi'(r L)[\![v]\!] \quad \text{on } S_v. \tag{67}$$

Inserting the above inequality into (64) leads to

$$\mathcal{E}(v) - \mathcal{E}(u^e) \geqslant \sum_{S_v} (\Phi'(r L) - \sigma_m^e)[\![v]\!] \quad \forall v \in \mathbb{B}_r(u^e). \tag{68}$$

Since $r \mapsto \Phi'(r L)$ is continuous and since $\lim_{r \downarrow 0} \Phi'(r L) = \sigma_c > \sigma_m^e$, we obtain the local stability of u^e.

Let us finally prove that the energy is not bounded from below when the loading is such that the bar is somewhere in traction in its elastic state, i.e. $\inf_{v \in \mathcal{D}} \mathcal{E}(v) = -\infty$ if $\max_{[0,L]} \sigma^e > 0$. The proof is very similar to that given in the case of the Griffith surface energy. Let $x_1 \in [0, L]$ be such that $\sigma^e(x_1) = \max_{[0,L]} \sigma^e > 0$ and let $\{v_n\}_{n \in \mathbb{N}}$ be the family of admissible fields defined in (48). By using (37), we get now

$$\mathcal{E}(v_n) = \mathcal{E}(u^e) + \Phi(nL)S - n\sigma^e(x_1)SL \leqslant \mathcal{E}(u^e) + G_c S - n\sigma^e(x_1)SL. \tag{69}$$

Hence $\mathcal{E}(v_n) \to -\infty$ as $n \to +\infty$. □

Let us now examine which are the states satisfying the first order stability conditions.

Proposition 7. *An admissible state u satisfies the first order stability conditions $\mathcal{E}_1(u; v) \geqslant 0$ for all the admissible directions v if and only if*

(1) $\sigma' S + f = 0$ in $[0, L] \setminus S_u$, $[\![\sigma]\!] = 0$ on S_u and $\sigma(L) = F$;
(2) $\sigma = \Phi'([\![u]\!])$ on S_u;
(3) $\sigma \leqslant \sigma_c$ on $[0, L] \setminus S_u$,

where $\sigma = \Psi'(u')$ denotes the associated stress field. Accordingly, the stress field is necessarily equal to the elastic stress field: $\sigma = \sigma^e$.

Moreover, if $\max_{[0,L]} \sigma^e \leqslant 0$, then the elastic state is the unique state verifying these first order conditions. But, if $0 < \max_{[0,L]} \sigma^e < \sigma_c$, then there exists an infinite number of states u verifying these three conditions because the jump set S_u can be chosen arbitrarily as a subset of $\Omega^+ = \{x \in [0, L]: 0 < \sigma^e(x) < \sigma_c\}$.

Remark 2. The first conditions are the equilibrium equation and the natural boundary condition. The second one says that the derivative of the surface energy density gives the cohesive force at the points of discontinuity. The third one requires that the stress is nowhere greater than the slope σ_c of the surface energy density at the origin. It generalizes the condition already obtained for the elastic state and allows to interpret σ_c as the critical stress of the material. It is really important to note that this condition is a part of the *first* order stability conditions. This condition rules out all the models of cohesive forces in which $\sigma_c = 0$, because in such a case the material cannot sustain any traction.

Proof. Since $\mathcal{E}_0(u; v) = \mathcal{E}(u)$ for every admissible direction, we have to consider all the admissible directions to derive the first order stability conditions. Moreover, the set of the admissible directions depends on u because of the non-interpenetration condition. Denoting it by $\mathcal{D}_u$, it is a punctured convex cone defined by

$$\mathcal{D}_u = \{v: v^-(0) = 0, \ v' \in L^p(0, L), \ [\![v]\!] \geqslant 0 \text{ on } S_v \setminus S_u\}. \tag{70}$$

Thus, in order that u is locally directionally stable, u must satisfy

$$\forall v \in \mathcal{D}_u, \quad \int_{[0,L] \setminus (S_u \cup S_v)} \sigma S v' \, dx + \mathcal{F}(v) + \sum_{S_u \cup S_v} \Phi'([\![u]\!]) S[\![v]\!] \geqslant 0. \tag{71}$$

By choosing continuous directions, i.e. v such that $S_v = \emptyset$, we deduce the equilibrium equation and the natural boundary condition by classical arguments of the Calculus of Variations. This condition (1) of the proposition ensures that $\sigma = \sigma^e$, what is due to the fact that there exists a unique statically admissible stress field in one dimension when one end is submitted to a soft device. Inserting this condition into (71) provides

$$\forall v \in \mathcal{D}_u, \quad \sum_{S_u \cup S_v} \left(\Phi'([\![u]\!]) - \sigma^e\right) S[\![v]\!] \geqslant 0. \tag{72}$$

By choosing v such that $S_v \subset S_u = \emptyset$, since the jump of v is arbitrary on S_u, we obtain the condition (2) of the proposition. By inserting it into (72), we obtain

$$\forall v \in \mathcal{D}_u, \quad \sum_{S_v \setminus S_u} \left(\sigma_c - \sigma^e\right) S[\![v]\!] \geqslant 0. \tag{73}$$

Since $[\![v]\!] \geqslant 0$ on $S_v \setminus S_u$, the above inequality is possible if and only if the condition (3) holds.

Let us note that if u satisfies the three conditions of the proposition, then

$$\mathcal{E}_1(u; v) = 0 \Longleftrightarrow S_v \subset S_u \cup \left\{ x \in [0, L]: \sigma^e(x) = \sigma_c \right\}. \tag{74}$$

If the bar is everywhere in compression, then $S_u = \emptyset$ because otherwise we should have $0 \geqslant \sigma^e = \Phi'([\![u]\!]) > 0$ on S_u by virtue of (2) and of the monotonicity of Ψ. If $0 < \max_{[0,L]} \sigma^e < \sigma_c$, then Ω^+ is a non-empty open subset of $[0, L]$. Since $\Phi'([\![u]\!])$ decreases continuously from σ_c to 0 when $[\![u]\!]$ grows from 0 to ∞, we may choose as many points of discontinuity as we want in Ω^+ and then u is simply given by

$$u^-(0) = 0, \qquad u' = \left(\Psi'\right)^{-1}\left(\sigma^e\right) \quad \text{in } [0, L] \setminus S_u, \qquad [\![u]\!] = \left(\Phi'\right)^{-1}\left(\sigma^e\right) \quad \text{on } S_u. \qquad \square \tag{75}$$

Thus, the first order stability conditions rule out all the loadings such that $\max_{[0,L]} \sigma^e > \sigma_c$, but otherwise they leave an infinite number of candidates when the bar is somewhere in traction. In fact, by virtue of the second order stability conditions all these candidates are unstable as it is proved in the following proposition.

Proposition 8. *It exists no other locally stable state than the elastic state.*

Proof. Let $u \neq u^e$ be an admissible state satisfying the three conditions of Proposition 7. (If such a u does not exist, there is nothing to prove.) Hence, $S_u \neq \emptyset$. Let $x_1 \in S_u$ and let v be the direction of norm 1 defined as follows:

$$v^-(0) = 0, \qquad [\![v]\!] = L \quad \text{on } S_v = \{x_1\}, \qquad v' = 0 \quad \text{in } [0, L] \setminus \{x_1\}. \tag{76}$$

Since $S_v \subset S_u$, then $\mathcal{E}_1(u; v) = 0$ and $\mathcal{E}_2(u; v) = \Phi''([\![u]\!](x_1)) S L^2 < 0$. Hence u is not locally stable in this direction by virtue of Proposition 3. $\square$

The results are slightly different if we replace the soft device at the end $x = L$ by a hard device prescribing the displacement instead of the force. The interested reader should refer to Charlotte et al. (2000) or Del Piero (1997) where a complete analysis of stability is made in the case where the bar is free of body forces, i.e. $f = 0$. The properties of stability of the discontinuous states are very dependent on the concavity assumptions made on the density surface energy, see Del Piero and Truskinovsky (2001).

4. The three-dimensional case

4.1. Surface energy assumptions

We consider now a three-dimensional homogeneous body, $N = 3$, made of an *isotropic* elastic-breakable material. Its surface energy density is of the Barenblatt type and then depends on both the jump of the displacement and the local orientation of the discontinuity surface. The isotropy assumption requires that

$$\Phi(\mathbf{Q}\boldsymbol{v}, \mathbf{Q}\boldsymbol{\delta}) = \Phi(\boldsymbol{v}, \boldsymbol{\delta}), \quad \forall \mathbf{Q} \in \mathrm{SO}^3, \ \forall \boldsymbol{v} \in \mathbb{S}^2, \ \forall \boldsymbol{\delta} \in \mathbb{R}^3, \tag{77}$$

where SO^3 denotes the group of rotations (i.e. the 3×3 matrices $\mathbf{Q}$ such that $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}$ and $\det \mathbf{Q} = 1$), and $\mathbb{S}^2$ denotes the unit sphere of $\mathbb{R}^3$, i.e. $\mathbb{S}^2 = \{\boldsymbol{v} \in \mathbb{R}^3: \|\boldsymbol{v}\| = 1\}$. This is possible if and only if the surface energy density only depends on the invariants of $(\boldsymbol{v}, \boldsymbol{\delta})$. Since $\boldsymbol{v}$ is of norm 1, these invariants are $\boldsymbol{\delta} \cdot \boldsymbol{v}$ and $\|\boldsymbol{\delta}\|$, or equivalently, $\boldsymbol{\delta} \cdot \boldsymbol{v}$ and $\|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{v}\boldsymbol{v}\|$. Moreover, the non-interpenetration condition requires that $\boldsymbol{\delta} \cdot \boldsymbol{v} \geqslant 0$. Consequently, there exists a function ϕ defined on $[0, \infty)^2$ such that the surface energy density reads as

$$\Phi(\boldsymbol{v}, \boldsymbol{\delta}) = \phi\left(\boldsymbol{\delta} \cdot \boldsymbol{v}, \|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{v}\boldsymbol{v}\|\right), \quad \forall \boldsymbol{v} \in \mathbb{S}^2, \ \forall \boldsymbol{\delta} \in \mathbb{R}^3: \boldsymbol{\delta} \cdot \boldsymbol{v} \geqslant 0. \tag{78}$$

The surface energy density vanishes when $\boldsymbol{\delta} = \mathbf{0}$ and hence $\phi(0, 0) = 0$. Since the surface energy density is of the Barenblatt type, $\Phi(\boldsymbol{v}, \boldsymbol{\delta})$ is a positive continuous function of $\boldsymbol{\delta}$ at given $\boldsymbol{v}$. So, ϕ is also a positive continuous function which is not necessarily differentiable, in particular near the origin $(0, 0)$. We make the following assumptions:

Hypothesis 1. The surface energy density $(\alpha, \beta) \mapsto \phi(\alpha, \beta)$ is a continuous function defined on $[0, \infty)^2$ which enjoys the following properties:

(1) ϕ is continuously differentiable at every point $(\alpha, \beta) \neq (0, 0)$, the two partial derivatives being denoted by $\phi_{,n}$ and $\phi_{,t}$:

$$\phi_{,n} = \frac{\partial \phi}{\partial \alpha}, \qquad \phi_{,t} = \frac{\partial \phi}{\partial \beta}. \tag{79}$$

(2) ϕ admits directional derivatives at the origin, i.e. there exists a positive, one-homogeneous function φ such that for all (α, β): $\alpha \geqslant 0$, $\beta \geqslant 0$, $(\alpha, \beta) \neq (0, 0)$,

$$0 < \varphi(\alpha, \beta) = \lim_{h \downarrow 0} \frac{1}{h} \phi(h\alpha, h\beta), \qquad \varphi(\lambda\alpha, \lambda\beta) = \lambda\varphi(\alpha, \beta), \quad \forall \lambda > 0. \tag{80}$$

In particular, we denote by σ_c and τ_c the partial derivatives:

$$\sigma_c = \varphi(1, 0) > 0, \qquad \tau_c = \varphi(0, 1) > 0. \tag{81}$$

When ϕ is differentiable at $(0, 0)$, then φ is linear and reads as

$$\varphi(\alpha, \beta) = \sigma_c \alpha + \tau_c \beta. \tag{82}$$

Let $\mathbf{v} \in \mathcal{D}_0$ and let $h > 0$. In view of the smoothness assumptions made on ϕ, the energy $\mathcal{E}(\mathbf{u}^e + h\mathbf{v})$ can be expanded in terms of h up to the second order:

$$\mathcal{E}(\mathbf{u}^e + h\mathbf{v}) = \mathcal{E}(\mathbf{u}^e) + h\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) + \mathrm{o}(h) \tag{83}$$

with

$$\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) = \int_{\Omega \setminus S_{\mathbf{v}}} \boldsymbol{\sigma}^e \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, \mathrm{d}x - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x - \int_{\partial_F \Omega} \mathbf{F} \cdot \mathbf{v} \, \mathrm{d}S + \int_{S_{\mathbf{v}}} \varphi\big([\![\mathbf{v}]\!] \cdot \boldsymbol{\nu}, \|[\![\mathbf{v}]\!] - [\![\mathbf{v}]\!] \cdot \boldsymbol{\nu}\boldsymbol{\nu}\|\big) \, \mathrm{d}S. \tag{84}$$

The equilibrium equations and the boundary conditions satisfied by $\boldsymbol{\sigma}^e$ imply then

$$\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) = \int_{S_{\mathbf{v}}} \big(\varphi\big([\![\mathbf{v}]\!] \cdot \boldsymbol{\nu}, \|[\![\mathbf{v}]\!] - [\![\mathbf{v}]\!] \cdot \boldsymbol{\nu}\boldsymbol{\nu}\|\big) - \boldsymbol{\sigma}^e \boldsymbol{\nu} \cdot [\![\mathbf{v}]\!]\big) \, \mathrm{d}S. \tag{85}$$

4.2. The stress yield criterion

We are now ready to prove the following

Proposition 9. *The elastic state is locally directionally stable only if the elastic stress field satisfies almost everywhere the following inequality*:

$$\boldsymbol{\sigma}^e \boldsymbol{\nu} \cdot \boldsymbol{\delta} \leqslant \varphi\big(\boldsymbol{\delta} \cdot \boldsymbol{\nu}, \|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{\nu}\boldsymbol{\nu}\|\big), \quad \forall \boldsymbol{\nu} \in \mathbb{S}^2, \ \forall \boldsymbol{\delta} \in \mathbb{R}^3 : \boldsymbol{\delta} \cdot \boldsymbol{\nu} \geqslant 0. \tag{86}$$

Furthermore, if the above inequality is strict everywhere, then the elastic state is really locally directionally stable.

Proof. Since $\mathcal{E}_0(\mathbf{u}^e; v) = \mathcal{E}(\mathbf{u}^e)$ $\forall v \in \mathcal{D}_0$, the elastic state is locally directionally stable only if it satisfies the first order stability conditions: $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) \geqslant 0$, $\forall \mathbf{v} \in \mathcal{D}_0$. Let $\mathbf{x}_0$ be a point of Ω located at a distance d of the boundary, $\boldsymbol{\nu} \in \mathbb{S}^2$, $\boldsymbol{\delta} \in \mathbb{R}^3$ such that $\boldsymbol{\delta} \cdot \boldsymbol{\nu} \geqslant 0$ and $\eta \in C_0^{\infty}(\mathbb{R})$ be a smooth positive function such that $\eta(0) = 1$ and whose support is included in $(-d, +d)$. Let $\mathbf{v}$ be the following admissible direction

$$\mathbf{v}(\mathbf{x}) = H\big((\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\nu}\big) \eta\big(\|\mathbf{x} - \mathbf{x}_0\|\big) \boldsymbol{\delta}, \tag{87}$$

where H denotes the Heaviside function, i.e. $H(x) = 0$ if $x \leqslant 0$, $H(x) = 1$ otherwise. Denoting by

$$P(\mathbf{x}_0, \boldsymbol{\nu}) = \big\{\mathbf{x} : (\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\nu} = 0\big\} \tag{88}$$

the hyperplane through $\mathbf{x}_0$ of normal vector $\boldsymbol{\nu}$, the field $\mathbf{v}$ is discontinuous across $P(\mathbf{x}_0, \boldsymbol{\nu})$,

$$S_{\mathbf{v}} \subset P(\mathbf{x}_0, \boldsymbol{\nu}) \cap \mathbb{B}(\mathbf{x}_0, d). \tag{89}$$

The jump of $\mathbf{v}$ is equal to $\boldsymbol{\delta}$ at $\mathbf{x}_0$ and more generally

$$[[\mathbf{v}]](\mathbf{x}) = \eta(\|\mathbf{x} - \mathbf{x}_0\|)\boldsymbol{\delta} \quad \text{on } S_{\mathbf{v}}. \tag{90}$$

By inserting these relations into (85) and by using the one-homogeneity of φ, the first order stability condition becomes

$$0 \leqslant \int\limits_{P(\mathbf{x}_0, \boldsymbol{\nu}) \cap \mathbb{B}(\mathbf{x}_0, d)} \big(\varphi(\boldsymbol{\delta} \cdot \boldsymbol{\nu}, \|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{\nu}\boldsymbol{\nu}\|) - \boldsymbol{\sigma}^e(\mathbf{x})\boldsymbol{\nu} \cdot \boldsymbol{\delta}\big)\eta(\|\mathbf{x} - \mathbf{x}_0\|)\,\mathrm{d}S. \tag{91}$$

Since η is an arbitrary positive smooth function, since the point $\mathbf{x}_0$ is arbitrary in Ω, since the unit normal vector $\boldsymbol{\nu}$ is arbitrary and since the vector $\boldsymbol{\delta}$ is arbitrary, the inequality (86) must hold almost everywhere in Ω.

Conversely, if the inequality (86) is strict everywhere, then (85) gives $\mathcal{E}_1(\mathbf{u}^e; \mathbf{v}) > 0$ for any admissible direction $\mathbf{v}$ and we conclude accordingly with Proposition 2. $\square$

We have thus obtained a criterion of initiation of crack in terms of the stress field. This criterion generalizes to the three dimensional context the one-dimensional criterion $\sigma^e \leqslant \sigma_c$. This stress yield is not specific to the elastic state, but it appears as a first order condition of stability for any state as we will see below.

4.3. The first order stability conditions of any state

Consider now any admissible state $\mathbf{u}$ with a jump set $S_{\mathbf{u}}$. We assume that $\mathbf{u}$ is piecewise smooth so that all the forthcoming calculations of the present subsection are valid. We introduce the following simplified notation:

$$[[u_n]] = [[\mathbf{u}]] \cdot \boldsymbol{\nu}, \qquad [[\mathbf{u}_t]] = [[\mathbf{u}]] - [[\mathbf{u}]] \cdot \boldsymbol{\nu}\boldsymbol{\nu}. \tag{92}$$

In the jump set $S_{\mathbf{u}}$, we can distinguish the part $S_{\mathbf{u}}^n$ where the jump is normal from the part $S_{\mathbf{u}}^t$ where the jump is tangential:

$$S_{\mathbf{u}}^n = \big\{\mathbf{x} \in S_{\mathbf{u}}: [[\mathbf{u}_t]] = \mathbf{0}\big\}, \qquad S_{\mathbf{u}}^t = \big\{\mathbf{x} \in S_{\mathbf{u}}: [[u_n]] = 0\big\}. \tag{93}$$

In the same way we introduce the normal and the tangential parts of the jump of the direction $\mathbf{v}$:

$$[[v_n]] = [[\mathbf{v}]] \cdot \boldsymbol{\nu}, \qquad [[\mathbf{v}_t]] = [[\mathbf{v}]] - [[\mathbf{v}]] \cdot \boldsymbol{\nu}\boldsymbol{\nu}. \tag{94}$$

In order that $\mathbf{u} + h\mathbf{v}$ satisfies the non-interpenetration condition for h sufficiently small, $\mathbf{v}$ must satisfy

$$[[v_n]] \geqslant 0 \quad \text{on } (S_{\mathbf{v}} \setminus S_{\mathbf{u}}) \cup S_{\mathbf{u}}^t. \tag{95}$$

Let us note also that the expansion of $\|[[\mathbf{u}_t + h\mathbf{v}_t]]\|$ is not the same on $S_{\mathbf{u}}^n$ as on $S_{\mathbf{u}} \setminus S_{\mathbf{u}}^n$:

$$\big\|[[\mathbf{u}_t + h\mathbf{v}_t]]\big\| = h\big\|[[\mathbf{v}_t]]\big\| \quad \text{on } S_{\mathbf{u}}^n, \tag{96}$$

$$\big\|[[\mathbf{u}_t + h\mathbf{v}_t]]\big\| = \|\mathbf{u}_t\| + h\mathbf{v}_t \cdot \frac{\mathbf{u}_t}{\|\mathbf{u}_t\|} + \mathrm{o}(h) \quad \text{on } S_{\mathbf{u}} \setminus S_{\mathbf{u}}^n. \tag{97}$$

The necessary first order stability conditions read then

$$0 \leqslant \int\limits_{\Omega \setminus (S_{\mathbf{u}} \cup S_{\mathbf{v}})} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}x - \int\limits_{\Omega} \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x - \int\limits_{\partial_F \Omega} \mathbf{F} \cdot \mathbf{v}\,\mathrm{d}S + \int\limits_{S_{\mathbf{v}} \setminus S_{\mathbf{u}}} \varphi\big([[v_n]], \big\|[[\mathbf{v}_t]]\big\|\big)\,\mathrm{d}S$$

$$+ \int\limits_{S_{\mathbf{u}}^n} \big(\phi_{,n}([[u_n]], 0)[[v_n]] + \phi_{,t}([[u_n]], 0)\big\|[[\mathbf{v}_t]]\big\|\big)\,\mathrm{d}S$$

$$+ \int\limits_{S_{\mathbf{u}} \setminus S_{\mathbf{u}}^n} \bigg(\phi_{,n}\big([[u_n]], \big\|[[\mathbf{u}_t]]\big\|\big)[[v_n]] + \phi_{,t}\big([[u_n]], \big\|[[\mathbf{u}_t]]\big\|\big)\frac{\mathbf{u}_t}{\|\mathbf{u}_t\|} \cdot \mathbf{v}_t\bigg)\,\mathrm{d}S, \tag{98}$$

for all the admissible directions $\mathbf{v}$, where $\boldsymbol{\sigma} = \Psi'(\boldsymbol{\varepsilon}(u))$ denotes the stress field of the state $\mathbf{u}$. Let us successively consider different types of directions.

• *Continuous directions*: $S_{\mathbf{v}} = \emptyset$. In this case, $\mathbf{v}$ belongs to the linear space $\mathcal{C}_0$ and (98) reduces to

$$\int_{\Omega \setminus S_{\mathbf{u}}} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, \mathrm{d}x = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x + \int_{\partial_F \Omega} \mathbf{F} \cdot \mathbf{v} \, \mathrm{d}S, \quad \forall \mathbf{v} \in \mathcal{C}_0, \tag{99}$$

which is equivalent to the equilibrium equations (including the continuity of the vector stress on $S_{\mathbf{u}}$) and the natural boundary conditions:

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = 0 \quad \text{in } \Omega \setminus S_{\mathbf{u}}, \qquad [\![\boldsymbol{\sigma}]\!] \boldsymbol{\nu} = \mathbf{0} \quad \text{on } S_{\mathbf{u}}, \qquad \boldsymbol{\sigma} \mathbf{n} = \mathbf{F} \quad \text{on } \partial_F \Omega. \tag{100}$$

• *Directions with a jump at a sound point*: $S_{\mathbf{v}} \cap S_{\mathbf{u}} = \emptyset$. Inserting the equilibrium equations into (98) and considering jumps only at sound points lead to

$$0 \leqslant \int_{S_{\mathbf{v}}} \big(\varphi([\![v_n]\!], \|[\![\mathbf{v}_t]\!]\|) - \boldsymbol{\sigma} \boldsymbol{\nu} \cdot [\![\mathbf{v}]\!]\big) \, \mathrm{d}S. \tag{101}$$

Since $\mathbf{v}$ has only to satisfy $[\![v_n]\!] \geqslant 0$, we recover the yield criterion for the stress field $\boldsymbol{\sigma}$: $\boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\delta} \leqslant \varphi(\boldsymbol{\delta} \cdot \boldsymbol{\nu}, \|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{\nu} \boldsymbol{\nu}\|)$, which must hold for all $\boldsymbol{\nu} \in \mathbb{S}^2$ and all $\boldsymbol{\delta} \in \mathbb{R}^3$ such that $\boldsymbol{\delta} \cdot \boldsymbol{\nu} \geqslant 0$. Owing to the one-homogeneity of ϕ, we may simply consider unit vector $\boldsymbol{\delta}$ and finally the yield stress criterion reads as

$$\boxed{\boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\delta} \leqslant \varphi\big(\boldsymbol{\delta} \cdot \boldsymbol{\nu}, \|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{\nu} \boldsymbol{\nu}\|\big), \quad \forall (\boldsymbol{\nu}, \boldsymbol{\delta}) \in \mathbb{S}^2 \times \mathbb{S}^2 : \boldsymbol{\delta} \cdot \boldsymbol{\nu} \geqslant 0.} \tag{102}$$

• *Directions with a jump on $S_{\mathbf{u}}$*: $S_{\mathbf{v}} \subset S_{\mathbf{u}}$. In such a case, after inserting the equilibrium equations, (98) reads as

$$0 \leqslant \int_{S_{\mathbf{u}}} -\boldsymbol{\sigma} \boldsymbol{\nu} \cdot [\![\mathbf{v}]\!] \, \mathrm{d}S + \int_{S_{\mathbf{u}}^n} \big(\phi_{,n}([\![u_n]\!], 0)[\![v_n]\!] + \phi_{,t}([\![u_n]\!], 0) \|[\![\mathbf{v}_t]\!]\|\big) \, \mathrm{d}S$$

$$+ \int_{S_{\mathbf{u}} \setminus S_{\mathbf{u}}^n} \left(\phi_{,n}([\![u_n]\!], \|[\![\mathbf{u}_t]\!]\|)[\![v_n]\!] + \phi_{,t}([\![u_n]\!], \|[\![\mathbf{u}_t]\!]\|) \frac{\mathbf{u}_t}{\|\mathbf{u}_t\|} \cdot \mathbf{v}_t\right) \mathrm{d}S \tag{103}$$

from which we deduce the relations giving the cohesive forces on the different part of $S_{\mathbf{u}}$. After decomposing $\boldsymbol{\sigma} \boldsymbol{\nu}$ into its normal and its tangential part,

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \Sigma \boldsymbol{\nu} + \mathbf{T}, \quad \mathbf{T} \cdot \boldsymbol{\nu} = 0, \tag{104}$$

we obtain

$$\text{On } S_{\mathbf{u}}^n: \Sigma = \phi_{,n}\big([\![u_n]\!], 0\big), \quad \|\mathbf{T}\| \leqslant \phi_{,t}\big([\![u_n]\!], 0\big), \tag{105}$$

$$\text{On } S_{\mathbf{u}}^t: \Sigma \leqslant \phi_{,n}\big(0, \|[\![\mathbf{u}_t]\!]\|\big), \quad \mathbf{T} = \phi_{,t}\big(0, \|[\![\mathbf{u}_t]\!]\|\big) \frac{\mathbf{u}_t}{\|\mathbf{u}_t\|}, \tag{106}$$

$$\text{On } S_{\mathbf{u}} \setminus \big(S_{\mathbf{u}}^n \cup S_{\mathbf{u}}^t\big): \Sigma = \phi_{,n}\big([\![u_n]\!], \|[\![\mathbf{u}_t]\!]\|\big), \quad \mathbf{T} = \phi_{,t}\big([\![u_n]\!], \|[\![\mathbf{u}_t]\!]\|\big) \frac{\mathbf{u}_t}{\|\mathbf{u}_t\|}. \tag{107}$$

The inequalities in (105) and (106) are due to the unilateral constraints $[\![v_n]\!] \geqslant 0$ on $S_{\mathbf{u}}^t$ and $\|[\![\mathbf{v}_t]\!]\| \geqslant 0$ on $S_{\mathbf{u}}^n$. The normal part (resp. the tangential part) of the cohesive force remains undetermined as long as the normal part (resp. the tangential part) of the jump is zero.

4.4. Analysis of the stress yield criterion

We establish in this subsection the most important properties of the yield stress criterion (102).

4.4.1. Case of a smooth surface energy density

Let us first consider the case where the surface energy density φ is differentiable at the origin. Then φ is linear and given by (82). The yield stress criterion becomes

$$\boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\delta} \leqslant \sigma_c \boldsymbol{\delta} \cdot \boldsymbol{\nu} + \tau_c \|\boldsymbol{\delta} - \boldsymbol{\delta} \cdot \boldsymbol{\nu} \boldsymbol{\nu}\| \tag{108}$$

and the inequality must hold for all unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\delta}$ such that $\boldsymbol{\delta} \cdot \boldsymbol{\nu} \geqslant 0$. Let $\boldsymbol{\nu}$ be a given unit vector and let $\boldsymbol{\tau}$ be a unit vector orthogonal to $\boldsymbol{\nu}$. Also decompose $\boldsymbol{\delta}$ in its normal and tangential part:

$$\boldsymbol{\delta} = \cos\theta\, \boldsymbol{\nu} + \sin\theta\, \boldsymbol{\tau}, \quad \theta \in [-\pi/2, \pi/2]. \tag{109}$$

Then (108) gives, $\forall(\boldsymbol{\nu}, \boldsymbol{\tau}) \in \mathbb{S}^2 \times \mathbb{S}^2 \colon \boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0$,

$$(\boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu} - \sigma_c)\cos\theta + \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\tau} \sin\theta - \tau_c|\sin\theta| \leqslant 0, \quad \forall\theta \in [-\pi/2, \pi/2]. \tag{110}$$

or, equivalently,

$$\max_{\boldsymbol{\nu} \in \mathbb{S}^2} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu} \leqslant \sigma_c, \qquad \max_{(\boldsymbol{\nu}, \boldsymbol{\tau}) \in \mathbb{S}^2 \times \mathbb{S}^2 \colon \boldsymbol{\nu} \cdot \boldsymbol{\tau} = 0} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\tau} \leqslant \tau_c \tag{111}$$

which are nothing but the criteria of maximal traction and maximal shear. We have thus obtained the following:

Proposition 10. *When the surface energy density ϕ is differentiable at the origin, the yield stress criterion given by the first order stability conditions consists in a maximal traction criterion and a maximal shear criterion. These criteria can be written in terms of the eigenvalues $(\sigma_1, \sigma_2, \sigma_3)$ of the stress tensor and read as*

$$\max_i \sigma_i \leqslant \sigma_c, \qquad \max_{i,j}(\sigma_i - \sigma_j) \leqslant 2\tau_c, \tag{112}$$

the critical traction σ_c and the critical shear τ_c being given by the partial derivatives of ϕ at $(0, 0)$.

Remark 3. It is worth noticing that these criteria were obtained under the only assumptions of isotropy and differentiability of the surface energy density. To obtain other criteria, we must relax at least one of these two assumptions. In the next subsection, we conserve the isotropy hypothesis and analyze what happens in the case of a only directionally differentiable surface energy.

4.4.2. Case of a non-smooth surface energy density

Let us consider the "general" case where φ is only one-homogeneous. We first prove the following

Proposition 11 *(Intrinsic curve). Let us assume that the surface energy density ϕ is only Gâteaux differentiable at the origin and that φ is one-homogeneous and continuous. Let $\boldsymbol{\sigma}$ be a given stress tensor and let $\boldsymbol{\nu}$ be a given unit vector. Decompose the stress vector $\boldsymbol{\sigma}\boldsymbol{\nu}$ into its normal and tangential parts*:

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \Sigma\boldsymbol{\nu} + T\boldsymbol{\tau}, \quad \boldsymbol{\tau} \in \mathbb{S}^2 \colon \boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0. \tag{113}$$

Then, the yield stress criterion is satisfied, i.e. the following inequality holds

$$\boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\delta} \leqslant \varphi\big(\boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}, \|\boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\tau}\|\big), \quad \forall\boldsymbol{\delta} \in \mathbb{S}^2 \colon \boldsymbol{\delta} \cdot \boldsymbol{\nu} \geqslant 0 \tag{114}$$

if and only if the stress vector (Σ, T) lies in the following convex set of the Mohr diagram:

$$|T| \leqslant \varphi_\star(\Sigma) \tag{115}$$

with

$$\varphi_\star(\Sigma) = \inf_{\lambda \geqslant 0}\big\{\varphi(\lambda, 1) - \lambda\Sigma\big\}. \tag{116}$$

Moreover the function $\varphi_\star$ giving the so-called intrinsic curve $|T| = \varphi_\star(\Sigma)$ enjoys the following properties:

(1) *The function $\varphi_\star$ is defined for $\Sigma \in (-\infty, \sigma_c)$, concave, continue, decreasing and $\lim_{\Sigma \to -\infty} \varphi_\star(\Sigma) = \tau_c = \varphi(0, 1)$. $\varphi_\star$ is non-negative for $\Sigma \in (-\infty, \sigma_c^\star]$ with $\sigma_c^\star \leqslant \sigma_c = \varphi(1, 0)$.*

(2) *The domain of the admissible (Σ, T) delimited by the intrinsic curve is convex, symmetric with respect to the axis $T = 0$, unbounded in the direction of negative normal stress and bounded by $\sigma_c^\star$ in the direction of positive normal stress, see Fig. 2.*

Proof. Let us decompose $\delta \in \mathbb{S}^2$ with $\delta \cdot \nu \geqslant 0$ into its normal and tangential parts:

$$\delta = \cos\theta \nu + \sin\theta \tau', \quad \tau' \in \mathbb{S}^2 \colon \tau' \cdot \nu = 0, \ \theta \in [0, \pi/2]. \tag{117}$$

Then, (114) reads as: $\forall \theta \in [0, \pi/2]$, and $\forall \tau' \in \mathbb{S}^2$ such that $\tau' \cdot \nu = 0$,

$$T\tau \cdot \tau' \sin\theta \leqslant \varphi(\cos\theta, \sin\theta) - \Sigma \cos\theta. \tag{118}$$

Since $|\tau \cdot \tau'| \leqslant 1$, the equality being true as $\tau' = \pm\tau$, we obtain

$$|T| \sin\theta \leqslant \varphi(\cos\theta, \sin\theta) - \Sigma \cos\theta, \quad \forall \theta \in [0, \pi/2]. \tag{119}$$

Putting $\theta = 0$ gives $\Sigma \leqslant \varphi(1, 0) = \sigma_c$. Choosing then $\theta \neq 0$ and using the one-homogeneity of φ lead to

$$|T| \leqslant \varphi(\lambda, 1) - \lambda \Sigma, \quad \forall \lambda \geqslant 0 \tag{120}$$

and (115) follows. Conversely, if (115) is satisfied, then (114) holds.

Let us now prove the properties of the function $\varphi_\star$. If we consider the function $\lambda \mapsto \bar{\varphi}(\lambda)$ defined on $\mathbb{R}$ by

$$\bar{\varphi}(\lambda) = \begin{cases} \varphi(\lambda, 1) & \text{if } \lambda \geqslant 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{121}$$

then we see that, by definition, $\varphi_\star = -\bar{\varphi}^\star$, $\bar{\varphi}^\star$ denoting the Legendre transform of $\bar{\varphi}$, see Ekeland and Temam (1976). Since $\bar{\varphi}^\star$ is convex by construction, $\varphi_\star$ is concave and continuous. If $\Sigma > \sigma_c$, consider the sequence $\lambda_n = n$. Then $\varphi_\star(\Sigma) \leqslant n(\varphi(1, 1/n) - \Sigma)$. Since $\lim_{n\to\infty} \varphi(1, 1/n) = \sigma_c$, we have $\varphi_\star(\Sigma) = -\infty$. If $\Sigma < \sigma_c$, then $\lambda \mapsto \varphi(\lambda, 1) - \lambda \Sigma$ is continuous and tends to $+\infty$ as λ tends to $+\infty$. Thus the infimum is reached (and finite).

To prove that $\varphi_\star$ is decreasing, let us consider $\Sigma_1 < \Sigma_2 < \sigma_c$ and let λ_1 and λ_2 be the points where the infimum is reached. We have

$$\varphi_\star(\Sigma_1) = \varphi(\lambda_1, 1) - \lambda_1 \Sigma_1 \leqslant \varphi(\lambda_2, 1) - \lambda_2 \Sigma_1, \tag{122}$$

$$\varphi_\star(\Sigma_2) = \varphi(\lambda_2, 1) - \lambda_2 \Sigma_2 \leqslant \varphi(\lambda_1, 1) - \lambda_1 \Sigma_2 \tag{123}$$

and hence $\varphi_\star(\Sigma_1) - \varphi_\star(\Sigma_2) \geqslant (\Sigma_2 - \Sigma_1)\lambda_1 \geqslant 0$.

Let us prove that $\lim_{\Sigma\to-\infty} \varphi_\star(\Sigma) = \tau_c$. Note first that $\varphi_\star(\Sigma) \leqslant \varphi(0, 1) - 0 \cdot \Sigma = \tau_c$ for all Σ. On the other hand, let $\Sigma_n = -n$ be the sequence converging to $-\infty$ and let λ_n be the associated sequence of minimizers. Then $\varphi_\star(-n) = \varphi(\lambda_n, 1) + n\lambda_n \leqslant \tau_c$, and, since φ is positive, $\lim_{n\to\infty} \lambda_n = 0$. Consequently, since $\tau_c \geqslant \varphi_\star(-n) \geqslant \varphi(\lambda_n, 1)$, we have $\lim_{n\to\infty} \varphi_\star(-n) = \tau_c$.

$\varphi_\star$ is neither automatically definite nor positive at $\Sigma = \sigma_c$, see the example below. If $\varphi_\star(\sigma_c) \geqslant 0$, then the domain of admissible stress vectors is $\{(\Sigma, T) \colon -\infty < \Sigma \leqslant \sigma_c, \ |T| \leqslant \varphi_\star(\Sigma)\}$. Else, by the continuity and the monotonicity of $\varphi_\star$ there exists $\sigma_c^\star$ such that $\varphi_\star(\sigma_c^\star) = 0$ and $\varphi_\star(\Sigma) < 0, \ \forall \Sigma > \sigma_c^\star$. In that case, the domain of admissible stress vectors is $\{(\Sigma, T) \colon -\infty < \Sigma \leqslant \sigma_c^\star, \ |T| \leqslant \varphi_\star(\Sigma)\}$.

The properties (2) are direct consequences of the first ones. $\quad\square$

Remark 4. In the case where φ is linear, we obtain

$$\varphi_\star(\Sigma) = \begin{cases} \tau_c & \text{if } \Sigma \leqslant \sigma_c, \\ +\infty & \text{otherwise} \end{cases} \tag{124}$$

and recover the criteria of maximal traction and maximal shear.

Along the proof of Proposition 11, σ and ν were fixed. For a given σ, when ν describes the unit sphere $\mathbb{S}^2$, the point (Σ, T) describes the domain delimited by the three Mohr circles. Consequently, in order that σ satisfy the yield stress criterion for all ν, the greatest Mohr circle must leave inside the domain delimited by the intrinsic curve. Finally, the set of admissible stress tensor is the convex hull of all greatest circles leaving inside that domain. That leads to the following

Proposition 12. *When the density surface energy ϕ is only directionally differentiable at the origin, the set of stress tensors σ satisfying the yield stress criterion* (102) *is such that*

$$\frac{\sigma_3 - \sigma_1}{2} \leqslant \varphi^\star\left(\frac{\sigma_1 + \sigma_3}{2}\right) \tag{125}$$

with

$$\varphi^{\star}(s) = \inf_{\theta \in [0,\pi/2]} \{\varphi(\cos\theta, \sin\theta) - s\cos\theta\}. \tag{126}$$

In (125)*, the σ_i's denote the ordered eigenvalues of the stress tensor* $\boldsymbol{\sigma}$*, i.e.* $\sigma_1 \leqslant \sigma_2 \leqslant \sigma_3$.
   This domain of admissible stress tensors is convex, unbounded in the direction of hydrostatic pressure.

Proof. Let $\omega \in [-\pi, \pi]$ and let (Σ, T) be the following point of the greatest Mohr circle:

$$\Sigma = \frac{\sigma_3 + \sigma_1}{2} + \frac{\sigma_3 - \sigma_1}{2}\cos\omega, \qquad T = \frac{\sigma_3 - \sigma_1}{2}\sin\omega. \tag{127}$$

Then (119) reads as: $\forall\theta \in [0, \pi/2]$ and $\forall\omega \in [-\pi, \pi]$,

$$\frac{\sigma_3 - \sigma_1}{2}\big(\sin\theta|\sin\omega| + \cos\theta\cos\omega\big) \leqslant \varphi(\cos\theta, \sin\theta) - \frac{\sigma_3 + \sigma_1}{2}\cos\theta. \tag{128}$$

Since the supremum of the right-hand side of the above inequality is reached for $\omega = \theta$, we obtain $\forall\theta \in [0, \pi/2]$,

$$\frac{\sigma_3 - \sigma_1}{2} \leqslant \varphi(\cos\theta, \sin\theta) - \frac{\sigma_3 + \sigma_1}{2}\cos\theta \tag{129}$$

and (125) follows. The remaining of the proposition is a consequence of the properties of the intrinsic curve. $\quad\square$

Remark 5. As in one-dimension, the asymmetric behavior between traction and compression is due to the non-interpenetration condition. The convexity of the domain of admissible stress tensors is a direct consequence of the stability conditions. This fundamental property is obtained without any assumption concerning the convexity of the surface energy density. The fact that the domain of admissible stress tensors is obtained from an intrinsic curve in the Mohr diagram (and hence does not depend on the intermediary stress eigenvalue σ_2) is a consequence both of the stability conditions and of the isotropy assumption. For anisotropic materials, more general convex sets could appear.

4.5. Examples

In all the examples below, ϕ is defined by

$$\phi(\alpha, \beta) = G_c\left(1 - \exp\left(-\frac{\varphi(\alpha, \beta)}{G_c}\right)\right), \tag{130}$$

with φ positive, continuous and one-homogeneous. Then ϕ is not Fréchet differentiable at $(0, 0)$, but its Gâteaux derivative is just φ.

4.5.1. φ convex
   Let us take φ as the following convex function

$$\varphi(\alpha, \beta) = 2\sqrt{\sigma_c^2\alpha^2 + \tau_c^2\beta^2} - \sigma_c\alpha - \tau_c\beta. \tag{131}$$

After some calculations, one gets

$$\varphi_{\star}(\Sigma) = \begin{cases} \tau_c & \text{if } \Sigma \leqslant -\sigma_c, \\ \tau_c\left(\sqrt{4 - \left(1 + \dfrac{\Sigma}{\sigma_c}\right)^2} - 1\right) & \text{if } -\sigma_c \leqslant \Sigma \leqslant \sigma_c. \end{cases} \tag{132}$$

Thus, $\varphi_{\star}$ is positive if and only if $\Sigma \leqslant (\sqrt{3} - 1)\sigma_c \equiv \sigma_c^{\star}$ and $\sigma_c^{\star}$ is the maximal traction that the material can sustained. The intrinsic curve, represented on Fig. 2, is made of a line segment and of an arc of ellipse.

4.5.2. φ concave
   Let us consider the case where φ is *concave* and continuous. Then the minimum of $\varphi(\cos\theta, \sin\theta) - s\cos\theta$ over $[0, \pi/2]$ is reached at the boundary, i.e. either at $\theta = 0$ or at $\theta = \pi/2$. Hence $\varphi^{\star}(s) = \min\{\sigma_c - s, \tau_c\}$ and we recover the maximal traction and the maximal shear criteria as in the case where φ were linear.
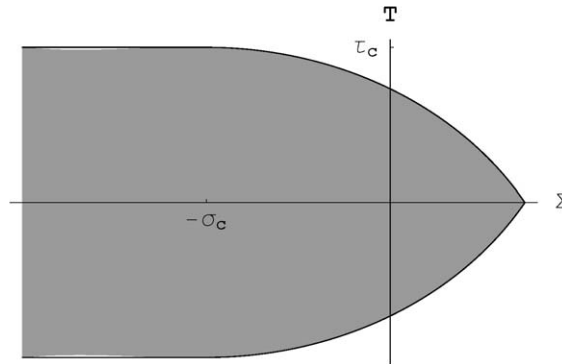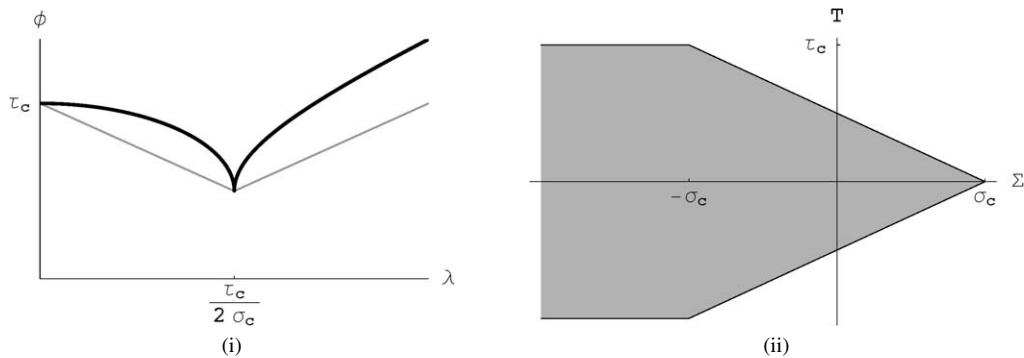
Fig. 2. The set of the admissible stress vectors in the Mohr diagram.

Fig. 3. (i) the graphs of the function $\lambda \mapsto \varphi(\lambda, 1)$ and of its convex envelop $\lambda \mapsto \varphi^{\star\star}(\lambda, 1)$. (ii) the corresponding intrinsic curve.

4.5.3. φ neither concave nor convex

Let us consider the following function φ:

$$\varphi(\alpha, \beta) = \tau_c \frac{\beta}{2} + \sqrt{\left| \sigma_c^2 \alpha^2 - \tau_c^2 \frac{\beta^2}{4} \right|} \tag{133}$$

which is neither convex nor concave. Therefore, $\varphi(\lambda, 1)$ is neither convex nor concave, and its convex envelop $\lambda \mapsto \varphi^{\star\star}(\lambda, 1)$ is made of two line segments, see Fig. 3. Then the minimization of $\lambda \mapsto \varphi(\lambda, 1) - \lambda \Sigma$ on $[0, \infty)$ is equivalent to minimize its convex envelope $\lambda \mapsto \varphi^{\star\star}(\lambda, 1) - \lambda \Sigma$, see Dacorogna (1992). We easily find

$$\varphi_\star(\Sigma) = \begin{cases} \tau_c & \text{if } \Sigma \leqslant -\sigma_c, \\ \dfrac{\tau_c}{2}\left(1 - \dfrac{\Sigma}{\sigma_c}\right) & \text{if } |\Sigma| \leqslant \sigma_c. \end{cases} \tag{134}$$

Then, $\sigma_c^\star = \sigma_c$ and the domain delimited by the intrinsic curve is represented on Fig. 3. When we consider the envelop of the greatest Mohr circles lying inside the intrinsic curve domain, the corners $(-\sigma_c, \pm \tau_c)$ disappear and finally the domain of the admissible stress tensors is given by

$$\varphi^\star(s) = \begin{cases} \tau_c & \text{if } s \leqslant \sigma_c - \sqrt{4\sigma_c^2 + \tau_c^2}, \\ \tau_c \dfrac{\sigma_c - s}{\sqrt{4\sigma_c^2 + \tau_c^2}} & \text{if } \sigma_c - \sqrt{4\sigma_c^2 + \tau_c^2} \leqslant s \leqslant \sigma_c. \end{cases} \tag{135}$$

5. Conclusion

It is important to recall that the yield stress criterion appears in our variational approach as a necessary first-order optimality condition, i.e. a condition involving only the first (directional) derivative(s) of the energy with respect to the displacement field. It is obtained by introducing a "small defect" in a neighborhood of any sound point of the body

and by evaluating the change that it induces in the energy at the first order. When this method is used with the Griffith surface energy, we just obtain the famous Griffith criterion $G \leqslant G_c$, whereas a stress criterion appears when we adopt a Barenblatt surface energy. In other words, we can symbolize the results given by the variational approach as follows:

Griffith's Energy + *Stability* $\Longrightarrow G \leqslant G_c$,

Barenblatt's Energy + *Stability* $\Longrightarrow \sigma \leqslant \sigma_c$.

Moreover, the domain of admissible stress tensors is necessarily convex and is deduced from the behavior near the origin of the surface energy density.

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