Mnacho Echenim
small evolutions : Patrick Reignier

Grenoble INP-Ensimag

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Outline

Evaluating how well an MLP is doing (supervised learning)

Problem we have a set of samples $S = \left\{ (\alpha^0_{(i)}, \rho_{(i)}) \;\middle|\; i = 1, \ldots, N \right\}$ and an MLP parameterized by a set of weights and biases collectively denoted by θ

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Goal: solve the optimization problem $\theta^* = \arg\min_\theta \mathcal{E}(S, \theta)$

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$$L_\delta = \begin{cases} \frac{(y-\widehat{y})^2}{2} & \text{if} |y - \widehat{y}| \leq \delta \\ \delta(|y - \widehat{y}| - \frac{\delta}{2}) & \text{otherwise} \end{cases} \quad (1)$$

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- $(y, \widehat{y}) \mapsto \log(1 + \exp(-y\widehat{y}))$ (logistic regression)
- $(y, \widehat{y}) \mapsto \max(0, y - \widehat{y})$ (binary classification)

Gradient descent

Goal: find the minimum of $\mathcal{E}(S, \theta)$ using gradient descent: if θ_k denotes the network parameters at iteration k then

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Issues:

- Computing a single gradient can be very long
- Vectorization is not possible for large samples

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  Using a single arbitrary sample gives us a noisy approximation of the actual gradient
- **Features**
  - Faster to compute
  - Can avoid local minima, saddle points (more on this later)

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- Problems arise when we are close to the optimal value θ^*: the noise becomes problematic
- Can this noise be reduced?

A trade-off: Mini-batch gradient descent

- **Principle:** use M arbitrary samples to optimize cost function

  The update rule becomes

  $$\theta_{k+1} \leftarrow \theta_k - \frac{\eta}{M} \sum_{j=1}^{M} \nabla_\theta \mathcal{C}(\theta_k, \rho_{(i_{k_j})}, \widehat{\rho_{(i_{k_j})}}),$$

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- Features
  - Less noisy approximation of real gradient
  - Computation cost can be controlled
    - Mini-batch size

Summary

Given an MLP and a sample set of size N:

Method	**Updates per epoch**	**Computations per update**
Gradient	1	N
SGD	N	1
Mini-batch	N/M	M

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About backpropagation

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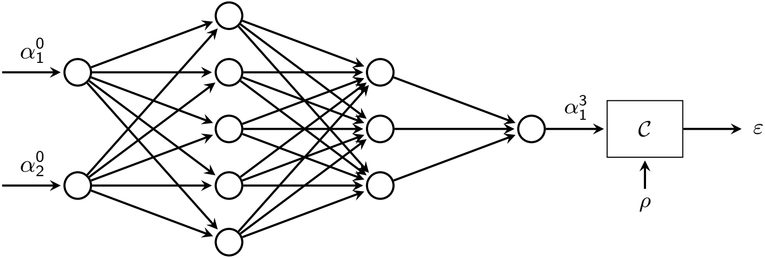
  - Very efficient
  - One of the reasons deep neural networks are so successful
- A special case of automatic differentiation
  - Generally presented with **computational graphs**
  - Modern ML frameworks implement the general case
  - Here, we focus on the derivation of backpropagation equations for neural networks

A high-level overview



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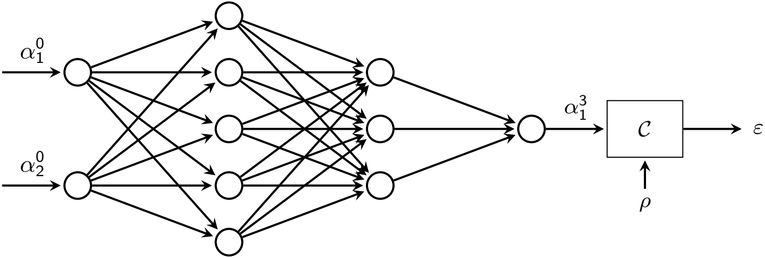

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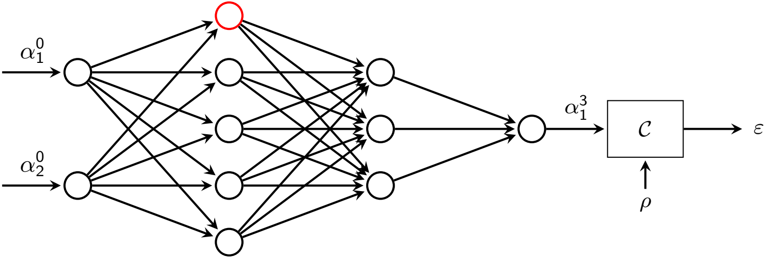

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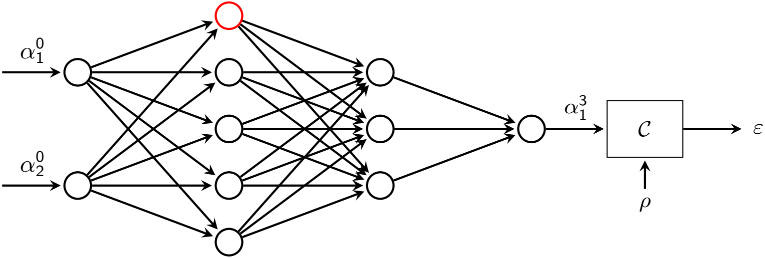

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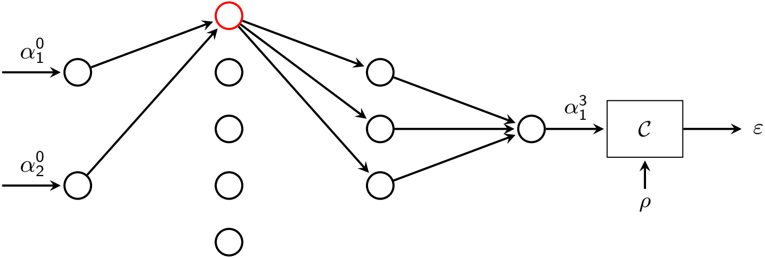

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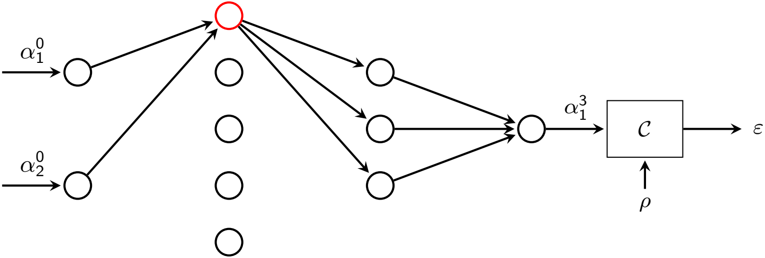

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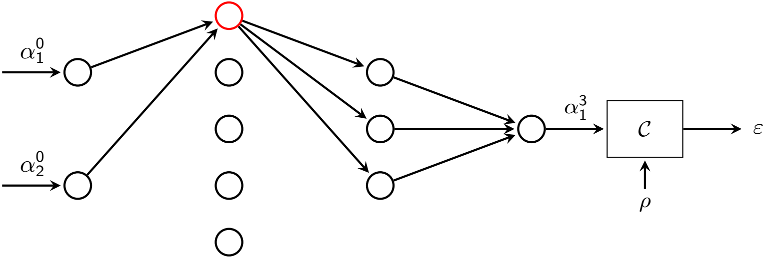

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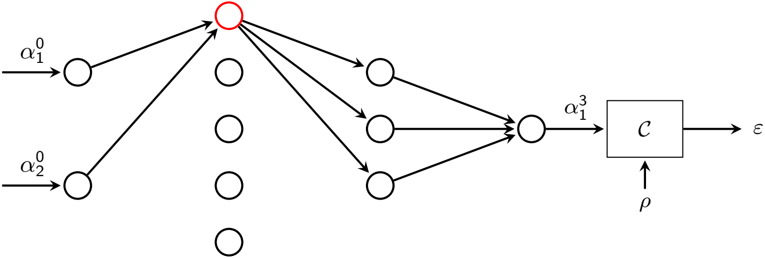

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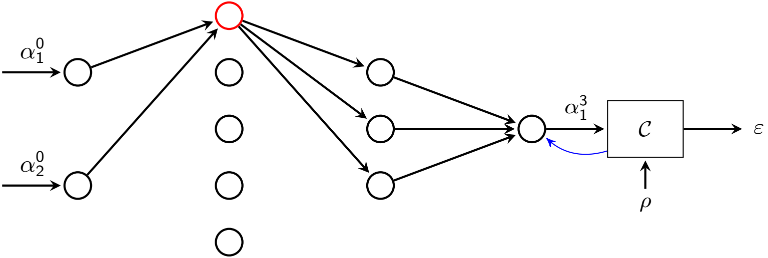

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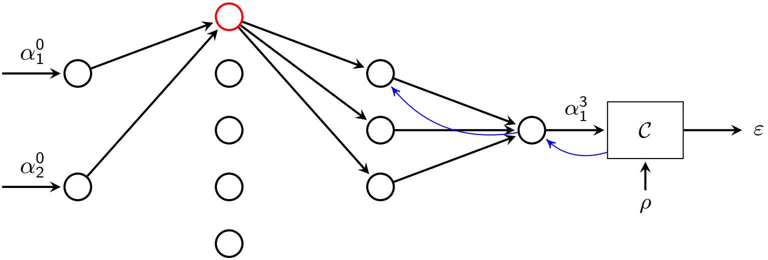

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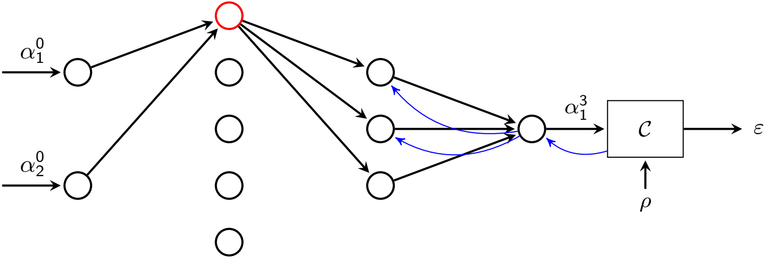

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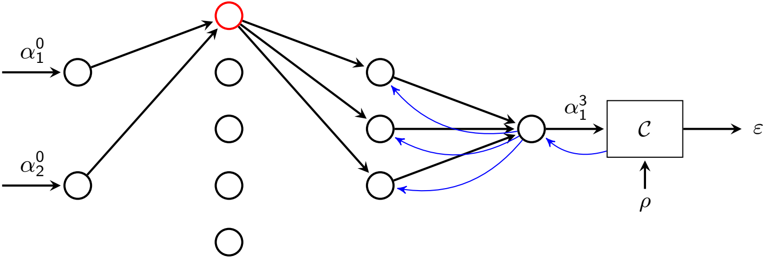

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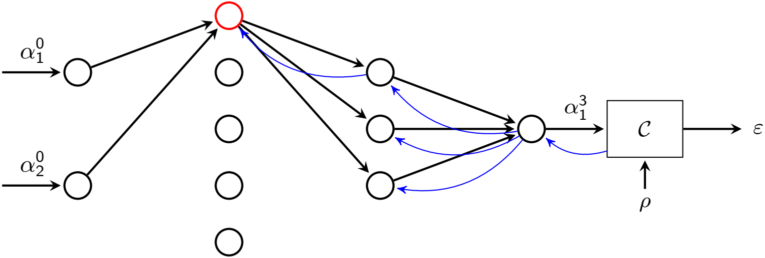

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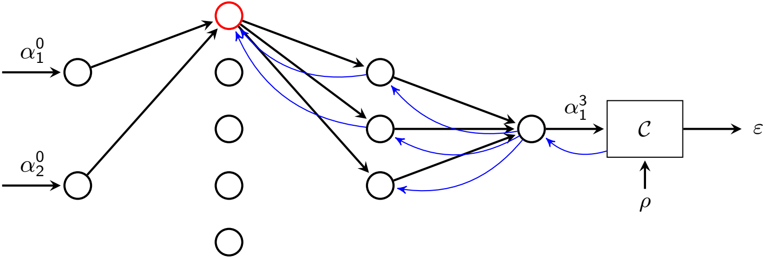

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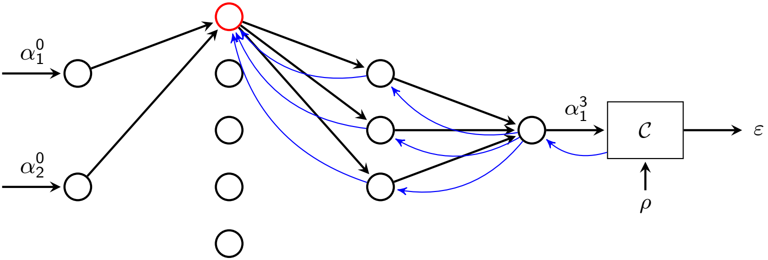
A high-level overview



- Assume $\varepsilon > 0$
- What is the contribution of ν_1^1 to this error?
  - Its parameters are $\omega_{1,1}^1$, $\omega_{2,1}^1$, β_1^1
  - The error can be viewed as $\varepsilon = \mathcal{C}(\omega_{1,1}^1, \omega_{2,1}^1, \beta_1^1)$
- Gradient descent:
  $\omega_{1,1}^1 \leftarrow \omega_{1,1}^1 - \nabla_{\omega_{1,1}^1}\mathcal{C}$,
  $\omega_{2,1}^1 \leftarrow \omega_{2,1}^1 - \nabla_{\omega_{2,1}^1}\mathcal{C}$,

- How are these partial derivatives computed efficiently?
- By *backpropagating* information

A word of warning

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- The presentation will be quite formal, to ensure each step is well understood
- This makes notations quite heavy
- An interesting exercise is to try to derive the rules by yourselves, using the lighter notations that are more standard

Some definitions

Given the matrices $M_1, \ldots, M_p$, we define the bloc matrix

$$\operatorname{diag}(M_1, \ldots, M_p) \stackrel{\text{def}}{=} \begin{pmatrix} M_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & M_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & M_p \end{pmatrix}$$

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Given $A, B \in \mathbb{R}^{n \times m}$, the **Hadamard product** of A and B, denoted by $A \odot B$, is the matrix such that $(A \odot B)_{i,j} = A_{i,j} \cdot B_{i,j}$ (pointwise multiplication)

The chain rule

The **Jacobian matrix** of a function $f: \begin{array}{ccc} \mathbb{R}^n & \rightarrow & \mathbb{R}^m \\ a & \mapsto & (f_1(a), \ldots, f_m(a))^{\mathrm{T}} \end{array}$

is the function that associates to $\alpha \in \mathbb{R}^n$ the matrix in $\mathbb{R}^{m \times n}$ denoted by $\frac{\partial f}{\partial a}(\alpha)$, where $\left(\frac{\partial f}{\partial a}(\alpha)\right)_{i,j} = \frac{\partial f_i}{\partial a_j}(\alpha)$

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Given: $f : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^m \\ a & \mapsto & f(a) \end{array}$ $\quad g : \begin{array}{ccc} \mathbb{R}^m & \to & \mathbb{R}^p \\ y & \mapsto & g(y) \end{array}$ $\quad h : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^p \\ a & \mapsto & g \circ f(a) \end{array}$

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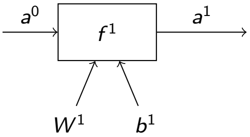
If f is differentiable at α and g is differentiable at $f(\alpha)$, then h is differentiable at α and

$$\frac{\partial h}{\partial a}(\alpha) = \frac{\partial g}{\partial y}(f(\alpha)) \cdot \frac{\partial f}{\partial a}(\alpha)$$

Some formalism



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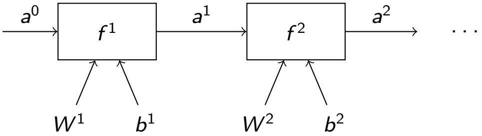

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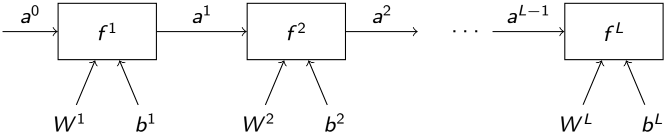

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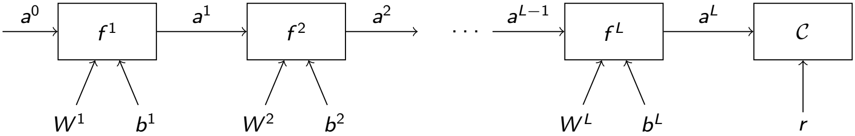

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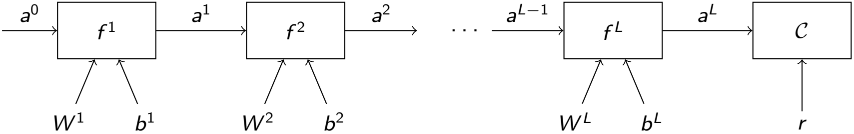

Where for $j = 1, \ldots, L$:

- $a^{j-1} \in \mathbb{R}^{n_{j-1}}$ represents the formal input of layer j

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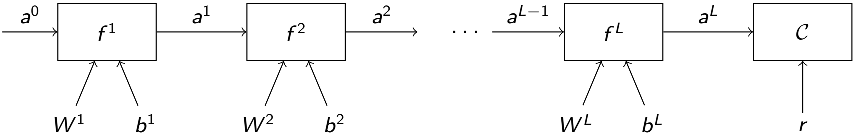

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- $b^j \in \mathbb{R}^{n_j}$ and $W^j = (w_1^j, \dots, w_{n_j}^j)$, where for $k = 1, \dots, n_j$, $w_k^j \in \mathbb{R}^{n_{j-1}}$ represents the weights for neuron k at layer j

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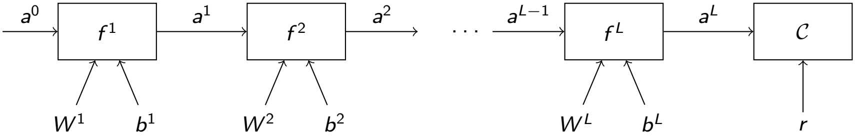

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- f^j represents the computation performed at layer j:

$$\begin{aligned} f^j \colon \ (\mathbb{R}^{n_{j-1}})^{n_j+1} \times \mathbb{R}^{n_j} &\to \mathbb{R}^{n_j} \\ a^{j-1}, W^j, b^j &\mapsto f^j(a^{j-1}, W^j, b^j) \end{aligned}$$

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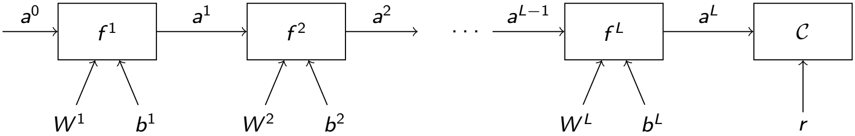

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$\mathcal{C}$ is the error function:
$$\begin{aligned} \mathcal{C} \colon \mathbb{R}^{n_L} \times \mathbb{R}^{n_L} &\to \mathbb{R}^+ \\ a^L, r &\mapsto \mathcal{C}(a^L, r) \end{aligned}$$

Reminders

- Parameters

Name	Formal	Actual	Dimension
Activation of layer j	a^j	α^j	$\mathbb{R}^{n_j}$

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- Partial derivatives

$$\frac{\partial f^j}{\partial a^{j-1}}(\alpha^{j-1}, \Omega^j, \beta^j) \quad \in \quad \mathbb{R}^{n_j \times n_{j-1}}$$

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- Partial derivatives

$$\begin{aligned} \frac{\partial f^j}{\partial a^{j-1}}(\alpha^{j-1}, \Omega^j, \beta^j) &\in \mathbb{R}^{n_j \times n_{j-1}} \\ \frac{\partial f^j}{\partial w_k^j}(\alpha^{j-1}, \Omega^j, \beta^j) &\in \mathbb{R}^{n_j \times n_{j-1}}, \text{ for } k = 1, \ldots, n_j \end{aligned}$$

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\frac{\partial f^j}{\partial b^j}(\alpha^{j-1}, \Omega^j, \beta^j) &\in \mathbb{R}^{n_j \times n_j}
\end{aligned}$$

What we want to compute

Let g^i represent the computation of the first i layers of the network:

$$
\begin{array}{rcrcl}
g^1 & : & a^0, W^1, b^1 & \mapsto & f_1(a^0, W^1, b^1) \\
g^i & : & a^0, W^1, b^1, \ldots, W^i, b^i & \mapsto & f^i(g^{i-1}(a^0, W^1, b^1, \ldots, W^{i-1}, b^{i-1}), W^i, b^i)
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Let $\mathcal{E}$ represent the output of the error function:

$$\mathcal{E} \quad : \quad a^0, W^1, b^1, \ldots, W^L, b^L, r \quad \mapsto \quad \mathcal{C}(g^L(a^0, W^1, b^1, \ldots, W^L, b^L), r)$$

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Output of i^{th} layer of the network, given the inputs α^0, Ω^1, β^1, ..., Ω^i, β^i:

$$\begin{array}{rcl} \alpha^i & = & g^i(\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^i, \beta^i) \\ & = & f^i(\alpha^{i-1}, \Omega^i, \beta^i) \end{array}$$

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Error of the network when expected result is ρ:

$$\begin{aligned} \epsilon &= \mathcal{E}(\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L, \rho) \\ &= \mathcal{C}(\alpha^L, \rho) \end{aligned}$$

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Goal

Compute $\frac{\partial \mathcal{E}}{\partial w_k^j}(\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L, \rho)$ and $\frac{\partial \mathcal{E}}{\partial b^j}(\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L, \rho)$

Using the chain rule

Chain rule on the error function:

$$\frac{\partial \mathcal{E}}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L, \rho) \quad = \quad \frac{\partial \mathcal{C}}{\partial w_k^j}(g^L(\alpha^0, \ldots, \Omega^L, \beta^L), \rho)$$

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Chain rule on the output of the network when $j = L$:

$$
\frac{\partial g^L}{\partial w_k^L}(\alpha^0, \ldots, \Omega^L, \beta^L) = \frac{\partial f^L}{\partial w_k^L}\left(g^{L-1}(\alpha^0, \ldots, \Omega^{L-1}, \beta^{L-1}), \Omega^L, \beta^L\right)
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Using the chain rule

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Using the chain rule (2)

Chain rule on the output of the network when $j < L$:

$$\frac{\partial g^L}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L) \quad = \quad \frac{\partial f^L}{\partial w_k^j}\left(g^{L-1}(\alpha^0, \ldots, \Omega^{L-1}, \beta^{L-1}), \Omega^L, \beta^L\right)$$

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Chain rule on the output of the network when $j < L$:

$$\begin{aligned}
\frac{\partial g^L}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L) &= \frac{\partial f^L}{\partial w_k^j}\left(g^{L-1}(\alpha^0, \ldots, \Omega^{L-1}, \beta^{L-1}), \Omega^L, \beta^L\right) \\
&= \frac{\partial f^L}{\partial a^{L-1}}(\alpha^{L-1}, \Omega^L, \beta^L) \cdot \frac{\partial g^{L-1}}{\partial w_k^j}(\alpha^0, \ldots, \Omega^{L-1}, \beta^{L-1})
\end{aligned}$$

Using the chain rule (2)

Chain rule on the output of the network when $j < L$:

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&= \left[\prod_{i=L}^{j+1} \frac{\partial f^i}{\partial a^{i-1}}(\alpha^{i-1}, \Omega^i, \beta^i)\right] \cdot \frac{\partial g^j}{\partial w_k^j}(\alpha^0, \dots, \Omega^j, \beta^j)
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\frac{\partial g^L}{\partial b^j}(\alpha^0, \ldots, \Omega^L, \beta^L) &= \left[\prod_{i=L}^{j+1} \frac{\partial f^i}{\partial a^{i-1}}(\alpha^{i-1}, \Omega^i, \beta^i)\right] \cdot \frac{\partial f^j}{\partial b^j}(\alpha^{j-1}, \Omega^j, \beta^j)
\end{aligned}
$$

Recap

Consider the inputs $\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L$ and the expected result ρ, and let

$$\mathcal{P}^{L+1} \stackrel{\text{def}}{=} \frac{\partial \mathcal{C}}{\partial a^L}(\alpha^L, \rho)$$

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Then for $j = 1, \ldots, L$, we have $\mathcal{P}^j \in \mathbb{R}^{1 \times n_{j-1}}$ and

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$$\frac{\partial \mathcal{E}}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L, \rho) = \frac{\partial \mathcal{C}}{\partial a^L}(\alpha^L, \rho) \cdot \frac{\partial g^L}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L)$$

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Consider the inputs $\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L$ and the expected result ρ, and let

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Recap

Consider the inputs $\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L$ and the expected result ρ, and let

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$$\frac{\partial \mathcal{E}}{\partial b^j}(\alpha^0, \ldots, \Omega^L, \beta^L, \rho)$$

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Consider the inputs $\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L$ and the expected result ρ, and let

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\frac{\partial \mathcal{E}}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L, \rho) &= \frac{\partial \mathcal{C}}{\partial a^L}(\alpha^L, \rho) \cdot \frac{\partial g^L}{\partial w_k^j}(\alpha^0, \ldots, \Omega^L, \beta^L) \\
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Recap

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$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial w_k^j}(\alpha^0, \dots, \Omega^L, \beta^L, \rho) &= \frac{\partial \mathcal{C}}{\partial a^L}(\alpha^L, \rho) \cdot \frac{\partial g^L}{\partial w_k^j}(\alpha^0, \dots, \Omega^L, \beta^L) \\
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&= \mathcal{P}^{j+1} \cdot \frac{\partial f^j}{\partial b^j}(\alpha^{j-1}, \Omega^j, \beta^j)
\end{aligned}$$

At layer j, we receive $\mathcal{P}^{j+1}$ and we need to compute $\frac{\partial f^j}{\partial a^{j-1}}(\alpha^{j-1}, \Omega^j, \beta^j)$, $\frac{\partial f^j}{\partial w_k^j}(\alpha^{j-1}, \Omega^j, \beta^j)$ and $\frac{\partial f^j}{\partial b^j}(\alpha^{j-1}, \Omega^j, \beta^j)$

Same recap, with simplified notations

Consider the inputs $\alpha^0, \Omega^1, \beta^1, \ldots, \Omega^L, \beta^L$ and the expected result ρ, and let

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$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial w^j_k} &= \frac{\partial \mathcal{C}}{\partial a^L} \cdot \frac{\partial g^L}{\partial w^j_k} \\ &= \mathcal{P}^{j+1} \cdot \frac{\partial f^j}{\partial w^j_k} \\ \frac{\partial \mathcal{E}}{\partial b^j} &= \end{aligned}$$

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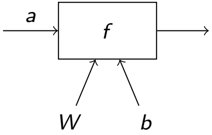
At layer j, we need to compute $\frac{\partial f^j}{\partial a^{j-1}}$, $\frac{\partial f^j}{\partial w_k^j}$ and $\frac{\partial f^j}{\partial b^j}$

Partial derivatives for a single layer

For a layer with n inputs and m neurons we have the following:

- $a \in \mathbb{R}^n$



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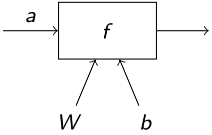
- $a \in \mathbb{R}^n$
- $b = (b_1, \ldots, b_m)^{\mathrm{T}}$ and $W = (w_1, \ldots, w_m)$, where for $p = 1, \ldots, m$, $w_p \in \mathbb{R}^n$



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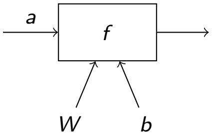

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- f is defined by:

$$
\begin{aligned}
f:\ (\mathbb{R}^n)^{m+1} \times \mathbb{R}^m &\to \mathbb{R}^m \\
a, W, b &\mapsto [h(a, w_1, b_1), \ldots, h(a, w_m, b_m)]^{\mathrm{T}}
\end{aligned}
$$

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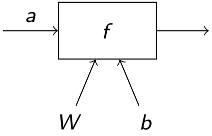

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- $h = \Phi \circ \Psi$, where

$$\begin{aligned} \Phi\colon \mathbb{R} &\to \mathbb{R} \\ z &\mapsto \Phi(z) \end{aligned} \quad \text{and} \quad \begin{aligned} \Psi\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} &\to \mathbb{R} \\ a, w, b &\mapsto w^{\mathrm{T}} a + b \end{aligned}$$

Partial derivatives for a single neuron

- Input: α (neuron input), β (bias) and ω (weights for the neuron)
  - Let ζ denote the net input of the neuron
  - $\zeta = \omega^{\mathrm{T}}\alpha + \beta = \Psi(\alpha, \omega, \beta)$

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$$\frac{\partial h}{\partial a} =$$

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- We have:

$$\frac{\partial h}{\partial a} = \frac{\partial \Phi}{\partial z}(\zeta) \cdot \frac{\partial \Psi}{\partial a}(\alpha, \omega, \beta)$$

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Partial derivatives for a single neuron

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Partial derivatives for a single neuron

- Input: α (neuron input), β (bias) and ω (weights for the neuron)
  - Let ζ denote the net input of the neuron
  - $\zeta = \omega^{\mathrm{T}}\alpha + \beta = \Psi(\alpha, \omega, \beta)$
- Output: $h(\alpha, \omega, \beta) = \Phi(\Psi(\alpha, \omega, \beta)) = \Phi(\zeta) = \Phi(\omega^{\mathrm{T}}\alpha + \beta)$
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&= \Phi'(\zeta)
\end{aligned}$$

Back to partial derivatives for a layer

$f: a, W, b \mapsto [h(a, w_1, b_1), \ldots, h(a, w_m, b_m)]^{\mathrm{T}}$
Input α, Ω, β, where $\Omega = (\omega_1, \ldots, \omega_m)$

$$\frac{\partial f}{\partial a} =$$

$$\frac{\partial f}{\partial w_k} =$$

$$\frac{\partial f}{\partial b_k} =$$

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Back to partial derivatives for an entire network

$\mathcal{P}^{L+1} =$ and $\mathcal{P}^{j} =$

$$\frac{\partial \mathcal{E}}{\partial w_k^j} =$$

$$\frac{\partial \mathcal{E}}{\partial b^j} =$$

Back to partial derivatives for an entire network

$$\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L} \quad \text{and} \quad \mathcal{P}^j =$$

$$\frac{\partial \mathcal{E}}{\partial w_k^j} \quad =$$

$$\frac{\partial \mathcal{E}}{\partial b^j} \quad =$$

Back to partial derivatives for an entire network

$$\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L} \quad \text{and} \quad \mathcal{P}^j = \mathcal{P}^{j+1} \cdot \frac{\partial f^j}{\partial a^{j-1}} =$$

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$$\frac{\partial \mathcal{E}}{\partial b^j} =$$

Back to partial derivatives for an entire network

$$\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L} \quad \text{and} \quad \mathcal{P}^j = \mathcal{P}^{j+1} \cdot \frac{\partial f^j}{\partial a^{j-1}} = \mathcal{P}^{j+1} \cdot \begin{pmatrix} \Phi'(\zeta_1^j). \left[\omega_1^j\right]^{\mathrm{T}} \\ \vdots \\ \Phi'(\zeta_{n_j}^j). \left[\omega_{n_j}^j\right]^{\mathrm{T}} \end{pmatrix}$$

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$$\frac{\partial \mathcal{E}}{\partial b^j} =$$

Back to partial derivatives for an entire network

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Back to partial derivatives for an entire network

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Back to gradient descent

- We are interested in computing
  - $\nabla_{w_k^j}\mathcal{E} \stackrel{\text{def}}{=} \left[\frac{\partial \mathcal{E}}{\partial w_k^j}\right]^{\mathrm{T}}$
  - $\nabla_{b^j}\mathcal{E} \stackrel{\text{def}}{=} \left[\frac{\partial \mathcal{E}}{\partial b^j}\right]^{\mathrm{T}}$

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- Weights are stored in a matrix: $\Omega^j = (\omega_1^j, \ldots, \omega_{n_j}^j) \in \mathbb{R}^{n_{j-1} \times n_j}$

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- If we define

$$\nabla_{W^j}\mathcal{E} \stackrel{\text{def}}{=} \begin{pmatrix} \nabla_{w_1^j}\mathcal{E} & \cdots & \nabla_{w_{n_j}^j}\mathcal{E} \end{pmatrix}$$

Back to gradient descent

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- Then parameters updates for stochastic gradient descent become

$$\begin{aligned} \Omega^j &\leftarrow \Omega^j - \eta \cdot \nabla_{W^j}\mathcal{E} \\ \beta_j &\leftarrow \beta_j - \eta \cdot \nabla_{b^j}\mathcal{E} \end{aligned}$$

Effective computations

$$\mathcal{P}^j =$$

$$\nabla_{W^j}\mathcal{E} =$$

$$\nabla_{b^j}\mathcal{E} =$$

Effective computations

$$\mathcal{P}^j = \mathcal{P}^{j+1} \cdot \begin{pmatrix} \Phi'(\zeta_1^j). \left[\omega_1^j\right]^{\mathrm{T}} \\ \vdots \\ \Phi'(\zeta_{n_j}^j). \left[\omega_{n_j}^j\right]^{\mathrm{T}} \end{pmatrix}$$

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Effective computations

$$
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& &= \left[\Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}}
\end{aligned}
$$

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$$= \left[\Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}}$$

$$\nabla_{W^j}\mathcal{E} = \left(\cdots \quad \underbrace{\left(0, \cdots, 0, \underbrace{\alpha^{j-1}.\Phi'(\zeta_k^j)}_{\text{column } k}, 0, \cdots, 0\right) \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}}_{k^{\text{th}} \text{ vector}} \quad \cdots \right)$$

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Effective computations

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$$= \left(\alpha^{j-1}.\left(\Phi'(\zeta_1^j) \cdot \mathcal{P}_1^{j+1}\right) \quad \cdots \quad \alpha^{j-1}.\left(\Phi'(\zeta_{n_j}^j) \cdot \mathcal{P}_{n_j}^{j+1}\right)\right)$$

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$$\nabla_{b^j}\mathcal{E} = \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}$$

Backpropagation rules

Let $\mathcal{B}^j \stackrel{\text{def}}{=} \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}} \in \mathbb{R}^{n_j \times 1}$ for $j = 1, \ldots, L-1$, so that

$$\mathcal{P}^j \;=\; \left[\Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}} \;=\; \left[\mathcal{B}^j\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}} \;=\; \left[\Omega^j \mathcal{B}^j\right]^{\mathrm{T}}$$

Backpropagation rules

Let $\mathcal{B}^j \stackrel{\text{def}}{=} \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}} \in \mathbb{R}^{n_j \times 1}$ for $j = 1, \ldots, L-1$, so that

$$\mathcal{P}^j \;=\; \left[\Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}} \;=\; \left[\mathcal{B}^j\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}} \;=\; \left[\Omega^j \mathcal{B}^j\right]^{\mathrm{T}}$$

Recall also that $\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L}$

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Recall also that $\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L}$

We obtain the following backpropagation rules:

$$\begin{aligned} \mathcal{B}^L &= \Phi'(\zeta^L) \odot \nabla_{a^L} \mathcal{C}(\alpha^L, \rho) \\ \mathcal{B}^j &= \Phi'(\zeta^j) \odot \left(\Omega^{j+1} \mathcal{B}^{j+1}\right) \quad \text{if } j < L \end{aligned}$$

Backpropagation rules

Let $\mathcal{B}^j \stackrel{\text{def}}{=} \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}} \in \mathbb{R}^{n_j \times 1}$ for $j = 1, \ldots, L-1$, so that

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Recall also that $\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L}$

We obtain the following backpropagation rules:

$$\begin{array}{rcll} \mathcal{B}^L & = & \Phi'(\zeta^L) \odot \nabla_{a^L}\mathcal{C}(\alpha^L, \rho) & \\ \mathcal{B}^j & = & \Phi'(\zeta^j) \odot \left(\Omega^{j+1}\mathcal{B}^{j+1}\right) & \text{if } j < L \\ \nabla_{W^j}\mathcal{E} & = & \alpha^{j-1} \cdot \left[\mathcal{B}^j\right]^{\mathrm{T}} & \text{for } j = 1, \ldots, L \end{array}$$

Backpropagation rules

Let $\mathcal{B}^j \stackrel{\text{def}}{=} \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}} \in \mathbb{R}^{n_j \times 1}$ for $j = 1, \ldots, L-1$, so that

$$\mathcal{P}^j \;=\; \left[\Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}} \;=\; \left[\mathcal{B}^j\right]^{\mathrm{T}} \cdot \left[\Omega^j\right]^{\mathrm{T}} \;=\; \left[\Omega^j \mathcal{B}^j\right]^{\mathrm{T}}$$

Recall also that $\mathcal{P}^{L+1} = \frac{\partial \mathcal{C}}{\partial a^L}$

We obtain the following backpropagation rules:

$$\begin{array}{rcll}
\mathcal{B}^L & = & \Phi'(\zeta^L) \odot \nabla_{a^L}\mathcal{C}(\alpha^L, \rho) & \\
\mathcal{B}^j & = & \Phi'(\zeta^j) \odot \left(\Omega^{j+1}\mathcal{B}^{j+1}\right) & \text{if } j < L \\
\nabla_{W^j}\mathcal{E} & = & \alpha^{j-1} \cdot \left[\mathcal{B}^j\right]^{\mathrm{T}} & \text{for } j = 1, \ldots, L \\
\nabla_{b^j}\mathcal{E} & = & \mathcal{B}^j & \text{for } j = 1, \ldots, L
\end{array}$$

Backpropagation and gradient computation

Input: A network with L layers
Input: α^0 that has been forward propagated
Input: ρ the expected output

$\mathcal{B}^L \leftarrow \Phi'(\zeta^L) \odot \nabla_{a^L}\mathcal{C}(\alpha^L, \rho)$;
$\left[\mathcal{P}^L\right]^{\mathrm{T}} \leftarrow \Omega^L \cdot \mathcal{B}^L$;
for $j \leftarrow L - 1$ **to** 1 **do**
  $\mathcal{B}^j \leftarrow \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}$;
  $\left[\mathcal{P}^j\right]^{\mathrm{T}} \leftarrow \Omega^j \cdot \mathcal{B}^j$;
end

Algorithm 1: Backpropagation algorithm

Backpropagation and gradient computation

Input: A network with L layers
Input: α^0 that has been forward propagated
Input: ρ the expected output

$\mathcal{B}^L \leftarrow \Phi'(\zeta^L) \odot \nabla_{a^L}\mathcal{C}(\alpha^L, \rho)$;
$\left[\mathcal{P}^L\right]^{\mathrm{T}} \leftarrow \Omega^L \cdot \mathcal{B}^L$;
for $j \leftarrow L - 1$ **to** 1 **do**
- $\mathcal{B}^j \leftarrow \Phi'(\zeta^j) \odot \left[\mathcal{P}^{j+1}\right]^{\mathrm{T}}$;
- $\left[\mathcal{P}^j\right]^{\mathrm{T}} \leftarrow \Omega^j \cdot \mathcal{B}^j$;

end

Algorithm 3: Backpropagation algorithm

Input: A network with L layers
Input: α^0 that has been forward propagated
Input: $(\mathcal{B}^1, \ldots, \mathcal{B}^L)$ that have been updated by backpropagation

for $j \in \{1, \ldots, L\}$ **do**
- $\mathrm{Gradient}(\Omega^j) \leftarrow \alpha^{j-1} \cdot \left[\mathcal{B}^j\right]^{\mathrm{T}}$;
- $\mathrm{Gradient}(\beta^j) \leftarrow \mathcal{B}^j$;

end

Algorithm 4: Gradient computation