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Outline

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Conclusion

The weights of neurons in a same layer are updated the same way, and the network cannot be trained efficiently

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Note

- Initialization schemes frequently depend on the activation function that is used
- For example, biases are sometimes initialized to a small value when using ReLU activations

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- See also Stanford lecture:
  `https://www.youtube.com/watch?v=wEoyxE0GP2M`

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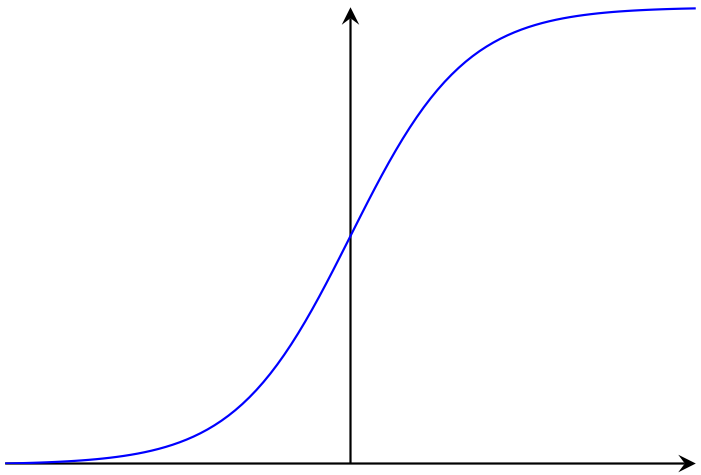
- Saturated neurons produce 0 gradient
- $\sigma'(x) \leq 0.25 \Longrightarrow$ vanishing gradient problem



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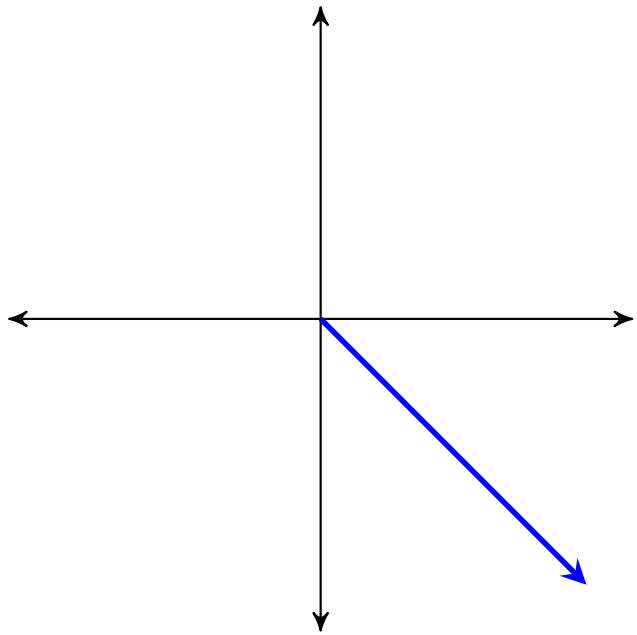

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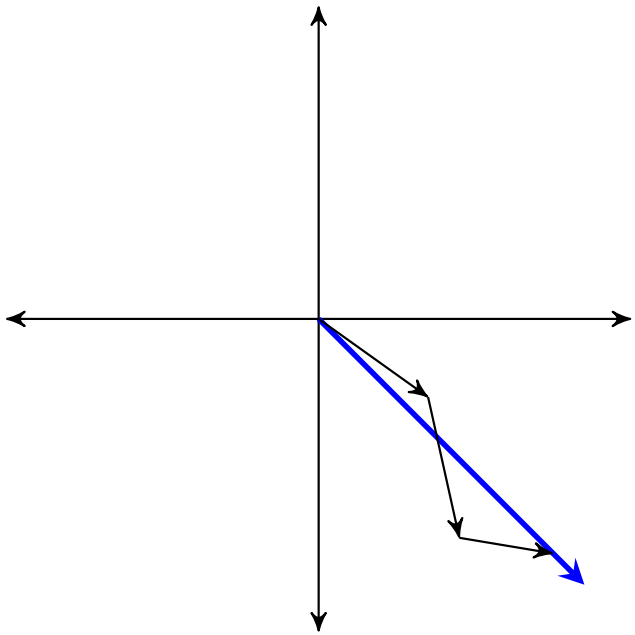

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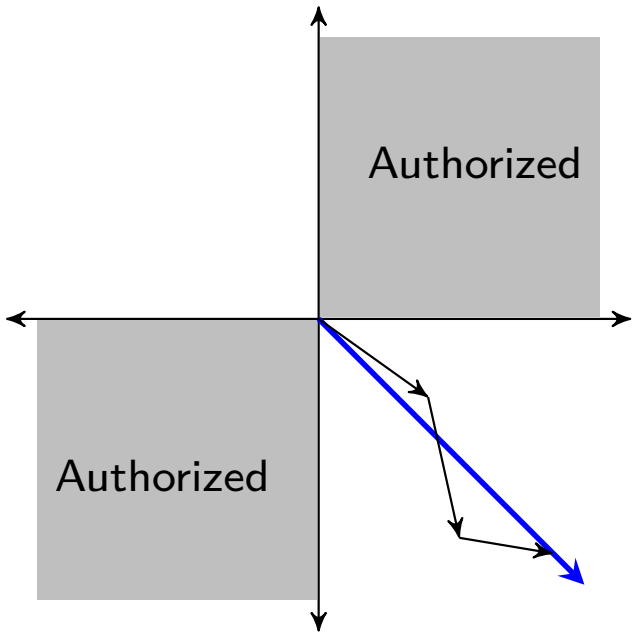

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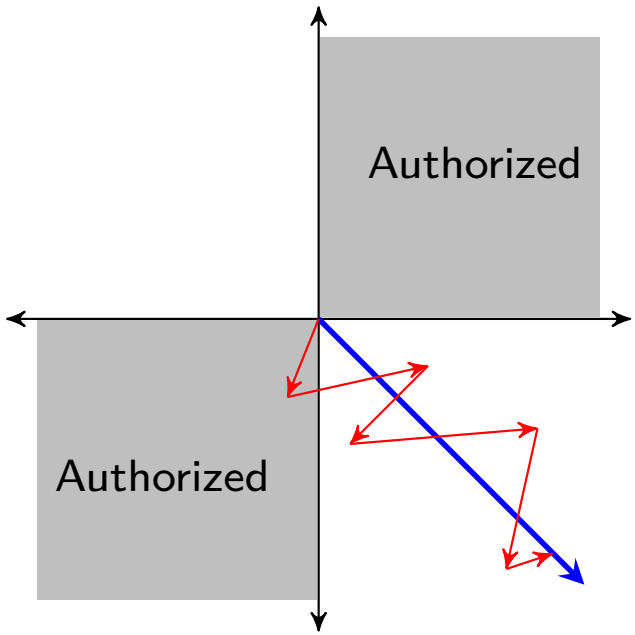
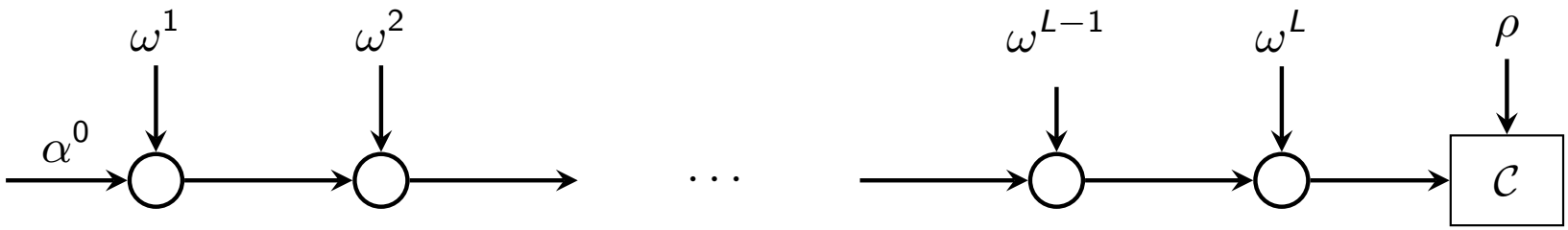


Vanishing gradients: illustration



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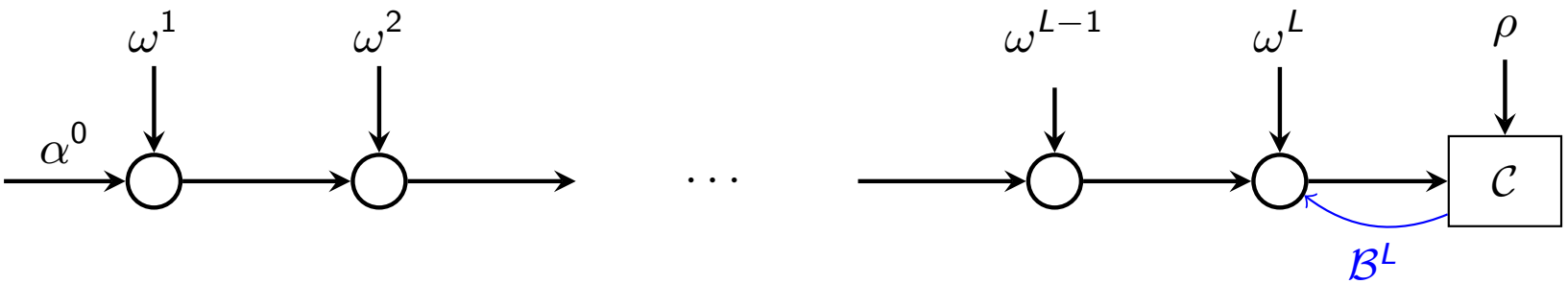

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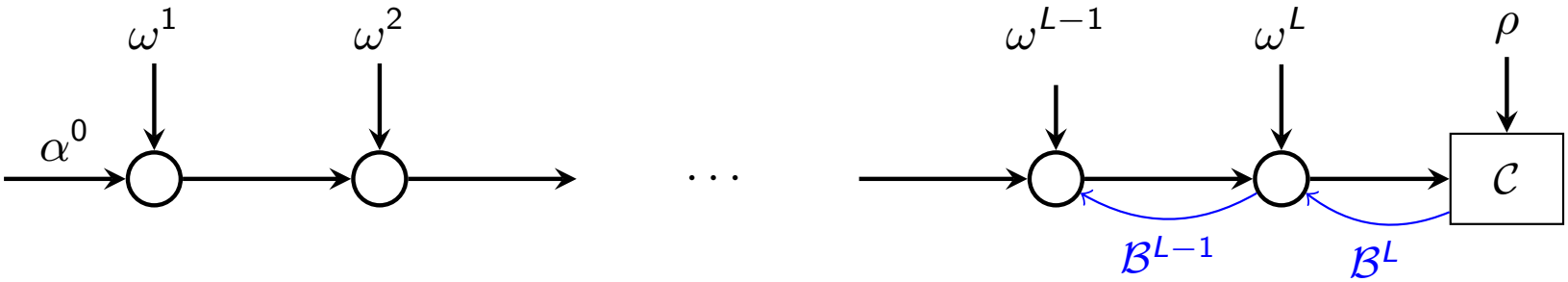

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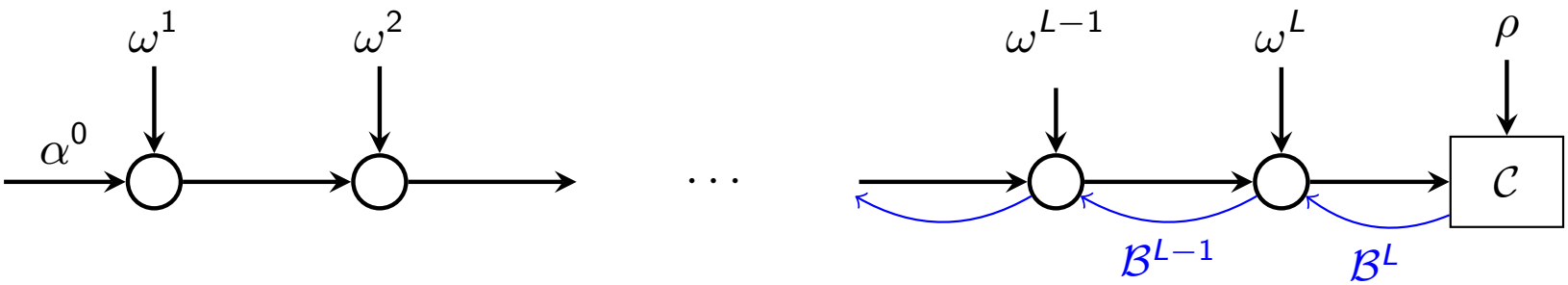

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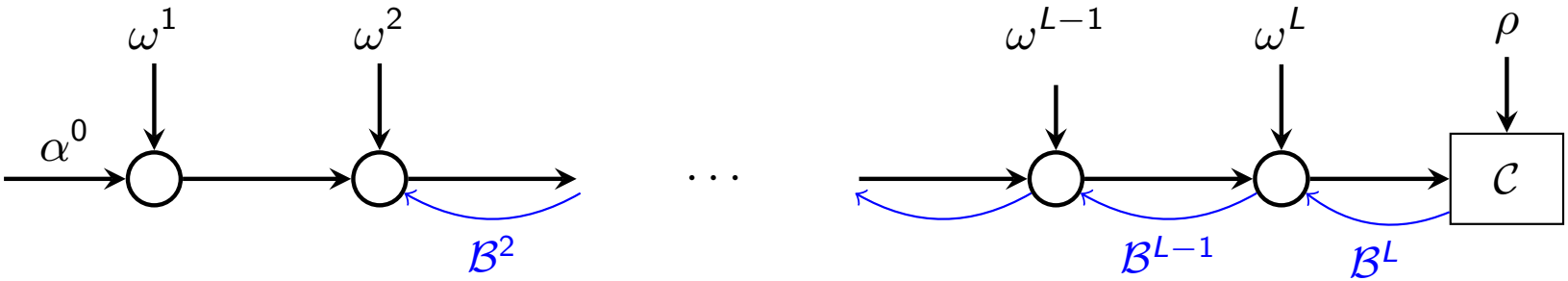

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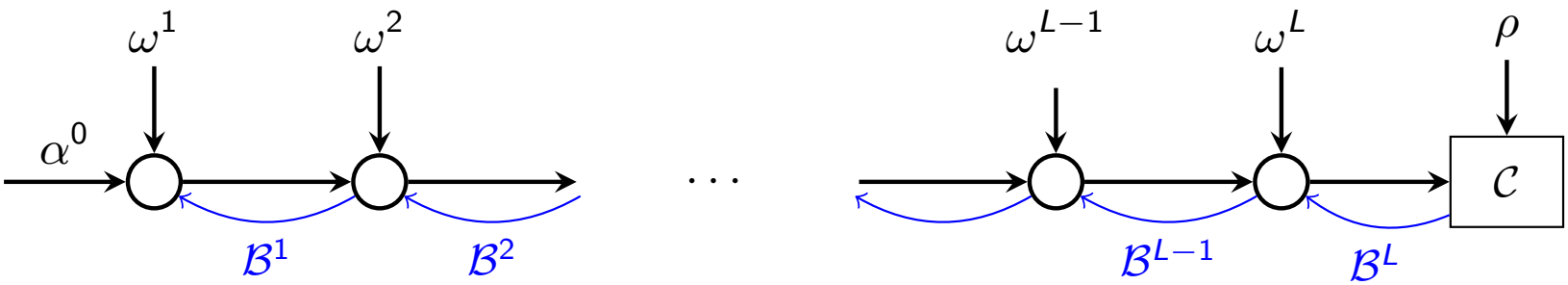

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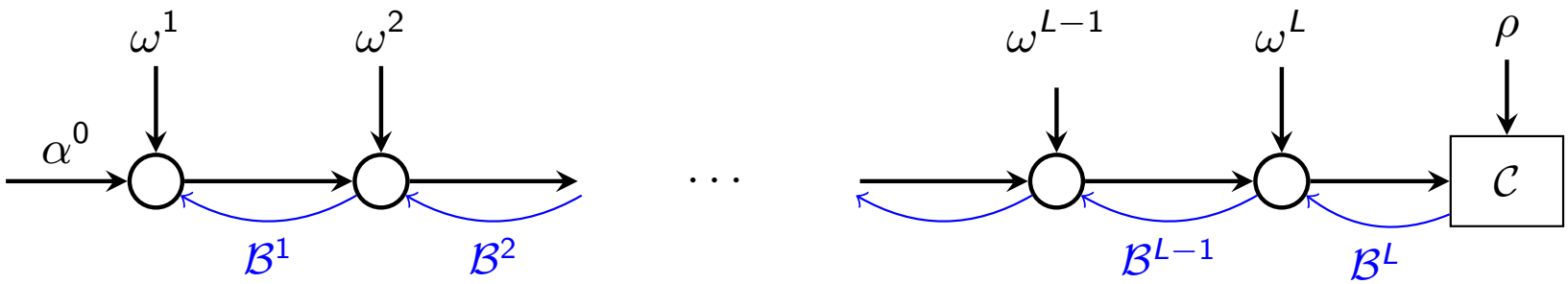

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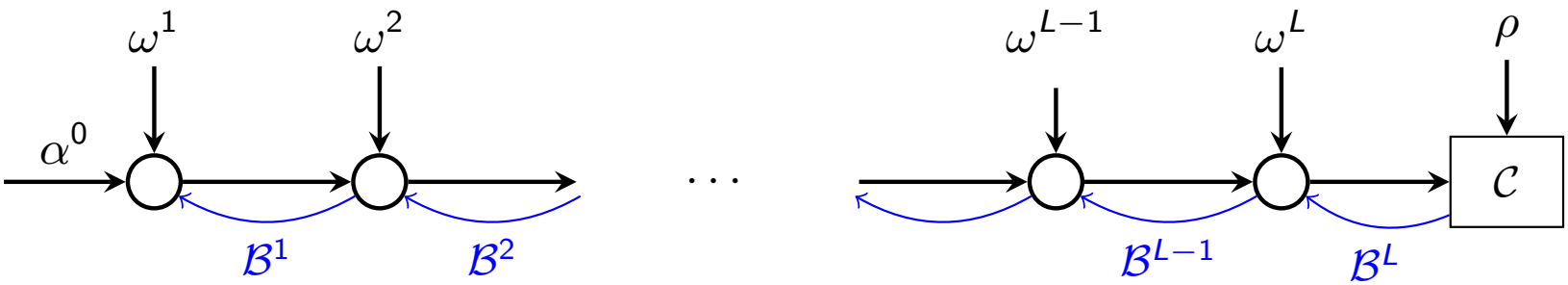

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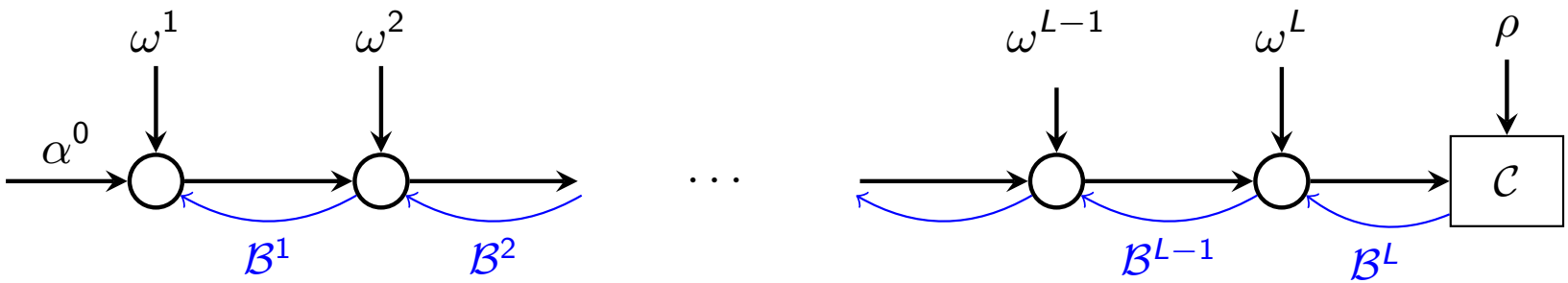

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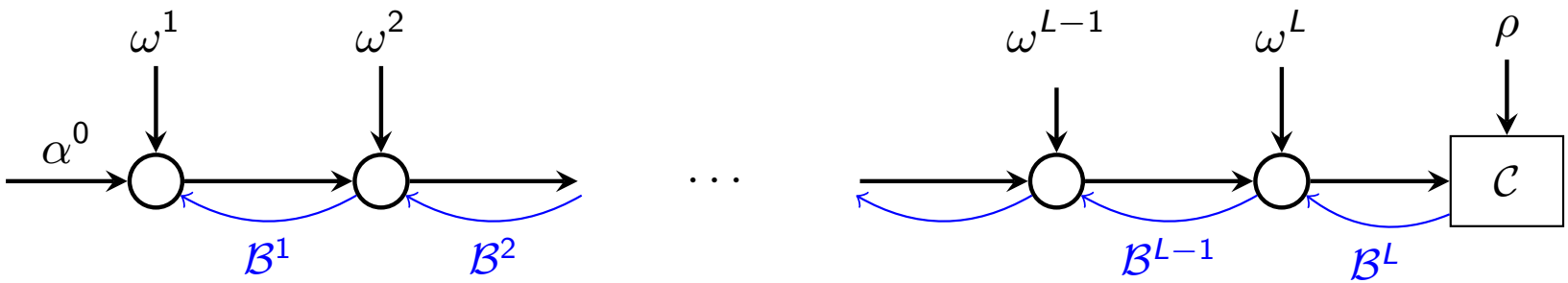

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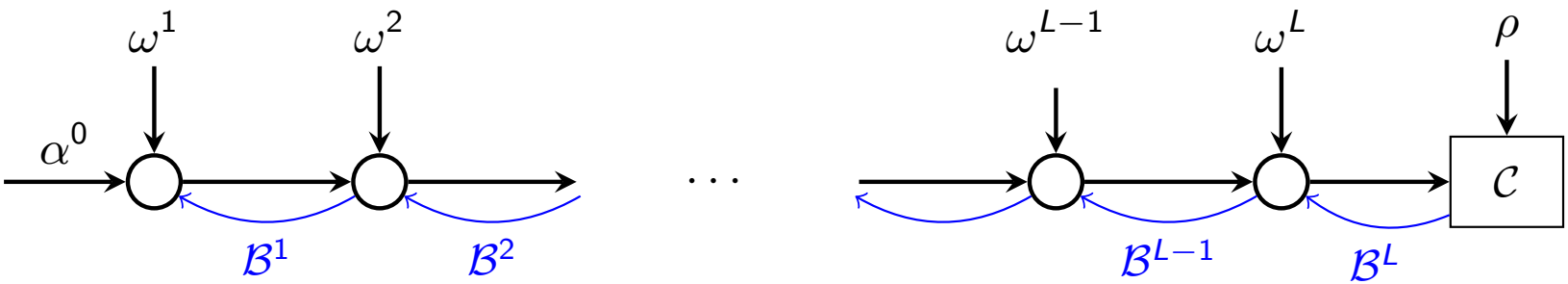

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- This is a general problem for very deep neural networks

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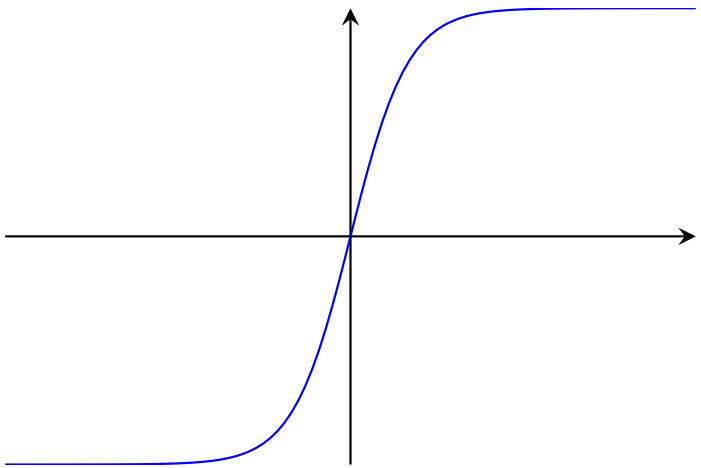

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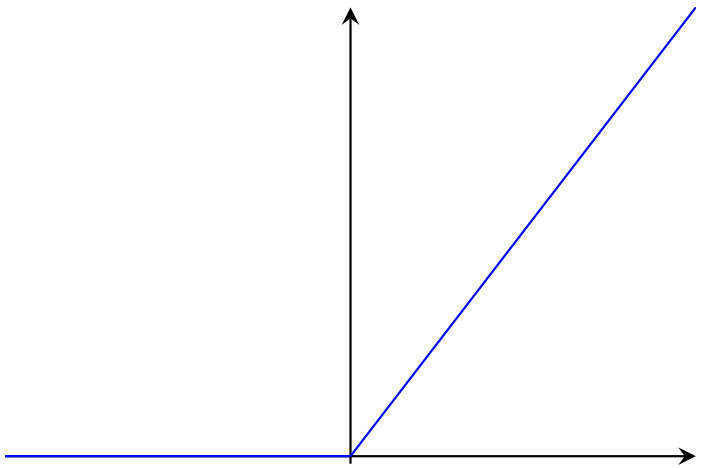

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- The gradient for this neuron will always be 0
- The neuron has no contribution and cannot be updated

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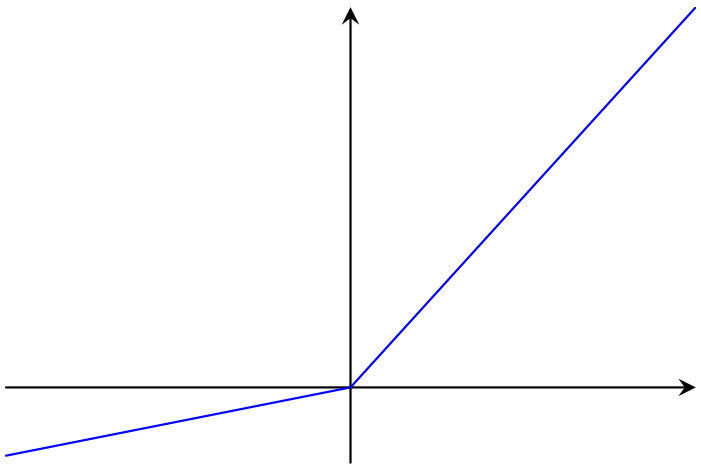

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  - Some seem more convenient depending on the amount of data at hand and the time that can be spent training the network

Outline

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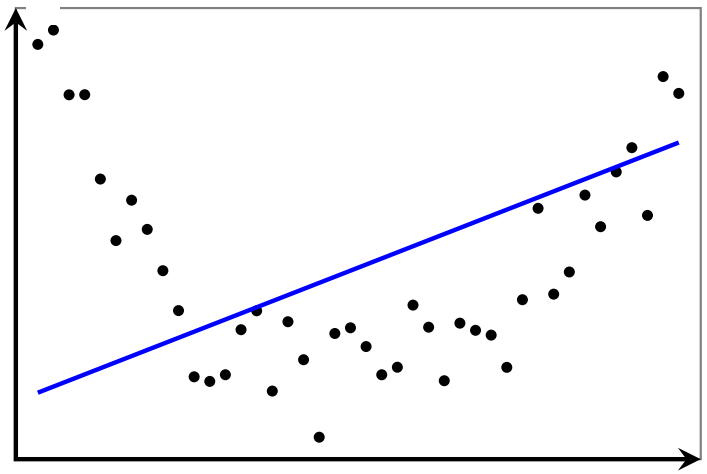

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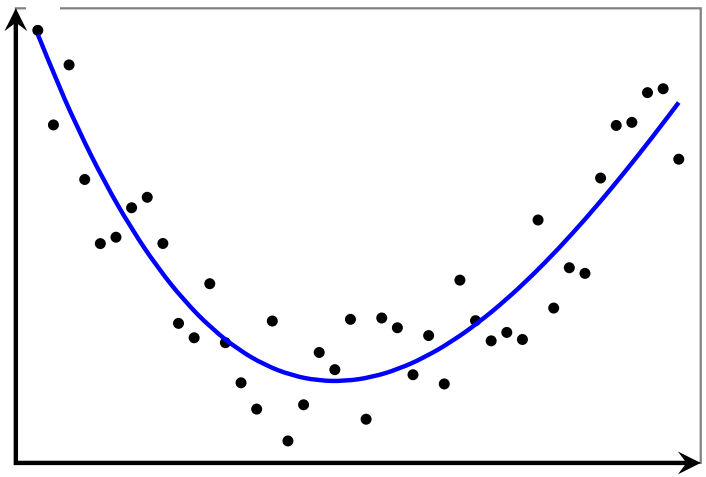

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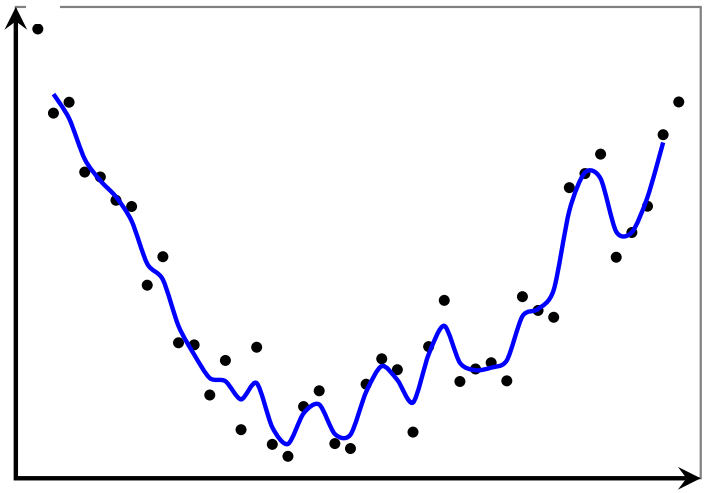

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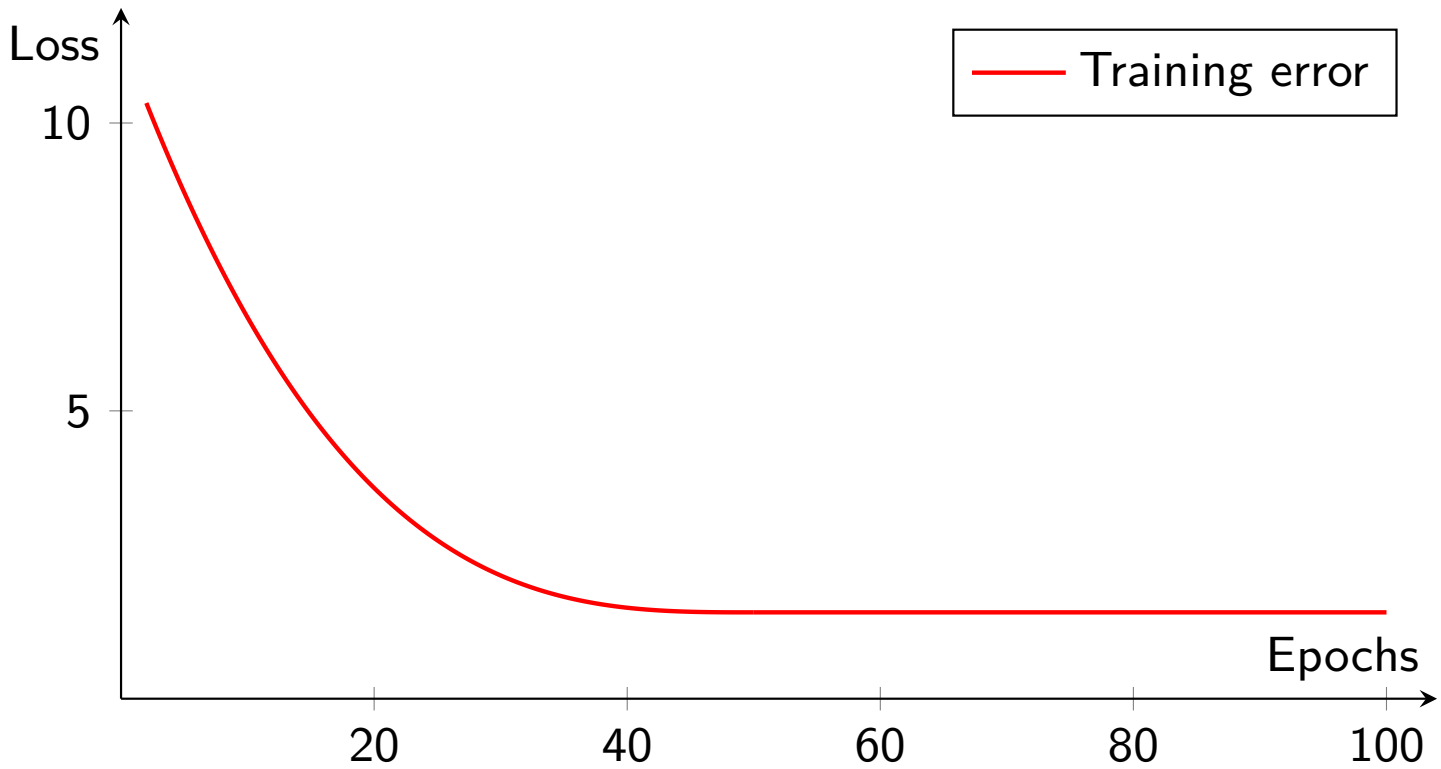

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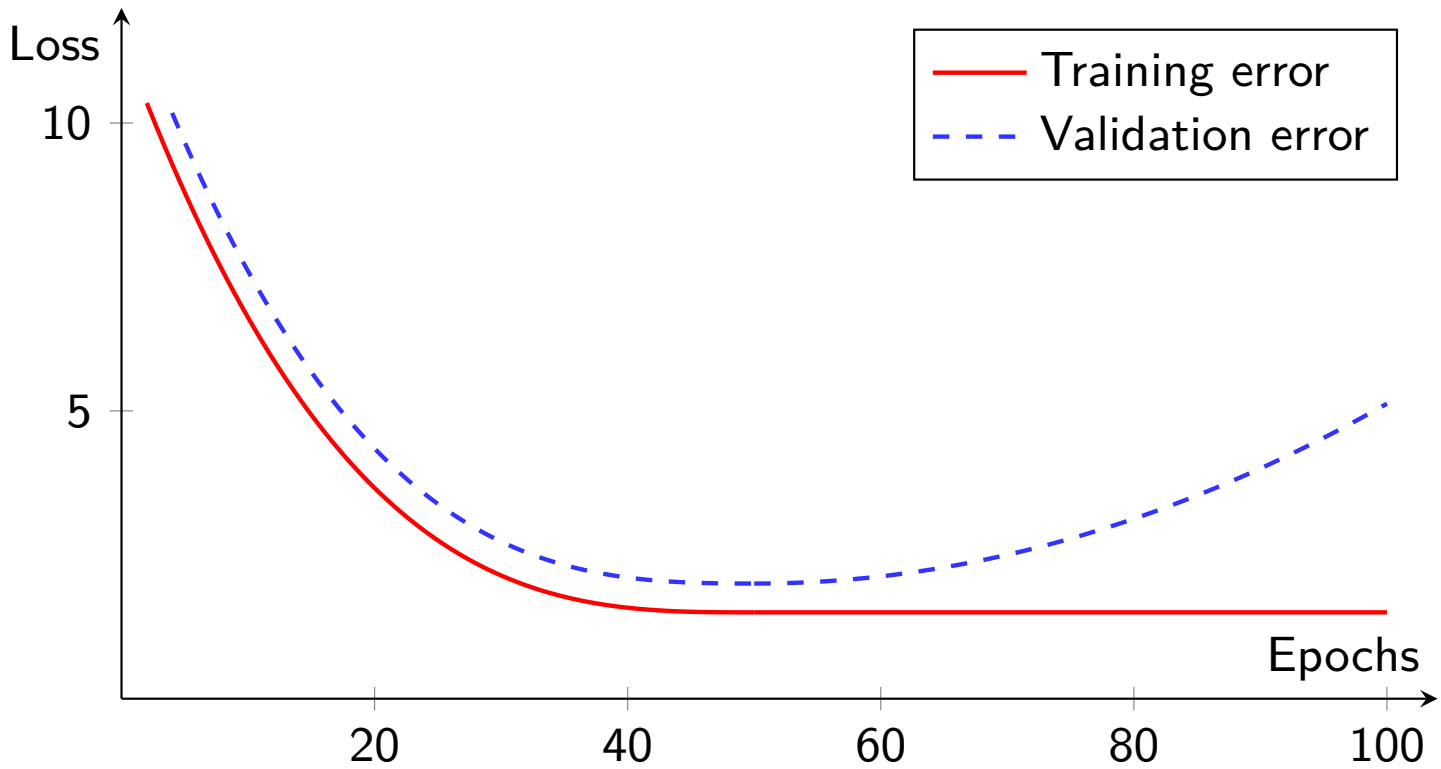

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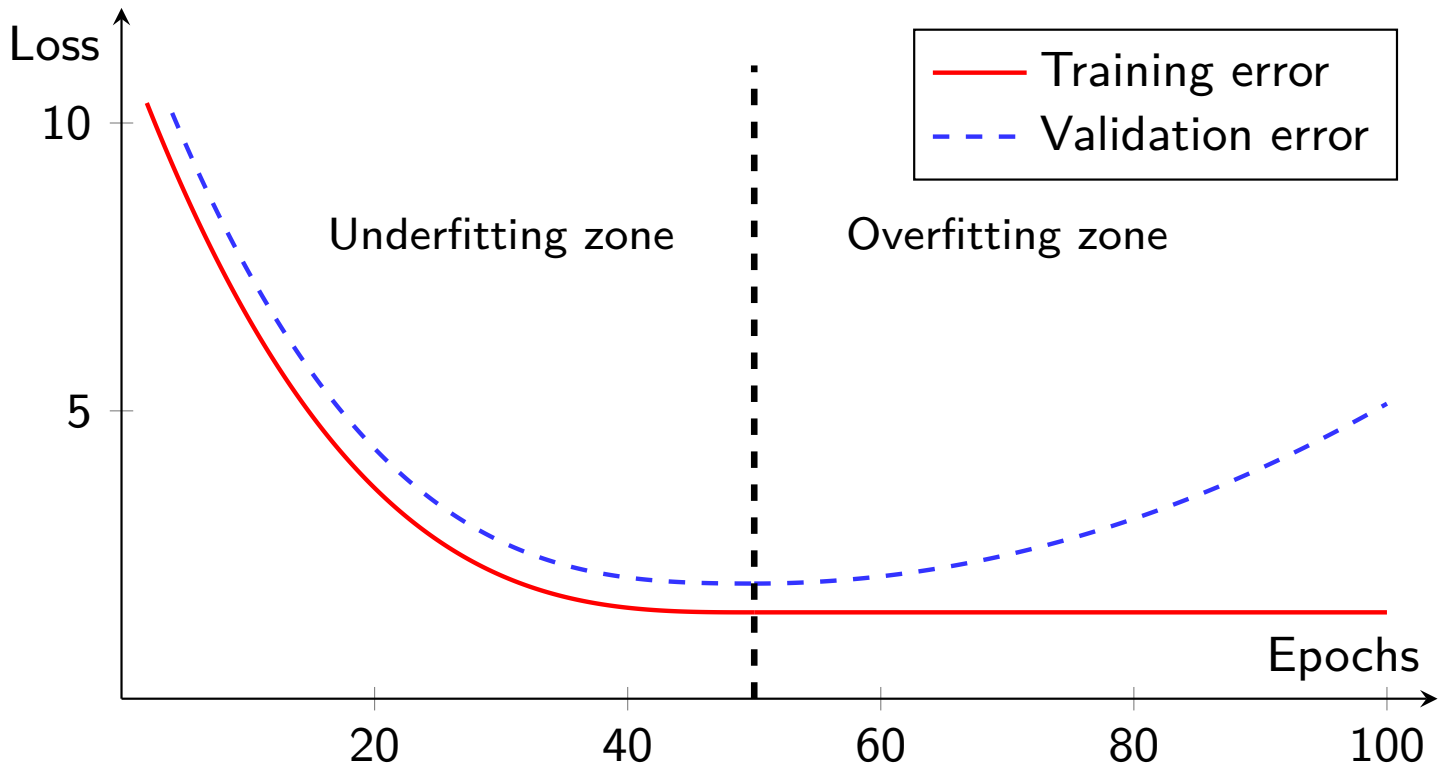

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Importance of regularization

In practice, deep learning architectures that give good results are large, and have had proper regularization techniques applied to them

Outline of this section

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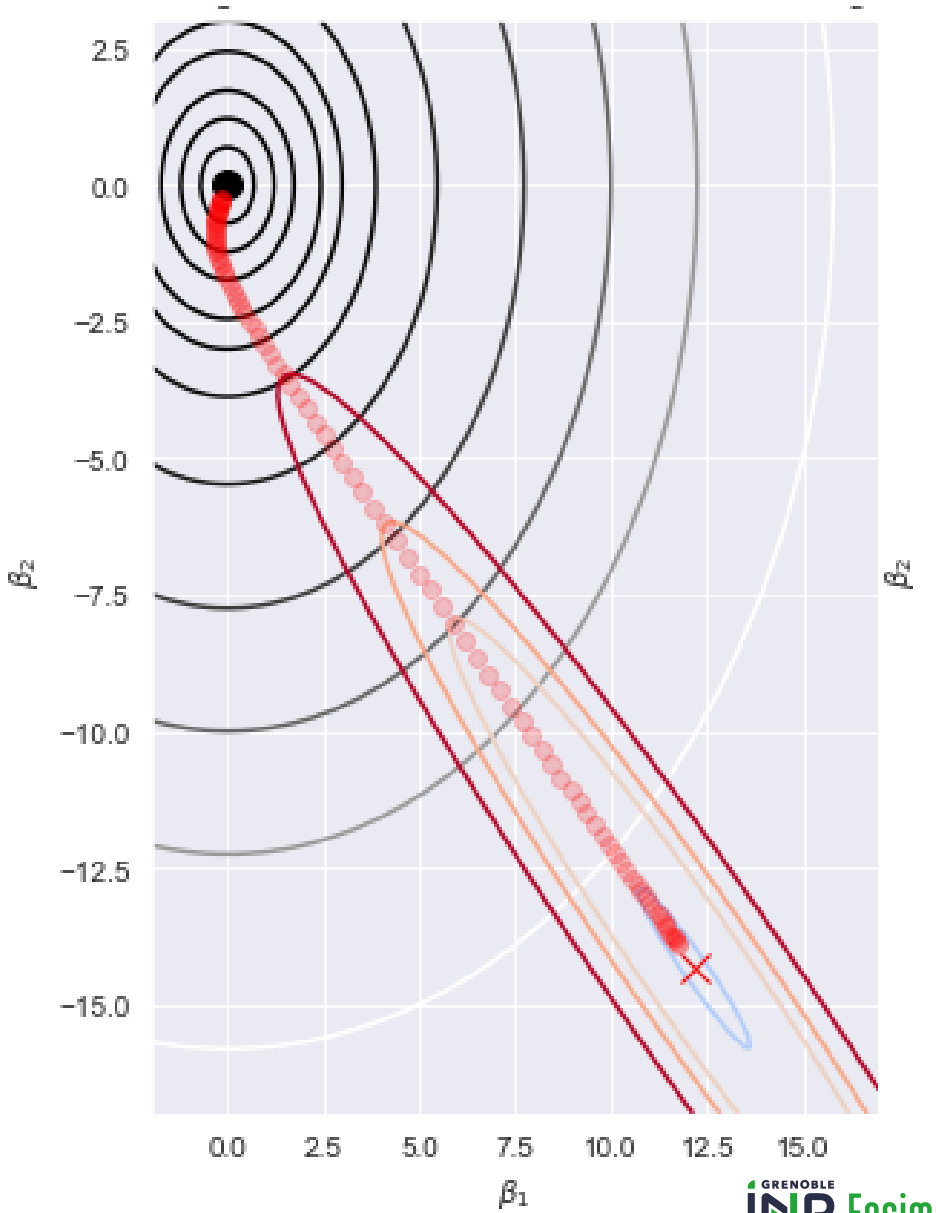

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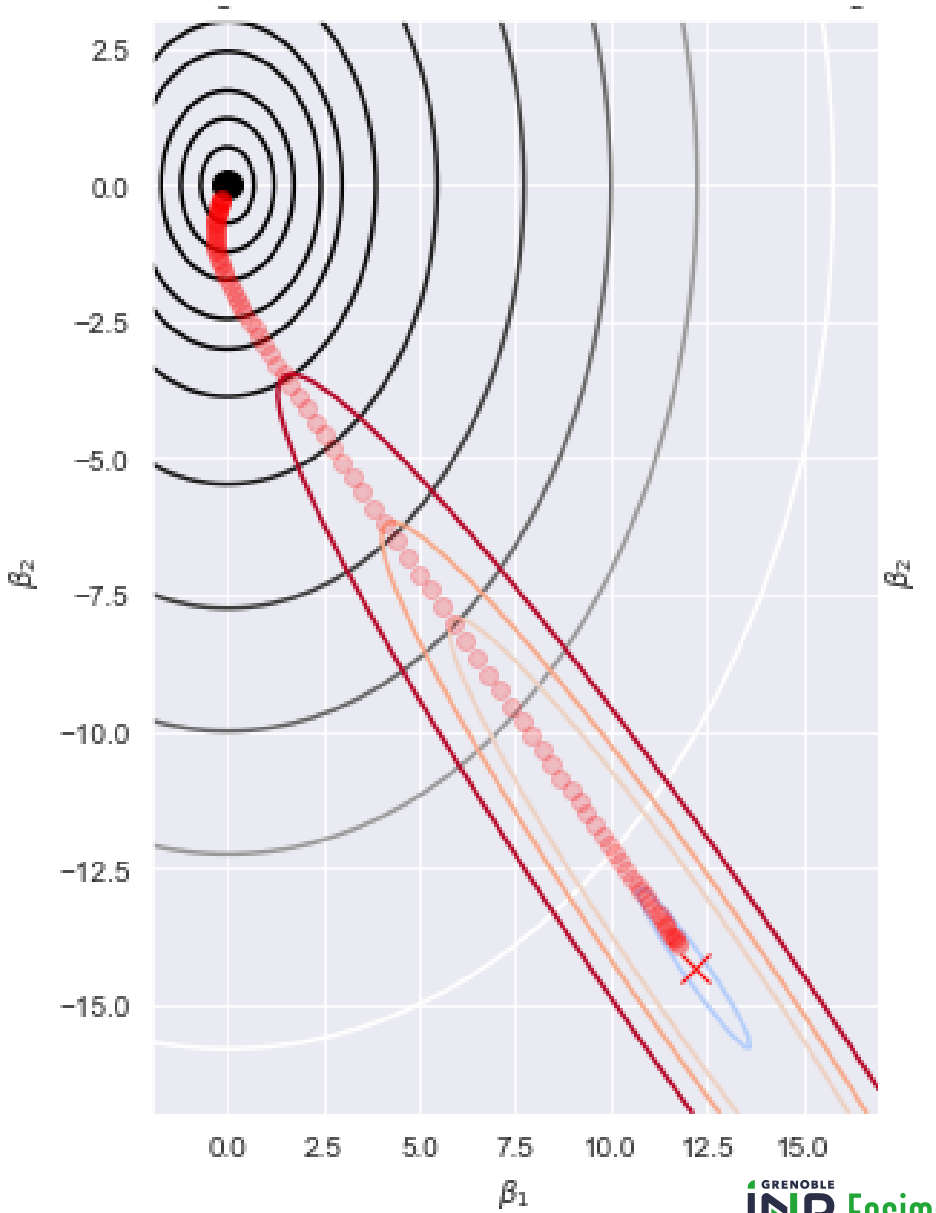

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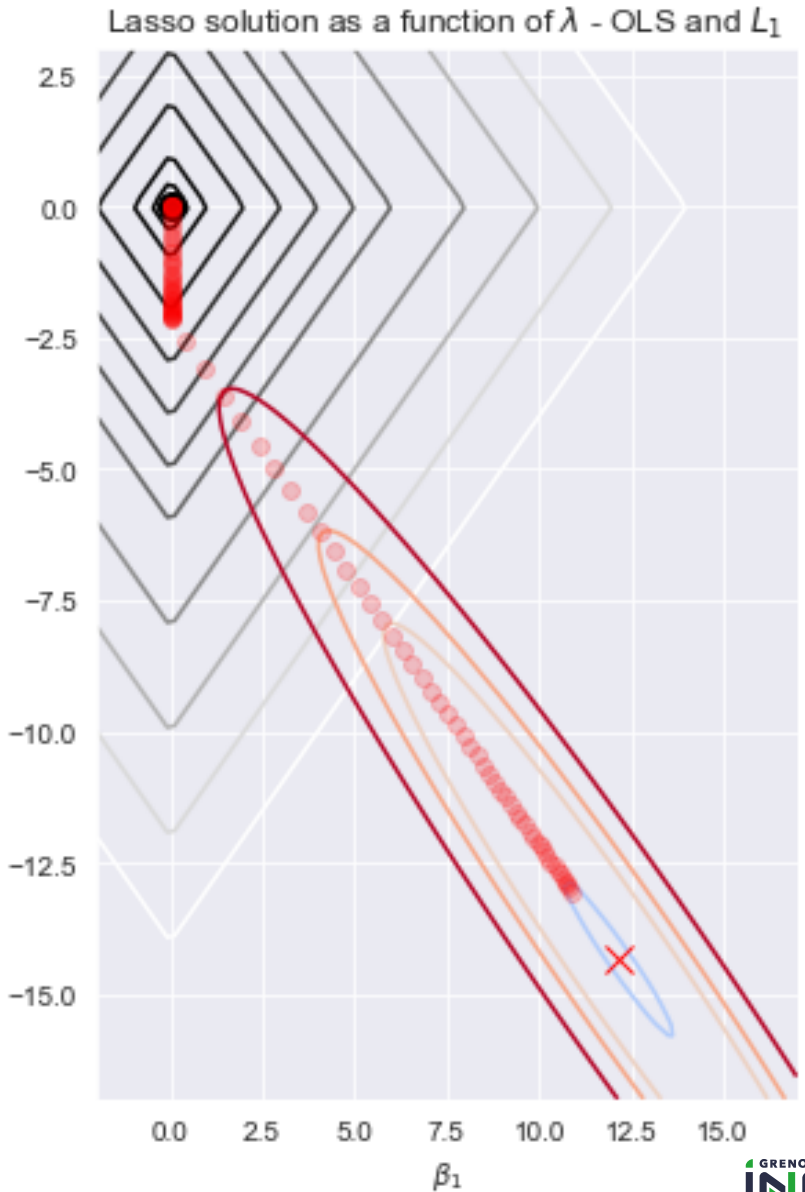

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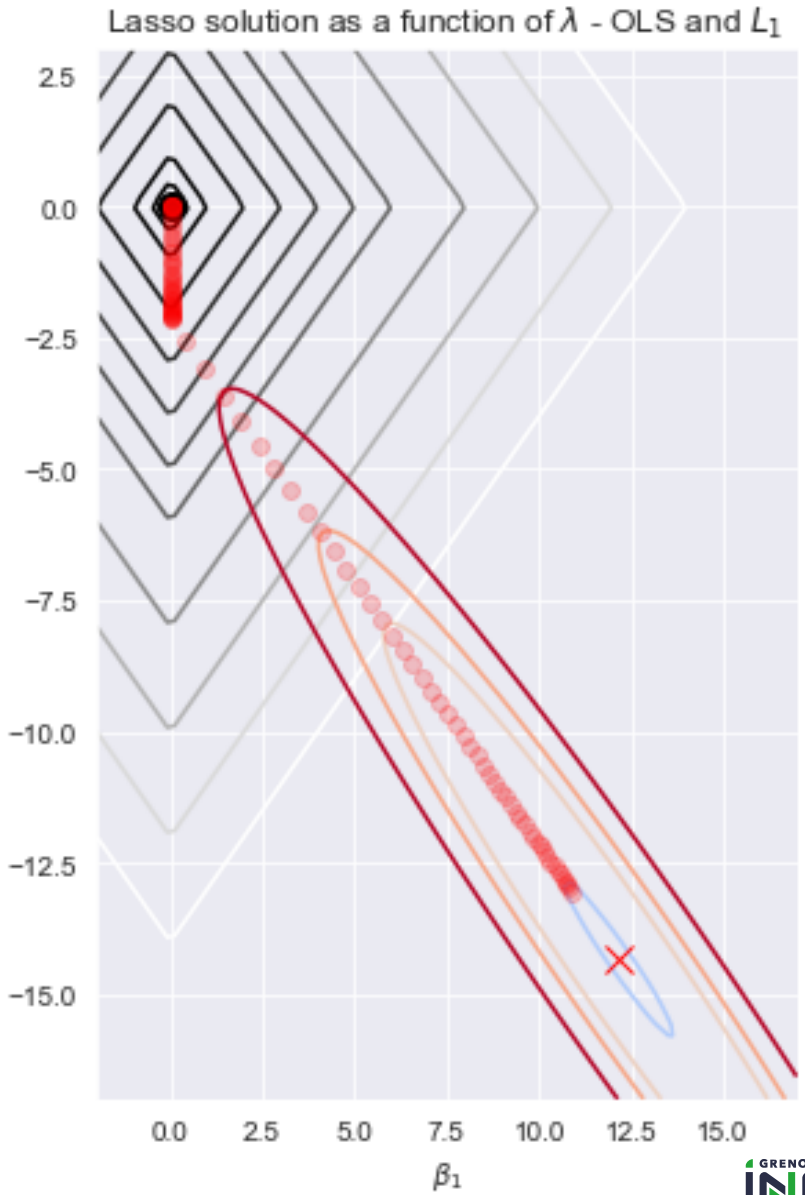

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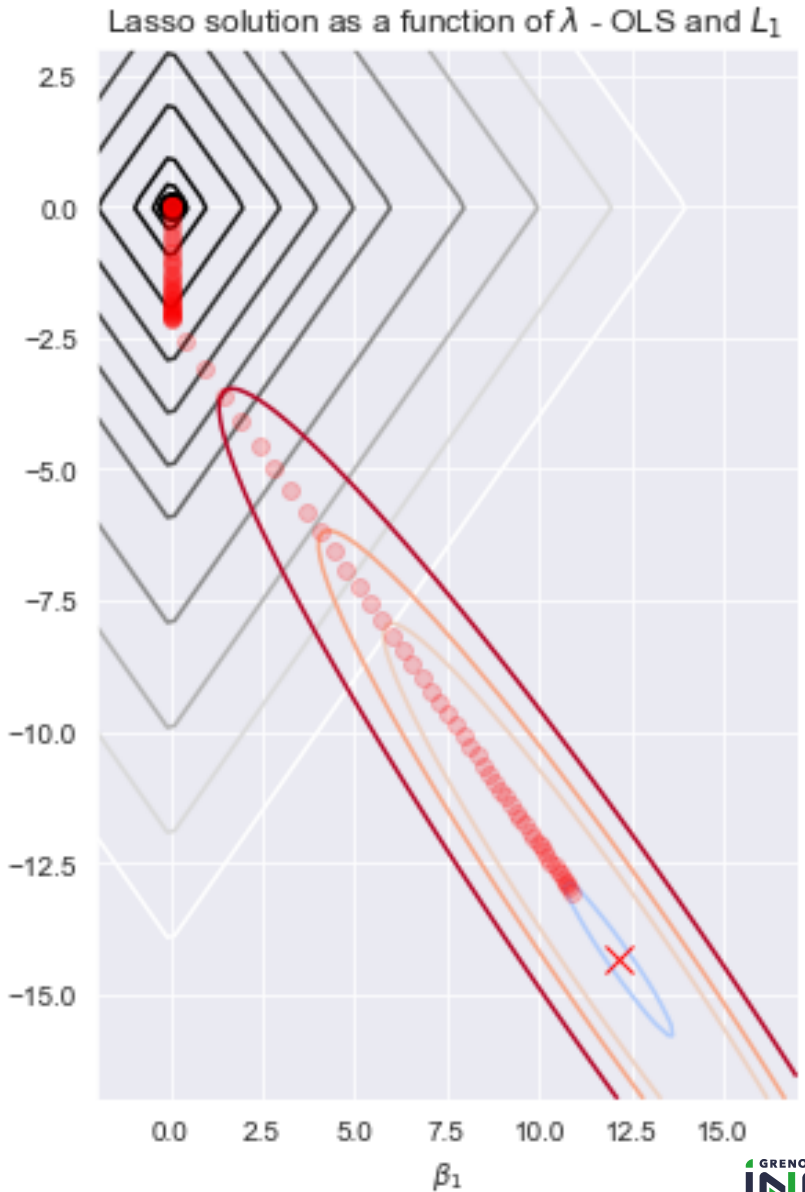

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- Adam appears to perform much better with weight decay than with L^2-regularization (Loshchilov & Hutter - Decoupled weight decay regularization. 2019)

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Input: n, number of steps between validation error evaluations
Input: p, number of observations of worsening validation errors before stop
Input: θ_0, initial parameters

$\theta \leftarrow \theta_0$, $\theta^* \leftarrow \theta$;
bestError $\leftarrow +\infty$;
$j \leftarrow 0$;
while $j < p$ **do**
    Perform n updates of θ;
    $j \leftarrow j + 1$;
    if *ValidationError*(θ) $<$ *bestError* **then**
        bestError $\leftarrow$ ValidationError(θ);
        $j \leftarrow 0$;
        $\theta^* \leftarrow \theta$;
    end
end
return θ^*

Outline of this section

Dropout: intuition

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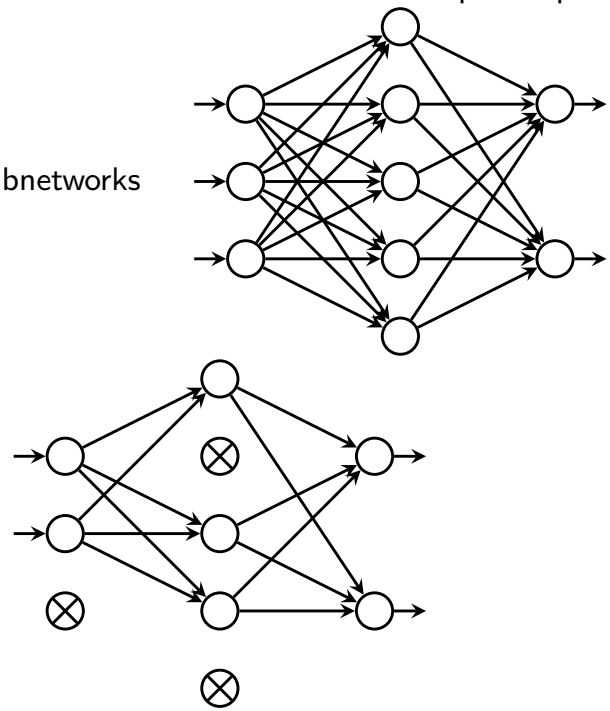

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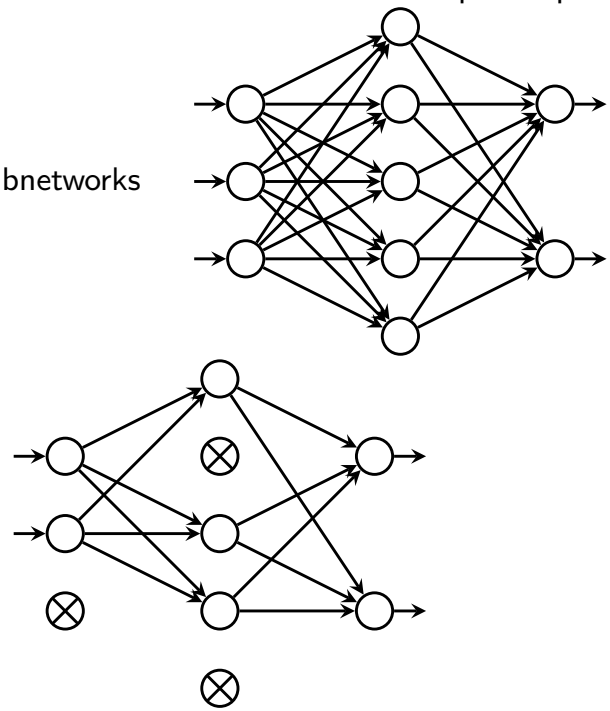

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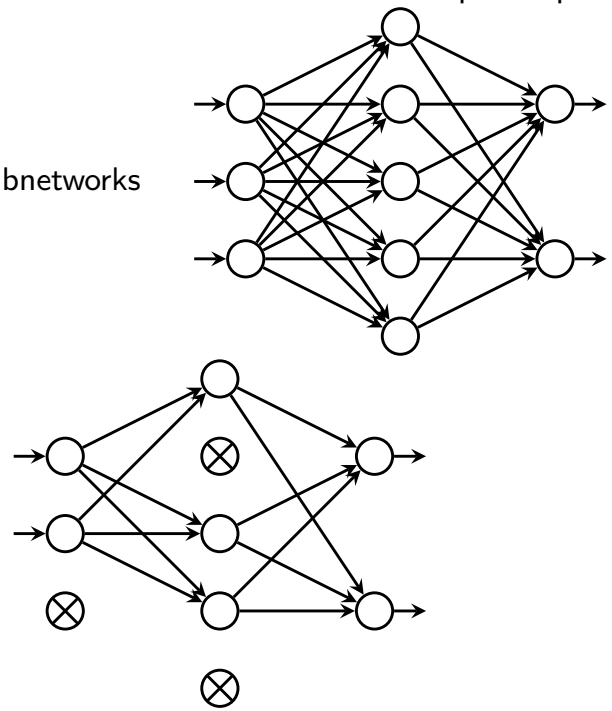

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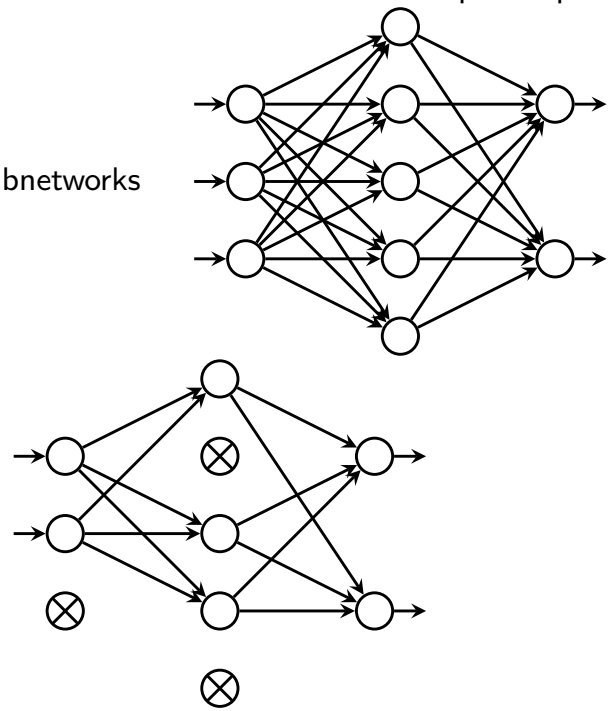

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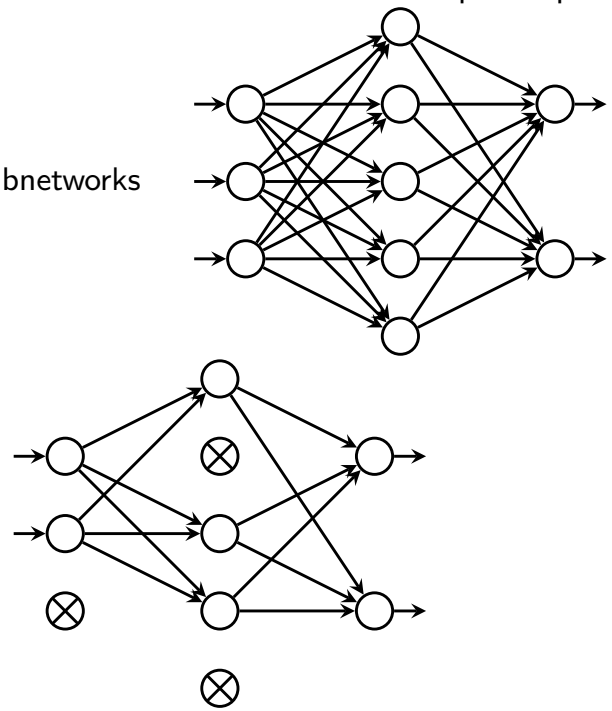

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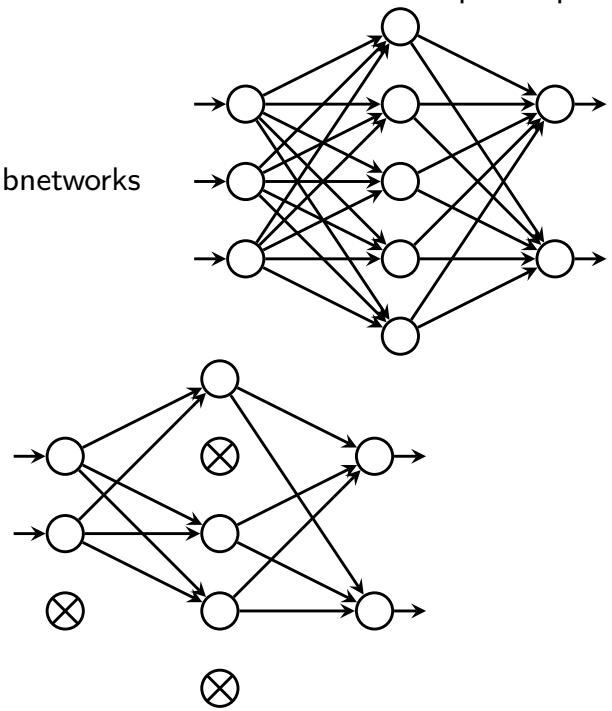
- Why not train these subnetworks separately and average their predictions during the test phase?
- Issues:
  - Which subnetworks are trained and how?

Dropout: intuition

- A network can be viewed as a compact representation of the set of all its subnetworks



- These subnetworks are not independent: they share weights
- Why not train these subnetworks separately and average their predictions during the test phase?
- Issues:
  - Which subnetworks are trained and how?
  - How are their predictions averaged?

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  - No theoretical argument for accuracy in the general case, but good results in practice

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  - Activation **per mini-batch** and **per neuron**:

$$\Phi_D(x) = \begin{cases} \frac{x}{p} & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

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