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**ORIGINAL ARTICLE**



**Fuzzy portfolio selection based on three‑way decision and cumulative prospect theory**

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**Abstract**

The goal of fuzzy portfolio selection is to make a combination of securities which can maximize the return or minimize the risk. Most of existing studies assumed that the investor has all the cash in hand and no securities position before portfolio optimization, which is sometimes inconsistent to reality. Besides, many studies are based on expected utility theory, which is in conflict with the behavior of some investors and may also lead to over-concentration of capital. Therefore, in this paper, we propose a fuzzy portfolio selection model based on three-way decision and cumulative prospect theory, which can mitigate the two shortcomings mentioned above. In this model, every action in the action set to the candidate securities is assigned to a prospect value and we can construct a tri -partition of the candidate securities according to three- way decision theory. To validate the effectiveness of our approach, we adopted two case studies on the basis of real market data. The experimented results prove that the using of three-way decision and cumulative prospect theory increases the investment return, meanwhile, reduces the risk for the investor.

**Keywords** Fuzzy portfolio selection · Three-way decision · Cumulative prospect theory · Expected utility theory

**1  Introduction**

Portfolio selection theory began with Markowitz’s seminal work [27], the core issue of which is to maximize the return or minimize the risk. In Ref. [27], Markowitz used the mean

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as an indicator of return and the variance as an indicator of risk. Different from Markowitz’s model which is based on random variables, Chen [ 3] used fuzzy variables to denote the uncertainty of future return and risk and obtained the optimal asset allocation by fuzzy portfolio optimization. In recent years, many scholars proposed different return and risk metrics to improve and perfect fuzzy portfolio selec-tion models [11, 12, 19, 34]. Works of these scholars greatly enrich the theoretical and applied value of portfolio selec-tion, however, most existing studies still suffer from two deficiencies.

The first deficiency is that their models assume that the investor has all the cash in hand and no securities position before the portfolio optimization. This assumption may sometimes contradict reality, which hinders the practical application of portfolio selection models to a certain extent. Therefore, a series of portfolio selection strategies have been proposed to alleviate this deficiency, most of which are online strategies, including following the winner, following the loser, meta-learning, pattern matching, and etc. [18]. The progresses of artificial intelligence have greatly improved the practical value of online portfolio selection strategies. However, the online portfolio selection strategies are aimed at very-short-term investment problems, i.e., high-frequency

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trading. And to the best of our knowledge, there are few studies solving the long-term portfolio selection problem considering the above deficiency. Based on the assump-tion that the investor already holds securities in a long-term investment, the portfolio selection model in this paper is developed by adopting three -way decision (3WD) theory, which is expected to provide comprehensive decision sup-port. Since the investor may act out hesitation for insufficient information in capital market, at this time a delayed decision inspired by 3WD might be a better option than the decision made in a hurry.

3WD [42], developed from classical rough set theory pro-posed by Pawlak, has aroused great interest in academia and industries related to decision making, rough set [40], granu-lar computing, attribute reduction [45]. The main concept of 3WD is to divide the decision space into three disjoint areas (positive, boundary and negative areas) by means of two thresholds, *α* and *β*. Based on the Bayesian risk minimiza-tion principle, decision rules of 3WD are generated from three disjoint areas, including acceptance, non-commit-ment and rejection. In the case of insufficient information, non-commitment is treated as a delayed decision, which is consistent with the human-made decision process. And in recent years, 3WD has been applied in many fields, includ-ing cost-sensitive learning [13], image recognition [20], cog-nitive concept learning [21], recommendation systems [ 24], medical diagnosis [41] and incremental clustering [43]. The three-way rules generated by the positive, boundary and negative regions can correspond with the decisions of buy, deferment and sell in portfolio selection. So it can be seen that the procedures of 3WD match with portfolio selection inherently.

Besides, introducing 3WD into portfolio selection can also be expected to hedge against the uncertainties of securi-ties future returns. Zhang [44] proposed the fuzzy-entropy-based game theoretic shadowed sets, a three- way approxi-mation of fuzzy sets, to find a balance method for two kinds uncertainties. And Mansour [26] noted that securities returns contain many aspects of uncertainty, such as ambiguity and vagueness. In order to describe the randomness and volatil-ity of the securities market, many scholars use random vari-ables or fuzzy variables to forecast the trend of securities returns [1, 17, 31]. No matter how advanced the forecasting techniques are, the forecasted results will always contain some subjective elements, such as the selection of probabil-ity distribution/membership function and the estimation of the related parameters like mean and variance. The unavoid-able fact is that these forecasts cannot be always correct and accurate [11 ]. Over reliance on the forecast results may lead to excessive risk concentration. What’s more, most of exist-ing methods involve only the buying and selling of securi-ties. By contrast, the using of 3WD, especially the consider-ation of the boundary operation can better hedge against the

forecast uncertainty to a certain extent. Therefore, it could be meaningful to build a 3WD-based portfolio selection model.

The second deficiency in most existing studies is that they are completely based on expected utility theory (EUT) . For a long time, EUT has been proved to be effec-tive in portfolio selection [2, 6]. However, Kahneman and Tversky found that the decision making behavior of some investors under risk and uncertainty does not fit well with EUT, and these findings became the source of prospect theory (PT) [15]. Following the work of Quiggin [29] and Schmeidler [30], Tversky and Kahneman extended PT to uncertain and risky prospects with a lot of outcomes, and introduced cumulative prospect theory (CPT) [33]. In CPT, it is assumed that the prospect value of a scenario is quantified by the probability weight function and the value function. CPT summarizes the phenomena of loss aver-sion and diminishing sensitivity. And the decision-maker will always choose the one with the highest prospect value among all the available options. The effectiveness of CPT in characterizing investors’ behavior has been verified by large amounts of experiments. Therefore, many scholars have used CPT to reveal the risk preference of decision-makers [8, 16, 39]. In this paper, CPT is utilized together with EUT to assess the decision-maker’s preference for each possible outcome.

Based on the above analysis, a novel fuzzy portfolio selection model based on 3WD and CPT is proposed in this paper. First of all, it is assumed that investors already have position in some securities before the portfolio optimization. Then the prospect value of the securities for each possible outcome is calculated. According to the calculation results, the securities can be classified into the three disjoint areas of 3WD. Finally, following the classification results, an opti-mal investment decision can be obtained by solving a given portfolio selection model.

The major contributions of this paper include the follow-ing: (1) 3WD is considered to handle the inaccurate forecast results of securities returns and the long-term investment problems with securities in hand. Experimental results dem-onstrate that the using of 3WD in portfolio selection guar-antees the diversity of capital allocation and achieves risk reduction; (2) CPT is used together with EUT to achieve a balance between maximizing the prospect value and expected return. Experimental results justify that the com-bination of CPT and EUT is effective in portfolio selection.

The remainder of this paper is organized as follows. Sec-tion 2 contains the preliminary knowledge which will be used in this paper. Section 3 defines the security classifica-tion rules based on 3WD and CPT. Section 4 builds four portfolio selection models and introduces the solution algo-rithm. In Sect. 5, two case studies based on real market data are performed to justify the effectiveness of the proposed methods. Finally, Sect. 6 is the conclusion of this paper.

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|  |  |  |  |  |  |
| **2 Preliminaries** | **Table 1** The loss function |  |  |  |  |
|  |  | *C* |  |
|  | matrix | |  |  |  |
| In this section, some preliminary knowledge of fuzzy set |  |  | *P* | *PN* |  |
| theory, 3WD and CPT will be introduced. | *A* | | *PP* |  |
|  |  | *B* | *BN* |  |
|  | *A* | | *BP* |  |
|  | *A* | | *NP* | *NN* |  |
|  |  |  | *N* |  |  |

**2.1  Fuzzy set theory**

In order to describe the volatility and non-statistical fea-ture of the securities market, fuzzy variables have been widely adopted to represent the possible securities returns in the future. In the fuzzy context, possibility is the most frequently used measure. However, it’s hard to show the chance level of a fuzzy event occurring for the possibil-ity measure is not self-dual. In order to construct a dual measure in fuzzy set theory, Liu and Liu [23] defined the self-dual set function, i.e., credibility measure.

Suppose that *ξ* is a fuzzy variable with membership function (*x*) . For any real number *R* ∈ ℜ, the credibility of event *r* is defined by Liu and Liu [23] as:

belongs to state , and the counterparts *PN* , *BN* , *NN* represent the losses incurred for taking the same actions while the element does not belong to state .

For an object *o*, *p*( [*o*]) and *p*( *C* [*o*]) denote the prob-abilities of *o* belongs to state or not, respectively. Then for object *o*, the expected losses of taking different actions are calculated by:

|  |  |
| --- | --- |
| R(*P* [*o*]) = *PP* × *p*( [*o*]) + *PN* × *p*( *C* [*o*]), |  |
| R(*B* [*o*]) = *BP* × *p*( [*o*]) + *BN* × *p*( *C* [*o*]), | (4) |
| R(*N* [*o*]) = *NP* × *p*( [*o*]) + *NN* × *p*( *C* [*o*]). |  |

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Cr{ | ≤ *r*} = | | 1 | [sup (*x*) + 1 − sup (*x*)]. | | | | | | | | (1) | According to the Bayesian decision rules, the principles of | | | | | | | | | | | | |  |
|  | minimum-loss can be summarized as: | | | | | | | | | | |  |  |  |
|  |  |  | 2 ≤*r* | |  |  |  | *>r* | |  |  |  |  |  |
| Liu and Liu [23] also defined the expect value of fuzzy vari- | | | | | | | | | | | | | (P0) IF R(*P* [*O*]) ≤ R(*B* [*O*]) AND R(*P* [*O*]) ≤ R(*N* [*O*]), | | | | | | | | | | | | |  |
|  |  |  | ∈ | ( | ) |  |  |  |  |  |  |  |  |
| able *ξ* as: | |  |  |  |  |  |  |  |  |  |  |  | THEN |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  | DECIDE | *O* |  | POS | | , |  |  |  |  |  |  |  |
|  | +∞ | | | |  | 0 | | |  |  |  |  | (B0) IF R(*B* [*O*]) | | | | ≤ R(*P* [*O*]) AND R(*B* [*O*]) ≤ R(*N* [*O*]), | | | | | | | | |  |
|  |  |  |  |  | (2) |  |  |  |  | ( |  |  |  | |  |  |  |  |  |
| E[ ] = �0 | |  | Cr{ | | ≥ *r*}*dr* − �−∞ Cr{ ≤ *r*}*dr*. | | | | | | | THEN |  | ∈ | ) | |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  | DECIDE | *O* |  | BND | | , |  |  |  |  |  |  |  |
| The credibility measurement is a self-dual function, i.e., | | | | | | | | | | | | | (N0) IF R(*N* | | [*O*]) | | ≤ R(*P* [*O*]) AND R(*N* | | | |  | [*O*]) ≤ R(*B* | | | [*O*]), |  |
| THEN |  |  | ∈ NEG( ). | | | |  |  |  |  |  |  |
|  | *r*} =1−Cr{ *>r*}. Furthermore, the credibility | | | | | | | | | | | |  | DECIDE | *O* |  |  | |  |  |  |  |  |  |  |  |
| Cr{ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| measurement satisfies the axioms of monotonicity, maxi- | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| mality, subadditivity and normality. And for any real number | | | | | | | | | | | | | **2.3  Cumulative prospect theory** | | | | | | | | | | |  |  |  |
| ∈ (0, 1), the notation of *γ*-level cut set of fuzzy variable *ξ* | | | | | | | | | | | | |  |  |  |
| is defined as [38]: | | | | |  |  |  |  |  |  |  |  | To provide a rational explanation of why financial deci- | | | | | | | | | | | | |  |
| [ (*x*)] = {*x* | |  |  |  |  |  |  | , |  |  |  | (3) |  |
| (*x*) | | | } = {*x* | [ | *x* | *x* |  | ]}. | sions investors made under uncertainty and risk conflict | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  | with classical economic theories, Tversky and Kanhne- | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  | man proposed PT [15] and CPT [33]. Suppose there are *M* | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  | options for decision-makers to choose from. Each option | | | | | | | | | | | | |  |
| **2.2  Classical three‑way decision theory** | | | | | | | | | | | |  | could lead to *k* outcomes, quantified by *x*1, *x*2, …, *xk* . The | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  | probability of outcome *xi* | | | | | | | is *pi* . In classical EUT, decision- | | | | | |  |
| The basic concepts and rules of 3WD were introduced by | | | | | | | | | | | | | makers will always choose the option with the highest | | | | | | | | | | | | |  |
| expected value, i.e., maximize | | | | | | | | | | *k* | × *pi*. | |  |
| Yao [42]. Suppose the state set is given by *U* = { , *C*}, | | | | | | | | | | | | | *i*=1 *xi* |  |
| which means an element is belong to state or not respec- | | | | | | | | | | | | | However, Tversky and Kahneman found out that inves- | | | | | | | | | | | | |  |
| tively. And the action set is given by A = {*aP* , *aB* , *aN* | | | | | | | | | | | | }, in | tors’ perception toward gains and losses are not linear. | | | | | | | | | | | | |  |
| which *aP* , *aB* , *aN* indicate to classify the elements into three | | | | | | | | | | | | | And the gains and losses are not absolute values, but are | | | | | | | | | | | | |  |
| disjoint areas: positive (POS( )), boundary (BND( )) and | | | | | | | | | | | | | related to the chosen of a reference point. Investors will | | | | | | | | | | | | |  |
| negative (NEG( )) areas respectively. The loss functions | | | | | | | | | | | | | act out loss aversion while confronting the same degree of | | | | | | | | | | | | |  |
| by taking different actions in different states are listed in | | | | | | | | | | | | | gains and losses. Simultaneously, they found that the prob- | | | | | | | | | | | | |  |
| Table 1, where *PP* , *BP* , *NP* represent the correspond- | | | | | | | | | | | | | abilities will be converted into decision weight, and inves- | | | | | | | | | | | | |  |
| ing losses of taking actions *aP* , *aB* , *aN* while the element | | | | | | | | | | | | | tors will magnify the low probability event and minify the | | | | | | | | | | | | |  |

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high probability event. The above formed two of the most important numerical principles in CPT: the value function and the probability weight function. Prototypical examples of these two functions given by Tversky and Kahneman are provided in Formulas (5) and (6).

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | (*x* − *x* ) , | | | | |  |  | *x > x* , | (5) |  |
|  |  | − ∗ (*x*0 − *x*) , | | | | |  |  | *x* ≤ *x*0. |  |
| *v*(*x*) = | |  | 0 | | |  | 0 | | |  |  |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  | *p* |  |  |  |  |  |
| *w*+(*pg*) = | |  |  |  | *g* |  |  | , |  |  |
|  |  |  |  | 1 | |  |  |
|  |  |  | (*pg* + (1 − *pg*) ) | | | | | | (6) |  |
|  |  |  |  |  | *l* | *pl* |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |
|  | *w*−(*pl*) = | |  |  |  |  |  | , |  |  |  |
|  |  |  |  | 1 |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | (*p* + (1 − *pl*) | | | | ) | | |  |  |
|  |  |  |  |  |

where it is normally suggested that = 0.88 , = 0.88 ,

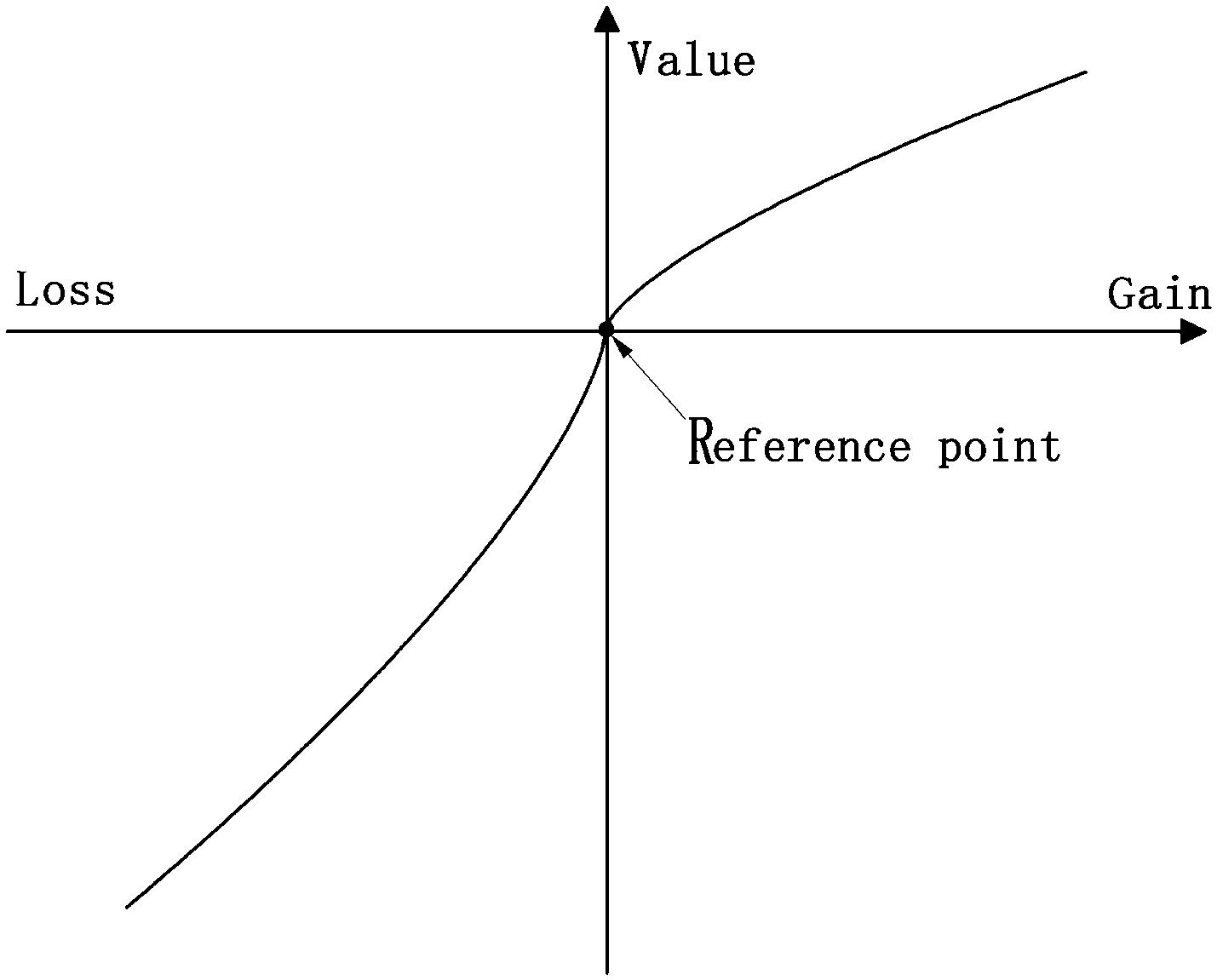
= 2.25, = 0.61, = 0.69, and the value of *θ* indicates the degree of loss aversion. Besides *x*0 is the value of the reference point, while *pg* and *pl* are the probability of gains and losses. The graphical representations of the value func-tion and the probability weight function are shown in Figs. 1 and 2. The combination of the weight function and the value function makes a sound explanation of why decision-makers always act out risk-seeking while facing high-probability losses and low-probability gains, and risk-avoiding while facing high-probability gains and low-probability losses.

In CPT, the preference of an option is evaluated by the corresponding prospect value. To calculate the prospect value of option X , the decision-maker firstly sorts all possible outcomes of option X in an ascend-ing order: from the greatest loss to the greatest gain, as

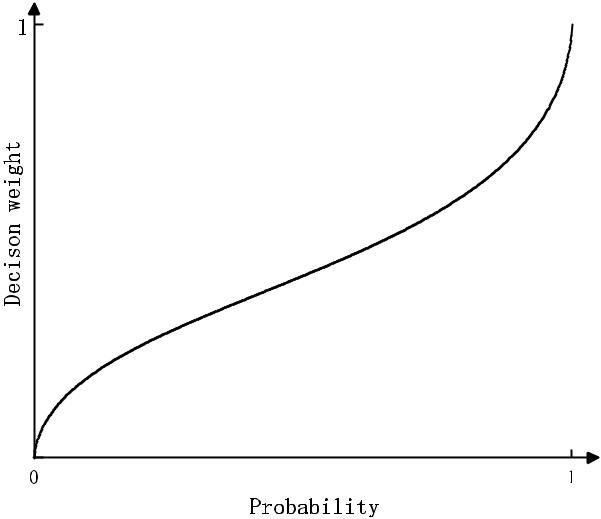
(*x*−*m*, *x*−*m*+1 … *x*−1, *x*1 … *xn*−1, *xn*) , where negative indices represent the loss, and positive indices represent the gain.

Suppose that the probability vector of the outcomes is

(*p*−*m*, *p*−*m*+1 … *p*−1, *p*1 … *pn*−1, *pn*), then, the decision weight for each outcome is calculated by:



**Fig. 2** An example of the weight function



|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | *w*−(*p* | | | | ), |  |  |  |  |  |  | i=-m, |  |
|  |  |  |  | −*m* |  |  |  |  |  |  |  |  |  |
|  |  | −( | |  | + ⋯ + ) − *w*−(*p* | | | | + ⋯ + *p* |  | ), | -m<i<0, |  |
| (*xi*) = | + |  | *p*−*m* |  | *pi* |  |  | −*m* | *i*−1 |  |  |  |
| *w*+ | |  |  | + |  |  | 0<i<n, |  |
|  | *w* |  | (*pi* + ⋯ + *pn*) − *w* | | | |  | (*pi*+1+⋯+*pn* ), | |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  | *w* (*pn*), | | | |  |  |  |  |  |  |  | i=n. |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  | (7) |  |
| Finally, the prospect value of option X is expressed as: | | | | | | | | | | | | |  |
|  | *n* | |  |  |  |  |  |  |  |  |  |  |  |
|  | *i m* | | | |  |  |  |  |  |  |  | (8) |  |
| V(X) = | |  |  | (*xi*) × *v*(*xi*). | | |  |  |  |  |  |  |

=−

**2.4  Three‑way decision based on cumulative prospect theory**

Considering the widespread application and success of CPT in decision making, Wang [37] combined CPT with 3WD theory to construct a CPT-3WD model. The model still consists of a state set *U* = { , *C*} and an action set A = {*a* , *a* , *a* }.

*P* *B* *N*

Besides, the elements of the state set and the action set in CPT-3WD share the same definitions with the corresponding elements in 3WD. In CPT, outcomes are utilized to assess the final state of investors’ wealthy. Outcomes represent the wealthy change of the decision- maker which resulted from taking aforementioned actions to elements in corre-sponding states, and Table 2 is the matrix of the outcome

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Table 2** The outcome matrix |  |  |  |  |
|  |  |  | *C* |  |
|  |  |  |  |  |  |
|  | *A* | | *PP* | *PN* |  |
|  |  |  | *P* |  |  |
|  | *A* | | *BP* | *BN* |  |
|  |  |  | *B* |  |  |
| **Fig. 1** An example of the value function | *A* | | *NP* | *NN* |  |
|  |  | *N* |  |  |
|  |  |  |  |  |

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|  |  |

*ij* (*i* = *P*, *B*, *N*, and *j* = *P*, *N*) in different states. It is supposed that the following conditions can be met in the real world [37]:

|  |  |  |  |
| --- | --- | --- | --- |
| *NP < BP* | *PP* | , |  |
| *PN < BN* | ≤ *NN* | . | (9) |

According to CPT, each outcome brings a decision value and shows the impact in the decision process. The decision value is influenced by the selection of the reference point. Suppose that the reference point has been pre-selected as

, where if the outcome *ij* *>* (*i* = *P*, *B*, *N*, and *j* = *P*, *N*), the decision-maker will regard it as a gain, and vice versa.



Substituting these outcomes and the reference point into the value function (in Formula (5)) yields Table 3 as below.

Following CPT, the conditional probabilities will be twisted into decision weights by the weight function (in Formula (7)), and then play a role in the decision-making process. Suppose the conditional probability of object *o* belongs to state is *p*( [*o*]), and *p*( *C* [*o*])denotes the conditional probabilityof object *o* does not belong to state . Furthermore, the degree to which conditional probabilities are twisted varies accord-ing to gains or losses. By comparing the values of *viP* with *viN* , all the conditions of the weight function *i*(*p*( [*o*]))and *i*(*p*( *C* [*o*])) (*i* = *P*, *B*, *N*) are written as follows:

V(*P* [*o*]) = *vPP* × *P*(*p*( [*o*])) + *vPN* × *P*(*p*( *C* [*o*])),

V(*B* [*o*]) = *vBP* × *B*(*p*( [*o*])) + *vBN* × *B*(*p*( *C* [*o*])), (11) V(*N* [*o*]) = *vNP* × *N* (*p*( [*o*])) + *vNN* × *N* (*p*( *C* [*o*])).

Eventually, the decision rules in CPT-3WD which are called the maximum-prospect-value rules are expressed below:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| (P1) |  |  |  |  | ∈ |  | [*O*]) | | | |  | [*O*]) | | |  | [*O*]), | | |  |
| IF V(*P* | [*O*]) | | | | V(*B* | AND V(*P* | V(*N* |  |
| THEN | DECIDE *O* | | | |  | POS( ), | | | | |  | |  |  |  | |  |  |  |
| (B1) |  | |  |  | ∈ |  | |  |  |  |  |  | [*O*]), | |  |
| IF V(*B* |  | [*O*]) ≥ V(*P* | | | |  | [*O*]) AND V(*B* | | | |  | [*O*]) ≥ V(*N* | | |  |  |
| THEN | DECIDE *O* | | | |  | BND( ), | | | | |  |  |  | [*O*]) ≥ V(*B* | | |  |  |  |
| (N1) IF V(*N* | |  |  | [*O*]) ≥ V(*P* | | |  |  | [*O*]) AND V(*N* | | | |  |  | [*O*]), |  |
| THEN |  | | |  | ∈ NEG( | | | |  | ). |  |  | |  |  | | |  |  |
|  | DECIDE *O* | | | |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |

**3  Security classification based on three‑way decision and cumulative prospect theory**

Based on the assumption that the investor already holds securities prior to the portfolio optimization, we proposed the portfolio selection model based on 3WD and CPT to achieve the goal of asset diversification and hedging against the uncertainties of future returns.

*w*+(*p*(**�**[*o*])),

1 − *w*+(*p*( *C***�**[*o*])),

1 − *w*−(*p*( *C***�**[*o*])),

*i*(*p*( **�**[*o*])) = *w*−(*p*( **�**[*o*])), *w*+(*p*(**�**[*o*])),

*w*−(*p*(**�**[*o*])),

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | − |  | *C* |  | **�** |  |  |
|  | 1 − *w*+(*p*( | | | | | [*o*])), |  |
|  | *w*+(*p*( *C* [*o*])), | | | | | |  |
|  |  | *w* (*p*( | |  | **�** | [*o*])), | |  |
| *i*(*p*( *C* [*o*])) = |  | + | − | *C* | **�** |  |  |
|  |  |  |  | [*o*])), |  |
| **�** |  | 1 − *w* (*p*( | | | | |  |
|  |  |  |  | **�** |  |  |  |
|  |  |  |  |  |  |  |
|  | *w*−(*p*( *C***�**[*o*])), | | | | | |  |
|  |  | *w* (*p*( | |  | **�** | [*o*])), | |  |
|  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

|  |  |  |
| --- | --- | --- |
| 0 | ≤ *viN* ≤ *viP*, | |
| 0 | ≤ *viP* *<* *viN* , | |
| *viN* ≤ *viP <* 0, | | |
| *viP < viN <* 0, | | |
| *viN <* 0 | | ≤ *viP*, |
| *viP <* 0 | | ≤ *viN* , |
| 0 | ≤ *viN* ≤ *viP*, | |
| 0 | ≤ *viP* *<* *viN* , | |
| *viN* ≤ *viP <* 0, | | |
| *viP < viN <* 0, | | |
| *viN <* 0 | | ≤ *viP*, |
| *viP <* 0 | | ≤ *viN* . |

(10)

Figure 3 shows how this portfolio selection model is built and solved in this paper. In Fig. 3, each rectangle color block represents a security and the area of each rec-tangle represents the investment proportion on the secu-rity. Step 1 is the calculation of the prospect value of the security for each listed action. And each security is painted with the color of the action corresponding to the maximum prospect value. Step 2 is the classification of securities into the three disjoint areas, and different actions will be per-formed on different areas. Step 3 is solving the portfolio selection model based on the classification results, where an optimal investment decision can be obtained.

First of all, our proposed portfolio selection model com-plies with the following premises. Supposing the future closing price of security *i* is estimated as fuzzy variable *pi* , and the present price is *pi* , the security *i* pays a divi-dend of *di* . So the return rate of security *i* in the given time

*p* +*d* −*p*�

The prospect values of taking different actions in the action set A = {*aP*, *aB*, *aN* } are listed below:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Table 3** The value function |  |  |  |  |
|  |  | *C* |  |
| matrix |  |  |  |  |
| *A* | | *VPP* = *V*( *PP* ) | *VPN* = *V*( *PN* ) |  |
|  |  | *P* |  |  |
| *A* | | *VBP* = *V*( *BP* ) | *VBN* = *V*( *BN* ) |  |
|  |  | *B* |  |  |
| *A* | | *VNP* = *V*( *NP* ) | *VNN* = *V*( *NN* ) |  |
|  |  | *N* |  |  |

horizon is defined by fuzzy variable *i* = *i* *i*� *i* . Suppos-



*pi*

ing that the membership function of fuzzy variable *i* is *i*(*x*) , and the *γ*-level cut set of fuzzy variable *i* is:

[ *i*(*x*)] = {*x* *i* (*x*) } = {*x* [*x* , *x* ]}. Then the condition, *X <* 0 *< X* , should be satisfied, which means the security

price may rise or fall in the future.

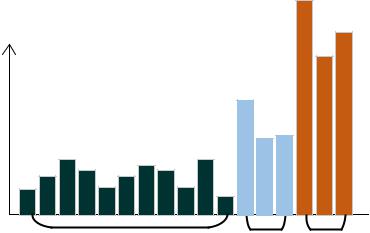
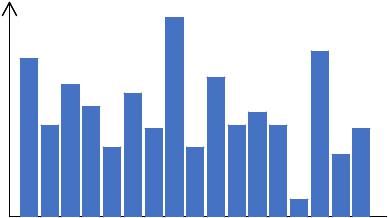
For the portfolio selection, supposing there are two states to indicate whether a security is worth investing, where means it is worth investing and *C* is not. The

1 3

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**Fig. 3** Steps of portfolio selec-tion based on 3WD and CPT

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Investmentproportion |  |  |  |  |
| 1 | 2 | 3 | ***Ċ*** | k |
|  |  |  |  | Secuity |
|  |  |  |  | number |

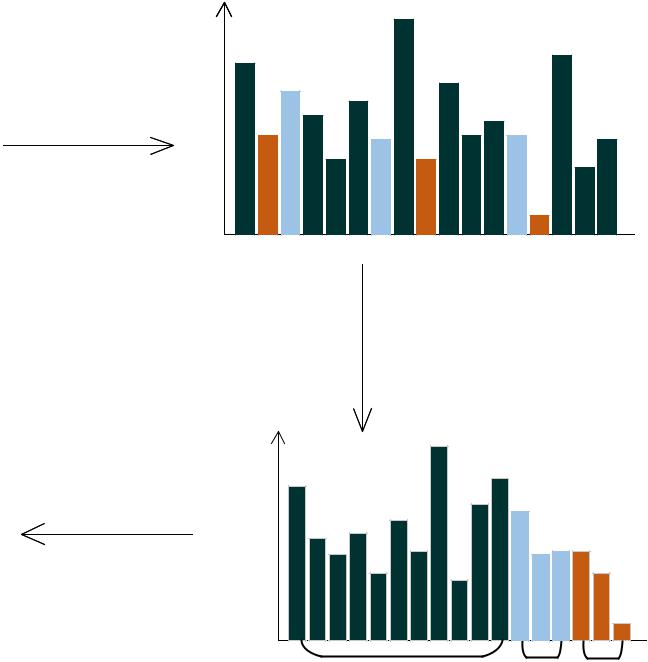


|  |
| --- |
| onpropor |

|  |
| --- |
| Investment |

|  |  |  |
| --- | --- | --- |
| Nega ve ***Ċ*** | Boundary Posi ve | |
| area | area | area |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Investmentproporon |  |  |  |  |  |  |  |
| Step 1 |  |  |  |  |  |  |  |
| Prospect value |  |  |  |  |  |  |  |
| calcula on |  |  |  |  |  |  |  |
|  | 1 | 2 | 3 | Securies | ***Ċ*** | k |  |
|  |  |  | classificaon | 2Step | **Secuity number** |  |
|  |  |  |  |  |  |  |
| Step 3 | Investment | proporon |  |  |  |  |  |
|  |  |  |  |  |  |  |
| Por olio |  |  |  |  |  |  |  |
| op miza on |  |  |  |  |  |  |  |



|  |  |  |
| --- | --- | --- |
| Nega ve ***Ċ*** | Boundary Posi ve | |
| area | area | area |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Table 4** The outcome matrix for |  |  |  |  |
|  |  | *C* |  |
| portfolio selection |  |  |  |  |
| *A* | |  |  |  |
|  | *P* | *PP* | *PN* |  |
| *A* | |  |  |  |
|  | *B* | *BP* | *BN* |  |
| *A* | |  |  |  |
|  | *N* | *NP* | *NN* |  |
|  |  |  |  |  |

decision-maker uses a return rate *x*0 as the reference point to evaluate whether a security is worth investing. *x*0 can be assigned to any reasonable value such as 0 or the risk-free rate. Then is the state while *x x*0 , and *C* is the state while *x* *<* *x*0 . The action set of portfolio selection is rep-resented by A = {*aP*, *aB*, *aN* }, where *aP* means to classify the security into the buying region (POS( )), while *aN* means to classify the security into the selling region (NEG( )). Furthermore, *aB* means to classify the security into the waiting region (BND( )) . Which denotes that the security should neither be bought or sold, i.e., the decision-maker w i l l w a i t a n d s e e . A n d t h e o u t c o m e �*ij* (*i* = P,B,N, and *j* = P,N) represents the investor’s

wealthy change by taking listed actions to the security in corresponding states. Thus the outcome matrix for portfo-lio selection is presented in Table 4.

Since the outcome *PP* denotes the outcome of classifying the security into the buying region in state , it is reasonable to use the weighted average from interval *x* *>* *x*0 to represent the *PP*. Same to this, *PN* can be represented by the weighted average from interval *x x*0 . To calculate the weighted

average from fuzzy interval, Dong [5] proposed a compu-tational algorithm based on *γ*-level cut of fuzzy set theory and interval analysis, which is called fuzzy weighted aver-age. This discretization method is capable to find an exact solution in a simple and efficient manner while containing the ambiguous information represented by the fuzzy variable as much as possible. Such that, this method is adopted by us, then the outcomes *PP* and *PN* are calculated as follow:

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  | + *x*0 | | |  |  |  |  |
|  | � | = | *x* | | , | |  |  |
| *PP* | 2 | | | |  | (12) |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| � | | = | *x* + *x* | | | | 0 | | . |  |
|  |  |  |  |  |  |
|  |  |  |  |  | |  |  |
|  | *PN* |  | 2 | | | |  |  |  |  |  |
|  |  |  |  |  |  |  |  |

Besides the outcomes *NP* and *NN* denote the outcomes of classifying the security into the selling region in state and *C* respectively, so it’s sound to let the following equationshold:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  | + *x*0 | | |  |  |  |  |
|  |  | � | = − |  | � | = − | *x* | | , | |  |  |
| *NP* |  | *PP* |  | 2 | | | |  | (13) |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |
|  |  | � = − | |  | � = − | | *x* + *x* | | | | 0 | | . |  |
| *NN* |  |  |  |  |  |  |  |
|  |  |  |  |  |  | |  |  |
|  |  |  | *PN* | |  | 2 | | | |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |

As mentioned before, *aB* represents the action of classify-ing the security into the waiting region, which will result in the investment proportion of the security remaining unchanged. Meanwhile, *aP* represents the action of classify-ing the security into the buying region, which will result

1 3

|  |  |
| --- | --- |
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|  |  |

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| in the increasing to the investment proportion of the secu- | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  | − *x*0 | | | | | | | | | |  |  |  | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| � | = *v*( | |  | � | ) = | |  |  | *x* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| rity. It is plausible to assume a proportional relationship | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | *v* | *PP* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | , | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  | 2 | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| *PP* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| exists between the outcomes of *aB* with *aP* | | | | | | | | | | | | | | | | | | | | | | | . Furthermore, | | | | | | | *PN* |  |  | *PN* | |  |  |  |  |  |  |  |  | **�** | | | | *x* | | |  | − *x*0 | | | | **�** | | |  | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| this assumption has been verified to be | | | | | | | | | | | | | | | | | | effective by Gao [7]. | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  | 2 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| Based on this assumption, the outcomes | | | | | | | | | | | | | | | | | | |  |  |  |  |  | and |  | could | | | | *v*� | = *v*( �) = − × | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  | , | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  | *BP* | |  | *BN* |  |  |  |  |  |  |  |  | **�** | | |  |  |  |  |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  | follows: | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | × | | | | |  | |  | + ( − 2) × *x*0 | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | *x* | |  |  |  |  |  |  |  |  | � | |  |
| be assigned as | | | | | |  | |  | | | |  | | | | | | | | | | | | | | | | | | | | � | |  | | |  |  | | | | | | | | | | | | | | | | | | | | | | | | | **�** | | , | | |  | | | | | | | *BP* *>* *x*0 , |  |
|  | | � | | | | | | | | | | | | | | | | | | | = *v*( | ) = | | | 2 | | | | | | | | | | | | | | | | | | | | | | | | |  | | | | | | |  |
|  | � |  | | | � |  | | | | | | |  | |  | | | | | | | | | | | | | | | *v* |  | |  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | | |  | | | | | | | | | | | | | | | | | | | | | | | |  | |  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| *BP* | | = × *PP* | | | | = | |  |  |  |  | × (*x* + *x*0), | | | | | |  |  |  |  |  |  |  |  |  |  |  |  | *BP* |  |  | *BP* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | (2 − ) × *x*0 − × | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | *x* | |  |  |  | � |  |
| *BN* � | | = × *PN* � | | | |  | = |  |  | | | × (*x* + *x* ), | | | | | |  |  |  |  |  |  |  |  | (14) | | | |  |  |  |  |  |  |  |  | − × | | | | | | | | | **�** | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | **�** | , |  | *BP* ≤ *x*0 , | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2 |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |  |
|  | | | | | | | | | |  | |  | | | | | 0 |  | | | | | | | | | | | | | | | | | | |  |  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
|  | | | | | | | | 2 | | | |  | |  | |  |  | | | | | | | | | | | | | | | | | | |  | | | | | | | | | | | | | | | | | | | | | | | | | | × *x* | | | | | |  |  | | |  |  | | | |  |
|  | | | | | | | |  | | | | |  | | | | | | | | | | | | | | | | | | |  | | | | | | | | | | | | | | | | | | | | | | | | | |  | | |  | | | |  |
|  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | � |  | | | � | |  |  |  | | | | | | | | | (2 − ) × *x*0 − | | | | | | | | | | | | | | | | |  | | | | | | | | |  |
|  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  | | |  |  | | | | | | | | |  | | | | |  |  | | | | | | | | |  |
| where (0 *< <* 0.5 is a risk control coefficient. The larger | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | *vBN* | = *v*( *BN* | | | |  | ) = − × **�** | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  | | |  |  |  |  |  |  |  |  |  |  | **�** , | | |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  | 2 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| the coefficient is, the more knowledge about the securities | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | � | = *v*( *NP*�) = − × **�** | | | | | | | | | | | | | | | | 3 × *x*0 + *x* | | | | | | | | | | | | |  |  |  | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| market the decision-maker acquires. | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  | *vNP* |  |  |  |  |  |  |  |  |  |  |  |  |  | **�** , | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2 | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  | In portfolio selection, the conditions, *x* | | | | | | | | | | | | | | | | | | | | |  | | *<* 0 | *x*0 |  | | | , |  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  | *<* | *x* |  |  | = *v*( *NN*�) = **�** | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | **�** | | |  | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| should be satisfied when the value of *γ* is | | | | | | | | | | | | | | | | | | | |  | | sufficiently small. | | | | | | | |  | − | | *x* | − 3 × *x*0 | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  | � |  |  |  |  |  |
| Only in this way, the value of the reference point *x*0 makes | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  | | |  |  |  |  |  |  | , |  |  |  |  |  |  |  |  |  |  | *NN* *>* *x*0 | | | | , |  |  |
| *v*� |  |  |  |  |  |  |  |  | 2 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| sense to the investor. Then the following formulas can be | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | *NN* |  |  |  |  |  |  |  | |  |  |  |  |  |  |  |  |  |  |  |  |  |  | *x* +3× *x*0 | | | | | | | | | | |  |  | |  |  |  |  |  |  |  | � |  |  |  |  |  |
|  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |  | | | | | | | | | | | | | | | | | | | | | | | | |  |
| easily obtained: | | | | | | | |  | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  | | − × | | | | | | | | | | |  | | |  | | | | | | | | | | **�** | |  | | , |  | | | | | *NN* ≤ *x*0 . | | | | |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |
| **�** | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  | | | |  | | | | | | | | | | | | | |  |
|  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| *PP* | | *> x*0 *> PN* | | | | | , | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | (17) |  |
| *BP* | | *> BN* | |  | , |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | On the other side, conditional probabilities are essential in | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| *x*0 *> BN* , | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | (15) | | | | the decision procedures of CPT-3WD, however, Mehlawat | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
|  | *>* | | *NP* | , | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | pointed out that it is convincing to use the credibility in fuzzy | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
|  | *NN* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | set theory to denote the conditional probability of an event | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| *x*0 *> NP* . | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | happening [28]. So the probability of security *i* belongs to | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| According to CPT, each outcome brings a decision value and | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | state is taken for Cr{*x x*0}, and the probability of secu- | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| rity *i* belongs to state *C* is taken for Cr{*x* *<* *x*0}. Such that | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |
| shows the impact in the decision process. The decision val- | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | the decision weights of | | | | | | | | | | | | | | | | | | | |  | *i*(Cr( [*o*])) and | | | | | | | | | | | | | | | | | | | |  | *i*(Cr( | | | | *C* |  | [*o*])) are |  |
| ues of outcomes | | | | | | | |  |  | � | | (*i* = P,B,N, and *j* = P,N) are calculated | | | | | | | | | | | | | | | | | |  |  |  |  |  |
| *ij* | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | | | and | | | | | | *V* | | (*i* = *P*, *B*, *N*) : | | | |  |
| by the following equations: | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  | calculated as follow by | | | | | | | | | | | | | | | | | | | |  | comparing *v* *P* | | | | | | | | | | | | | | | |  | *IN* |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | |  |  | *i* | | |  |  |  |  |  |  |  |  |  |  |  |  |  |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | ( � | − *x* | ) , | � | *> x* , | |  |  |
| *v*�= *v*(� | | ) = | *ij* | 0 |  | *ij* | 0 |  |  |  |
|  |  |  |  | ≤ *x*0 |  | (16) |  |
| *ij* | *ij* |  | − × (*x*0 − *ij*�) , | | | *ij*� | , |  |

where = 0.88, = 0.88, = 2.25.

Therefore, the value function matrix for portfolio selection can be obtained and presented in Table 5.

In which,

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Table 5** The value function |  |  |  |  |
|  |  | *C* |  |
| matrix for portfolio selection |  |  |  |  |
| *A* | | *V* | *V* |  |
|  | *P* | *PP* | *PN* |  |
| *A* | | *V* | *V* |  |
|  | *B* | *BP* | *BN* |  |

*w*+(Cr(**�**[*o*])),

1 − *w*+(Cr( *C***�**[*o*])),

1 − *w*−(Cr( *C***�**[*o*])),

*i*(Cr( **�**[*o*])) = *w*−(Cr( **�**[*o*])), *w*+(Cr(**�**[*o*])),

*w*−(Cr(**�**[*o*])),

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | − |  |  | *C* |  | **�** |  |  |
|  | 1 − *w*+(Cr( | | | | | | [*o*])), |  |
|  | *w*+(Cr( *C* [*o*])), | | | | | | |  |
|  |  | *w* (Cr( | | |  | **�** | [*o*])), | |  |
| *i*(Cr( *C* [*o*])) = |  | + | − |  | *C* | **�** |  |  |
|  |  |  |  |  | [*o*])), |  |
| **�** |  | 1 − *w* |  | (Cr( | | | |  |
|  |  |  |  |  | **�** |  |  |  |
|  |  |  |  |  |  |  |  |
|  | *w*−(Cr( *C***�**[*o*])), | | | | | | |  |
|  |  | *w* (Cr( | | |  | **�** | [*o*])), | |  |
|  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

|  |  |  |  |
| --- | --- | --- | --- |
| 0 ≤ *v*� | | ≤ *v*� | , |
|  | *iN* | *iP* |  |
| 0 ≤ *v*� | | *< v*� | , |
|  | *iP* | *iN* |  |
| *v*� | ≤ *v*� | *<* 0, | |
| *iN* | *iP* | |  |
| *v*� | *< v*� | *<* 0, | |
| *iP* | *iN* | |  |
| *v*� | *<* 0 | ≤ *v*� | , |
| *iN* |  | *iP* |  |
| *v*� | *<* 0 | ≤ *v*� | , |
| *iP* |  | *iN* |  |
| 0 ≤ *v*� | | ≤ *v*� | , |
|  | *iN* | *iP* |  |
| 0 ≤ *v*� | | *< v*� | , |
|  | *iP* | *iN* |  |
| *v*� | ≤ *v*� | *<* 0, | |
| *iN* | *iP* | |  |
| *v*� | *< v*� | *<* 0, | |
| *iP* | *iN* | |  |
| *v*� | *<* 0 | ≤ *v*� | , |
| *iN* |  | *iP* |  |
| *v*� | *<* 0 | ≤ *v*� . | |
| *iP* |  | *iN* |  |
|  |  | (18) | |

|  |  |  |  |
| --- | --- | --- | --- |
| *A* | | *V* | *V* |
|  | *N* | *NP* | *NN* |
|  |  |  |  |

Therefore, the following formulas can be obtained based on the conclusions in Formulas (15), (17) and (18):

1 3

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|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | *P* | (Cr( [*o*])) = *w*+ | | | | | | | (Cr( ) [*o*]), | | | | | | | | | |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | *C* | | |  |  |  | − |  |  | *C* | |  |  |  |  | ]) | |  |  |  |  |  |  |  |
| *P*(Cr( | | |  |  |  | [*o*])) = *w* | | |  | (Cr( | | − ) | |  |  | [*oC* | | | , | |  | � |  | � |  |  |
|  |  |  |  |  |  |  |  |  | 1 − *w* Cr( | | | | | | | | | |  | [*o*]), | *v* ≤ *v <* 0, | | | | |  |
| *B*(Cr( [*o*])) = | | | | | | | |  |  |  |  |  |  |  | |  |  |  |  |  |  | *BN* |  | *BP* |  |  |
| − |  | + |  |  |  |  |  |  |  |  |  | � |  |  | � |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  | | [*o*])), | | | | | | *vBN* | | *<* 0 *< vBP*, | | |  |
|  |  |  |  |  |  |  | *w* (Cr( | | | | | | |  |  |
| *B*(Cr( *C* [*o*])) = *w*−(Cr( *C* [*o*])), | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |
|  |  |  |  |  | [*o*])) = *w* | | |  |  |  | ( [ ])) | | | | | | |  |  |  |  |  |  |  |  |  |
| *N* (Cr( | | | |  |  | (Cr+ | | |  | *o C* | | | |  | , |  |  | � |  |  |  | � |  |
| *N* (Cr( *C* | | | | | |  |  |  |  | *w* Cr( | | | | |  |  |  | |  |  | *v <* 0 *< v* , | | | | |  |
|  | [*o*])) = |  |  |  |  |  | [*o*]), | | |  |
|  |  |  |  | − |  |  |  |  |  |  |  |  |  | � |  | � |  | *NN* |  |
|  |  |  |  |  |  |  | *w* (Cr( *C* [*o*])), | | | | | | | | | | | | | | *NP* | |  |  | ≤ 0. |  |
|  |  |  |  |  |  |  | *vNP < vNN* | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | (19) |  |

Then according to Sect. 2.4, the prospect values of taking different actions are listed below.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| V�(*P* | | [*i*]) = *v*� | | × (Cr( [*i*])) + *v*� | | | × (Cr( *C* | |  | [*i*])), |  |  |
|  |  |  | *PP* | *P* |  | *PN* | *P* |  | |  |  |  |
|  |  |  |  |  |  |  |  |  |
| V�(*B* [*i*]) = *v*� | | | | × | (Cr( [*i*])) + *v*� | | × | (Cr( *C* |  | [*i*])), |  |  |
|  |  | | *BP* | *B* |  | *BN* | *B* |  |  | | (20) |  |
|  |  |  |  |  | *C* |  |
| � | (*N* |  | � | × *N* (Cr( | | � |  |  |  |  |  |
| V |  | [*i*]) = *vNP* | [*i*])) + *vNN* | × *N* (Cr( [*i*])). | | | |  |  |

Finally, the candidate securities can be classified into the POS( ) , BND( ) , NEG( ) areas. The classification rules are listed below:

loss in the worst-case scenario. Theoretically, VaR is more sensitive to general investors [ 36]. Suppose fuzzy variable L*i* denotes the loss of security *i* that an investor may suffer, then the VaR of L*i* under confidence level (1 − ) is:

|  |  |
| --- | --- |
| VaR1− (L*i* ) = sup{ Cr(L*i*)}. | (21) |

Most of existing studies assumed that all the capital is in cash and no securities are held before the portfolio optimi-zation. Unfortunately, this assumption sometimes contradict reality, and investors may already own securities. Suppose there are *n* securities for the investor to choose from. The investment proportion of each security before the portfo-lio optimization is *xi* , and the investment proportion after implementing an investment decision is *xi*. Fuzzy variable *i* represents the future return rate of security *i* and its expected value is *E*( *i*), and fuzzy variable L*i* represents the possi-ble loss of security *i* in the future, obviously, L*i* = − *i* for

* = 1, 2, …*n*.

Then in Sect. 4.2 , four portfolio selection models are proposed. All of the four models are based on the assump-tion that the investor put the safety of their investment first, so only when VaR is less than a predefined constraint

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| (P3) if V�(*P* [*i*]) | | V�(*B* [*i*]) and V�(*P* [*i*]) | | V�(*N* [*i*]), then decide *i* ∈ POS( ) , | | | | | | | |  |  |
|  | V� | V� | V� | V�( | [ | | ]) |  | ∈ |  | ( ) |  |  |
| (B3) if | �(*B* [*i*]) ≥ | �(*P* [*i*]) and | �(*B* [*i*]) ≥ | � *N* |  | *i* |  | , then decide *i* |  | BND |  | , |  |
|  |  |  |  |  | | [*i*]), then decide *i* ∈ NEG( ) . | | | | | | |  |
| (N3) if V (*N* [*i*]) ≥ V (*P* [*i*]) and V (*N* [*i*]) ≥ V (*B* | | | | |  |  |

**4  Mathematical modeling and solution algorithm**

Based on the classification results, the work of portfolio optimization could be done. In this section, four portfolio selection models are proposed and the algorithm to solve these models is introduced. This section also covers some background descriptions.

level *S* will they chase to maximize returns. Of course, the model can be transformed without difficulty to mini-mize the VaR subjects to a given expected return level, we omit this part due to space limits. Finally, in Sect. 4.3, the algorithm to solve the four portfolio selection models is introduced. For the sake of simplicity, transaction costs are not considered in this paper.

**4.1 Background descriptions**

Following the works of [ 34], we use the expected value of fuzzy variables as the revenue metric, and the fuzzy Value-at-Risk (VaR) as the risk metric. Specifically, VaR is defined as the maximum possible loss that an investor may suffer over a certain period of time, given a predefined confidence level. This indicator expresses the risk that an investor faces as a specific number for greater clarity [14]. In contrast to other risk metrics such as variance, entropy and semi-variance, investors could specify different con-fidence levels and then use VaR to obtain the possible

**4.2  Portfolio selection modeling**

In this subsection, four portfolio selection models are intro-duced, including Buy&Hold, Expected value-VaR, model based on classical 3WD and model based on 3WD and CPT.

**4.2.1  Buy&Hold portfolio selection model (B&H‑PS)**

B&H-PS is often used as a benchmark model in portfo-lio selection [10]. In this model, no change will happen to the investment proportion of securities held by the investor. Because of its simplicity, B&H-PS is often used to compare performance with other models. The mathematical model of B&H-PS is:

1 3

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| International Journal of Machine Learning and Cybernetics (2022) 13:293–308\ | | | | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 301 |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | | | | | | | | | | | | | | | | | | | | | |  |  |
|  | max E[*x*1 × 1 + *x*2 × 2 + ⋯ + *xn* × *n*] | | | | | | | | | | | | | | | | | | | | | | | **4.2.4  Portfolio selection model based on three‑way** | | | | | | | | | | | | | | | | | | | | | |  |  |
|  |  |  | **decision and cumulative prospect theory** | | | | | | | | | | | | | | | | | | | |  |  |
| s.t. | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| *xi* = *xi*�≥0, *i* =1, 2… *n* | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  | **(3WD&CPT‑PS)** | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | (22) |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | *n* | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | Following the security classification rules in Sect. 3, the | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  | *xi* | | | = 1. | | | |  |  |  |  |  |  |  |  |  |  |  |  | mathematical model of 3WD&CPT-PS is: | | | | | | | | | | | | | | | | | | | | | |  |  |
|  |  | *i*=1 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | max E[*x*1 × 1 + *x*2 × 2 + ⋯ + *xn* × *n*] | | | | | | | | | | | | | | | | | | | | |  |  |
|  |  | **�** | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | s.t. | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| **4.2.2  Expected value‑VaR portfolio selection model (EV‑PS)** | | | | | | | | | | | | | | | | | | | | | | | |  | VaR1− [*x*1 × L1 + *x*2 × L2 + | | | | | | | | | | | | | | | | ⋯ | + *xn* × L*n*] *<* *S* | | |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| In conventional fuzzy portfolio selection models, the goal is | | | | | | | | | | | | | | | | | | | | | | | |  |  | *x* | | *i* | ≥ | | 0, | |  | *i* =1, 2… *n* | | | | | | |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| to maximize the investment returns at a given level of risk. | | | | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  | *n* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | (25) |  |
| In [36], Wang proposes a novel fuzzy portfolio selection | | | | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  | *xi* =1 | | | | |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  | *i*=1 | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| model with Value-at-Risk, which is adopted by us and used | | | | | | | | | | | | | | | | | | | | | | | |  |  | **�** | | | | | � | |  | ≥ 0, | |  |  |  |  |  |  |  |  |  |  |  |  |
| to show the potential effectiveness of 3WD and CPT. So the | | | | | | | | | | | | | | | | | | | | | | | |  |  | (*xi* − *xi* ) | | | | | | |  | *i* ∈POSC( ) | | | | | |  |  |  |  |  |  |
|  |  | ( |  |  |  | − | �) = | | | |  |  | ∈ | |  |  | ( ) | |  |  |  |  |  |
| mathematical model of EV-PS is: | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  | *xi* | | |  | *xi* | |  | ≤ | 0, | *i* |  |  |  | BNDC |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | (*xi* | | | | − *x*�) | | |  | 0, | *i* ∈NEGC( ), | | | | | | |  |  |  |  |  |
|  | max E[*x* | | | |  | × | | | |  | + *x* | | × | |  | + ⋯ + *x* | |  | × | | |  | ] |  |  |  |  |  |  |  | *i* | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| s.t. | |  |  |  | 1 |  |  |  |  | 1 |  | 2 |  |  | 2 |  |  | *n* |  |  |  | *n* |  |  |  |  |  |  |  |  |  |  | ), BNDC( | | | | |  | | ) and NEGC | | | ( |  | ) are three disjoint | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | where POSC( | | | | | | | |  |  |  |  |  |
| VaR1− [*x*1 × L1 + *x*2 × L2 + ⋯ + *xn* × L*n*] *<* *S* | | | | | | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  | areas in 3WD based on CPT. | | | | | | | | | | | | | | | | |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  | *xi* | | ≥ 0, | | | | |  |  | *i* =1, 2… *n* | | | | | |  |  |  |  |  |  | (23) | **4.3** | | **Solution algorithm** | | | | | | | | | | | | | |  |  |  |  |  |  |  |  |
|  |  |  | *n* | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  | *xi* | | | = 1. | | | |  |  |  |  |  |  |  |  |  |  |  |  | At | | Sect. | | | | 4.1, | | | | we | | discussed the | | | | | | | definition of | | | VaR |  |
|  |  | *i*=1 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | and how to calculate VaR of a single fuzzy vari- | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | **�** | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | able. However, since the future returns of candidate | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | securities | | | | | | | may | | | | be | not | | | totally independent, it is | | | | | | | not |  |
| **4.2.3  Portfolio selection model based on classical** | | | | | | | | | | | | | | | | | | | | | | | | appropriate to use the aforementioned method to cal- | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | **three‑way decision (3WD‑PS)** | | | | | | | | | | | | | | | | |  |  |  |  |  |  |
|  |  |  |  |  |  |  | culate VAR1− [*X*1 × L1 + *X*2 × L2 + ⋯ + *XN* × L*N*] | | | | | | | | | | | | | | | | | | | | | | and |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| To make a comparison with the portfolio selection model | | | | | | | | | | | | | | | | | | | | | | | | E[*x*1 × 1 + *x*2 × 2 + ⋯ + *xn* × *n*]. To solve this problem, | | | | | | | | | | | | | | | | | | | | | | |  |
| Liu and Iwamura [22] and Liu and Liu [23] proposed a dis- | | | | | | | | | | | | | | | | | | | | | | |  |
| based on 3WD and CPT, we also adopt the model based on | | | | | | | | | | | | | | | | | | | | | | | |  |
| cretization method named fuzzy simulation to obtain an | | | | | | | | | | | | | | | | | | | | | | |  |
| classical 3WD. The mathematical model of 3WD-PS is: | | | | | | | | | | | | | | | | | | | | | | | |  |
| approximation of VaR and expected value, which can be | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  | max E[*x*1 × 1 + *x*2 × 2 + ⋯ + *xn* × *n*] | | | | | | | | | | | | | | | | | | | | | | | adopted in this paper. | | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |
| s.t. | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | Besides, the proposed models are complicated nonlinear | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | ⋯ | |  |  |  | optimization which cannot be solved directly by conven- | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | VaR1− [*x*1 × L1 + *x*2 × L2 + | | | | | | | | | | | | | | | | |  |  | + *xn* × L*n*] *<* *S* | | | tional methods or existing softwares. Hakli [9] proved that | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | *x* | | ≥ | |  | 0, | |  |  | *i* =1, 2… *n* | | | | | |  |  |  |  |  |  |  | heuristic algorithms could be efficient to solve sophisticated | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  | *i* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | *n* | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | optimization problems and be capable of obtaining suffi- | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  | *xi* | | | = 1 | | | |  |  |  |  |  |  |  |  |  |  |  | (24) | ciently accurate solutions. Among them, the particle swarm | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | *i*=1 | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | optimization (PSO) algorithm explores the search space | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | **�** | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | � |  |  |  |  | *i* ∈POS( ) | | | | |  |  |  |  |  |  | and obtains the optimal solution by adjusting the speed and | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  | (*xi* − *xi* ) ≥ 0, | | | | | | | | | | |  |  |  |  |  |  |  |
|  |  | ( |  | − | | |  | �) = | | | |  |  | ∈ | |  | ( ) | |  |  |  |  |  | position of the particles according to the population intelli- | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  | *xi* |  |  |  | *xi* | |  |  |  | 0, | *i* |  |  | BND |  |  |  |  |  |  |  |  |
|  |  | (*xi* | | − *x*�) ≤ | | | | | | | | 0, | *i* ∈NEG( ), | | | | | |  |  |  |  |  | gence. Due to its easy-implementation and fast convergence | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  |  |  |  | *i* |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | features, PSO has been widely-used in many optimization | | | | | | | | | | | | | | | | | | | | | | |  |
|  |  |  |  |  |  | ( ) | | |  |  |  |  | ( ) |  |  |  | ( ) | | |  |  |  |  | problems [4, 25, 32]. | | | | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |
| where POS | | | | | |  |  |  | , BND | | | |  | and NEG | | | |  |  | are three disjoint areas | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  | Recently, to overcome the local convergence problem | | | | | | | | | | | | | | | | | | | | | |  |
| in 3WD. | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | of conventional PSO, Wang proposed an improved PSO | | | | | | | | | | | | | | | | | | | | | | |  |

by employing the particle restart position and escape

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speed [35]. Considering the feasibility and effectiveness of this improved PSO algorithm to solve portfolio selec-tion issues has been proved, so this algorithm is used in this paper to obtain the solutions of the proposed models. The solution process of this algorithm can be summarized as follows:

**[Step 1]** Particle initialization: A swarm of*m*particles areinitialized, and the *k*th particle P*k* is a *n* × 1 real- valued matrix which represents the position of particle *k* in vir-tual *n*-dimensional space that indicates the portfolio deci-sion in portfolio selection:

|  |  |
| --- | --- |
| P*k* → [*x*1,*k*, *x*2,*k*, … , *xn*,*k*]T, | (26) |
| where ∀*k* , *i*=1 *xi*,*k* = 1, *xi*,*k* | 0 and *xi*,*k* represents the |
| *n* |  |

investment portion of security *i*.

**[Step 2]** Particle adjustment: The randomly generatedparticles probably can not satisfy the constraints of VaR values. Therefore, each unqualified particle is revised to meet the VaR value constraints by the following iterative approach.

[*Step 2.1*] *k* = 1

[*Step 2.2*] Fuzzy simulation is used to calculate

VaR1− [*x*1,*k* × L1 + *x*2,*k* × L2 + ⋯ + *xn*,*k* × L*n*]. If this

value is greater than the upper limit *S*, then go to Step 2.3, else go to Step 2.4.

[*Step 2.3*] The *k*th particle is infeasible, the position of

this particle is re-initialized and go to Step 2.2

[*Step 2.4*] If *k* = *m*, these *m* particles are feasible and

go to Step 3 for subsequent optimization, else *k* = *k* + 1

and go to Step 2.2.

**Table 7** Predetermined parameter values for case study 1

|  |  |  |
| --- | --- | --- |
| Symbol | Meaning | Value |
|  |  |  |
| *S* | The upper limit of VaR | 0.1 |
| 1 − | The confidence level of VaR | 0.9 |
| *ω* | The risk control coefficient | 0.05 |
| *γ* | The -level cut for fuzzy variable | 0.003 |
| *X*0 | The reference return rate | 0.02 |

**[Step 3]** Fitness calculation: Using fuzzy simulation tocalculate the objective function to acquire the fitness value of each particle. Then we initialize personal best position P*bests*, personal best fitness value P*value*, global best position G*bests* and global best fitness value G*value*. **[Step 4]** Particle update: Based on theP*bests*andG*bests*in Step 3, each particle position can be updated as follow:

|  |  |  |
| --- | --- | --- |
| V*K* | = × V*K* + *C*1 × *RAND*(0, 1) × (P*BESTS* − P*K* ) |  |
|  | + *C*2 × *RAND*(0, 1) × (G*BESTS* − P*K* ), | (27) |
| P*K* | = P*K* + V*K* , |  |

where V*k* represents the velocity of particle *k*, *ω* is the inertia weight, *C*1 and *C*2 are the swarm learning rates and *rand*(0, 1) denotes a randomly generated value in (0, 1).**[Step 5]** Feasibility check: Every newly generated particleis checked to see if it meets the constraints in its port-folio selection model and regenerate invalid ones using Formula (27).

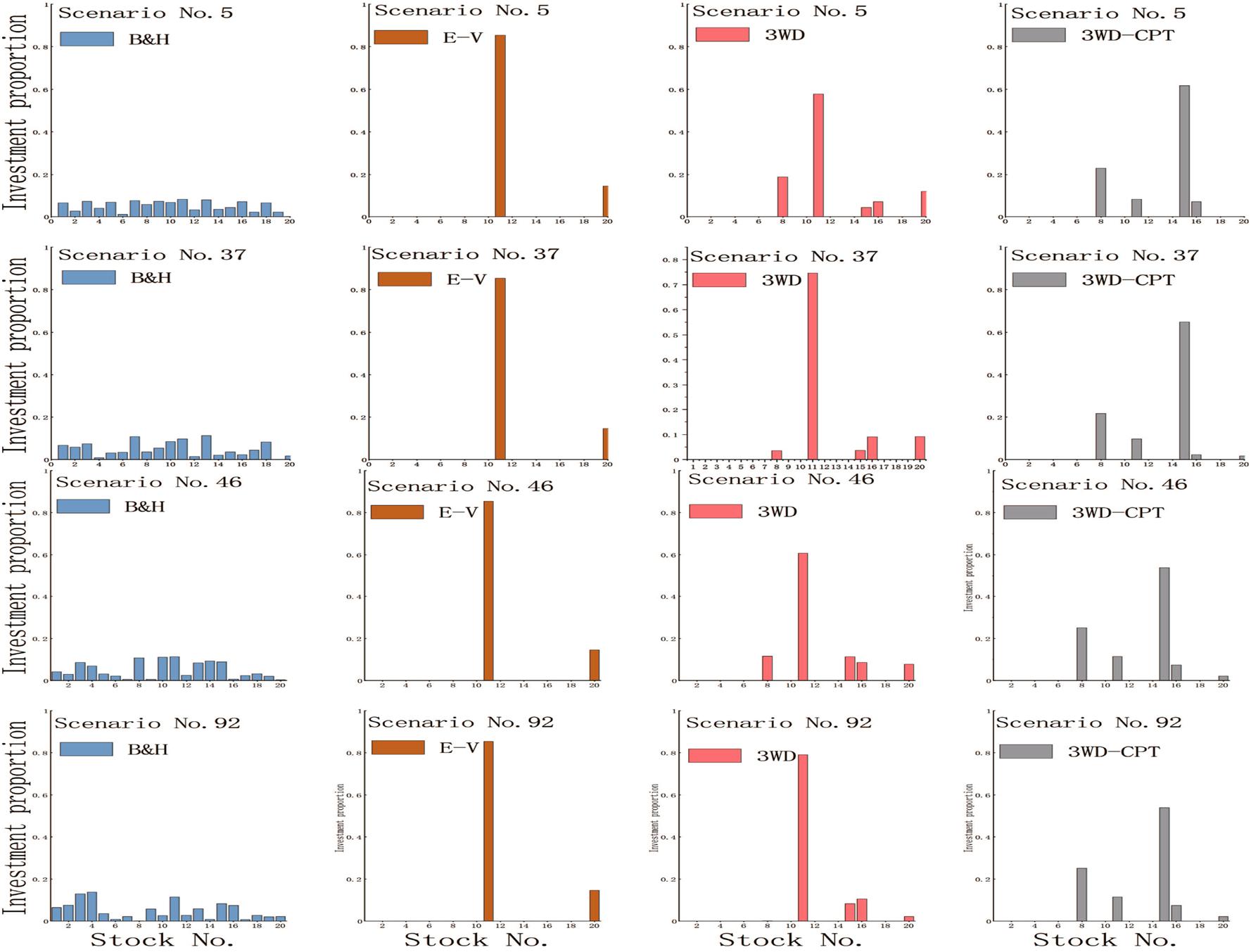
**[Step 6]** Swarm iteration: Step 3 5 are iterated for a pre-defined number of times. After the last iteration, G*value* is taken as the optimal result of the whole problem, and its corresponding position G*bests* is the final portfolio deci-sion.

**Table 6** Fuzzy forecast of stocks returns

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Stock no. | Abbreviation | Fuzzy returns | Expected value | Stock no. | Abbreviation | Fuzzy returns | Expected value |
|  |  |  |  |  |  |  |  |
| 1 | UNH | (− 0.121, − 0.013, 0.031, | 0.007 | 11 | DIS | (− 0.132, 0.145, 0.300) | 0.090 |
|  |  | 0.130) |  |  |  |  |  |
| 2 | HD | (− 0.116, − 0.032, 0.017, | 0.006 | 12 | WBA | (− 0.177, − 0.037, 0.213) | − 0.027 |
|  |  | 0.155) |  |  |  |  |  |
| 3 | MCD | (− 0.063, − 0.010, 0.009, | 0.001 | 13 | AXP | (− 0.105, − 0.003, 0.191) | 0.004 |
|  |  | 0.065) |  |  |  |  |  |
| 4 | MSFT | (− 0.098, 0.019, 0.036, 0.125) | 0.020 | 14 | TRV | (− 0.127, − 0.009, 0.204) | − 0.002 |
| 5 | GS | (− 0.164, − 0.037, 0.006, | − 0.026 | 15 | AAPL | (− 0.161, 0.074, 0.536) | 0.086 |
|  |  | 0.092) |  |  |  |  |  |
| 6 | V | (− 0.171, − 0.041, 0.201) | 0.011 | 16 | CSCO | (− 0.223, 0.102, 0.603) | 0.096 |
| 7 | MMM | (− 0.136, 0.025, 0.170) | 0.007 | 17 | PFE | (− 0.157, − 0.067, 0.271) | − 0.028 |
| 8 | BA | (− 0.081, 0.063, 0.381) | 0.075 | 18 | JPM | (− 0.131, 0.017, 0.201) | 0.009 |
| 9 | CAT​ | (− 0.062, − 0.002, 0.117) | 0.003 | 19 | KO | (− 0.185, 0.042, 0.211) | 0.010 |
| 10 | JNJ | (− 0.144, − 0.061, 0.090) | − 0.052 | 20 | IBM | (− 0.321, 0.107, 0.801) | 0.107 |
|  |  |  |  |  |  |  |  |

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| --- | --- |
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|  |  |



**Fig. 4** Stocks position in case study 1

The readers may refer to [35] for the details of the algorithm.

**5 Numerical examples**

In this section, two cases are studied to show the effective-ness of the proposed 3WD&CPT portfolio selection model. All the experiments were implemented with Python on a Dell U8S9M95 3.2 GHz personal computer.

**5.1  Case study 1**

In this case, 20 stocks were randomly selected from the Dow Jones Industrial Average. Based on the observation of his-torical data as well as experts knowledge, the future returns of candidate stocks described as fuzzy variables are listed in Table 6, where (*a*, *b*, *c*) denotes triangular fuzzy variables and (*a*, *b*, *c*, *d*) indicates trapezoidal fuzzy variables. And the time span is from 01-01-2019 to 03 -31-2019, including 60 trading days. Furthermore, there are a number of parameters whose values need to be predetermined by investors and are listed in Table 7.

In Table 7, the values of *S* and 1 − vary from person to person, which express the risk appetite of a specific deci-sion-maker. And we set *S* = 0.1, 1 − = 0.9, which follow the suggestion given in [36].

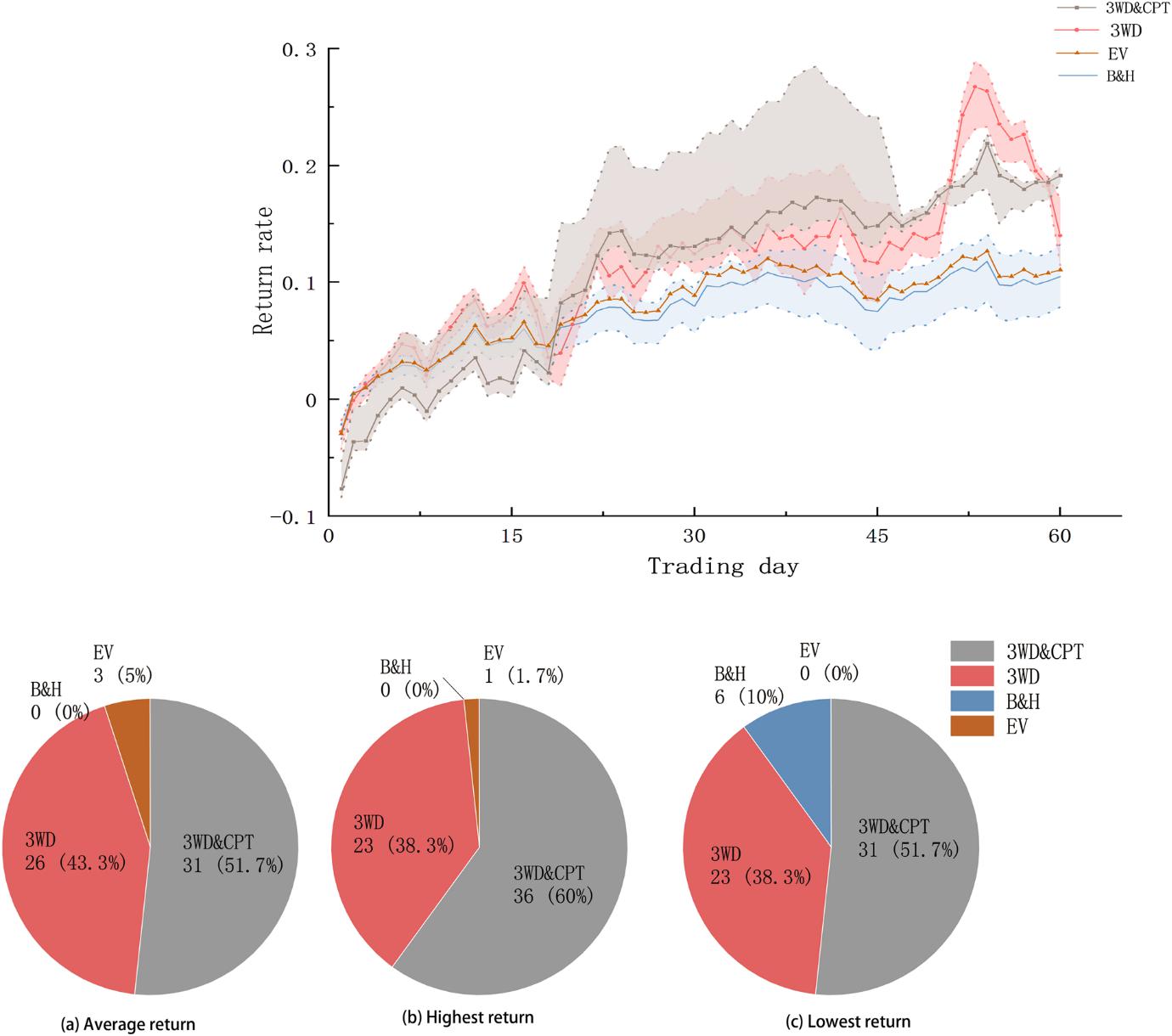
Besides, the risk control coefficient *ω* is set as 0.05, which is based on the historical data as well as experts knowledge. The value of *γ* in the *γ*-level cut is set as 0.003 after a number of trails. Finally, the value of the reference return rate *x*0 is set as 0.02, which is based on experts knowledge and the prediction of the average rate of return in the stock market in the given time horizon.

Note that the initial capital allocation *xi*� (*i* = 1, 2 … *n*) (i.e., the investment proportion of each stock before the port-folio optimization) is important to the optimal result, which however, differs from case to case. Therefore, without losing generality, we generate 100 stochastic capital allocation sce-narios via Python programs, each of which represents a stock position before the portfolio optimization. Then each initial capital allocation scenario is considered as the input of the four portfolio selection models described in Sect. 4.2, thus four optimal investment decisions are obtained by the inves-tors respectively. Four initial capital allocation scenarios are randomly selected among them, and the corresponding

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**Fig. 5** The error band chart ofinvestment return rates in case study 1



**Fig. 6** Number of days has best performance on three indicators in case study 1

optimal investment decisions obtained by the proposed port-folio selection models are plotted in Fig. 4.

As can be seen from Fig. 4, compared to B&H-PS which spreads the capital over all the stocks, EV-PS concentrates all the capital on two stocks, i.e., stock No.11 and 20, which leads to high risk as well as high returns. By contrast, the stocks position of 3WD- PS and 3WD&CPT-PS show that the using of 3WD avoids excessive concentration of capital allocation, thus reducing the investment risk. At the same time, the introduction of CPT changes the investors’ prefer-ence for stocks.

Then we place the all the optimal investment decisions on the real stock market to get their return performances. And we plot the return performances of the four portfolio

selection models on an error band chart, as depicted in Fig. 5.

The x-axis of Fig. 5 represents the trading day and y-axis represents the return rate. Each color block rep-resents the investment performance of the corresponding portfolio selection model in the real stock market, where the solid line represents the average return, the dashed line at the upper border represents the maximum return, and the dashed line at the lower border represents the mini-mum return. Particularly, since the EV-PS yields the same optimal investment decision regardless of the initial capital allocation scenarios, the return rate of the EV-PS is a solid line with no blocks of color.

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| --- | --- |
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|  |  |

**Fig. 7** The error band chart ofinvestment return rates in case study 2

**Table 8** Predetermined parameter values for case study 2

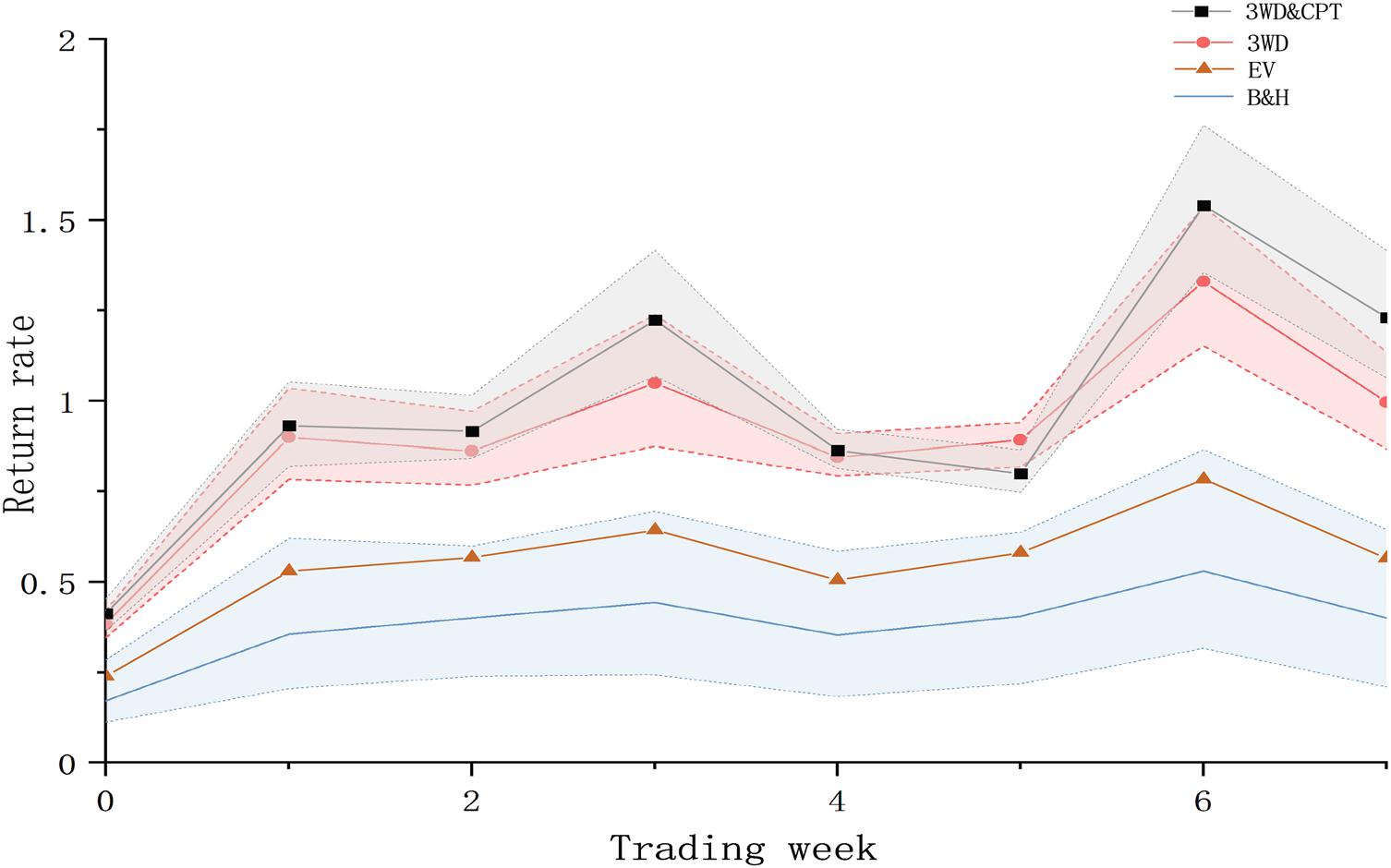
|  |  |  |  |
| --- | --- | --- | --- |
|  | Symbol | Meaning | Value |
|  |  |  |  |
|  | S | The upper limit of VaR | 0.1 |
| 1 − | | The confidence level of VaR | 0.9 |
|  | *ω* | The risk control coefficient | 0.1 |
|  | *γ* | The -level cut for fuzzy variable | 0.003 |
|  | *X*0 | The reference return rate | 0.08 |
|  |  |  |  |

In Fig. 5, all four models do not lose money most of the time because the 20 stocks in the stock pool rose over the given time horizon. The B&H-PS has the lowest aver-age return because it did not make any optimization on the initial investment proportion. The EV-PS has a higher average return than B&H- PS because EV-PS achieved a concentration of capital on the high return stocks. By con-trast, the investment performances of both 3WD-PS and 3WD&CPT-PS are better than B&H-PS and EV- PS owing to the diversification of investment options, risk reduction and return enhancement achieved by classifying stocks.

To make a detailed comparison of the four models, we count the number of days in which the four models appear to have the greatest average return rate, plotted as the pie chart in Fig. 6a. We also count the number of days in which the four models appear to have the greatest highest return and greatest lowest return rate likewise, as shown in Fig. 6b, c, respectively.

The average return and the highest return could be viewed as the portfolio profit-chasing evaluation index, meanwhile the lowest return could be viewed as a portfolio risk-avoidance evaluation index. So from the perspective of

the indicator average return and highest return, Fig. 6 dem-onstrates that the EV-PS achieves higher return rate than B&H-PS, meanwhile the indicator of lowest return shows that the EV-PS also results in risk concentration. And the 3WD&CPT-PS and 3WD-PS perform better than the B&H-PS and EV-PS, so it can be concluded that the introduction of 3WD with portfolio selection achieves risk reduction and returns enhancement simultaneously. Furthermore, Fig. 6 demonstrates that the proposed 3WD&CPT-PS has the best investment performance on all the three indicators, which shows that the use of CPT leads to capital focusing on the better stocks, i.e., security No.8 and 15. Therefore, it can be conclude that the combination of 3WD and CPT with portfolio selection is effective.



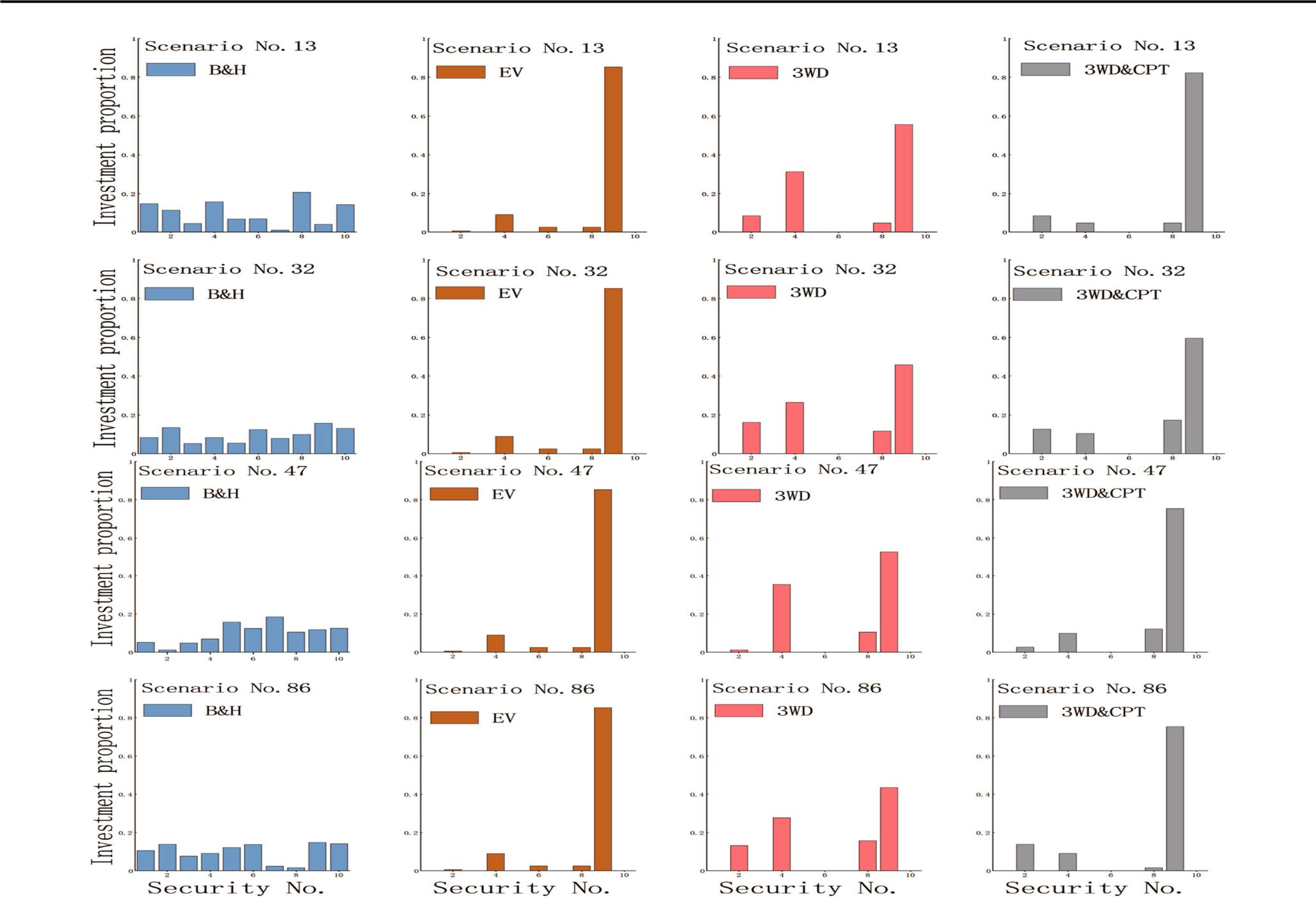
**5.2  Case study 2**

In case study 2, a portfolio selection problem consisting of a group of 10 stocks from the New York Stock Exchange is discussed, the forecast return of the stocks in the time span of 8 weeks are described as triangular or Gaussian ( *i* , *i* ) fuzzy variables, which are listed in [34]. Similarly, there are a number of parameters whose values need to be prede-termined by investors and are listed in Table 8. Generally, the values of these parameters follow the same settings in case study 1.

Again we first generate 100 stochastic initial capital allo-cation scenarios via Python program, then the same works as to case study 1 are done in case study 2. Four initial capital scenarios are randomly selected, and Fig. 7 depicts the opti-mal investment decisions obtained from the four portfolio selection models. Fig. 7 demonstrates that compared to the

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**Fig. 8** Stocks position in case study 2

capital concentration of EV-PS, in which the investment proportion of stock No.9 is greater than 80% , the classifica-tion capabilities realized by the use of 3WD achieve capital diversification and risk reduction further.

And Fig. 8 is the error band chart of the investment per-formances in the real stock market of the four portfolio selection models. Since it’s only accessible to the closing price of the stock every Friday, the error band chart contains only 8 time nodes. It is clearly that the investment perfor-mances of the 3WD-PS and 3WD&CPT-PS are better than the B&H-PS and EV-PS on each time node.

Similar to case study 1, we count the number of weeks in which the four portfolio selection models appear to have the greatest average return, the greatest highest return and the greatest lowest return in case study 2. On all three indicators, the same results emerge, with 3WD&CPT-PS performing the best at seven time nodes, accounting for 87.5%. Mean-while, 3WD- PS performs best at one time node, accounting for 12.5% on all three indicators. Therefore, it can be con-cluded that the combination of 3WD and CPT is effective.

It can be found that the volatility of case study 2 is much greater than that of case study 1 by observing the stock mar-ket data. The experimental results prove that the portfolio

selection model based on three-way decision and cumulative prospect theory is effective in both these two cases. There-fore, it might be concluded that our model is stable to dif-ferent investment problems.

**6 Conclusion**

Based on the assumption that investors already hold secu-rities in hand in long-term portfolio selection, three-way decision and cumulative prospect theory are used to build the portfolio selection model in this paper. Three-way deci-sion theory is used to classify candidate securities into the three disjoint regions (buying, selling and waiting regions), different actions (buy, sell and wait) are implemented into each corresponding region. And cumulative prospect theory is used together with expected utility theory to assess the investors’ preference toward the candidate securities. Two case studies on the basis of real stock market data prove that the introduction of three-way decision theory meets the goal of effectiveness diversification of investment. The real market experiments results prove that the proposed portfolio selection model based on three-way decision and cumulative

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prospect theory can increase investment return while reduc-ing investment risk, so the proposed portfolio selection model is effective.

Since the future trends of securities prices are reflec-tions of various factors, viewing the risk/expected-return level as a static value may not fit well with some investors’ behavior. So it is obliged to update the forecast of securi-ties return rate by the investment result of last period, and adjust the portfolio to these forecast results. Therefore, it would be an interesting and important research direction to combine three-way decision and cumulative prospect theory with multi-period portfolio selection. Besides, con-sidering the inherent similarity between portfolio selec-tion, the method proposed in this paper may be useful to applied in these fields, such as energy planning and medical diagnosis.

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