Fast Multidimensional Signal Processing with Shearlab.jl

Héctor Andrade Loarca
(github: **arsenal9971**)
Notebook and Beamer:
https://github.com/arsenal997/Shearlab.jl/presentations/PyData

TU Berlin, BMS

2^{nd} of July, 2017

What is a signal?

Our definition

Function (or something that can be represented as) that contains information about the behavior or attributes of some phenomenon. It can be digital (discrete) or analog (continuous).

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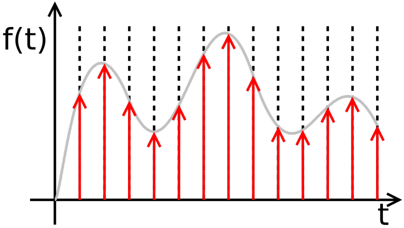

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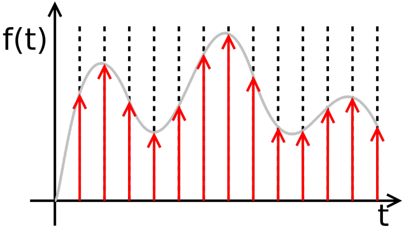

Figure: Digital and continuous one-dimensional signals

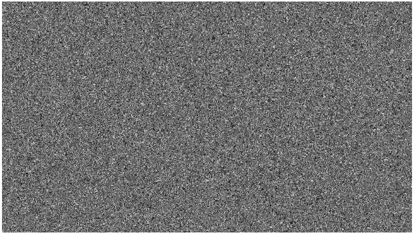

Figure: White noise, not a signal

Sparse representations of signals

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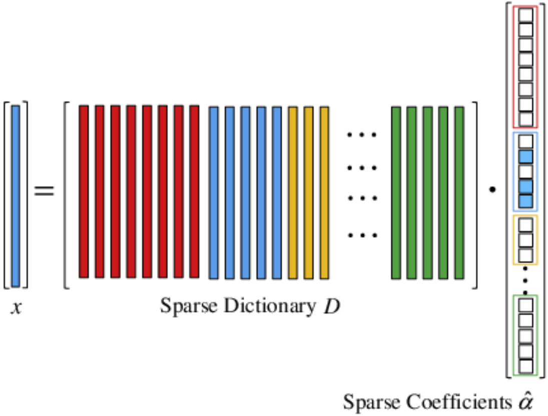
- Goal: Find the right dictionary to represent optimally our data.



Fourier Transform (Fourier, 1822)

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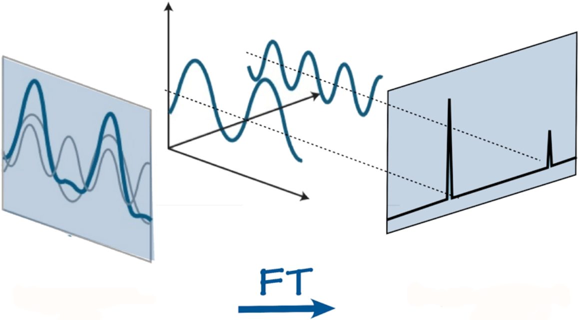

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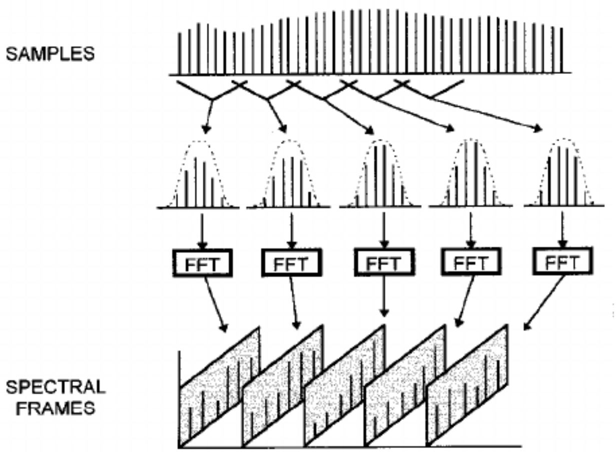

Wavelet Transform (Morlet and Grossman, 1984)

$$\mathcal{W}_\psi f(a,b) = \int_{\mathbb{R}} f(t) a^{-\frac{1}{2}} \overline{\psi\left(\frac{t-b}{a}\right)} dt$$
$$= (f * D_a \overline{\psi}^*)(b), \quad (a,b) \in \mathbb{R}^+ \times \mathbb{R}$$

where

$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

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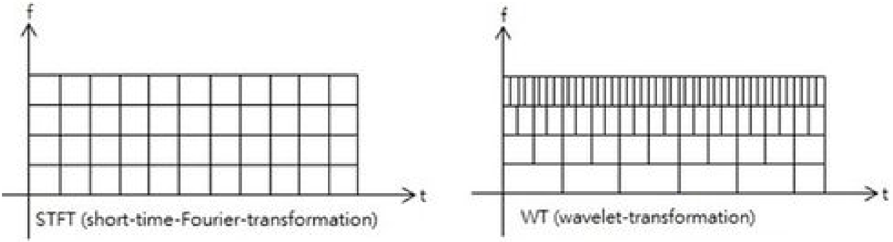

Cartoon-like functions

Definition

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0,1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $||f_i||_{C^2} \leq 1$ and $\text{supp} f_i \subset [0,1]^2$ for $i = 0, 1$.

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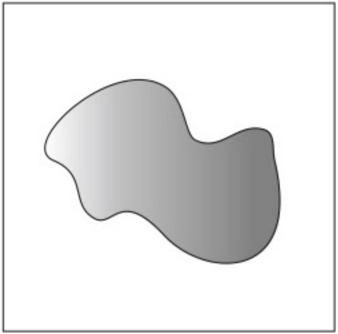

Optimal error for 2D signals

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{R}^2(\mathbb{R}^2)$ is

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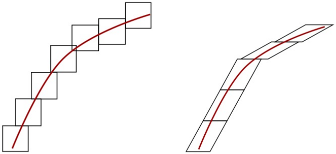

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- *Scaling*

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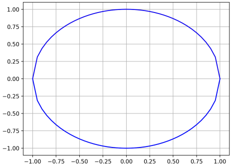

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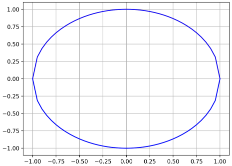

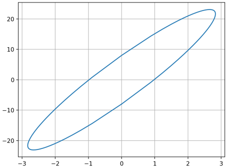

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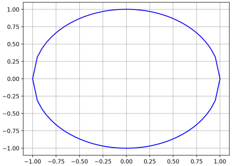

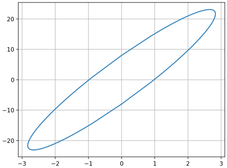

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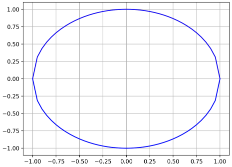

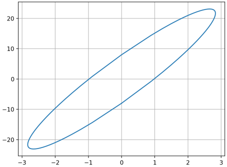

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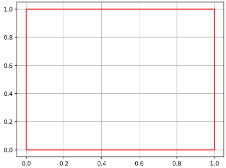

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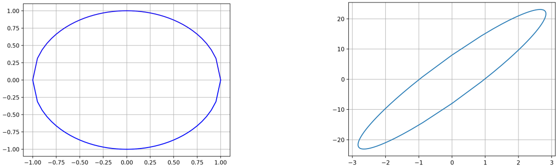

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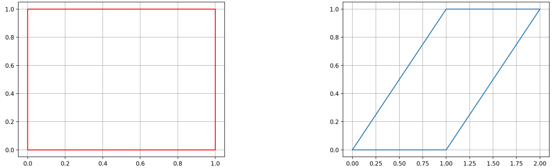

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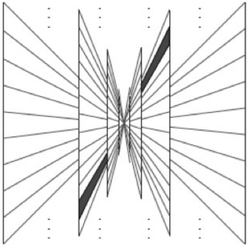

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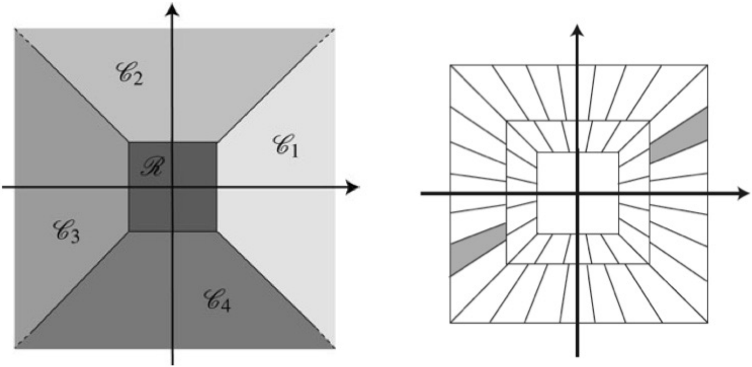

(a) (b)

Separable and non-separable generator

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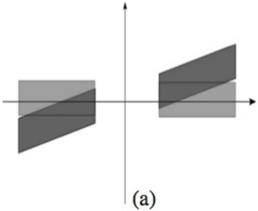

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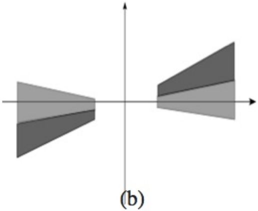

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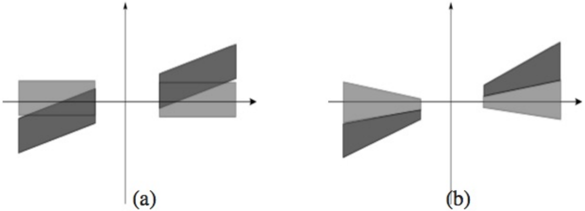

- Best N-term approximation error

$$\sigma_N(f, \{\psi_{j,k,m}\}_{j,k,m}) \sim N^{-1}(\log(N))^{3/2}$$

Current software

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- **Julia**
  - **Shearlab.jl** (H. Andrade, TU Berlin) https://github.com/arsenal9971/Shearlab.jl

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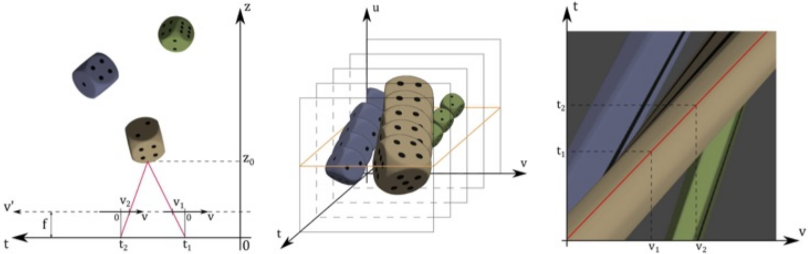
- Support of multithreading and painless GPU processing with ArrayFire.jl.

Current application: Light Field Recovery



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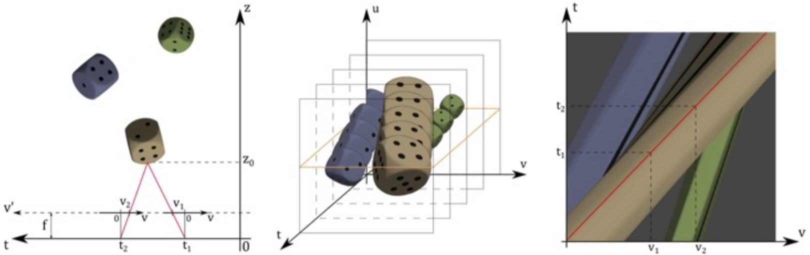

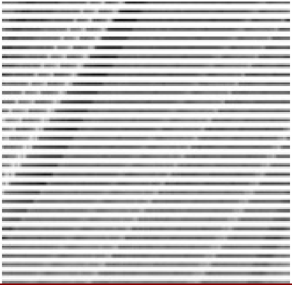

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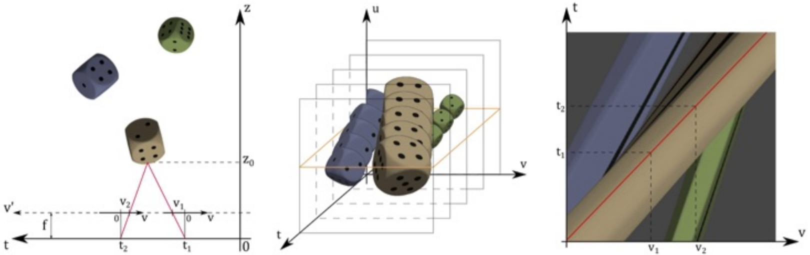

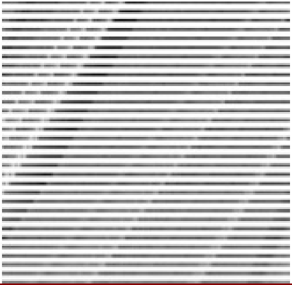

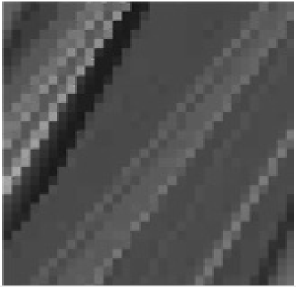

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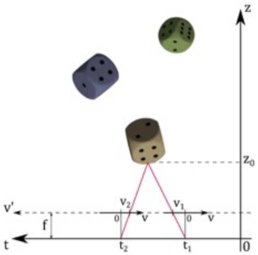

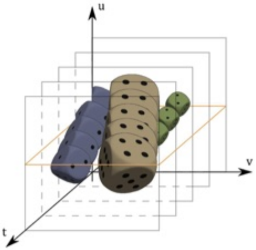

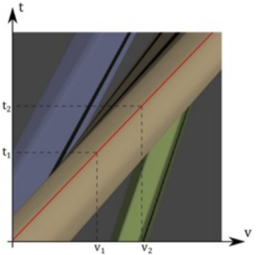

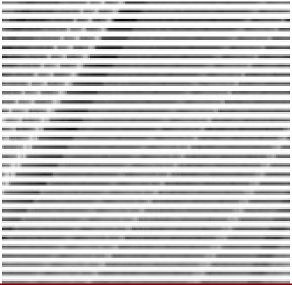

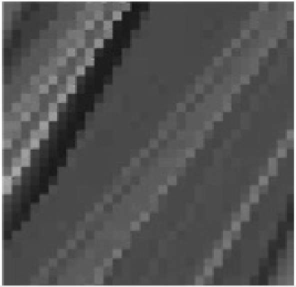

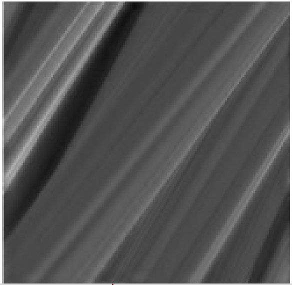

Thanks!

Questions?