Tema 3

• Să se calculeze divergența și rotorul următoarelor câmpuri vectoriale

1) $\vec{v} = \underbrace{x^2 e^{yz}}_{P(x,y,z)} \vec{i} + \underbrace{[y^2 + \ln(xz)]}_{Q(x,y,z)} \vec{j} + \underbrace{\frac{z^2}{x+2z^3}}_{R(x,y,z)} \vec{k}$

$div(\vec{v}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= 2xe^{yz} + 2y + \ln(xz) + \frac{2z(x+2z^3) - z^2 6z^2}{(x+2z^3)^2}$

$rot(\vec{v}) = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$= \vec{i}\left(0 - \frac{1}{\cancel{x}z}\cancel{x}\right) - \vec{j}\left(\frac{-z^2 \cdot 1}{(x+2z^3)^2} - x^2 e^{yz} y\right)$

$+ \vec{k}\left(\frac{1}{x\cancel{z}}\cancel{z} - x^2 e^{yz} z\right)$

$= \vec{i}\left(-\frac{1}{z}\right) - \vec{j}\left(\frac{-z^2}{(x+2z^3)^2} - x^2 e^{yz} y\right) + \vec{k}\left(\frac{1}{x} - x^2 e^{yz} z\right)$

2) $\vec{v} = \underbrace{y\cos(xz^2)}_{P(x,y,z)} \vec{i} + \underbrace{y^2\cos(xz)}_{Q(x,y,z)} \vec{j} + \underbrace{x\sin(y^2 z)}_{R(x,y,z)} \vec{k}$

$div(\vec{v}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y(-\sin(xz^2)z^2) + 2y\cos(xz)$

$+ x\cos(y^2 z)y^2$

$$\vec{rot}(\vec{v}) = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\cancel{= \vec{i}\left(\cos(xz^2) + y^2(-\sin(xz)x)\right) - \vec{j}\left(y(-\sin(xz^2)z^2)\right) +}$$

$$\cancel{+\vec{-j}\,(\sin y}$$

$$= \vec{i}\left(x\cos(y^2z)2yz + y^2(-\sin(xz)x)\right) - \vec{j}\left(\sin(y^2z) - y(-\sin(xz)\,2xy)\right) + \vec{k}\left(+y^2\sin(xz)z - \cos(xz^2)\right)$$

3) $\vec{v} = \underbrace{6xy}_{P(x,y,z)}\vec{i} + \underbrace{(x^2+\ln y)}_{Q(x,y,z)}\vec{j} + \underbrace{z^2\ln z}_{R(x,y,z)}\vec{k}$

$$\mathrm{div}(\vec{v}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 6y + \frac{1}{y} + 2z\ln z + z^2\frac{1}{z}$$

$$= \frac{6y^2+1}{y} + z\left(2\ln z + \frac{1}{z}\right)$$

$$\vec{rot}(\vec{v}) = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2x - 6x)$$

$$= -4x\vec{k}$$

4) $\vec{v} = \underbrace{(e^x - xz)}_{P(x,y,z)}\vec{i} + \underbrace{(ze^{xy} - e^{2y})}_{Q(x,y,z)}\vec{j} - \underbrace{xyz^4}_{R(x,y,z)}\vec{k}$

$\operatorname{div}(\vec{v}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= e^x - z + ze^{xy}x - e^{2y}2 + -4xy^3$

$\operatorname{rot}(\vec{v}) = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$= \vec{i}(-xz^4 - e^{xy}) - \vec{j}(-yz^4 + x) + \vec{k}(ze^{xy}y - 0)$

• Să se calculeze gradientul următoarelor câmpuri scalare

1) $f(x,y,z) = e^{xyz} + \sin(x+2y+3z)$

$\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$= (e^{xyz}yz + \cos(x+2y+3z)(2y+1))\vec{i} + (e^{xyz}xz + \cos(x+2y+3z)2)\vec{j}$

$+ (e^{xyz}xy + \cos(x+2y-3z)3)\vec{k}$

2) $f(x,y,z) = (2x+3y)e^{xyz} - \ln(x^2+y+3z)$

$\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = (2e^{xyz} + (2x+3y)e^{xyz}yz$

$- \frac{1}{x^2+y+3z} \cdot 2x)\vec{i} + (3e^{xyz} + (2x+3y)e^{xyz}xz - \frac{1}{x^2+y+3z} \cdot)\vec{j} +$

$+\left((2x+3y)e^{xyz}xy-\frac{1\cdot 3}{x^2+y+3z}\right)\vec{k}$

3) $f(x,y,z)=\operatorname{arctg}(x+2y+3z)$

$\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}=\frac{1}{(x+2y+3z)^2+1}\vec{i}+$

$\frac{1}{(x+2y+3z)^2+1}\cdot 2\,\vec{j}+\frac{1}{(x+2y+3z)^2+1}\cdot 6z\,\vec{k}$

• Să se calculeze derivata următoarelor câmpuri scalare după direcția vectorului $\vec{v}$

1) $f(x,y,z)=x^2y^3z^4 \qquad \vec{v}=2\vec{i}+2\vec{j}-\vec{k}$

$|\vec{v}|=\sqrt{4+4+1}=\sqrt{9}=3$

$\vec{v_1}=\frac{2}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{1}{3}\vec{k}$

$\cos\alpha=\frac{2}{3}, \ \cos\beta=\frac{2}{3}, \ \cos\gamma=-\frac{1}{3}$

$\frac{df}{d\vec{v}}=\cos\alpha\frac{\partial f}{\partial x}+\cos\beta\frac{\partial f}{\partial y}+\cos\gamma\frac{\partial f}{\partial z}$

$=\frac{2}{3}(2xy^3z^4)+\frac{2}{3}(3x^2y^2z^4)-\frac{1}{3}(4x^2y^3z^3)$

2) $f(x,y,z) = \sin(xyz)$ $\quad \vec{v} = \vec{i} - \vec{j} - \vec{k}$

$|\vec{v}| = \sqrt{3}$

$\cos\alpha = \frac{1}{\sqrt{3}}$, $\cos\beta = -\frac{1}{\sqrt{3}}$, $\cos\gamma = -\frac{1}{\sqrt{3}}$

$$\frac{df}{d\vec{v}} = \cos\alpha \frac{df}{dx} + \cos\beta \frac{df}{dy} + \cos\gamma \frac{df}{dz}$$

$$= \frac{1}{\sqrt{3}}\left(\cos(xyz)\, yz\right) - \frac{1}{\sqrt{3}}\left(\cos(xyz)\, xz\right) - \frac{1}{\sqrt{3}}\left(\cos(xyz)\, xy\right)$$

3) $f(x,y,z) = \frac{xy}{z}$ $\quad \vec{v} = 3\vec{i} - 5\vec{k}$

$|\vec{v}| = 6$

$\cos\alpha = 2$, $\cos\beta = 0$, $\cos\gamma = -\frac{5}{6}$

$$\frac{df}{d\vec{v}} = \cos\alpha \frac{df}{dx} + \cos\beta \frac{df}{dy} + \cos\gamma \frac{df}{dz}$$

$$\cancel{= \frac{2y}{z}} +$$

$$= 2\frac{y}{z} + \frac{5}{6}\frac{xy}{z^2}$$

• Ecuații cu derivate parțiale de ordin I

1) $yz\frac{\partial u}{\partial x} + xz\frac{\partial u}{\partial y} + xy\frac{\partial u}{\partial z} = 0$

Sistemul simetric

$$\frac{\cancel{x}\,dx}{yz} = \frac{\cancel{y}\,dy}{xz} = \frac{\cancel{z}\,dz}{xy}$$

$$\frac{dx}{yz} = \frac{dy}{xz} \qquad \frac{dy}{xz} = \frac{dz}{xy}$$

$$x\,dx = y\,dy \;\Big|\int \qquad y\,dy = z\,dz \;\Big|\int$$

$$\int x\,dx = \int y\,dy \qquad \int y\,dy = \int z\,dz$$

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1}{2} \qquad \frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$x^2 = y^2 + c_1 \qquad y^2 = z^2 + c_2$$

$$c_1 = x^2 - y^2 \qquad c_2 = y^2 - z^2$$

$$u(x,y,z) = \Phi\left(x^2 - y^2,\; y^2 - z^2\right)$$

2) $xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = z^2$

$$xz\frac{\partial u}{\partial x} + yz\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{z^2}$$

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}} \Big| \int \qquad \frac{dy}{y\cancel{z}} = \frac{dz}{z\cancel{z}} \Big| \int$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \qquad \int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\ln x = \ln y + \ln c_1 \qquad \ln y = \ln z + \ln c_2$$

$$\ln c_1 = \ln x - \ln y \qquad c_2 = \frac{y}{z}$$

$$\ln c_1 = \ln\left(\frac{x}{y}\right)$$

$$u(x,y,z) = \left(\frac{x}{y}, \frac{y}{z}\right)$$

$$c_1 = \frac{x}{y}$$

$$\cancel{z \cdot \frac{dx}{xz} = z \cdot \frac{dy}{yz} = (x+y)\frac{dz}{z^2}}$$

$$\cancel{= zdx + zdy + (x+y)dz}$$

3) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + (x+y)\frac{\partial u}{\partial z} = 0$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x+y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \qquad \frac{dx + dy - dz}{0}$$

$$\ln x = \ln y + \ln c_1 \qquad dx + dy - dz = 0$$

$$c_1 = \frac{x}{y} \qquad x + y - z = c_2$$

4 $(y+z)\frac{\partial z}{\partial x}+(z+x)\frac{\partial z}{\partial y}=x+y$

$(y+z)\frac{\partial u}{\partial x}+(z+x)\frac{\partial u}{\partial y}+(x+y)\frac{\partial u}{\partial z}=0$

$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$

6

6.)

5. $x(y^2-z^2)\frac{\partial u}{\partial x}+y(z^2-x^2)\frac{\partial u}{\partial y}+z(x^2-y^2)\frac{\partial u}{\partial z}=0$

$\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$

$x\,dx+y\,dy+z\,dz=0$

$d(x^2+y^2+z^2)=0$

$x^2+y^2+z=C_1$

6

6.) $$\frac{dx}{z^2 - y^2} = \frac{^{-z)}\,dy}{z} = \frac{^{y)}\,dz}{-y}$$

$$\frac{dy}{z} = \frac{dz}{-y}$$

$$-y\,dy = z\,dz$$

$$-\int y\,dy = \int z\,dz$$

$$-y^2 = z^2 + C_1$$

$$C_1 = \cancel{\ldots} - y^2 - z^2$$

$$\cancel{z^2 - y^2 - z^2 + y^2}$$

$$dx - z\,dy - y\,dz = 0$$

$$dx - d(zy) = 0$$

$$d(x - zy) = 0$$

$$x - zy = C_2$$

7) $\begin{cases} \sqrt{x}\dfrac{\partial u}{\partial x} + \sqrt{y}\dfrac{\partial u}{\partial y} + \sqrt{z}\dfrac{\partial u}{\partial z} = 0 \\ z = 1 \\ u(x,y,z) = x - y \end{cases}$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \qquad \sqrt{y} - \sqrt{z} = C_2$$

$$2\sqrt{x} = 2\sqrt{y} + 2C$$

$$C_1 = \sqrt{x} - \sqrt{y}$$

$$\begin{cases} \sqrt{y} - \sqrt{z} = C_2 \Rightarrow C_2 = \sqrt{y} - 1 \Rightarrow \sqrt{y} = C_2 + 1 \Rightarrow y = (C_2+1)^2 \\ \sqrt{x} - \sqrt{y} = C_1 \Rightarrow \sqrt{x} - C_2 + 1 = C_1 \\ z = 1 \qquad \Rightarrow x = (C_1 + C_2 + 1)^2 \\ u(x,y,z) = x - y \Rightarrow (C_1 + C_2 + 1)^2 - (C_2 + 1)^2 \end{cases}$$

$$= C_1^2 + 2C_1C_2 + 2C_1 + \cancel{C_2^2} + \cancel{2C_2 + 1}$$

$$\cancel{-C_2^2} \cancel{-2C_2} \cancel{-1}$$

$$= (\sqrt{x} - \sqrt{y})^2 + 2(\sqrt{x} - \sqrt{y})(\sqrt{y} - \sqrt{z}) + 2(\sqrt{x} - \sqrt{y})$$

8) $\begin{cases} (x^2+y^2)\dfrac{\partial u}{\partial x} + 2xy\dfrac{\partial u}{\partial y} + xz\dfrac{\partial u}{\partial z} = 0 \\ z = 1 \\ x^2+y^2 = 1 \end{cases}$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{xz}$$

$$\frac{dy}{2\cancel{x}y} = \frac{dz}{\cancel{x}z}$$

$$\int \frac{dy}{2y} = \frac{dz}{z}$$

$$\frac{1}{2}\ln y = \ln z + \ln c$$

$$c_1 = \frac{\sqrt{y}}{z}$$

$$9\begin{cases}(z-y)\dfrac{\partial u}{\partial x}+z\dfrac{\partial u}{\partial y}+y\dfrac{\partial u}{\partial z}=0\\ u(0,y,z)=2y(y-z)\end{cases}$$

$$\frac{dx}{z-y}=\frac{dy}{z}=\frac{dz}{y}$$

$$\frac{dy}{z}=\frac{dz}{y}\qquad dx-dy+dz=0$$

$$d(x-y+z)=0$$

$$zdz=ydy\qquad x-y+z=C_2$$

$$C_1=z^2-y^2$$

$$z^2-y^2=C_1\Rightarrow \cancel{\sqrt{C_2+y}-\sqrt{y}}\quad (C_2+y)^2-y^2=C_1$$

$$x-y+z=C_2\Rightarrow -y+z=C_2\Rightarrow z=C_2+\cancel{z}\,y$$

$$u(0,y,z)=2y/(y-z)$$

$$C_2^2+\cancel{y^2}+2C_2y-\cancel{y^2}=C_1$$

$$C_2^2+2C_2y=C_1$$

$$y=\frac{C_1-C_2^2}{2C_2}\Rightarrow u(0,y,z)=\frac{C_1-C_2^2}{C_2}\left(\frac{C_1-C_2^2}{C_2}-z\right)$$

$$\frac{z^2-y^2-(x-y+z)^2}{x-y+z}\left(\frac{z^2-y^2-(x-y+z)^2}{x-y+z}-z\right)$$

10) $$\begin{cases} xz\dfrac{\partial z}{\partial x} - yz\dfrac{\partial z}{\partial y} = x^2-y^2 \\ z=1 \\ x^2-y^2-2xy=1 \end{cases}$$

• Sisteme de ecuatii diferentiale

5) $$\begin{cases} x' = -x+y+z \\ y' = x-y+z \\ z' = x+y+z \end{cases} \qquad A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det(A-rI)=0 \Leftrightarrow \begin{vmatrix} -1-r & 1 & 1 \\ 1 & -1-r & 1 \\ 1 & 1 & 1-r \end{vmatrix}$$

$= (-1-r)^2(1-r)+1+1+1+r \cancel{-1}+r \cancel{-1}+r$

$= -r^3-r^2+r+1+1+1+1+r+r+r$

$= -r^3-r^2+4r+4 \qquad D_4=\{\pm1,\pm2,\pm4\}$

schema lui horner

	-1	-1	4	4
-1	1	-2	6	-2
1	1	$\boxed{0}$		
			10	
-2	1	-3		
2	1	1	6	

-4	1	3
-4	1	-5

$\lambda=1 \Rightarrow r=0$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Pt $\lambda = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$\begin{cases} -a_1 + a_2 + a_3 = 0 \\ a_1 - a_2 + a_3 = 0 \\ a_1 + a_2 + a_3 = 0 \end{cases}$$

Fie $B = \begin{pmatrix} -1 & 1 \\ +1 & 1 \end{pmatrix}$ $\det B = -2 \neq 0$

mp a_2 a_3

ms a_1

$$\begin{cases} -a_2 + a_3 = -\alpha \\ a_2 + a_3 = -\alpha \end{cases}$$

$\cancel{-a_1 + a_2 = -\alpha}$

$\cancel{a_1 - a_2 = -\alpha}$

$2a_3 = -2\alpha$

$a_3 = -\alpha \Rightarrow -a_2 - \alpha = -\alpha \Rightarrow a_2 = 0$

Fie $\alpha = C_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow C_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

1) $\begin{cases} x' = 4x - 3y \\ y' = 3x + 4y \end{cases}$ $A = \begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix}$

$\det(A - rI) = \begin{vmatrix} 4-r & -3 \\ 3 & 4-r \end{vmatrix}$

$= (4-r)^2 + 9$

$= 16 - 8r + r^2 + 9$

$= r^2 - 8r + 25$

$\Delta = 64 - 100 = -36 < 0$

$r_1 = \frac{8 + 6i}{2} = 4 + 3i$

$r_2 = 4 - 3i$

$\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 4 + 3i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\cancel{4a_1} - 3a_2 = \cancel{4a_1} + 3a_1 i$

$3a_1 + \cancel{4a_2} = \cancel{4a_2} + 3a_2 i$

$\begin{cases} -3a_2 = +3a_1 i \\ 3a_1 = 3a_2 i \end{cases}$ $\begin{cases} -a_2 = a_1 i \\ a_1 = a_2 i - \end{cases}$