5. SOLVING ALGEBRAIC EQUATIONS. CALCULATINGEIGENVALUES AND EIGENVECTORS

Objectives of the paper:

- Fixing knowledge regarding the solving of algebraic equations, both numerically and symbolically, using the Matlab programming environment,
- Recapitulation of elements regarding eigenvalues and eigenvectors,
- Fixing knowledge about calculating eigenvalues and eigenvectors, as well as the singular values and thecondition number with respect to the inversion of a matrix, using the Matlab programming environment, by studying some examples and solving some problems.

It is recommended to go through Annex M5 before studying paragraphs 5.1 and 5.2.

5.1.Elements regarding the solving of algebraic equations in Matlab. Elements regarding the determination of eigenvalues and eigenvectors in Matlab

Solvingalgebraic equations

An algebraic equation with the unknown $x$of form or brought to the form:

$$a_0 \cdot x^n + a_1 \cdot x^{n-1} + \ldots + a_{n-1} \cdot x + a_n = 0$$

with $a_i \in \boldsymbol{R}$, $i=0,1,\ldots,n$, is defined inby specifying the vector of the coefficients in descending order according to the powers of the unknowns:

```
c = [a0   a1   ...   an-1   an ]
```

Solving the equations abovemeans determining the roots of the polynomial:

$$P_n = a_0 \cdot X^n + a_1 \cdot X^{n-1} + \ldots + a_{n-1} \cdot X + a_n$$

In order to determine the roots of a polynomial P_n and the solutions of an algebraic equation, in Matlab the function roots can be used, which receives the vector c of coefficients as argument:

```
roots(c)
```

The function roots returns,in the set of complex numbers, the solutions of algebraic equations (respectively, all polynomials roots). Matlab can also be used to solve algebraic equations with parameters. For this purpose, the toolbox *Symbolic Math* is used, whichwas briefly presented in the paragraph 4.1.

Determination of the eigenvalues and eigenvectors

For calculating the eigenvalues of a matrix A the Matlab *eig* function is used, with the syntax:

```
val =eig(A)
```

The function returns a column vector *val* that contains the eigenvalues of the matrix A. The same Matlab `eig` function can also be used to determine a unit eigenvector (i.e, with Euclidean norm 1) for each eigenvalue. For this purpose, the function is used with the syntax:

```
[vec,val] =eig(A)
```

where *val* represents a diagonal matrix that contains the eigenvalues of matrix A on the principal diagonal, and *vec* is a matrix whose columns are the unit eigenvectors of the eigenvalues from *val*, such that the column *i* of the matrix *vec* represents the eigenvector associated to the eigenvalue $val_{i,i}$, i=1,2,...,n, with nbeing the order of the matrixA.

Calculation of singular values

For calculating the singular values of a matrix A the Matlab function *svd* is used, with the syntax:

```
vs =svd(A)
```

The function returns a column vectorvs that contains the singular values of the matrix A.

Calculation ofthe condition number with respect to the inversion of a matrix

For calculating the condition number with respect to the inversion of a matrix A the Matlab function *cond* is used, with the syntax:

```
nrc =cond(A)
```

Matlab can also be used to calculate other matrix condition numbers. A more performant Matlab function that calculates the conditioning number of a matrix is the function *rcond*, having the same syntax as the function *cond*. Thus, a system of linear equation whose coefficient matrix is A is well-conditioned, if *rcond(A)* is almost 1 and ill-conditioned if *rcond(A)* is almost 0.

5.2. Examples

Example 5.1: Solve the algebraic equation:

$$x^3 - 3x^2 - x + 3 = 0$$

Solution: The following Matlab program sequence is executed:

```
>> c=[1 -3 -1 3]; % the coefficients vector
>>disp('The solutions of the equation: '), sol=roots(c)
```

```
The solutions of the equation:

sol =
     3.0000
    -1.0000
     1.0000
```

It results that the equation has three solutions: x_1=3, x_2=-1 andx_3=1. It can be observed that this equation has only real solutions.

Example 5.2: Solve the algebraic equation:

$$2x + 3 = 6 + \sqrt{x-1}$$

in the set of real numbers.

Solution: Solving the problem consists of following these steps:

1. Separateinto a member of the equation the radical, the other terms being grouped in the other member. The equation becomes:

$$2x-3=\sqrt{x-1}$$

Examining the above equation, the following comments result:

- the condition for the existence of the radical is: $x \geq 1$;
- the radical of a real number is positive, in particular, the right member is positive; therefore the left member must be positive, that is: $x \geq 1.5$.

Conclusion: the solutions must have values greater than or equal to 1.5.

2. The equation is brought to the form$f(x)$ = 0, with$f$ – polynomial, by exponentiation and by moving all the terms to the left member. It results:

$$4x^2-13x+10=0$$

3. The above equation is solved in the Matlab environment and those results are extracted that meet the condition obtained in step 1. For this purpose the following Matlab program sequence is executed:

```
rad=roots([4 -13 10]);
sol=[];
for i=1:length(rad)
    if imag(rad(i))==0 & rad(i)>=1.5
        sol=[sol rad(i)];
    end
    end
disp('The solutions of the algebraic equation:')
format short g
sol
```

After the sequence above is executed:

```
The solutions of the algebraic equation:
sol=       2
```

The equation has only one real solution, namely: x = 2.

Example 5.3: Determine thereal strictly positive solutions (Re x_i>0) of the algebraic equation:

$$-45x^2+x^7+11x^3+x^6+4x^4-50x-2x^5=0$$

Solution: The following Matlab program sequence is executed (for example, M-file):

```
% the equation coefficients vector
c=[1 1 -2 4 11 -45 -50 0];
% solving the equation in C
rad=roots(c);
% extracting the solutions with the real part>0
j=1;
```

```
for i=1:length(rad)
    if real(rad(i))>0
        sol(j)=rad(i); j=j+1;
    end
end
disp('The solutions that have the real part strictly positive:')
sol
```

After the sequence above is executed:

```
The solutions that have the real part strictly positive:
sol=     2.0000
         1.0000 + 2.0000i
         1.0000 - 2.0000i
```

Example 5.4: Solve the following algebraic equations:

a) $$\frac{ax-b}{ax+b}=e^{-a},$$

in relation to the unknown x, respectively in relation to the unknown b;

b) $$\frac{ax^2+2b}{bx^2-2a}=\frac{b}{a},$$ in relation to the unknown x.

Solution: Since symbolic solving in Matlab returns the solution in the most favorable case, it is necessary to identify the compatibility situations before solving the equations symbolically.

a) Discussions:

- The equation with the unknown x presents the following points of discussion:

(I.x) $a=0$ and $b=0$ is not a possible case (the denominator of the equation being0);

(II.x) $a=0$, $b\neq 0$ lead to an impossible equation $-1=1$;

(III.x) $a\neq 0$, $b=0$ is not a possible case (equation becomes: $1=e^{-a}$, which contradictsa≠0);

(IV.x) $a\neq 0$ and$b\neq 0$. The equation is solved. For this the following Matlab program sequence is executed:

```
% defining the symbolic objects
syms x a b;
% expressing the left member of the equation brought to the form
% f(x)=0
f=a*(exp(-a)-1)*x+b*(exp(-a)+1);
%solving the equation is implicitly in relation with the variable x
solx=solve(f)
```

After the sequence above is executed, the following solution is obtained:

```
solx = -b*(exp(-a)+1)/a/(exp(-a)-1)
```

It has to be verified if for each $a \neq 0$ and each $b \neq 0$, the solution x fulfills the conditionthat the denominator $a \cdot x + b \neq 0$. Calculate:

$$ax + b = \frac{2b}{1 - e^{-a}}$$

which obviously is not null. So the calculated solution is valid.

- In the case of the unknownb, the following cases exist:

(I.b) $a=0$ or $x=0$ lead to an impossible equation $-1=e^{-a}$, namely $-1>0$, which is false;

(II.b) it has to be verified if, for $a \neq 0$ and $x \neq 0$, the solution b which is obtained does not cancel the equation denominator, namely $ax+b \neq 0$.

In order to obtain the solution, the following Matlab sequence is executed:

```
% defining the symbolic objects
syms x a b;
% expressing the left member of the equation brought to the form
% f(x)=0
f=(a*x-b)/(a*x+b)-exp(-a);
% solving the equation in relation with the variable b
solb=solve(f,b)
```

The following symbolic expression is obtained:

```
solb = -a*x*(-1+exp(-a))/(1+exp(-a))
```

The verification is done $a \cdot x + b \neq 0$:

$$ax + b = \frac{2 \cdot a \cdot x}{1 + e^{-a}}$$

which is not null for every $a \neq 0$ and $x \neq 0$. So the symbolic expression obtained is the solution.

b) <u>Discussions</u>:

(I) $a$cannot bezero;

(II) in the case $b=0$ the equation has a single solution, $x=0$;

(III) for $a=b$ and $a=-b, b \neq 0$, an impossible equation is obtained,which is reduced to $a=-a$, which is not possible, $a$being unable to be 0;

(IV) for the rest, the solutions of the equation are determine by executing the following Matlab sequence:

```
% defining the symbolic objects
syms x a b;
% expressing the left member of the equation brought to the form
% f(x)=0
f=(a*x^2+2*b)/(b*x^2-2*a)-b/a;
% solving the equation in relation with the variable x
solx=solve(f,x)
```

The following solutions are obtained:

```
solx =2/(a^2-b^2)*(-(a^2-b^2)*a*b)^(1/2)
-2/(a^2-b^2)*(-(a^2-b^2)*a*b)^(1/2)
```

Comments. 1.If the expression under the radical, $(-a^2+b^2)ab$, is strictly negative, the two solutions of the equation are complex numbers;

2. It is easy to verify that the denominator bx^2-$2\cdot a$ is not canceled for the obtained solutions.

Example 5.5: The following square matrix of order 3 is considered:

$$A=\begin{bmatrix} 13 & -10 & 12 \\ -11 & 14 & 10 \\ 0 & 12 & -13 \end{bmatrix}$$

Determine:

a) the eigenvalues of the matrix$A$;

b) for each eigenvalue of A a unit eigenvector;

c) the singular values of matrix$A$;

d) the condition number with respect to the inversion of matrix A.

Solution: The following Matlab program sequence is executed:

```
clear, clc

A=[13 -10 12; -11 14 10; 0 12 -13];
disp('the eigenvalues of A:')
lambda=eig(A)
% determining the unit eigenvectors
[vec,val]=eig(A);
for i=1:3
    disp(['the unit eigenvector corresponding to the'...
    'eigenvalue' num2str(lambda(i)) ': '])
    disp(vec(:,i))
end

% calculating the singular values
disp(blanks(1)')
disp('The singular values of A:')
val_sing=svd(A)

% calculating the condition number
disp(blanks(1)')
disp('The condition number with respect to theinversion of A')
nr=cond(A)
```

After the sequence above is executed, the following is obtained:

```
the eigenvalues of A:
lambda =
   23.5829
    9.2073
  -18.7902
```

```
the unit eigenvector corresponding to the eigenvalue 23.5829:
   -0.4782
    0.8345
    0.2737
the unit eigenvector corresponding to the eigenvalue 9.2073:
   -0.6320
   -0.6818
   -0.3684
the unit eigenvector corresponding to the eigenvalue - 18.7902:
    -0.4303
    -0.3923
     0.8130

The singular values of A:
val_sing =
    26.8248
    18.9470
     8.0275

The condition number with respect to the inversion of A
nr=       3.3416
```

Comments: 1. The Matlab function *num2str* converts a number into a string formed of the of the digits and the decimal point of that number.

2. The call *disp(blanks(n)')*, where *n* is a natural non-zero number, determines the display of n empty lines.

Example 5.6: Solve the following system of linear equations:

$$\begin{cases} x_1 + 0.5\cdot x_2 + 0.3333\cdot x_3 + 0.25\cdot x_4 = 0.1 \\ 0.5\cdot x_1 + 0.3333\cdot x_2 + 0.25\cdot x_3 + 0.2\cdot x_4 = 0 \\ 0.3333\cdot x_1 + 0.25\cdot x_2 + 0.2\cdot x_3 + 0.1667\cdot x_4 = 0.1 \\ 0.25\cdot x_1 + 0.2\cdot x_2 + 0.1667\cdot x_3 + 0.1429\cdot x_4 = 0 \end{cases}$$

Is this system well-conditioned orill-conditioned? Compare the obtained solution with the solution of the following system:

$$\begin{cases} x_1 + 0.5\cdot x_2 + 0.3333\cdot x_3 + 0.25\cdot x_4 = 0.1 \\ 0.5\cdot x_1 + 0.3333\cdot x_2 + 0.25\cdot x_3 + 0.2\cdot x_4 = 0 \\ 0.3333\cdot x_1 + 0.25\cdot x_2 + 0.2\cdot x_3 + 0.1667\cdot x_4 = 0.1 \\ 0.25\cdot x_1 + 0.2\cdot x_2 + 0.1667\cdot x_3 + 0.1436\cdot x_4 = 0 \end{cases}$$

Solution: It can be seen that a second system was obtained from the first system, by modifying the coefficient of the 4th equation corresponding to the unknown x_4. Therefore, the matrix of the coefficients of the second system can be obtained by replacing the element on the line 4 – column 4 with a new value.

It is also more convenient to enter the coefficient matrix lines on separate code lines. In this case, the matrix line separator is ENTER instead of the usual separator „;”.

The following Matlab program sequence is executed:

```
% coefficient matrix
A=[1       0.5    0.33330.25
     0.5    0.3333 0.25   0.2
     0.3333 0.25   0.2    0.1667
     0.25   0.2    0.1667 0.1429];
% constant terms vector
b=[0.1; 0; 0.1; 0];
% system solution
x=A\b
% condition number of A
nr_cond=rcond(A)

% coefficient matrix of system 2
A1=A; A1(4,4)=0.1436;
% system 2 solution
x1=A1\b

%comparison of the solutions
dif=abs(x-x1)
```

After the sequence above is executed, the following is obtained:

```
x=      32.8794
    -360.8831
     858.5994
    -554.0360
nr_cond =
        2.7586e-005

x1=     12.2531
    -119.2262
     261.8065
    -159.2004

dif=     20.6263
        241.6569
        596.7930
        394.8356
```

It can be noticed that thecondition number of matrix$A$ is close to zero. So the system is ill-conditioned. The same observation results from the comparison of the solutions of the two systems:to a small modification of a coefficient of the matrix A – the modulus of the difference between the values that differ is$7\cdot 10^{-4}$ – a large change in the solution is obtained– for example, for the unknown $x_3$the difference modulus is over 500.

5.3. Problems to solve

P5.1. Solve the algebraic equations:

a) $2x^5 - 3x^4 + 3x^2 - 10x - 8 = 0$, in ***C***;

b) $2x + \sqrt{1 - x^2} + 1 = x^2 + 3x + 2$, in***R***.

P5.2. Write a program that receives as argument the vector of the coefficients of an algebraic equation and returns the vector of the real solutions of the equation.

P5.3. Write a program that receives as argument the vector of the coefficients of an algebraic equation and returns the vector of the complex solutions with a modulus greater than one.

P5.4. Solve in***R*** the following algebraic equation with the unknown*x*:

$$m - x + \frac{n}{x} = m \cdot x + 1.$$

P5.5. Considering the matrix:

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

Determine the eigenvalues of the matrix. How many eigenvectors correspond to each determined eigenvalue? Display a minimum of three eigenvectors for each determined eigenvalue.

P5.6. Write a program that receives as argument a square matrix and returns theminimal singular value, the maximumsingular value and the condition number with respect to the inversion of matrix.

P5.7. Determine whether the following systems are well-conditioned or ill-conditioned:

$$\text{a)} \begin{cases} 2x_1 - 3x_2 = 7 \\ -6x_1 + 8x_2 - x_3 = -5 \\ 3x_2 + 4x_3 = 1 \end{cases}$$

$$\text{b)} \begin{cases} 190x + 7y = 4 \\ 2x + 200y = -2 \end{cases}$$

ANNEX M5. ELEMENTS REGARDING THE SOLVING OF ALGEBRAIC EQUATIONS. ELEMENTS REGARDING EIGENVALUES AND EIGENVECTORS

M5.1. Algebraic equations

It is called an ***algebraic equation (with the unknown x)*** anequation with the form:

$f(x) = 0$

where$f$ is a non-zero polynomial, or a reducible equation (through algebraic operations) to this.

If the polynomial f has a degree of n, it is said that ***the degree of the algebraic equation is n***.

The roots of the polynomial$f$ are called ***solutions of the algebraic equation f(x)=0***.

Usually determining the real solutions of an algebraic equation is of interest.

M5.2. Eigenvalues. Eigenvectors

Let$A$ be a square matrixof order$n$ withreal or complex elements.

A number$\lambda \in \mathbf{C}$ is called ***an eigenvalue of matrix A*** if there exists an n-dimensional vector x ($x \in \mathbf{R}^n$ or $x \in \mathbf{C}^n$), non-zero, such that:

$A\,x = \lambda x$

In this case, the non-zero vector$x$ is called ***an eigenvector of matrix A associated to the eigenvalue λ***.

The above relationship can also be written in the form:

$(A - \lambda I_n)x = 0$

The last relationship represents the matrix form of a system of homogeneous linear equations. Because from the definition of the eigenvector it results that this systemalso allows nontrivial solutions, thesystem is compatible andundetermined. So the system has an infinity of solutions, which leads to the conclusion that an infinity of eigenvectors correspond to each eigenvalue. The determinant of the system is:

$det(A - \lambda I_n) = 0$

The equation $det(A - \lambda \cdot I_n) = 0$represents an algebraic equation with the unknown λ and is called the ***characteristic equation of the matrix A***. The$n$ degree polynomial $P_n(\lambda) = det(A - \lambda \cdot I_n)$, is called the ***characteristic polynomial of the matrixA***.

The set of all the eigenvalues of the matrix A iscalled the ***spectrum of the matrix A***, and it is noted $\sigma(A)$. The number $\rho(A)=max|\lambda|$, $\lambda \in \sigma(A)$ is called ***the spectral radius of matrix A***.

The eigenvectors associatedto two distinct eigenvalues are linearly independent. If the matrix A of the order n has n linearly independent eigenvectors, then it is called ***a simple matrix*** or ***non-defective matrix***. Otherwise, matrix A is called ***a defective matrix***.

Singular values

Theorem. If A is a square matrix of the order n, then there exist the orthogonal square matrices of the order n, U and V, and a diagonal matrix Σ of the order n with positive elements, that satisfy the relation:

$$A = U \cdot \Sigma \cdot V^T$$

The diagonal elements of matrix Σ are called ***the singular values of matrix A***.

A few propertiesof the singular values are:
1. The rank of matrix A is equal to the number of non-zero singular values.
2. The singular values of matrix A are the square roots of the eigenvalues of the symmetric matrix $A^T \cdot A$.

Condition number

The ratio between the largest non-zero singular value and the lowest non-zero singular value of the matrix A is called ***thecondition number with respect to the inversion of matrixA***.

The condition number of a matrix A characterizes the sensitivity of a system of linear equations for which A is the coefficient matrix.

It is said that ***a systemof linear equations is ill-conditioned*** if minor changes of the system coefficients lead to large changes to the solution. Otherwise, it is said that the system is ***well-conditioned***. If the condition number of the coefficients matrix of a system of linear equationsis large, then the system is ill-conditioned.