Lucrare de control

subiectul B

rândul R1

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1. Determinați inf, sup, min, max pt mulțimea A

$$A = \left\{ \frac{p}{p+q} \;\middle|\; p, q \in \mathbb{N}^* \right\}$$

2. Determinați mulțimea pct. limită

$$x_n = \left(\frac{n+3}{n+1}\right)^{n \cdot \cos\frac{n\pi}{3}}$$

3. Calculați

$$\lim_{x \to 0} \frac{x^3 - tg^3 x}{x^5}$$

1. $A = \left\{ \frac{p}{p+q} \mid p, q \in \mathbb{N}^* \right\}$

$\frac{p}{p+q} < \cancel{\frac{p+q}{p+q}}$

$p < p+q \Rightarrow q > 0 \Rightarrow \frac{p}{p+q}$ - subunitară

$p, q \in \mathbb{N}^* \Rightarrow \frac{p}{p+q} > 0$ $\quad \Rightarrow \frac{p}{p+q} \in [0, 1]$

$p = q = 1 \Rightarrow \frac{p}{p+q} = \frac{1}{2}$

$\cancel{\frac{p}{p+q} > \frac{1}{2} \Rightarrow \frac{p}{p+q} - \frac{1}{2} > 0}$

$\cancel{\frac{2p - p - q}{2(p+q)} > 0}$

Fie 1 - ~~inf A~~ sup A

$\cancel{\frac{p-q}{2(p+q)} > 0}$

$\frac{p}{p+q} > 1 \Rightarrow$ $\quad p > p+q \Rightarrow q < 0$ (F) $\Rightarrow (\forall) x \in A, x \le 1$

$(\forall) \varepsilon > 0, \exists y \in A$ a.î. $y > M - \varepsilon$

Fie $y \in A$, $y = \frac{p}{p+q}$

$y > 1 - \varepsilon$

$\frac{p}{p+q} > 1 - \varepsilon \Rightarrow p > p + q - \varepsilon(p+q)$

$q < \varepsilon(p+q)$

$p, q \in \mathbb{N}^*$

$\Rightarrow 1 = \sup A$ $\quad \Rightarrow \nexists \min A$

Fie 0 - inf A <=> $\begin{cases} (i)\ (\forall)\ x \in A,\ x \geq 0 \\ (ii)\ (\forall)\ \varepsilon > 0\ \exists\ y \in A \text{ a.î. } y < \varepsilon \end{cases}$

(i) - trivial dem

$\frac{p}{p+q} \geq 0 \Rightarrow$ $\left.\begin{array}{l} p \geq 0 \\ p \geq 0 \\ p+q > 0 \\ p, q \in A \end{array}\right\} \Rightarrow \frac{p}{p+q} \geq 0$

ii) $(\forall)\ \varepsilon > 0,\ \exists\ y \in A$ a.î. $y < \varepsilon$

$\frac{p}{p+q} < \varepsilon \Rightarrow$ $\left.\begin{array}{l} p < \varepsilon(p+q) \\ p+q \in \mathbb{N}^* \end{array}\right\} \Rightarrow (A)$

$\Rightarrow 0$ - inf A $\cancel{\exists}$ min A $\cancel{\exists}$

inf A = 0 min A $\not\exists$

sup A = 1 max A $\not\exists$

3.

$$l = \lim_{x\to 0} \frac{x^3 - tg^3 x}{x^5} = \lim_{x\to 0}\left(\frac{x^3}{x^5} - \frac{tg^3 x}{x^5}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{tg^3 x}{x^3}\cdot\frac{1}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2} - \left(\frac{tg\, x}{x}\right)^3\cdot\frac{1}{x^2}\right)$$

$$\cancel{\lim_{x\to 0}\frac{tg\, x}{x} \overset{[\frac{0}{0}]}{\underset{l'H}{=}} \lim_{x\to 0}\frac{\frac{1}{\cos^2 x}}{1} = 1}$$

$$\cancel{l = \lim_{x\to 0}\left(\frac{1}{x^2} - 1\cdot\frac{1}{x^2}\right) = \lim_{x\to 0}\frac{0}{x^2} = 0}$$

$$\lim_{x\to 0}\frac{1}{x^2}\left(1 - \left(\frac{tg\, x}{x}\right)^3\right) \overset{[\frac{0}{0}]}{\underset{l'H}{=}}$$

$$= \lim_{x\to 0}\frac{-3\left(\frac{tg\, x}{x}\right)^2\cdot\left(\frac{\frac{x}{\cos^2 x} - tg\, x}{x^2}\right)}{x}$$

$$= \lim_{x\to 0} -\frac{3}{x}\underbrace{\left(\frac{tg\, x}{x}\right)^2}_{1}\cdot\left(\frac{x}{x^2\cos^2 x} - \frac{tg\, x}{x^2}\right)$$

$$\overset{[\frac{0}{0}]}{\underset{l'H}{=}} \lim_{x\to 0} -\frac{3}{x}\cdot\left(\frac{x}{x^2\cos^2 x} - \frac{tg\, x}{x^2}\right) = 1$$

$2.\ x_n = \left(\frac{n+3}{n+1}\right)^{n \cdot \cos\frac{n\pi}{3}}$

$n = 3k_n$

$$x_{3k_n} = \left(\frac{3k_n+3}{3k_n+1}\right)^{3k_n \cdot \cos\frac{3k_n\pi}{3}} = \left(\frac{3k_n+3}{3k_n+1}\right)^{3k_n \cdot \cos k_n\pi}$$

$$= \left(\frac{3k_n+3}{3k_n+1}\right)^{3k_n \cdot (-1)^{k_n}}$$

$k_n = 2p$

$$x_{6p} = \left(\frac{6p+3}{6p+1}\right)^{6p}$$

$$\lim_{p\to\infty}\left(\frac{6p+3}{6p+1}\right)^{6p} = \lim_{p\to\infty}\left[\left(1+\frac{2}{6p+1}\right)^{\frac{6p+1}{2}}\right]^{\frac{2}{6p+1}\cdot 6p}$$

$$= e^2$$

$k_n = 2p+1$

$$x_{6p+3} = \left(\frac{6p+6}{6p+4}\right)^{(6p+3)\cdot(-1)} = \left(\frac{6p+4}{6p+6}\right)^{6p+3}$$

$$\lim_{p\to\infty}\left(\frac{6p+4}{6p+6}\right)^{6p+3} = \lim_{p\to\infty}\left[\left(1+\frac{-2}{6p+6}\right)^{\frac{6p+6}{-2}}\right]^{\frac{-2}{6p+6}\cdot 6p+3}$$

$$= e^{-2}$$

$n = 3k + 1$

$$x_{3k+1} = \left(\frac{3k+4}{3k+2}\right)^{(3k+1)\cdot\cos\frac{(3k+1)\pi}{3}}$$

$$= \left(\frac{3k+4}{3k+2}\right)^{(3k+1)\cdot\cos\left(k\pi+\frac{\pi}{3}\right)}$$

$$= \left(\frac{3k+4}{3k+2}\right)^{(3k+1)\cdot\cancel{\left(-\cos\frac{\pi}{3}\right)}(-1)^{k}\cdot\cos\frac{\pi}{3}}$$

$k = 2p$

$$x_{6p+1} = \left(\frac{6p+4}{6p+2}\right)^{(6p+1)\cdot\frac{1}{2}}$$

$$\lim_{p\to\infty}\left(\frac{6p+4}{6p+2}\right)^{(6p+1)\cdot\frac{1}{2}} = \lim_{p\to\infty}\left[\left(1+\frac{2}{6p+2}\right)^{\frac{6p+2}{2}}\right]^{\frac{2}{6p+2}\cdot\frac{6p+1}{2}}$$

$$= e$$

$k = 2p+1$

$$x_{6p+4} = \left(\frac{6p+7}{6p+5}\right)^{(6p+4)\cdot\left(-\frac{1}{2}\right)} = \left(\frac{6p+5}{6p+7}\right)^{\frac{6p+4}{2}}$$

$$\lim_{p\to\infty}\left(\frac{6p+5}{6p+7}\right)^{\frac{6p+4}{2}} = \lim_{p\to\infty}\left(1+\frac{-2}{6p+7}\right)^{\frac{6p+7}{-2}\cdot\frac{-2}{6p+7}\cdot\frac{6p+4}{2}}$$

$$= e^{-1} = \frac{1}{e}$$

$n = 3k + 2$

$$x_{3k+2} = \left(\frac{3k+5}{3k+3}\right)^{(3k+2)\cdot\cos\frac{(3k+2)\pi}{3}} = \left(\frac{3k+5}{3k+3}\right)^{(3k+2)\cdot(-1)^{k}\cos\frac{2\pi}{3}}$$

$$\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2}$$

$k = 2p$

$$x_{6p+2} = \left(\frac{6p+5}{6p+3}\right)^{(6p+2)\cdot\left(-\frac{1}{2}\right)}$$

$$\lim_{p\to\infty} x_{6p+2} = \lim_{p\to\infty}\left(\frac{6p+5}{6p+3}\right)^{\frac{(6p+2)}{-2}} =$$

$$= \lim_{p\to\infty}\left[\left(1 + \frac{2}{6p+3}\right)^{\frac{6p+3}{2}}\right]^{\frac{2}{6p+3}\cdot\frac{6p+2}{-2}}$$

$$= e^{-1} = \frac{1}{e}$$

$k = 2p+1$

$$x_{6p+5} = \left(\frac{6p+8}{6p+6}\right)^{\frac{6p+5}{2}} =$$

$$\lim_{p\to\infty}\left[\left(1 + \frac{2}{6p+6}\right)^{\frac{6p+6}{2}}\right]^{\frac{2}{6p+6}\cdot\frac{6p+5}{2}} = e^{1} = e$$

$$LIM(x_n) = \left\{\frac{1}{e^2}, \frac{1}{e}, e, e^2\right\}$$