Title: Flying With Differential Equations

SAMPLE SOLUTION

Part 1.

A. We can model this situation as one where a periodic external force, $F_0 \sin wt$ (vibration of the propeller), is applied to a spring-mass system (the wing). Newton's Second Law yields:

$$m\,a = \_\,F = m\,y''(t) = -k\,y(t) + F_0 \sin wt \qquad \textbf{(1)}$$

Initial conditions: $y(0) = 0,\ y'(0) = 0$

where m = mass of the wing, k = spring constant, w = period of the forcing function, $F_0 = F(0)$, and $y(t)$ = displacement of the wing's center of mass from its resting position.

We can rewrite equation (1) as:

$$m\,y''(t) + k\,y(t) = F_0 \sin wt \qquad \textbf{(2)}$$

Initial conditions: $y(0) = 0,\ y'(0) = 0$

B. To find a general solution for equation (2), we look for both the particular and complementary solutions. These are the solutions to the nonhomogeneous and homogeneous problems respectively. First, the complementary solution is found by solving the homogeneous equation:

$$m\,y''(t) + k\,y(t) = 0$$

whose characteristic equation is $r^2 + (k/m) = 0$. The roots of the characteristic equation are $r = \pm i\sqrt{\dfrac{k}{m}}$.

We find the complementary solution to be:

$$y_c(t) = c_1 \cos \sqrt{\frac{k}{m}}\ t + c_2 \sin \sqrt{\frac{k}{m}}\ t$$

and, since $w_0 = \sqrt{\dfrac{k}{m}}$, this leads to:

$$y_c(t) = c_1 \cos w_0 t + c_2 \sin w_0 t.$$

Second, looking for a particular solution, we try $y_p(t) = A \sin wt$. Substituting, we obtain:

$$m(-A\,w^2 \sin wt) + k\,(A \sin wt) = F_0 \sin wt\,.$$

So, $\qquad A\,(-mw^2 + k) = F_0$

and therefore, $\qquad A = F_0 / (-mw^2 + k) = F_0 / [m\,(k/m - w^2)]$.

Letting w_0 = the natural period of the spring-mass system = $\sqrt{\dfrac{k}{m}}$, we see that:

$$A = F_0 / [m\,(w_0^2 - w^2)]$$

and therefore, $\qquad y_p(t) = [F_0 / [m\,(w_0^2 - w^2)]] \sin wt$.

Therefore, the general solution is the linear combination of the particular and complementary solutions:

$$y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + [F_0 / [m\,(w_0^2 - w^2)]] \sin wt$$

Using the initial conditions of $y(0) = 0$ and $y'(0) = 0$:

$$y(0) = c_1 + 0 + 0 = 0$$
$$\longrightarrow c_1 = 0$$

$$y'(0) = 0 + w_0\,c_2 + w\,F_0 / [m(w_0^2 - w^2)] = 0$$

$$\longrightarrow c_2 = -(w/w_0) F_0 / [m(w_0^2 - w^2)]$$

Our solution becomes:

$y(t) = -(w/w_0)[F_0/[m(w_0^2 - w^2)]] \sin w_0 t + [F_0/[m(w_0^2 - w^2)]] \sin wt$

$y(t) = [F_0/[m(w_0^2 - w^2)]] [\sin wt - (w/w_0) \sin w_0 t]$
(3)

C. Letting k = 13,600 N/m, m = 850 kg, w = 8, and F_0 = 1,550 N, we can substitute into equation **(3)** and obtain the equation of motion (don't forget $w_0 = \sqrt{\frac{k}{m}}$).

Since: $w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{13{,}600}{850}} = 4$,

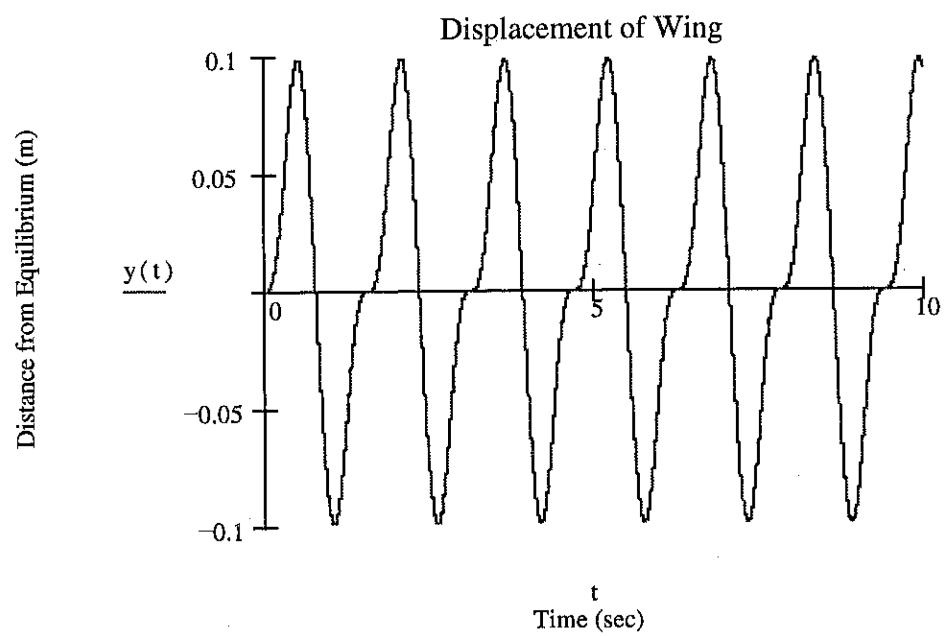
$F_0/[m(w_0^2 - w^2)] = -0.03799$ and $w/w_0 = 8/4 = 2$.

Therefore: $y(t) = [-0.03799] [\sin 8t + 2 \sin 4t]$

$y(t) = 0.07598 \sin 4t - 0.03799 \sin 8t$

D. Plotting this solution, we obtain the following graph

Graph of $y(t) = 0.07598 \sin 4t - 0.03799 \sin 8t$

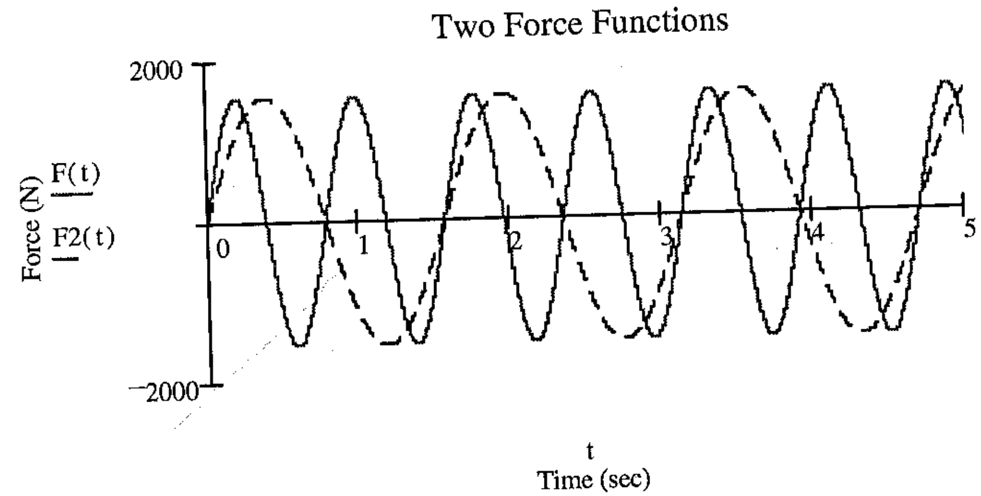


E. As time (t) grows large, we see the center-of-mass of the wing is oscillating on periodic cycle. The maximum and minimum points are equidistant from the axis defined by the wing at rest. Therefore, if the wing is stable through one complete cycle it should remain stable as long as no other forces are applied.

Part 2.

For this requirement, we simulate the slowing down of the propeller by changing the forcing function to $F_2(t) = 1550 \sin 4t$.

A. Plotting the two forcing functions ($F(t)$ and $F_2(t)$) on a calculator or computer, we obtain the following graphs.



We can see that the principle difference between the two functions is their periods. $F(t) = 1{,}550 \sin 8t$ has a period of $\pi/4$, while $F_2(t) = 1{,}550 \sin 4t$ has a period of $\pi/2$. The first function, $F(t)$, completes two cycles in the time the second function, $F_2(t)$, completes only one. The amplitude of the two functions is the same (1550).

B. Using the new forcing function, $F_2(t)$, with all other conditions remaining the same, the equation of motion becomes:

$$850\, y''(t) + 13{,}600\, y(t) = 1550 \sin 4t$$

Solving this differential equation we note that now $w = w_0$. The homogeneous part of the solution remains the same:

$$y_c(t) = c_1 \cos 4t + c_2 \sin 4t$$

But, in finding the characteristic equation of the nonhomogeneous part, we are led to a solution ($y_p(t) = c_3 \cos 4t + c_4 \sin 4t$) which is absorbed by the homogeneous solution. Therefore, we multiply this expression by the variable t and obtain the form for the particular solution:

$$y_p(t) = c_3 t \cos 4t + c_4 t \sin 4t$$

We see:

$$y_p'(t) = -4 c_3 t \sin 4t + c_3 \cos 4t + 4 c_4 t \cos 4t + c_4 \sin 4t$$

$$= (c_3 + 4c_4 t) \cos 4t + (c_4 - 4c_3) \sin 4t$$

$$y_p''(t) = -16 c_3 t \cos 4t - 4 c_3 \sin 4t - 4 c_3 \sin 4t - 16 c_4 t \sin 4t$$

$$+ 4 c_4 \cos 4t + 4 c_4 \cos 4t$$

$$= 8 [(c_4 - 2c_3 t)(\cos 4t) - (c_3 + 2c_4 t)(\sin 4t)]$$

Now, substituting back into the original equation:

$$850[8 [(c_4 - 2c_3 t)(\cos 4t) - (c_3 + 2c_4 t)(\sin 4t)]] + 13{,}600[(c_3 t)(\cos 4t) + (c_4 t)(\sin 4t)] = 1550 \sin 4t$$

To solve for the coefficients we group into like terms:

$$(6800\, c_4)\cos 4t + (-6800\, c_3 - 1500)\sin 4t = 0$$

Substituting $t = 0$, we see that $c_4 = 0$. This makes sense since the forcing function has no cosine term and there are only derivatives of even order. The coefficient of the cosine term in this equation should be zero.

Letting t be any other value we see that $(-6800\, c_3 - 1500)$ must always be zero. Therefore, $c_3 = -0.2205$.

The particular solution becomes: $y_p(t) = -0.2205 t \cos 4t$

Forming the general solution by combining the complementary and particular solutions, we obtain:

$$y(t) = c_1 \cos 4t + c_2 \sin 4t - 0.2205 t \cos 4t$$

Now use the initial conditions, $y(0) = 0$ and $y'(0) = 0$, to solve for the coefficients:

$$y(t) = c_1 \cos 4t + c_2 \sin 4t - 0.2205 t \cos 4t$$

$$y(0) = c_1 + 0 - 0 = 0$$

$$c_1 = 0$$

$$y'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t + 0.882 t \sin 4t - 0.2205 \cos 4t$$

$$y'(0) = 0 + 4c_2 + 0 - 0.2205 = 0$$

$$c_2 = 0.055125$$

Substituting into the general solution:

$$y(t) = 0.055125 \sin 4t - 0.2205 t \cos 4t$$

C. The following graph is obtained by plotting the resultant motion equation above:

Graph of $y(t) = 0.055125 \sin 4t - 0.2205 t \cos 4t$

Solving this differential equation we note that now $w = w_0$. The homogeneous part of the solution remains the same:

$$y_c(t) = c_1 \cos 4t + c_2 \sin 4t$$

But, in finding the characteristic equation of the nonhomogeneous part, we are led to a solution ($y_p(t) = c_3 \cos 4t + c_4 \sin 4t$) which is absorbed by the homogeneous solution. Therefore, we multiply this expression by the variable t and obtain the form for the particular solution:

$$y_p(t) = c_3 t \cos 4t + c_4 t \sin 4t$$

We see:

$$y_p'(t) = -4 c_3 t \sin 4t + c_3 \cos 4t + 4 c_4 t \cos 4t + c_4 \sin 4t$$

$$= (c_3 + 4c_4 t) \cos 4t + (c_4 - 4c_3) \sin 4t$$

$$y_p''(t) = -16 c_3 t \cos 4t - 4 c_3 \sin 4t - 4 c_3 \sin 4t - 16 c_4 t \sin 4t$$

$$+ 4 c_4 \cos 4t + 4 c_4 \cos 4t$$

$$= 8 [(c_4 - 2c_3 t)(\cos 4t) - (c_3 + 2c_4 t)(\sin 4t)]$$

Now, substituting back into the original equation:

$$\begin{aligned} &850[8 [(c_4 - 2c_3 t)(\cos 4t) - (c_3 + 2c_4 t)(\sin 4t)]] \\ &\quad + 13{,}600[(c_3 t)(\cos 4t) + (c_4 t)(\sin 4t)] \\ &= 1550 \sin 4t \end{aligned}$$

To solve for the coefficients we group into like terms:

$$(6800\, c_4)\cos 4t + (-6800\, c_3 - 1500)\sin 4t = 0$$

Substituting $t = 0$, we see that $c_4 = 0$. This makes sense since the forcing function has no cosine term and there are only derivatives of even order. The coefficient of the cosine term in this equation should be zero.

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Now use the initial conditions, $y(0) = 0$ and $y'(0) = 0$, to solve for the coefficients:

$$y(t) = c_1 \cos 4t + c_2 \sin 4t - 0.2205 t \cos 4t$$

$$y(0) = c_1 + 0 - 0 = 0$$

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$$y'(0) = 0 + 4c_2 + 0 - 0.2205 = 0$$

$$c_2 = 0.055125$$

Substituting into the general solution:

$$y(t) = 0.055125 \sin 4t - 0.2205 t \cos 4t$$

C. The following graph is obtained by plotting the resultant motion equation above:

Graph of $y(t) = 0.055125 \sin 4t - 0.2205 t \cos 4t$