Interdisciplinary Lively Application Project

Title: Flying With Differential Equations

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Mathematics Classifications: Differential Equations, Calculus

Disciplinary Classification: Mechanical & Aeronautical Engineering,

Prerequisite/Corequisite Skills:
1. Modeling with Differential Equations
2. Solving Constant Coefficient, Nonhomogeneous Differential Equations
3. Numerical Solution of 1st-order Differential Equation (Euler's Method and/or a Runge-Kutta Method)

Physical Concepts Examined:
1. Forced Vibrations
2. Mechanical Resonance
3. Motion under Gravity

Materials Available:
1. Problem Statement (4 Parts); Student
2. Sample Solution (4 Parts); Instructor
3. Notes for the Instructor

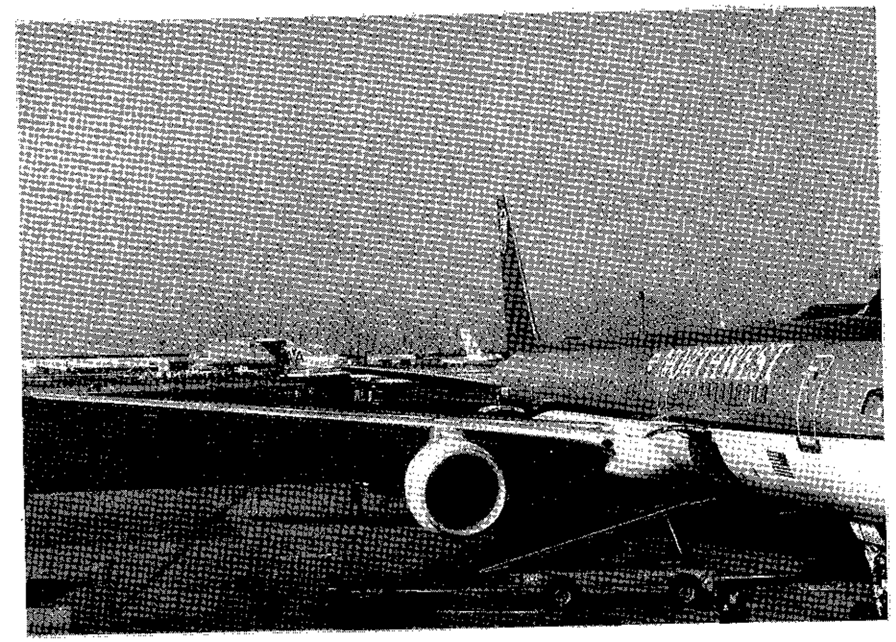
Computing Requirements:
1. Tools for iterating a difference equation (derived from Euler or Runge-Kutta Methods), such as a spreadsheet)
2. Numerical differential equations solver (optional)

Background Information

> If you have ever looked out a window while in flight, you have probably observed that wings on an airplane are not perfectly rigid. A reasonable amount of flex or flutter is not only tolerated but necessary to prevent the wing from snapping like a piece of peppermint stick candy. In late 1959 and early 1960 two commercial plane crashes involving a relatively new model of prop-jet occurred, illustrating the destructive effects of large mechanical oscillations. ... After a massive technical investigation, the problem was eventually traced in each case to an outboard engine and engine housing. Roughly, it was determined that when each plane surpassed a critical speed of approximately 400 mph, a propeller and engine began to wobble, causing a gyroscopic force, which could not be quelled or damped by the engine housing. This *external vibrational force* was then transferred to the already oscillating wing. This, in itself, need not have been destructively dangerous since aircraft wings are designed to withstand the stress of unusual and excessive forces. ... But unfortunately, after a short period of time during which the engine wobbled rapidly, *the frequency of the impressed force actually slowed* to a point at which it approached and finally coincided with the maximum frequency of wing flutter. The amplitudes of wing flutter became large enough to snap the wing. (Zill, 1989, p. 219)

In this problem, we will examine the phenomena which caused the airplane wing in the above scenario to snap. This phenomena is called mechanical resonance. Any structure or mechanical system is susceptible to damage by the forces of resonance. The following excerpt presents a clear and interesting discussion of resonance.

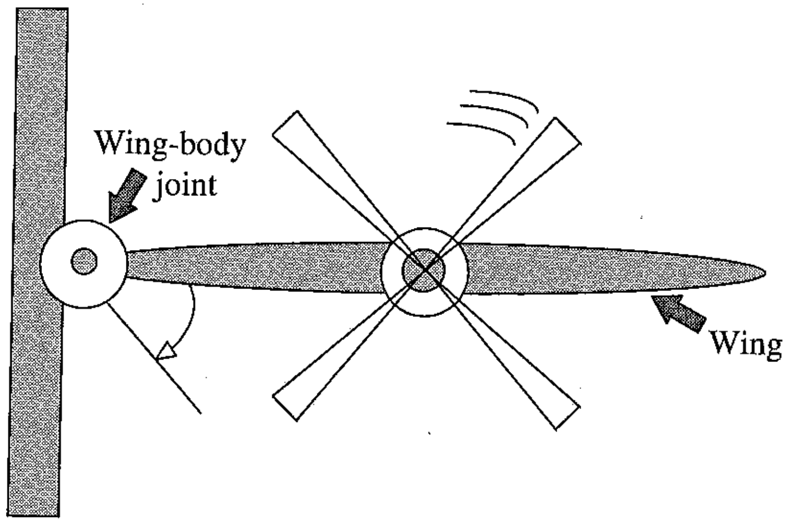
> When the frequency of a periodic external force applied to a mechanical system is related in a simple way to the natural frequency of the system, mechanical resonance may occur which builds up the oscillations to such tremendous magnitudes that the system may fall apart. A company of soldiers marching in step across a bridge may in this manner cause the bridge to collapse even though the bridge would have been strong enough to carry many more soldiers had they marched out of step. For this reason soldiers [are] required to "break step" [when] crossing a bridge. In an analogous manner, it may be possible for a musical note of proper characteristic frequency to shatter a glass. Because of the great damages which may thus occur, mechanical resonance is in general something which needs to be avoided, especially by the engineer in designing structure or vibrating systems. (Spiegel, 1981, p. 239)



We can use differential equations to construct a simple model of wing flutter. The *center-of-mass* of an object is the point at which we can consider the entire mass of an object to be represented. It is the balance point (in terms of mass) of the object. If we restrict our investigation to the fluttering motion that takes place at the wing's center-of-mass, we can build a simple model of this situation. By restricting our attention to this point, we can then think of the wing as a *spring-mass system*. The *spring* is the wing-body joint that allows the wing's center-of-mass to move up and down in a fluttering motion. The forcing function is the external vibrational force that comes from the wobbling propeller. Remember -- any motion at the wing's center-of-mass will be magnified out at the tip of the wing!

Some Assumptions

For the sake of solving this simple model, we assume that the wing has a mass of m kg and that the wing-body joint will act like a spring with a *spring constant* of k N/m (Newtons per meter). The motion of the wing's center-of-mass is actually a curved arc; however, since we are working with long wingspans, we can simplify the situation by assuming that the center-of-mass moves up and down in a straight line. Assume that, before the propeller begins to wobble, the wing is at rest. Also, assume that any forces which tend to damp the motion of the wing are negligible.



We want to analyze and understand this wing flutter phenomenon, and, as part of our company's design team, we need you to conduct the following step-by-step analysis to determine several of the critical issues of this phenomenon.

Part 1.

Starting at *time* = 0 ($t = 0$), assume the propeller begins to vibrate with a force equal to $F(t) = F_0 \sin wt$.

A. Find the equation of motion for the wing's center-of-mass. Be sure to include the appropriate initial conditions you would need to solve this differential equation.

B. Solve your equation of motion.

C. Now let $m = 850$ kg, k = 13,600 N/m, $w = 8$, and $F_0 = 1{,}550$ N, and write the solution using these values.

D. Plot your displacement function using your calculator or a computer. Include this plot in your submission.

E. Describe the motion of the center-of-mass as t (time) grows large.

Part 2.

Recall from the excerpt above that just before wing failure the frequency of the external force caused by the wobbling propeller actually slowed down. Let's simulate that slowing down by changing our forcing function to $F_2(t) = 1550 \sin 4t$ N.

A. Plot the two forcing functions ($F(t)$ and $F_2(t)$) and include with your submission. What is the principal difference in the two functions?

B. Find the equation of motion using the new forcing function (letting all other conditions remain the same). Again, make sure that you include the appropriate initial conditions. Solve your new equation of motion for the displacement function.

C. Plot this displacement function on your calculator or computer and include with your submission.

D. Describe what happens as t grows large. What consequences does this have for the wing?

Part 3.

Let's now assume that the wing does finally snap. The wing itself is now in free fall towards the earth. Further assume that the wing does not tumble, but instead remains mostly horizontal as it falls. Several forces may be acting upon the wing as it falls. We will consider two: resistive drag force (F_d), and force due to gravity (F_g). (Assume that down is the positive direction.)

The drag force results from collisions between falling objects (in this case the wing) and the air molecules. A common sub-model for drag forces proposes that the force from drag is proportional to some power of the object's velocity. We choose a two-term model here:

$$F_d = -k_1 v - k_2 v^2.$$

(One can think of this as the first two terms of a Taylor Series representation for the drag function.)

The sub-model for force due to gravity should be familiar:

$$F_g = mg.$$

For the purposes of this problem, we will use $k_1 = 10.5$, and $k_2 = 6.5$.

A. Model the velocity of the free-falling wing with a differential equation. Classify the differential equation (order, linearity, homogeneity) and be sure to include the appropriate initial conditions.

B. You notice that this type of differential equation is not as easy to solve using analytic methods. Therefore, you must use a numerical method to approximate the solution. Choose a suitable numerical technique and step size that will allow you to determine the velocity of the wing 4 seconds after it snaps.

C. Assuming its initial vertical velocity is zero, what is the velocity of the wing after it has fallen 50 meters?

D. What is the wing's terminal velocity?

Part 4.

Think critically about the model used in Requirements 1 and 2.

A. What forces are at work on the real plane wing that we did not include in our model. (Hint: Look at the differential equation you used in Requirement 2. What type of motion is being modeled?)

B. What other factors or issues might an engineer want to analyze in his investigation of wing flutter?

C. How would you model flex in the aircraft wing (e.g., with a system of differential equations)?

REFERENCES

Spiegel, M. R. (1981). *Applied differential equations.* Englewood Cliffs, N. J.: Prentice Hall.

Zill, D. G. (1989). *A first course in differential equations with applications.* Boston: PWS-Kent.