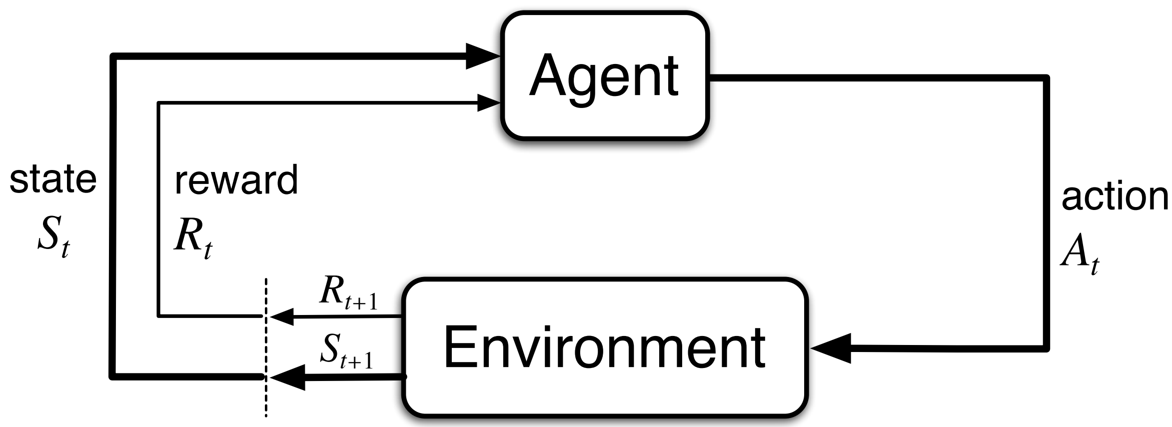
Chapter 3: The Reinforcement Learning Problem (Markov Decision Processes, or MDPs)

Objectives of this chapter:

- present Markov decision processes—an idealized form of the AI problem for which we have precise theoretical results
- introduce key components of the mathematics: value functions and Bellman equations

The Agent-Environment Interface



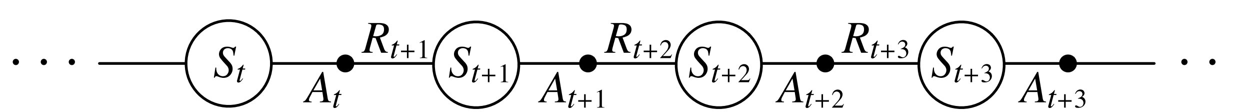
Agent and environment interact at discrete time steps: $t = 0, 1, 2, 3, \ldots$

Agent observes state at step t: $S_t \in \mathcal{S}$

produces action at step t : $A_t \in \mathcal{A}(S_t)$

gets resulting reward: $R_{t+1} \in \mathcal{R} \subset \mathbb{R}$

and resulting next state: $S_{t+1} \in \mathcal{S}^+$



Markov Decision Processes

- If a reinforcement learning task has the Markov Property, it is basically a **Markov Decision Process (MDP)**.
- If state and action sets are finite, it is a **finite MDP**.
- To define a finite MDP, you need to give:
  - **state and action sets**
  - one-step "dynamics"

$$p(s', r|s, a) = \mathbf{Pr}\{S_{t+1} = s', R_{t+1} = r \mid S_t = s, A_t = a\}$$

  - there is also:

$$p(s'|s, a) \doteq \Pr\{S_{t+1} = s' \mid S_t = s, A_t = a\} = \sum_{r \in \mathcal{R}} p(s', r|s, a)$$

$$r(s, a) \doteq \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r|s, a)$$

The Agent Learns a Policy

Policy at step t = π_t =

a mapping from states to action probabilities

$\pi_t(a \mid s) =$ probability that $A_t = a$ when $S_t = s$

Special case - *deterministic policies*:

$\pi_t(s) =$ the action taken with prob=1 when $S_t = s$

- Reinforcement learning methods specify how the agent changes its policy as a result of experience.
- Roughly, the agent's goal is to get as much reward as it can over the long run.

The Meaning of Life
(goals, rewards, and returns)

Return

Suppose the sequence of rewards after step t is:

$$R_{t+1}, R_{t+2}, R_{t+3}, \ldots$$

What do we want to maximize?

At least three cases, but in all of them,

we seek to maximize the **expected return**, $E\left\{G_t\right\}$, on each step t.

- Total reward, G_t = sum of all future reward in the episode
- Discounted reward, G_t = sum of all future *discounted* reward
- Average reward, G_t = average reward per time step

Episodic Tasks

Episodic tasks: interaction breaks naturally into episodes, e.g., plays of a game, trips through a maze

In episodic tasks, we almost always use simple *total reward*:

$$G_t = R_{t+1} + R_{t+2} + \cdots + R_T ,$$

where T is a final time step at which a **terminal state** is reached, ending an episode.

Continuing Tasks

Continuing tasks: interaction does not have natural episodes, but just goes on and on...

In this class, for continuing tasks we will always use *discounted return*:

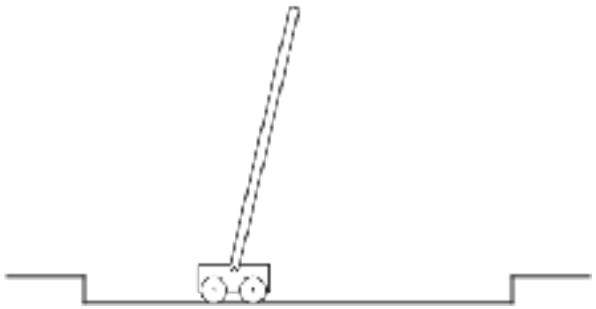
$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1},$$

where γ, $0 \le \gamma \le 1$, is the **discount rate**.

shortsighted $0 \leftarrow \gamma \rightarrow 1$ farsighted

Typically, $\gamma = 0.9$

An Example: Pole Balancing



Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track

As an **episodic task** where episode ends upon failure:

reward $= +1$ for each step before failure

$\Rightarrow$ return $=$ number of steps before failure

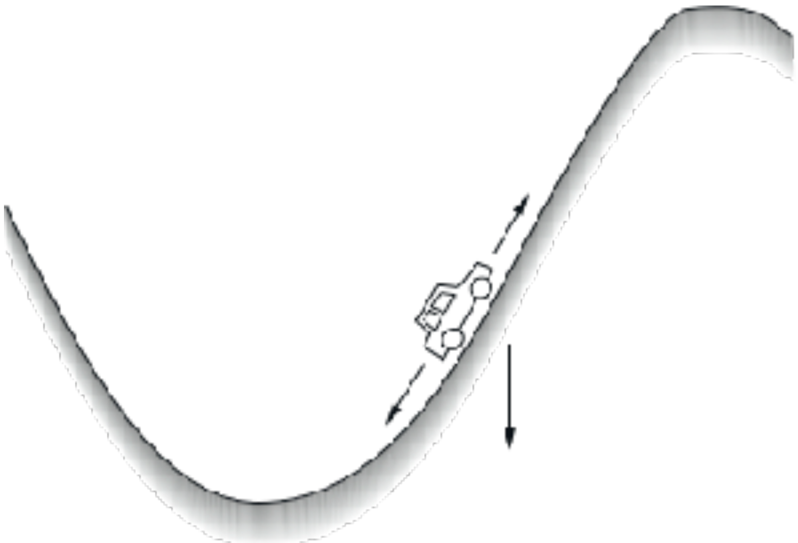
As a **continuing task** with discounted return:

reward $= -1$ upon failure; 0 otherwise

$\Rightarrow$ return $= -\gamma^k$, for k steps before failure

In either case, return is maximized by avoiding failure for as long as possible.

Another Example: Mountain Car



Get to the top of the hill as quickly as possible.

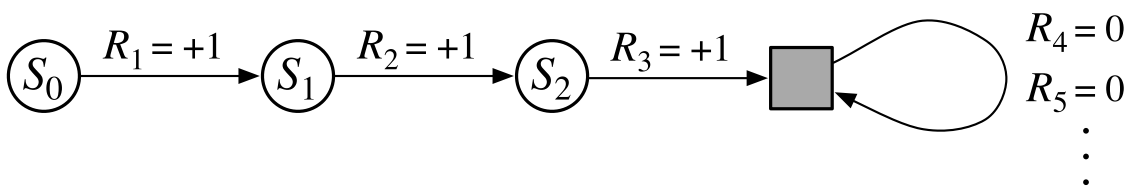
reward = −1 for each step where **not** at top of hill

⇒ return = − number of steps before reaching top of hill

Return is maximized by minimizing number of steps to reach the top of the hill.

A Trick to Unify Notation for Returns

- In episodic tasks, we number the time steps of each episode starting from zero.
- We usually do not have to distinguish between episodes, so instead of writing $S_{t,j}$ for states in episode j, we write just S_t
- Think of each episode as ending in an absorbing state that always produces reward of zero:



- We can cover all cases by writing $G_t = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$,

where γ can be 1 only if a zero reward absorbing state is always reached.

Rewards and returns

- The objective in RL is to maximize long-term future reward
- That is, to choose A_t so as to maximize $R_{t+1}, R_{t+2}, R_{t+3}, \ldots$
- But what exactly should be maximized?
- The discounted *return* at time t:

the *discount rate*

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0,1)$$

γ	Reward sequence	Return
0.5(or any)	1 0 0 0…	1
0.5	0 0 2 0 0 0…	0.5
0.9	0 0 2 0 0 0…	1.62
0.5	-1 2 6 3 2 0 0 0…	2

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

- Suppose $\gamma = 0.5$ and the reward sequence is

$$R_1 = 1, R_2 = 6, R_3 = -12, R_4 = 16, \text{ then zeros for } R_5 \text{ and later}$$

- What are the following returns?

$$G_4 = 0 \quad G_3 =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

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- What are the following returns?

$$G_4 = 0 \quad G_3 = 16 \quad G_2 = -4 \quad G_1 =$$

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- What are the following returns?

$$G_4 = 0 \quad G_3 = 16 \quad G_2 = -4 \quad G_1 = 4 \quad G_0 =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

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- What are the following returns?

$$G_4 = 0 \quad G_3 = 16 \quad G_2 = -4 \quad G_1 = 4 \quad G_0 = 3$$

- Suppose $\gamma = 0.5$ and the reward sequence is all 1s.

$$G =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

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$$G = \frac{1}{1 - \gamma}$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

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- Suppose $\gamma = 0.5$ and the reward sequence is all 1s.

$$G = \frac{1}{1 - \gamma} = 2$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0,1)$$

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$$G = \frac{1}{1-\gamma} = 2$$

- Suppose $\gamma = 0.5$ and the reward sequence is

$$R_1 = 1, R_2 = 13, R_3 = 13, R_4 = 13, \text{ and so on, all 13s}$$

$$G_2 =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

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- Suppose $\gamma = 0.5$ and the reward sequence is

$$R_1 = 1, R_2 = 13, R_3 = 13, R_4 = 13, \text{ and so on, all 13s}$$

$$G_2 = 26 \quad G_1 =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0,1)$$

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$$G_2 = 26 \quad G_1 = 26 \quad G_0 = 14$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

- Suppose $\gamma = 0.5$ and the reward sequence is

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$$G_2 = 26 \quad G_1 = 26 \quad G_0 = 14$$

- And if $\gamma = 0.9$?

$$G_1 =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0,1)$$

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$$G_2 = 26 \quad G_1 = 26 \quad G_0 = 14$$

- And if $\gamma = 0.9$?

$$G_1 = 130 \quad G_0 =$$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0,1)$$

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$$G_2 = 26 \quad G_1 = 26 \quad G_0 = 14$$

- And if $\gamma = 0.9$?

$$G_1 = 130 \quad G_0 = 118$$

4 value functions

	state values	action values
prediction	v_π	q_π
control	v_*	q_*

- All theoretical objects, mathematical ideals (expected values)

- Distinct from their estimates:

$$V_t(s) \qquad Q_t(s, a)$$

Values are *expected* returns

- The value of a state, given a policy:

$$v_\pi(s) = \mathbb{E}\{G_t \mid S_t = s, A_{t:\infty} \sim \pi\} \qquad v_\pi : \mathcal{S} \to \Re$$

- The value of a state-action pair, given a policy:

$$q_\pi(s,a) = \mathbb{E}\{G_t \mid S_t = s, A_t = a, A_{t+1:\infty} \sim \pi\} \qquad q_\pi : \mathcal{S} \times \mathcal{A} \to \Re$$

- The optimal value of a state:

$$v_*(s) = \max_\pi v_\pi(s) \qquad v_* : \mathcal{S} \to \Re$$

- The optimal value of a state-action pair:

$$q_*(s,a) = \max_\pi q_\pi(s,a) \qquad q_* : \mathcal{S} \times \mathcal{A} \to \Re$$

- Optimal policy: π_* is an optimal policy if and only if

$$\pi_*(a|s) > 0 \text{ only where } q_*(s,a) = \max_b q_*(s,b) \qquad \forall s \in \mathcal{S}$$

  - in other words, π_* is optimal iff it is *greedy* wrt q_*

4 value functions

	state values	action values
prediction	v_π	q_π
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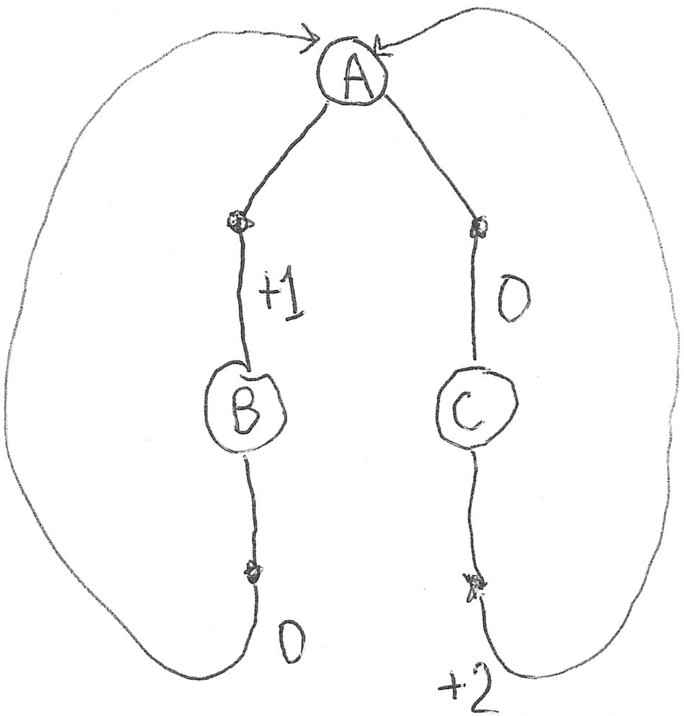
- All theoretical objects, mathematical ideals (expected values)

- Distinct from their estimates:

$$V_t(s) \qquad Q_t(s,a)$$

optimal policy example



What policy is optimal?

A: left

B: Right C: Other

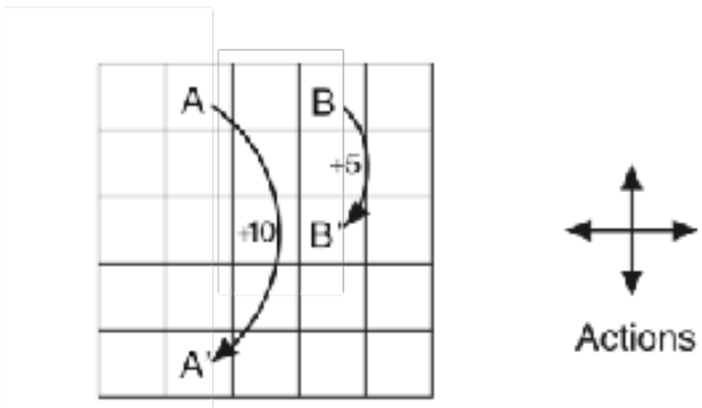
If $\gamma = 0$?

If $\gamma = .99$

If $\gamma = \frac{1}{2}$?

Gridworld

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.



(a)

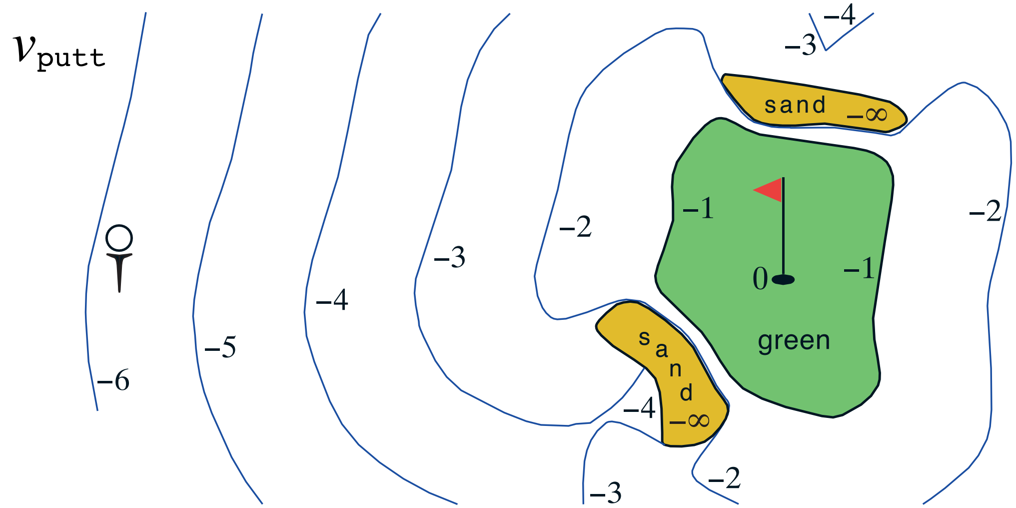
3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

State-value function for equiprobable random policy; $\gamma = 0.9$

Golf

- State is ball location
- Reward of –1 for each stroke until the ball is in the hole
- Value of a state?
- Actions:
  - `putt` (use putter)
  - `driver` (use driver)
- `putt` succeeds anywhere on the green



What we learned so far

- Finite Markov decision processes!
  - States, actions, and rewards
  - And returns
  - And time, discrete time
  - They capture essential elements of life — state, causality
- The goal is to optimize expected returns
  - returns are *discounted sums* of *future* rewards
- Thus we are interested in *values* — expected returns
- There are four value *functions*
  - state vs state-action values
  - values for a policy vs values for the optimal policy

Optimal Value Functions

- For finite MDPs, policies can be **partially ordered**:

$$\pi \geq \pi' \quad \text{if and only if} \quad v_\pi(s) \geq v_{\pi'}(s) \quad \text{for all } s \in \mathcal{S}$$

- There are always one or more policies that are better than or equal to all the others. These are the **optimal policies**. We denote them all π_*.

- Optimal policies share the same **optimal state-value function**:

$$v_*(s) = \max_\pi v_\pi(s) \quad \text{for all } s \in \mathcal{S}$$

- Optimal policies also share the same **optimal action-value function**:

$$q_*(s,a) = \max_\pi q_\pi(s,a) \quad \text{for all } s \in \mathcal{S} \text{ and } a \in \mathcal{A}$$

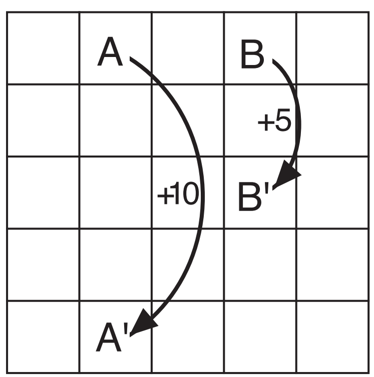
This is the expected return for taking action a in state s and thereafter following an optimal policy.

Why Optimal State-Value Functions are Useful

Any policy that is greedy with respect to v_* is an optimal policy.

Therefore, given v_*, one-step-ahead search produces the long-term optimal actions.

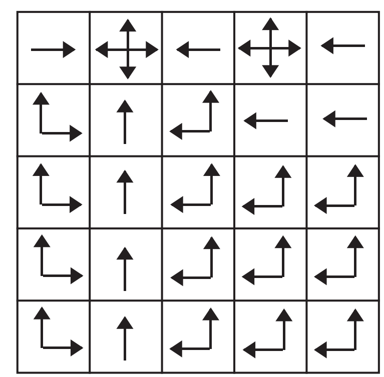
E.g., back to the gridworld:



a) gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

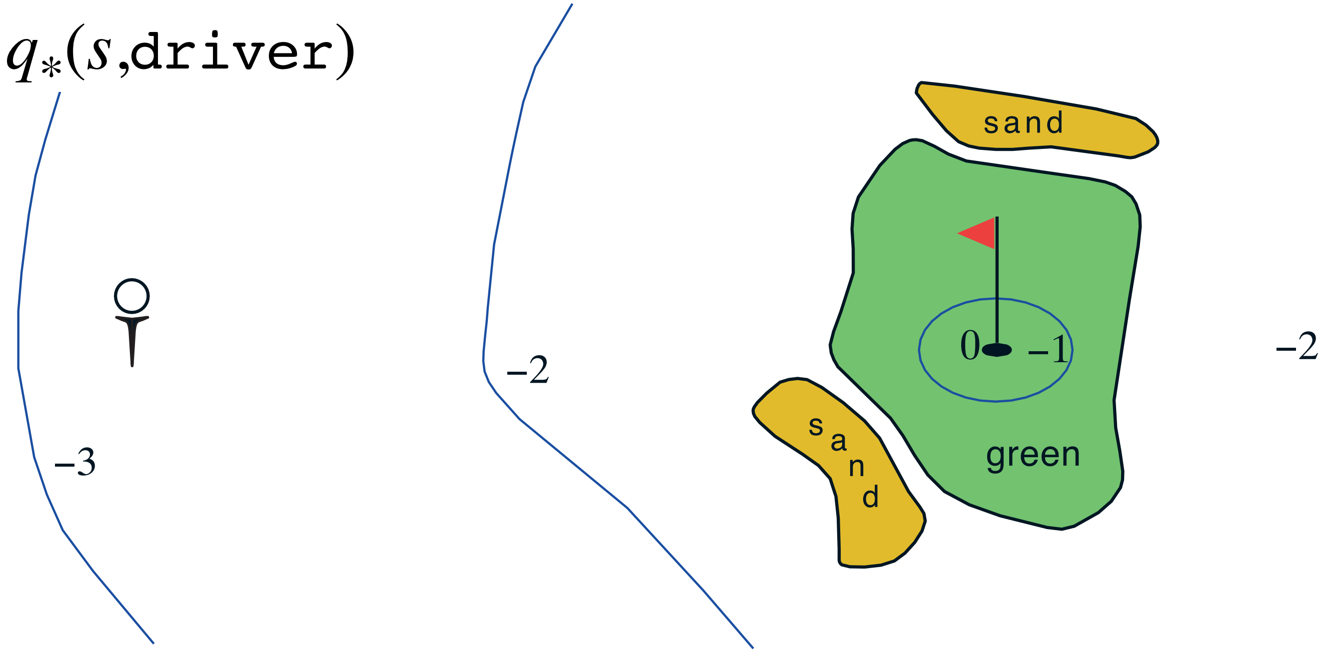
b) v_*



c) π_*

Optimal Value Function for Golf

- We can hit the ball farther with `driver` than with `putter`, but with less accuracy
- $q_*(s,\texttt{driver})$ gives the value or using `driver` first, then using whichever actions are best



What About Optimal Action-Value Functions?

Given q_*, the agent does not even have to do a one-step-ahead search:

$$\boxed{\pi_*(s) = \arg\max_a q_*(s,a)}$$

Value Functions
x 4

Bellman Equations
x 4

Bellman Equation for a Policy π

The basic idea:

$$
\begin{aligned}
G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\
&= R_{t+1} + \gamma \left(R_{t+2} + \gamma R_{t+3} + \gamma^2 R_{t+4} + \cdots \right) \\
&= R_{t+1} + \gamma G_{t+1}
\end{aligned}
$$

So:

$$
\begin{aligned}
v_\pi(s) &= E_\pi \left\{ G_t \,\middle|\, S_t = s \right\} \\
&= E_\pi \left\{ R_{t+1} + \gamma v_\pi \left(S_{t+1} \right) \,\middle|\, S_t = s \right\}
\end{aligned}
$$

Or, without the expectation operator:

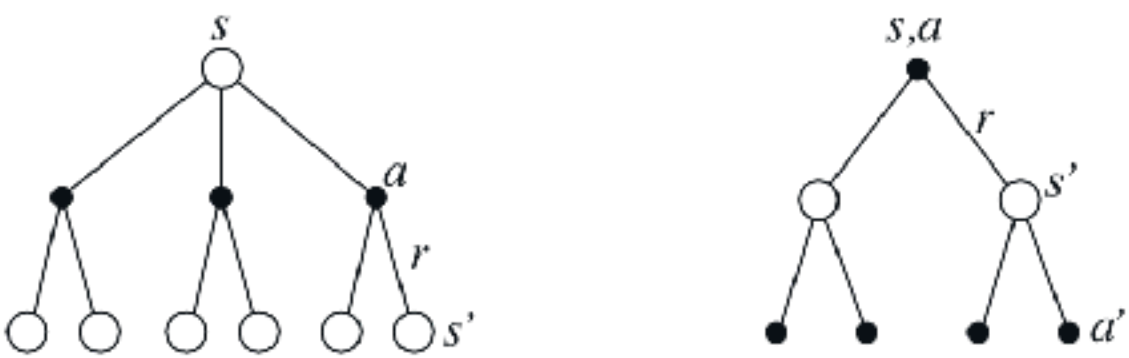
$$
v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) \Big[r + \gamma v_\pi(s') \Big]
$$

More on the Bellman Equation

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]$$

This is a set of equations (in fact, linear), one for each state.
The value function for π is its unique solution.

Backup diagrams:



for v_π for q_π

Gridworld

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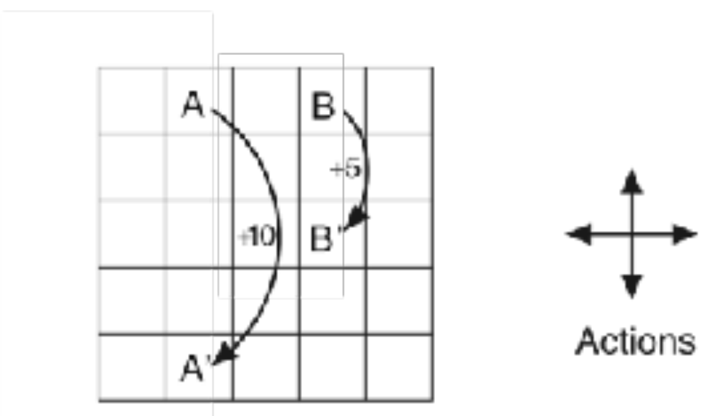
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.



(a)

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

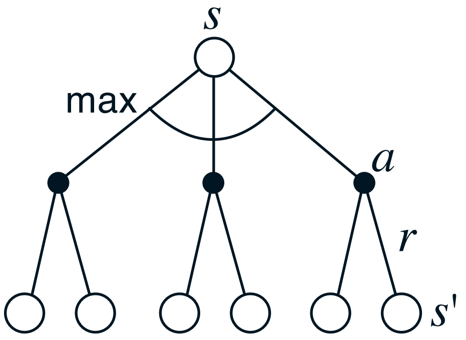
State-value function for equiprobable random policy;
$\gamma = 0.9$

Bellman Optimality Equation for v_*

The value of a state under an optimal policy must equal the expected return for the best action from that state:

$$\begin{aligned}
v_*(s) &= \max_a q_{\pi_*}(s, a) \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s', r} p(s', r|s, a)\big[r + \gamma v_*(s')\big].
\end{aligned}$$

The relevant backup diagram:

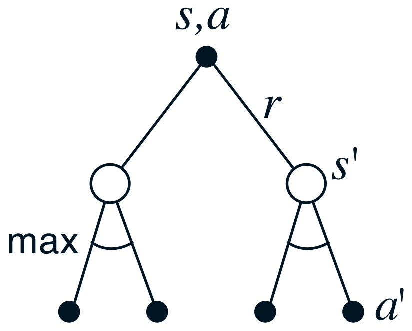


v_* is the unique solution of this system of nonlinear equations.

Bellman Optimality Equation for q_*

$$
\begin{aligned}
q_*(s,a) &= \mathbb{E}\Big[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1},a') \;\Big|\; S_t = s, A_t = a\Big] \\
&= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma \max_{a'} q_*(s',a')\Big].
\end{aligned}
$$

The relevant backup diagram:



q_* is the unique solution of this system of nonlinear equations.

Solving the Bellman Optimality Equation

- Finding an optimal policy by solving the Bellman Optimality Equation requires the following:
  - accurate knowledge of environment dynamics;
  - we have enough space and time to do the computation;
  - the Markov Property.
- How much space and time do we need?
  - polynomial in number of states (via dynamic programming methods; Chapter 4),
  - BUT, number of states is often huge (e.g., backgammon has about 10^{20} states).
- We usually have to settle for approximations.
- Many RL methods can be understood as approximately solving the Bellman Optimality Equation.

Summary

- Agent-environment interaction
  - States
  - Actions
  - Rewards
- Policy: stochastic rule for selecting actions
- Return: the function of future rewards agent tries to maximize
- Episodic and continuing tasks
- Markov Property
- Markov Decision Process
  - Transition probabilities
  - Expected rewards
- Value functions
  - State-value function for a policy
  - Action-value function for a policy
  - Optimal state-value function
  - Optimal action-value function
- Optimal value functions
- Optimal policies
- Bellman Equations
- The need for approximation