# Mathematical Problem Solving Final Portfolio

*Math 101, Fall 2022*

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Diversity summary

In short, I would say that my solutions are well diversified. In almost every category there is at least 1 to 2 solutions that involve the strategy, tactic, technique, or concept. It is clear that many solutions involved certain categories more, like Diophantine equations–which of course will– and generalization. I had a lot of fun using factoring and sequences in this class so the proofs are also skewed in that direction too.

# IC Solutions which obtained a $C^3$

HW1 IC 9/28 10. Get your hands dirty, recurrence relations, Diophantine equations
Find infinitely many solutions in natural numbers to the equation

$$x^2 + y^2 + z^2 = 3xyz$$

By setting y and z equal to 1, we are able to solve for x.

$$x^2 + 1 + 1 = 3x$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$$x = 2, 1$$

Interestingly, we can take the solution here where x = 2 and use that to set the next y and z values, 1 and 2.

$$x^2 + 4 + 1 = 6x$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

$$x = 5, 1$$

We are able to follow this pattern of setting the next y and z values as the solutions of previous x values. As we follow this pattern, solutions will turn out as 1, 5, 13, 34, 89, and every other Fibonacci number. Infinite solutions of $x^2 + y^2 + z^2 = 3xyz$ will be 1, 1, F(2n+1) where F gives you a number in the Fibonacci series given an index.

HW1 IC 9/30 18. Factor tactic, arithmetic and geometric sequences and series, inequalities, primes and divisibility
Prove that $6|7^n - 1$ for $n \in \mathbf{N}$
$\underline{Pf.}$ Base case: n = 1

$$6|7^1 - 1$$

$$= 6|6$$

Inductive hypothesis: Suppose that $6|7^n - 1$ for some n. Let us consider n+1.

$$6|7^{n+1} - 1$$

$$= 6|7 * (7^n) - 1$$

$$= 6|7 * (7^n) - 7 + 6$$

$$= 6|7 * (7^n - 1) + 6$$

We know $7^n - 1$ is divisible by 6 by the inductive hypothesis. Any multiple of this will be divisible by 6, and a multiple of six added to this will be divisible by six. Thus $6|7^n - 1$ for all $n \in \mathbf{N}\Box$

HW3 IC Problem 10/10 41. Generalization, Invariance principle

No, given 7 quarters all initially heads, flipping any four at a time, it is impossible to get them all to be tails. Let us consider this from a numerical standpoint, where h = 7 and tails = 0. There are are a set number of operations. We can flip 4 of the same coin and take 4 from one to the other. Next we can flip 3 of one side and 1 of the other, like $h = 3, t = 4 \rightarrow h = 1, t = 6$. And technically flipping 2 of either one, which does nothing. All of these operations can only change h or t by an even number. However, given that we need t to be 7, this will be impossible, as you cannot add even numbers together to get an odd number.

HW3 IC Problem 10/14 53. Specialization, factor tactic, arithhmetic and geometric sequences and series, geometry

Suppose we have an equilateral triangle with lattice points (0,0), (a,b), and (c,d), where a,b,c,d $\in \mathbb{Z}$.

$$a^2 + b^2 = c^2 + d^2 = (a-c)^2 + (b-d)^2 = 2ac + 2bd$$

Using Fermat's identity, consider

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

Let $S = a^2 + b^2 = c^2 + d^2 = 2(ac + bd)$.

ac + bd = $\frac{S}{2}$

$$S^2 = \frac{S^2}{4} + (ad - bc)^2$$

$$4 * S^2 = S^2 + 4 * (ad - bc)^2$$

$$3 * S^2 = 4 * (ad - bc)^2$$

This contradicts the statement that all points are lattice points however. $4 * (ad - bc)^2$ is a perfect square, however, $3 * S^2$ cannot be a perfect square. Thus no equilateral triangle exists on lattice points.$\square$

HW4 IC 10/17 58. Wishful thinking, arithmetic and geometric sequences and series, polynomials, Pascal's triangle and the binomial theorem

Consider the binomial expansion $(1 + x)^n$. Doing this, we get a sequence $x^n + \binom{n}{1}x^{n-1} + ... + 1$. If we take the derivative, we get $nx^{n-1} + \binom{n}{1} * nx^{n-2} + ...$ which is the original sum from the beginning. Meaning, by setting x equal to 1, we can get the sum of the coefficients, which would the the derivative of $(1 + x)^n$, $n(1 + x)^{n-1}$, and the formula of the sum would be $n2^{n-1}$.

HW6 10/31 IC 101. Generalization, counting in two different ways, generating functions, Diophantine equations

Given $a, b, c, d \in \mathbb{N} > 0$, a + b + c + d = 12, and 2 numbers a and b are always odd, find how many solutions there are.

For this we can create a simple generating function. We will set it up as so.

$$(x^1 + x^3 + x^5 + x^7 + x^9)^2 * (x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)^2$$

Here, we are considering all four variables' choices. a and b are represented by first parentheses, as they can only be odd numbers up to 9, and c and d are represented by the second. We will simplify.

$$= (x^2 + 2x^4 + 3x^6 + 4x^8 + 5x^{10}) * (x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 7x^8 + 8x^9 + 9x^{10})$$

Despite it being possible for powers to go up past 10, they are not relevant in the calculation, as in the end result we really only care about the coefficient of $x^{12}$. So, each side has been truncated where in actuality there would have been terms such as $x^{18}$ or $4x^{12}$ on the left side. Simplifying further, and only paying attention to the ways $x^{12}$ can be formed, we get:

$$9x^{12} + 14x^{12} + 15x^{12} + 12x^{12} + 5x^{12} = 55x^{12}$$

So there are 55 solutions to the given equation.

HW6 10/31 IC 103 Not sure which categories this proof fits under.
We will show that this equation is true using induction.
Base case: n = 1, r = 1

$$\binom{1}{0} + \binom{2}{1} = \binom{3}{1}$$
$$1 + 2 = 3\checkmark$$

Suppose this equation holds true up to n-1, r-1

$$\binom{n}{0} + \binom{n+1}{1} + ... + \binom{n+r-1}{r-1} + \binom{n+r}{r}$$
$$= \binom{n+r}{r-1} + \binom{n+r}{r}$$
$$= \frac{(n+r)!}{(r-1)!(n+1)!} + \frac{(n+r)!}{r!n!}$$
$$= \frac{r(n+r)!}{r!(n+1)!} + \frac{(n+1)(n+r)!}{r!(n+1)!}$$
$$= \frac{(n+r)!(n+r+1)}{r!(n+1)!}$$
$$= \frac{(n+r+1)!}{r!(n+1)!}$$
$$= \binom{n+r+1}{r}$$

□

HW7 11/9 IC 134. Formulate intermediate goals, factor tactic, primes and divisibility, Diophantine equations

Considering that n must be even, we can say n = 2k for some k. Let us substitute this into the expression.

$$
\begin{aligned}
&20^n + 16^n - 3^n - 1\\
&= 20^{2k} + 16^{2k} - 3^{2k} - 1\\
&= 400^k + 256^k - 9^k - 1
\end{aligned}
$$

In order to show that 323 divides this expression, what we really need to do is show that its constituents, 17 and 19, divide it, which is what we will do.

$$
\begin{gathered}
(400-9)|(400^k - 9^k)\\
17|(400-9)\\
17|(400^k-9^k)\\
(256-1)|(256^k-1)\\
17|(256-1)\\
17|256^k-1
\end{gathered}
$$

In this way we show that 17 divides through the whole expression. In a similar manner we can show that this is true for 19 as well.

$$
\begin{gathered}
19|(400-1)|(400^k-1)\\
19|(256-9)|(256^k-9^k)
\end{gathered}
$$

Since both 17 and 19 divide the entire expression properly, this would imply that 323 divides the whole expression as well. □

HW9 IC 11/14 137. Extremal principle, factor tactic, polynomials, Diophantine equations

Find all positive integers n such that $2^4+2^7+2^n$ is a perfect square. In other words, $144+2^n$ is equal to some square number $k^2, k \in \mathbb{N}$.

$$
\begin{gathered}
2^4+2^7+2^n = k^2\\
144+2^n = k^2\\
144-k^2 = -2^n\\
k^2-144 = 2^n\\
(k-12)(k+12) = 2^n
\end{gathered}
$$

As we manipulate this equation, we can find that $(k-12)(k+12) = 2^n$. What this equation essentially says is that we can find solutions to n if k-12 and k+12 are simultaneously powers of 2. We find our first solution for n at n=8, or k=20. However, in order to find further solutions for n, we will need to find powers of two that are a distance of exactly 24 apart. This will never be possible for any other value of k other than 20.

As we consider powers of two beyond (k-12)=8, the distance between successive powers will grow greater and greater and thus it'd be impossible to find another solution for n. Thus the sole solution to the expression being a perfect square will be n=8. □

# OC Solutions which obtained a $C^3$

HW3 OC Problem 10/12 35. Generalization, invariance principle, graph theory

Some student must have miscounted their number of friends. Let us interpret one of the statements. "Three students said 4 people are their friends." In this sentence, what this means graphically is that there is a one way connection from this student to another. However, what it means to truly be friends is for there to be a two-way connection. In other words, if everyone in the class correctly counted, every connection will be two-way, and thus there should be an even amount of connections in the class. However, counting up these connections, it sadly added up to 181, meaning at least 1 student needs to reevaluate their situation.

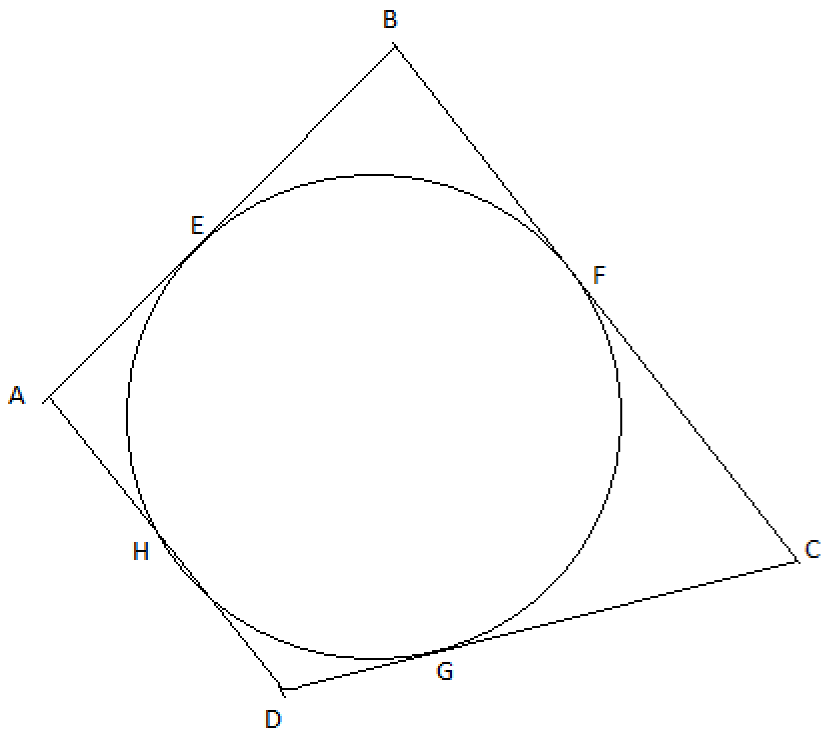
HW3 OC Problem 10/12 28. Graph theory

No, the new chess piece, the camel, cannot get to a square adjacent to it. Let us consider the coordinates (0,0). The camel can be adjacent if it can get to (1,0), (0,1), (-1,0), or (0,-1). When the camel moves, both coordinates are simultaneously even or odd, as you can only add or subtract odd numbers to both on each move. However, all adjacent coordinates have an odd and even number. Thus the camel will never be able to reach an adjacent square to (0,0). $\square$

HW9 OC 11/21 102. Find and exploit symmetries, congruence, geometry

Let A be the point, B and C the points where the tangent lines intersect the circle, and O the center of the circle. Two triangles form, $\triangle AOC$ and $\triangle AOB$. Both triangles share AO, have right angles, and $CO \cong BO$ since those are radii of the circle. Therefore by the Hypotenuse-Leg theorem, $\triangle AOB \cong \triangle AOC$, and $AC \cong AB$, what we really want.

Suppose we have a quadrilateral with an inscribed circle. Let us call the quadrilateral $\square ABCD$. There are intersections E, F, G, H between any two consecutive points in $\square ABCD$ where the quadrilateral intersects the circle. We will say E is between AB, F between BC, G between CD, and H between AD.




We can show that $AE \cong AH, EB \cong BF, FC \cong GC$, and $GD \cong DH$.

$$AE + EB + CG + GD = AH + HD + BF + FC$$

$$\implies AB + CD = AD + BC$$

# Putnam Solutions that received $C^3$

I received no $C^3$ Putnam solutions unfortunately.

# IC Solutions being resubmitted

HW6 10/31 IC 102. Generalization, relax conditions, counting in two different ways

Given an 8x8 chess board, we can select a square. We cannot select a square on the same row or column, so we remain with essentially a 7x7 grid. There are 49 choices per square to choose as a second option.

Supposing we chose the second square first, it is possible to choose the first square afterwards, meaning that for every way to choose two squares, there are two ways to do it. Meaning there are 64*49/2 ways to choose 2 squares on the board, or 32*49 ways.

HW 7 11/7 IC 123. Congruence modulo

Prove $a^2+(a+1)^2+(a+2)^2+(a+3)^2+(a+4)^2$ isn't a square number.

What I will do to prove this is simplify this expression and show that it cannot be square due to how it simplifies at a certain point. Suppose that such an expression is equal to a square number.

$$\begin{aligned} &a^2+(a+1)^2+(a+2)^2+(a+3)^2+(a+4)^2 \\ &= a^2+a^2+2a+1+a^2+4a+4+a^2+6a+9+a^2+8a+16 \\ &= 5a^2+20a+30 \\ &= 5(a^2+4a+6) \end{aligned}$$

Here, we can use modular arithmetic to show that $a^2+4a+6$ won't be divisible by 5. Let us consider first all the residues of $a^2 mod 5$. When we consider these from 0, 1, 2, 3, and 4, we find that the residues will equal 0, 1, 4, 4, 1 respectively. Furthermore, integers beyond 4 to be plugged into a are irrelevant, since they will be equal to their input mod 5. Let's observe the other residues for 4a and 6.

$$\begin{aligned} inputs &: 0,1,2,3,4 \\ a^2 \mod 5 &= 0,1,4,4,1 \\ 4a \mod 5 &= 0,4,3,2,1 \\ 6 \mod 5 &= 1,1,1,1,1 \end{aligned}$$

As we know, adding the inputs, calculating mod, and finding the output mod 5 will be equal to adding the outputs afterwards and modding them. I.e,

$$\begin{aligned} x \mod 5 &= a \\ y \mod 5 &= b \\ x+y \mod 5 &= (a+b) \mod 5 \end{aligned}$$

In other words, we can add our results for $a^2$, 4a, and 6 to find all possible residue classes mod 5.

$$a^2+4a+6 \mod 5 = 1,1,3,2,3$$

As we can see, none of these residue classes contain 0, meaning $a^2+4a+6$ will never be divisible by 5. QED

HW5 IC 10/28 95. Pascal's triangle and the binomial theorem, get your hands dirty

Let us consider this sequence not as the literal fractions first, but simply as $a_1, a_2, a_3, ....$ Doing so, we can see a better relation with what happens when we take repeated averages. Below will be some rows of the original sequence and subsequent repeated averages of the sequence and sequences.

$$a_1|a_2|a_3|a_4|a_5$$
$$\frac{a_1+a_2}{2}|\frac{a_2+a_3}{2}|\frac{a_3+a_4}{2}|\frac{a_4+a_5}{2}$$
$$\frac{a_1+2a_2+a_3}{4}|\frac{a_2+2a_3+a_4}{4}|\frac{a_3+2a_4+a_5}{4}$$
$$\frac{a_1+3a_2+3a_3+a_4}{8}|\frac{a_2+3a_3+3a_4+a_5}{8}$$
$$\frac{a_1+4a_2+6a_3+4a_4+a_5}{16}$$

In this case, we took the repeated average of the sequence $\frac{1}{n}$ up to n = 5. As one can see, the numerator of the result looks precisely like a binomial expansion, divided by 2 to the power of how many repeated averages were taken, in this case 4. For my conjecture, I believe the general equation for sequence up to n will be:

$$\frac{1}{2^{n-1}}\sum_{k=1}^{n}(\binom{n-1}{k-1}*\frac{1}{k})$$
$$=\frac{1}{2^{n-1}}\sum_{k=1}^{n}\binom{n-1}{k}$$
$$=\frac{1}{2^{n-1}}*\frac{1}{n}\sum_{k=1}^{n}\binom{n-1}{k}*n$$
$$=\frac{1}{2^{n-1}}*\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}$$
$$=\frac{1}{2^{n-1}}*\frac{1}{n}(\sum_{k=0}^{n}\binom{n}{k}-1)$$
$$=\frac{1}{2^{n-1}}*\frac{1}{n}(\sum_{k=0}^{n}\binom{n}{k}-1)$$
$$=\frac{1}{2^{n-1}}*\frac{1}{n}(2^n-1)$$
$$\frac{2}{n}-\frac{1}{2^{n-1}n}<\frac{2}{n}$$

$\square$

# OC Solutions being resubmitted

HW5 OC 10/24 65. Relax conditions, polynomials, primes and divisibility

In order to find the remainder of $f(x) = x^{4016} - 2x^{2009}$, we can write a generalized formula that will represent the remainder. When dividing f(x) by $x^2 - 1$, one can assume that there will be a quotient and a remainder. The quotient can be represented as g(x). The remainder must be in the form of Ax + B, a first degree polynomial, as the divisor was of second degree so it would not make sense for the remainder to carry a second degree term. Thus we write the equation as so:

$$x^{4016} - 2x^{2009} = (x^2 - 1)g(x) + Ax + B$$

First we evaluate f(x) at 0 to find B.

f(0) = 0 = B

Knowing B, we can evaluate f(x) at x=1 to solve for A, replacing B in the previous equation with 0.

f(1) = -1 = A

Thus our remainder, Ax+B is -1*x+0, or -x.

HW2 IC 10/7 31. Pigeonhole principle, relax conditions, formulate intermediate goals

Prove that, for an infinite plane where every lattice point can be either red, green, or blue, it is possible to find a rectangle will all vertices the same color.

Pf. Consider 4 points in a row. Given that there are 3 colors to choose from, by the pigeonhole principle at least 2 points must be the same color. It is possible to create 81 different permutations in which at least 2 colors will always be the same in a row of 4 points. If we consider 82 rows, that would mean at least 2 rows will be identical, with 2 of the same color by the pigeonhole principle.

Now we have two identical rows that each have 2 repeating colors. Let us suppose the four points on these rows $R_x$ and $R_y$ are RRGB, without loss of generality. Furthermore let's choose points to make this convenient. Suppose that the first row starts at the origin and the points go out. Each of the four points for some row $R_x$ would be the points (0, x), (1, x), (2, x), and (3, x). That would mean for row $R_x$, (0, x) and (1, x) are red. Likewise for $R_y$ (0, y) and (1, y) are red.

Set A = (0, x), B = (1, x), C = (0, y), D = (1, y)

Slope of AB = $\frac{x-x}{1-0} = 0$, Slope of CD = $\frac{y-y}{1-0} = 0$

So AB $\parallel$ CD

Slope of AC = $\frac{y-x}{0-0} = \infty$, Slope of BD = $\frac{y-x}{0-0} = \infty$

In other words, AC and BD are completely vertical lines, and AC $\parallel$ BD. Likewise AB and CD are completely horizontal lines that are parallel. Thus $AB \perp BC \perp CD \perp AD$ $\Box$

HW6 11/4 OC 79. Generalization, get your hands dirty, formulate intermediate goals, partitions and bijections, principle of inclusion-exclusion

Given 8 cars in a parking lot, we want to see the chance of a car with width two being able to park. We could solve this using unfriendly subsets of cardinality 4. The amount of unfriendly subsets of cardinality 4 of the set $\{1, 2, 3, ..., 12\}$ will be all the ways a sports vehicle can't park in the parking lot. Subtract that from all the different ways we can have 8 cars parked in the parking lot and we have the amount of ways a sports vehicle can park in the lot.

To calculate a few, consider a first unfriendly subset (1,3,5,7). We can increment the last element by 1 5 times to get 5 more unfriendly subsets. We can then increment the third element by 1, set the fourth equal to the third plus one to get (1,3,6,8). We can increment the fourth element again to get more. Similarly, once the third element reaches 10, we can then increment the second element by 1, set the third equal to the second element + 2, and the fourth equal to the third element + 2 and follow the same algorithm again. Following this procedure over and over again, we can find that there are 21+15+10+6+3+1 unfriendly sets, or 56. $\binom{12}{4=495}$. $\frac{495-56}{495} = \frac{439}{495}$, the chance that a sports vehicle can park in the parking lot.

# Putnam Solutions being resubmitted

HW9 IC 11/23 PP8. Relax conditions, formulate intermediate goals, primes and divisibility

Show that any rational number can be rewritten as the fraction of the factorials of prime numbers.

If we actually consider what the question is really saying, proving this would be the same thing as saying that every prime number can also be rewritten as the quotient of prime factorials, we would just need to multiply them together to achieve the original $\frac{a}{b}$.

Proposition: All prime numbers can be rewritten as the quotient of factorials of prime numbers.

Base step: p = 2

2 = $\frac{2!3!}{3!}$

Inductive hypothesis: Suppose that for this proposition will hold for all prime numbers less than p, where p > 2. Consider the following:

p = $\frac{p!}{(p-1)!}$

p! is already the factorial of a prime number since that's how we set it up. Furthermore, any factor of (p-1)! will be less than p as the greatest possible prime factor would be p-1 hypothetically. Thus the inductive hypothesis holds. Therefore arbitrary p can also be written as the quotient of factorials of primes.

Thus the original claim is true. □

# New IC Solutions

# New OC Solutions

# New Putnam Solutions

12/2 PP X Generalization, arithmetic

I wasn't sure what the exact number of this question was since the problems were never posted for December second. However, I do remember the exact question.

The problem statement involved a basketball player who averaged a 60% free throw accuracy in the first half of the season. In the second half of the season, said basketball player reached an 80% free throw accuracy. Did the basketball player have to have exactly a 70% accuracy at any point? The question was some variant of this.

The answer is no. This question is trying to relate the Intermediate Value Theorem to discrete points. However, the IVT requires the function to be continuous. The function relating the running average is not continuous. A possible point in which the average could cross over 70% could be as so. Say the basketball player hit 69 free throws out of 100. They can then proceed to successfully get the next 4 free throws in a row, and they will never have gotten exactly 70% accuracy. □

11/28 PP 15. Find a penultimate step, formulate intermediate goals, congruence, recurrence relations

For this problem, when I refer to "numbers" or "all numbers", I am talking about numbers in the natural numbers.

Let us observe a fact. Given a number n, n+5 will also be in the set. We can see this because if n is in the set, $(n+5)^2$ is in the set. Then because $(n+5)^2$ is in the set, (n+5) is in the set. Meaning all numbers in the congruence class 2 mod 5 are in S.

Let's use this result to get other elements in the set. Next, we can get $49 \in S$ with $(2+5)^2$. We also know 64 is in S since 64 = 49 + 5 + 5 + 5. Since 64 is in S, 16 is in S, 8 is in S, and 4 is in S, since the square root of a number is in S.

Since 4 is in S, 9 is in S, and because the square root is in S, 3 is in S as well. So far, we currently have these congruence classes in S: 2 mod 5, 3 mod 5, 4 mod 5. We will not be able to get 1 mod 5 but we can get 1 mod 5 / {1}, or the congruence class 1 mod 5 not including 1. $4^2 = 16, 36 = 16 + 5 + 5 + 5 + 5$, $6 = \sqrt{36}$.

So, the set S will be composed of all numbers in modulo classes 2 mod 5, 3 mod 5, 4 mod 5, and 1 mod 5, not including 1. There will be no numbers that are multiples of 5 in S. It is impossible to square any number from the other 4 modulo classes and end up with a number in the modulo class 0 mod 5. 1*1 = 1, 2*2 = 4, 3*3 = 4, 4*4 = 1. QED