1 Семинарский лист 2

Задача 1.1.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}},\ a_n = \frac{1}{n\sqrt[n]{n}}$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 : \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^0 = 1 \implies a_n \sim \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ сходится} \implies \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}} \text{ по признаку сравнения.}$$

Задача 1.2.

$$\sum_{n=2}^{\infty} \frac{2\cdot 5\cdot 8\cdot \ldots \cdot (3n-4)}{3^n \cdot n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(3(n+1)-4)\cdot 3^n \cdot n!}{3^{n+1}\cdot (n+1)!} = \frac{3(n+1)-4}{3(n+1)} = \frac{3n-1}{3n+3} = \frac{1-\frac{\frac{1}{3}}{n}}{1+\frac{1}{n}} = \left(1-\frac{1}{3n}\right)\left(1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right) =$$

$$= 1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)-\frac{\frac{1}{3}}{n}+\frac{\frac{1}{3}}{n^2}-O\left(\frac{1}{3n^3}\right) = 1-\frac{\frac{4}{3}}{n}+O\left(\frac{1}{n^2}\right) \implies \begin{cases} p = \dfrac{4}{3} \\ \delta = 1 \end{cases} \implies \text{ряд сходится по признаку Гаусса.}$$

Задача 1.3.

$$\sum_{n=2}^{\infty} \frac{1}{n\ln n},\ f(x) = \frac{1}{x\ln x}$$

$$f'(x) = -\frac{\ln x + 1}{x^2\ln^2 x} = 0 \iff \begin{cases} x \neq 0 \\ x \neq 1 \\ x = \dfrac{1}{e} \end{cases} \implies f(x) \text{ монотонно убывает при } x > 1$$

$$f(n+t) \leqslant a_n \leqslant f((n-1)+t)$$

$$\int_n^{n+1} f(x)dx \leqslant a_n \leqslant \int_{n-1}^{n} f(x)dx$$

$$\int_2^{N+1} f(x)dx \leqslant \sum_{n=2}^{N} a_n \leqslant \int_1^N f(x)dx$$

$$\int \frac{1}{x\ln x}dx = \begin{bmatrix} t = \ln x \\ e^t = x \\ dx = e^t dt \end{bmatrix} = \int \frac{e^t}{t\cdot e^t}dt = \int \frac{1}{t}dt = \ln t + C = \ln\ln x + C$$

$$\int_2^{N+1} f(x)dx = \ln\ln(N+1) - \ln\ln 2$$

$$\int_1^N f(x)dx = \ln\ln N - 0 = \ln\ln N$$

Ответ: $\ln\ln(N+1) - \ln\ln 2 \leqslant \sum_2^N \frac{1}{n\ln n} \leqslant \ln\ln N$

Задача 1.4.

$$\sum_{n=1}^{N} \frac{\ln n}{n^2} = S - \frac{\ln N}{N} + o\left(\frac{\ln N}{N}\right)$$

$$q_n = S - \sum_{n=1}^{N} \frac{\ln n}{n^2}$$

$$p_n = \frac{\ln N}{N}$$

По теореме Штольца: $\lim\limits_{nto\infty} \frac{q_n}{p_n} = \lim\limits_{nto\infty} \frac{q_{n+1}-q_n}{p_{n+1}-p_n} = \lim\limits_{n\to\infty} \frac{S-S_{n+1}-S+S_n}{\frac{\ln(N+1)}{N+1}-\frac{\ln N}{N}} = \frac{-\frac{\ln(N+1)}{(N+1)^2}\cdot(N(N+1))}{N\ln(N+1)-(N+1)\ln N} =$

$$= \frac{N\ln(N+1)}{(N+1)\ln N - N\ln(N+1)} = \frac{\frac{N\ln(N+1)}{N+1}}{\ln N + N\ln\frac{N}{N+1}} = [\text{По правилу Лопиталя}] = \frac{\ln(N+1)-\frac{N}{N+1}}{}$$